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Haezendonck-Goovaerts risk measures and Orlicz quantiles

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Abstract

In this paper, we will study the well-known Haezendonck-Goovaerts risk measures on their natural domain, that is on Orlicz spaces and, in particular, on Orlicz hearts. We will provide a dual representation as well as the optimal scenario in such a representation and investigate the properties of the minimizer x_α^* (that we will call Orlicz quantile) in the definition of the Haezendonck-Goovaerts risk measure. Since Orlicz quantiles fail to satisfy an internality property, bilateral Orlicz quantiles are also introduced and analyzed.

Keywords: Orlicz premiums; Haezendonck risk measures; Conditional Value at Risk; quantiles; expectiles.

JEL Classification: G22

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1 Introduction

There has been recently an increasing interest in Haezendonck risk measures, both from the mathematical and from the statistical point of view. As it was suggested in the IME conference in Trieste, we adopt the terminology of 'Haezendonck-Goovaerts' risk measures, in order to better acknowledge the contribution of both authors in their seminal paper (Haezendonck and Goovaerts, 1982).

From a mathematical point of view, Haezendonck-Goovaerts risk measures are interesting because they are the simplest example of coherent risk measures

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that are not comonotonically additive; equivalently, they cannot in general be expressed as distortion risk measures (see Goovaerts et al., 2010, for further interesting discussions on the relationships between Haezendonck-Goovaerts and distortion risk measures). Moreover, as we will discuss in details in the following, Haezendonck-Goovaerts risk measures are naturally defined on Orlicz spaces, that are becoming more and more popular in the field of risk measures because of a well established duality theory that generalize the standard L^p theory (see for example Cheridito and Li, 2009, and the references therein).

It is well known that the simplest case of Haezendonck-Goovaerts risk measure is the Conditional Value at Risk (Rockafellar et al., 2000), corresponding to the Young function $\Phi(x) = x$; other special cases are also studied explicitly, e.g. the power case $\Phi(x) = x^p$ in Dentcheva et al. (2010).

This class of risk measures is probably the most well-studied non-comonotone extension of the CVaR.

From the statistical point of view, some very general results about the properties of the plug-in estimator of coherent risk measures can be found in Belomestny and Krätschmer (2010), and the specific applications to the case of Haezendonck-Goovaerts risk measures are discussed in details in Ahn and Shyamalkumar (2011b) and Krätschmer and Zahle (2011).

The first aim of this work is to systematically apply the results of Cheridito and Li (2009) to the special case of Haezendonck-Goovaerts risk measures, in order to generalize the dual representation that was given in the L^∞ case in Bellini and Rosazza Gianin (2008a). Moreover, it is possible to determine explicitly the 'optimal scenario' $Q_{\alpha,X}$, that is the probability measure (depending on α and on the risk X) such that $\pi_\alpha(X) = E_{Q_{\alpha,X}}[X]$. We will also show that the construction of the Haezendonck-Goovaerts risk measure as an inf-convolution of an Orlicz norm and a suitably chosen functional, that we discussed in Bellini and Rosazza Gianin (2008a), still holds when the domain is the Orlicz heart M^Φ .

The second aim is to investigate the first order conditions for the problem that defines the Haezendonck-Goovaerts premium

$$\inf_{x \in \mathbb{R}} \left\{ x + H_\alpha \left((X - x)^+ \right) \right\}.$$

Several expressions have been introduced in the literature (see Goovaerts et al., 2010, Ahn and Shyamalkumar, 2011b, Nam et al., 2011).

In general, the minimizer x^* cannot be characterized by a single equation but only by a couple of inequalities, that are obtained by computing the right and left derivatives with the aid of the Gateaux derivative of the norm in the Orlicz heart (see Kosmol and Müller-Wichards, 2011).

The third aim is to analyse the properties of the minimizer(s) x^* , that we call 'Orlicz quantile'. As it was already pointed out (see for example Dentcheva et al., 2010, or Ahn and Shyamalkumar, 2011a), it does not satisfy an internality property, in the sense that it may happen that $x^* < \text{ess. inf}(X)$.

In order to overcome this difficulty, we propose the alternative definition of 'bilateral Orlicz quantiles', that are defined as the solutions of

$$\inf_{x \in \mathbb{R}} \left\{ \alpha H \left((X - x)^+ \right) + (1 - \alpha) H \left((X - x)^- \right) \right\}$$

. In this case it is easy to show that the internality property is indeed satisfied. It is instructive to compare the bilateral Orlicz quantiles with the M -quantiles of the statistical literature, introduced by Breckling and Chambers (1988), that are the solutions of

$$\inf_{x \in \mathbb{R}} \left\{ \alpha E \left(\Phi (X - x)^+ \right) + (1 - \alpha) E \left(\Phi (X - x)^- \right) \right\}$$

However, in contrast with the usual quantiles, bilateral Orlicz quantiles do not satisfy a monotonicity property with respect to the usual stochastic order, so their interpretation as risk measures is obviously questionable.

The paper is organized as follows. In Section 2, we recall and give some preliminary definitions and results on Orlicz spaces and on Haezendonck-Goovaerts premiums. Section 3 is devoted to the dual representation and to the optimal scenario of Haezendonck-Goovaerts risk measures as well as to their representation as an inf-convolution. In Section 4, the minimization problem and the first order condition are faced, while in Section 5 Orlicz quantiles and bilateral Orlicz quantiles are studied in details.

2 Haezendonck-Goovaerts risk measures on Orlicz spaces

We recall the basic definitions on Orlicz spaces following Edgar and Sucheston (1992) and Cheridito and Li (2008, 2009).

Let $\Phi : [0, +\infty) \rightarrow [0, +\infty)$ be convex with

$$\Phi(0) = 0, \quad \Phi(1) = 1 \quad \text{and} \quad \lim_{x \rightarrow +\infty} \Phi(x) = +\infty.$$

It then follows that Φ is continuous and strictly increasing on $\{\Phi > 0\}$.

The convex conjugate of Φ is defined as

$$\Psi(y) \triangleq \sup_{x \geq 0} \{xy - \Phi(x)\}, \quad y \geq 0.$$

The Orlicz space is

$$L^\Phi \triangleq \left\{ X : E \left[\Phi \left(\frac{|X|}{a} \right) \right] < +\infty \text{ for some } a > 0 \right\},$$

while the Orlicz heart is its subset

$$M^\Phi \triangleq \left\{ X : E \left[\Phi \left(\frac{|X|}{a} \right) \right] < +\infty \text{ for every } a > 0 \right\}.$$

The Luxemburg norm of $X \in L^\Phi$ is given by

$$\|X\|_\Phi \triangleq \inf \left\{ a > 0 : E \left[\Phi \left(\frac{|X|}{a} \right) \right] \leq 1 \right\}$$

and is a Fatou norm that makes L^Φ a Banach lattice with the natural pointwise a.s.-ordering (see Edgar and Sucheston, 1992). Moreover, when $X \in M^\Phi$ the Luxemburg norm solves the equation:

$$E \left[\Phi \left(\frac{|X|}{\|X\|_\Phi} \right) \right] = 1. \quad (1)$$

This is always when Φ satisfies the so-called Δ_2 condition, since in this case $M^\Phi = L^\Phi$.

The Orlicz norm of $X \in L^\Phi$ is defined as

$$\|X\|_\Phi^* \triangleq \sup_{\|Y\|_\Psi \leq 1} E[|XY|],$$

where Ψ is the convex conjugate of Φ . It is well known (see Edgar and Sucheston, 1992, and Rao and Ren, 1991) that the Luxemburg and the Orlicz norm are equivalent and that the normed dual space of $(M^\Phi, \|\cdot\|_\Phi)$ is $(L^\Psi, \|\cdot\|_\Psi^*)$.

In our previous works (Bellini and Rosazza Gianin, 2008a and 2008b), we considered Haezendonck-Goovaerts risk measures with domain L^∞ , in order to ensure the validity of equation (1) for any X and Φ . In the present work, we consider as domain L^Φ or M^Φ (as it was already done in Haezendonck and Goovaerts, 1982). The extension is very natural since M^Φ is the closure of L^∞ in the Luxemburg norm (see again Edgar and Sucheston, 1992).

We recall below some basic definitions.

Definition 1 For any $X \in L_+^\Phi$ and $\alpha \in [0, 1)$, the Orlicz premium is defined as

$$H_\alpha(X) \triangleq \inf \left\{ a > 0 : E \left[\Phi \left(\frac{X}{a} \right) \right] \leq 1 - \alpha \right\} = \|X\|_{\Phi_\alpha},$$

with $\Phi_\alpha \triangleq \frac{\Phi}{1-\alpha}$.

Definition 2 For any $X \in L^\Phi$ and $\alpha \in [0, 1)$, the Haezendonck-Goovaerts premium is defined as

$$\pi_\alpha(X) \triangleq \inf_{x \in \mathbb{R}} \left\{ x + H_\alpha \left((X - x)^+ \right) \right\} \triangleq \inf_{x \in \mathbb{R}} \pi_\alpha(X, x)$$

and the corresponding Haezendonck-Goovaerts risk measure is defined as

$$\rho_\alpha(X) \triangleq \pi_\alpha(-X) = \inf_{x \in \mathbb{R}} \left\{ x + H_\alpha \left((-X - x)^+ \right) \right\}.$$

We summarize in the following Proposition some results that in the L^∞ case have been given in Bellini and Rosazza Gianin (2008a).

Proposition 3 *Let $X \in L^\Phi$, $\alpha \in (0, 1)$ and $x \in \mathbb{R}$.*

(a) $\pi_\alpha(X, x)$ is finite, convex, and satisfies $\lim_{x \rightarrow +\infty} \pi_\alpha(X, x) = +\infty$. Furthermore, $\lim_{x \rightarrow -\infty} \pi_\alpha(X, x) = +\infty$ for $\alpha \neq 0$, while $\lim_{x \rightarrow -\infty} \pi_\alpha(X, x) \geq E[X]$ for $\alpha = 0$;

(b) if $\alpha \neq 0$ the infimum in the definition of $\pi_\alpha(X)$ is always attained;

(c) if Φ is strictly convex and $X \in M^\Phi$, then $\pi_\alpha(X, x)$ is strictly convex for $x < \text{ess sup}(X)$ (or for any $x \in \mathbb{R}$ if X is unbounded);

(d) if $\alpha \neq 0$, Φ is strictly convex and $X \in M^\Phi$, then the minimum in the definition of $\pi_\alpha(X)$ is unique;

(e) the Haezendonck-Goovaerts premium $\pi_\alpha : L^\Phi \rightarrow \mathbb{R}$ is law invariant and coherent.

Proof. (a) If $X \in L^\Phi$ then $(X - x)^+ \in L^\Phi$ and $H_\alpha((X - x)^+) < +\infty$; the convexity is trivial. As in the L^∞ case, it is easy to check that for $X \in L_+^\Phi$

$$H_\alpha(X) \geq \frac{E[X]}{\Phi^{-1}(1-\alpha)}.$$

It follows that $\pi_\alpha(X, x) \geq x + \frac{E[(X-x)^+]}{\Phi^{-1}(1-\alpha)} \geq x + \frac{E[X-x]}{\Phi^{-1}(1-\alpha)} = \frac{E[X]}{\Phi^{-1}(1-\alpha)} + x \left(1 - \frac{1}{\Phi^{-1}(1-\alpha)}\right)$. Hence: $\lim_{x \rightarrow -\infty} \pi_\alpha(X, x) = +\infty$ for $\alpha \neq 0$ (since Φ is normalized); while $\lim_{x \rightarrow -\infty} \pi_\alpha(X, x) \geq E[X]$ for $\alpha = 0$.

When $x \rightarrow +\infty$ we have that $\pi_\alpha(X, x) \geq x + \frac{E[(X-x)^+]}{\Phi^{-1}(1-\alpha)} \rightarrow +\infty$ from the dominated convergence theorem.

(b) From the convexity and (a) we have that the infimum is attained for $\alpha \neq 0$.

(c) If Φ is strictly convex and $X \in M^\Phi$, then H_α is strictly convex (see Kosmol and Müller-Wichards, 2011, for a throughout discussion; the same property is not true in L^Φ without additional hypothesis). It follows that when $x > y$ and $(X - x)^+ \neq 0$

$$\begin{aligned} H_\alpha \left[(X - \lambda x - (1-\lambda)y)^+ \right] &\leq H_\alpha \left[\lambda(X - x)^+ + (1-\lambda)(X - y)^+ \right] \\ &< \lambda H_\alpha((X - x)^+) + (1-\lambda)H_\alpha((X - y)^+), \end{aligned}$$

from which the strict convexity of $\pi_\alpha(X, x)$ follows immediately; (d) is a trivial consequence of (c).

(e) The proof of the coherence and law invariance of π_α on L^Φ can be driven similarly to Bellini and Rosazza Gianin (2008a) in the L^∞ case. ■

3 Dual representations

The Haezendonck-Goovaerts risk measure ρ_α is a particular case of the so-called Transformed Norm Risk measures introduced by Cheridito and Li (2009) as a sub-family of cash-additive hulls (Filipovic and Kupper, 2007). In the

terminology of Cheridito and Li (2009), a Transformed Norm Risk measure on M^Φ has the following form:

$$\inf_{s \in \mathbb{R}} \{F(\|H(s - X)\|_G) - s\},$$

where

- $F : [0, +\infty) \rightarrow (-\infty, +\infty]$ is a left-continuous, increasing and convex function such that $\lim_{x \rightarrow +\infty} F(x) = +\infty$;
- $G : [0, +\infty) \rightarrow [0, +\infty)$ is increasing and convex, with $G(0) = 0$ and $\lim_{x \rightarrow +\infty} G(x) = +\infty$;
- $H : (-\infty, +\infty) \rightarrow [0, +\infty)$ is an increasing and convex function such that $\lim_{x \rightarrow +\infty} H(x) = +\infty$.

The Haezendonck-Goovaerts risk measure corresponds to the case $F(x) = x$, $G = \Phi_\alpha$ and $H(x) = x^+$. Ψ_α will denote the convex conjugate of Φ_α . From the general results of Cheridito and Li (2008, 2009) we obtain the following.

Proposition 4 *Let $\alpha \in (0, 1)$ and $X \in M^\Phi$. Let $\mathcal{D}^\Psi \triangleq \{\varphi \in L_+^\Psi : E[\varphi] = 1\}$.*

(a) *The infimum in the definition of the Haezendonck-Goovaerts premium*

$$\pi_\alpha(X) = \inf_{x \in \mathbb{R}} \left\{ x + \left\| (X - x)^+ \right\|_{\Phi_\alpha} \right\}$$

is attained in some $x^ = x_X^*$.*

(b) *The dual representation of the Haezendonck-Goovaerts risk measure is given by*

$$\rho_\alpha(X) = \max_{Q \in \mathcal{Q}} E_Q[-X] \quad (2)$$

with

$$\mathcal{Q} = \left\{ Q \text{ prob. meas.} : \frac{dQ}{dP} \in \mathcal{D}^\Psi \text{ and } \left\| \frac{dQ}{dP} \right\|_{\Psi_\alpha}^* \leq 1 \right\}.$$

(c) *If Φ is differentiable and if $\left\| (X - x)^+ \right\|_{\Phi_\alpha}$ is differentiable in x_X^* , then*

$$\pi_\alpha(X) = E_{Q_X}[X],$$

where

$$\frac{dQ_X}{dP} = \frac{\Phi' \left(\frac{(X - x^*)^+}{\left\| (X - x^*)^+ \right\|_{\Phi_\alpha}} \right) \mathbf{1}_{\{X > x^*\}}}{E \left[\Phi' \left(\frac{(X - x^*)^+}{\left\| (X - x^*)^+ \right\|_{\Phi_\alpha}} \right) \mathbf{1}_{\{X > x^*\}} \right]}. \quad (3)$$

Therefore,

$$\pi_\alpha(X) = \frac{E \left[X \Phi' \left(\frac{(X-x^*)^+}{\|(X-x^*)^+\|_{\Phi_\alpha}} \right) \mathbf{1}_{\{X > x^*\}} \right]}{E \left[\Phi' \left(\frac{(X-x^*)^+}{\|(X-x^*)^+\|_{\Phi_\alpha}} \right) \mathbf{1}_{\{X > x^*\}} \right]}.$$

(d) ρ_α is Lipschitz-continuous and nonexpansive with respect to the $\|\cdot\|_{\Phi_\alpha}$ -norm, i.e.

$$|\rho_\alpha(X) - \rho_\alpha(Y)| \leq \|X - Y\|_{\Phi_\alpha}$$

Proof. (a) As already pointed out, $\rho_\alpha(X) \triangleq \pi_\alpha(-X)$ is a transformed norm risk measure with $F(x) = x$, $G = \Phi_\alpha$ and $H(x) = x^+$. In order to prove (a) it is therefore sufficient to verify that the hypothesis of Lemma 5.2 of Cheridito and Li (2009) are satisfied. We recall them for completeness:

(FGH1) there exist $x \in \mathbb{R}$ and $\varepsilon > 0$ such that $F\left(\frac{H(x)+\varepsilon}{G^{-1}(1)}\right) < +\infty$;

(FGH2) $\lim_{x \rightarrow +\infty} \{F(H(x)) - xG^{-1}(1)\} = +\infty$.

(FGH1) is clearly satisfied. In our setting, (FGH2) can be rewritten as

$$\lim_{x \rightarrow +\infty} \{x^+ - x\Phi_\alpha^{-1}(1)\} = +\infty.$$

Since $\Phi_\alpha(1) = \frac{1}{1-\alpha} > 1$, we have $\lim_{x \rightarrow +\infty} \{x^+ - \Phi_\alpha^{-1}(1)x\} = +\infty$. The thesis is therefore guaranteed by the aforementioned result of Cheridito and Li (2009).

(b) Since in our setting $F(x) = x$ and $H(x) = x^+$, from Section 5.3.1 of Cheridito and Li (2009) we have that the minimal penalty function in the dual representation (2) is given by $F^*\left(\left\|\frac{dQ}{dP}\right\|_{\Psi_\alpha}^*\right)$, where, in our case, the conjugate of F is the indicator function of $\left\{\left\|\frac{dQ}{dP}\right\|_{\Psi_\alpha}^* \leq 1\right\}$.

(c) By Theorem 5.2 of Cheridito and Li (2009) and by the coherence of ρ_α , it follows that

$$\rho_\alpha(Y) = E_{Q_Y}[-Y], \quad \forall Y \in M^\Phi,$$

where

$$\begin{aligned} \frac{dQ_Y}{dP} &= \eta \cdot h(y^* - Y) \in \mathcal{D}^\Psi \\ \eta &= \frac{g\left(\frac{H(y^*-Y)}{\|H(y^*-Y)\|_G}\right)}{E\left[g\left(\frac{H(y^*-Y)}{\|H(y^*-Y)\|_G}\right)h(y^*-Y)\right]}, \end{aligned}$$

$y^* \in \arg \min_y \left\{ \|(y - Y)^+\|_{\Phi_\alpha} - y \right\}$ and g and h denote the right-derivatives of G and H , respectively.

In our setting: $g = \Phi'_\alpha = \frac{\Phi'}{1-\alpha}$, $h(x) = \mathbf{1}_{\{x>0\}}$, $X = -Y$ and $x^* = -y^*$, hence

$$\begin{aligned}\eta &= \frac{\Phi'_\alpha \left(\frac{(X-x^*)^+}{\|(X-x^*)^+\|_{\Phi_\alpha}} \right)}{E \left[\Phi'_\alpha \left(\frac{(X-x^*)^+}{\|(X-x^*)^+\|_{\Phi_\alpha}} \right) \mathbf{1}_{\{X>x^*\}} \right]} = \frac{\Phi' \left(\frac{(X-x^*)^+}{\|(X-x^*)^+\|_{\Phi_\alpha}} \right)}{E \left[\Phi' \left(\frac{(X-x^*)^+}{\|(X-x^*)^+\|_{\Phi_\alpha}} \right) \mathbf{1}_{\{X>x^*\}} \right]} \\ \frac{dQ_X}{dP} &= \frac{\Phi' \left(\frac{(X-x^*)^+}{\|(X-x^*)^+\|_{\Phi_\alpha}} \right) \mathbf{1}_{\{X>x^*\}}}{E \left[\Phi' \left(\frac{(X-x^*)^+}{\|(X-x^*)^+\|_{\Phi_\alpha}} \right) \mathbf{1}_{\{X>x^*\}} \right]}.\end{aligned}$$

(d) It is an immediate consequence of Corollary 4.7 in Cheridito and Li (2009) and item (b). ■

Part (a) of the thesis is reported for completeness and can be derived directly as in Proposition 3. If $\alpha = 0$ it is not possible to guarantee the existence of a minimum, as shown in Example 15 in Bellini and Rosazza Gianin (2008a). Item (d) implies the lower semi-continuity of ρ_α with respect to the $\|\cdot\|_{\Phi_\alpha}$ -norm. A direct proof of it will be given in the Appendix.

Since the minimum in the definition of π_α is attained (by Proposition 4(a)), it follows from Theorem 4.3 and Corollary 4.2 of Cheridito and Li (2009) that for any $X \in M^\Phi$

$$\pi_\alpha(X) = \max_{Q \in \mathcal{D}^\Psi} \left\{ E_Q[X] - \pi_\alpha^* \left(\frac{dQ}{dP} \right) \right\},$$

where $\pi_\alpha^* : L^\Psi \rightarrow \mathbb{R} \cup \{+\infty\}$ is the convex conjugate of π_α , i.e. $\pi_\alpha^* \left(\frac{dQ}{dP} \right) \triangleq \sup_{X \in M^\Phi} \{E_Q[X] - \pi_\alpha(X)\}$.

In Bellini and Rosazza Gianin (2008a), we showed also that π_α can be expressed as an inf-convolution of two properly chosen functionals on L^∞ ; we state an analogous result on L^Φ .

We recall that the inf-convolution $f \square g$ of two functionals $f, g : L^\Phi \rightarrow \mathbb{R} \cup \{+\infty\}$ is defined as

$$(f \square g)(X) \triangleq \inf_{Y \in L^\Phi} \{f(X - Y) + g(Y)\}, \quad X \in L^\Phi.$$

(see Rockafellar, 1970, or Barrieu and El Karoui, 2005, among many others).

Proposition 5 *Let $\pi_\alpha : M^\Phi \rightarrow \mathbb{R}$ be the Haezendonck-Goovaerts premium with $\alpha \in (0, 1)$.*

Then $\pi_\alpha = f \square g$, where

$$f(X) \triangleq \|X^+\|_{\Phi_\alpha} = H_\alpha(X^+) \quad ; \quad g(X) \triangleq \begin{cases} x; & \text{if } X = x, \text{ } P\text{-a.s.} \\ +\infty; & \text{otherwise} \end{cases},$$

and

$$\pi_\alpha^* \left(\frac{dQ}{dP} \right) = f^* \left(\frac{dQ}{dP} \right) + g^* \left(\frac{dQ}{dP} \right) = \begin{cases} 0; & \text{if } E_Q[W] \leq \|W\|_{\Phi_\alpha} \text{ for any } W \in M_+^\Phi \\ +\infty; & \text{otherwise} \end{cases} \quad (4)$$

The first statement holds also on L^Φ .

Proof. $\pi_\alpha = f \square g$ is straightforward.

By proceeding as in Rockafellar (1970), $\pi_\alpha^*(Y) = f^*(Y) + g^*(Y)$ for any $Y \in L^\Psi$. On one hand, as in Bellini and Rosazza Gianin (2008a) it is easy to check that

$$g^*(Y) = \begin{cases} 0; & \text{if } E[Y] = 1 \\ +\infty; & \text{otherwise} \end{cases}.$$

On the other hand,

$$f^*(Y) = \sup_{X \in M^\Phi} \{E[XY] - f(X)\} = \sup_{X \in M^\Phi} \{E[XY] - \|X^+\|_{\Phi_\alpha}\}$$

and, since $X \in M^\Phi$ implies that $\lambda X \in M^\Phi$ for any $\lambda \geq 0$,

$$\begin{aligned} f^*(Y) &= \sup_{\lambda \geq 0} \sup_{\lambda X \in M^\Phi} \left\{ E[\lambda XY] - \|(\lambda X)^+\|_{\Phi_\alpha} \right\} \\ &= \sup_{\lambda \geq 0} \left\{ \lambda \sup_{X \in M^\Phi} \left\{ E[XY] - \|X^+\|_{\Phi_\alpha} \right\} \right\} \\ &= \begin{cases} 0; & \text{if } E[XY] \leq \|X^+\|_{\Phi_\alpha} \text{ for any } X \in M^\Phi \\ +\infty; & \text{otherwise} \end{cases}, \end{aligned} \quad (5)$$

where (5) is due to positive homogeneity of the Luxemburg norm.

For any $Q \in \mathcal{D}^\Psi$ the following are equivalent: (i) $E_Q[X] \leq \|X^+\|_{\Phi_\alpha}$ for any $X \in M^\Phi$; (ii) $E_Q[X] \leq \|X\|_{\Phi_\alpha}$ for any $X \in M_+^\Phi$.

(i) $\Rightarrow$ (ii) is trivial. (ii) $\Rightarrow$ (i): take any $X \in M^\Phi$. Then $E_Q[X] = E_Q[X^+] - E_Q[X^-] \leq E_Q[X^+] \stackrel{(ii)}{\leq} \|X^+\|_{\Phi_\alpha}$, hence the thesis.

Hence, (4) follows from the arguments above. ■

4 First order conditions

In this section we derive first order conditions for the minimizer(s) $x^* = x_{\alpha, X}^*$ in the definition of the Haezendonck-Goovaerts premium

$$\pi_\alpha(X) = \inf_{x \in \mathbb{R}} \left\{ x + H_\alpha \left((X - x)^+ \right) \right\}. \quad (6)$$

From our preceding results, we know that if $\alpha \in (0, 1)$ the infimum is always attained, and if Φ is strictly convex and $X \in M^\Phi$ then the minimizer is unique. In the case $\Phi(x) = x$ the set of minimizers coincides with set of the usual α -quantiles (see for example Rockafellar and Uryasev, 2000).

Proposition 6 *Let Φ be differentiable, $\alpha \in (0, 1)$ and $X \in M^\Phi$. Then any minimizer $x^* = x_{\alpha, X}^*$ in (6) such that $x^* \neq \text{ess. sup}(X)$ satisfies the following conditions:*

$$\begin{cases} E \left[\left(\frac{(X-x^*)^+}{\|(X-x^*)^+\|_{\Phi_\alpha}} - \mathbf{1}_{\{X \geq x^*\}} \right) \Phi'_\alpha \left(\frac{(X-x^*)^+}{\|(X-x^*)^+\|_{\Phi_\alpha}} \right) \right] \leq 0 \\ E \left[\left(\frac{(X-x^*)^+}{\|(X-x^*)^+\|_{\Phi_\alpha}} - \mathbf{1}_{\{X > x^*\}} \right) \Phi'_\alpha \left(\frac{(X-x^*)^+}{\|(X-x^*)^+\|_{\Phi_\alpha}} \right) \right] \geq 0 \end{cases} \quad (7)$$

Proof. By Proposition 4(a), we know that the infimum in (6) is attained. From the convexity of $\pi_\alpha(X, x)$, any minimizer x^* of (6) has to satisfy the following condition:

$$\frac{\partial^- \pi_\alpha}{\partial x}(X, x^*) \leq 0 \leq \frac{\partial^+ \pi_\alpha}{\partial x}(X, x^*).$$

It remains therefore to compute $\frac{\partial^- \pi_\alpha}{\partial x}(X, \cdot)$ and $\frac{\partial^+ \pi_\alpha}{\partial x}(X, \cdot)$.

Set $g(x) \triangleq (X - x)^+$. Take now $h > 0$. Set

$$G_h^X(x) \triangleq \frac{g(x+h) - g(x)}{h} = \begin{cases} 0; & X \leq x \\ \frac{x-X}{h} \in (-1, 0); & x < X < x+h \\ -1; & X \geq x+h \end{cases}.$$

We have therefore that $\lim_{h \rightarrow 0^+} G_h^X(x) = -\mathbf{1}_{\{X > x\}} \triangleq G^X(x)$ pointwise. By Proposition 2.1.10 of Edgar and Sucheston (1992), $G_h^X(x) \rightarrow G^X(x)$ in L^Φ if and only if $E[\Phi(k|G_h^X(x) - G^X(x)|)] \rightarrow 0$, as $h \rightarrow 0^+$, for any $k > 0$. Since

$$\begin{aligned} 0 &\leq E[\Phi(k|G_h^X(x) - G^X(x)|)] \\ &= E\left[\Phi\left(k \frac{x+h-X}{h} \mathbf{1}_{\{x < X < x+h\}}\right)\right] \\ &= \Phi(0)P(X \notin (x, x+h)) + E\left[\Phi\left(k \frac{x+h-X}{h} \mathbf{1}_{\{x < X < x+h\}}\right) \mathbf{1}_{\{x < X < x+h\}}\right] \\ &\leq \Phi(k)P(x < X < x+h) \rightarrow 0, \end{aligned}$$

the convergence in L^Φ is guaranteed.

Take now $h < 0$. Let

$$R_h^X(x) \triangleq \frac{g(x+h) - g(x)}{h} = \begin{cases} 0; & X \leq x+h \\ \frac{X-x-h}{h} \in (-1, 0); & x+h < X < x \\ -1; & X \geq x \end{cases}.$$

We have that $\lim_{h \rightarrow 0^-} R_h^X(x) = -\mathbf{1}_{\{X \geq x\}} \triangleq R^X(x)$ pointwise and in L^Φ .

Hence,

$$g'_-(x) = -\mathbf{1}_{\{X \geq x\}}; \quad g'_+(x) = -\mathbf{1}_{\{X > x\}}.$$

By Chapter 8 of Kosmol and Müller-Wichards (2011), it is well known that, when Φ is differentiable and $X \in M^\Phi$ (hence the same holds for Φ_α), the derivative of H_α with respect to the direction Z is given by

$$\lim_{h \rightarrow 0} \frac{H_\alpha(X + hZ) - H_\alpha(X)}{h} = \frac{E \left[Z \Phi'_\alpha \left(\frac{X}{H_\alpha(X)} \right) \right]}{E \left[\frac{X}{H_\alpha(X)} \Phi'_\alpha \left(\frac{X}{H_\alpha(X)} \right) \right]}, \quad (8)$$

when X is not P -a.s. null.

Consider now $x^* \in \arg \min \pi_\alpha(X, x)$ such that $x^* \neq \text{ess. sup}(X)$. By applying (8) with directions $g'_-(x^*)$ and $g'_+(x^*)$, we obtain that when $x^* \neq \text{ess. sup}(X)$

$$\frac{\partial^- \pi_\alpha}{\partial x}(X, x^*) = 1 + \frac{E \left[-\mathbf{1}_{\{X \geq x^*\}} \Phi'_\alpha \left(\frac{(X - x^*)^+}{\|(X - x^*)^+\|_{\Phi_\alpha}} \right) \right]}{E \left[\frac{(X - x^*)^+}{\|(X - x^*)^+\|_{\Phi_\alpha}} \Phi'_\alpha \left(\frac{(X - x^*)^+}{\|(X - x^*)^+\|_{\Phi_\alpha}} \right) \right]}$$

and

$$\frac{\partial^+ \pi_\alpha}{\partial x}(X, x^*) = 1 + \frac{E \left[-\mathbf{1}_{\{X > x^*\}} \Phi'_\alpha \left(\frac{(X - x^*)^+}{\|(X - x^*)^+\|_{\Phi_\alpha}} \right) \right]}{E \left[\frac{(X - x^*)^+}{\|(X - x^*)^+\|_{\Phi_\alpha}} \Phi'_\alpha \left(\frac{(X - x^*)^+}{\|(X - x^*)^+\|_{\Phi_\alpha}} \right) \right]}.$$

Since $x^* < \text{ess. sup}(X)$, it follows that $E \left[\frac{(X - x^*)^+}{\|(X - x^*)^+\|_{\Phi_\alpha}} \Phi'_\alpha \left(\frac{(X - x^*)^+}{\|(X - x^*)^+\|_{\Phi_\alpha}} \right) \right] > 0$, hence the thesis follows easily. ■

Since the infimum is attained in (6) for any $\alpha \in (0, 1)$ and $X \in M^\Phi$ (see Proposition 4(a)), when Φ is differentiable we may conclude that either (7) has solution or the minimum is attained in $x_{\alpha, X}^* = \text{ess. sup}(X)$.

Remark 7 When $\Phi'(0) = 0$, then $\frac{\partial^- \pi_\alpha}{\partial x}(X, x^*) = \frac{\partial^+ \pi_\alpha}{\partial x}(X, x^*)$ and the condition (7) can be simplified as follows:

$$E \left[\frac{(X - x^*)^+}{\|(X - x^*)^+\|_{\Phi_\alpha}} \Phi'_\alpha \left(\frac{(X - x^*)^+}{\|(X - x^*)^+\|_{\Phi_\alpha}} \right) \right] = E \left[\Phi'_\alpha \left(\frac{(X - x^*)^+}{\|(X - x^*)^+\|_{\Phi_\alpha}} \right) \right]. \quad (9)$$

Example 8 (Particular case: $\Phi(x) = x$) For $\Phi(x) = x$ and a not necessarily continuous random variable X , condition (7) reduces to

$$\begin{cases} E \left[\left(\frac{(X - x^*)^+}{\|(X - x^*)^+\|_1} (1 - \alpha) - \mathbf{1}_{\{X \geq x^*\}} \right) \frac{1}{1 - \alpha} \right] \leq 0 \\ E \left[\left(\frac{(X - x^*)^+}{\|(X - x^*)^+\|_1} (1 - \alpha) - \mathbf{1}_{\{X > x^*\}} \right) \frac{1}{1 - \alpha} \right] \geq 0 \end{cases},$$

hence

$$P(X < x^*) \leq \alpha \leq P(X \leq x^*),$$

that is x^* is an α -quantile. This case corresponds to the Conditional Value at Risk case and the result is coherent with the one of Rockafellar and Uryasev (2000, 2002).

Obviously, if X is a continuous random variable then $x_{\alpha, X}^* = F_X^{-1}(\alpha)$.

Example 9 (Particular case: $\Phi(x) = x^p, p > 1$) Consider $\Phi(x) = x^p, p > 1$, and a not necessarily continuous random variable X . Since $\Phi'(0) = 0$, $\frac{\partial^- \pi_\alpha}{\partial x}(X, x) = \frac{\partial^+ \pi_\alpha}{\partial x}(X, x)$ and the condition (9) reduces to

$$E \left[\frac{(X - x^*)^+}{\|(X - x^*)^+\|_{\Phi_\alpha}} \Phi'_\alpha \left(\frac{(X - x^*)^+}{\|(X - x^*)^+\|_{\Phi_\alpha}} \right) \right] = E \left[\Phi'_\alpha \left(\frac{(X - x^*)^+}{\|(X - x^*)^+\|_{\Phi_\alpha}} \right) \right],$$

hence to

$$\|(X - x^*)^+\|_{p-1} = (1 - \alpha)^{\frac{1}{p(p-1)}} \|(X - x^*)^+\|_p.$$

In the L^p case, the first order condition has been also derived in Dentcheva et al. (2010), Goovaerts et al. (2010) and Nam et al. (2011).

Remark 10 Thanks to Proposition 6 and Remark 7, we provide an alternative proof of the characterization of the optimal scenario in (3).

Suppose that x^* satisfies $\frac{\partial^- \pi_\alpha}{\partial x}(X, x^*) = \frac{\partial^+ \pi_\alpha}{\partial x}(X, x^*) = 0$, that is

$$E \left[\frac{(X - x^*)^+}{\|(X - x^*)^+\|_{\Phi_\alpha}} \Phi'_\alpha \left(\frac{(X - x^*)^+}{\|(X - x^*)^+\|_{\Phi_\alpha}} \right) \right] = E \left[\mathbf{1}_{\{X > x^*\}} \Phi'_\alpha \left(\frac{(X - x^*)^+}{\|(X - x^*)^+\|_{\Phi_\alpha}} \right) \right]. \quad (10)$$

Since $(X - x)^+ = (X - x)\mathbf{1}_{\{X > x^*\}}$, the equation above reduces to

$$E \left[\frac{X - x^*}{\|(X - x^*)^+\|_{\Phi_\alpha}} \frac{\mathbf{1}_{\{X > x^*\}} \Phi'_\alpha \left(\frac{(X - x^*)^+}{\|(X - x^*)^+\|_{\Phi_\alpha}} \right)}{E \left[\mathbf{1}_{\{X > x^*\}} \Phi'_\alpha \left(\frac{(X - x^*)^+}{\|(X - x^*)^+\|_{\Phi_\alpha}} \right) \right]} \right] = 1. \quad (11)$$

Set now

$$\frac{dQ_X^*}{dP} \triangleq \frac{\mathbf{1}_{\{X > x^*\}} \Phi'_\alpha \left(\frac{(X - x^*)^+}{\|(X - x^*)^+\|_{\Phi_\alpha}} \right)}{E \left[\mathbf{1}_{\{X > x^*\}} \Phi'_\alpha \left(\frac{(X - x^*)^+}{\|(X - x^*)^+\|_{\Phi_\alpha}} \right) \right]}.$$

Trivially, $\frac{dQ_X^*}{dP} \geq 0$ and $E \left[\frac{dQ_X^*}{dP} \right] = 1$.

Furthermore, $\frac{dQ_X^*}{dP} \in L^\Psi$. The last statement can be shown as in the proof of Theorem 5.2 in Cheridito and Li (2009).

From (11) it follows that $H_\alpha \left((X - x^*)^+ \right) = \left\| (X - x^*)^+ \right\|_{\Phi_\alpha} = E_{Q_X^*} [X - x^*]$,
so

$$\pi_\alpha(X) = x^* + H_\alpha \left((X - x^*)^+ \right) = x^* + E_{Q_X^*} [X - x^*] = E_{Q_X^*} [X].$$

Hence, Q_X^* is the optimal scenario in (3).

5 Orlicz quantiles

By analogy with the case $\Phi(x) = x$ where the set of the minimizers of

$$\pi_\alpha(X, x) = x + H_\alpha \left((X - x)^+ \right) \quad (12)$$

coincides with the set of α -quantiles of X , it is very natural to define any

$$x_{\alpha, X}^* \in \arg \min \pi_\alpha(X, x) \quad (13)$$

as an α -Orlicz quantile of X .

Similar extensions of the notion of quantile have been proposed in the quantile regression literature (see for example the book of Koenker, 2005 for a review), giving rise to the notions of expectiles (Newey and Powell, 1987), L^p quantiles (Chen, 1996), M -quantiles (Breckling and Chambers, 1988). All these generalized quantiles are based on the minimization of a possibly asymmetric expected loss function. On the contrary, Orlicz quantiles are based on the minimization of the functional (12), that depends on the Luxemburg norm H_α that in general cannot be represented as an expected loss.

Here below we provide a non exhaustive list of properties of the usual quantiles shared also by Orlicz quantiles.

Proposition 11 (*Properties of Orlicz quantiles*) Suppose that Φ is strictly convex, $\alpha \in (0, 1)$ and $X \in M^\Phi$.

- (a) $x_{\alpha, X}^*$ is cash additive, positively homogeneous and law invariant;
- (b) if $X = c$ P -a.s., then $x_{\alpha, X}^* = c$;
- (c) if X is bounded from above, then $x_{\alpha, X}^* \leq \text{ess. sup}(X)$.

Proof. (a) Consider an arbitrary $c \in \mathbb{R}$. On one hand, $\pi_\alpha(X + c) = H_\alpha \left((X - \tilde{x})^+ \right) + \tilde{x} + c$ (by taking $\tilde{x} = x_{\alpha, X+c}^* - c$). On the other hand, by cash additivity of π_α ,

$$H_\alpha \left((X - x_{\alpha, X}^*)^+ \right) + x_{\alpha, X}^* + c = \pi_\alpha(X) + c = \pi_\alpha(X + c) = H_\alpha \left((X - \tilde{x})^+ \right) + \tilde{x} + c.$$

Hence, the uniqueness of $x_{\alpha, X}^*$ implies therefore that $x_{\alpha, X+c}^* = x_{\alpha, X}^* + c$.

Consider now any $X \in M^\Phi$ and any $\beta > 0$. By positive homogeneity of π_α and of H_α , it is easy to check $x_{\alpha, \beta X}^* = \beta \cdot x_{\alpha, X}^*$. The case where $\beta = 0$ is straightforward.

Consider now X, Y such that $X \sim Y$. By law invariance of π_α and H_α , it follows immediately that $x_{\alpha, X}^* = x_{\alpha, Y}^*$.

(b) may be checked easily.

(c) Take $Y \leq 0$. From $\pi_\alpha(Y) = H_\alpha\left((Y - x_{\alpha, Y}^*)^+\right) + x_{\alpha, Y}^* \leq 0$ and $H_\alpha\left((Y - x_{\alpha, Y}^*)^+\right) \geq 0$, it follows immediately $x_{\alpha, Y}^* \leq 0$. If X is bounded from above, then $Y = X - \text{ess. sup}(X) \leq 0$. By the previous argument and item (a), it follows that $0 \geq x_{\alpha, Y}^* = x_{\alpha, X}^* - \text{ess. sup}(X)$. ■

Unfortunately, as it was already pointed out in Dentcheva et. al (2010), Orlicz quantiles do not satisfy an internality property, that is it may happen that $x_{\alpha, X}^* < \text{ess. inf}(X)$, as in the following example (see also Dentcheva et al., 2010, Remark 2).

Example 12 Let $\Phi(x) = x^2$ and X uniform on $[0, 1]$. The first order condition (7) becomes

$$\|(X - x^*)^+\|_1 = (1 - \alpha)^{\frac{1}{2}} \|(X - x^*)^+\|_2 \quad (14)$$

for $x^* < \text{ess. sup}(X)$. It is easy to check that if $\alpha \in (0, \frac{1}{4})$ the only solution of (14) is given by

$$x^* = \frac{\alpha - \sqrt{\frac{\alpha}{3}(1 - \alpha)}}{2\alpha} < 0. \quad (15)$$

In conclusion: although $0 \leq X \leq 1$, the minimizer $x_{\alpha, X}^*$ is negative for any $\alpha \in (0, \frac{1}{4})$.

See also Ahn and Shyamalkumar (2011a) in which a lower bound for x^* is provided. Since $x_{\alpha, X}^*$ does not satisfy an internality property, it follows immediately that it fails to be monotone in X , i.e. $X \leq Y$ P -a.s. does not necessarily imply $x_{\alpha, X}^* \leq x_{\alpha, Y}^*$. Indeed, just compare the uniform variable X of the previous example with $Y = 0$. Trivially, $Y \leq X$ but $x_{\alpha, X}^* < 0 = x_{\alpha, Y}^*$ for $\alpha < \frac{1}{4}$.

So it seems that the interpretation of $x_{\alpha, X}^*$ as a generalized quantile is quite problematic. Two natural alternative that still include the usual quantiles as special cases are the following:

$$B_\alpha(X, x) \triangleq \alpha H\left((X - x)^+\right) + (1 - \alpha) H\left((X - x)^-\right) \quad (16)$$

$$M_\alpha(X, x) \triangleq \alpha E\left[\Phi(X - x)^+\right] + (1 - \alpha) E\left[\Phi(X - x)^-\right], \quad (17)$$

where $H \triangleq H_0$.

We call any

$$x_{\alpha, X}^* \in \arg \min B_\alpha(X, x) \quad (18)$$

a *bilateral Orlicz quantile* (the letter ' B ' standing for 'bilateral').

The minimizers of $M_\alpha(X, x)$ are known in the statistical literature as M -quantiles (see Breckling and Chambers, 1988); we are investigating their properties as risk measures in a parallel work.

The properties of $B_\alpha(X, x)$ are very similar to those of $\pi_\alpha(X, x)$.

Proposition 13 For any $X \in M^\Phi$ and any $\alpha \in (0, 1)$:

- (a) $B_\alpha(X, x)$ is finite and convex in x ;
- (b) the infimum in (18) is attained;
- (c) the set of minimizers in (18) is an interval;
- (d) if Φ is strictly convex, then the minimizer is unique.

Proof. The proofs are completely similar to those of Proposition 3. ■

Any bilateral Orlicz quantiles has the following properties (in particular, they do satisfy an internality property).

Proposition 14 (Properties of Bilateral Orlicz quantiles)

Suppose that Φ is strictly convex, $\alpha \in (0, 1)$ and $X \in M^\Phi$.

- (a) $x_{\alpha, X}^*$ is cash additive, positively homogeneous and law invariant;
- (b) $x_{\alpha, X}^*$ is nondecreasing in α ;
- (c) if $X = c$ P -a.s., then $x_{\alpha, X}^* = c$;
- (d) internality: $x_{\alpha, X}^* \in [\text{ess. inf}(X), \text{ess. sup}(X)]$.

Proof. The proofs of items (a) and (c) are similar to those of Proposition 11. It remains to verify internality. Consider now an arbitrary $X \in L^\infty$. For $x \geq \text{ess. sup}(X)$, $\pi_\alpha(X, x) = \alpha H((X - x)^+) + (1 - \alpha) H((X - x)^-) = (1 - \alpha) H(x - X)$ is increasing in x ; for $x \leq \text{ess. inf}(X)$, on the contrary, $\pi_\alpha(X, x) = \alpha H(X - x)$ is decreasing in x . Hence, $x_{\alpha, X}^* \in [\text{ess. inf}(X), \text{ess. sup}(X)]$. For X unbounded from above and/or below, the thesis follows immediately from the arguments above.

(b) Since the left and right derivatives

$$\frac{\partial^- B_\alpha}{\partial x}(X, x) = \alpha \frac{\partial^- H((X - x)^+)}{\partial x} + (1 - \alpha) \frac{\partial^- H((X - x)^-)}{\partial x}$$

$$\frac{\partial^+ B_\alpha}{\partial x}(X, x) = \alpha \frac{\partial^+ H((X - x)^+)}{\partial x} + (1 - \alpha) \frac{\partial^+ H((X - x)^-)}{\partial x}$$

and $\frac{\partial^- H((X - x)^+)}{\partial x}, \frac{\partial^+ H((X - x)^+)}{\partial x} \leq 0$ while $\frac{\partial^- H((X - x)^-)}{\partial x}, \frac{\partial^+ H((X - x)^-)}{\partial x} \geq 0$, both $\frac{\partial^- B_\alpha}{\partial x}(X, x)$ and $\frac{\partial^+ B_\alpha}{\partial x}(X, x)$ are decreasing in α . The thesis follows therefore immediately. ■

As shown in the following result, also bilateral Orlicz quantiles can be identified by means of a first order condition.

Proposition 15 Let Φ be differentiable with $\Phi'(0) = 0$ and $X \in M^\Phi$. Then any minimizer x^* in (18) such that $x^* \neq \text{ess. inf}(X), \text{ess. sup}(X)$ satisfies the following condition:

$$\alpha \frac{E \left[\Phi' \left(\frac{(X - x^*)^+}{\|(X - x^*)^+\|_\Phi} \right) \right]}{E \left[\frac{(X - x^*)^+}{\|(X - x^*)^+\|_\Phi} \Phi' \left(\frac{(X - x^*)^+}{\|(X - x^*)^+\|_\Phi} \right) \right]} = (1 - \alpha) \frac{E \left[\Phi' \left(\frac{(X - x^*)^-}{\|(X - x^*)^-\|_\Phi} \right) \right]}{E \left[\frac{(X - x^*)^-}{\|(X - x^*)^-\|_\Phi} \Phi' \left(\frac{(X - x^*)^-}{\|(X - x^*)^-\|_\Phi} \right) \right]}. \quad (19)$$

Proof. The proof is similar to the one of Proposition 6. ■

Example 16 For $\Phi(x) = x^2$, the condition (19) simplifies to

$$\alpha \frac{E[(X - x^*)^+]}{\|(X - x^*)^+\|_2} = (1 - \alpha) \frac{E[(X - x^*)^-]}{\|(X - x^*)^-\|_2}.$$

It is interesting to compare this equation with the definition of the expectiles of Newey and Powell (1987), that would be simply

$$\alpha E[(X - x^*)^+] = (1 - \alpha) E[(X - x^*)^-].$$

Quite surprisingly, also bilateral Orlicz quantiles fail to be monotone in X (or, equivalently, they are not monotone with respect to the usual stochastic order $\leq_{st}$), as shown by the following counterexample.

Example 17 Consider $\Phi(x) = x^2$, $\alpha = 0.5$ and $X = \begin{cases} 0, & \frac{1}{4} \\ 1, & \frac{1}{4} \\ 2, & \frac{1}{4} \\ k, & \frac{1}{4} \end{cases}$, with $k > 2$.

It is easy to check that the first order condition has exactly one solution $x^* \in (1, 2)$ that is the solution of

$$\frac{(k - x^*) + (2 - x^*)}{\sqrt{(k - x^*)^2 + (2 - x^*)^2}} = \frac{x^* + (x^* - 1)}{\sqrt{x^{*2} + (x^* - 1)^2}},$$

hence

$$x^* = \frac{k}{k - 1}.$$

When $k \rightarrow +\infty$ we have that the distribution of X is strictly increasing in the usual stochastic order $\leq_{st}$, but the corresponding x^* is decreasing.

6 Appendix

In this Appendix, we will provide a direct proof of lower semi-continuity of π_α with respect to the $\|\cdot\|_{\Phi_\alpha}$ -norm.

Proposition 18 Let $\alpha \in (0, 1)$. The Haezendonck-Goovaerts premium $\pi_\alpha : M^\Phi \rightarrow \mathbb{R}$ is $\|\cdot\|_{\Phi_\alpha}$ -lower semicontinuous.

Proof. Take any sequence $(X_n)_{n \geq 0} \subseteq M^\Phi$ converging to some X in $\|\cdot\|_{\Phi_\alpha}$. Hence $X \in M^\Phi$.

By Proposition 4(a), the minimum in the definition of $\pi_\alpha(X)$ is attained in some $x^* \in \mathbb{R}$, hence

$$\begin{aligned}\pi_\alpha(X) &= x^* + H_\alpha\left((X - x^*)^+\right) \\ &\leq \liminf_n \left\{ x_n^* + H_\alpha\left((X - x_n^*)^+\right) \right\} \\ &= \liminf_n \left\{ x_n^* + \left\| (X - x_n^*)^+ \right\|_{\Phi_\alpha} \right\},\end{aligned}$$

where $x_n^* \in \arg \min_x \left\{ x + H_\alpha\left((X_n - x)^+\right) \right\}$. Furthermore,

$$\begin{aligned}\left\| (X - x_n^*)^+ \right\|_{\Phi_\alpha} &\leq \left\| (X - X_n)^+ + (X_n - x_n^*)^+ \right\|_{\Phi_\alpha} \\ &\leq \left\| (X - X_n)^+ \right\|_{\Phi_\alpha} + \left\| (X_n - x_n^*)^+ \right\|_{\Phi_\alpha}.\end{aligned}$$

By $\left\| (X - X_n)^+ \right\|_{\Phi_\alpha} \leq \|X - X_n\|_{\Phi_\alpha} \rightarrow 0$ and by the arguments above, it follows that

$$\begin{aligned}\pi_\alpha(X) &\leq \liminf_n \left\{ x_n^* + \left\| (X - x_n^*)^+ \right\|_{\Phi_\alpha} \right\} \\ &\leq \liminf_n \left\{ x_n^* + \left\| (X - X_n)^+ \right\|_{\Phi_\alpha} + \left\| (X_n - x_n^*)^+ \right\|_{\Phi_\alpha} \right\} \\ &= \liminf_n \left\{ \left\| (X - X_n)^+ \right\|_{\Phi_\alpha} \right\} + \liminf_n \left\{ x_n^* + \left\| (X_n - x_n^*)^+ \right\|_{\Phi_\alpha} \right\} \\ &= \liminf_n \pi_\alpha(X_n),\end{aligned}$$

that is lower semi-continuous of π_α with respect to $\|\cdot\|_{\Phi_\alpha}$. ■

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