Mann iteration with power means

G.I. Bischi, F. Cavalli & A. Naimzada


To link to this article: http://dx.doi.org/10.1080/10236198.2015.1080252

Published online: 03 Nov 2015.

Submit your article to this journal

Article views: 59

View related articles

View Crossmark data
Mann iteration with power means

G.I. Bischi\textsuperscript{a,1}, F. Cavalli\textsuperscript{b,*} and A. Naimzada\textsuperscript{b,2}

\textsuperscript{a}Department of Economics, Society, Politics, University of Urbino, via Saffi 42, 61029 Urbino, Italy; \textsuperscript{b}Department of Economics, Management and Statistics, University of Milano-Bicocca, U6 Building, Piazza dell’Ateneo Nuovo 1, 20126 Milano, Italy

(Received 26 February 2015; accepted 31 July 2015)

We analyse the recurrence $x_{n+1} = f(z_n)$, where $z_n$ is a weighted power mean of $x_0, \ldots, x_n$, which has been proposed to model a class of non-linear forward-looking economic models with bounded rationality. Under suitable hypotheses on weights, we prove the convergence of the sequence $x_n$. Then, to simulate a fading memory, we consider exponentially decreasing weights. Since, in this case, the resulting recurrence does not fulfil the hypotheses of the previous convergence theorem, it is studied by reducing it to an equivalent two-dimensional autonomous map, which shares the asymptotic behaviours with a particular one-dimensional map. This allows us to prove that a long memory with sufficiently large weights has a stabilizing effect. Finally, we numerically investigate what happens when the memory ratio is not sufficiently large to provide stability, showing that, depending on the power mean and the memory ratio, either a delayed or early cascade of flip bifurcations occurs.

Keywords: forward-looking models; learning; Mann iterations; non-autonomous difference equations

1. Introduction

The canonical economic theory assumes that agents have perfect rationality: they have both the skills to exploit information achieved in the economic system and the ability to compute all actions needed to reach an optimal solution. Learning characterizes models in which agents with bounded rationality try to reconstruct key elements of the economic system using information available from the past experience.

For example, if the economic agents have to make forecasts about future, e.g. prices or taxes or incomes, in doing that, they collect and analyse past data. In the early literature, for instance in the cobweb and Cournot oligopoly models, the static expectation hypothesis was widely used, with agents expecting that the next-period price will be at the same level of the current price. The adaptive expectation hypothesis became popular in the 1960s and 1970s. In this case, agents’ belief about the next-period price corresponds to a linear convex combination of currently observed price and predicted price. Based on the observation that static and adaptive expectations imply a poor set of information and limited computational skills, in the 1980s, the rational expectation hypothesis played a prominent role in economic theory. Under such assumption, agents are not supposed to make systematic forecasting errors, as they are assumed to have a full knowledge of the economic system and all relevant information in order to make the optimal choice. The rational expectation hypothesis has been criticized for the unrealistic informational and computational requirements and for the contrast with the observed human behaviour in

\*Corresponding author. Email: fausto.cavalli@unimib.it

© 2015 Taylor & Francis
laboratory experiments, for example by Sargent [25] and Conlisk [12]. Recently, a more realistic view about the forecasting activity has been proposed, in which agents act like statisticians or econometricians, collecting a large set of data from the past observations and using sophisticated algorithms, such as regressions, sample means and recursive least squares, in order to form their expectations. Our contribution belongs to this strand of research and tries to study the evolution of the system when it is supposed that economic agents have enough data and computational capabilities for forecasting future prices using weighted power means. Our way of modelling forecasting processes is intensely interweaved with the mathematical notion of Mann iteration, namely an iterative scheme of the form

\[ x_{n+1} = f(z_n), \quad (1) \]

where \( f : I \rightarrow I, I = [a, b] \subseteq \mathbb{R}^+ \), and \( z_n \) is the arithmetic mean of all the previous values \( x_i, 0 \leq i \leq n \)

\[ z_n = \frac{1}{n+1} \sum_{k=0}^{n} a_{nk} x_k \quad (2) \]

with

\[ \sum_{k=0}^{n} a_{nk} = 1. \quad (3) \]

Such an iteration scheme has been used to model economic and social systems with agents who have not perfect foresight, so they learn from the past experiences using all the available information (i.e. present and past data), in order to calculate the expected values of future states. If \( n \) represents discrete time periods and \( x_n \) the value of the state variable in period \( n \), \( z_n \) can be interpreted as the expected value (see the contributions of Bray [11], Lucas [20], Balasko and Royer [3], Bischi and Naimzada [9], Barucci [4], and Foroni et al. [14]). Starting from the seminal paper of Mann [21], iterations (1) have been studied by many authors, among others, Borwein and Borwein [10], Rhoades [23], Aicardi and Invernizzi [1], Bischi and Gardini [7], Bischi et al. [8]. For example, Bray in [11] proposed a recurrence of the form (1), with \( z_n \) given by a uniform arithmetic mean

\[ z_n = \frac{1}{n+1} \sum_{k=0}^{n} x_k, \]

as a learning mechanism. In this case, the Mann iteration coincides with the Cesáro iteration, whose dynamics are very simple since for each \( x_0 \in I \), the resulting sequence \( \{x_n\} \) converges to a fixed point of \( f \), as shown by Franks and Marzec in [15]. This suggests a strong stabilizing effect of a distributed uniform memory, since any kind of dynamics more complex than convergence towards a fixed point of \( f \) is excluded, being the existence of more than one fixed point of \( f \) in \( I \) the only possibility of non-trivial dynamics, as in such situation, different basins of attraction must be considered.

In this work, we propose a generalization of (2) expressed by the power mean

\[ z_n = \left( \sum_{k=0}^{n} a_{nk} x_k^s \right)^{1/s}, \quad s \neq 0. \quad (4) \]
The arithmetic mean (2) is a special case of (4) when \( s = 1 \), but other commonly used algebraic means can be obtained from (4), such as the weighted quadratic mean for \( s = 2 \) and the weighted harmonic mean for \( s = -1 \). Furthermore, the weighted geometric mean is obtained as a limiting case for \( s \to 0 \), since

\[
\lim_{s \to 0} \left( \frac{\sum_{k=0}^{n} a_{nk} x_k^s}{\prod_{k=0}^{n} x_k^{a_{nk}}} \right) = \prod_{k=0}^{n} x_k^{a_{nk}}.
\]

Of course, if \( s < 0 \), the further condition \( x_i > 0 \) for each \( i \) should be verified. For a detailed description of the properties of such means, as well as their applications, we refer to the book by Vajani [26], Chapter 6.

This study is motivated by the possibility that some learning mechanism can be expressed by the iteration scheme (1) with algebraic means of the form (4) with \( s \neq 1 \) (an example is given in Section 4).

This paper is organized as follows. In Section 2, the iteration scheme (1) with (4) is reduced to a first-order non-autonomous recurrence, and some convergence results are given, which generalize the results of Mann [21] and Borwein and Borwein [10], where only the arithmetic mean (2) is considered. In Section 3, the power mean (4) is considered with weights decreasing as terms of a geometric progression. In Section 4, we study an example coming from the literature on overlapping generations’ (OLG) models with a learning mechanism based on power iteration means.

2. Convergence of recurrences with power means

In what follows, we assume that the weights are obtained as

\[
a_{nk} = \frac{\omega_k^{(n)}}{W_n}, \quad \omega_k^{(n)} \geq 0,
\]

where, for each \( n \geq 0 \), the \((n+1)\)-dimensional vector of non-negative weights

\[
\omega^{(n)} = \{ \omega_0^{(n)}, \omega_1^{(n)}, \ldots, \omega_n^{(n)} \}
\]

defines the relative influence of each state \( x_k, k = 0, \ldots, n \), in the computation of the average \( z_n \), and

\[
W_n = \sum_{k=0}^{n} \omega_k^{(n)},
\]

so that (3) is satisfied.

In this section we assume, as in [10,23], that at each \( n \), the vector of relative weights is obtained by adding the last component without any change of the previous ones, i.e. from \( \omega^{(n)} = (\omega_0, \omega_1, \ldots, \omega_n) \), we obtain \( \omega^{(n+1)} = (\omega_0, \omega_1, \ldots, \omega_n, \omega_{n+1}) \). In this case, we have

\[
W_{n+1} = W_n + \omega_{n+1}.
\]
The iterative scheme (1) with a power mean (4) becomes

\[
\begin{aligned}
  x_{n+1} &= f(z_n) \quad \text{with} \quad z_n = \left( \frac{1}{s} \sum_{k=0}^{n} \frac{\omega_k}{W_n} x_k^s \right)^{1/s}, \\
  W_n &= \sum_{k=0}^{n} \omega_k, \quad s \neq 0,
\end{aligned}
\]

(6)

where the continuous function \( f \), mapping the compact set \( I \) into itself, has at least one fixed point in \( I \).

Recurrence (6) with \( s = 1 \) is a Mann iteration, for which the following classical result holds.

**Theorem 2.1 [21].** Let \( s = 1 \) and \( W_n \to \infty \). If either of the sequences \( \{x_n\} \) and \( \{z_n\} \) converges, then the other also converges to the same point, and their common limit is a fixed point of \( f \).

In [10,23], a Mann iteration (1) is reduced to the following non-autonomous iteration, called the **segmenting Mann iteration**

\[
  z_{n+1} = (1 - t_n)z_n + t_n f(z_n),
\]

(7)

where \( z_0 = x_0 \in I \), and

\[
  t_n = \frac{\omega_{n+1}}{W_{n+1}}.
\]

(8)

From \( \{z_n\} \), the sequence of states \( \{x_n\} \) can be easily obtained as the images of \( z_n \) under \( f \)

\[
  x_{n+1} = f(z_n).
\]

(9)

The following result is proved by Borwein and Borwein in [10].

**Theorem 2.2 [10].** Suppose that \( \{t_n\} \) tends to zero. Then, the sequence \( \{z_n\} \) converges.

In this section we generalize these theorems to the case of power means with \( s \neq 1 \). This can be easily done once the iterative scheme (6) is put into a recursive form, for the expected variables \( z_n \), similar to (7). In fact, even provided that \( s \neq 1 \), from (6), we get

\[
  z_{n+1} = \left( \sum_{k=0}^{n} \frac{\omega_k}{W_{n+1}} x_k^s + \frac{\omega_n+1}{W_{n+1}} x_{n+1}^s \right)^{1/s} = \left( \frac{W_n}{W_{n+1}} \sum_{k=0}^{n} \frac{\omega_k}{W_n} x_k^s + \frac{\omega_n+1}{W_{n+1}} x_{n+1}^s \right)^{1/s},
\]

from which, by using the definition (8) of \( t_n \) and the identity (5), we obtain what we shall call the **generalized segmenting Mann iteration**

\[
  z_{n+1} = F(n, z_n) = \left( (1 - t_n)z_n^s + t_n f(z_n)^s \right)^{1/s}.
\]

(10)

Also in this case, the iterative process described by the non-autonomous first-order difference equation is equivalent to the iterative process (6), in the sense that given an
initial condition $z_0 = x_0$, the sequence of expected values obtained from (10) is the same as that obtained from (6) (and the sequence of states is given by (9)).

We recall that a fixed point (or stationary state) of iteration (6) is defined as a value $x^* \in \mathcal{R}$, such that if $x_0 = x^*$, then (6) generates the sequence $x_n = x^*$ for each $n \geq 0$. The following results are straightforward.

**Proposition 2.3.** Regarding iterations (6) and (10), we have that

(i) $x^*$ is a fixed point of iteration (6) if and only if it is a fixed point of the function $f$.

(ii) $z^*$ is a fixed point of (10) if and only if it is a fixed point of $f$.

We recall that a fixed point (or stationary state) of the non-autonomous difference Equation (10) is defined as a value $z^*$, such that $F(n, z^*) = z^*$ for each $n$.

The following proposition generalizes the above quoted theorems.

**Proposition 2.4.** Considering (6) and (10), we have that

(i) If $W_n \to \infty$, then the sequence $\{x_n\}$ defined in (6) converges if and only if the sequence $\{z_n\}$ in (10) converges, and the two sequences converge to a common limit, which is a fixed point of $f$.

(ii) If in (10) $\{t_n\}$ is a positive sequence that tends to zero, then the sequence $\{z_n\}$ is convergent.

**Proof.** (i) First we prove that, under the assumption $W_n \to \infty$, if $x_n$ is convergent, then also $z_n$ converges to the same limit.

Let $x_n \to q > 0$ (the case $q = 0$ will be treated separately). Then, for each $s$, $x_n^s \to q^s$, i.e. for each $\varepsilon > 0$, an $N > 0$ exists, such that

$$q^s - \varepsilon < x_n^s < q^s + \varepsilon \quad \text{for } n > N. \tag{11}$$

Now we prove that $(z_n^s - q^s) \to 0$, i.e. $z_n^s \to q^s$, which implies that $z_n \to q$. For $n > N$, we have

$$(z_n^s - q^s) = \sum_{k=0}^{n} \frac{\omega_k}{W_n} x_k^s - q^s = \sum_{k=0}^{N} \frac{\omega_k}{W_n} x_k^s + \sum_{k=N+1}^{n} \frac{\omega_k}{W_n} x_k^s - q^s$$

$$= \frac{1}{W_n} \sum_{k=0}^{N} \omega_k x_k^s + \frac{W_n - W_N}{W_n} \sum_{k=N+1}^{n} \frac{\omega_k}{W_n} x_k^s - q^s.$$

From the right inequality in (11), we have

$$(z_n^s - q^s) \leq \frac{1}{W_n} \sum_{k=0}^{N} \omega_k x_k^s + \frac{W_n - W_N}{W_n} (q^s + \varepsilon) - q^s, \tag{12}$$

because

$$\sum_{k=N+1}^{n} \frac{\omega_k}{W_n} = 1.$$
Similarly, from the left inequality in (11), we have

$$
(z_n^s - q^s) \geq \frac{1}{W_n} \sum_{k=0}^{N} \omega_k x_k^s + \frac{W_n - W_N}{W_n} (q^s - \varepsilon) - q^s.
$$

Since $W_n \to \infty$ and the $\omega_k$ are bounded, from (12) follows that

$$
\lim_{n \to \infty} (z_n^s - q^s) \leq \varepsilon,
$$

and, from (13),

$$
\lim_{n \to \infty} (z_n^s - q^s) \geq \varepsilon.
$$

Since $\varepsilon$ is arbitrarily small, (14) and (15) prove that $\lim_{n \to \infty} (z_n^s - q^s) = 0$.

We consider now the case $x_n \to 0$. If $s > 0$, the previous arguments can be applied with no substantial modifications. If $s < 0$, since the $x_n$ are supposed to be positive, we have that $x_n^s \to +\infty$, i.e. for each $M > 0$, an $N > 0$ exists, such that $x_n^s > M$ for $n > N$. For $n > N$, we have

$$
z_n^s = \sum_{k=0}^{N} \frac{\omega_k}{W_n} x_k^s + \sum_{k=N+1}^{n} \frac{\omega_k}{W_n} x_k^s > M \sum_{k=N+1}^{n} \frac{\omega_k}{W_n},
$$

and, since $M$ can be arbitrarily large, this implies that $z_n^s \to +\infty$, from which, since $s < 0$, we have $z_n \to 0$.

To complete this part of the proof, it remains to show that the common limit is a fixed point of $f$. Indeed, since $f$ is continuous, from $z_n \to q$, it follows that $f(z_n) \to f(q)$. However, $x_{n+1} = f(x_n)$, so that $q = f(q)$.

We assume now that $z_n$ converges, and prove that $x_n$ also converges to the same limit. If $z_n \to r$, then $x_n \to f(r)$ because $f$ is continuous. From the previous argument, it must also be $z_n \to f(r)$, which implies that $r = f(r)$.

(ii) Since, for $z_0 \in I = [a, b] \subset R^+$, the whole sequence $\{z_n\}$ is contained in $I$, it has at least one limit point. We show that it is unique. From (10), it can be rewritten as

$$
z_n^s - z_n^s = t_n \left( f(z_n)^s - z_n^s \right),
$$

we deduce that, since $t_n \to 0$, $z_n$ and $f(z_n)$ are bounded, for each $\varepsilon > 0$, a $m > 0$ exists such that

$$
|z_n^s - z_m^s| < \varepsilon \quad \text{for} \quad n > m.
$$

Following the argument used by Borwein and Borwein [10], let us assume, for the sake of contradiction, that $\xi$ and $\eta$, with $a \leq \xi < \eta \leq b$, are two distinct limit points. A consequence of this assumption is that $f(z) = z$ for each $z \in (\xi, \eta)$. In fact, let $c$ be a point such that $\xi < c < \eta$. If $f(c) > c$, then, by the continuity of $f$, a $\delta \in (0, c)$ exists, such that

$$
f(z) > z \quad \text{whenever} \quad |z - c| < \delta.
$$

Since $\eta$ is a limit point for $\{z_n\}$, a $N > m$ exists, such that $|z_N - \eta| < (\eta - c)$, which implies that $z_N > c$. It follows that $z_n > c$ for each $n > N$. To prove this, we separately analyse the
cases of positive and of negative $s$. Consider first $s > 0$. If $c < z_N < c + \delta$, from (8), it follows that $f(z_N) > z_N$, which gives, since $s > 0$, $[f(z_N)]^s > z_N^s$. From (6), it follows that $z_{N+1}^s > z_N^s$ (remember that $t_n > 0$), and this implies $z_{N+1} > z_N$ because $s > 0$. If $z_N \geq c + \delta$, we have $z_N^s \geq (c + \delta)^s$, so that

$$z_{N+1}^s - c^s = z_{N+1}^s - z_N^s + z_N^s - c^s \geq z_{N+1}^s - z_N^s + (c + \delta)^s - c^s$$

$$> -\varepsilon + (c + \delta)^s - c^s$$

(19)

where (17) has been used. Since $\delta < c$, from the binomial series, we have

$$(c + \delta)^s = c^s + s\delta c^{s-1} + \frac{s(s-1)}{2} \delta^2 c^{s-2} + \frac{s(s-1)(s-2)}{3!} \delta^3 c^{s-3} + \ldots$$

so that $(c + \delta)^s - c^s > s\delta c^{s-1}$ for $s \geq 1$, and $(c + \delta)^s - c^s > s\delta c^{s-2}(c - (1 - s)\delta/2)$ for $0 < s < 1$. Thus, if for $s \geq 1$ we take $0 < \varepsilon < s\delta c^{s-1}$ or, for $0 < s < 1$, $0 < \varepsilon < s\delta c^{s-2}(c - ((1 - s)/2)\delta)$, (19) gives $z_{N+1}^s - c^s > 0$, which, for $s > 0$, implies $z_{N+1} > c$.

We consider now $s < 0$. If $c < z_N < c + \delta$, from (18) it follows that $f(z_N) > z_N$, which gives, since $s < 0$, $[f(z_N)]^s < z_N^s$. From (16), it follows that $z_{N+1}^s < z_N^s$, which implies $z_{N+1} > z_N$ because $s < 0$. If $z_N \geq c + \delta$, we have $z_N^s \leq (c + \delta)^s$, so that

$$z_{N+1}^s - c^s = z_{N+1}^s - z_N^s + z_N^s - c^s \leq z_{N+1}^s - z_N^s + (c + \delta)^s - c^s$$

$$< \varepsilon + (c + \delta)^s - c^s,$$

(20)

where (17) has been used. From the binomial series with $s < 0$, we have $(c + \delta)^s - c^s < s\delta c^{s-1}$, so that if we take $0 < \varepsilon < -s\delta c^{s-1}$, (20) gives $z_{N+1}^s - c^s < 0$, which, for $s < 0$, implies again $z_{N+1} > c$.

Hence, by induction, $z_n > c$ for $n \geq N$ against the assumption that $\xi < c$ is a limit point of $\{z_n\}$.

If $f(c) < c$, a similar reasoning contradicts the assumption that $\eta$ is a limit point. Thus, $f(c) = c$ for each $\xi < c < \eta$.

Now, if for a given $\bar{n}$ we have $\xi < z_{\bar{n}} < \eta$, then $z_{\bar{n}+1} = z_{\bar{n}}$, and so $z_n = z_{\bar{n}}$ for each $n \geq \bar{n}$, which contradicts the fact that $\bar{\xi}$ and $\bar{\eta}$ are both limit points. If this is not the case, since $\{z_n\}$ cannot oscillate out of the interval $(\bar{\xi}, \bar{\eta})$ because of (17), taking $\varepsilon < (\eta - \bar{\xi})$, it remains $z_n > \eta$ or $z_n < \xi$ for each $n$, and, again, this excludes the possibility that $\xi < \eta$ be both limit points. Therefore, $\{z_n\}$ converges to its unique limit point.

Of course, if $f$ has a unique fixed point $x^* \in I$, then it is globally attracting in $I$, i.e. $x_n \to x^*$ for each $x_0 \in I$.

A typical example in which these propositions can be applied is that of a uniform power mean, i.e. with equal weights $\omega_k = \omega$ for any $k$. In fact, in this case, we have $t_n = 1/(n+1) \to 0$ and $W_n \to \infty$. This constitutes a generalization of the result by Franks and Marzec [15] on the Cesàro iteration, since it includes the uniform arithmetic mean for $s = 1$, the uniform harmonic mean for $s = -1$, the uniform geometric mean for $s \to 0$, and so on.
3. Asymptotic dynamics with exponentially decreasing weights

In this section we study the asymptotic dynamics arising when exponentially decreasing weights are assumed. These are often used in applications, since they describe, as suggested by Friedman in [16], agents which ‘form their expectations according to a weighted estimation procedure which exponentially discounts older observations’, i.e. an exponentially fading memory. In this case, some assumptions of propositions discussed in the previous section are not satisfied, and more complex asymptotic dynamics can be obtained. The results of this section generalize to the case of power means the results given by Bischi and Gardini [6,7] and by Bischi et al. [8] on Mann iterations, which can be reduced to two-dimensional maps.

Exponentially decreasing weights can be defined by setting, at each \(n\), in the vector of relative weights, a fixed value to the weight of the last state, say \(v^n(\theta) = v^0(\theta) = 1\), while the values of the previous ones are obtained, so that the ratio between two successive weights is fixed, say \(v^n(\theta)/v^{n+1}(\theta) = \rho\). So, from \(v^n(\theta) = (\rho^{n-1}, \rho^{n-2}, ..., \rho, 1)\), we obtain, more concisely,

\[ v^n(\theta) = \rho^{n-k}, \quad 0 \leq k \leq n. \]

With these weights, the following relation holds:

\[ W_{n+1} = 1 + \rho W_n, \quad (21) \]

and the recurrence with fading memory becomes

\[
\begin{align*}
    x_{n+1} &= f(z_n) \quad \text{with} \quad z_n = \left( \sum_{k=0}^{n} \frac{\rho^{n-k} x^k_s}{W_n} \right)^{1/s}, \\
    W_n &= \sum_{k=0}^{n} \rho^{n-k} = \frac{1 - \rho^{n+1}}{1 - \rho}, \quad s \neq 0.
\end{align*}
\]

As already stressed in Section 1, these weights are often used in economic modelling (see Gandolfo et al. [18], Aicardi and Invernizzi [1]) since, with a memory ratio \(\rho \in (0, 1)\), they represent the realistic assumption of an exponentially fading memory (see Friedman [16], Radner [22]). Let us first show that the relation (21) allows us to obtain, also in this case, a generalized segmenting Mann iteration. In fact, we have

\[
    z_{n+1} = \left( \frac{1}{W_{n+1}} \left( \sum_{k=0}^{n} \rho \rho^{n-k} x^k_s + x^s_{n+1} \right) \right)^{1/s} = \left( \frac{\rho W_n}{W_{n+1}} z^n_s + \frac{1}{W_{n+1}} [f(z_n)]^s \right) \left( \frac{1}{W_{n+1}} \right)^{1/s}
\]

and defining

\[ t_n = \frac{1}{W_{n+1}} = \frac{1 - \rho}{1 - \rho^{n+1}}, \quad (23) \]

and making use of the identity (21), we get the required non-autonomous difference Equation (10). When \(\rho \geq 1\) (non-decreasing memory), the main results of Section 2 can be applied, without substantial changes, also to the case of geometric weights. In what follows, we consider the more realistic case of memory ratio \(\rho \in (0, 1)\), giving exponentially fading memory, for which the propositions of Section 2 do not apply,
because the sequence of partial sums $W_n$ converges to $W^* = 1/(1 - \rho)$ and, consequently, sequence $t_n$, defined in (23), is not convergent to zero, being $t_n \to (1 - \rho)$. For $\rho = 0$ (no memory of the past), the problem reduces to the study of the dynamics of an ordinary one-dimensional map $x_{n+1} = f(x_n)$. Since, as it is well known, the asymptotic dynamics of this iteration may be periodic of period $k \geq 1$, or even chaotic, depending on the shape of the function $f$, we can expect complex dynamics also for $\rho > 0$.

We saw in Section 2 that the only possible fixed points of the generalized Mann iteration are the fixed points of the function $f$. One may ask whether also different asymptotic states, such as $k$-cycles, $k \geq 2$, are related to $k$-cycles of the map $f$. The answer is no.

Indeed, if $0 < \rho < 1$, and a $k$-cycle of (10) exists, then, in general, it is not a $k$-cycle of map $f$. However, such cycles are related to those of another one-dimensional (autonomous) map. This can be intuitively justified on the basis of the observation that the sequences of the time-dependent coefficients on the right-hand side of (23) are convergent, since $t_n \to (1 - \rho)$, so that the right-hand side of (10) possesses an autonomous limiting form

$$z_{n+1} = g_\rho(z_n), \quad g_\rho(z) = (\rho z^s + (1 - \rho)|f(z)|^s)^{1/s}.$$  

(24)

It is natural to conjecture that the asymptotic behaviour of (10) is related to that of the map $g_\rho(z)$, and this can be rigorously proved by making use of a two-dimensional map. Let us note, in fact, that the sequence of the partial sums $W_n$ of the geometric weights can be defined recursively by (1), and this allows us to obtain a two-dimensional map $(z_{n+1}, W_{n+1}) = T(z_n, W_n)$ defined as

$$T: \begin{cases}
z_{n+1} = \left(\frac{\rho W_n z_n^s}{1 + \rho W_n} + \frac{1}{1 + \rho W_n} |f(z_n)|^s\right)^{1/s}, \\
W_{n+1} = 1 + \rho W_n.
\end{cases}$$  

(25)

This map is equivalent to (10) if the initial condition is taken with $W_0 = 1$, i.e.

$$(z_0, W_0) = (x_0, 1), \quad x_0 \in I.$$  

(26)

In fact, in such a case, the sequence $\{z_n\}$ given by (25) coincides with the sequence obtained from the generalized segmenting Mann iteration (10) related to the same initial condition $z_0$. In other words, the projection on the $z$-axis of an orbit of the map $T$ (with the initial condition as in (26)) is the orbit of the non-autonomous iterative process (10).

The map (25) is triangular, i.e. a map with the structure $T(z, W) = (T_1(z, W), T_2(W))$. We notice that the map $T$ is not defined on the points of the line of equation $W = -1/\rho$; however, since the initial conditions have to be taken on the line $W = 1$, we shall consider the restriction of $T$ to the half-plane $W > -1/\rho$. Moreover, this half-plane is mapped into itself by $T$, because the second difference equation in (25) gives an increasing sequence (the partial sums of the geometric series starting from $W = 1$) that always converges to the limit

$$W^* = \frac{1}{1 - \rho}.$$  

This also implies that the line $W = W^*$ is mapped into itself by $T$ (i.e. it is a trapping set), and it is globally attracting for $T$ in the half-space $W > -1/\rho$ (which means that for any
point in the domain $W > -1/\rho$, the limit set of its orbit belongs to the trapping line $W = W^*$. In particular, any initial condition (26) has an orbit that is bounded in the rectangle $S = I \times J$, with $J = [1, W^*]$, and the limit set of the orbit belongs to the segment of $S$ on the line $W = W^*$, which is an invariant set of the restriction of $T$ to the line $W = W^*$, namely of the one-dimensional map $g_\rho(z)$ given in (24).

The considerations given above prove the following proposition.

**Proposition 3.1.** Let $f : I \to I$, $0 < \rho < 1$, $g_\rho$ defined in (24) and $T$ defined in (25). Then,

(i) the orbits of the non-autonomous Equation (10) are in one-to-one correspondence with the orbits of the autonomous two-dimensional map $T$ associated with an initial condition on the line $W = 1$;

(ii) the invariant sets of $T$ belong to the line $W = W^*$;

(iii) the invariant sets of $T$ and those of $g_\rho$ are in one-to-one correspondence;

(iv) an invariant set of $T$ is attracting (respectively repelling) if and only if the corresponding invariant set of $g_\rho$ is attracting (respectively repelling).

Now we investigate whether the knowledge of stability/instability of the cycles of the map $g_\rho$ may be useful in order to decide on the existence and on the stability of cycles for the non-autonomous recurrence (10).

An answer to this question can be obtained from an analysis of the global properties of $T$. In fact, from the properties of the limiting map $g_\rho$ we know the local properties of $T$ near the asymptotic line $W = W^*$ but, since the initial conditions for $T$ must be taken on the line $W = 1$, we need a global study of the map $T$ in order to obtain information on the properties of the non-autonomous Equation (10). The following proposition gives an answer to this question.

**Proposition 3.2.** Let $A$ be a $k$-cycle, $k \geq 1$, of the map $g_\rho(z)$, $0 < \rho < 1$. Then,

(i) if $A$ is attracting, or attracting from one side, for the limiting map $g_\rho$, then it is an attracting cycle for the non-autonomous process (10), and hence $f(A)$ is an attracting set of the iteration (22);

(ii) the basin of attraction $D$ of the attractor $f(A)$ of (22) is given by the intersection of the two-dimensional basin, say $\hat{D}$, of the cycle $\hat{A} = A \times \{W^*\}$ of the map $T$ (located on the trapping line $W = W^*$) with the line of initial conditions $W = 1$, i.e. $\hat{D} \cap \{W = 1\} = D \times \{1\}$.

In this proposition, the term attracting $k$-cycle, for the process with memory, means that the process generated by (22) converges asymptotically to the cycle starting from a set of initial conditions of measure greater than zero. It can be noticed that the attracting sets are not, in general, invariant sets (as usual for the non-autonomous processes). This means that, starting from a point of an attracting $k$-cycle, the sequence $\{x_n\}$ generated by (22) may not converge to the $k$-cycle, i.e. the basin of a given attractor may not contain the points of the cycle itself (see Bischi and Naimzada [9], Bischi and Gardini [7]).

If we consider (22) with $z_n = x_n$ (no memory case), then its asymptotic behaviour is indeed given by the study of the map $f(z)$. Conversely, if we consider exponentially fading memory, limit sets of (22) must be searched among the invariant sets of another one-dimensional autonomous map, the limiting map $g_\rho$ defined in (24). However, we remark that their basins of attraction can only be determined through a global study of the two-
dimensional map $T$. As we have already observed, only the fixed points of the map $g_\rho$ coincide with the fixed points of the map $f$, whereas the other invariant sets, $k$-cycles or chaotic sets, are in general different.

Of course, the shape of the map $g_\rho$ depends on that of $f$. From the definition (24), the function $g_\rho(z)$ is a power mean of $z$ and $f(z)$, so for each $z \in I$,

$$\min(z,f(z)) \leq g_\rho(z) \leq \max(z,f(z)).$$

This means that the graph of $g_\rho$ always belongs to the area between the bisector and the graph of $f$, and the graphs of $f$ and $g_\rho$ intersect at the common fixed points.

The derivative of the function $g_\rho$ is

$$g'_\rho(z) = (\rho z^s + (1 - \rho)[f(z)]^s)^{(1-1)/s}(\rho z^s - 1 + (1 - \rho)[f(z)]^s - f'(z)),$$

and if $z^*$ is a positive fixed point of $f$, it becomes

$$g'_\rho(z^*) = \rho + (1 - \rho)f'(z^*),$$

which implies

$$\min(1,f'(z^*)) \leq g'_\rho(z^*) \leq \max(1,f'(z^*)).$$

If $z^* = 0$, i.e. $f(0) = 0$, then $g'_\rho(z^*)$ is not defined. However, in this case,

$$\lim_{z \to 0} g'_\rho(z) = (\rho + (1 - \rho)f'(0))^{1/s}$$

so (29) holds even for $z^* = 0$.

If $-1 < f'(z^*) < 1$, so that $z^*$ is an attracting fixed point of the map $f$, then (29) implies $-1 < g'_\rho(z^*) < 1$, thus $z^*$ is attracting for the map $g_\rho$ too. If $|f'(z^*)| > 1$, so that $z^*$ is a repelling fixed point of $f$, then $z^*$ may be attracting or repelling for $g_\rho$. In particular, if $f'(z^*) > 1$, then $z^*$ is repelling also for $g_\rho$ since from (3), we have $1 < g'_\rho(z^*) < f'(z^*)$, while $f'(z^*) < -1$ gives $f'(z^*) < g'_\rho(z^*) < 1$, and in this case, $z^*$ may be attracting for $g_\rho$.

More exactly, if $f'(z^*) < -1$, let $\tilde{\rho} \in (0,1)$ be defined as

$$\tilde{\rho} = \frac{f'(z^*) + 1}{f'(z^*) - 1}.$$ 

Then, the sufficient condition for the stability of the fixed point of the map $g_\rho$, $|g'_\rho(z^*)| < 1$, which, from (28), can be written as $-(1 + \tilde{\rho})/(1 - \tilde{\rho}) < f'(z^*) < 1$, is satisfied for $\tilde{\rho} < \rho < 1$, i.e. with a sufficiently strong memory. These arguments are summarized in the following proposition, which also states the stabilizing effect of a strong memory.

**Proposition 3.3.** Let $z^*$ be a fixed point of $f$.

(i) If $|f'(z^*)| < 1$, then also $|g'_\rho(z^*)| < 1$ for each $\rho \in (0,1)$.

(ii) If $f'(z^*) < -1$, a value $\tilde{\rho} \in (0,1)$ exists, given by (30), such that $|g'_\rho(z^*)| < 1$ for $\rho < \tilde{\rho} < 1$.

(iii) If $f'(z^*) > 1$, then also $g'_\rho(z^*) > 1$. 
This proposition allows us to distinguish, among the fixed points of the map \( f \), those attracting for the process with a sufficiently strong memory (in particular with a uniform memory, obtained in the limiting case \( \rho \to 1 \)).

4. Power mean learning mechanism in an OLG model

In this section we apply the learning mechanisms studied in the previous sections to an economic system modelled by a law expressed in the forward-looking form

\[ x_n = f(x_{n+1}^{(e)}), \]

where \( x_{n+1}^{(e)} \) represents the expected value of the state variable \( x \) for the next time period. The example that we consider belongs to the family of OLG models, which, among others, was studied by Samuelson [24], Diamond [13], Gale [17], Benhabib and Day [5], Azariadis [2]. In the setting, we consider that population is constant in time, and assume that the economy is characterized by a single perishable consumption good and an asset called money. In each time period \( n \), money is exchanged against good at a price \( p_n \). Economic agents are consumers of a single type; hence, we will only consider one agent as the representative of the whole population. The representative agent lives two periods, in which she is referred to as young (period 1) and old (period 2), respectively, and she possesses time-invariant endowments \( w_1 \) and \( w_2 \) of the good for each period of life.

We suppose that the preferences about current (\( c_n \)) and future (\( c_{n+1} \)) consumptions are given by the following additive lifetime utility function:

\[ U(x_1, x_2) = u(c_n) + v(c_{n+1}), \]

in which both \( u \) and \( v \) are assumed to be twice differentiable, increasing and strictly concave real functions.

In the economy, there is a nominal quantity of money, denoted by \( M \), exogenously determined by the government, which is used to transfer wealth from one period to the next one in the following way: the young consumer saves part of her first period endowment using money, and then she consumes her second period endowment and the saving when old. In this framework, it is supposed that all the money at the beginning of each period is held by the old agent.

The young consumer, on the base of the expected price \( p_{n+1}^e \), must choose the level of consumption of the two periods of life and the nominal amount of money \( m_d \geq 0 \) to save for the second period of life. This can be achieved by solving the optimization problem

\[
\max [u(c_n) + v(c_{n+1})] \text{ under constraints } \begin{cases} \ p_n c_n + m_d = p_n w_n, \\ p_{n+1}^e c_{n+1} = p_{n+1}^e w_2 + m_d, \end{cases}
\]

which straightforwardly leads to

\[ p_{n+1}^e u'(w_2 - \frac{m_d}{p_n}) = p_n v'(w_2 + \frac{m_d}{p_{n+1}^e}).\]  

(32)

If we introduce the demand optimal excess of the good

\[ z_i = c_i - w_i, \quad i = 1, 2, \]
we can write the money demand as

$$m_d(p_n, p_{n+1}^e) = -p_n z_1 \left( \frac{p_n}{p_{n+1}^e} \right) = p_{n+1}^e z_2 \left( \frac{p_n}{p_{n+1}^e} \right).$$

The excess demand of the good by the old consumer during the period $n$ is $M/p_n$; so, in any given period $n$, a competitive equilibrium is described by equating the demand and the supply in both the good and the money market, namely

$$z_1 \left( \frac{p_n}{p_{n+1}^e} \right) + \frac{M}{p_n} = 0, \quad m_d(p_n, p_{n+1}^e) = M.$$ 

Since, by Walras’ Law, the two previous equilibrium conditions are equivalent, we can rewrite (32) as

$$p_{n+1}^e \{w_1 - \frac{M}{p_n}\} - p_n \{w_2 + \frac{M}{p_{n+1}^e}\} = 0.$$ 

The previous relation can be expressed in a compact form as $G(p_n, p_{n+1}^e) = 0$, from which, assuming that $\partial G/\partial p_n \neq 0$, we can obtain an equation of the form (1). In order to run the dynamics of the model, we still have to specify how the agents form expectations about the next period price. A possible choice is given by a relation of the form

$$p_{n+1}^e = \psi(p_{n-1}, p_{n-2}, \ldots, p_0),$$ \hspace{1cm} (33)

in which $\psi$ is a prevision function, which must satisfy

$$x = \psi(x, x, \ldots, x).$$ \hspace{1cm} (34)

In particular, if we assume that the agents have a long memory and the capabilities to compute the power mean expressed in (4), we obtain the general iterative mechanism of the power means. We remark that the no-memory case $\rho = 0$ actually corresponds to the so-called myopic expectations.

In what follows, we will focus on three different examples of the previous OLG model, obtained by considering different utility functions, and we study through simulations the effect of power means with fading memory. These three examples differ for the monotonicity of the map (31), which is either increasing, decreasing or unimodal (see Figure 1). The main goal is to investigate the case of exponentially fading memory.

In the first example, we consider Cobb–Douglas utility functions

$$u(c_n) = \alpha_1 \log c_n, \quad v(c_{n+1}) = \alpha_2 \log c_{n+1},$$ \hspace{1cm} (35)

where $\alpha_i > 0, i = 1, 2$ and $\alpha_1 + \alpha_2 = 1$, which gives the linear temporary equilibrium map

$$p_n = f_1(p_{n+1}^e) = \frac{w_2}{w_1} p_{n+1}^e + \frac{2M}{w_1}.$$ \hspace{1cm} (36)

If we consider the Samuelson case (see [24]) and assume $w_1 > w_2$, the right-hand side of (36) is increasing with respect to $p_{n+1}^e$ and, provided that $p_{n+1}^e$ satisfies (33) and (34), has
the unique positive equilibrium

\[ p^* = \frac{2M}{w_1 - w_2}. \]

The plot of map (36) for \( w_1 = 1, w_2 = 0.5 \) and \( M = 1 \) is reported in Figure 1. Since we assume \( w_2/w_1 < 1 \), we have that equilibrium is unconditionally stable for both myopic and power mean expectations, i.e. for any \( \rho \) and \( s \). As there is just one equilibrium, the dynamics are always convergent to \( p^* \), and they can only be different with respect to how quickly they approach, up to a desired precision, the equilibrium, namely their speed of convergence. Setting \( w_1 = 1, w_2 = 0.5, M = 1 \), in the left plot of Figure 2, we compare the time series of \( p_n \) obtained for \( \rho = 0 \) and for \( \rho = 0.5 \) and \( s = -1, 1, 5 \). The fastest convergence is achieved by the process without memory, and increasing the value of power \( s \) seems to improve the speed of convergence. Similarly, if we set \( s = 1 \) and compare the time series obtained for different values of \( \rho \), we can see that as the weight increases, the convergence speed becomes more and more slow (see the left plot of Figure 2).

Figure 2. Time series of (36) without memory (solid line) and with memory (dashed, dotted and dash-dotted lines) for different exponents \( s \) (left plot) and memory ratios \( \rho \) (right plot). As \( s \) increases, the convergence becomes more and more fast, while increasing \( \rho \) reduces the convergence speed.
The remaining examples are inspired by utility functions studied by Benhabib and Day in [5]. First, we consider

\[ u(c_n) = \log c_n, \quad v(c_{n+1}) = \frac{c_n^{1-\alpha}}{1 - \alpha}, \]  

where \( \alpha \neq 1 \). We remark that in (37), functions \( u \) and \( v \) are swapped with respect to the example proposed in [5].

Assuming \( v_1 = v, v_2 = 0 \), the resulting temporary equilibrium function is given by

\[ p_n = f_2(p^\rho_{n+1}) = \frac{M}{\omega_1} + \frac{M^\alpha}{\omega_1(p^\rho_{n+1})^{\alpha-1}}. \]  

We notice that, for \( \alpha > 1 \), the right-hand side of (38) is decreasing with respect to \( p^\rho_{n+1} \) (see the middle plot of Figure 1, obtained by setting \( w_1 = 1, M = 1 \) and \( \alpha = 1.6 \)), so it has a unique equilibrium \( p^* \), the expression of which, however, cannot be analytically obtained. Moreover, if no memory is considered (\( \rho = 0 \)), we have that (38) can only converge to either the stable equilibrium or a period-2 cycle. If \( \rho < 1 \) is sufficiently large, from Proposition 3.3 we have that the power mean iteration scheme allows \( p_n \) to converge for any value of \( s \) to the equilibrium. In Figure 3, we compare the iterates obtained by setting \( M = 1, \omega = 1 \) and \( \alpha = 50 \) for \( \rho = 0.6 \) and \( s = -1, 1, 5 \). We remark that with myopic expectations, we would have a period-2 cycle, in which \( p_n \) alternates between values, respectively, close to 1 and 2. Introducing a suitable weight allows for converging prices, with a more and more slow convergence as the power \( s \) increases. Similarly, increasing \( \rho \) leads to a slow convergence, which, however, for small values of \( \rho \) exhibits an oscillating behaviour.

Now we investigate the effect on \( p_n \) of varying \( \rho \), obtained by setting again \( M = 1, \omega = 1 \) and \( \alpha = 50 \). We already noticed that if \( \rho = 0 \), the price dynamic can exhibit at most a period-2 cycle. However, introducing a positive memory ratio \( \rho > 0 \), we can have an initial sequence of period doublings leading to chaos, which, when \( \rho \) is further increased, develops a sequence of period halvings leading to convergence towards the equilibrium for \( \rho > 0.47 \), as shown in Figure 4 for \( s = 2 \). Such phenomena are known as bubbling (see [19]). This means that for \( \rho < \hat{\rho} \), fading memory can also introduce an
initial, with respect to $r$, complexity increasing. We also remark that such increasing of complexity does not occur for each power $s$, as shown by the red bifurcation diagram in Figure 4. The simulations that we performed indicate that large values of $s$ produce qualitatively more complex dynamic scenarios. If we keep $r$ fixed and compare the behaviour of $p_n$ on varying $a$, we can notice that, if $r \approx \bar{r}$, even if, as predicted by Proposition 3.3, the loss of stability of the equilibrium occurs for the same value of $a$ for any power $s$, the subsequent route to chaos can be different. For example, the second period doubling occurs for different values of $a$, as shown in Figure 5, which once more suggests that qualitatively simpler behaviours are induced by larger values of $s$ (we point out that only one branch of the period-4 bifurcation is actually visible from Figure 5, as that arising from the value close to 1 is very flat).

To investigate this aspect, we numerically computed the value of $a$ at which the second period doubling occurs, giving rise to the period-4 cycle. To this end, we define function $\hat{\alpha}_4 : \mathcal{R} \to (1, +\infty]$ that, for any given $s$, provides the infimum of the values of bifurcation parameter $a$ for which a period-4 cycle occurs. If, for a given $s$, a period-4 cycle never occurred, we would have $\hat{\alpha}_4(s) = +\infty$. We computationally estimate $\hat{\alpha}_4(s)$ and report the results in Figure 6, in which we can see that $\hat{\alpha}_4$ is a decreasing function, which implies that we have an increasingly retarded route towards chaos for small values of $s$ (we remark that the larger the $\hat{\alpha}_4$, the later the period-4 cycle, and the subsequent cascade of flip bifurcations occurs).

In the last example, we set

$$u(c_n) = bc_n, \quad v(c_{n+1}) = -ae^{-c_{n+1}}, \quad \tag{39}$$

where $a, b$ are positive constants. For any initial endowments $w_1, w_2$, the resulting temporary equilibrium function is given by

$$p_n = f_3\left(p_{n+1}^c\right) = \frac{\left(p_{n+1}^c\right)^2}{ae^{-c_{n+1}}} \quad \tag{40}$$

Figure 4. (Colour online) Bifurcation diagrams for (38) on varying $\rho$, with power means prediction function, for powers $s = 2$ (black) and $s = -2$ (red). Even if without memory at most a period-2 cycle is possible; when $\rho > 0$, both a qualitatively similar level of complexity ($s = -2$) or a complete sequence of period doubling/halving ($s = 2$) are possible.
which, for $a > 1$, has the unique equilibrium

$$p^* = \frac{M}{\log (a)}.$$  

Function $f_3$ is unimodal, as shown on the right plot of Figure 1, obtained for $a = 2$ and $M = 1$. As predicted by Proposition 3.3, the dynamic with power means is convergent for $\rho > \bar{\rho}$ defined by (30), which, since $f'(p^*) = 1 - \log (a)$, is $\bar{\rho} = (\log (a) - 2)/(\log (a))$. As the bifurcation parameter $a$ varies, differently from (38), the dynamic generated by (40) with myopic expectations ($\rho = 0$) consists in a complete cascade of flip bifurcations towards chaos. Introducing a non-null memory ratio, the power mean iteration scheme is able to stabilize the equilibrium for $a \in (1, \exp (2/(1 - \rho)))$, and so setting $a = \exp (2/(1 - \rho))$, for any value of $s$, a flip bifurcation occurs. However, for (40)
also, the subsequent behaviour depends on $s$, as qualitatively described by the bifurcation diagrams in Figure 7. As for (38), we define the function $\hat{a}_4 : \mathcal{R} \to (1, +\infty)$, so that $\hat{a}_4(s)$ represents the infimum of the set of values of $a$ for which a period-4 cycle occurs. We compute $\hat{a}_4$ numerically for $\rho = 0.1$ and report the results in Figures 8 and 9, distinguishing between $s < \hat{s}$ and $s > \hat{s}$ with $\hat{s} \approx -0.648$. When $s$ is sufficiently small, increasing $s$ corresponds to an increase of $\hat{a}_4$, as reported in Figure 8. Conversely, when $s > -0.648$, the behaviour seems to be more complicated, as reported in Figure 9, from which we can notice that, when $s$ lies in a right neighbourhood of $\hat{s}$, the threshold $\hat{a}_4(s)$ is much smaller than that for $s \to \hat{s}^-$. To better understand the discrepancy between the behaviours with $s < \hat{s}$ and $s > \hat{s}$, we refer to the bifurcation diagrams reported in Figure 10. Looking at the behaviour of $\hat{a}_4$ reported in Figure 8, we expect that $\hat{a}_4(s)$ be increasingly larger than 250 for $s > \hat{s}$.

Figure 7. (Colour online) Bifurcation diagrams of (40), in which both myopic expectations (black) and memory ratio with two different values of $s$ (red, blue) are considered.

Figure 8. (Left plot) Function $\hat{a}_4(s)$ for $s < -0.648$ obtained through simulations. In this subdomain, the function is increasing. (Right plot) Bifurcation diagrams for three different values of $s$. 

Figure 10. Bifurcation diagrams for three different values of $s$. 

Figure 9. Function $\hat{\alpha}_4(s)$ for $s > -0.648$ obtained through simulations. For these values of $s$, $\hat{\alpha}_4(s)$ is non-monotonic.

Figure 10. Bifurcation diagrams for model (40) for $\rho = 0.1, s = -0.6$ (first row) and $s = -0.5$ (second row). Left column: for small values of $a$, after the first period doubling, a bubble develops, consisting of either a period-4 cycle (top plot) or a complex sequence of period doubling/halving, leading to a period-2 cycle again. Right column: for large values of $a$, the period-2 cycle evolves through a sequence of period doublings.
Indeed, if we look at bifurcation diagrams reported on the right column of Figure 10, we can see that a period-4 indeed occurs for increasingly larger values of $a$ for $s > \hat{s}$ too. However, if we compute the same bifurcation diagrams for small values of $a$ (left column plots of Figure 10), we can see that a new, transient, increase of complexity occurs, giving rise to a bubble. For values of $s$ sufficiently close to $\hat{s}$, we have only a couple of period doubling/halving appears, while slightly increasing $s$, an increasingly complex sequence of period doublings and halvings occurs. In any case, it seems that the function $\hat{a}_4$ attains its minimum value when an (approximate) geometric mean is considered ($s \approx 0$).

We found that $\hat{a}_4$ shows a behaviour similar to those reported in Figures 8 and 9, and also when different parameter settings or values of $\rho$ are considered. However, we remark that the discriminating value $\hat{s}$ seems to depend on $\rho$.

Even if the results that we reported considering (36), (38) and (40) are only numerical, we can indeed say that, when the more realistic case of fading memory $\rho < 1$ is considered, the choice of a particular power mean is relevant. The value of $s$ affects not only the number of iterations required to approach the equilibrium (and this would indeed be true also for $\rho \geq 1$ and for the averaging processes considered in Section 2), but can also give rise to very different behaviours when stability is lost. In particular, the increasing in complexity can occur for very different values of the parameters. Moreover, the most natural choice of arithmetic mean could not be the most efficient one, as shown, for instance, by the last example, where the values close to $\hat{s}$ allow for the most retarded route towards chaos.

An analytical investigation of the previous phenomena is beyond the purposes of the present work; however, it is indeed one of the research aims that we want to focus on, to better understand the behaviour of the period-4 appearance and of the further complexity increasing with respect to the shape of the recurrence function (monotone, unimodal, multimodal) and to investigate the conditions under which bubbles appear.

5. Conclusions

In this paper, we studied an iterative scheme of the form $x_{n+1} = f(z_n)$, where $z_n$ is a weighted power mean of all the previous state variables $x_0, \ldots, x_n$. Our contribution extends some existing results about arithmetic mean only to a general class of commonly used algebraic means (including arithmetic, quadratic, harmonic and geometric means).

These iterative schemes can be used to model learning mechanisms in economic and social systems, where the agents use all the available past data to compute the expected values by some averaging method.

A particular distribution of weights, exponentially decreasing like the terms of a geometric series of ratio $\rho$, has been used to investigate the effects of a fading memory on the asymptotic properties of the discrete process. This has been obtained through the reduction of the problem to the study of an equivalent two-dimensional triangular map whose asymptotic behaviour is governed by a one-dimensional map.

This allows us to state that the presence of a strong memory, i.e. with a memory ratio suitably close to 1, has a stabilizing effect. Conversely, if the memory ratio is too small, we may not have convergence. In this case, we numerically investigated the route towards chaos, focusing on three particular examples arising from OLG models. The computational analysis suggests that the choice of the particular exponent $s$ and the memory ratio $\rho$ conditions the (retarded or early) appearance of period-4 cycles and the subsequent cascades of flip bifurcations. We aim to analyse such aspects in future research.
Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

This work was performed within the framework of COST Action IS1104 ‘The EU in the new economic complex geography: models, tools and policy evaluation’ and under the auspices of GNFM, Gruppo Nazionale di Fisica Matematica (Italy).

Notes

1. Email: gian.bischi@uniurb.it
2. Email: ahmad.naimzada@unimib.it

References


