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Footnote Information

Incomplete financial markets with real assets and wealth-dependent credit limits

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1 **Abstract** In this paper we analyze the effects of restricted participation in a two-period
2 general equilibrium model with incomplete financial markets and two key elements:
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13 frictions resulting in an inefficient allocation the regulation of markets may have a
14 Pareto-improving effect on the economy.

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15 **Keywords** General equilibrium · Restricted participation · Financial markets ·
 16 Generic regularity · Real assets · Pareto suboptimality

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18 1 Introduction

19 In this paper we analyze a two-period general equilibrium model with incom-
 20 plete financial markets and two key elements: (1) the competitive trading of real
 21 assets, where real assets have payouts in terms of vectors of commodities, and (2)
 22 household-specific inequality constraints that restrict participation in the financial
 23 markets.

24 A formal introduction of time and uncertainty in a general equilibrium model dates
 25 back to Chapter 7 of "Theory of Value" by [Debreu \(1959\)](#). There, it is observed
 26 that enlarging the concept of commodity in order to include not only its physical and
 27 chemical characteristics, but also its location in space, time and state of the world, it
 28 is possible to show that existence and Pareto optimality of equilibria still hold true. In
 29 other words, those standard general equilibrium results can be obtained assuming the
 30 presence of markets in which all commodities, in the loose sense defined by Debreu,
 31 can be exchanged.

32 [Arrow \(1964\)](#)'s contribution is to observe that, while it is hard to believe that
 33 assumption is satisfied, it can probably be assumed that there exist sufficiently diversi-
 34 fied and numerous available assets which are able to behave as well as those plethora
 35 of markets: indeed, equilibrium allocations in a market à la Debreu are the same as
 36 equilibria with commodity markets opening only inside each state of the world and
 37 enough available assets. On the other hand, it can also be claimed that in a quite uncer-
 38 tain world, available assets are not enough to insure against each possible future event:
 39 financial markets and, therefore, commodity à la Debreu markets, may be incomplete.
 40 That observation led to an important literature which focuses on the validity, in that
 41 more general and realistic environment, of the three main results in general equilib-
 42 rium theory: existence, Pareto optimality and generic regularity of equilibria. While
 43 the first result is a sort of consistency check of the model and the second one is one
 44 of the main goal of economic analysis, the importance of the third, apparently only
 45 technical, result lies in the fact that it is the preliminary, indispensable tool for the
 46 analysis of existence of second best equilibria and of the chance of success of eco-
 47 nomic policies—see for example Chapter 15 in [Villanacci et al. \(2002\)](#). Indeed, the
 48 validity of the three above results was analyzed in the incomplete market framework
 49 with respect to some abstract types of assets, usually denoted in the literature as nom-
 50 inal, numeraire and real assets. Nominal assets promise to deliver units of account;
 51 real assets a bundle of goods; and numeraire assets just some amount of a given good,
 52 the so called numeraire good.

53 Each of the three main results of the general equilibrium model fails to hold true in
 54 one or more of the above defined types of incomplete market frameworks, as roughly
 55 summarized in the following table

	Existence	Generic local uniqueness	Pareto optimality
Numeraire assets	Yes	Yes*	No*
Nominal assets	Yes	No*	No*
Real assets	Yes*	Yes*	No*

(1)

where “Yes” means “it is always insured”, “Yes*” means “it is generically,¹ but not always insured”, and “No*” means “it is generically not insured”.²

Two important constructive criticisms to the model of incomplete markets, concerning the absence of restricted participation and of default possibilities, led to more general and realistic models [see for instance [Seghir and Torres-Martinez \(2011\)](#) and [Geanakoplos and Zame \(2014\)](#)].

“While there might be some disagreement over whether, in a modern developed economy, financial markets are actually incomplete, there can hardly be any disagreement over whether at least some economic agents are variously constrained in transacting on those financial markets” ([Cass 1992](#)). In other words, financial markets may be complete, but households surely face a personalized and restricted access to financial markets.

The literature on possibility of default removes the assumption that agents always honor their financial obligations. This literature has a basic link with the literature on restricted participation. It is indeed usually assumed that some commodity and/or financial collateral is required to partially counteract the damages of default. The corresponding conditions to be added to the standard model with incomplete markets make the model of default (and collateral) as a particular model of restricted participation with real assets. While existence and (lack of) Pareto optimality have been proved in such contexts, the existing literature has little to say about generic regularity in those models. The general goal of our paper is to study that problem.

Indeed, apart from the seminal paper by [Radner \(1972\)](#), “universal”³ existence proofs can be found in [Geanakoplos and Zame \(2014\)](#) in the case of collateral constraints, as well as in [Seghir and Torres-Martinez \(2011\)](#) and in [Gori et al. \(2013\)](#) in the case of financial constraints depending on some endogenous variables. The main limitation of those results is that the analytical methods used there seem unable to verify other equilibrium properties. By contrast, the technique we use to guarantee generic regularity also verifies the generic existence⁴ of an equilibrium, the generic (Pareto) suboptimality of the equilibrium allocations, and ensures that numerical methods [see [Kubler \(2007\)](#), for example] can be used to compute equilibria.

To the best of our knowledge, the only model of restricted participation with real assets in which generic regularity is proven is the one by [Polemarchakis and Siconolfi](#)

¹ “Generically” means in an open and full measure subset of the finite dimensional economy space.

² Table 1 and the comments above are taken from [Villanacci et al. \(2002\)](#), the reader is referred to for further discussion.

³ By universal existence of equilibria, we mean existence for any element in the economy space. As already recalled, existence or regularity is generic if it holds in a large (to be appropriately defined) subset of that space.

⁴ Consistently with the above quoted papers, we do conjecture that equilibria exist for every economy.

(1997). Yet, in that paper, the restriction sets for the asset choices are difficult to interpret. Specifically, each household is exogenously associated with a linear subspace of the possible wealth transfers. Its restriction set is then described by the orthogonal projection of that subspace on the (price dependent) image of the return matrix. We are unable to provide an economic interpretation of that modeling choice and the authors themselves provide no explanation at all.

What we offer is an analysis of economically meaningful constraints on households' participation in financial markets and a general approach to determine which constraints can be analyzed in a real asset setting.

The participation constraints described in our model, consistent with what occurs in the market for bank loans, impose an upper bound, called credit limit, on a household's future debt in all states of uncertainty in the final period. A credit limit is the maximum amount that a household is allowed to borrow.⁵ It is an assessment of a prospective borrower to determine the likelihood that she will default on debts. For example, it is the most that a credit card company will allow a card holder to take out on a credit card in a given period of time. This limit is based on the household's credit risk, which depends on a range of factors, such as her employment stability, level of income, level of debt, credit history. Households with a higher expected wealth will generally be loaned more money, that is, have a higher credit limit. In other words, there are two components to credit worthiness. One is a borrower's current and projected ability to repay a loan or offer of credit. This can be determined by looking at things like income, other debts the borrower is carrying, expenses, and future employment opportunities. Another issue is the borrower's inclination to repay debts, a surely more complicated matter. Some negative signals on that inclination are repeated delinquencies on other debts, sluggish repayment of loans, and other entries in a borrower's credit history. One tool lenders can use to quickly assess credit worthiness is to consult a credit rating agency. These agencies monitor consumers and keep track of their financial activities to generate a credit score. Such rating is often used as a device to set credit limits. Credit limits provide some important advantages. For example, they serve to protect the borrower from borrowing too much money, and they also protect the lender from being over exposed to borrower's getting too far into debt. Credit limits may however cause some disadvantages. For example, they may severely restrict the borrowing capacity of the borrower, reducing the value of items that they can purchase. As a result, credit limits can also reduce the potential income received by the lender because borrowers are limited in what they can borrow.

Our goal is to model the above economic observations in a general equilibrium model with two periods of time and absence of default. To formalize both components describing how credit limits are determined (credit risk and inclination to repay) we proceed as follows. The amount each household can borrow must be such that the household can repay what is due in all states in the final period. Additionally, each

⁵ Explanations on how credit limits work may be found even on web-pages for non-specialists on financial markets, such as <http://www.wisegeek.com/what-is-a-credit-limit.htm>, <http://www.wisegeek.com/what-is-credit-worthiness.htm>, http://en.wikipedia.org/wiki/Credit_limit or <http://www.creditorweb.com/definition/credit-limit.html>. We believe this can be taken as a signal of their wide diffusion in real world markets for loans, and thus we think that their effects on the households' wealth are worth studying.

household is assumed not being able or willing to consume less than a given proportion of her wealth in each state in the second period. Those requirements lead to a household specific wealth-dependent credit limit, formalized in the borrowing constraint in (7).⁶

For our model, we show generic existence of equilibria and generic regularity, the latter being an indispensable tool to both describe equilibria and to prove several important normative properties of equilibria.

The first property we prove is that the equilibria are generically suboptimal. Additionally, we conjecture that a form of generic constrained suboptimality holds. We are verifying this conjecture in a companion paper, in which we restrict attention to the significant set of economies in which a sufficiently high number of participation constraints are binding. Generically in that set of economies, these equilibria are Pareto improvable through a local change of the participation constraints. The general strategy that is used in that framework is described in [Carosi et al. \(2009\)](#).

In the current paper, we describe a more direct approach to constrained suboptimality for a specific economy. Our contribution is in line with a general viewpoint dating back to what is usually mentioned as Lipsey–Lancaster theory of second best—see [Lipsey and Lancaster \(1956\)](#), according to which an increase in the level of a market imperfection may lead to a Pareto improvement. In the framework of incomplete markets, [Hart \(1975\)](#) showed that decreasing the number of available assets, and thus increasing the incompleteness of the markets, may increase efficiency. We rephrase and show the Lipsey–Lancaster claim in our model of restricted participation on financial markets. Indeed, we present a robust example in the economy space whose associated equilibria are such that using a more restrictive credit policy results in a Pareto improvement.

Without exaggerating the importance of a robust example, we do believe that the presented one supports two simple ideas. First of all, it shows the importance of general equilibrium analysis, i.e., the fact that price effects are able to more than welfare compensate a reduction of available choices. Moreover, it suggests, contrary to what is often cited as economic wisdom in the popular press, that in a setting with frictions resulting in an inefficient allocation, the regulation of markets may have a Pareto-improving effect on the economy.

We think our analysis contains both technically and economically significant features. We first discuss our technical contributions to the analysis of the problem of generic existence and generic regularity of equilibria.

The seminal contribution for generic existence in a model with real assets is the paper by [Duffie and Shafer \(1985\)](#). To prove generic existence, they first set the dimension of the return space equal to the number of available assets, define the resulting equilibrium a “pseudo equilibrium”, and show the existence of such an equilibrium. Finally, they prove that these pseudo equilibria are true equilibria for a generic subset

⁶ We stress that there is no history explicitly modeled in our two-period framework. Indeed, in our model the parameters describing the the borrower’s inclination to repay debts are exogenously determined, implicitly assuming that an unmodeled device like a rating agency has been able to determine the household specific credit limits, taking into account their credit history. Moreover our model analyzes neither default nor physical, financial collateral requirements even though we consider our paper as a first step in that direction. Finally, we observe that our framework does not apply to other institutional environments, like secondary security markets.

of household endowments and asset payouts. With the inequality constraints that we use to model households' restricted participation, the household demand functions are in general not C^1 , and therefore the equilibrium manifold is not C^1 either. This fact prevents us from using the smooth analysis arguments in Duffie and Shafer (1985). Rather, we employ a fixed point argument based on the approach by Dierker (1974) for the Walrasian model and later generalized and formalized by Husseini et al. (1990) for the incomplete markets model with real assets. To the best of our knowledge, we provide the first application of the methodology in Husseini et al. (1990).

Duffie and Shafer (1985) use a "fixed dimension return space" approach⁷ with results in terms of the *kernel* of a well chosen linear function. In our model, characterizing equilibria using that approach would allow us to verify the existence of appropriately defined pseudo equilibria, but we are then unable to show that the pseudo equilibria are generically true equilibria.⁸ To circumvent this problem, we take a different approach by presenting a natural characterization of equilibria in terms of the *image* of an appropriately chosen linear function.

For those interested in the technical aspects of our proof,⁹ we preview the strategy used to obtain the generic existence result. As previously discussed, once the definition of equilibrium is introduced, we define a type of equilibrium in which we fix the dimension of the return space. Then, as done in the approach followed by Duffie and Shafer (1985), we use a Mr. 1 trick,¹⁰ i.e., we get rid of the explicit presence of the financial side of the economy using the introduction of a specific household, Mr. 1, who behaves as a Walrasian consumer. After showing that the latter two concepts are equivalent, we prove that they are a "true" equilibrium if a (standard) full rank condition of the return matrix holds true—see Proposition 9.

The technical reason to introduce two different types of auxiliary equilibria is described by the following logic. Using the approach by Dierker (1974) in the form of the theorem proved by Husseini et al. (1990), we are able to show the existence of a Mr. 1 equilibrium. Yet, as mentioned above, we are then unable to complete the next step as we are unable to verify that the projection function from the equilibrium set to the economy space is proper. This is a required step for the genericity argument. We can verify properness by using the equivalent concept of (normalized price) symmetric equilibrium in Definition 6.

We now discuss the economically significant features of both our model and our proof methodology. In terms of the type of participation constraints we employ, we believe they are realistic and economically meaningful. They are meant to represent the market for bank loans. Consider that legal requirements (or uncontractable social norms) are present that guarantee households a base level of consumption expenditure.

⁷ This terminology is borrowed from Bich and Cornet (1997).

⁸ Specifically, it is not clear how to show properness of the projection from the equilibrium set to the economy space, since it is not possible to uncouple two multipliers and prove that they converge separately.

⁹ In order not to overburden the present paper, we chose to report only the proofs of Theorems 13 and 14, the two major results of our paper. The interested reader may find the proofs of all results in the working paper version (Hoelle et al. 2012).

¹⁰ The terminology "Mr. 1 equilibrium" was introduced in the seminal existence paper by David Cass—see Cass (1984)—and commonly used since then.

206 Thus, for all states of uncertainty, a household is only able to repay previous debts
 207 if this leaves the household with at least this base level of consumption expenditure.
 208 Knowing this, the financial markets only permit borrowing up the point where the
 209 household is able to repay the loan and not be reduced to consumption expenditure
 210 below the base level.

211 To show the economic relevance of the constraints, we consider a simple example of
 212 our model. For the particular economy chosen, the equilibrium allocation can be Pareto
 213 improved by tightening the participation constraints in some states, without loosening
 214 the participation constraints in any other states, for all households. This result may seem
 215 counter-intuitive, but demonstrates the importance of general equilibrium price effects
 216 in financial markets. Thus, restricting credit access may in fact make all households
 217 better off.

218 More generally, the analysis presented in the paper provides what we believe
 219 are crucial conditions on the *type* of constraints for which generic regularity
 220 can be verified, at least following what currently seems to be the only suc-
 221 cessful approach: the fixed dimension return space approach. As will be dis-
 222 cussed further in Sect. 2,¹¹ both the kernel approach [that is used in [Duffie and](#)
 223 [Shafer \(1985\)](#)] and the image approach (that we use here) require that the con-
 224 straints on the financial side of the economy are rewritten in terms of constraints
 225 on the real side, specifically in terms of the values of the excess demands in
 226 each state. In the former approach, the financial side simply disappears from
 227 the household maximization problems. In the latter one, we must introduce ficti-
 228 tious asset demands and we recognize that constraints imposed on the fictitious
 229 asset demands may not be equivalent to constraints imposed on the true asset
 230 demands.

231 Future research is required to confirm this conjecture about the type of con-
 232 straints that can be employed in models with real assets. Given this negative result,
 233 any attempts to obtain regularity for interesting models of collateral and default
 234 may be in vain. The reason is that any known approaches to modeling collateral
 235 and default involve restrictions that differ from the types of restrictions that we
 236 described above as being successful. Again, future research is required in this direc-
 237 tion.

238 The rest of the paper is organized as follows. In Sect. 2, we present the set up
 239 of the model. In Sect. 3, we introduce some equivalent definitions of equilibria with
 240 fixed dimension of the return space. In Sect. 4, we state the results of existence of
 241 these equilibria, together with the generic existence, generic regularity and generic
 242 suboptimality of true equilibria. Section 5 contains the numerical example and the
 243 Appendix collects the proofs of Theorems 13 and 14.

244 2 Set up of the model

245 Our model builds on the standard two-period, general equilibrium model of pure
 246 exchange with uncertainty. In the commodity markets, $C \geq 2$ different physical

¹¹ See especially the part immediately after condition (13).

Author Proof

247 commodities are traded, denoted by $c \in \mathcal{C} = \{1, 2, \dots, C\}$. In the final period,
 248 only one among $S \geq 1$ possible states of the world, denoted by $s \in \{1, 2, \dots, S\}$,
 249 will occur. The initial period is denoted $s = 0$ and we define the set of all states
 250 $\mathcal{S} = \{0, 1, \dots, S\}$ and the set of uncertain states $\mathcal{S}' = \{1, \dots, S\}$. In the initial
 251 period, asset markets open and $A \geq 1$ assets are traded, denoted by $a \in \mathcal{A} =$
 252 $\{1, 2, \dots, A\}$.

253 We assume $A \leq S$. Finally, there are $H \geq 2$ households, denoted by $h \in \mathcal{H} =$
 254 $\{1, 2, \dots, H\}$. The time structure of the model is as follows: in the initial period,
 255 households exchange commodities and assets, and consumption takes place. In the
 256 final period, uncertainty is resolved, households honor their financial obligations,
 257 exchange commodities, and then consume commodities.

258 We denote $x_h^c(s) \in \mathbb{R}_{++}$ and $e_h^c(s) \in \mathbb{R}_{++}$ as the consumption and the endow-
 259 ment of commodity c in state s by household h , respectively.¹² We define $\mathbf{x}_h(s) =$
 260 $(x_h^c(s))_{c \in \mathcal{C}} \in \mathbb{R}_{++}^C$, $\mathbf{x}_h = (\mathbf{x}_h(s))_{s \in \mathcal{S}} \in \mathbb{R}_{++}^G$, $\mathbf{x} = (\mathbf{x}_h)_{h \in \mathcal{H}} \in \mathbb{R}_{++}^{GH}$ and similarly
 261 $\mathbf{e}_h(s) \in \mathbb{R}_{++}^C$, $\mathbf{e}_h \in \mathbb{R}_{++}^G$, $\mathbf{e} \in \mathbb{R}_{++}^{GH}$, where $G = C(S + 1)$.

262 Household h 's preferences are represented by a utility function $u_h : \mathbb{R}_{++}^G \rightarrow \mathbb{R}$.
 263 As in most of the literature on smooth economies we assume that, for every $h \in \mathcal{H}$,

264
$$u_h \in C^2(\mathbb{R}_{++}^G); \tag{2}$$

265 for every $\mathbf{x}_h \in \mathbb{R}_{++}^G$, $\mathbf{D}u_h(\mathbf{x}_h) \gg \mathbf{0}$; $\tag{3}$

266 for every $\mathbf{v} \in \mathbb{R}^G \setminus \{0\}$ and $\mathbf{x}_h \in \mathbb{R}_{++}^G$, $\mathbf{v} \mathbf{D}^2 u_h(\mathbf{x}_h) \mathbf{v} < 0$; $\tag{4}$

267 for every $\underline{\mathbf{x}}_h \in \mathbb{R}_{++}^G$, $\{\mathbf{x}_h \in \mathbb{R}_{++}^G : u_h(\mathbf{x}_h) \geq u_h(\underline{\mathbf{x}}_h)\}$ is closed in the Euclidean
 268 topology of \mathbb{R}^G . $\tag{5}$

270 Let us denote by \mathcal{U} the set of vectors $\mathbf{u} = (u_h)_{h \in \mathcal{H}}$ of utility functions satisfying
 271 assumptions (2), (3), (4), and (5). We denote by $p^c(s) \in \mathbb{R}_{++}$ the price of commodity
 272 c in state s , by $q^a \in \mathbb{R}$ the price of asset a and by $b_h^a \in \mathbb{R}$ the quantity of asset a held
 273 by household h . Moreover we define $\mathbf{p}(s) = (p^c(s))_{c \in \mathcal{C}} \in \mathbb{R}_{++}^C$, $\mathbf{p} = (\mathbf{p}(s))_{s \in \mathcal{S}} \in$
 274 \mathbb{R}_{++}^G , $\mathbf{q} = (q^a)_{a \in \mathcal{A}} \in \mathbb{R}^A$, $\mathbf{b}_h = (b_h^a)_{a \in \mathcal{A}} \in \mathbb{R}^A$, $\mathbf{b} = (\mathbf{b}_h)_{h \in \mathcal{H}} \in \mathbb{R}^{AH}$.

275 We denote by $y^{a,c}(s) \in \mathbb{R}$ the units of commodity c delivered by one unit of asset a
 276 in state s and we define $\mathbf{y}^a(s) = (y^{a,c}(s))_{c \in \mathcal{C}} \in \mathbb{R}^C$, $\mathbf{y}(s) = (\mathbf{y}^a(s))_{a \in \mathcal{A}} \in \mathbb{R}^{CA}$, $\mathbf{y} =$
 277 $(\mathbf{y}(s))_{s \in \mathcal{S}'} \in \mathbb{R}^{CAS}$.¹³ Note in particular that, in state s , asset a promises to deliver a
 278 vector $\mathbf{y}^a(s)$ of commodities.

279 For any $m, n \in \mathbb{N} \setminus \{0\}$, let $\mathbb{M}(m, n)$ be the set of real $m \times n$ matrices and $\mathbb{M}^f(m, n)$
 280 be the set of real $m \times n$ matrices with full rank. Define the return matrix function as
 281 follows

282
$$\mathcal{R} : \mathbb{R}_{++}^G \times \mathbb{R}^{CAS} \rightarrow \mathbb{M}(S, A),$$

¹² Given $\mathbf{v}, \mathbf{w} \in \mathbb{R}^N$, we denote by $\mathbf{v} \gg \mathbf{w}$, $\mathbf{v} \geq \mathbf{w}$ and $\mathbf{v} > \mathbf{w}$ the standard binary relations between vectors. Also the definitions of the sets \mathbb{R}_+^N and \mathbb{R}_{++}^N are the common ones.

¹³ We consider possibly negative yields. Notice however that all the results we obtain are still valid in the case of positive yields.

$$\begin{aligned}
 & \mathbf{p}, \mathbf{y} \mapsto \begin{bmatrix} \mathbf{p}(1)\mathbf{y}^1(1) & \dots & \mathbf{p}(1)\mathbf{y}^a(1) & \dots & \mathbf{p}(1)\mathbf{y}^A(1) \\ \vdots & & \ddots & & \ddots & \vdots \\ \mathbf{p}(s)\mathbf{y}^1(s) & \dots & \mathbf{p}(s)\mathbf{y}^a(s) & \dots & \mathbf{p}(s)\mathbf{y}^A(s) \\ \vdots & & \ddots & & \ddots & \vdots \\ \mathbf{p}(S)\mathbf{y}^1(S) & \dots & \mathbf{p}(S)\mathbf{y}^a(S) & \dots & \mathbf{p}(S)\mathbf{y}^A(S) \end{bmatrix} = \begin{bmatrix} \mathbf{r}^1(\mathbf{p}, \mathbf{y}) \\ \vdots \\ \mathbf{r}^s(\mathbf{p}, \mathbf{y}) \\ \vdots \\ \mathbf{r}^S(\mathbf{p}, \mathbf{y}) \end{bmatrix}.
 \end{aligned}$$

For future use we also define, for every $\mathbf{p} \in \mathbb{R}_{++}^G$,

$$\Phi(\mathbf{p}) = \begin{bmatrix} \mathbf{p}(0) & & & & \\ & \mathbf{p}(1) & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \mathbf{p}(S) \end{bmatrix} \in \mathbb{M}(S + 1, G).$$

Mimicking what happens in the market for bank loans, the constraints that we impose are credit limits which bound the amount of future debt of the borrower, proportionally to his/her credit worthiness. In fact, we say that the amount household h can borrow, i.e., $\mathbf{q}(-\mathbf{b}_h)$, must be such that the household can repay what is due in all states in the final period. That amount due is

$$\mathbf{r}^s(\mathbf{p}, \mathbf{y})(-\mathbf{b}_h), \quad \forall s \in \mathcal{S}'. \tag{6}$$

Additionally, we assume that each household will not be able (due to legal restrictions) or will not be willing to consume less than a given proportion $\gamma_h(s)$ of its wealth in state $s \in \mathcal{S}'$. In other words, there is a base level of consumption expenditure required for all households:

$$\gamma_h(s) \mathbf{p}(s) \mathbf{e}_h(s).$$

Then we require that the amount due in (6) has to be smaller than the difference between a household's endowment level and the base level of consumption expenditure:

$$(1 - \gamma_h(s)) \mathbf{p}(s) \mathbf{e}_h(s).$$

Therefore, defining $\alpha_h(s) = 1 - \gamma_h(s)$, the borrowing constraints we impose are¹⁴:

$$\forall h \in \mathcal{H}, \quad \forall s \in \mathcal{S}', \quad -\mathbf{r}^s(\mathbf{p}, \mathbf{y}) \mathbf{b}_h \leq \alpha_h(s) \mathbf{p}(s) \mathbf{e}_h(s), \tag{7}$$

where $\alpha_h(s) \in (0, 1)$, $\forall h \in \mathcal{H}$ and $\forall s \in \mathcal{S}'$.

¹⁴ Notice that our model can easily incorporate the case in which in some states some households may not be compelled to satisfy any borrowing constraint. This can be obtained just enlarging the set of the admissible parameters $\alpha_h(s)$ from $(0, 1)$ to any open set containing $[0, 1]$. Indeed, the case $\alpha_h(s) \geq 1$ corresponds to the situation in which for household h is state s no financial constraint is imposed. That extension is possible because we do not use the restriction $0 < \alpha_h(s) < 1$ in any step of our proofs. Observe also that the case $\alpha_h(s) \leq 0$ corresponds instead to the situation in which household h is not trusted to repay any debt in state s : we can deal with that framework, as well.

Including the participation constraint parameters along with the parameters governing the asset structure and the household endowments and preferences, we define the set of economies as

$$\mathcal{E} = \mathbb{R}_{++}^{GH} \times \mathcal{U} \times \mathbb{R}^{CAS} \times (0, 1)^{SH},$$

with generic element $(\mathbf{e}, \mathbf{u}, \mathbf{y}, \boldsymbol{\alpha})$, where $\boldsymbol{\alpha} = (\alpha_h)_{h \in \mathcal{H}} = (\alpha_h(s))_{h \in \mathcal{H}, s \in \mathcal{S}'}$.

Definition 1 A vector $(\mathbf{x}^*, \mathbf{p}^*, \mathbf{b}^*, \mathbf{q}^*) \in \mathbb{R}_{++}^{GH} \times \mathbb{R}_{++}^G \times \mathbb{R}^{AH} \times \mathbb{R}^A$ is an **equilibrium** for the economy $(\mathbf{e}, \mathbf{u}, \mathbf{y}, \boldsymbol{\alpha}) \in \mathcal{E}$ if

- $\forall h \in \mathcal{H}, (\mathbf{x}_h^*, \mathbf{b}_h^*)$ solves the following problem: given $(\mathbf{p}^*, \mathbf{q}^*, \mathbf{e}, \mathbf{u}, \mathbf{y}, \boldsymbol{\alpha})$

$$\begin{aligned} & \max_{(\mathbf{x}_h, \mathbf{b}_h) \in \mathbb{R}_{++}^G \times \mathbb{R}^A} u_h(\mathbf{x}_h) \\ & \text{s.t. } \mathbf{p}^*(0) (\mathbf{x}_h(0) - \mathbf{e}_h(0)) + \mathbf{q}^* \mathbf{b}_h \leq 0 \end{aligned} \tag{8a}$$

$$\mathbf{p}^*(s) (\mathbf{x}_h(s) - \mathbf{e}_h(s)) - (\mathbf{p}^*(s) \mathbf{y}^a(s))_{a=1}^A \mathbf{b}_h \leq 0, \quad \forall s \in \mathcal{S}' \tag{8b}$$

$$- (\mathbf{p}^*(s) \mathbf{y}^a(s))_{a=1}^A \mathbf{b}_h \leq \alpha_h(s) \mathbf{p}^*(s) \mathbf{e}_h(s), \quad \forall s \in \mathcal{S}' \tag{8c}$$

- $(\mathbf{x}^*, \mathbf{b}^*)$ satisfies the market clearing conditions

$$\sum_{h=1}^H (\mathbf{x}_h^* - \mathbf{e}_h) = \mathbf{0} \tag{9}$$

and

$$\sum_{h=1}^H \mathbf{b}_h^* = \mathbf{0}. \tag{10}$$

Define

$$\mathbf{p}^\backslash(s) = (p^c(s))_{c \neq 1} \in \mathbb{R}_{++}^{C-1}, \text{ for any } s \in \mathcal{S}, \text{ and } \mathbf{p}^\backslash = (\mathbf{p}^\backslash(s))_{s \in \mathcal{S}} \in \mathbb{R}_{++}^{G-(S+1)},$$

and similarly, for any $h \in \mathcal{H}, \mathbf{x}_h^\backslash(s), \mathbf{x}_h^\backslash, \mathbf{e}_h^\backslash(s)$ and \mathbf{e}_h^\backslash .

Remark 2 Observe that the number of admissible price normalizations for the equilibrium concept presented above is $S + 1$ (one for each spot) and there are $S + 1$ Walras' laws. Therefore, the number of significant equations [i.e., conditions (9) and (10) "without $S + 1$ Walras' laws"] is equal to the number of significant variables (i.e., spot by spot normalized good prices \mathbf{p}^\backslash and asset prices \mathbf{q}).

The above observations are formalized in the following definition.

Definition 3 A vector $(\mathbf{x}^*, \mathbf{p}^\backslash, \mathbf{b}^*, \mathbf{q}^*) \in \mathbb{R}_{++}^{GH} \times \mathbb{R}_{++}^{G-(S+1)} \times \mathbb{R}^{AH} \times \mathbb{R}^A$ is a **normalized equilibrium** for the economy $(\mathbf{e}, \mathbf{u}, \mathbf{y}, \boldsymbol{\alpha}) \in \mathcal{E}$ if

- 332 1. $\forall h \in \mathcal{H}$, $(\mathbf{x}_h^*, \mathbf{b}_h^*)$ solves Problem (8) given $(\mathbf{p}^*, \mathbf{q}^*, (\mathbf{e}, \mathbf{u}, \mathbf{y}, \boldsymbol{\alpha}))$, where $\mathbf{p}^* =$
 333 $(1, \mathbf{p}^{\setminus*}(s))_{s=0}^S$;
 334 2. \mathbf{b}^* satisfies market clearing conditions (10) and

335
$$\sum_{h=1}^H (\mathbf{x}_h^{\setminus*} - \mathbf{e}_h^{\setminus*}) = \mathbf{0}.$$

336 *Remark 4* We do distinguish between different normalizations because showing our
 337 results requires to carefully keep track of different normalizations for some of the
 338 introduced definitions of equilibria. Observe however that, similarly to what done
 339 below in the case of equilibria with fixed dimension of the return space in Proposition
 340 8, it is easy to prove that an equilibrium according to Definition 1 and its normalized
 341 version in Definition 3 are allocation equivalent.

342 Let's further consider the restrictions we chose to analyze and the technical reasons
 343 why these constraints can be analyzed using our proof methodology. From Duffie and
 344 Shafer (1985), the standard way to tackle the problem of generic existence of equilibria
 345 in the real asset model is to fix the dimension of the return space so that the return
 346 matrix does not suffer a drop in rank. We briefly describe this process. Define

347
$$\Phi^1(\mathbf{p}) = \begin{bmatrix} \mathbf{p}(1) & & \\ & \ddots & \\ & & \mathbf{p}(S) \end{bmatrix} \quad \text{and} \quad \mathbf{z}_h^1 = \mathbf{x}_h^1 - \mathbf{e}_h^1,$$

348 with $\mathbf{x}_h^1 = (\mathbf{x}_h(s))_{s \in \mathcal{S}'} \in \mathbb{R}_{++}^{CS}$ and $\mathbf{e}_h^1 = (\mathbf{e}_h(s))_{s \in \mathcal{S}'} \in \mathbb{R}_{++}^{CS}$.

349 In this fixed dimension return space approach, the budget constraints in the final
 350 period (8.2) can be equivalently expressed as:

351
$$\Phi^1(\mathbf{p}) \mathbf{z}_h^1 \in L, \tag{11}$$

352 where L is an A dimensional subspace of \mathbb{R}^S . Condition (11) is equivalent to any of
 353 the following conditions:

1.

354
$$\exists \mathbf{M}(L) \in \mathbb{M}^f(S - A, S) \text{ such that } \mathbf{M}(L) \cdot \Phi^1(\mathbf{p}) \mathbf{z}_h^1 = \mathbf{0}, \tag{12}$$

355 where $\mathbf{M}(L)$ is such that $\ker \mathbf{M}(L) = L$;

2.

356
$$\exists \mathbf{N}(L) \in \mathbb{M}^f(S, A) \text{ and } \exists \mathbf{b}_h \in \mathbb{R}^A \text{ such that } \Phi^1(\mathbf{p}) \mathbf{z}_h^1 = \mathbf{N}(L) \cdot \mathbf{b}_h, \tag{13}$$

357 where $\mathbf{N}(L)$ is such that $\text{Im } \mathbf{N}(L) = L$.

Duffie and Shafer (1985) use condition 12). Here, we use condition (13) as well. With either condition, it is not clear how to impose constraints directly on \mathbf{b}_h . In the first condition, \mathbf{b}_h does not appear. In regard to the second condition, we show that for a “fictitious” (regular) equilibrium, up to permutations of states, there exists $\mathbf{E} \in \mathbb{M}(S - A, A)$ such that

$$\begin{bmatrix} \mathbf{I} \\ \mathbf{E} \end{bmatrix} \mathbf{b}_h = \mathcal{R}(\mathbf{p}, \mathbf{y}) \mathbf{b}'_h = \begin{bmatrix} \mathbf{R}^*(\mathbf{p}, \mathbf{y}) \\ \widehat{\mathbf{R}}(\mathbf{p}, \mathbf{y}) \end{bmatrix} \mathbf{b}'_h,$$

where $\mathbf{R}^*(\mathbf{p}, \mathbf{y})$ has full rank, \mathbf{b}_h is the asset demand in a fictitious equilibrium, and \mathbf{b}'_h is the asset demand in the true equilibrium. Therefore, again up to permutations, we get that¹⁵

$$\mathbf{b}'_h = [\mathbf{R}^*(\mathbf{p}, \mathbf{y})]^{-1} \mathbf{b}_h.$$

The above condition indicates that imposing restrictions on the fictitious equilibrium asset demand \mathbf{b}_h does not imply that the same restrictions will hold for the true asset demand \mathbf{b}'_h . For example, the restriction $\mathbf{b}_h \geq \mathbf{0}$ does not imply that $\mathbf{b}'_h = [\mathbf{R}^*(\mathbf{p}, \mathbf{y})]^{-1} \mathbf{b}_h \geq \mathbf{0}$. That explains why the fixed dimension return space approach is likely not applicable for restrictions written directly in terms of \mathbf{b}_h .

On the other hand, constraints on the physical quantity b_h^a have little meaning as the future yields depend upon future commodity prices. So constraints could be written for each asset on the value $q^a b_h^a$, but this presupposes that either lenders are not able to gain information about the other assets in a household’s portfolio or do not care about this information.

The latter assumption is absurd as future repayment likelihoods depend upon all asset positions of a household, while the former assumption imposes an unrealistic information gap in this market for bank loans. Thus, it appears more economically meaningful to consider constraints imposed upon the payouts of all assets of a household in the final period.¹⁶

Finally, our methods allow us to conjecture that [similarly to Polemarchakis and Siconolfi (1997)] restricting excess demand in all states to a linear subspace of the column span of the returns matrix also suffices to guarantee generic existence and regularity. The restriction that we have in mind is:

$$\Phi^1(\mathbf{p}) \mathbf{z}_h^1 \in L_h(\mathbf{p}), \quad (14)$$

where $L_h(\mathbf{p})$ is a household-specific price-dependent return space, which is a linear subspace of L . The constraints of this form (14) fit with the fixed dimension return

¹⁵ The asset demand in the fictitious equilibrium is equal to the asset demand in the true equilibrium, up to a change of basis. The elements of the basis are the columns of $\mathbf{R}^*(\mathbf{p}, \mathbf{y})$, which is an invertible matrix with A states following the permutation of states.

¹⁶ Observe that the methods of this paper would be equally effective in obtaining generic existence and regularity, if we were to consider participation constraints as inequalities on initial period portfolio value $\mathbf{q}\mathbf{b}_h = -\mathbf{p}(0)(\mathbf{x}_h(0) - \mathbf{e}_h(0))$.

390 space approach, as we can simply replace L in the previous analysis with $L_h(\mathbf{p})$. In
 391 fact, similar restrictions have been considered by Balasko et al. (1990), in the case of
 392 nominal assets. We do not consider constraints (14) any further in the present paper,
 393 but we are working with them in a companion paper.

394 **3 Equilibria with fixed dimension of the return space**

395 As explained in the previous sections, we present some definitions of equilibria in
 396 which the dimension of the feasible wealth transfer space L is fixed and equal to the
 397 number of available assets, A .

398 The main difference between the concept of pseudo equilibrium in Duffie and Shafer
 399 (1985) and the one proposed below in Definitions 5 and 6 is that in the former the
 400 space L appearing in the household maximization problem is written as the *kernel* of
 401 a linear function, while in our household maximization problem [see (16)] the space
 402 L is the *image* of a linear function.

403 Below, after introducing some preliminary definitions and facts, we present three
 404 equivalent definitions of equilibria that are useful for our analysis. Indeed, as explained
 405 in Sect. 1, the different steps in the proofs of Theorems 13 and 14 require a different
 406 definition of equilibrium.

407 We denote by $\mathcal{G}_{A,S}$ the set of A dimensional vector subspaces of \mathbb{R}^S . It can be shown
 408 that $\mathcal{G}_{A,S}$ is a Hausdorff, compact, and second countable (and therefore sequentially
 409 compact) metric space and also a C^∞ abstract manifold of dimension $A(S - A)$.¹⁷

410 Denoting by Σ the set of permutations of $\{1, \dots, S\}$, with generic element $\sigma \in \Sigma$,
 411 by \mathbf{P}_σ the corresponding permutation matrix and by \mathbf{I}_M the M -dimensional identity
 412 matrix, then for every $L \in \mathcal{G}_{A,S}$, there exists $\sigma^{-1} \in \Sigma$, a neighborhood $V_{\sigma^{-1}}$ of L ,
 413 and a diffeomorphism¹⁸

414
$$\psi_{\sigma^{-1}} : V_{\sigma^{-1}} \rightarrow \mathbb{M}(S - A, A) \tag{15}$$

415 such that $L = \text{Im } \mathbf{P}_\sigma \begin{bmatrix} -\psi_{\sigma^{-1}}(L) \\ \mathbf{I}_A \end{bmatrix} = \ker [\mathbf{I}_{S-A} \mid \psi_{\sigma^{-1}}(L)] \cdot \mathbf{P}_{\sigma^{-1}}$.

416 Define

417
$$\mathbf{A}_h = \begin{bmatrix} \alpha_h(1) & & \\ & \ddots & \\ & & \alpha_h(S) \end{bmatrix}.$$

418 **Definition 5** A vector $(\mathbf{x}^*, \mathbf{p}^*, \mathbf{b}^*, \mathbf{q}^*, L^*) \in \mathbb{R}_{++}^{GH} \times \mathbb{R}_{++}^G \times \mathbb{R}^{AH} \times \mathbb{R}^A \times \mathcal{G}_{A,S}$ is a
 419 **symmetric equilibrium** for the economy $(\mathbf{e}, \mathbf{u}, \mathbf{y}, \boldsymbol{\alpha}) \in \mathcal{E}$ if

- 420 1. $\forall h \in \mathcal{H}$, $(\mathbf{x}_h^*, \mathbf{b}_h^*)$ solves the following problem:
 421 given $(\mathbf{p}^*, \mathbf{q}^*, L^*, \mathbf{e}, \mathbf{u}, \mathbf{y}, \boldsymbol{\alpha})$

¹⁷ See Kato (1995), page 198.

¹⁸ From now on, for ease of notation, we will simply write ψ in place of $\psi_{\sigma^{-1}}$. Notice that we chose to start with σ^{-1} instead of σ because in this way the definitions of equilibria below get simplified.

$$\max_{(\mathbf{x}_h, \mathbf{b}_h) \in \mathbb{R}_{++}^G \times \mathbb{R}^A} u_h(\mathbf{x}_h)$$

$$s.t. - \mathbf{p}^*(0) (\mathbf{x}_h(0) - \mathbf{e}_h(0)) - \mathbf{q}^* \mathbf{b}_h = 0 \tag{16a}$$

$$-\Phi^1(\mathbf{p}^*) (\mathbf{x}_h^1 - \mathbf{e}_h^1) + \mathbf{P}_\sigma \begin{bmatrix} -\psi(L^*) \\ \mathbf{I}_A \end{bmatrix} \mathbf{b}_h = \mathbf{0} \tag{16b}$$

$$\mathbf{P}_\sigma \begin{bmatrix} -\psi(L^*) \\ \mathbf{I}_A \end{bmatrix} \mathbf{b}_h + \mathbf{A}_h \cdot \Phi^1(\mathbf{p}^*) \mathbf{e}_h^1 \geq \mathbf{0} \tag{16c}$$

2. $(\mathbf{x}^*, \mathbf{b}^*)$ satisfies market clearing conditions (9) and (10);

3. $\text{Im} \mathcal{R}(\mathbf{p}^*, \mathbf{y}) \subseteq L^*$, i.e.,

$$\text{vec} [\mathbf{I}_{S-A} \mid \psi(L^*)] \cdot \mathbf{P}_{\sigma-1} \cdot \mathcal{R}(\mathbf{p}^*, \mathbf{y}) = \mathbf{0}. \tag{17}$$

Define $\Delta_{++}^{G-1} = \{\mathbf{p} \in \mathbb{R}_{++}^G : \sum_{s=0}^S \sum_{c=1}^C p^c(s) = 1\}$ and, for any $h \in \mathcal{H}$, $\mathbf{e}_h^\diamond = (e_h^1(s))_{s \in \mathcal{S}} \in \mathbb{R}_{++}^{S+1}$, $\mathbf{e}_h^{\setminus(01)} = (e_h^c(s))_{(s,c) \neq (0,1)} \in \mathbb{R}_{++}^{G-1}$, $\mathbf{x}_h^{\setminus(01)} = (x_h^c(s))_{(s,c) \neq (0,1)} \in \mathbb{R}_{++}^{G-1}$. Moreover, $\mathbf{1}_N$ denotes an N dimensional vector whose components are all equal to 1; if no confusion arises, we will write $\mathbf{1}$ in the place of $\mathbf{1}_N$.

As in the case of equilibria presented in Definition 1, we formalize the possibility of normalizing prices and the validity of Walras' laws in Definition 6 below.

Definition 6 A vector $(\mathbf{x}^*, \mathbf{p}^*, \mathbf{b}^*, \mathbf{q}^*, L^*) \in \mathbb{R}^{GH} \times \mathbb{R}_{++}^{G-(S+1)} \times \mathbb{R}^{AH} \times \mathbb{R}^A \times \mathcal{G}_{A,S}$ is a **normalized symmetric equilibrium** for the economy $(\mathbf{e}, \mathbf{u}, \mathbf{y}, \boldsymbol{\alpha}) \in \mathcal{E}$ if

1. $\forall h \in \mathcal{H}$, $(\mathbf{x}_h^*, \mathbf{b}_h^*)$ solves Problem (16) given $(\mathbf{p}^*, \mathbf{q}^*, L^*, (\mathbf{e}, \mathbf{u}, \mathbf{y}, \boldsymbol{\alpha}))$, where $\mathbf{p}^* = (1, \mathbf{p}^{\setminus*}(s))_{s=0}^S$;
2. \mathbf{b}^* satisfies market clearing conditions (10) and

$$\sum_{h=1}^H (\mathbf{x}_h^{\setminus*} - \mathbf{e}_h^{\setminus*}) = \mathbf{0};$$

3. Condition (17) holds true.

We now introduce the needed definition of Mr. 1 equilibrium.

Definition 7 A vector $(\mathbf{x}^*, \mathbf{p}^*, L^*) \in \mathbb{R}_{++}^{GH} \times \Delta_{++}^{G-1} \times \mathcal{G}_{A,S}$ is a **Mr. 1 equilibrium**¹⁹ for the economy $(\mathbf{e}, \mathbf{u}, \mathbf{y}, \boldsymbol{\alpha}) \in \mathcal{E}$ if

- 1a. $\forall h \in \mathcal{H} \setminus \{1\}$, \mathbf{x}_h^* solves the following problem:
given $(\mathbf{p}^*, L^*, \mathbf{e}, \mathbf{u}, \mathbf{y}, \boldsymbol{\alpha})$,

$$\max_{\mathbf{x}_h \in B_h^{\setminus b}(\mathbf{p}^*, L^*)} u_h(\mathbf{x}_h) \tag{18}$$

¹⁹ Among the various kinds of equilibria with fixed dimension of the return space we introduce, the one that bears the most resemblance to the original concept in Duffie and Shafer (1985) is Definition 7. However, we have elected to call it ‘‘Mr. 1 equilibrium’’ instead of ‘‘pseudo equilibrium’’ to highlight the main difference between this notion and the notion of symmetric equilibrium in Definition 5.

449 where $B_h^{\setminus \mathbf{b}}(\mathbf{p}^*, L^*) =$

450 $\left\{ \mathbf{x}_h \in \mathbb{R}_{++}^G : \exists \mathbf{b}_h \in \mathbb{R}^A \text{ such that}$

451 $-\mathbf{p}^*(0) (\mathbf{x}_h(0) - \mathbf{e}_h(0)) - \mathbf{1} \cdot \mathbf{P}_\sigma \begin{bmatrix} -\psi(L^*) \\ \mathbf{I}_A \end{bmatrix} \mathbf{b}_h = 0 \quad (19a)$

452 $-\Phi^1(\mathbf{p}^*) (\mathbf{x}_h^1 - \mathbf{e}_h^1) + \mathbf{P}_\sigma \begin{bmatrix} -\psi(L^*) \\ \mathbf{I}_A \end{bmatrix} \mathbf{b}_h = \mathbf{0} \quad (19b)$

453 $\left. \mathbf{P}_\sigma \begin{bmatrix} -\psi(L^*) \\ \mathbf{I}_A \end{bmatrix} \mathbf{b}_h + \mathbf{A}_h \Phi^1(\mathbf{p}^*) \mathbf{e}_h^1 \geq \mathbf{0} \right\} \quad (19c)$

454 1b. \mathbf{x}_1^* solves the following problem:
 455 given $(\mathbf{p}^*, L^*, \mathbf{e}, \mathbf{u}, \mathbf{y}, \boldsymbol{\alpha})$,

456 $\max_{\mathbf{x}_1 \in \mathbb{R}_{++}^G} u_1(\mathbf{x}_1)$

457 $s.t. -\mathbf{p}^*(0) (\mathbf{x}_1(0) - \mathbf{e}_1(0)) - \mathbf{1} \cdot \Phi^1(\mathbf{p}^*) (\mathbf{x}_1^1 - \mathbf{e}_1^1) = 0 \quad (20a)$

458 $\Phi^1(\mathbf{p}^*) (\mathbf{x}_1^1 - \mathbf{e}_1^1) + \mathbf{A}_1 \Phi^1(\mathbf{p}^*) \mathbf{e}_1^1 \geq \mathbf{0} \quad (20b)$

459 2. \mathbf{x}^* satisfies market clearing conditions

460
$$\sum_{h=1}^H (\mathbf{x}_h^{*(01)} - \mathbf{e}_h^{*(01)}) = \mathbf{0}; \quad (21)$$

461 3. Condition (17) holds true.

462 Definitions 5, 6, and 7 are in fact “allocation equivalent”, as stated below.²⁰

463 **Proposition 8** For a given economy $(\mathbf{e}, \mathbf{u}, \mathbf{y}, \boldsymbol{\alpha}) \in \mathcal{E}$, the following statements are
 464 equivalent:

- 465 1. \mathbf{x} is a symmetric equilibrium allocation;
 466 2. \mathbf{x} is a normalized symmetric equilibrium allocation;
 467 3. \mathbf{x} is a Mr. 1 equilibrium allocation.

468 Proposition 9 describes the relationship between equilibria with fixed dimension
 469 of the return space and “true” equilibria.

470 **Proposition 9** If $(\mathbf{x}^*, \mathbf{p}^*, \mathbf{b}^*, \mathbf{q}^*, L^*) \in \mathbb{R}_{++}^{GH} \times \mathbb{R}_{++}^{G-(S+1)} \times \mathbb{R}^{AH} \times \mathbb{R}^A \times \mathcal{G}_{A,S}$ is
 471 a normalized symmetric equilibrium for the economy $(\mathbf{e}, \mathbf{u}, \mathbf{y}, \boldsymbol{\alpha}) \in \mathcal{E}$ and

472 $\text{rank} \mathcal{R}(\mathbf{p}^*, \mathbf{y}) = A, \quad (22)$

²⁰ This result is formally proven in Hoelle et al. (2012).

Author Proof

473 where $\mathbf{p}^* = (1, \mathbf{p}^{\setminus*}(s))_{s=0}^S$, then there exist \mathbf{b}^{**} and \mathbf{q}^{**} such that $(\mathbf{x}^*, \mathbf{p}^{\setminus*}, \mathbf{b}^{**}, \mathbf{q}^{**})$
 474 is a normalized equilibrium for \mathcal{E} .

475 *Proof* See Hoelle et al. (2012).

476 4 Generic existence, regularity, and suboptimality

477 In this section, we first show existence of an Mr. 1 equilibrium—see Theorem 13.
 478 Then, we can obtain the generic existence of a true equilibrium, after showing the
 479 generic regularity and generic full rank condition of the return matrix for normalized
 480 symmetric equilibria—see Theorem 14.

481 As a preliminary step towards the application of a Brouwer like fixed point theorem
 482 to prove Theorem 13—see Appendix A.1—we show some basic properties of the
 483 demand function associated with Definition 7.

484 Omitting for simplicity the dependence on utility functions, define

$$\begin{aligned}
 485 \quad & \beta_1 : \Delta_{++}^{G-1} \times \mathbb{R}_{++}^G \times (0, 1)^S \rightrightarrows \mathbb{R}_{++}^G, \\
 486 \quad & \beta_1(\mathbf{p}, \mathbf{e}_1, \boldsymbol{\alpha}_1) = \left\{ \mathbf{x}_1 \in \mathbb{R}_{++}^G : -\mathbf{p}(\mathbf{x}_1 - \mathbf{e}_1) \geq 0 \right. \\
 487 \quad & \quad \left. \Phi^1(\mathbf{p})(\mathbf{x}_1^1 - \mathbf{e}_1^1) + \mathbf{A}_1 \Phi^1(\mathbf{p}) \mathbf{e}_1^1 \geq \mathbf{0} \right. \\
 488 \quad & \quad \left. u_1(\mathbf{x}_1) - u_1\left(\frac{1}{2}\mathbf{e}_1\right) \geq 0 \right\},
 \end{aligned}$$

489 and for every $h \in \mathcal{H} \setminus \{1\}$

$$\begin{aligned}
 490 \quad & \beta_h : \Delta_{++}^{G-1} \times \mathbb{R}_{++}^G \times (0, 1)^S \times \mathcal{G}_{A,S} \rightrightarrows \mathbb{R}_{++}^G \times \mathbb{R}^A, \\
 491 \quad & \beta_h(\mathbf{p}, \mathbf{e}_h, \boldsymbol{\alpha}_h, L) = \left\{ (\mathbf{x}_h, \mathbf{b}_h) \in \mathbb{R}_{++}^G \times \mathbb{R}^A : \right. \\
 492 \quad & \quad -\mathbf{p}(0)(\mathbf{x}_h(0) - \mathbf{e}_h(0)) - \mathbf{1} \cdot \mathbf{P}_\sigma \begin{bmatrix} -\psi(L) \\ \mathbf{I}_A \end{bmatrix} \mathbf{b}_h \geq 0 \\
 493 \quad & \quad -\Phi^1(\mathbf{p})(\mathbf{x}_h^1 - \mathbf{e}_h^1) + \mathbf{P}_\sigma \begin{bmatrix} -\psi(L) \\ \mathbf{I}_A \end{bmatrix} \mathbf{b}_h \geq 0 \\
 494 \quad & \quad \mathbf{P}_\sigma \begin{bmatrix} -\psi(L) \\ \mathbf{I}_A \end{bmatrix} \mathbf{b}_h + \mathbf{A}_h \Phi^1(\mathbf{p}) \mathbf{e}_h^1 \geq \mathbf{0} \\
 495 \quad & \quad \left. u_h(\mathbf{x}_h) - u_h\left(\frac{1}{2}\mathbf{e}_h\right) \geq 0 \right\}.
 \end{aligned}$$

497 *Remark 10* It is obvious that if $(\mathbf{x}_h, \mathbf{b}_h)$ is a solution to

$$498 \quad \max_{(\mathbf{x}_h, \mathbf{b}_h) \in \mathbb{R}_{++}^G \times \mathbb{R}^A} u_h(\mathbf{x}_h) \quad \text{s.t.} \quad (\mathbf{x}_h, \mathbf{b}_h) \in \beta_h(\mathbf{p}, \mathbf{e}_h, \boldsymbol{\alpha}_h, L), \quad (23)$$

499 for some $h \in \mathcal{H} \setminus \{1\}$, then \mathbf{x}_h is a solution to (18); and conversely if \mathbf{x}_h is a solution
 500 to (18), then there exists \mathbf{b}_h such that $(\mathbf{x}_h, \mathbf{b}_h)$ is a solution to (23).

501 **Lemma 11** For any $h \in \mathcal{H}$, β_h is nonempty valued, convex valued, compact valued,
 502 closed and lower hemi continuous.

503 *Proof* See Hoelle et al. (2012).

504 **Proposition 12** The demand correspondences associated with Problems (18) and (20)
 505 are continuous functions.

506 *Proof* It follows from Remark 10, Lemma 11, the Maximum Theorem, and assumption
 507 (4).

508 **Theorem 13** For every economy, a Mr. 1 equilibrium exists.

509 *Proof* The proof is presented in Appendix A.1.

510 Consider the Hausdorff topological vector space

$$511 \quad \mathcal{V} = \mathbb{R}_{++}^{GH} \times \left[C^2(\mathbb{R}_{++}^G) \right]^H \times \mathbb{R}^{CAS} \times (0, 1)^{SH}, \quad (24)$$

512 endowed with the product topology of the natural topologies on each of the spaces
 513 in the Cartesian product. In particular, on the C^2 function space, we consider the C^2
 514 compact-open topology. Assume that $\mathcal{E} \subseteq \mathcal{V}$ is endowed with the topology induced
 515 by \mathcal{V} .

516 **Theorem 14** There exists an open dense set $\mathcal{D} \subseteq \mathcal{E}$ such that, for any $(\mathbf{e}, \mathbf{u}, \mathbf{y}, \boldsymbol{\alpha}) \in$
 517 \mathcal{D} , there is a (positive) finite number of associated normalized equilibria which locally
 518 smoothly depend on the elements of \mathcal{D} .

519 *Proof* First of all, observe that from Proposition 8 and Theorem 13, a normalized
 520 symmetric equilibrium exists. Moreover, from Proposition 9, it is enough to show that
 521 generically rank condition (22) does hold true. The strategy of the proof is then to
 522 consider normalized symmetric equilibria and proceed through the following steps:

- 523 1. The associated extended equilibrium system²¹ is such that border line cases are
- 524 rare;
- 525 2. The return matrix has generic full rank;
- 526 3. The associated projection from the equilibrium set to the economy space is proper;
- 527 4. Apply a simplified version of the implicit function theorem given in Theorem 2.3
- 528 of Glöckner (2006).

529 Each of the above steps is formalized and proven in Appendix A.2.

530 The theorem below states the typical inefficiency of equilibria.

²¹ By extended system associated with a given definition of equilibrium, we mean the related system of Lagrange conditions of households' maximization problems and market clearing conditions.

531 **Theorem 15** *If $A < S$, then there exists an open and dense set $\tilde{\mathcal{D}} \subseteq \mathcal{E}$ such that,*
 532 *for every $(\mathbf{e}, \mathbf{u}, \mathbf{y}, \boldsymbol{\alpha}) \in \tilde{\mathcal{D}}$, every corresponding equilibrium allocation is not Pareto*
 533 *optimal.*

534 The proof of the above theorem follows a standard argument and therefore it is
 535 omitted.

536 Observe that in the statement of the above theorem, the qualification $A < S$ is indeed
 537 a necessary condition. If it were the case $A = S$, starting from a regular economy in
 538 the complete market model with an associated Pareto optimal equilibrium, and then
 539 adding “insignificant” constraints, it would be immediate to construct an open set of
 540 economies in the restricted participation model with the property that at least one
 541 associated equilibrium is still Pareto optimal.

542 5 A numerical example

543 Given the proof of generic regularity of equilibria in Theorem 14, we can now compute
 544 an equilibrium of our model using algorithms that utilize the theory of differential
 545 topology. Specifically, the two equilibria computed in this section are numerically
 546 determined using homotopy methods, i.e., the HOMPACK suite of subroutines for
 547 Fortran 90, and Kubler (2007). These methods require generic regularity to work
 548 successfully.

549 With these two equilibria, a comparative statics analysis yields interesting con-
 550 clusions. In particular, the example shows that by tightening credit constraints,
 551 an anonymous planner intervention can actually effect a Pareto improvement. The
 552 planner intervention works through adjustments in the parameters governing the
 553 participation restriction (7): $(\alpha_h(s))_{h \in \mathcal{H}, s \in \mathcal{S}'} \in (0, 1)^{SH}$. For this example, these
 554 parameters are household independent, so they are simply $(\alpha(s))_{s \in \mathcal{S}'} \in (0, 1)^S$.
 555 The planner intervention is also household independent and its tools are given by
 556 $\tau(s) \in \left(-1, -1 + \frac{1}{\alpha(s)}\right)$, $s \in \mathcal{S}'$, so that the new parameters in (7) are defined as

$$557 \quad \hat{\alpha}(s) = (1 + \tau(s)) \cdot \alpha(s) \in (0, 1), \quad \forall s \in \mathcal{S}'.$$

558 Obviously, an intervention with $\boldsymbol{\tau} = (\tau(s))_{s \in \mathcal{S}'} = \mathbf{0}$, where $\mathbf{0} = (0, \dots, 0) \in \mathbb{R}^S$,
 559 implies no change in either the parameters or the resulting equilibrium. Define the
 560 equilibrium obtained following planner intervention as $(\hat{\mathbf{x}}, \hat{\mathbf{p}}, \hat{\mathbf{b}}, \hat{\mathbf{q}})$, in contrast to the
 561 original equilibrium $(\mathbf{x}, \mathbf{p}, \mathbf{b}, \mathbf{q})$ prior to planner intervention.

562 The example in this section demonstrates the following fact. For some values
 563 $\tau(s) \leq 0$, $s \in \mathcal{S}'$, the resulting equilibrium allocation $\hat{\mathbf{x}}$ Pareto dominates the original
 564 equilibrium allocation \mathbf{x} .

565 That is, for this particular economy, more regulation on the credit markets is
 566 employed in order to make all households better off. Notice that, due to the generic
 567 regularity result and the way the algorithm works, the example is robust to perturbation.

568 The economy is defined by:

- 569 – $H = 3$ households;
- 570 – $C = 2$ commodities traded in each state;

- 571 – $S = 4$ possible states of uncertainty tomorrow;
- 572 – $A = 2$ real assets.

573 The household endowments are given by:

	$e_h^1(0)$	$e_h^2(0)$	$e_h^1(1)$	$e_h^2(1)$	$e_h^1(2)$	$e_h^2(2)$	$e_h^1(3)$	$e_h^2(3)$	$e_h^1(4)$	$e_h^2(4)$
574 $h = 1$	1	7	1	0.5	7	1	7.5	6	1	7
$h = 2$	2	4	7	6.5	1	7	2	0.5	8	2
$h = 3$	9	1	4	5	4	4	2.5	5.5	3	3
Sum	12	12	12	12	12	12	12	12	12	12

575 The household utility functions are given by:

$$576 \quad u_h(x_h) = \gamma_h(0) \cdot \log(x_h^1(0)) + (1 - \gamma_h(0)) \cdot \log(x_h^2(0))$$

$$577 \quad + \frac{1}{4} \sum_{s \in \mathcal{S}'} \left[\gamma_h(s) \cdot \log(x_h^1(s)) + (1 - \gamma_h(s)) \cdot \log(x_h^2(s)) \right],$$

578 where

	$\gamma_h(0)$	$\gamma_h(1)$	$\gamma_h(2)$	$\gamma_h(3)$	$\gamma_h(4)$
579 $h = 1$	2/3	1/2	1/4	3/4	1/2
$h = 2$	1/3	1/4	3/4	1/2	1/4
$h = 3$	2/3	3/4	1/2	1/4	3/4

580 The assets are real assets, so each asset has payouts in terms of a vector of commodities
 581 in each state $s \in \mathcal{S}'$. These vector of payouts are given by:

	States\Assets	$a = 1$	$a = 2$
582	$s = 1$	(4, 0.5)	(0.5, 3.6)
	$s = 2$	(3.9, 0.5)	(3.7, 0.5)
	$s = 3$	(0.5, 3.8)	(0.5, 3.8)
	$s = 4$	(0.5, 3.7)	(3.9, 0.5)

583 Finally, the parameters (identical for all households) governing the participation
 584 restriction (7) are given by:

$$585 \quad \alpha(1) = 0.033$$

$$586 \quad \alpha(2) = 0.020$$

$$587 \quad \alpha(3) = 0.029$$

$$588 \quad \alpha(4) = 0.033.$$

589 For this economy, the equilibrium²² is given by:

²² A unique equilibrium is guaranteed by our use of the Cobb–Douglas utility functions.

590 – Consumption

	$x_h^1(0)$	$x_h^2(0)$	$x_h^1(1)$	$x_h^2(1)$	$x_h^1(2)$	$x_h^2(2)$	$x_h^1(3)$	$x_h^2(3)$	$x_h^1(4)$	$x_h^2(4)$
591 $h = 1$	2.966	3.680	0.881	0.871	4.951	3.240	8.275	5.441	2.017	5.422
$h = 2$	1.615	4.778	6.266	7.305	3.133	4.811	1.338	0.988	6.107	4.084
$h = 3$	7.426	3.546	4.851	3.822	3.913	3.946	2.383	5.569	3.873	2.487

592 – Assets

	b_h^1	b_h^2
593 $h = 1$	0.334	-0.314
$h = 2$	0.225	-0.240
$h = 3$	-0.556	0.550

594 – Prices

595 $p^1(0) = 1$	$p^2(0) = 0.622$
$p^1(1) = 1$	$p^2(1) = 0.973$
$p^1(2) = 1$	$p^2(2) = 0.982$
$p^1(3) = 1$	$p^2(3) = 1.205$
$p^1(4) = 1$	$p^2(4) = 0.781$
$q^1 = 5.886$	$q^2 = 5.941$

596 – Utility values

$$\begin{aligned}
 597 \quad u_1(\mathbf{x}_1) &= 2.2473 \\
 598 \quad u_2(\mathbf{x}_2) &= 2.4142 \\
 599 \quad u_3(\mathbf{x}_3) &= 3.1676. \qquad (25)
 \end{aligned}$$

600 Above is the original equilibrium. Following planner intervention, we will obtain
 601 a new equilibrium. The planner intervenes according to:

$$\begin{aligned}
 602 \quad \tau(1) &= 0 \\
 603 \quad \tau(2) &= -0.22 \\
 604 \quad \tau(3) &= 0 \\
 605 \quad \tau(4) &= -0.17.
 \end{aligned}$$

606 This means that the parameters $\hat{\alpha}(1)$ and $\hat{\alpha}(3)$ remain unchanged compared to $\alpha(1)$
 607 and $\alpha(3)$, but $\hat{\alpha}(2)$ is 22 % lower compared to $\alpha(2)$ and $\hat{\alpha}(4)$ is 17 % lower compared
 608 to $\alpha(4)$:

$$\begin{aligned}
 609 \quad \hat{\alpha}(1) &= 0.033 \\
 610 \quad \hat{\alpha}(2) &= 0.016 \\
 611 \quad \hat{\alpha}(3) &= 0.029 \\
 612 \quad \hat{\alpha}(4) &= 0.028.
 \end{aligned}$$

613 The credit constraints have just been tightened.

614 The equilibrium following planner intervention (again, unique²³) is:

615 • Consumption

	$\hat{x}_h^1(0)$	$\hat{x}_h^2(0)$	$\hat{x}_h^1(1)$	$\hat{x}_h^2(1)$	$\hat{x}_h^1(2)$	$\hat{x}_h^2(2)$	$\hat{x}_h^1(3)$	$\hat{x}_h^2(3)$	$\hat{x}_h^1(4)$	$\hat{x}_h^2(4)$
616 $h = 1$	3.029	3.624	0.866	0.859	4.841	3.328	8.302	5.395	2.096	5.392
$h = 2$	1.619	4.784	6.215	7.363	3.233	4.717	1.327	1.007	6.005	4.174
$h = 3$	7.358	3.595	4.917	3.776	3.923	3.952	2.366	5.596	3.897	2.428

617 • Assets

	\hat{b}_h^1	\hat{b}_h^2
618 $h = 1$	0.301	-0.284
$h = 2$	0.237	-0.251
$h = 3$	-0.537	0.533

619 • Prices

620 $\hat{p}^1(0) = 1$	$\hat{p}^2(0) = 0.627$
$\hat{p}^1(1) = 1$	$\hat{p}^2(1) = 0.973$
$\hat{p}^1(2) = 1$	$\hat{p}^2(2) = 0.985$
$\hat{p}^1(3) = 1$	$\hat{p}^2(3) = 1.180$
$\hat{p}^1(4) = 1$	$\hat{p}^2(4) = 0.795$
$\hat{q}^1 = 6.233$	$\hat{q}^2 = 6.305$

621 • Utility values

$$\begin{aligned}
 622 \quad u_1(\hat{\mathbf{x}}_1) &= 2.2603 \\
 623 \quad u_2(\hat{\mathbf{x}}_2) &= 2.4259 \\
 624 \quad u_3(\hat{\mathbf{x}}_3) &= 3.1686.
 \end{aligned}
 \tag{26}$$

625 Comparing the utility values in (25) and (26), a Pareto improvement has been
 626 achieved. The utility increases are 0.58 % for household $h = 1$, 0.48 % for household
 627 $h = 2$, and 0.03 % for household $h = 3$.

628 We now explain the intuition behind this Pareto improvement. Taken in isolation,
 629 a binding constraint of the form (7) for a single household h and for a particular state
 630 $s \in \mathcal{S}'$ has well-established properties. A reduction in the parameter $\alpha_h(s)$ restricts
 631 the budget set for household h , because the constraint has become tighter. This results
 632 in lower utility for household h .

633 However, consider what happens, as in the above example, when a reduction in the
 634 parameter $\alpha_h(s)$ results in constraints binding in some states in which they previously
 635 did not bind. Specifically, the following table illustrates this endogenous effect for the
 636 above example:

²³ Again, a unique equilibrium is guaranteed by our use of the Cobb–Douglas utility functions.

Constraint (7) is binding in states		
	Before intervention	After intervention
$h = 1$	$s = 4$	$s = 4$
$h = 2$	$s = 3$	$s = 3$ and $s = 4$
$h = 3$	$s = 1$	$s = 2$

As can be seen from the table, for households $h = 2$ and $h = 3$, different constraints are binding after the intervention compared to before the intervention. When the states of binding constraints “switch” following an intervention, the property described above for isolated constraints is no longer valid. In particular, two effects now play a leading role in determining the equilibrium. First, portfolio effects are present as households adjust their portfolios across the states of uncertainty where now the constraints may bind for different states. Second, general equilibrium effects are present, whereby one household’s adjustments to the newly binding constraints must affect the other households, through the relative commodity prices and asset prices, in order for the market clearing conditions to be satisfied.

Appendix A

A.1: Proof of Theorem 13

The proof of Theorem 13 requires some preliminary results before proceeding.

Define $\Delta_+^{G-1} = \{\mathbf{p} \in \mathbb{R}_+^G : \sum_{s=0}^S \sum_{c=1}^C p^c(s) = 1\}$ and $\Pi^{G-1} = \{\mathbf{p} \in \mathbb{R}_+^G : \sum_{s=0}^S \sum_{c=1}^C p^c(s) = 1\}$.

In what follows, we take for given an economy $(\mathbf{e}, \mathbf{u}, \mathbf{y}, \alpha)$.

From Proposition 12, we can define the following continuous functions.

$$\begin{aligned} \mathbf{x}_h &: \Delta_{++}^{G-1} \times \mathcal{G}_{A,S} \rightarrow \mathbb{R}^G, \quad \text{for } h \in \mathcal{H}, \\ x_1(\mathbf{p}, L) &= \arg \max(20), \\ x_h(\mathbf{p}, L) &= \arg \max(18), \quad \text{for } h \in \mathcal{H} \setminus \{1\}, \end{aligned}$$

and

$$z : \Delta_{++}^{G-1} \times \mathcal{G}_{A,S} \rightarrow \mathbb{R}^G, \quad (\mathbf{p}, L) \mapsto \sum_{h \in \mathcal{H}} (x_h(\mathbf{p}, L) - \mathbf{e}_h). \quad (27)$$

Define also

$$\psi : \Delta_+^{G-1} \times \mathcal{G}_{A,S} \rightarrow \mathbb{R}^{SA}, \quad (\mathbf{p}, L) \mapsto \mathcal{B}(\mathbf{p}, \mathbf{y}). \quad (28)$$

We say that a vector (\mathbf{p}^*, L^*) is a reduced Mr. 1 equilibrium for the economy $(\mathbf{e}, \mathbf{u}, \mathbf{y}, \alpha) \in \mathcal{E}$, if there exists \mathbf{x}^* such that $(\mathbf{x}^*, \mathbf{p}^*, L^*)$ is a Mr. 1 equilibrium for that economy.

665 **Proposition 16** A vector (\mathbf{p}^*, L^*) is a reduced Mr. 1 equilibrium for the economy
 666 $(\mathbf{e}, \mathbf{u}, \mathbf{y}, \alpha) \in \mathcal{E}$ if

- 667 1. $z(\mathbf{p}^*, L^*) = \mathbf{0}$ and
 668 2. $\langle \psi(\mathbf{p}^*, L^*) \rangle \subseteq L^*$.

669 In the next result we list some properties of the function z in (27) we will need in
 670 the proof of Lemma 19.

- 671 **Lemma 17** 1. z is continuous;
 672 2. z satisfies Walras' law;
 673 3. z is bounded from below;
 674 4. z satisfies the boundary condition, i.e., if $(\mathbf{p}^{[n]}, L^{[n]}) \rightarrow (\bar{\mathbf{p}}, \bar{L})$ with $\bar{\mathbf{p}} \in \partial \Delta_+^{G-1}$,
 675 then $\|z(\mathbf{p}^{[n]}, L^{[n]})\| \rightarrow \infty$.

- 676 *Proof* 1. It follows from Proposition 12.
 677 2. It follows from household budget constraints.
 678 3. From market clearing, for every s, c , $z^c(s)$ is bounded below by $-\sum_{h \in \mathcal{H}} e_h^c(s)$.
 679 4. It follows from the budget constraint and the strict monotonicity of u_h .

680 In the proof of Theorem 13 we are going to use the following result in [Husseini](#)
 681 [et al. \(1990\)](#).

682 **Theorem 18** (A Grassmannian Brouwer-like fixed point theorem) Let H^N be an N -
 683 dimensional affine subspace, $C \subset H^N$ a compact convex subset with nonempty relative
 684 interior and let

685
$$\Phi : C \times \mathcal{G}_{A,S} \rightarrow H^N, \quad \Psi : C \times \mathcal{G}_{A,S} \rightarrow \mathbb{R}^{AS}$$

686 be continuous functions such that $\Phi(\partial C, L) \subseteq C, \forall L \in \mathcal{G}_{A,S}$. Then there exists $(\bar{\mathbf{p}}, \bar{L})$
 687 such that

688
$$\Phi(\bar{\mathbf{p}}, \bar{L}) = \bar{\mathbf{p}}, \quad \langle \Psi(\bar{\mathbf{p}}, \bar{L}) \rangle \subseteq \bar{L}.$$

689 A crucial role in the application of the above theorem is played by the following
 690 lemma. We present the proof of the lemma in the case, analyzed in the present paper,
 691 in which the return space is described as a Grassmannian manifold. In fact, [Husseini](#)
 692 [et al. \(1990\)](#) presented instead the proof in the case of Stiefel manifolds.

693 **Lemma 19** There exists a continuous function $\varphi : \Delta_+^{G-1} \times \mathcal{G}_{A,S} \rightarrow [0, 1]$ such that
 694 the function

695
$$\phi : \Delta_+^{G-1} \times \mathcal{G}_{A,S} \rightarrow \Pi^{G-1}$$
 defined by

696
$$\phi(\mathbf{p}, L) = \varphi(\mathbf{p}, L)(p^c(s) + p^c(s)z^c(s)(\mathbf{p}, L)_{s,c} + (1 - \varphi(\mathbf{p}, L))\mathbf{u}, \quad (29)$$

697 where $\mathbf{u} = (\frac{1}{G}, \dots, \frac{1}{G}) \in \mathbb{R}^G$, satisfies

- 698 1. $\phi(\partial\Delta_+^{G-1}, L) \subseteq \Delta_+^{G-1}, \forall L \in \mathcal{G}_{A,S};$
 699 2. $\phi(\mathbf{p}, L) = \mathbf{p} \Leftrightarrow z(\mathbf{p}, L) = \mathbf{0}.$ ²⁴

700 *Proof* Define

$$701 \quad V_j = \left\{ (\mathbf{p}, L) \in \Delta_{++}^{G-1} \times \mathcal{G}_{A,S} : z_j(\mathbf{p}, L) > 0, p_j < \frac{1}{G} \right\}, \quad j \in \{1, \dots, G\}$$

702 and $K = \left(\Delta_{++}^{G-1} \times \mathcal{G}_{A,S} \right) \setminus \left(\bigcup_{j=1}^G V_j \right)$. We are going to prove that K is closed in
 703 $\Delta_+^{G-1} \times \mathcal{G}_{A,S}$. Since $\mathcal{G}_{A,S}$ is a metric space, also $\Delta_+^{G-1} \times \mathcal{G}_{A,S}$ is a metric space.
 704 Thus it is enough to prove that K is sequentially closed, i.e., that the limit point of any
 705 convergent sequence of elements of K belongs to K .

706 Rewriting K as follows

$$707 \quad K = \left\{ (\mathbf{p}, L) \in \Delta_{++}^{G-1} \times \mathcal{G}_{A,S} : \forall j \in \{1, \dots, G\}, z_j(\mathbf{p}, L) \leq 0 \text{ or } p_j \geq \frac{1}{G} \right\}$$

708 and recalling that $\mathcal{G}_{A,S}$ is compact, and thus closed, it is clear that the only way
 709 in which the limit point $(\bar{\mathbf{p}}, \bar{L})$ of a sequence $(\mathbf{p}^{[n]}, L^{[n]})$ of elements of K does
 710 not belong to K is that $\bar{\mathbf{p}} \in \partial\Delta_+^{G-1}$. However this is prevented by the boundary
 711 condition and the continuity of z on K . Indeed, if $\bar{\mathbf{p}} \in \partial\Delta_+^{G-1}$, then there exists
 712 $j \in \{1, \dots, G\}$ such that $\bar{p}_j = 0$. Hence there exists $\bar{n} \in \mathbb{N}$ such that, for every
 713 $n \geq \bar{n}$, $p_j^{[n]} < \frac{1}{G}$. By definition of K and recalling that z is bounded from below,
 714 we then have $-\sum_{h \in \mathcal{H}} e_h(j) \leq z_j(\mathbf{p}^{[n]}, L^{[n]}) \leq 0$ and thus, by the continuity of z ,
 715 it holds that $-\sum_{h \in \mathcal{H}} e_h(j) \leq z_j(\bar{\mathbf{p}}, \bar{L}) \leq 0$. On the other hand, by the boundary
 716 condition, $z_j(\mathbf{p}^{[n]}, L^{[n]}) \rightarrow +\infty$. The contradiction is found.

717 Notice that $K \cap \left(\partial\Delta_+^{G-1} \times \mathcal{G}_{A,S} \right) = \emptyset$ and that $\partial\Delta_+^{G-1} \times \mathcal{G}_{A,S}$ is closed in
 718 $\Delta_+^{G-1} \times \mathcal{G}_{A,S}$. Recalling that any metric space is normal²⁵ and that on normal spaces
 719 the Urysohn Lemma²⁶ applies, there exists a continuous function $\varphi : \Delta_+^{G-1} \times \mathcal{G}_{A,S} \rightarrow$
 720 $[0, 1]$ such that $\varphi(K) = 1$ and $\varphi(\partial\Delta_+^{G-1} \times \mathcal{G}_{A,S}) = 0$. Let us then check that the
 721 function ϕ in (29) has Π^{G-1} as codomain and satisfies 1. and 2.

722 As regards the codomain of ϕ , fix $(\mathbf{p}, L) \in \Delta_+^{G-1} \times \mathcal{G}_{A,S}$. Then $\sum_{j=1}^G p_j = 1$ and
 723 recalling that z obeys Walras' law, it holds that

²⁴ Notice that, although z in (27) is defined only on $\Delta_{++}^{G-1} \times \mathcal{G}_{A,S}$, the function ϕ is defined on $\Delta_+^{G-1} \times \mathcal{G}_{A,S}$ because, by construction, $\phi(\partial\Delta_+^{G-1} \times \mathcal{G}_{A,S}) = 0$ and thus $\phi(\partial\Delta_+^{G-1} \times \mathcal{G}_{A,S}) = \mathbf{u}$.

²⁵ A topological space X is called normal if for any pair of closed disjoint subsets C_1 and C_2 of X there exists a pair of open disjoint subsets O_1 and O_2 of X , with $O_1 \supset C_1$ and $O_2 \supset C_2$.

²⁶ We recall that the Urysohn lemma says that given two disjoint closed subsets C_1 and C_2 of a normal space X , there exists a continuous function $f : X \rightarrow [0, 1]$ such that $f(C_1) = 0$ and $f(C_2) = 1$.

$$\begin{aligned}
 \sum_{j=1}^G \phi_j(\mathbf{p}, L) &= \sum_{j=1}^G \left(\varphi(\mathbf{p}, L)(p_j + p_j z_j(\mathbf{p}, L)) + (1 - \varphi(\mathbf{p}, L)) \frac{1}{G} \right) \\
 &= \varphi(\mathbf{p}, L) \sum_{j=1}^G (p_j + p_j z_j(\mathbf{p}, L)) + (1 - \varphi(\mathbf{p}, L)) \sum_{j=1}^G \frac{1}{G} \\
 &= \varphi(\mathbf{p}, L) \sum_{j=1}^G p_j z_j(\mathbf{p}, L) + 1 = 1,
 \end{aligned}$$

i.e., $\phi(\Delta_+^{G-1} \times \mathcal{G}_{A,S}) \subseteq \Pi^{G-1}$, as desired.

In regard to 1., as we already know that $\phi(\partial\Delta_+^{G-1} \times \mathcal{G}_{A,S}) \subseteq \Pi^{G-1}$, in order to show that $\phi(\partial\Delta_+^{G-1} \times \mathcal{G}_{A,S}) \subseteq \Delta_+^{G-1}$, it is enough to check that $\phi_j(\mathbf{p}, L) \geq 0$, for every $(\mathbf{p}, L) \in \partial\Delta_+^{G-1} \times \mathcal{G}_{A,S}$ and $j \in \{1, \dots, G\}$. Since $\varphi(\partial\Delta_+^{G-1} \times \mathcal{G}_{A,S}) = 0$, it holds that, for $(\mathbf{p}, L) \in \partial\Delta_+^{G-1} \times \mathcal{G}_{A,S}$, $\phi_j(\mathbf{p}, L) = \frac{1}{G} \geq 0$.

Let us finally prove 2. Assume that $z(\mathbf{p}, L) = \mathbf{0}$ and show that $\phi(\mathbf{p}, L) = \mathbf{p}$. If $z(\mathbf{p}, L) = \mathbf{0}$ then $z_j(\mathbf{p}, L) \leq 0$ for every j and so $(\mathbf{p}, L) \in K$. Hence, $\varphi(\mathbf{p}, L) = 1$ and thus $\phi_j(\mathbf{p}, L) = p_j + p_j z_j(\mathbf{p}, L) = p_j$, for every j , as desired.

Assume now that $\phi(\mathbf{p}, L) = \mathbf{p}$ and show that $z(\mathbf{p}, L) = \mathbf{0}$. Notice that

$$\Delta_+^{G-1} \times \mathcal{G}_{A,S} = \left(\partial\Delta_+^{G-1} \times \mathcal{G}_{A,S} \right) \cup K \cup \left(\bigcup_{j=1}^G V_j \right)$$

and that $\partial\Delta_+^{G-1} \times \mathcal{G}_{A,S}$, K and $\bigcup_{j=1}^G V_j$ are pairwise disjoint. If $(\mathbf{p}, L) \in \Delta_+^{G-1} \times \mathcal{G}_{A,S}$, then there are three cases to consider, i.e., $(\mathbf{p}, L) \in \partial\Delta_+^{G-1} \times \mathcal{G}_{A,S}$, $(\mathbf{p}, L) \in K$ and $(\mathbf{p}, L) \in V_{j^*}$, for some $j^* \in \{1, \dots, G\}$. We claim that only in the second case it may happen that $\phi(\mathbf{p}, L) = \mathbf{p}$. Indeed, in the first case $\phi(\mathbf{p}, L) = (\frac{1}{G}, \dots, \frac{1}{G}) \notin \partial\Delta_+^{G-1}$. In the third case, by definition of V_{j^*} , we would have

$$\begin{aligned}
 p_{j^*} &= \phi_{j^*}(\mathbf{p}, L) = \varphi(\mathbf{p}, L)(p_{j^*} + p_{j^*} z(j^*)(\mathbf{p}, L)) + (1 - \varphi(\mathbf{p}, L)) \frac{1}{G} \\
 &> \varphi(\mathbf{p}, L)(p_{j^*} + p_{j^*} z(j^*)(\mathbf{p}, L)) + (1 - \varphi(\mathbf{p}, L)) p_{j^*} \\
 &= p_{j^*} + \varphi(\mathbf{p}, L) p_{j^*} z(j^*)(\mathbf{p}, L) \geq p_{j^*},
 \end{aligned}$$

a contradiction. Thus $\phi(\mathbf{p}, L) = \mathbf{p}$ only if $(\mathbf{p}, L) \in K$ and in this case, as $\varphi(\mathbf{p}, L) = 1$, it follows that, for every $j \in \{1, \dots, G\}$, $p_j = \phi_j(\mathbf{p}, L) = p_j + p_j z_j(\mathbf{p}, L)$, from which, since $p_j > 0$, we have $z_j(\mathbf{p}, L) = 0$, as desired.

748 *Proof of Theorem 13* We want to apply Theorem 18, identifying Φ and Ψ with ϕ in
 749 (29) and ψ in (28), respectively. Π^{G-1} is an affine subspace of \mathbb{R}^G and $\Delta_+^{G-1} \subset \Pi^{G-1}$
 750 is clearly a compact, convex subset with nonempty relative interior. ϕ is a continuous
 751 function from Lemma 17 and from the fact that φ is a continuous function from Lemma
 752 19. Again from the latter lemma, we have that $\phi(\partial\Delta_+^{G-1}, L) \subseteq \Delta_+^{G-1}$, $\forall L \in \mathcal{G}_{A,S}$.
 753 Finally, from Theorem 18 and Proposition 16, the desired result follows. \square

754 A.2: Proof of Theorem 14

755 Let \mathcal{V} be a topological Hausdorff vector space, $V \subseteq \mathcal{V}$ be an open set and $f : V \rightarrow \mathbb{R}^n$
 756 be a function. We say that $f \in C^0(V, \mathbb{R}^n)$ if f is continuous, while $f \in C^1(V, \mathbb{R}^n)$
 757 if it is continuous, there exists the limit

$$758 \quad df(\mathbf{v}, \mathbf{w}) = \lim_{\varepsilon \rightarrow 0} \frac{f(\mathbf{v} + \varepsilon \mathbf{w}) - f(\mathbf{v})}{\varepsilon}, \quad \forall \mathbf{v} \in V, \mathbf{w} \in \mathcal{V},$$

759 and the function $df : V \times \mathcal{V} \rightarrow \mathbb{R}^n$ is continuous.

760 Given any (not necessarily open) set $X \subseteq \mathcal{V}$ and $f : X \rightarrow \mathbb{R}^n$, we say $f \in$
 761 $C^0(X, \mathbb{R}^n)$ if f is continuous with respect to the topology induced by \mathcal{V} on X , while,
 762 as in the finite dimensional setting, $f \in C^1(X, \mathbb{R}^n)$ if for every $\mathbf{v}_0 \in X$ there exists an
 763 open neighborhood of \mathbf{v}_0 in \mathcal{V} , say $V(\mathbf{v}_0)$, and a function $\bar{f} : V(\mathbf{v}_0) \rightarrow \mathbb{R}^n$ such that
 764 $\bar{f} \in C^1(V(\mathbf{v}_0), \mathbb{R}^n)$ and, for every $\mathbf{x} \in V(\mathbf{v}_0) \cap X$, $f(\mathbf{x}) = \bar{f}(\mathbf{x})$.

765 Those definitions allow to state the following implicit function theorem which is a
 766 simplified version of Theorem 2.3 in Glöckner (2006).²⁷

767 **Theorem 20** *Let us consider $f : O \times V \rightarrow \mathbb{R}^n$, where O is an open subset of \mathbb{R}^n*
 768 *and V is an open subset of a topological Hausdorff vector space \mathcal{V} . Assume $f \in$*
 769 *$C^1(O \times V, \mathbb{R}^n)$ and let $(\mathbf{x}_0, \mathbf{v}_0) \in O \times V$ such that $f(\mathbf{x}_0, \mathbf{v}_0) = \mathbf{0}$ and $\mathbf{D}_x \mathbf{f}(\mathbf{x}_0, \mathbf{v}_0)$ is*
 770 *invertible.²⁸ Then there exist $O(\mathbf{x}_0) \subseteq O$ open neighborhood of \mathbf{x}_0 , $V(\mathbf{v}_0) \subseteq V$ open*
 771 *neighborhood of \mathbf{v}_0 and $g : V(\mathbf{v}_0) \rightarrow O(\mathbf{x}_0)$ such that*

- 772 1. $g \in C^1(V(\mathbf{v}_0), O(\mathbf{x}_0))$,
- 773 2. $g(\mathbf{v}_0) = \mathbf{x}_0$,
- 774 3. $\{(\mathbf{x}, \mathbf{v}) \in O(\mathbf{x}_0) \times V(\mathbf{v}_0) : f(\mathbf{x}, \mathbf{v}) = \mathbf{0}\} = \{(\mathbf{x}, \mathbf{v}) \in O(\mathbf{x}_0) \times V(\mathbf{v}_0) : \mathbf{x} = g(\mathbf{v})\}$.

²⁷ We stress that we need such more sophisticated result, rather than the classical implicit function theorem—see for instance Lang (1983), page 131—because one of the factors of the Cartesian product in the domain of the “equilibrium function” we deal with in Theorem 14 is given by the set of twice continuously differentiable utility functions. The topology that set is commonly endowed with is the C^2 compact-open topology—see for example Hirsch (1976), pages 34–35. Unfortunately, that topology is not generated by a norm [see again Hirsch (1976), page 35] and thus the standard implicit function theorem cannot be applied in our framework. On the other hand, the space of our utility functions with the C^2 compact-open topology is a topological Hausdorff vector space and therefore the theorem by Glöckner can be used instead.

²⁸ Note that if $f \in C^1(O \times V, \mathbb{R}^n)$ then, for every $\mathbf{v} \in V$, $f(\cdot, \mathbf{v}) : O \rightarrow \mathbb{R}^n$, $\mathbf{x} \mapsto f(\mathbf{x}, \mathbf{v})$, belongs to $C^1(O, \mathbb{R}^n)$ and thus, for every $(\mathbf{x}, \mathbf{v}) \in O \times V$, $\mathbf{D}_x \mathbf{f}(\mathbf{x}, \mathbf{v})$ is well defined.

775 *Proof of Theorem 14 Step 1.*

776 Define, for each $\sigma \in \Sigma$,

777
$$\mathcal{E}_\sigma = \mathbb{R}_{++}^{GH} \times \mathbb{R}^{(S+1)H} \times \mathbb{R}^{AH} \times \mathbb{R}^{SH} \times \mathbb{R}_{++}^G \times \mathbb{R}^A \times V_{\sigma^{-1}}, \quad (30)$$

778 with generic element

779
$$\xi = ((\mathbf{x}_h, \lambda_h, \mathbf{b}_h, \mu_h)_{h \in \mathcal{H}}, \mathbf{p}, \mathbf{q}, L) = (\mathbf{x}, \lambda, \mathbf{b}, \mu, \mathbf{p}, \mathbf{q}, L),$$

780 and the function

781
$$\mathcal{F}_\sigma : \mathcal{E}_\sigma \times \mathcal{E} \rightarrow \mathbb{R}^{\dim(\mathcal{E}_\sigma)},$$

782
$$\mathcal{F}_\sigma(\xi, \mathbf{e}, \mathbf{u}, \mathbf{y}, \alpha) = \left[\begin{array}{l} \mathbf{D}_{\mathbf{x}_h(s)} \mathbf{u}_h(\mathbf{x}_h) - \lambda_h(s) \mathbf{p}(s) \\ -\Phi(\mathbf{p})(\mathbf{x}_h - \mathbf{e}_h) + \begin{bmatrix} -\mathbf{q} \\ \mathbf{P}_\sigma \begin{bmatrix} -\psi(L) \\ \mathbf{I}_A \end{bmatrix} \end{bmatrix} \mathbf{b}_h \\ \lambda_h \begin{bmatrix} -\mathbf{q} \\ \mathbf{P}_\sigma \begin{bmatrix} -\psi(L) \\ \mathbf{I}_A \end{bmatrix} \end{bmatrix} + \mu_h \mathbf{P}_\sigma \begin{bmatrix} -\psi(L) \\ \mathbf{I}_A \end{bmatrix} \\ \min \left\{ \mu_h, \mathbf{P}_\sigma \begin{bmatrix} -\psi(L) \\ \mathbf{I}_A \end{bmatrix} \mathbf{b}_h + \mathbf{A}_h \Phi^1(\mathbf{p}) \mathbf{e}_h^1 \right\} \\ \sum_{h=1}^H (\mathbf{x}_h^1 - \mathbf{e}_h^1) \\ \sum_{h=1}^H \mathbf{b}_h \\ p^1(s) - 1 \\ \text{vec} [\mathbf{I}_{S-A} \mid \psi(L)] \cdot \mathbf{P}_{\sigma^{-1}} \cdot \mathcal{R}(\mathbf{p}, \mathbf{y}) \end{array} \right] \quad (31)$$

783 where $\psi : V_{\sigma^{-1}} \rightarrow \mathbb{M}(S - A, A)$ is the diffeomorphism in (15), with $V_{\sigma^{-1}} \subseteq \mathcal{G}_{A,S}$
 784 open.

785 For simplicity and without loss of generality, from now on we consider the case
 786 $\mathbf{P}_\sigma = \mathbf{I}$, so that \mathcal{F}_σ becomes $\mathcal{F} : \mathcal{E} \times \mathcal{E} \rightarrow \mathbb{R}^{\dim(\mathcal{E})}$.

787 We now show that border line cases are rare. For every $h \in \mathcal{H}$, we define $\mathcal{S}_h^1, \mathcal{S}_h^2$
 788 and $\widehat{\mathcal{S}}_h^1$ so that $\{1, \dots, S\} = \mathcal{S}_h^1 \cup \mathcal{S}_h^2$, with $(\mathcal{S}_h^1 \setminus \widehat{\mathcal{S}}_h^1) \cap \mathcal{S}_h^2 = \emptyset$ and $\widehat{\mathcal{S}}_h^1 \subseteq \mathcal{S}_h^1$,
 789 in order to have

$$\begin{aligned}
 s \in \mathcal{S}_h^1 \setminus \widehat{\mathcal{S}}_h^1 &\Rightarrow \mathbf{m}(s) \cdot \mathbf{b}_h + \alpha_h(s) \cdot \mathbf{p}(s) \mathbf{e}_h(s) = 0 \\
 s \in \widehat{\mathcal{S}}_h^1 &\Rightarrow \mathbf{m}(s) \cdot \mathbf{b}_h + \alpha_h(s) \cdot \mathbf{p}(s) \mathbf{e}_h(s) = 0 \quad \text{and} \quad \mu_h(s) = 0 \\
 s \in \mathcal{S}_h^2 &\Rightarrow \mu_h(s) = 0,
 \end{aligned}$$

where we have denoted by $\mathbf{m}(s)$ the s -th row of $\begin{bmatrix} -\psi(L) \\ \mathbf{I}_A \end{bmatrix}$.

Define $\hat{\mathbf{y}} = (y^{a,1}(s))_{a \in \mathcal{A}, s \in \{1, \dots, S-A\}}$ and the full rank matrix

$$\mathbf{B} = \begin{bmatrix} p^1(1) & & & & & & & & \\ & \ddots & & & & & & & \\ & & p^1(1) & & & & & & \\ & & & \ddots & & & & & \\ & & & & p^1(S-A) & & & & \\ & & & & & \ddots & & & \\ & & & & & & p^1(S-A) & & \end{bmatrix}$$

The computation of the desired (partial) Jacobian matrix is presented in the table below, where the following conventions are adopted.

(a) The symbol \odot denotes a matrix which is insignificant for our argument, while no symbol means 0. For size convenience, just in the table below, we set

$$\mathbf{R}(\mathbf{q}) = \begin{bmatrix} -\mathbf{q} \\ -\psi(L) \\ \mathbf{I} \end{bmatrix} \quad \text{and} \quad \mathbf{R} = \begin{bmatrix} -\psi(L) \\ \mathbf{I} \end{bmatrix}.$$

- (b) The $*$ next to a matrix indicates that it is a full row rank matrix.
- (c) The desired full rank result is obtained as follows. In each super-row, use the starred matrix to clean up that super-row, being sure that in that super-column there are only zero matrices. An order in which the appropriate elementary (super) column operations have to be performed is the one indicated in the last column of the table.

Author Proof

806 where, for $h \in \mathcal{H}$, we have set \mathbf{Z}_h^1 equal to the square diagonal matrix with elements
 807 $\mathbf{p}(s)\mathbf{e}_h(s)$, for $s \in \mathcal{S}_h^1$, on the diagonal. Moreover

808
$$\Phi(\mathbf{p}^1) = \begin{bmatrix} p^1(0) & & & \\ & p^1(1) & & \\ & & \ddots & \\ & & & p^1(S) \end{bmatrix}.$$

809 Step 2.

810 After having shown that border line cases are rare, we are going to prove that in a
 811 full measure subset of $\mathbb{R}_{++}^{GH} \times \mathbb{R}^{CAS}$, the return matrix $\mathcal{R}(\mathbf{p}, \mathbf{y})$ has full rank. Actually,
 812 we are going to show that its square A -dimensional submatrix $\widehat{\mathcal{R}}(\mathbf{p}, \mathbf{y})$ has full rank,
 813 where

814
$$\widehat{\mathcal{R}}(\mathbf{p}, \mathbf{y}) = \begin{bmatrix} \mathbf{p}(S-A+1)\mathbf{y}^1(S-A+1) \cdots \mathbf{p}(S-A+1)\mathbf{y}^a(S-A+1) \cdots \mathbf{p}(S-A+1)\mathbf{y}^A(S-A+1) \\ \vdots \\ \mathbf{p}(S-A+a)\mathbf{y}^1(S-A+a) \cdots \mathbf{p}(S-A+a)\mathbf{y}^a(S-A+a) \cdots \mathbf{p}(S-A+a)\mathbf{y}^A(S-A+a) \\ \vdots \\ \mathbf{p}(S)\mathbf{y}^1(S) \cdots \mathbf{p}(S)\mathbf{y}^a(S) \cdots \mathbf{p}(S)\mathbf{y}^A(S) \end{bmatrix},$$

816 by showing that 0 is a regular value for $(F, G) : \mathcal{E} \times \mathcal{E} \times \mathbb{R}^A \rightarrow \mathbb{R}^{\dim(\mathcal{E})+(A+1)}$,
 817 where

818
$$G : \mathcal{E} \times \mathcal{E} \times \mathbb{R}^A \rightarrow \mathbb{R}^{A+1}, \quad (\xi, \mathbf{e}, \mathbf{u}, \mathbf{y}, \alpha, \mathbf{d}) \mapsto (\mathbf{d} \cdot \widehat{\mathcal{R}}(\mathbf{p}, \mathbf{y}), \mathbf{d}\mathbf{d} - 1).$$

819 Calling \mathbf{T} the Jacobian matrix in the previous page, we then have to show that the
 820 following matrix

821
$$\begin{bmatrix} \mathbf{T} & \star & \mathbf{0} \\ \mathbf{0} & \mathbf{N} & \widehat{\mathcal{R}}(\mathbf{p}, \mathbf{y}) \\ \mathbf{0} & \mathbf{0} & 2\mathbf{d}^T \end{bmatrix}$$

822 has full rank, where the last two columns are the derivatives with respect to $\widehat{\mathbf{y}}$ and \mathbf{d} ,
 823 respectively, with $\widehat{\mathbf{y}}$ defined hereinafter.

824 As $\mathbf{d} = (d(S-A+a))_{a \in \mathcal{A}}$ is such that $\mathbf{d}\mathbf{d} = 1$, then there exists $\bar{a} \in \mathcal{A}$ such that
 825 $d(S-A+\bar{a}) \neq 0$. Then we set $\widehat{\mathbf{y}} = (y^{a,1}(S-A+\bar{a}))_{a \in \mathcal{A}} \in \mathbb{R}^A$. Notice that

826
$$\mathbf{d} \cdot \widehat{\mathcal{R}}(\mathbf{p}, \mathbf{y}) = \begin{bmatrix} \sum_{j=1}^A d(S-A+j)\mathbf{p}(S-A+j)\mathbf{y}^1(S-A+j) \\ \vdots \\ \sum_{j=1}^A d(S-A+j)\mathbf{p}(S-A+j)\mathbf{y}^a(S-A+j) \\ \vdots \\ \sum_{j=1}^A d(S-A+j)\mathbf{p}(S-A+j)\mathbf{y}^A(S-A+j) \end{bmatrix}$$

827 and thus

828
$$\mathbf{N} = \begin{bmatrix} d(S - A + \bar{a})p^1(S - A + \bar{a}) & & & \\ & \ddots & & \\ & & d(S - A + \bar{a})p^1(S - A + \bar{a}) & \\ & & & \ddots \\ & & & & d(S - A + \bar{a})p^1(S - A + \bar{a}) \end{bmatrix}$$

830 which has clearly full rank. This concludes the proof of the step.

831 Step 3.

832 The proof is given in Proposition 21 below.

833 Step 4.

834 Apply Theorem 20. □

835 Recalling the definition of \mathcal{E}_σ in (30), we rewrite the function $\mathcal{F}_\sigma : \mathcal{E}_\sigma \times \mathcal{E} \rightarrow$
 836 $\mathbb{R}^{\dim(\mathcal{E}_\sigma)}$ in (31) as

837
$$\mathcal{F}_\sigma(\xi, \mathbf{e}, \mathbf{u}, \mathbf{y}, \alpha)$$

838
$$= \begin{bmatrix} \mathbf{D}_{\mathbf{x}_h(s)} \mathbf{u}_h(\mathbf{x}_h) - \lambda_h(s) \mathbf{p}(s) \\ -\mathbf{p}(0)(\mathbf{x}_h(0) - \mathbf{e}_h(0)) - \mathbf{q} \mathbf{b}_h \\ -\mathbf{p}(s)(\mathbf{x}_h(s) - \mathbf{e}_h(s)) - \sum_{a=1}^A m_{\sigma^{-1}(s)a} b_h^a, \quad \sigma^{-1}(s) \in \{1, \dots, S - A\} \\ -\mathbf{p}(s)(\mathbf{x}_h(s) - \mathbf{e}_h(s)) + b_h^{\sigma^{-1}(s) - (S - A)}, \quad \sigma^{-1}(s) \in \{S - A + 1, \dots, S\} \\ -\lambda_h(0) q^{\sigma^{-1}(s) - (S - A)} - \sum_{\sigma^{-1}(s)=1}^{S-A} (\lambda_h(s) + \mu_h(s)) m_{\sigma^{-1}(s)(\sigma^{-1}(s) - (S - A))} + \\ + \lambda_h(s) + \mu_h(s), \quad \sigma^{-1}(s) \in \{S - A + 1, \dots, S\} \\ \min \left\{ \mu_h, \mathbf{P}_\sigma \begin{bmatrix} -\psi(L) \\ \mathbf{I}_A \end{bmatrix} \mathbf{b}_h + \mathbf{A}_h \Phi^1(\mathbf{p}) \mathbf{e}_h^1 \right\} \\ \sum_{h=1}^H (\mathbf{x}_h^\lambda - \mathbf{e}_h^\lambda) \\ \sum_{h=1}^H \mathbf{b}_h \\ p^1(s) - 1 \\ \text{vec} [\mathbf{I}_{S-A} \mid \psi(L)] \cdot \mathbf{P}_{\sigma^{-1}} \cdot \mathcal{R}(\mathbf{p}, \mathbf{y}) \end{bmatrix}$$

839 where $L \in V_{\sigma^{-1}}$ and $\psi(L) = (m_{sa})_{s \in \{1, \dots, S - A\}, a \in \mathcal{A}} \in \mathbb{M}(S - A, A)$.

840 We also recall that a function $f : A \rightarrow B$, with A and B topological spaces, is
 841 proper if, for every $K \subseteq B$ compact set, $f^{-1}(K) \subseteq A$ is compact as well. We also
 842 recall that any proper and continuous function is closed, i.e., it maps closed sets onto
 843 closed sets.

844 **Proposition 21** \mathcal{F}_σ is continuous on $\mathcal{E}_\sigma \times \mathcal{E}$ and

845
$$\pi : \bigcup_{\sigma \in \Sigma} \mathcal{F}_\sigma^{-1}(\mathbf{0}) \rightarrow \mathcal{E}, \quad (\xi, \mathbf{e}, \mathbf{u}, \mathbf{y}, \alpha) \mapsto \pi(\xi, \mathbf{e}, \mathbf{u}, \mathbf{y}, \alpha) = (\mathbf{e}, \mathbf{u}, \mathbf{y}, \alpha)$$

Author Proof

846 *is proper.*

847 *Proof* The continuity of \mathcal{F}_σ is immediate. In order to show that π is proper, we
 848 have to prove that each sequence $(\xi^{[n]}, \mathbf{e}^{[n]}, \mathbf{u}^{[n]}, \mathbf{y}^{[n]}, \alpha^{[n]})_{n \in \mathbb{N}}$ in $\bigcup_{\sigma \in \Sigma} \mathcal{F}_\sigma^{-1}(\mathbf{0})$,
 849 such that $(\mathbf{e}^{[n]}, \mathbf{u}^{[n]}, \mathbf{y}^{[n]}, \alpha^{[n]})$ converges in \mathcal{E} , admits a converging subsequence in
 850 $\bigcup_{\sigma \in \Sigma} \mathcal{F}_\sigma^{-1}(\mathbf{0})$. Since $\mathcal{G}_{A,S}$ is sequentially compact, let us assume that

851
$$(\mathbf{e}^{[n]}, \mathbf{u}^{[n]}, \mathbf{y}^{[n]}, \alpha^{[n]}, L^{[n]}) \rightarrow (\bar{\mathbf{e}}, \bar{\mathbf{u}}, \bar{\mathbf{y}}, \bar{\alpha}, \bar{L}) \in \mathcal{E} \times \mathcal{G}_{A,S}.$$

852 Therefore there exists $\sigma \in \Sigma$ such that $\bar{L} \in V_{\sigma-1}$ and for sufficiently large n , $L^{[n]} \in$
 853 $V_{\sigma-1}$, too. Without loss of generality we can assume that $\mathbf{P}_\sigma = \mathbf{I}$, so that \mathcal{F}_σ simply
 854 becomes

855 $\mathcal{F}(\xi, \mathbf{e}, \mathbf{u}, \mathbf{y}, \alpha)$

$$= \left[\begin{array}{l} \text{(32a)} \quad \mathbf{D}_{\mathbf{x}_h(s)} \mathbf{u}_h(\mathbf{x}_h) - \lambda_h(s) \mathbf{p}(s) \\ \quad - \mathbf{p}(0)(\mathbf{x}_h(0) - \mathbf{e}_h(0)) - \mathbf{q} \mathbf{b}_h \\ \text{(32b)} \quad - \mathbf{p}(s)(\mathbf{x}_h(s) - \mathbf{e}_h(s)) - \sum_{a=1}^A m_{sa} b_h^a, \quad s \in \{1, \dots, S-A\} \\ \quad - \mathbf{p}(s)(\mathbf{x}_h(s) - \mathbf{e}_h(s)) + b_h^{s-(S-A)}, \quad s \in \{S-A+1, \dots, S\} \\ \text{(32c)} \quad - \lambda_h(0) q^a - \sum_{s=1}^{S-A} (\lambda_h(s) + \mu_h(s)) m_{sa} + \lambda_h(S-A+a) + \mu_h(S-A+a) \\ \text{(32d)} \quad \min \left\{ \mu_h, \left[\begin{array}{c} -\psi(L) \\ \mathbf{I}_A \end{array} \right] \mathbf{b}_h + \mathbf{A}_h \Phi^1(\mathbf{p}) e_h^1 \right\} \\ \text{(32e)} \quad \sum_{h=1}^H (\mathbf{x}_h - \mathbf{e}_h) \\ \text{(32f)} \quad \sum_{h=1}^H \mathbf{b}_h \\ \text{(32g)} \quad p^1(s) - 1 \\ \text{(32h)} \quad \text{vec} [\mathbf{I}_{S-A} \mid \psi(L)] \cdot \mathcal{R}(\mathbf{p}, \mathbf{y}) \end{array} \right] \tag{32}$$

857

858 Then it suffices to show that, up to a subsequence, $(\xi^{[n]})_{n \in \mathbb{N}}$ converges to a certain
 859 $\bar{\xi} \in \mathcal{E}$: indeed the condition $\mathcal{F}(\bar{\xi}, \bar{\mathbf{e}}, \bar{\mathbf{u}}, \bar{\mathbf{y}}, \bar{\alpha}) = \mathbf{0}$ follows by the continuity of \mathcal{F} . As
 860 we are going to use a diagonal argument, every time we say that a sequence converges
 861 we mean it has a converging subsequence. Let us start with the convergence of $\mathbf{x}^{[n]}$.
 862 For a fixed $h \in \mathcal{H}$, we know that, for every $n \in \mathbb{N}$, $(\mathbf{x}_h^{[n]}, \mathbf{b}_h^{[n]})$ is solution to the
 863 problem

864
$$\max_{(\mathbf{x}_h, \mathbf{b}_h)} u_h^{[n]}(\mathbf{x}_h)$$

865
$$\text{s.t. } - \mathbf{p}^{[n]}(0) (\mathbf{x}_h(0) - \mathbf{e}_h^{[n]}(0)) - \mathbf{q}^{[n]} \mathbf{b}_h = 0 \tag{33}$$

866
$$- \Phi^1(\mathbf{p}^{[n]}) (\mathbf{x}_h^1 - \mathbf{e}_h^{[n]1}) + \left[\begin{array}{c} -\psi(L^{[n]}) \\ \mathbf{I}_A \end{array} \right] \mathbf{b}_h = \mathbf{0} \tag{34}$$

867
$$\left[\begin{array}{c} -\psi(L^{[n]}) \\ \mathbf{I}_A \end{array} \right] \mathbf{b}_h + \mathbf{A}_h^{[n]} \Phi^1(\mathbf{p}^{[n]}) \mathbf{e}_h^{[n]1} \geq \mathbf{0} \tag{35}$$

Author Proof

868 and then, since $(\mathbf{e}_h^{[n]}, \mathbf{0})$ belongs to the constraint set, it has to be $u_h^{[n]}(\mathbf{x}_h^{[n]}) \geq u_h^{[n]}(\mathbf{e}_h^{[n]})$.
 869 Since $(\mathbf{e}_h^{[n]})_{n \in \mathbb{N}}$ converges to $\bar{\mathbf{e}}_h \in \mathbb{R}_{++}^G$, it holds that the compact set $E_h = \{\mathbf{e}_h^{[n]}\}_{n \in \mathbb{N}} \cup$
 870 $\{\bar{\mathbf{e}}_h\}$ is a subset of \mathbb{R}_{++}^G and we have

$$871 \quad u_h^{[n]}(\mathbf{x}_h^{[n]}) \geq u_h^{[n]}(\mathbf{e}_h^{[n]}) \geq \min_{\mathbf{x}_h \in E_h} u_h^{[n]}(\mathbf{x}_h) \geq \min_{\mathbf{x}_h \in E_h} \bar{u}_h(\mathbf{x}_h) - \varepsilon^{[n]},$$

872 for a suitable sequence $(\varepsilon^{[n]})_{n \in \mathbb{N}}$ in \mathbb{R}_{++} such that $\varepsilon^{[n]} \rightarrow 0$ if $n \rightarrow \infty$, by the
 873 definition of the topology on $C^2(\mathbb{R}_{++}^G)$. Indeed we can define, for every $n \in \mathbb{N}$,

$$874 \quad \varepsilon^{[n]} = \max_{\mathbf{w} \in E_h} |u_h^{[n]}(\mathbf{w}) - \bar{u}_h(\mathbf{w})|.$$

875 Let $\mathbf{x}_h^* \in E_h$ be such that $\min_{\mathbf{x}_h \in E_h} \bar{u}_h(\mathbf{x}_h) = \bar{u}_h(\mathbf{x}_h^*)$, and let $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^G$
 876 and $\delta > 0$ be small enough such that $\mathbf{x}_h^* - 2\delta\mathbf{1} \in \mathbb{R}_{++}^G$. Obviously, since by (3),
 877 $\bar{u}_h(\mathbf{x}_h^*) > \bar{u}_h(\mathbf{x}_h^* - \delta\mathbf{1})$, there exists n_1 such that $n \geq n_1$ implies $\bar{u}_h(\mathbf{x}_h^*) - \varepsilon^{[n]} \geq$
 878 $\bar{u}_h(\mathbf{x}_h^* - \delta\mathbf{1})$ and thus, for every $n \geq n_1$,

$$879 \quad u_h^{[n]}(\mathbf{x}_h^{[n]}) \geq \bar{u}_h(\mathbf{x}_h^* - \delta\mathbf{1}). \tag{36}$$

880 Of course, because of the validity of $S + 1$ Walras' laws in our model, we can also
 881 assume that, for every $n \geq n_1$,

$$882 \quad \mathbf{0} \ll \mathbf{x}_h^{[n]} \leq \sum_{h=1}^H \mathbf{e}_h^{[n]} \leq \sum_{h=1}^H \bar{\mathbf{e}}_h + \mathbf{1}.$$

883 Our purpose now is to prove that for infinite values of n it is $\bar{u}_h(\mathbf{x}_h^{[n]}) \geq \bar{u}_h(\mathbf{x}_h^* - 2\delta\mathbf{1})$.
 884 Let $\hat{\mathbf{x}}_h \in [\mathbf{0}, \sum_{h=1}^H \bar{\mathbf{e}}_h + \mathbf{1}]$ be a cluster point of $(\mathbf{x}_h^{[n]})_{n \geq n_1}$. Then we can assume
 885 $\mathbf{x}_h^{[n]} \rightarrow \hat{\mathbf{x}}_h$. Consider any $\tilde{\mathbf{x}}_h \in \mathbb{R}_{++}^G$ such that $\bar{u}_h(\tilde{\mathbf{x}}_h) = \bar{u}_h(\mathbf{x}_h^* - 2\delta\mathbf{1})$. If we take n
 886 large enough, by (36), it is $u_h^{[n]}(\mathbf{x}_h^{[n]}) - u_h^{[n]}(\tilde{\mathbf{x}}_h) \geq 0$. Then, for n sufficiently large,

$$887 \quad 0 \leq u_h^{[n]}(\mathbf{x}_h^{[n]}) - u_h^{[n]}(\tilde{\mathbf{x}}_h) \leq \mathbf{D}_{\mathbf{x}_h} u_h^{[n]}(\tilde{\mathbf{x}}_h)(\mathbf{x}_h^{[n]} - \tilde{\mathbf{x}}_h) =$$

$$888 \quad = (\mathbf{D}_{\mathbf{x}_h} u_h^{[n]}(\tilde{\mathbf{x}}_h) - \mathbf{D}_{\mathbf{x}_h} \bar{u}_h(\tilde{\mathbf{x}}_h))(\mathbf{x}_h^{[n]} - \tilde{\mathbf{x}}_h) + \mathbf{D}_{\mathbf{x}_h} \bar{u}_h(\tilde{\mathbf{x}}_h)(\mathbf{x}_h^{[n]} - \tilde{\mathbf{x}}_h).$$

889 Taking the limit as $n \rightarrow \infty$ in the previous inequality, we get

$$890 \quad \mathbf{D}_{\mathbf{x}_h} \bar{u}_h(\tilde{\mathbf{x}}_h)(\hat{\mathbf{x}}_h - \tilde{\mathbf{x}}_h) \geq 0.$$

891 Then

$$892 \quad \hat{\mathbf{x}}_h \in \bigcap_{\tilde{\mathbf{x}}_h \in \{\mathbf{y} \in \mathbb{R}_{++}^G : \bar{u}_h(\mathbf{y}) = \bar{u}_h(\mathbf{x}_h^* - 2\delta\mathbf{1})\}} \{\mathbf{y} \in \mathbb{R}^G : \mathbf{D}_{\mathbf{x}_h} \bar{u}_h(\tilde{\mathbf{x}}_h)(\mathbf{y} - \tilde{\mathbf{x}}_h) \geq 0\}. \tag{37}$$

893 Since the right hand side of (37) is exactly $\{\mathbf{y} \in \mathbb{R}^G : \bar{u}_h(\mathbf{y}) \geq \bar{u}_h(\mathbf{x}_h^* - 2\delta\mathbf{1})\}$, which
 894 is a subset of \mathbb{R}_{++}^G by (5), then $\widehat{\mathbf{x}}_h \in \mathbb{R}_{++}^G$ and the proof is complete. As regards the
 895 convergence of $\lambda^{[n]}$, from (32a), (32g) and (35) we find that, for every $h \in \mathcal{H}$ and
 896 $s \in \mathcal{S}$,

$$897 \quad \lambda_h^{[n]}(s) = \mathbf{D}_{\mathbf{x}_h^1(s)} \mathbf{u}_h^{[n]}(\mathbf{x}_h^{[n]}) \rightarrow \mathbf{D}_{\mathbf{x}_h^1(s)} \bar{\mathbf{u}}_h(\bar{\mathbf{x}}_h) = \bar{\lambda}_h(s) \in \mathbb{R}_{++},$$

898 since $\mathbf{D}_{\mathbf{x}_h^1(s)} \mathbf{u}_h^{[n]} \rightarrow \mathbf{D}_{\mathbf{x}_h^1(s)} \bar{\mathbf{u}}_h$ uniformly on compact subsets of \mathbb{R}_{++}^G . Then, from (32a)
 899 and (3), it follows that, for every $s \in \mathcal{S}$,

$$900 \quad \mathbf{p}^{[n]}(s) = \frac{\mathbf{D}_{\mathbf{x}_h(s)} \mathbf{u}_h^{[n]}(\mathbf{x}_h^{[n]})}{\lambda_h^{[n]}(s)} \rightarrow \frac{\mathbf{D}_{\mathbf{x}_h(s)} \bar{\mathbf{u}}_h(\bar{\mathbf{x}}_h)}{\bar{\lambda}_h(s)} = \bar{\mathbf{p}}(s) \in \mathbb{R}_{++}^C$$

901 and thus $(\mathbf{p}^{[n]})_{n \in \mathbb{N}}$ converges to an element $\bar{\mathbf{p}} \in \mathbb{R}_{++}^G$.

902 By (32b) we then immediately get the convergence of $\mathbf{b}_h^{[n]}$ to an element $\bar{\mathbf{b}}_h \in \mathbb{R}^A$.

903 Let us now check the convergence of $\mu_h^{[n]}$. Let us set $\mathcal{S}'_h = \{s \in \mathcal{S} : \mu_h^{[n]}(s) \rightarrow 0\}$
 904 and $\mathcal{S}''_h = \mathcal{S} \setminus \mathcal{S}'_h$. We have only to show that $\mu_h^{[n]}(s)$ is convergent for $s \in \mathcal{S}''_h$.
 905 From (32c) it follows that

$$906 \quad \lambda_h^{[n]}(0) \mathbf{q}^{[n]} \mathbf{b}_h^{[n]} = - \sum_{s=1}^{S-A} \sum_{a=1}^A (\lambda_h^{[n]}(s) + \mu_h^{[n]}(s)) m_{sa}^{[n]} b_h^{[n]a}$$

$$907 \quad + \sum_{a=1}^A \left(\lambda_h^{[n]}(S-A+a) + \mu_h^{[n]}(S-A+a) \right) b_h^{[n]a} = - \sum_{s=1}^{S-A} \sum_{a=1}^A \lambda_h^{[n]}(s) m_{sa}^{[n]} b_h^{[n]a}$$

$$908 \quad + \sum_{a=1}^A \lambda_h^{[n]}(S-A+a) b_h^{[n]a} - \sum_{s \in \{1, \dots, S-A\} \cap \mathcal{S}'_h} \sum_{a=1}^A \mu_h^{[n]}(s) m_{sa}^{[n]} b_h^{[n]a}$$

$$909 \quad + \sum_{s \in \{S-A+1, \dots, S\} \cap \mathcal{S}'_h} \mu_h^{[n]}(s) b_h^{[n](s-(S-A))} - \sum_{s \in \{1, \dots, S-A\} \cap \mathcal{S}''_h} \sum_{a=1}^A \mu_h^{[n]}(s) m_{sa}^{[n]} b_h^{[n]a}$$

$$910 \quad + \sum_{s \in \{S-A+1, \dots, S\} \cap \mathcal{S}''_h} \mu_h^{[n]}(s) b_h^{[n](s-(S-A))}.$$

911 If for $s \in \mathcal{S}''_h$, $\mu_h^{[n]}$ is bounded above, then it admits a convergent subsequence and
 912 we are done. Suppose otherwise. Then, $\mu_h^{[n]}$ is not bounded above and there exists a
 913 subsequence converging to $+\infty$. Notice that if $s \in \mathcal{S}''_h$ there exists $n(s) \in \mathbb{N}$ such
 914 that $\mu_h^{[n]} > \mathbf{0}$, for all $n \geq n(s)$ and thus $\sum_a m_{sa}^{[n]} b_h^{[n]a} = \alpha_h^{[n]}(s) \mathbf{p}^{[n]}(s) \mathbf{e}_h^{[n]}(s)$, if $s \in$
 915 $\{1, \dots, S-A\}$ and $b_h^{[n](s-(S-A))} = -\alpha_h^{[n]}(s) \mathbf{p}^{[n]}(s) \mathbf{e}_h^{[n]}(s)$, if $s \in \{S-A+1, \dots, S\}$.
 916 Set then $n^* = \max\{n(s) : s \in \mathcal{S}''_h\}$ and for $n \geq n^*$ the above expression becomes

$$\begin{aligned}
 & - \sum_{s=1}^{S-A} \sum_{a=1}^A \lambda_h^{[n]}(s) m_{sa}^{[n]} b_h^{[n]a} + \sum_{a=1}^A \lambda_h^{[n]}(S-A+a) b_h^{[n]a} \\
 & - \sum_{s \in \{1, \dots, S-A\} \cap \mathcal{S}'_h} \sum_{a=1}^A \mu_h^{[n]}(s) m_{sa}^{[n]} b_h^{[n]a} + \sum_{s \in \{S-A+1, \dots, S\} \cap \mathcal{S}'_h} \mu_h^{[n]}(s) b_h^{[n](s-(S-A))} \\
 & - \sum_{s \in \{1, \dots, S-A\} \cap \mathcal{S}''_h} \mu_h^{[n]}(s) \alpha_h^{[n]}(s) \mathbf{p}^{[n]}(s) \mathbf{e}_h^{[n]}(s) \\
 & + \sum_{s \in \{S-A+1, \dots, S\} \cap \mathcal{S}''_h} \mu_h^{[n]}(s) (-\alpha_h^{[n]}(s) \mathbf{p}^{[n]}(s) \mathbf{e}_h^{[n]}(s)) = - \sum_{s=1}^{S-A} \sum_{a=1}^A \lambda_h^{[n]}(s) m_{sa}^{[n]} b_h^{[n]a} \\
 & + \sum_{a=1}^A \lambda_h^{[n]}(S-A+a) b_h^{[n]a} - \sum_{s \in \{1, \dots, S-A\} \cap \mathcal{S}'_h} \sum_{a=1}^A \mu_h^{[n]}(s) m_{sa}^{[n]} b_h^{[n]a} \\
 & + \sum_{s \in \{S-A+1, \dots, S\} \cap \mathcal{S}'_h} \mu_h^{[n]}(s) b_h^{[n](s-(S-A))} - \sum_{s \in \mathcal{S}''_h} \mu_h^{[n]}(s) \alpha_h^{[n]}(s) \mathbf{p}^{[n]}(s) \mathbf{e}_h^{[n]}(s)
 \end{aligned}$$

and thus from (32b) we obtain

$$\begin{aligned}
 & -\lambda_h^{[n]}(0) \mathbf{p}^{[n]}(0) (\mathbf{x}_h^{[n]}(0) - \mathbf{e}_h^{[n]}(0)) = \lambda_h^{[n]}(0) \mathbf{q}^{[n]} \mathbf{b}_h^{[n]} = - \sum_{s=1}^{S-A} \sum_{a=1}^A \lambda_h^{[n]}(s) m_{sa}^{[n]} b_h^{[n]a} \\
 & + \sum_{a=1}^A \lambda_h(S-A+a) b_h^{[n]a} - \sum_{s \in \{1, \dots, S-A\} \cap \mathcal{S}'_h} \sum_{a=1}^A \mu_h^{[n]}(s) m_{sa}^{[n]} b_h^{[n]a} \\
 & + \sum_{s \in \{S-A+1, \dots, S\} \cap \mathcal{S}'_h} \mu_h^{[n]}(s) b_h^{[n](s-(S-A))} - \sum_{s \in \mathcal{S}''_h} \mu_h^{[n]}(s) \alpha_h^{[n]}(s) \mathbf{p}^{[n]}(s) \mathbf{e}_h^{[n]}(s).
 \end{aligned}$$

Letting $n \rightarrow \infty$ we find

$$\begin{aligned}
 & \sum_{s=1}^{S-A} \sum_{a=1}^A \bar{\lambda}_h(s) \bar{m}_{sa} \bar{b}_h^a - \sum_{a=1}^A \bar{\lambda}_h(S-A+a) \bar{b}_h^a - \bar{\lambda}_h(0) \bar{\mathbf{p}}(0) (\bar{\mathbf{x}}_h(0) - \bar{\mathbf{e}}_h(0)) \\
 & = - \sum_{s \in \mathcal{S}''_h} \bar{\mu}_h(s) \bar{\alpha}_h(s) \bar{\mathbf{p}}(s) \bar{\mathbf{e}}_h(s)
 \end{aligned}$$

and thus, if $\bar{\mu}_h(s) = +\infty$ for some s , we would find that the left hand side should be $-\infty$, which is impossible, as all its terms are finite. Thus $\bar{\mu}_h(s) \in \mathbb{R}$, for every s , as desired.

Finally, from (32c) we easily get that also q^a is convergent, for every $a \in \mathcal{A}$. The proof is complete.

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