ON THE EXISTENCE OF SOLUTIONS TO A SPECIAL VARIATIONAL PROBLEM

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Abstract. In this paper we establish an existence and regularity result for solutions to the problem

$$\text{minimize} \int_\Omega L(|\nabla u(x)|)\, dx \quad \text{on } \{u : u - u_0 \in W_0^{1,1}(\Omega)\}$$

for boundary data that are constant on each connected component of the boundary of Ω. The Lagrangean L belongs to a class that contains both extended valued Lagrangeans and Lagrangeans with linear growth. Regularity means that the sulution u is Lipschitz continuous and that, in addition, $\|L'(|\nabla \tilde{w}|)\|_\infty$ is bounded.

1. Introduction

In the present paper, Ω is a bounded open subset of $\mathbb{R}^N$, with a C^2 boundary $\partial\Omega$, whose connected components will be denoted by Γ_i. We consider the special problem of minimizing

$$\int_\Omega L(|\nabla u(x)|)\, dx \tag{1}$$

on $u \in \phi + W_0^{1,1}(\Omega)$, where we assume that the restriction of the boundary data ϕ to each Γ_i is a constant, k_i. The simplest example of this situation is provided by an annulus in $\mathbb{R}^2$, when ϕ is constant on each of the two radii. The minimal surface problem offers an example of a minimization problem of this type where the solution to the minimum problem either does not exist or it exists but fails to be Lipschitz continuous: in fact, the special nature of this problem, in itself, prevents the validity of the mean curvature condition, which is a sufficient condition to ensure the existence of solutions in the case of minimal surfaces [3],[4].

Our purpose here is to prove *existence* and *regularity* of solutions. Existence is meant in $\phi + W_0^{1,1}(\Omega)$. By regularity, we mean more than Lipschitz continuity: for the class of problems we wish to discuss, where L can be extended valued, the Lipschitz continuity of the solution, by itself, would not be relevant enough. In fact, for instance, consider the case where the Lagrangean L is $L(t) = \frac{1}{4}\frac{1}{1-|t|}$ for $|t| < 1$, $= +\infty$ elsewhere: then, every w that makes the integral functional finite, independent on whether it is a solution or not, has to be Lipschitz continuous. A possible notion of regularity for the case of extended-valued Lagrangeans would be to require both that $\nabla \tilde{w} \in L^\infty$ and $L'(|\nabla \tilde{w}|) \in L^1$. In this paper we seek, and we prove, more: we will call regular a map w when $\|\nabla \tilde{w}\|_{L^\infty}$ and $\|L'(|\nabla \tilde{w}|)\|_{L^\infty}$ are both bounded. Our Theorem 1 provides existence of solutions for a broad class of

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Lagrangeans, containing Lagrangeans of linear growth and Lagrangeans that are extended valued: in fact, no specific growth assumption appears: L has to belong to a class of convex functions, invariant by polarity, defined through the requirement of continuity of the function itself and of its polar.

2. Main results

$\overline{\mathbb{R}}$ is $\mathbb{R} \cup +\infty$. We denote by $Dom(L)$ the effective domain of L. By L^* we mean the *polar* or *Legendre transform* of L, a (possibly extended valued) convex function [6].

By $\pi(x)$ we denote the subset of $\partial\Omega$ of points nearest to $x \in \Omega$, i.e., $\pi(x) = \{y \in \partial\Omega : |x - y| = dist(x, \partial\Omega)\}$. We shall consider points x sufficiently close to $\partial\Omega$ to have an unique projection on it. Set $c_i(y)$ be the curvatures of $\partial\Omega$ at y, define r_i by $c_i(y) = -\frac{1}{r_i(y)}$: from the smoothness of $\partial\Omega$ we have that

$$r^* = \inf_i \inf_y |r_i(y)| > 0. \tag{2}$$

It is well known, [2] , [1], that $\Delta d(x) = \sum_1^{N-1} \frac{1}{r_i(\pi(x)) + d(x)}$.

In what follows, $Lip_0(\Omega)$ is the linear space of Lipschitz continuous functions vanishing at the boundary of Ω. We shall consider the problem of minimizing (1) on $\phi + Lip_0(\Omega)$ and on $\phi + W_0^{1,1}(\Omega)$. A function w is called *regular* if both $\|\nabla w\|_\infty$ and $\|L'(|\nabla w|)\|_\infty$ are bounded.

We consider the class $\mathbb{CC}$ of convex Lagrangeans defined below.

Definition 1. *$L : \mathbb{R} \to \overline{\mathbb{R}}$ is in $\mathbb{CC}$ if it is a (possibly extended valued) symmetric, convex function, twice differentiable on the interior of its effective domain, and with $L'' > 0$ there, and such that $0 \in \mathrm{int}Dom(L)$ and $L(0) = L'(0) = 0$. In addition, both L and L^* are continuous.*

The assumption $L(0) = 0$ is satisfied by any convex function, possibly by adding a constant, that has no effect on the minimization problem; the assumption $L'(0) = 0$ is needed to have differentiability at 0; $L'' > 0$ assures the uniqueness of the solution.

The minimal surface functional, where $L(t) = \sqrt{1+t^2} - 1$, satisfies all the requirements for being in $\mathbb{CC}$ with the only exception that its polar, i.e. the convex function $L^*(p) = -\sqrt{1-p^2} + 1$ for $|p| \le 1$, $= +\infty$ elsewhere, is not continuous (as an extended-valued function.) Hence, Theorem 1, that follows, does not apply to the minimal surface functional, as it has to be: in fact, it is well known that the claim of the theorem is false in this case.

We have the following proposition

Proposition 1. *Let L be in $\mathbb{CC}$. Then L^* is in $\mathbb{CC}$.*

Proof. From the definition of polar, it follows that $L^*(0) = 0$. From $(L^*)'(L'(0)) = 0$ follows that $(L^*)'(0) = 0$; moreover, for $\hat{p}$ in $\mathrm{int}(Dom(L^*))$, we have that $(L^*)''$ exists and is positive: in fact, fix $\hat{p}$ and set $\hat{\xi} = (L^*)'(\hat{p})$. Write

$$\frac{(L^*)'(p) - (L^*)'(\hat{p})}{p - \hat{p}} = \frac{(L^*)'(L'(x)) - (L^*)'(L'(\hat{x}))}{L'(x) - L'(\hat{x})} = \frac{x - \hat{x}}{L'(x) - L'(\hat{x})}$$

to obtain that $(L^*)''(\hat{p}) = \frac{1}{L''((L^*)'(\hat{p}))}$. □

Being $\mathbb{CC}$ invariant under polarity, does not, in itself, contain any specific growth assumption.

The following proposition will be of use. Recall that, by assumption, $L'(0) = 0$: hence, by the notation $L'(|\xi|)\frac{\xi}{|\xi|}$, we mean $L'(|\xi|)\frac{\xi}{|\xi|}$, when $\xi \neq 0$, and 0, when $\xi = 0$.

Proposition 2. *Let L be in $\mathbb{CC}$, let w, a solution to the problem of minimizing (1) on $\phi + Lip_0(\Omega)$, be regular. Then the Euler Lagrange equation holds, i.e., for every η in $W_0^{1,1}(\Omega)$, we have*

$$\int_\Omega L'(|\nabla u(x)|)\langle \frac{\nabla u(x)}{|\nabla u(x)|}, \nabla \eta(x)\rangle \, dx = 0 \tag{3}$$

Proof. The assumption of regularity implies that, in either case when $Dom(L) = (-\ell, +\ell)$ or $Dom(L) = (-\infty, +\infty)$, there exist K and δ such that $|\nabla u(x)| \leq K$ and $(-K-\delta, K+\delta) \subset\subset Dom(L)$, so that L' is uniformly bounded on $(-K-\delta, K+\delta)$. For fixed η in $C_c^\infty(\Omega)$, for every ε small, $|\nabla u(x) + \varepsilon \nabla \eta(x)| < K + \delta$, and one can pass to the limit under the integral sign. This establishes the validity of (3) for η in $C_c^\infty(\Omega)$. Moreover, $\nabla L(|\nabla \tilde{w}|) \in L^\infty(\Omega)$, so that equation (3) holds also for $\eta \in W_0^{1,1}(\Omega)$. □

The following is our main result. Recall that Γ_i are the connected components of $\partial\Omega$.

Theorem 1. *Let L be in $\mathbb{CC}$. Let each restriction of the boundary datum ϕ to Γ_i be constant. Then, whenever ϕ is regular, a solution $\tilde{w}$ to the problem of minimizing (1) on $u \in \phi + W^{1,1}(\Omega)$, exists, is unique and it is regular.*

Notice that, in the case where L takes values in $\mathbb{R}$, then, no matter what the values k_i are, the restriction of ϕ to $\partial\Omega$ can be extended to Ω so as to make ϕ regular.

The following Lemmas will be used in the Proof. Opposite to Theorem 1, the validity of Lemma 1 does not depend on any special condition on the behaviour of ϕ on the boundary of Ω.

Lemma 1 (A Maximum Principle for extended valued Lagrangeans)**.** *Let $L : \mathbb{R} \to \overline{\mathbb{R}}$ be convex and continuous, with $Dom(L) = (-\ell, +\ell)$, and let it be twice differentiable on its effective domain, with $L'' > 0$ there. Let $\tilde{w}$ be a solution to the problem of minimizing*

$$\int_\Omega L(|\nabla u(x)|)\, dx \tag{4}$$

on $\phi + W_0^{1,1}(\Omega)$, where ϕ is Lipschitz continuous of Lipschitz constant $\lambda < \ell$. Then, for no $x^* \in \Omega$ there exists $y^* \in \pi(x^*)$, such that $|\tilde{w}(x^*) - \phi(y^*)| = \ell|x^* - y^*|$.

Proof. Assume that such x^* and y^* exist. Let $g = g(t)$ be defined by

$$L'(g(t)) = L'(\frac{\ell}{2})\frac{1}{t^{N-1}}. \tag{5}$$

The map g is the solution to the differential equation

$$-\frac{L''}{L'}(v(r))\frac{dv}{dr} = \frac{N-1}{r} \tag{6}$$

satisfying $v(1) = \frac{\ell}{2}$; in particular, $g'(t) < 0$. We have that $g(t) \to \ell^-$ as $t \to 0^+$: hence, fix $\lambda < \Lambda < \ell$ and let $\delta < 1$ be such that $t < \delta$ implies $g(t) > \Lambda$. Let D be the diameter of Ω and choose $\tilde{r}^0$ so large that $\frac{D}{\tilde{r}^0} < \delta$.

Set $w(x)=\int_0^{|x-y^*|} g(\frac{s}{\tilde r^0})ds+\phi(y^*)$: w is a pointwise solution to

$$\mathrm{div}_x \nabla_\xi L(|\nabla v(x)|)=0 \tag{7}$$

such that $w(y^*)=\phi(y^*)$ and such that, on $\{|x-y^*|=\tilde r^0\}$, we have $\nabla w(x)=\frac{\ell}{2}\frac{x-y^*}{|x-y^*|}$; we claim that it is also a solution to the Euler Lagrange equation, in the sense that, for every η Lipschitz continuous with $\eta(y)=0$ when $y\in\partial\Omega$, we have

$$\int_\Omega \langle \nabla_\xi L(|\nabla w(x)|),\nabla\eta(x)\rangle dx=\int_\Omega L'(\frac{\ell}{2})\frac{1}{|x-y^*|^{N-1}}\langle\frac{x-y^*}{|x-y^*|},\nabla\eta(x)\rangle dx=0.$$

Set E^+ be the open set $\{x:\eta(x)>0\}$ and analogously for E^-. We shall prove that

$$\int_{E^+}\frac{1}{|x-y^*|^{N-1}}\langle\frac{x-y^*}{|x-y^*|},\nabla\eta(x)\rangle dx \text{ and } \int_{E^+}\frac{1}{|x-y^*|^{N-1}}\langle\frac{x-y^*}{|x-y^*|},\nabla\eta(x)\rangle dx$$

are zero. Consider polar coordinates (ω,r) centered at y^*: the intersection of a half line $L_c=\{\omega=c,r\geq 0\}$ with the open set E^+ can be described as $\{\omega=c;r\in\cup_i(\alpha_i(c),\beta_i(c))\}$ where some or all of the $\{\omega=c;\alpha_i(c)\}$ and of the $\{\omega=c;\beta_i(c)\}$ can belong to $\partial\Omega$. On almost every half line, the derivative of the map $r\to\eta(\omega' r)$ equals a.e. $\langle\omega,\nabla\eta(\omega' r)\rangle$ so that

$$\int_{E^+}\frac{1}{|x-y^*|^{N-1}}\langle\frac{x-y^*}{|x-y^*|},\nabla\eta(x)\rangle dx=\int_{|\omega|=1}\left(\Sigma_i\int_{\alpha_i(c)}^{\beta_i(c)}\frac{c}{r^{N-1}}(\frac{d}{dr}(\eta(r\omega)))r^{N-1}\,dr\right)d\omega.$$

For each i, $\eta(\alpha_i(c),c)=0$ and the same is true at $\{\omega=c;\beta_i(c)\}$. Hence we obtain that the last integral is zero. Thus we prove the claim, and the convexity of L implies that w, a solution to the Euler Lagrange equation, is also a solution to the minimization problem.

Let $y\in\overline{\Omega}$: then $w(y)-w(y^*)=\int_0^{|y-y^*|}g(\frac{s}{\tilde r^0})ds$ and, by our choice of $\tilde r^0$, $\frac{|y-y^*|}{\tilde r^0}\leq\frac{D}{\tilde r^0}<\delta$ and we infer $g(\frac{s}{\tilde r^0})>\Lambda$, so that $w(y)-w(y^*)>\Lambda|y^*-y|$. On the other hand, for $y\in\partial\Omega$, we have $|\tilde w(y)-\tilde w(y^*)|=|\phi(y)-\phi(y^*)|\leq\lambda|y^*-y|$; hence, on $\partial\Omega$, one has $w\geq\tilde w$. By assumption, at the point x^* we have $|\tilde w(x^*)-\phi(y^*)|=\ell|y^*-x^*|$ while $w(x^*)-\phi(y^*)=\int_0^{|x^*-y^*|}g(\frac{s}{\tilde r^0})ds<\ell|y^*-x^*|$. Hence the set $\{x\in\Omega:\tilde w(x)>w(x)\}$ is an open set containing x^*: on the connected component containing x^* there are two solutions with the same boundary data, a contradiction to the assumption of strict convexity of L. □

Lemma 2. *Let $L:\mathbb{R}\to\mathbb{R}$ be convex and twice differentiable. Then, $\int^\infty tL''(t)dt=+\infty$ if and only if $\int^\infty t\frac{L''(t)}{L'(t)}dt=+\infty$.*

Proof. a): only if.

Since L' is increasing, we have

$$\int_\alpha^\beta t\frac{L''(t)}{L'(t)}dt\geq\frac{1}{L'(\beta)}\int_\alpha^\beta tL''(t)dt$$

and both $\lim_{\beta\to\infty}(L'(\beta))=+\infty$ and $\lim_{\beta\to\infty}\int_\alpha^\beta tL''(t)dt=+\infty$. By l'Hopital rule,

$$\lim_{\beta\to\infty}\frac{1}{L'(\beta)}\int_\alpha^\beta tL''(t)dt=\lim_{\beta\to\infty}\frac{\beta L''(\beta)}{L''(\beta)}=+\infty.$$

b): if.

L' is bounded below by a positive constant. □

Proof of Theorem 1. $Dom(L^*)$ is the open interval from $-\ell^*$ to $+\ell^*$, with ℓ^* possibly $= \infty$.

The proof will consist in the construction of a suitable barrier. To prove Lipschitz continuity of solutions, affine barriers have been used, as in [5], through the bounded slope condition, as well as barriers of more general form, as in [1], [2], for the minimal surface case. The construction of our barrier will differ from those appearing in the papers quoted above.

We shall first prove the existence of a solution to the minimization problem (1) on $\phi + Lip_0(\Omega)$. Let $d^* > 0$ be so small that $d^* < r^*$, where r^* is defined in (2) so that for $dist(x, \partial\Omega) \le d^*$, we have that $\pi(x)$ is single valued; set $d(x) = |x - \pi(x)|$. Fix i: we shall define a barrier on $\{x \in \Omega : dist(x, \partial\Omega_i) \le d^*\}$.

When $v(x) = \psi(d(x))$, one obtains:

(8)
$$\mathrm{div}_x \nabla_\xi L(|\nabla v(x)|) = L'(|\psi'(d(x))|)\mathrm{sign}(\psi'(d(x)))\,\Delta d(x) + \psi''(d(x))L''(|\psi'(d(x))|)$$
.

i) Case $Dom(L) = (-\infty, +\infty)$.

Let $v^* = \sup_{y,z\in\partial\Omega} |\phi(z) - \phi(y)|$: as it is well known, when $\tilde{w}$ is a solution, for $x \in \Omega$ and each i, $|\tilde{w} - k_i| \le v^*$. Set $D = \sup\{|\Delta(x)| : x \in \Omega, d(x) \le d^*\}$: from the assumptions of regularity of $\partial\Omega$ and the choice of d^*, D is finite.

Let ψ be a solution to

$$L'(\psi'(d))\,D + \psi''(d)L''(\psi'(d) = 0 \tag{9}$$

For generic ℓ_1 and ℓ_2, we have

$$\int_{\ell_1}^{\ell_2} \psi'(\ell)d\ell = \int_{\psi'(\ell_1)}^{\psi'(\ell_2)} \psi' \frac{d\ell}{d\psi'} d\psi'$$

and, from $\psi''(\ell(\psi') = \frac{1}{\ell'(\psi')}$ we obtain

$$\psi(\ell_2) - \psi(\ell_1) = \int_{\ell_1}^{\ell_2} \psi'(\ell)d\ell = \int_{\psi'(\ell_2)}^{\psi'(\ell_1)} \psi' \frac{L''(\psi')}{DL'(\psi')} d\psi'. \tag{10}$$

From the assumption of continuity of L^*, we have:

$$\lim_{\beta\to\infty} \int_0^\beta tL''(t)\,dt = \lim_{\beta\to\infty} \left(tL'(t)|_0^\beta - \int_0^\beta L''(t)\,dt \right) = \lim_{\beta\to\infty} L^*(L'(\beta))$$
$$= \lim_{p\to\ell^*} L^*(p) = +\infty.$$

Applying Lemma 2, we obtain

$$\lim_{\beta\to\infty} \int_0^\beta t\frac{L''(t)}{L'(t)} dt = +\infty.$$

Hence, for every α, there exists $\beta(\alpha)$ such that

$$\int_\alpha^{\beta(\alpha)} t\frac{L''(t)}{DL'(t)} dt = v^*.$$

Fix $\tilde{\alpha}$ such that $\tilde{\alpha}d^* \ge v^*$. Let $\tilde{\psi}$ be the solution to equation (9) satisfying the initial conditions

$$\tilde{\psi}(0) = 0; \tilde{\psi}'(0) = \beta(\tilde{\alpha})$$

and define d^{**} setting $\tilde{\psi}'(d^{**}) = \tilde{\alpha}$. From equation (10) we obtain

$$\tilde{\psi}(d^{**}) = \int_{\psi'(d^{**})}^{\psi'(0)} \psi' \frac{L''(\psi')}{DL'(\psi')} d\psi' = v^*.$$

Call $\Omega_i^{**} = \{x \in \Omega; d(x, \partial\Omega_i) \le d^{**}\}$ and call $\partial_i^{**} = \{x \in \Omega; d(x, \partial\Omega_i) = d^{**}\}$. We have obtained that, on ∂_i^{**}, the map $\tilde{v} = k_i + \tilde{\psi}(d(x))$ is such that $\tilde{v}(x) \ge \tilde{w}(x)$. Moreover, $\tilde{\psi}''$, defined by (9), is negative so that, on the interval $(0, d^{**})$, $\tilde{\psi}' \ge \tilde{\alpha}$. Hence, we have $v^* = \tilde{\psi}(d^{**}) \ge \tilde{\alpha} d^{**}$, while, from the choice of $\tilde{\alpha}$, $\tilde{\alpha} d^* \ge v^*$, so that $d^* \ge d^{**}$. In addition, being $\tilde{\psi}' > 0$, from (9), we obtain

$$\begin{aligned}\mathrm{div}_x \nabla_\xi L(|\nabla \tilde{v}(x)|) &= L'(|\tilde{\psi}'(d(x))|)\mathrm{sign}(\tilde{\psi}'(d(x)))\, \Delta d(x) + \tilde{\psi}''(d(x)) L''(|\tilde{\psi}'(d(x))|) \\ &= L'(\tilde{\psi}'(d(x)))\, \Delta d(x) + \tilde{\psi}''(d(x)) L''(\tilde{\psi}'(d(x))) \le 0.\end{aligned}$$

On $\{x \in \Omega : d(x) < d^{**}\}$, $\tilde{v}$ is a Lipschitz continuous supersolution. In the case $\ell = \infty$, this proves the existence of a solution to the minimization problem (1) on $\phi + Lip_0(\Omega)$.

ii) Case $Dom(L) = (-\ell, +\ell)$. By the assumption of continuity, $\lim_{t\to\ell} L(t) = +\infty$.

Let $\tilde{w}$ be a solution to problem (1) on $\phi + W^{1,1}(\Omega)$: such a solution exists and it is Lipschitz continuous.

a) We claim that there exits $\Lambda^* < \ell$ such that, when $d(x, \partial\Omega_i) = d^*$, $|\tilde{w}(x) - k_i| \le \Lambda^* d^*$: in fact, otherwise, let (x_n) be such that $d(x_n) = d^*$ and $|\tilde{w}(x) - k_i| \to \ell d^*$; a subsequence converges to x^*, and $\tilde{w}(x^*) - k_i = \ell d^*$, a contradiction to lemma 1.

We consider g, as defined by (5), and we seek r^0 such that, setting

$$\tilde{\psi}(r) = k_i + \int_{r^*}^{r} g(\frac{s}{r_0})\, ds,$$

we have $\tilde{\psi}(r^* + d^*) \ge \Lambda^* d^* + k_i$, i.e., such that

$$\frac{1}{d^*} \int_{r^*}^{r^*+d^*} \tilde{\psi}'(s) ds = \frac{1}{d^*} \int_{r^*}^{r^*+d^*} g(\frac{s}{r^0}) ds \ge \Lambda^* \tag{11}$$

It is enough to consider δ such that $t \le \delta$ implies $g(t) > \Lambda^*$, and to choose r^0 so large that $\frac{r^*+d^*}{r^0} < \delta$: with this choice, (11) holds.

To define a barrier near $\partial\Omega_i$, consider $v(d) = \tilde{\psi}(d + r^*)$ (so that $v'(d) = \tilde{\psi}'(d + r^*) = g(\frac{d+r^*}{r_0})$, $v''(d) = g'(\frac{d+r^*}{r_0})\frac{1}{r_0}$ and, in particular, $v'(d) > 0$) and set $\tilde{v}(x) = v(d(x))$. On $\partial\Omega_i$, $\tilde{v} = v(0) = \tilde{\psi}(r^*) = k_i$ and, when $dist(x, \partial\Omega_i) = d^*$, recalling (11) and the choice of Λ^* in point a), we have $\tilde{v}(x) = \tilde{\psi}(r^* + d^*) \ge d^* \Lambda^* + k_i \ge \tilde{w}(x)$. Hence, to show that $\tilde{v}$ is an upper barrier, it is left to show that it is a supersolution, i.e., that, for $x \in \Omega$, we have $\mathrm{div}_x \nabla_\xi L(\|\nabla v(x)\|) \le 0$.

Fix x, and recall that $\Delta d(x) = \sum_1^{N-1} \frac{1}{r_i(\pi(x)) + d(x)}$. From $0 < r^* \le |r_i(\pi(x))|$ we obtain

$$\sum_1^{N-1} \frac{1}{r_i(\pi(x)) + d(x)} \le \sum_1^{N-1} \frac{1}{|r_i(\pi(x))| + d(x)} \le \frac{N-1}{r^* + d(x)}$$

hence

$$\mathrm{div}_x \nabla_\xi L(|\nabla \tilde{v}(x)|) =$$

$$L'(g(\frac{d(x) + r^*}{r_0})) \sum_1^{N-1} \frac{1}{r_i(\pi(x)) + d(x)} + \frac{1}{r_0} g'(\frac{d(x) + r^*}{r_0}) L''(g(\frac{d(x) + r^*}{r_0}))$$

$$\leq \frac{1}{r_0}[L'(g(\frac{d(x)+r^*}{r_0}))\frac{r_0(N-1)}{r^*+d(x)} + g'(\frac{d(x)+r^*}{r_0})L''(g(\frac{d(x)+r^*}{r_0}))] = 0,$$

since g satisfies (6).

The above ends the proof that $\tilde{v}$ is a barrier, thus showing the existence of a solution on $\phi + Lip_0(\Omega)$ [2].

iii) Points i) and ii) above prove the existence of a solution $\tilde{w}$ in the class of Lipschitz continuous functions. We wish to show that $\tilde{w}$ is a solution in $\phi + W_0^{1,1}(\Omega)$. By Proposition 2, the Euler Lagrange equation holds. From the convexity of L, one obtains that

$$\int_\Omega L(|\nabla\tilde{w}(x) + \nabla\eta(x)|)\,dx \geq \int_\Omega L(|\nabla\tilde{w}(x)|)\,dx + \int_\Omega \langle \nabla L(|\nabla\tilde{w}(x)|), \nabla\eta(x)\rangle dx$$

thus proving that $\tilde{w}$ is a solution in $\phi + W_0^{1,1}(\Omega)$. □

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