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The values distribution in a competing shares financial market model
A financial market model with endogenous fundamental values through imitative behavior

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In this paper, we propose a financial market model with heterogeneous speculators, i.e., optimistic and pessimistic fundamentalists that, respectively, overestimate and underestimate the true fundamental value due to ambiguity in the stock market, which prevents them from relying on the true fundamental value in their speculations. Indeed, we assume that agents use in its place fundamental values determined by an imitative process. Namely, in forming their beliefs, speculators consider the relative profits realized by optimists and pessimists and update their fundamental values proportionally to those relative profits. Moreover, differently from the majority of the literature on the topic, the stock price is determined by a nonlinear mechanism that prevents divergence issues. For our model, we study, via analytical and numerical tools, the stability of the unique steady state, its bifurcations, as well as the emergence of complex behaviors. We also investigate multistability phenomena, characterized by the presence of coexisting attractors.

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The past decades have seen a growing interest in behavioral financial market models, i.e., models in which, in addition to the description of the mechanisms governing the updating of the financial variables, it is assumed that speculators are not fully rational, but that they rather use simple decisional rules and continuously update their beliefs and/or decisional rules on the basis of evolutionary selection and learning (see Ref. 13).

In particular, starting from Ref. 6, it is often assumed that traders choose between two behavioral forecasting rules concerning the future development of the stock price: fundamentalism and chartism. Fundamentalists (or contrarians), believing that stock prices will return to their fundamental value, buy stocks in undervalued markets and sell stocks in overvalued markets, while chartists (or trend followers) behave in the opposite manner.

Inside the heterogeneous agent literature, only few papers deal with heterogeneous fundamentalists.\textsuperscript{9,10,16,19,21,22,24,27} Our paper belongs to such strand of literature and, in particular, like in Refs. 9 and 19, we assume that optimistic (pessimistic) fundamentalists systematically overestimate (underestimate) the reference value due to a belief bias. However, differently from those papers, the belief biases in our model are not exogenous, but are rather determined by an imitative process. Indeed, we suppose that due to the ambiguity in the financial market generated by the uncertainty about the future stock price, agents do not rely on the true fundamental value, but that in forming their beliefs about the reference value, speculators consider the relative profits realized by optimists and pessimists and update their fundamental values proportionally to those relative profits.

I. INTRODUCTION

In this paper, we propose a financial market model with heterogeneous speculators, i.e., optimistic and pessimistic fundamentalists that, respectively, overestimate and underestimate the true fundamental value due to ambiguity in the stock market, which prevents them from relying on the true fundamental value in their speculations. Indeed, we assume that agents use in its place fundamental values determined by an imitative process.

The papers in the related financial market literature on belief biases that bear more resemblance to ours are Refs. 9 and 19, even if crucial differences are also present.

More precisely, in Ref. 9 both the optimistic and pessimistic belief biases and the perceived fundamental value are exogenously determined, while in Ref. 19 the agents perceive an endogenous fundamental value. Differently from those papers, in the present model even the belief biases are not exogenous, but are rather determined by an imitative process. Namely, in forming their beliefs, speculators consider the relative profits realized by optimists and pessimists and update their fundamental values proportionally to those relative profits. Such kind of updating mechanism bears resemblances to the so-called “Proportional Imitation Rules” in Ref. 25, whose main features read as

- follow an imitative behavior, i.e., change actions only through imitating others;
imitate an individual that performed better with a probability that is proportional to how much better this individual performed.

Moreover, our updating mechanism is similar to the switching mechanism in Ref. 2, used in Ref. 9, too.

In regard to the literature on endogenous fundamental value and/or endogenous beliefs about it, we also recall,\textsuperscript{4,7,8,16} where the fundamental value of assets is assumed to evolve as a random walk,\textsuperscript{27} where, due to mistakes in information processing, time variations in both the value of the fundamental and in its perceived value are allowed;\textsuperscript{15} where the fundamental value of stocks is assumed to be constant, but time variations in the beliefs about the fundamental are allowed.

Another important feature of our model consists in assuming, unlike the majority of the literature on the topic, the stock price to be determined by a nonlinear mechanism, so as to prevent divergence issues. More precisely, similarly to Ref. 20, the price dynamics are described by a sigmoidal price adjustment mechanism characterized by the presence of two asymptotes that bound the price variation and thus the dynamics.

For our model, we study, via analytical and numerical tools, the stability of the unique steady state, its bifurcations, as well as the emergence of complex behaviors. We also investigate multistability phenomena, characterized by the presence of coexisting attractors.

More precisely, after analytically deriving the expression of the unique steady state and the stability conditions with respect to the parameter describing the intensity of the imitative process ($\beta$) and with respect to the parameter describing the maximum possible degree of optimism and pessimism ($\Delta$), we study the possible dynamics arising when using $\beta$ and $\Delta$ as bifurcation parameters. The results we find in the two ways are analogous. In particular, we show that when the market maker price adjustment parameter ($\gamma$) is small enough and thus the isolated financial market, obtained for $\beta = 0$ or $\Delta = 0$ is stable, an increasing value for $\beta$ or $\Delta$ may be destabilizing. When instead $\gamma$ is sufficiently large and the isolated financial market is unstable or even chaotic, increasing $\beta$ or $\Delta$ may be stabilizing. If however one of such two parameters is too large, this may lead to instability again. When the isolated financial market is unstable, it is also possible that increasing $\beta$ or $\Delta$ reduces the complexity of the dynamics, but without reaching a complete stabilization. We provide an economic interpretation of our results, explaining the rules governing the dynamics of price and of fundamental values. Finally, we illustrate how rich multistability phenomena may arise for our model, characterized for instance by the presence of four coexisting attractors.

The remainder of the paper is organized as follows. In Section II, we introduce the model. In Section III, we perform the corresponding stability analysis. In Section IV, we illustrate the role of the main parameters on the stability of the system and we present the bifurcation analysis. In Section V, we interpret the dynamics we observe from an economic viewpoint. In Section VI, we analyze a scenario characterized by multistability phenomena. Finally, in Section VII, we discuss the results and propose some possible extensions of the model.

II. THE MODEL

As mentioned in Sec. I, we are interested in analyzing a financial market model with heterogeneous agents. Hence, the two crucial aspects to be described in this scenario are the behavioral rules of speculators and the mechanism of price formation (see Ref. 13). More precisely, we will deal with two groups of fundamentalists (i.e., agents that believing that stock prices will return to their fundamental value, buy stocks in undervalued markets and sell stocks in overvalued markets), whose behavior is captured by the dynamic motions of the corresponding perceived fundamental values. In particular, we will consider optimists, that overestimate the true fundamental value, and pessimists, that instead underestimate it. Indeed, in our model, after gathering all the orders and computing excess demand, as usual the market maker sets the stock price for the next period; the main innovation we introduce concerns the fundamentalists’ decisional mechanism and consists in considering, instead of a fixed fundamental value, time-varying beliefs about the fundamental value, as the result of an imitative process. Namely, we assume that due to the ambiguity in the financial market generated by the uncertainty about the future stock price, agents do not rely on the true fundamental value but that they rather consider the relative profits realized by optimists and pessimists and, still remaining optimists or pessimists, update their fundamental values proportionally to those relative profits.

The majority of the existing literature on behavioral financial market models deals with a linear price adjustment mechanism. Even when a nonlinear price adjustment mechanism is considered, authors usually consider a multiplicative formulation (see, for instance, Refs. 23, 26, and 28), which admits $P = 0$ as steady state. However, both the linear and the multiplicative formulations do not impose any bound on the price variation and thus may allow overreaction phenomena, which in turn lead to instability and/or divergence issues. Differently from such approaches, we consider a nonlinear price adjustment mechanism which determines a bounded price variation in every time period. In particular, we assume that the adjustment mechanism is $S$-shaped, and thus we specify the price variation as

$$P(t + 1) - P(t) = \gamma a_2 \left( \frac{a_1 + a_2}{a_1 \exp(-D(t)) + a_2} - 1 \right),$$

where $\gamma > 0$ represents the market maker price adjustment reactivity and $D(t)$, whose expression will be specified in (2.5), reflects the orders placed by the two groups of fundamentalists at time $t$. Moreover, $a_1$ and $a_2$ are positive parameters playing the role, together with $\gamma$, of horizontal asymptotes. Indeed, with the choice in (2.1), $P(t + 1) - P(t)$ is increasing in $D(t)$ and vanishes when $D(t) = 0$; moreover, $P(t + 1) - P(t)$ is bounded from below by $-\gamma a_2$ (obtained...
when $D(t) \to -\infty$ and from above by $\gamma a_1$ (obtained when $D(t) \to +\infty$). Hence, the price variations in (2.1) are gradual and the presence of the two horizontal asymptotes prevents the dynamics of the stock market from diverging and helps avoiding negativity issues. As regards the specific shape of the price adjustment mechanism in our work, instead of describing the price variation through a piecewise linear map, we chose to deal with the sigmoidal function in (2.1), which is differentiable and thus simplifies the mathematical treatment of the model. Moreover, the presence of $a_2$ also in front of the sigmoid allows obtaining, in addition to the upper bound $\gamma a_1$, also a lower bound ($-\gamma a_2$) for the price variation depending on the parameters $a_1$ and $a_2$. In particular, increasing (decreasing) such parameters, we obtain an increase (decrease) in the possible price variations. We finally observe that differently from the majority of the literature on the topic, in which it is assumed that the behavior of the market maker is symmetric with respect to variations in excess demand that have opposite signs but coincide in absolute value, we allow $a_1$ and $a_2$ to be possibly different, so that we can deal with more general frameworks in which the market maker can react in a different manner to a positive or to a negative excess demand.

We recall that the adjustment mechanism on the r.h.s. of (2.1), determining a bounded variation of a given variable, has been already considered in Refs. 17–20. More precisely, in Refs. 17–19, we imposed a bound on the variation of the output variable in macroeconomic models, both without (see Ref. 17) and with (see Refs. 18 and 19) the stock market sector, while in Ref. 20, we imposed a bound on the price variation in a financial market model.

The model we are going to study reads as follows:

\[
\begin{align*}
X(t + 1) &= f \frac{e^\beta \sigma_X(t + 1)}{1 + e^{-\beta(X(t) - \sigma_X(t + 1))}} + F \frac{e^\beta \sigma_X(t + 1)}{1 + e^{-\beta(X(t) - \sigma_X(t + 1))}} \\
Y(t + 1) &= F \frac{e^\beta \sigma_Y(t + 1)}{1 + e^{-\beta(Y(t) - \sigma_Y(t + 1))}} + \bar{f} \frac{e^\beta \sigma_Y(t + 1)}{1 + e^{-\beta(Y(t) - \sigma_Y(t + 1))}} \\
P(t + 1) &= P(t) + \gamma a_2 \left( \frac{a_1 + a_2}{a_1 e^{-\gamma \sigma_Y(X(t) - P(t)) + (1 - \gamma \sigma_Y(Y(t) - P(t))} + a_2} - 1 \right)
\end{align*}
\]

or, equivalently,

\[
\begin{align*}
X(t + 1) &= f \frac{1}{1 + e^{-\beta(X(t) - \sigma_X(t + 1))}} + F \frac{1}{1 + e^{-\beta(X(t) - \sigma_X(t + 1))}} \\
Y(t + 1) &= F \frac{1}{1 + e^{-\beta(Y(t) - \sigma_Y(t + 1))}} + \bar{f} \frac{1}{1 + e^{-\beta(Y(t) - \sigma_Y(t + 1))}} \\
P(t + 1) &= P(t) + \gamma a_2 \left( \frac{a_1 + a_2}{a_1 e^{-\gamma \sigma_Y(X(t) - P(t)) + (1 - \gamma \sigma_Y(Y(t) - P(t))} + a_2} - 1 \right),
\end{align*}
\]

where $X(t)$ ($Y(t)$) describe the fundamental value of the pessimists (optimists), who always underestimate (overestimate) the true fundamental value $F$, i.e., $X(t) < F < Y(t)$. Moreover, $f$ and $\bar{f}$ represent, respectively, the lower bound and the upper bound of the ranges in which the fundamental values $X(t)$ and $Y(t)$ may vary, for which we assume that $0 < f < F < \bar{f}$. Indeed, by construction it holds that for every $t \geq 1$, $X(t) \in (f, F)$ and $Y(t) \in (F, \bar{f})$. We stress that the formulation above implies that agents do not rely on the true fundamental value because, by the ambiguity in the financial market, even if traders know such value $F$, they cannot rely on it. On the contrary, in each time period, agents remain always pessimists, and then their fundamental value lies in the interval between the lower bound $f$ and the true fundamental value $F$, or always optimists, and then their fundamental value lies in the interval between $F$ and the upper bound $\bar{f}$.

The variable $P(t)$ describes the stock price and $\sigma_X(t + 1)$ and $\sigma_Y(t + 1)$ are the profits for the two types of speculators, defined, respectively, as

\[
\sigma_X(t + 1) = (P(t + 1) - P(t))\sigma_X(X(t) - P(t))
\]

and

\[
\sigma_Y(t + 1) = (P(t + 1) - P(t))\sigma_Y(Y(t) - P(t)),
\]

where $\sigma_X$ and $\sigma_Y$ are positive parameters representing the reactivities of optimistic and pessimistic agents, respectively. Moreover, $\beta \geq 0$ represents the intensity of the imitative process, in which agents, still remaining pessimists or optimists, proportionally imitate those who obtain higher profits. In particular, when $\beta = 0$, then $X(t + 1) \equiv \frac{1}{2} (f + F)$ and $Y(t + 1) \equiv \frac{1}{2} (F + \bar{f})$ and thus there is no imitation. When instead $\beta \to +\infty$, then if $\sigma_X > \sigma_Y$, i.e., it is more profitable to be pessimists, then $X(t + 1) \to f$ and $Y(t + 1) \to F$, i.e., both variables tend towards their lowest possible value, while if $\sigma_X < \sigma_Y$, then $X(t + 1) \to F$ and $Y(t + 1) \to \bar{f}$. Notice that the coefficient
which describes the relative profits of traders of type $i$, coincides with the share in the next period of agents of type $i$ in the discrete choice model (see Refs. 1 and 2). Furthermore, assuming a normalized population of size one, $\omega \in (0, 1)$ represents the fraction of the population composed by pessimists, so that total excess demand reads as

$$D(t) = \omega \sigma_X(X(t) - P(t)) + (1 - \omega) \sigma_Y(Y(t) - P(t)), \quad (2.5)$$

where the orders placed by pessimists and optimists are weighted with their corresponding shares. Finally, as already explained, $\gamma > 0$ is the market maker price adjustment reactivity parameter, and $a_1$ and $a_2$ are two positive parameters bounding, together with $\gamma$, the price variation, and thus the dynamics.

In order to simplify our analysis, we assume that $f$ and $\bar{f}$ lay at the same distance $\Delta$ from $F$, i.e., that $f = F - \Delta$ and $\bar{f} = F + \Delta$. In this manner, $\Delta \geq 0$ describes the maximum possible degree of pessimism and optimism and it may be used as bifurcation parameter in our analytical and numerical results about local stability. Indeed, in Sections III and IV, we will use $\beta$ and $\Delta$ as bifurcation parameters.

Using the notation just introduced, and noticing that the weight coefficients in (2.2) sum up to 1, we may rewrite that system as

$$\begin{align*}
X(t+1) &= F - \Delta \left( \frac{1}{1 + e^{-\beta(\pi_X(t+1) - \pi_Y(t+1))}} \right), \\
Y(t+1) &= F + \Delta \left( \frac{1}{1 + e^{\beta(\pi_X(t+1) - \pi_Y(t+1))}} \right), \\
P(t+1) &= P(t) + \gamma a_2 \left( \frac{a_1 + a_2}{a_1 e^{-(\omega \sigma_X(X(t) - P(t)) + (1 - \omega) \sigma_Y(Y(t) - P(t)))} + a_2} - 1 \right). 
\end{align*} \quad (2.6)$$

We then start our analysis by studying the steady states of (2.6).

**Proposition 2.1** System (2.6) has a unique steady state in $(X^*, Y^*, P^*) = \left( F - \frac{\Delta}{2}, F + \frac{\Delta}{2}, F - \frac{\Delta}{2} \right)$. Proof. The expression of the steady state can be easily found by noticing that, in equilibrium, the last equation in (2.6) implies that $P^* = \frac{\omega \sigma_X X^* + (1 - \omega) \sigma_Y Y^*}{\omega \sigma_X + (1 - \omega) \sigma_Y}$ and that $\pi_X = \pi_Y = 0$, so that $X^* = F - \frac{\Delta}{2}$ and $Y^* = F + \frac{\Delta}{2}$. Inserting such expressions in $P^*$, we get the desired conclusion. □

Hence, the steady state values for $X$ and $Y$ are symmetric with respect to $F$ and they lie at the middle points of the intervals in which they may, respectively, vary. In particular, when $\Delta = 0$ we find $X^* = Y^* = P^* = F$, like in the classical framework without belief biases and imitation, in which there are two groups of agents, of size $\omega$ and $1 - \omega$ and, respectively, characterized by the reactivities $\sigma_X$ and $\sigma_Y$, which use $F$ as fundamental value. As we shall see in Sections III and IV, in this case, the system inherits the stability/instability of the financial market. We observe that also when $\sigma_X = \sigma_Y$ and $\omega = \frac{1}{2}$, we find $P^* = F$, even if now $X^* \neq F \neq Y^*$.

We finally stress that it is possible to rewrite $P^*$ as

$$P^* = \frac{\omega \sigma_X F - \frac{\Delta}{2} + (1 - \omega) \sigma_Y (F + \frac{\Delta}{2})}{\omega \sigma_X + (1 - \omega) \sigma_Y}.$$
in the sense that the two systems generate the same trajectories.

Hence, for simplicity, in Section III, we will deal with \( G \) to analytically derive the stability conditions for our model, while for the numerical simulations, we will rely on the original formulation in (2.6), in order to illustrate the behavior of all variables. We stress however that we also derived the stability conditions for the three-dimensional system in (2.6) using the method in Ref. 11, obtaining exactly the same results. For sake of brevity, we preferred to report only the computations for the simpler two-dimensional case.

III. STABILITY ANALYSIS

Since for our analytical results we will deal with the two-dimensional framework in (2.7), we observe that similarly to what done in Proposition 2.2, it is possible to prove that (2.7) has a unique fixed point in \((X^*,P^*)\) when

\[
(F - \frac{\Delta}{2}, F - \frac{\Delta}{2} + \frac{\Lambda}{2 \Lambda + (1 - \omega) \sigma})
\]

We are going to derive the local stability conditions for our system at the steady state by using the well-known Jury conditions (see Ref. 14).

To such aim, we need to compute the Jacobian matrix for \( G \) in correspondence to \((X^*,P^*)\), which reads as

\[
J_G(X^*,P^*) = \begin{bmatrix}
\frac{\Delta^2 \beta \gamma \sigma_X \sigma_Y}{4} & \frac{\Delta^2 \beta \gamma \sigma_X \sigma_Y}{4} \\
\frac{\gamma (\omega \sigma_X + (1 - \omega) \sigma_Y)}{4} & 1 - \frac{\gamma (\omega \sigma_X + (1 - \omega) \sigma_Y)}{4}
\end{bmatrix},
\]

where we set \( \gamma = \frac{\gamma \sigma_X (1 - \omega) \sigma_Y}{\gamma (\omega \sigma_X + (1 - \omega) \sigma_Y)} \).

Denoting by \( \det(J) \) and \( \text{tr}(J) \) the determinant and the trace of the above Jacobian matrix, respectively, the Jury conditions read as follows

(i) \( \det(J) < 1 \)

(ii) \( 1 + \text{tr}(J) + \det(J) > 0 \)

(iii) \( 1 - \text{tr}(J) + \det(J) > 0 \).

In our framework, we have

\[
\det(J) = \frac{\beta \Delta^2 \gamma \sigma_X \sigma_Y}{4},
\]

\[
\text{tr}(J) = \frac{\beta \Delta^2 \gamma \sigma_X \sigma_Y}{4} + 1 - \frac{\gamma (\omega \sigma_X + (1 - \omega) \sigma_Y)}{4},
\]

and thus, making \( \beta \) explicit, when \( \Delta \neq 0 \) (i)–(iii) are fulfilled if

\[
2 \left( \frac{\gamma (\omega \sigma_X + (1 - \omega) \sigma_Y)}{4} - 2 \right) < \beta < \frac{4}{\gamma (\omega \sigma_X + (1 - \omega) \sigma_Y)}.
\]

We stress that usually in the literature increasing the parameter, strongly related to our \( \beta \), describing the intensity of choice in the switching mechanism between different decisional rules has just a destabilizing effect (see for instance Ref. 13), while for us it may also be stabilizing (cf. Figures 2 and 3). In fact, also in Ref. 9, two stability thresholds for the intensity of choice parameter were found, corresponding, respectively, to a flip and a Hopf bifurcation. On the other hand, before the flip bifurcation in that paper, some numerical simulations we performed suggest that the system diverges and thus that bifurcation would not lead to complex behaviors: as we shall see in Section IV, in our framework, the flip bifurcation is instead preceded by periodic and possibly chaotic motions. Also in Ref. 5, two stability thresholds for a different parameter (i.e., the population weighted reaction coefficient of the fundamentalists) were detected. However, as recalled in Ref. 9, in Ref. 5 too, the authors, in the numerical simulations, focus their attention on the case in which stability of the steady state is lost through a Hopf bifurcation.

In (i)–(iii), it is also possible to make \( \Delta \) explicit, finding that when \( \beta \neq 0 \), the stability conditions read as

\[
\frac{2 \left( \frac{\gamma (\omega \sigma_X + (1 - \omega) \sigma_Y)}{4} - 2 \right)}{\gamma (\omega \sigma_X + (1 - \omega) \sigma_Y)} < \Delta < \frac{4}{\gamma (\omega \sigma_X + (1 - \omega) \sigma_Y)},
\]

or as

\[
\gamma (\omega \sigma_X + (1 - \omega) \sigma_Y) < 2 \text{ and}
\]

\[
\Delta < \frac{2}{\gamma (\omega \sigma_X + (1 - \omega) \sigma_Y)}.
\]

We stress that the above achievements are valid when \( \beta \neq 0 \), i.e., when the dynamics of \( G \) are “truly” two-dimensional. When instead \( \beta = 0 \) or \( \Delta = 0 \), the dynamics are generated by the financial market only and the latter is locally asymptotically stable at \( P^* = F \) if

\[
\gamma < \frac{2}{\omega \sigma_X + (1 - \omega) \sigma_Y} = \gamma^*.
\]

Indeed, at \( \gamma^* \) a flip bifurcation occurs, as the canonical conditions in Ref. 12 are fulfilled. This shows, in particular, that \( \gamma \) has a destabilizing effect on the isolated financial market. Notice that due to the nonlinearity of the price adjustment mechanism, differently from Ref. 9, we do not face...
divergence issues when the isolated financial market is unstable. Hence, the flip bifurcation opens for us a route to chaos, not to divergence (see Figures 2, 3, 7, and 8).

We stress that rewriting (3.1) as

$$\frac{2\omega}{\sigma_t \Delta^2} + \frac{2(1 - \omega)}{\sigma_x \Delta^2} - \frac{4}{\gamma \sigma_x \sigma_t \Delta^2} < \beta < \frac{4}{\gamma \sigma_x \sigma_t \Delta^2}$$

and (3.2) as

$$\sqrt{\frac{2\omega}{\Delta} + \frac{2(1 - \omega)}{\sigma_x \beta}} < \Delta < \frac{2}{\sqrt{\gamma \sigma_x \beta}}$$

the destabilizing role of $\tilde{\gamma}$, and thus, in particular, of $\gamma$ becomes evident. Since by (3.3), $\tilde{\gamma}$ has a destabilizing effect also on the isolated financial market, we may explain the presence of two thresholds for stability in (3.1) and (3.2) as follows. When $\tilde{\gamma}$ is large enough, the isolated financial market is unstable and, through the imitative process, small but nonnull values for $\beta$ and $\Delta$ allow the transmission of such turbulence to the imitative process and thus to the dynamics of the fundamental values. When $\beta$ and $\Delta$ increase further, looking at the first two equations in (2.6), we observe that intermediate values for $\beta$ dampen large profits and this makes the fundamental values for both pessimists and optimists stabilize on the mean value they may assume, i.e., on their steady state values; similarly, intermediate values for $\Delta$ make the fundamental values in the next period lie not too distant from $F$ and this again dampens large profits. On the other hand, when $\beta$ and $\Delta$ are too large, they become destabilizing. Indeed, large values for $\beta$ represent a high degree of nervousness in the imitative mechanism, leading to erratic fluctuations in the fundamental values; large values for $\Delta$ make the distances between the fundamental values and $F$ increase, allowing for large profits and thus to complex dynamics for the fundamental values.

We also notice that when $\tilde{\gamma}$ is too small, the left threshold value in (3.1) and the radicand in (3.2) become negative and thus only sufficiently large values for $\beta$ and $\Delta$ may destabilize the system.

In the numerical simulations in Sections IV–VI, for simplicity, we will focus on frameworks with $\sigma_x = \sigma_f = 1$ and $\omega = 0.5$. Indeed, as we shall see, the most significant form of heterogeneity in the model is represented by the different attitudes of agents towards the reference value rather than by such reactivity parameters and the numerosity of the two groups.

When $\sigma_x = \sigma_f = 1$ and $\omega = 0.5$, the expression for the steady state becomes

$$(X^*, Y^*, P^*) = \left(F - \frac{\Delta}{2}, F + \frac{\Delta}{2}, F\right).$$

Moreover, the stability conditions with respect to $\beta$ for $\Delta \neq 0$ read as

$$\frac{2(\tilde{\gamma} - 2)}{\tilde{\gamma} \Delta^2} < \beta < \frac{4}{\tilde{\gamma} \Delta^2}, \tag{3.4}$$

while the stability conditions with respect to $\Delta$, when $\beta \neq 0$ read as

$$\sqrt{\frac{2(\tilde{\gamma} - 2)}{\tilde{\gamma} \Delta^2}} < \Delta < \frac{2}{\sqrt{\tilde{\gamma}} \beta}, \tag{3.5}$$

if $\tilde{\gamma} \geq 2$, or as

$$\Delta < \frac{2}{\sqrt{\tilde{\gamma}} \beta}, \tag{3.6}$$

if $\tilde{\gamma} < 2$.

When $\beta = 0$ or $\Delta = 0$, the stability conditions for the isolated financial market at $P^* = F$ simply read as $\tilde{\gamma} < 2$.

In Section IV, we will split our numerical analysis according to the different scenarios, in terms of stability, obtainable when using $\beta$ and $\Delta$ as bifurcation parameters. As we shall see, the two parameters allow to observe the same kinds of dynamical frameworks and thus our exposition will be quite symmetric. More precisely, we will consider various frameworks for the stock price dynamics when $\beta = 0$ (in Section IV A) and $\Delta = 0$ (in Section IV B), and then we will, respectively, increase the parameters describing the intensity of the imitative process and the maximum possible degree of optimism and pessimism, in order to observe the qualitative changes on the dynamics of our system resulting when joining the stock price mechanism with the imitative process.

IV. BIFURCATION ANALYSIS

A. The role of $\beta$

In this subsection, we will take $\beta$ as bifurcation parameter, similarly to Ref. 9, where the intensity of choice for the switching mechanism is used. However, differently from that context, in which only the destabilizing role of $\beta$ is illustrated, we split our analysis into three different dynamical frameworks.

1. First scenario: Destabilizing role of $\beta$

We start by presenting a framework in which increasing the value of $\beta$ leads to instability, and, in particular, to the occurrence of quasiperiodic motions.

According to (3.4), for the parameter configuration considered in Figure 1, our system is locally asymptotically stable at $(X^*, Y^*, P^*) = (1.6, 2.4, 2)$ for $\beta \in \left(\frac{2(\tilde{\gamma} - 2)}{\tilde{\gamma} \Delta^2}, \frac{4}{\tilde{\gamma} \Delta^2}\right)$, $\cap [0, +\infty) = (-9.375, 12.5) \cap [0, +\infty) = (0, 12.5)$.

In more detail, the threshold for the flip bifurcation is $\beta = \frac{2(\tilde{\gamma} - 2)}{\tilde{\gamma} \Delta^2} = -9.375 < 0$, and thus is not visible in Figure 1, as it makes no economic sense, while the Hopf bifurcation occurs for $\beta = \frac{4}{\tilde{\gamma} \Delta^2} = 12.5$.

As already recalled, such framework is in line with the numerical simulations in Ref. 9.

2. Second scenario: Mixed role of $\beta$

We now present a framework in which intermediate values for $\beta$, neither too small nor too large, reduce the complexity of the system, until a complete stabilization of the
dynamics. We stress that such scenario is obtained from the previous one just by increasing the value for \( \gamma \), which indeed by (3.3) has a destabilizing effect on the financial market, when isolated.

According to (3.4), for the parameter configuration considered in Figure 2, our system is locally asymptotically stable at \((X^*, Y^*, P^*) = (1.6, 2.4, 2)\) for \( \beta \in \left(\frac{2(\gamma-2)}{2\Delta}, \frac{4}{3\Delta}\right) \) = (0.625, 2.5). In particular, the flip bifurcation occurs for \( \beta = \frac{2(\gamma-2)}{2\Delta} = 0.625 \), while the Hopf bifurcation occurs for \( \beta = \frac{4}{3\Delta} = 2.5 \).

As argued in Section III, an increasing value for the intensity of the imitative process (or, more generally, for the intensity of choice) has usually just a destabilizing effect, while for us it may also be stabilizing. This happens because, when \( \beta \) is positive but close to 0, through the imitative process, the instability of the financial market gets transmitted to the dynamics of the fundamental values, which inherit the period-two cycle of the isolated financial market. Increasing values for \( \beta \) intensify the oscillations due to optimism and pessimism, but when \( \beta \) is sufficiently large, positive and negative excess demands for the two groups of agents balance out in the aggregate excess demand and this causes smaller price oscillations, which in turn make the profit differential decrease and this leads to smaller variations for the fundamental values of optimists and pessimists. When \( \beta \) increases further, agents become however very reactive in updating the fundamental values and this causes the emergence of complex, quasiperiodic dynamics.

Still in such “mixed” scenario, we now show that when choosing \( a_1 \neq a_2 \), complex dynamics can occur also for small values of \( \beta \), and not just for values of that parameter for which the Hopf bifurcation has already occurred, like in Figure 2. This happens when the isolated financial market (that we obtain for \( \beta = 0 \)) displays chaotic dynamics, so that such feature gets transmitted to the whole system, as long as the parameter governing the intensity of the imitative process is not too large. We illustrate such framework in Figure 3, where we also observe a homoclinic bifurcation for \( \beta \) positive but very close to 0. Moreover, we stress that the diverging trajectories that would follow the flip bifurcation in the case of a linear price adjustment mechanism (see Ref. 9) are limited by the asymptotes \( a_1 \) and \( a_2 \), setting the trajectories on a period-two cycle. In particular, due to the coexistence between the period-two cycle and the fixed point, the trajectories visit the former or the latter according to the chosen initial condition.

For a deeper analysis of the role of \( a_1 \) and \( a_2 \), we refer the interested reader to Ref. 20.

In more detail, according to (3.4), for the parameter configuration considered in Figure 3, our system is locally asymptotically stable at \((X^*, Y^*, P^*) = (1.6, 2.4, 2)\) for \( \beta \in \left(\frac{2(\gamma-2)}{2\Delta}, \frac{4}{3\Delta}\right) \approx (1.117, 2.008) \). In particular, the flip bifurcation occurs for \( \beta = \frac{2(\gamma-2)}{2\Delta} \approx 1.117 \), while the Hopf bifurcation occurs for \( \beta = \frac{4}{3\Delta} \approx 2.008 \).

3. Third scenario: No stabilization with \( \beta \)

We finally present a framework in which sufficiently large values for \( \beta \) may reduce the complexity of the system, leading to the existence of stable periodic orbits, but are not able to guarantee a complete stabilization of the dynamics.
According to (3.4), for the parameter configuration considered in Figure 4, the stability conditions at \((X^*, Y^*, P^*) = (1.6, 2.4, 2)\) would read as
\[
1.823 \approx \frac{2(\tilde{\gamma} - 2)}{\gamma \Delta^2} < \beta < \frac{4}{\gamma \Delta^2} \approx 1.302.
\]
Hence, they are satisfied for no values of \(\beta\).

We stress that the numerical configurations considered in Figures 1–4 were characterized by increasing values for \(\gamma\) (indeed, we have \(\tilde{\gamma} = 0.5\) in Figure 1, \(\tilde{\gamma} = 2.5\) in Figure 2, \(\tilde{\gamma} \approx 3.112\) in Figure 3, and \(\tilde{\gamma} \approx 3.466\) in Figure 4), in line with the analytical results in Section III, according to which \(\tilde{\gamma}\) has a destabilizing effect. The same remark applies to the ordering of the numerical configurations we will consider in Section IV B, where the overall complexity gradually increases (indeed, we have \(\tilde{\gamma} = 0.5\) in Figure 6, \(\tilde{\gamma} = 2.25\) in Figure 7, \(\tilde{\gamma} \approx 3.223\) in Figure 8, and \(\tilde{\gamma} \approx 4.144\) in Figure 9).

In order to illustrate the destabilizing role of \(\tilde{\gamma}\) and, in particular, of \(\gamma\), for the isolated financial market, in Figure 5, we report the bifurcation diagram for \(P\) with respect to \(\gamma\) when \(\beta = 0\), showing that at \(\gamma \approx 2.769\), i.e., at \(\tilde{\gamma} = 2\), in agreement with the analytical results from Section III, \(P^* = 2\) loses its stability through a flip bifurcation and then undergoes a cascade of period-doubling bifurcations leading to chaos. We stress that the parameter configuration considered in Figure 5 is the same used in Figures 3 and 4, except for \(\beta\) there varying in suitable intervals, while \(\gamma\) is there kept fixed at \(\gamma = 4.31\) and \(\gamma = 4.8\), respectively. Notice that for both such values of \(\gamma\), Figure 5 confirms the chaotic behavior found for \(\beta = 0\) at the left end of Figures 3 and 4.

B. The role of \(\Lambda\)

In this subsection, we will use \(\Delta\) as bifurcation parameter and, similarly to what done in Section IV A, we will split our analysis in three different dynamical frameworks.

1. First scenario: Destabilizing role of \(\Lambda\)

We start by presenting a framework in which increasing the value of \(\Delta\) leads to instability, and, in particular, to the occurrence of quasiperiodic motions.

Since \(\tilde{\gamma} = 0.5 < 2\), according to (3.6), for the parameter configuration considered in Figure 6, our system is locally asymptotically stable at \((X^*, Y^*, P^*) = (1.3 - \frac{\Delta}{2}, 1.3 + \frac{\Delta}{2}, 1.3)\) for \(\Delta \in (-\infty, \frac{2}{\sqrt{\tilde{\gamma}}}] \cap [0, F] \approx (-\infty, 0.894) \cap [0, 1.3] = [0, 0.894]\). In particular, the threshold for the flip bifurcation would be \(\Delta = \sqrt{\frac{25 - 21}{4}} = \sqrt{0.6}\), while the Hopf bifurcation occurs for \(\Delta = \sqrt{\frac{25 - 21}{4}} \approx 0.894\).

2. Second scenario: Mixed role of \(\Lambda\)

We now present a framework in which intermediate values for \(\Delta\), neither too small nor too large, reduce the complexity of the system, until a complete stabilization of the dynamics. We stress that, like for \(\beta\), such scenario is obtained from the previous one just by increasing the value for \(\gamma\).

Since \(\tilde{\gamma} = 2.25 > 2\), according to (3.5), for the parameter configuration considered in Figure 7, our system is locally asymptotically stable at \((X^*, Y^*, P^*) = (1.3 - \frac{\Delta}{2}, 1.3 + \frac{\Delta}{2}, 1.3)\) for \(\Delta \in (-\infty, \frac{2}{\sqrt{\tilde{\gamma}}}] \cap [0, F] \approx (-\infty, 0.894) \cap [0, 1.3] = [0, 0.894]\). In particular, the threshold for the flip bifurcation would be \(\Delta = \sqrt{\frac{25 - 21}{4}} = \sqrt{0.6}\), while the Hopf bifurcation occurs for \(\Delta = \sqrt{\frac{25 - 21}{4}} \approx 0.894\).
bifurcation occurs for $\Delta = \frac{2\sqrt{\tilde{\gamma} - 2}}{\tilde{\gamma} - 1} \approx 1.323$, while the Hopf bifurcation occurs for $\Delta = \frac{2}{\sqrt{\tilde{\gamma}} \beta} \approx 1.575$.

3. Third scenario: No stabilization with $\Delta$

We finally present a framework in which intermediate values for $\Delta$ may reduce the complexity of the system, leading to the existence of stable periodic orbits, but are not able to guarantee a complete stabilization of the dynamics.

Since $\tilde{\gamma} \approx 4.144 > 2$, according to (3.5), for the parameter configuration considered in Figure 9, the stability conditions at $(X^*, Y^*, P^*) = \left(4 - \frac{3}{2}, 4 + \frac{3}{2}, 4\right)$ would read as

$$1.438 \approx \frac{2(\tilde{\gamma} - 2)}{\tilde{\gamma} \beta} < \Delta < \frac{2}{\sqrt{\tilde{\gamma}} \beta} \approx 1.389.$$ 

Hence, they are satisfied for no values of $\Delta$.

V. ECONOMIC INTERPRETATION OF THE RESULTS

In the present section, starting from some time series, we will try to explain the rules governing the dynamics of the stock price and of the fundamental values. In particular, we will investigate how the former depends on excess demand in the previous period and the latter on the difference of profits between pessimists and optimists at the beginning of the same period.

A. The dynamics of price

Making reference to the time series for $X$, $Y$, and $P$ in Figure 10, let us start noticing that since $\sigma_X = \sigma_Y = 1$ and $\omega = 0.5$, then the expression for the excess demand at time $t$ reads as

$$ED(i) = 0.5(X(i) - P(i)) + 0.5(Y(i) - P(i)).$$ (5.1)

We now focus our attention on two particular instants of time, denoted by $i$ and $j$ in Figure 10, at which the three variables display a different behavior.
Indeed, at $t = \bar{t}$ it holds that $P(t) > Y(t) > X(t)$, and thus $ED(t) < 0$. Recalling the stock price equation in (2.6), that using (5.1) may be rewritten as

$$P(t + 1) = P(t) + \gamma a_2 \left( \frac{a_1 + a_2}{a_1 e^{-ED(t)} + a_2} - 1 \right), \quad (5.2)$$

and noticing that the term inside the parentheses is negative, we find $P(t + 1) < P(t)$, in agreement with Figure 10.

Looking now at $t = \bar{t}$, where it holds that $Y(t) > P(t) > X(t)$ and $|Y(t) - P(t)| > |X(t) - P(t)|$, we find $ED(t) > 0$ and thus, by (5.2), it follows that $P(t + 1) > P(t)$, like happens in Figure 10.

**B. The dynamics of fundamental values**

Making reference to the time series in Figure 11 for $X$, $Y$, $P$, and the difference $\pi_X - \pi_Y$ between the profits of pessimists and optimists, let us start observing that since $a_X = a_Y = 1$, then, by (2.3) and (2.4), the expression for $\pi_X - \pi_Y$ at time $t + 1$ reads as

$$\pi_X(t + 1) - \pi_Y(t + 1) = (P(t + 1) - P(t))(X(t) - Y(t)). \quad (5.3)$$

Notice that by (5.2) and the expression for the excess demand in (5.1), both $P(t + 1)$ and the profit differential in (5.3) may be deduced at time $t$.

Focusing again our attention on the same time instants $\bar{t}$ and $\bar{t}$, we considered also in Figure 10, let us analyze how do the fundamentals of pessimists and optimists evolve starting from those two points.

At $t = \bar{t}$, it holds that $P(t + 1) < P(t)$ and, since $X(t) - Y(t)$ is negative, by (5.3) it follows that $\pi_X(t + 1) - \pi_Y(t + 1) > 0$, i.e., the profits realized by pessimists are higher than the profits realized by optimists. This increases the pessimism in the whole population, and thus we observe that $X(t + 1) < X(t)$ and $Y(t + 1) < Y(t)$, i.e., the fundamental values for both pessimists and optimists decrease.

At $t = \bar{t}$, it holds instead that $P(t + 1) > P(t)$ and hence by (5.3) we have that $\pi_X(t + 1) - \pi_Y(t + 1) < 0$, so that the profits realized by pessimists are now lower than the profits realized by optimists. This increases the overall optimism and thus we observe that $X(t + 1) > X(t)$ and $Y(t + 1) > Y(t)$, i.e., the fundamental values for both pessimists and optimists increase.

**VI. SOME MULTISTABILITY PHENOMENA**

In the present section, we illustrate and analyze a setting particularly rich in multistability phenomena, characterized by the coexistence of cyclic attractors of various periods with different chaotic attractors, in one or more pieces. We stress that we found several other parameter configurations for which interesting multistability phenomena arise. However, for sake of brevity, we chose to report the one in Figure 12 only, where we take $\beta$ as bifurcation parameter and where we observe a period-three cycle, which at $\beta = 0.3$ starts coexisting with two chaotic attractors in one piece, the former of which splits into two pieces, while the latter, after subdividing into two pieces, too, is followed first by a stable fixed point and then by periodic or chaotic attractors, in one or more pieces.

In particular, by (3.4), for the parameter configuration considered in Figure 12 we find that the fixed point $(X^*, Y^*, P^*) = (2.2, 3, 2.6)$ is locally asymptotically stable for $\beta \in \left(\frac{2(\bar{t} - 1)}{\gamma}, \frac{4}{\gamma}\right) \approx (1.394, 1.730)$, and that it undergoes a flip bifurcation for $\beta = \frac{2(\bar{t} - 1)}{\gamma} \approx 1.394$ and a Hopf
bifurcation for $\beta = \frac{4}{7} \approx 1.730$. We stress that on the left of the flip bifurcation, we do not see any period-two cycle because it exists just for a small interval of values for $\beta$. This is a signal of the divergence issues that we would face without introducing our nonlinear price adjustment mechanism and which would prevent us from observing the interesting dynamics we find for $\beta < 1.394$.

Moreover, we show in Figure 13 the phase portrait in the $(X, P)$-plane corresponding to the previous bifurcation diagram for $\beta = 2.5658$, where we find the coexistence among a chaotic attractor in two pieces (in blue), a chaotic attractor in ten pieces (in green), a period-three cycle (in red), and a period-twelve cycle (in pink). We remark that the period-twelve cycle is not visible in Figure 12.

VII. CONCLUSIONS AND POSSIBLE EXTENSIONS

In the present paper, we proposed a financial market model with heterogeneous speculators, i.e., optimistic and pessimistic fundamentalists that, respectively, overestimate and underestimate the true fundamental value due to ambiguity in the stock market that prevents them from relying on the true fundamental value in their speculations. Indeed, we assumed that agents use in their place fundamental values determined through an updating mechanism based on the relative profits realized by optimists and pessimists, which recalls the “Proportional Imitation Rules” in Ref. 25. In this manner, both the belief biases and the perceived fundamental values are endogenously determined, differently from the papers in the financial market literature on belief biases that bear more resemblance to ours, i.e., Refs. 9 and 19. Namely, in Ref. 9, both the optimistic and pessimistic belief biases and the perceived fundamental value are exogenously determined, while, in Ref. 19, the agents perceive an endogenous fundamental value, but the belief biases are exogenous.

Moreover, unlike the majority of the literature on the topic, the stock price is for us determined by a nonlinear mechanism, already considered in Ref. 20, which prevents divergence issues.

For our model we studied, via analytical and numerical tools, the stability of the unique steady state, its bifurcations, as well as the emergence of complex behaviors. We also investigated multistability phenomena, characterized by the presence of coexisting attractors.

Of course, our model may be extended in various directions and indeed we mean it as a first step in view of investigating the consequences of heterogeneity from an evolutionary viewpoint. More precisely, in the next pages, we will briefly describe some suitable ways to generalize the present framework, we are going to deal with in future papers.

A. Endogenous switching mechanism

A first interesting extension of our model can be obtained by adding to the present structure an endogenous switching mechanism, similar for instance to the one in Ref. 9, so that agents may not only update their fundamental values according to the relative profits realized by optimists and pessimists, however still remaining optimists and pessimists, but can also switch to the other group of speculators, if they performed better.

In such new framework, when setting $\beta = 0$ it holds that $X(t + 1) = F - \frac{X}{2}$ and $Y(t + 1) = F + \frac{X}{2}$. Hence, in this case, there is no imitation in the updating of the fundamental values that indeed are fixed. Thus, when $\beta = 0$, we enter the framework in Ref. 9 with bias $a = \frac{X}{2}$ except for the presence of our nonlinear price adjustment mechanism which replaces the linear price equation in Ref. 9. When instead $\beta \neq 0$, the fundamental values are no more constant and we generalize the setting in Ref. 9, with the only exception of our nonlinear price equation.

B. Relaxing the symmetry condition

A second very natural extension of our model, in view also of increasing the heterogeneity among agents, may consist in considering (possibly) different degrees of optimism and pessimism for agents, i.e., allowing $\Delta X \neq \Delta Y \in [0, F]$ in System (2.6), that becomes

$$
\begin{align*}
X(t + 1) &= F - \Delta X \left( \frac{1}{1 + e^{-\beta (x_{1}(t+1)-x_{1}(t+1))}} \right) \\
Y(t + 1) &= F + \Delta Y \left( \frac{1}{1 + e^{-\beta (y_{2}(t+1)-y_{2}(t+1))}} \right) \\
P(t + 1) &= P(t) + \gamma a_2 \left( \frac{a_1 + a_2}{a_1 e^{-\beta (x_{1}(t)-P(t)) + (1-a_1) a_2 (y_{2}(t)-P(t))} + a_2} \right).
\end{align*}
$$

(7.1)
In this case, instead of the result in Proposition 2.2, for all \( t \geq 1 \) it holds that

\[
Y(t + 1) = X(t + 1) + \Delta X \frac{1}{1 + e^{\alpha(t+1)-\eta(t+1)}} + \Delta Y \frac{1}{1 + e^{\beta(t+1)-\eta(t+1)}}.
\]

Hence, it is no more possible to reduce the dimensionality of our problem, as the three variables \( X, Y, \) and \( P \) are now linearly independent and the dynamics generated by System (7.1) are “truly” three-dimensional.

C. Introducing unbiased fundamentalists and chartists

A third possible extension may concern, similarly to Refs. 3 and 9, the introduction in our model of a group of unbiased fundamentalists (or contrarians) and a group of unbiased chartists (or trend followers), whose degrees of optimism and pessimism are null, and which use the true fundamental value in their speculations. The aim is that of investigating the effects of such further groups of agents on the dynamics of the system, in order to check whether, like in Ref. 9, the former group has a stabilizing role, i.e., its presence makes the stability region become larger, while the latter group is destabilizing.

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