ON THE EXISTENCE OF GROUND STATE SOLUTIONS TO NONLINEAR SCHRÖDINGER EQUATIONS WITH MULTISINGULAR INVERSE-SQUARE ANISOTROPIC POTENTIALS

By

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Abstract. A class of nonlinear Schrödinger equations with critical power-nonlinearities and potentials exhibiting multiple anisotropic inverse square singularities is investigated. Conditions on strength, location, and orientation of singularities are given for the minimum of the associated Rayleigh quotient to be achieved, both in $\mathbb{R}^N$ and in bounded domains.

1 Introduction and statement of the main results

This paper is concerned with the following class of nonlinear Schrödinger equations with a critical power-nonlinearity and a potential exhibiting multiple anisotropic inverse square singularities:

$$(1) \qquad \begin{cases} -\Delta v - \sum_{i=1}^k \frac{h_i\left(\frac{x-a_i}{|x-a_i|}\right)}{|x-a_i|^2}\, v = v^{2^*-1}, \\ v > 0 \quad \text{in } \mathbb{R}^N \setminus \{a_1, \dots, a_k\}, \end{cases}$$

where $N \geq 3$, $k \in \mathbb{N}$, $h_i \in C^1(\mathbb{S}^{N-1})$, $(a_1, a_2, \dots, a_k) \in \mathbb{R}^{kN}$, $a_i \neq a_j$ for $i \neq j$, and $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent.

The interest in such a class of equations arises in nonrelativistic molecular physics. Inverse square potentials with anisotropic coupling terms turn out to describe the interaction between electric charges and dipole moments of molecules; see [16]. In crystalline matter, the presence of many dipoles leads one to consider multisingular Schrödinger operators of the form

$$(2) \qquad -\Delta - \sum_{i=1}^k \frac{\lambda_i\,(x-a_i)\cdot \mathbf{d}_i}{|x-a_i|^3},$$

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where $\lambda_i > 0$, $i = 1, \dots, k$, is proportional to the magnitude of the i-th dipole and $\mathbf{d}_i$, $i = 1, \dots, k$, is the unit vector giving the orientation of the i-th dipole.

Schrödinger equations and operators with isotropic inverse-square singular potentials have been largely investigated in the literature, both for the case of a single pole [1, 13, 15, 19, 21] and for multiple singularities [2, 5, 6, 8, 9, 12]. The anisotropic case was first considered in [21], where the problem of existence of ground state solutions to (1) was discussed for $k = 1$. In [10], an asymptotic formula for solutions to an equation associated with dipole-type Schrödinger operators near the singularity was established. We also mention that positivity, localization of binding and essential self-adjointness properties of a class of Schrödinger operators with many anisotropic inverse-square singularities were investigated in [11].

Ground state solutions to (1), i.e., solutions with the smallest energy, can be obtained through minimization of the associated Rayleigh quotient

$$S(h_1, h_2, \dots, h_k) = \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\mathcal{Q}(u)}{\left(\int_{\mathbb{R}^N} |u|^{2^*} dx\right)^{2/2^*}}, \tag{3}$$

where $\mathcal{D}^{1,2}(\mathbb{R}^N)$ denotes the closure of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} |\nabla u|^2 \, dx\right)^{1/2},$$

and $\mathcal{Q} : \mathcal{D}^{1,2}(\mathbb{R}^N) \to \mathbb{R}$ is the quadratic form associated to the left-hand side of equation (1), viz.,

$$\mathcal{Q}(u) := \int_{\mathbb{R}^N} |\nabla u|^2 dx - \sum_{i=1}^k \int_{\mathbb{R}^N} \frac{h_i\left(\frac{x - a_i}{|x - a_i|}\right)}{|x - a_i|^2} u^2(x) \, dx. \tag{4}$$

Positive minimizers of (3) suitably rescaled give rise to weak $\mathcal{D}^{1,2}(\mathbb{R}^N)$-solutions to (1); by the Brezis–Kato Theorem [3] and standard elliptic regularity theory, these turn out to be classical solutions in $\mathbb{R}^N \setminus \{a_1, \dots, a_k\}$.

The present paper means to extend to problems (1) and (3) the analysis performed in [12] in the case of locally isotropic inverse square potentials (i.e., for all h_i's constant), proving conditions on the strength, location and orientation of singularities for their solvability.

A necessary condition for the existence of positive classical solutions to (1) in $\mathbb{R}^N \setminus \{a_1, \dots, a_k\}$ is that $\mathcal{Q}$ be positive semidefinite in $\mathcal{D}^{1,2}(\mathbb{R}^N)$.

Proposition 1.1. *A necessary condition for the solvability of problem* (1) *is that the quadratic form* $\mathcal{Q}(u)$ *defined in* (4) *be positive semidefinite, i.e.,*

$$\mathcal{Q}(u) \geq 0 \quad \textit{for all } u \in \mathcal{D}^{1,2}(\mathbb{R}^N).$$

A necessary condition on the angular coefficients h_i for the positive semidefiniteness of the quadratic form can be expressed in terms of the first eigenvalues of the associated Schrödinger operators on the sphere. Indeed, for any $h \in C^1(\mathbb{S}^{N-1})$, let $\mu_1(h)$ be the first eigenvalue of the operator $-\Delta_{\mathbb{S}^{N-1}} - h(\theta)$ on $\mathbb{S}^{N-1}$, i.e.,

$$\mu_1(h) = \min_{\psi \in H^1(\mathbb{S}^{N-1}) \setminus \{0\}} \frac{\int_{\mathbb{S}^{N-1}} |\nabla_{\mathbb{S}^{N-1}} \psi(\theta)|^2 \, dV(\theta) - \int_{\mathbb{S}^{N-1}} h(\theta) \psi^2(\theta) \, dV(\theta)}{\int_{\mathbb{S}^{N-1}} \psi^2(\theta) \, dV(\theta)}.$$

A necessary (but not sufficient) condition for the quadratic form defined in (4) to be positive semidefinite is that

(5)
$$\mu_1(h_i) \geq -\left(\frac{N-2}{2}\right)^2, \quad \text{for all } i = 1, \dots, k, \quad \text{and} \quad \mu_1\left(\textstyle\sum_{i=1}^k h_i\right) \geq -\left(\frac{N-2}{2}\right)^2;$$

see [10].

In particular, condition (5) is necessary for solvability of problem (1). In this paper, we consider multisingular anisotropic potentials with angular terms satisfying the stronger assumption

(6)
$$\mu_1(h_i) > -\left(\frac{N-2}{2}\right)^2, \quad \text{for all } i = 1, \dots, k, \quad \text{and} \quad \mu_1\left(\textstyle\sum_{i=1}^k h_i\right) > -\left(\frac{N-2}{2}\right)^2.$$

In [11, Proposition 1.2], condition (6) was shown to be necessary for the quadratic form $\mathcal{Q}$ to be positive definite, i.e., for

$$\mu(h_1, \dots, h_k, a_1, \dots, a_k) := \inf_{\mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\mathcal{Q}(u)}{\|u\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)}} > 0. \tag{7}$$

On the other hand, (6) is not sufficient for the validity of (7), see [11, Example 1.5]. However, if (6) holds, then (7) turns out to be necessary for the solvability of (1).

Proposition 1.2. *If* (6) *holds and* (1) *admits a positive* $\mathcal{D}^{1,2}(\mathbb{R}^N)$*-solution, then* (7) *is necessarily satisfied.*

Due to the above proposition, in order to look for solutions to (1), we assume that the quadratic form $\mathcal{Q}$ is positive definite. The dependence of positivity of the quadratic form on the location and orientation of dipoles has been deeply investigated in [11], where conditions on the h_i's and a_i's ensuring the validity of (7) can be found. If $\mathcal{Q}(u)$ is positive definite, then Sobolev's inequality implies that

$$S(h_1, h_2, \dots, h_k) \geq \mu(h_1, \dots, h_k, a_1, \dots, a_k) \, S > 0,$$

where S is the best constant in the classical Sobolev inequality, i.e.,

$$S = \inf_{\mathcal{D}^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\|u\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)}}{\|u\|^2_{L^{2^*}(\mathbb{R}^N)}}.$$

Problems (1) and (3) have been treated by Terracini in [21] in the one-dipole case $k = 1$. For $h \in C^1(\mathbb{S}^{N-1})$, let

$$S(h) := \inf_{u\in\mathcal{D}^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} \big[|\nabla u(x)|^2 - \frac{h(x/|x|)}{|x|^2}u^2(x)\big]\,dx}{\big(\int_{\mathbb{R}^N}|u|^{2^*}\big)^{2/2^*}}. \tag{8}$$

Let us recall from [21] the following existence result for the one-dipole type problem.

Theorem 1.3 ([21, Proposition 5.3 and Theorem 0.2])**.** *Let $h \in C^1(\mathbb{S}^N)$ such that $\mu_1(h) > -\big(\frac{N-2}{2}\big)^2$ and*

$$\begin{cases} \max_{\mathbb{S}^{N-1}} h > 0, & \textit{if } N \geq 4,\\ \int_{\mathbb{S}^{N-1}} h \geq 0, & \textit{if } N = 3. \end{cases} \tag{9}$$

Let $S(h)$ be defined in (8)*. Then $S(h) < S$ and $S(h)$ is achieved.*

The main difficulty in the minimization of the Rayleigh quotient in (3) is due to the lack of compactness of the embeddings $\mathcal{D}^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ and $\mathcal{D}^{1,2}(\mathbb{R}^N) \hookrightarrow L^2\big(\mathbb{R}^N, |x|^{-2}h\big(x/|x|\big)dx\big)$, where, for $h \in L^\infty(\mathbb{S}^{N-1})$, $L^2\big(\mathbb{R}^N, |x|^{-2}h\big(x/|x|\big)dx\big)$ is the the weighted Lebesgue space endowed with the norm

$$\left(\int_{\mathbb{R}^N} |x|^{-2}h\big(x/|x|\big)u^2(x)\,dx\right)^{1/2}.$$

Such a lack of compactness could produce non-convergence of minimizing sequences and non-attainability of the infimum of the Rayleigh quotient in some cases. In [12], several configurations for which the infimum of the Rayleigh quotient is not attained are produced in the isotropic case, i.e., for all h_i constant; e.g. the infimum in (3) is not attained if the coefficients h_i are positive constants or if $k = 2$ and h_1 and h_2 are constant.

A careful analysis of the behavior of minimizing sequences performed through the P. L. Lions Concentration-Compactness Principle [17, 18] clarifies the possible reasons for lack of compactness: concentration of mass at some non-singular point, at one of the singularities or at infinity; see Theorem 4.1. Extending analogous results of [12] for the isotropic case, Theorem 1.4 below provides sufficient conditions for minimizing sequences to stay at an energy level which is strictly below all the energy thresholds at which the compactness can be lost. The proof is based on a comparison between levels and is carried out by testing the energy functional associated to (1) with solutions to (8). On the other hand, while in the isotropic case the solutions to (8) are completely classified and can be explicitly written; in the anisotropic case such an explicit form is not available. We overcome this difficulty by exploiting the asymptotic analysis of the behavior near the singularities of solutions performed in [10], which allows us to estimate the behavior of minimizing sequences and to force their level to stay in the recovered compactness range.

From now on, for every $h \in C^1(\mathbb{S}^{N-1})$, we denote by $\mu_1(h)$ the first eigenvalue of the operator $-\Delta_{\mathbb{S}^{N-1}} - h(\theta)$ on $\mathbb{S}^{N-1}$ and by ψ_1^h the associated positive L^2-normalized eigenfunction, and set

$$\sigma_h := -\frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2}\right)^2 + \mu_1(h)}. \tag{10}$$

Theorem 1.4. *For* $i = 1, \dots, k$, *let* $a_i \in \mathbb{R}^N$, $a_i \neq a_j$ *for* $i \neq j$, *and* $h_i \in C^1(\mathbb{S}^N)$ *satisfy* (7). *If*

$$S(h_k) = \min\{S(h_j) : j = 1, \dots, k\}, \tag{11}$$

$$h_k \text{ satisfies } (9), \tag{12}$$

$$\begin{cases} \displaystyle\sum_{i=1}^{k-1} \frac{h_i\left(\frac{a_k - a_i}{|a_k - a_i|}\right)}{|a_k - a_i|^2} > 0, \\ \qquad\qquad \text{if } \mu_1(h_k) \geq -\left(\frac{N-2}{2}\right)^2 + 1, \\ \displaystyle\sum_{i=1}^{k-1} \int_{\mathbb{R}^N} \frac{h_i\left(\frac{x}{|x|}\right)\left[\psi_1^{h_k}\left(\frac{x+a_i-a_k}{|x+a_i-a_k|}\right)\right]^2}{|x|^2 |x+a_i-a_k|^{2(\sigma_{h_k}+N-2)}} > 0, \\ \qquad\qquad \text{if } -\left(\frac{N-2}{2}\right)^2 < \mu_1(h_k) < -\left(\frac{N-2}{2}\right)^2 + 1, \end{cases} \tag{13}$$

$$S(h_k) \leq S\left(\sum_{i=1}^{k} h_i\right), \tag{14}$$

then the infimum in (3) *is achieved and problem* (1) *admits a solution in* $\mathcal{D}^{1,2}(\mathbb{R}^N)$.

We note that $S(h) = S(h \circ A)$ for any $h \in C^1(\mathbb{S}^N)$ and any orthogonal matrix $A \in O(N)$. Hence condition (14) is satisfied, for example, if there exists an orthogonal matrix $A \in O(N)$ such that

$$\sum_{i=1}^{k} h_i(\theta) \leq h_k(A(\theta)), \quad \text{for all } \theta \in \mathbb{S}^{N-1}.$$

Let us describe in more detail the case in which the singularities are generated by electric dipoles, i.e., $h_i(\theta) = \lambda_i \theta \cdot \mathbf{d}_i$, for some $\lambda_i > 0$ and $\mathbf{d}_i \in \mathbb{R}^N$ with $|\mathbf{d}_i| = 1$. For any $\lambda > 0$ and $\mathbf{d} \in \mathbb{R}^N$ with $|\mathbf{d}| = 1$, let

$$\mu_1^\lambda = \min_{\psi \in H^1(\mathbb{S}^{N-1})\setminus\{0\}} \frac{\int_{\mathbb{S}^{N-1}} |\nabla_{\mathbb{S}^{N-1}} \psi(\theta)|^2 \, dV(\theta) - \lambda \int_{\mathbb{S}^{N-1}} (\theta \cdot \mathbf{d}) \psi^2(\theta) \, dV(\theta)}{\int_{\mathbb{S}^{N-1}} \psi^2(\theta) \, dV(\theta)}$$

be the first eigenvalue of the operator $-\Delta_{\mathbb{S}^{N-1}} - \lambda\,(\theta \cdot \mathbf{d})$ on $\mathbb{S}^{N-1}$. By rotation invariance, it is easy to verify that the above minimum does not depend on $\mathbf{d}$. Moreover, condition (6) can be explicitly expressed as a bound on the dipole magnitudes; indeed,

$$\mu_1^\lambda > -\left(\frac{N-2}{2}\right)^2 \quad \text{if and only if} \quad \lambda < \frac{1}{\Lambda_N}$$

where Λ_N is the best constant in the dipole Hardy-type inequality, i.e.,

$$\Lambda_N := \sup_{u \in \mathcal{D}^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} \frac{x \cdot \mathbf{d}}{|x|^3} \, u^2(x) \, dx}{\int_{\mathbb{R}^N} |\nabla u(x)|^2 \, dx};$$

see [10]. By rotation invariance, Λ_N does not depend on the unit vector $\mathbf{d}$ and, by the classical Hardy's inequality, $\Lambda_N < 4/(N-2)^2$. For every $\lambda > 0$, let us denote

$$\sigma^\lambda := -\frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2}\right)^2 + \mu_1^\lambda}.$$

Corollary 1.5. *For* $i = 1, \dots, k$, *let* $a_i \in \mathbb{R}^N$, $a_i \neq a_j$ *for* $i \neq j$, $\mathbf{d}_i \in \mathbb{R}^N$ *with* $|\mathbf{d}_i| = 1$, *and*

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k < \Lambda_N^{-1}.$$

Assume that the quadratic form

$$u \mapsto \int_{\mathbb{R}^N} |\nabla u(x)|^2 dx - \sum_{i=1}^{k} \frac{\lambda_i \, (x - a_i) \cdot \mathbf{d}_i}{|x - a_i|^3} \, u^2(x) \, dx$$

is positive definite and that

$$(15)\qquad \begin{cases} \displaystyle\sum_{i=1}^{k-1} \frac{\lambda_i\,\mathbf{d}_i\cdot\frac{a_k-a_i}{|a_k-a_i|}}{|a_k-a_i|^2} > 0, \\ \qquad\qquad\qquad\qquad \text{if } \mu_1^{\lambda_k} \geq -\big(\frac{N-2}{2}\big)^2+1, \\ \\ \displaystyle\sum_{i=1}^{k-1}\int_{\mathbb{R}^N} \frac{\lambda_i\frac{x}{|x|}\cdot\mathbf{d}_i\Big[\psi_1^{\lambda_k\theta\cdot\mathbf{d}_k}\big(\frac{x+a_i-a_k}{|x+a_i-a_k|}\big)\Big]^2}{|x|^2|x+a_i-a_k|^{2(\sigma^{\lambda_k}+N-2)}} > 0, \\ \qquad\qquad\qquad\qquad \text{if } -\big(\frac{N-2}{2}\big)^2 < \mu_1^{\lambda_k} < -\big(\frac{N-2}{2}\big)^2+1, \end{cases}$$

$$(16)\qquad \Big|\sum_{i=1}^{k}\lambda_i\mathbf{d}_i\Big| \leq \lambda_k.$$

Then the infimum

$$\inf_{u\in\mathcal{D}^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N}|\nabla u(x)|^2dx - \sum_{i=1}^k\frac{\lambda_i\,(x-a_i)\cdot\mathbf{d}_i}{|x-a_i|^3}\,u^2(x)\,dx}{\big(\int_{\mathbb{R}^N}|u|^{2^*}dx\big)^{2/2^*}}$$

is achieved and the problem

$$(17)\qquad \begin{cases} -\Delta u - \sum_{i=1}^k\frac{\lambda_i\,(x-a_i)\cdot\mathbf{d}_i}{|x-a_i|^3}\,u = u^{2^*-1} \\ u>0 \quad \text{in } \mathbb{R}^N\setminus\{a_1,\dots,a_k\} \end{cases}$$

admits a solution in $\mathcal{D}^{1,2}(\mathbb{R}^N)$.

With respect to the isotropic case, the possibility of orientating the dipoles helps in finding the balance between the strength and the locations of the singularities required in assumptions (15) and (16). Consider, for example, the case of two dipoles $k=2$. Assume that $0<\lambda_1\leq\lambda_2$, λ_2 is small and N is large in such a way that the associated quadratic form is positive definite and $\mu_1^{\lambda_2}\geq-\big(\frac{N-2}{2}\big)^2+1$. Then condition (15) becomes

$$(a_2-a_1)\cdot\mathbf{d}_1>0,$$

while (16) becomes

$$\mathbf{d}_1\cdot\mathbf{d}_2 < -\frac{\lambda_1}{2\lambda_2}.$$

In this case, if the first dipole $\lambda_1\mathbf{d}_1$ is fixed at point a_1, (15) gives a constraint on the location of the second dipole while (16) gives a condition on its orientation. In particular, it is possible to construct many configurations ensuring the existence of ground state solutions to (17), unlike the isotropic case where problem (1) with $k = 2$ and h_1 and h_2 constants has no ground state solutions, as observed in [12, Theorem 1.3].

In bounded domains, concentration of mass at infinity is no longer possible and an existence result similar to Theorem 1.4 can be obtained without assumption (14).

Theorem 1.6. *Assume that Ω is a bounded smooth domain, $\{a_i\}_{i=1}^k \subset \Omega$, $h_i \in C^1(\mathbb{S}^N)$, $i = 1, \dots, k$, such that the quadratic form*

$$\mathcal{Q}_\Omega(u) =:= \int_\Omega |\nabla u(x)|^2 dx - \sum_{i=1}^k \int_{\mathbb{R}^N} \frac{h_i\Big(\frac{x-a_i}{|x-a_i|}\Big)}{|x-a_i|^2} u^2(x)\, dx \tag{18}$$

is positive definite in $H_0^1(\Omega)$, h_k satisfies (9)*, $S(h_k) = \min\{S(h_j) : j = 1, \dots, k\}$,*

$$\mu_1(h_k) \geq -\left(\frac{N-2}{2}\right)^2 + 1, \quad \textit{and} \quad \sum_{i=1}^{k-1} \frac{h_i\Big(\frac{a_k-a_i}{|a_k-a_i|}\Big)}{|a_k-a_i|^2} > 0.$$

Then the infimum in

$$S_\Omega(h_1, h_2, \dots, h_k) = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\mathcal{Q}_\Omega(u)}{\|u\|^2_{L^{2^*}(\Omega)}} \tag{19}$$

is achieved and equation

$$\begin{cases} -\Delta u - \sum_{i=1}^k \frac{h_i\big(\frac{x-a_i}{|x-a_i|}\big)}{|x-a_i|^2}\, u = u^{2^*-1} \\ u > 0 \quad \textit{in } \Omega \setminus \{a_1, \dots, a_k\}, \quad u = 0 \quad \textit{on } \partial\Omega \end{cases} \tag{20}$$

admits a solution in $H_0^1(\Omega)$.

The further assumption $\mu_1(h_k) \geq -\big(\frac{N-2}{2}\big)^2 + 1$ of Theorem 1.6 is not technical but quite natural when working in bounded domains. Indeed, it plays the role of a critical dimension for Brezis–Nirenberg type problems in bounded domains; see [4, 15].

The paper is organized as follows. Section 2 contains the proofs of Propositions 1.1 and 1.2. In section 3 some interaction estimates are first deduced and then applied to a comparison of energy levels of minimizing sequences. Section 4 provides a local Palais–Smale condition which is used to prove Theorem 1.4 and Corollary 1.5. Finally, in section 5 we analyze the problem in bounded domains.

Notation. We list some notation used throughout the paper.

- $B(a, r)$ denotes the ball $\{x \in \mathbb{R}^N : |x - a| < r\}$ in $\mathbb{R}^N$ with center at a and radius r.

- For any $A \subset \mathbb{R}^N$, χ_A denotes the characteristic function of A.
- S is the best constant in the Sobolev inequality $S\|u\|^2_{L^{2^*}(\mathbb{R}^N)} \leq \|u\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)}$.
- ω_N denotes the volume of the unit ball in $\mathbb{R}^N$.
- $O(N)$ denotes the group of orthogonal $N \times N$ matrices.

2 Necessity of the positivity of the quadratic form

In the present section we discuss the necessity of the positivity of the quadratic form for the solvability of (1), by proving Propositions 1.1 and 1.2.

Proof of Proposition 1.1. Let u be a positive classical solution to (1) in $\mathbb{R}^N \setminus \{a_1, \dots, a_k\}$. For any $\phi \in C_c^\infty(\mathbb{R}^N \setminus \{a_1, \dots, a_k\})$, by testing equation (9) with ϕ^2/u we obtain

$$2\int_{\mathbb{R}^N} \frac{\phi}{u}\,\nabla\phi\cdot\nabla u\,dx - \int_{\mathbb{R}^N} \frac{\phi^2}{u^2}\,|\nabla u|^2\,dx \\ - \sum_{i=1}^k \int_{\mathbb{R}^N} \frac{h_i\big(\frac{x-a_i}{|x-a_i|}\big)}{|x-a_i|^2}\,u^2(x)\,dx - \int_{\mathbb{R}^N} \phi^2 u^{2^*-2}\,dx = 0.$$

From the elementary inequality $2\,\frac{\phi}{u}\,\nabla\phi\cdot\nabla u - \frac{\phi^2}{u^2}\,|\nabla u|^2 \leq |\nabla\phi|^2$, we deduce

$$\mathcal{Q}(\phi) \geq \int_{\mathbb{R}^N} \phi^2 u^{2^*-2}\,dx \geq 0 \quad \text{for all } \phi \in C_c^\infty(\mathbb{R}^N \setminus \{a_1, \dots, a_k\}).$$

From the density of $C_c^\infty(\mathbb{R}^N \setminus \{a_1, \dots, a_k\})$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ (see [7, Lemma 2.1]), we obtain that $\mathcal{Q}$ is positive semidefinite in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. □

Proof of Proposition 1.2. Assume that (6) holds and let $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ be a positive $\mathcal{D}^{1,2}(\mathbb{R}^N)$-solution to (1). By Proposition 1.1, it follows that

$$\mu(h_1, \dots, h_k, a_1, \dots, a_k) \geq 0,$$

where $\mu(h_1, \dots, h_k, a_1, \dots, a_k)$ has been defined in (7). Let us assume by contradiction that $\mu(h_1, \dots, h_k, a_1, \dots, a_k) = 0$. From [11, Proposition 4.1], $\mu(h_1, \dots, h_k, a_1, \dots, a_k) = 0$ is attained by some $v \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, $v \geq 0$ a.e. in $\mathbb{R}^N$, $v \not\equiv 0$, which then satisfies

$$-\Delta v - \sum_{i=1}^k \frac{h_i\big(\frac{x-a_i}{|x-a_i|}\big)}{|x-a_i|^2}\,v = 0 \quad \text{weakly in } \mathcal{D}^{1,2}(\mathbb{R}^N).$$

Testing the above equation with u, we obtain that

$$\int_{\mathbb{R}^N} u^{2^*-1}(x)v(x)\,dx = 0,$$

which contradictes the positivity of u. □

3 Interaction estimates and comparison of energy levels

By Theorem 1.3, for every function $h \in C^1(\mathbb{S}^N)$ verifying $\mu_1(h) > -\big(\frac{N-2}{2}\big)^2$ and (9), there exists some $\phi_h \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, $\phi_h \geq 0$, $\phi_h \not\equiv 0$, such that ϕ_h attains $S(h)$, i.e.,

$$S(h) = \frac{\int_{\mathbb{R}^N} \Big[|\nabla \phi_h(x)|^2 - \frac{h(x/|x|)}{|x|^2}\phi_h^2(x)\Big]\,dx}{\big(\int_{\mathbb{R}^N} |\phi_h|^{2^*}\big)^{2/2^*}}, \tag{21}$$

and solves

$$-\Delta \phi_h - \frac{h(x/|x|)}{|x|^2}\phi_h = \phi_h^{2^*-1}, \quad \text{in } \mathbb{R}^N. \tag{22}$$

Moreover, Kelvin's transform $w_h(x) := |x|^{-(N-2)}\phi_h(x/|x|^2)$ solves

$$-\Delta w_h - \frac{h(x/|x|)}{|x|^2}w_h = w_h^{2^*-1} \quad \text{in } \mathbb{R}^N.$$

From [10], it follows that, with σ_h defined in (10), the functions

$$x \mapsto \frac{\phi_h(x)}{|x|^{\sigma_h}\psi_1^h(x/|x|)}, \quad x \mapsto \frac{w_h(x)}{|x|^{\sigma_h}\psi_1^h(x/|x|)} = \frac{\phi_h(x/|x|^2)}{|x|^{\sigma_h+N-2}\psi_1^h(x/|x|)}$$

are continuous in $\mathbb{R}^N$ and admit positive limits as $|x| \to 0$, i.e.,

$$\begin{aligned} c_0^h &:= \lim_{|x|\to 0} \frac{\phi_h(x)}{|x|^{\sigma_h}\psi_1^h(x/|x|)} \in (0,+\infty) \quad \text{and} \\ c_\infty^h &:= \lim_{|x|\to +\infty} \frac{\phi_h(x)}{|x|^{-\sigma_h-N+2}\psi_1^h(x/|x|)} \in (0,+\infty). \end{aligned} \tag{23}$$

Hence there exists a positive constant $C(h) > 0$ such that

$$\frac{1}{C(h)}\frac{|x|^{\sigma_h}}{1+|x|^{2\sigma_h+N-2}} \leq \phi_h(x) \leq \frac{C(h)\,|x|^{\sigma_h}}{1+|x|^{2\sigma_h+N-2}}, \quad \text{for all } x \in \mathbb{R}^N \setminus \{0\}. \tag{24}$$

For any $\mu > 0$, we denote $\phi_\mu^h(x) := \mu^{-(N-2)/2}\phi_h(x/\mu)$.

Lemma 3.1. *Let $h, k \in C^1(\mathbb{S}^N)$ such that h satisfies* (9) *and*

$$\mu_1(h) > -\Big(\frac{N-2}{2}\Big)^2 + 1.$$

Then, for every $a \in \mathbb{R}^N \setminus \{0\}$,

$$\int_{\mathbb{R}^N} \phi_h^2(x)\,dx \in (0,+\infty)$$

and

$$\int_{\mathbb{R}^N} \frac{k\big(\frac{x-a}{|x-a|}\big)}{|x-a|^2}|\phi_\mu^h(x)|^2\,dx = \mu^2\left[\frac{k\big(\frac{-a}{|a|}\big)}{|a|^2}\int_{\mathbb{R}^N}\phi_h^2(x)\,dx + o(1)\right]$$

as $\mu \to 0^+$.

Proof. From (24) and the assumption $\mu_1(h) > -\big(\frac{N-2}{2}\big)^2 + 1$, it follows that $\phi_h \in L^2(\mathbb{R}^N)$. We have

$$\int_{\mathbb{R}^N} \frac{k\big(\frac{x-a}{|x-a|}\big)}{|x-a|^2} |\phi_\mu^h(x)|^2\,dx = \mu^2 \int_{|x|<\frac{|a|}{2\mu}} \frac{k\big(\frac{\mu x-a}{|\mu x-a|}\big)}{|\mu x-a|^2} \phi_h^2(x)\,dx + \mu^{-N+2} \int_{|x+a|\geq\frac{|a|}{2}} \frac{k\big(\frac{x}{|x|}\big)}{|x|^2} \phi_h^2\Big(\frac{x+a}{\mu}\Big)\,dx. \tag{25}$$

Since

$$\left|\chi_{B\left(0,\frac{|a|}{2\mu}\right)}(x) \frac{k\left(\frac{\mu x-a}{|\mu x-a|}\right)}{|\mu x-a|^2}\right| \leq \frac{4}{|a|^2}\|k\|_{L^\infty(\mathbb{S}^{N-1})}$$

and $\phi_h \in L^2(\mathbb{R}^N)$, we deduce from the Dominated Convergence Theorem that

$$\lim_{\mu\to0^+} \int_{|x|<\frac{|a|}{2\mu}} \frac{k\big(\frac{\mu x-a}{|\mu x-a|}\big)}{|\mu x-a|^2}\phi_h^2(x)\,dx = \lim_{\mu\to0^+}\int_{\mathbb{R}^N} \chi_{B\left(0,\frac{|a|}{2\mu}\right)}(x) \frac{k\big(\frac{\mu x-a}{|\mu x-a|}\big)}{|\mu x-a|^2}\phi_h^2(x)\,dx = \frac{k\left(\frac{-a}{|a|}\right)}{|a|^2}\int_{\mathbb{R}^N}\phi_h^2(x)\,dx. \tag{26}$$

Moreover, from (24) and $\mu_1(h) > -\big(\frac{N-2}{2}\big)^2 + 1$, it follows that

$$\left|\mu^{-N}\int_{|x+a|\geq\frac{|a|}{2}} \frac{k\big(\frac{x}{|x|}\big)}{|x|^2}\phi_h^2\Big(\frac{x+a}{\mu}\Big)\,dx\right| \leq \mu^{2\sigma_h+N-4}\|k\|_{L^\infty(\mathbb{S}^{N-1})}(C(h))^2 \int_{|x+a|\geq\frac{|a|}{2}} \frac{1}{|x|^2|x+a|^{2(\sigma_h+N-2)}}\,dx = o(1) \quad \text{as } \mu\to0^+. \tag{27}$$

The conclusion then follows from (25), (26), and (27). □

Lemma 3.2. *Let $h, k \in C^1(\mathbb{S}^N)$ such that h satisfies* (9)*, and*

$$\mu_1(h) = -\Big(\frac{N-2}{2}\Big)^2 + 1.$$

Then, for every $a \in \mathbb{R}^N \setminus \{0\}$,

$$N\omega_N \frac{|\log\mu|}{|C(h)|^2}(1+o(1)) \leq \int_{|x|<\frac{|a|}{2\mu}} \phi_h^2(x)\,dx \leq N\omega_N|C(h)|^2\,|\log\mu|\,(1+o(1)), \quad \textit{as } \mu\to0^+, \tag{28}$$

where ω_N is the volume of the standard unit N-ball, and

$$\int_{\mathbb{R}^N} \frac{k\big(\frac{x-a}{|x-a|}\big)}{|x-a|^2}|\phi_\mu^h(x)|^2\,dx = \mu^2\left(\frac{k\left(\frac{-a}{|a|}\right)}{|a|^2} + o(1)\right)\left[\int_{|x|<\frac{1}{\mu}}\phi_h^2(x)\,dx\right] \tag{29}$$

as $\mu \to 0^+$.

Proof. Estimate (28) follows from (24) and direct calculations. We have

(30)
$$\begin{aligned}&\int_{\mathbb{R}^N}\frac{k\big(\frac{x-a}{|x-a|}\big)}{|x-a|^2}|\phi^h_\mu(x)|^2\,dx\\ &=\mu^2\bigg[\frac{k\big(\frac{-a}{|a|}\big)}{|a|^2}\int_{|x|<\frac{|a|}{2\mu}}\phi_h^2(x)\,dx+\int_{|x|<\frac{|a|}{2\mu}}k\Big(\frac{\mu x-a}{|\mu x-a|}\Big)\bigg(\frac{1}{|\mu x-a|^2}-\frac{1}{|a|^2}\bigg)\phi_h^2(x)\,dx\\ &\qquad+\frac{1}{|a|^2}\int_{|x|<\frac{|a|}{2\mu}}\bigg(k\Big(\frac{\mu x-a}{|\mu x-a|}\Big)-k\Big(\frac{-a}{|a|}\Big)\bigg)\phi_h^2(x)\,dx\\ &\qquad+\mu^{-N}\int_{|x+a|\geq\frac{|a|}{2}}\frac{k\big(\frac{x}{|x|}\big)}{|x|^2}\phi_h^2\Big(\frac{x+a}{\mu}\Big)\,dx\bigg].\end{aligned}$$

Since
$$\bigg|\frac{1}{|\mu x-a|^2}-\frac{1}{|a|^2}\bigg|\leq\frac{4}{|a|^4}\big(\mu^2|x|^2+2\mu|a||x|\big)\quad\text{for } |x|<\frac{|a|}{2\mu},$$
it follows from (24) that

$$\int_{|x|<\frac{|a|}{2\mu}}k\Big(\frac{\mu x-a}{|\mu x-a|}\Big)\bigg(\frac{1}{|\mu x-a|^2}-\frac{1}{|a|^2}\bigg)\phi_h^2(x)\,dx=O(1)\quad\text{as } \mu\to 0^+. \tag{31}$$

Since $k\in C^1(\mathbb{S}^N)$, for some positive constant C depending on k

$$\begin{aligned}\bigg|k\Big(\frac{\mu x-a}{|\mu x-a|}\Big)-k\Big(\frac{-a}{|a|}\Big)\bigg|&\leq C\,\bigg|\frac{\mu x-a}{|\mu x-a|}-\frac{-a}{|a|}\bigg|\\ &=\frac{C\sqrt{2}}{\sqrt{|\mu x-a|}}\sqrt{|\mu x-a|-|a|+\mu\,\frac{a\cdot x}{|a|}}\\ &\leq\frac{C\sqrt{2}}{\sqrt{|\mu x-a|}}\sqrt{2\mu|x|}\leq\frac{2\,C\sqrt{2}\,\sqrt{\mu}\sqrt{|x|}}{\sqrt{|a|}}\quad\text{for } |x|<\frac{|a|}{2\mu};\end{aligned}$$

hence, from (24), it follows that

$$\int_{|x|<\frac{|a|}{2\mu}}\bigg(k\Big(\frac{\mu x-a}{|\mu x-a|}\Big)-k\Big(\frac{-a}{|a|}\Big)\bigg)\phi_h^2(x)\,dx=O(1)\quad\text{as } \mu\to 0^+. \tag{32}$$

From (24), we deduce that

$$\begin{aligned}&\bigg|\mu^{-N}\int_{|x+a|\geq\frac{|a|}{2}}\frac{k\big(\frac{x}{|x|}\big)}{|x|^2}\phi_h^2\Big(\frac{x+a}{\mu}\Big)\,dx\bigg|\\ &\qquad\leq\|k\|_{L^\infty(\mathbb{S}^{N-1})}(C(h))^2\int_{|x+a|\geq\frac{|a|}{2}}\frac{1}{|x|^2|x+a|^N},\end{aligned}$$

hence

$$\mu^{-N}\int_{|x+a|\geq\frac{|a|}{2}}\frac{k\big(\frac{x}{|x|}\big)}{|x|^2}\phi_h^2\Big(\frac{x+a}{\mu}\Big)\,dx=O(1)\quad\text{as } \mu\to 0^+. \tag{33}$$

From (28), (30), (31), (32), and (33) it follows that

$$\int_{\mathbb{R}^N} \frac{k\big(\frac{x-a}{|x-a|}\big)}{|x-a|^2}|\phi_\mu^h(x)|^2\,dx = \mu^2\left(\frac{k\big(\frac{-a}{|a|}\big)}{|a|^2}+o(1)\right)\left[\int_{|x|<\frac{|a|}{2\mu}}\phi_h^2(x)\,dx\right] \tag{34}$$

as $\mu \to 0^+$. From (24) and the assumption $\mu_1(h) = -\big(\frac{N-2}{2}\big)^2+1$, we obtain

$$\begin{aligned}\left|\int_{|x|<\frac{|a|}{2\mu}}\phi_h^2(x)\,dx - \int_{|x|<\frac{1}{\mu}}\phi_h^2(x)\,dx\right| &\le N\omega_N(C(h))^2\left|\int_{\frac{1}{\mu}}^{\frac{|a|}{2\mu}} r^{-1}dr\right|\\ &= N\omega_N(C(h))^2\left|\log\frac{|a|}{2}\right|;\end{aligned}$$

hence, taking into account that, under the assumption $\mu_1(h) = -\big(\frac{N-2}{2}\big)^2+1$, $\phi_h \notin L^2(\mathbb{R}^N)$,

$$\int_{|x|<\frac{|a|}{2\mu}}\phi_h^2(x)\,dx = \int_{|x|<\frac{1}{\mu}}\phi_h^2(x)\,dx + O(1) = (1+o(1))\int_{|x|<\frac{1}{\mu}}\phi_h^2(x)\,dx \tag{35}$$

as $\mu \to 0^+$. The conclusion (29) follows from (34) and (35). □

Lemma 3.3. *Let $h, k \in C^1(\mathbb{S}^N)$ such that h satisfies* (9) *and*

$$-\Big(\frac{N-2}{2}\Big)^2 < \mu_1(h) < -\Big(\frac{N-2}{2}\Big)^2+1.$$

Then, for every $a \in \mathbb{R}^N\setminus\{0\}$ and $A \in O(N)$ such that $Ae_1 = a/|a|$, with $e_1 = (1,0,\dots,0)\in\mathbb{R}^N$,

$$\begin{aligned}&\int_{\mathbb{R}^N} \frac{k\big(\frac{x-a}{|x-a|}\big)}{|x-a|^2}|\phi_\mu^h(x)|^2\,dx\\ &\quad= \mu^{2\sigma_h+N-2}\left[(c_\infty^h)^2\int_{\mathbb{R}^N}\frac{k\big(\frac{x}{|x|}\big)\Big[\psi_1^h\big(\frac{x+a}{|x+a|}\big)\Big]^2}{|x|^2|x+a|^{2(\sigma_h+N-2)}}\,dx+o(1)\right]\\ &\quad= \frac{\mu^{2\sigma_h+N-2}}{|a|^{2\sigma_h+N-2}}\left[(c_\infty^h)^2\int_{\mathbb{R}^N}\frac{(k\circ A)\big(\frac{x}{|x|}\big)\Big[(\psi_1^h\circ A)\big(\frac{x+e_1}{|x+e_1|}\big)\Big]^2}{|x|^2|x+e_1|^{2(\sigma_h+N-2)}}\,dx+o(1)\right]\end{aligned}$$

as $\mu\to 0^+$, where c_∞^h is defined in (23)*.*

Proof. A direct calculation yields

$$\begin{aligned}&\int_{\mathbb{R}^N} \frac{k\big(\frac{x-a}{|x-a|}\big)}{|x-a|^2}|\phi_\mu^h(x)|^2\,dx\\ &\quad= \mu^{2\sigma_h+N-2}\int_{\mathbb{R}^N}\frac{k\big(\frac{x}{|x|}\big)\Big[\psi_1^h\big(\frac{x+a}{|x+a|}\big)\Big]^2}{|x|^2|x+a|^{2(\sigma_h+N-2)}}\cdot\frac{\phi_h^2\big(\frac{x+a}{\mu}\big)}{\big|\frac{x+a}{\mu}\big|^{2(2-\sigma_h-N)}\Big[\psi_1^h\big(\frac{x+a}{|x+a|}\big)\Big]^2}\,dx.\end{aligned} \tag{36}$$

From (24), it follows that the function

$$x \mapsto \frac{\phi_h^2\left(\frac{x+a}{\mu}\right)}{\left|\frac{x+a}{\mu}\right|^{2(2-\sigma_h-N)}\left[\psi_1^h\left(\frac{x+a}{|x+a|}\right)\right]^2}$$

is bounded a.e. in $\mathbb{R}^N$ uniformly with respect to $\mu > 0$, whereas (23) implies that, for a.e. $x \in \mathbb{R}^N$,

$$\lim_{\mu\to 0} \frac{\phi_h^2\left(\frac{x+a}{\mu}\right)}{\left|\frac{x+a}{\mu}\right|^{2(2-\sigma_h-N)}\left[\psi_1^h\left(\frac{x+a}{|x+a|}\right)\right]^2} = (c_\infty^h)^2. \tag{37}$$

Since the assumption $\mu_1(h) < -\left(\frac{N-2}{2}\right)^2 + 1$ ensures that

$$x \mapsto \frac{k\left(\frac{x}{|x|}\right)\left[\psi_1^h\left(\frac{x+a}{|x+a|}\right)\right]^2}{|x|^2|x+a|^{2(\sigma_h+N-2)}} \in L^1(\mathbb{R}^N),$$

from (36), (37), and the Dominated Convergence Theorem we deduce that

$$\int_{\mathbb{R}^N} \frac{k\left(\frac{x-a}{|x-a|}\right)}{|x-a|^2}|\phi_\mu^h(x)|^2\,dx = \mu^{2\sigma_h+N-2}\left[(c_\infty^h)^2\int_{\mathbb{R}^N}\frac{k\left(\frac{x}{|x|}\right)\left[\psi_1^h\left(\frac{x+a}{|x+a|}\right)\right]^2}{|x|^2|x+a|^{2(\sigma_h+N-2)}}\,dx + o(1)\right]$$

as $\mu \to 0$. By the change of variable $x = |a|Ay$, we obtain

$$\int_{\mathbb{R}^N}\frac{k\left(\frac{x}{|x|}\right)\left[\psi_1^h\left(\frac{x+a}{|x+a|}\right)\right]^2}{|x|^2|x+a|^{2(\sigma_h+N-2)}}\,dx = |a|^{-N-2\sigma_h+2}\int_{\mathbb{R}^N}\frac{(k\circ A)\left(\frac{y}{|y|}\right)\left[\psi_1^h\left(A\left(\frac{y+e_1}{|y+e_1|}\right)\right)\right]^2}{|y|^2|y+e_1|^{2(\sigma_h+N-2)}}\,dx,$$

thus completing the proof. □

The interaction estimates provided by Lemmas 3.1, 3.2, and 3.3 allow us to compare the ground state level of the multisingular problem with the ground state level of the single dipole problem.

Proposition 3.4. *Let* $h_i \in C^1(\mathbb{S}^N)$, $i = 1,\dots,k$ *and* $j \in \{1,2,\dots,k\}$. *We assume that* h_j *verifies* (9), *and one of the following assumptions is satisfied*

$$\mu_1(h_j) \geq -\left(\frac{N-2}{2}\right)^2 + 1 \quad \textit{and} \quad \sum_{\substack{i=1\\ i\neq j}}^{k}\frac{h_i\left(\frac{a_j-a_i}{|a_j-a_i|}\right)}{|a_j-a_i|^2} > 0, \tag{38}$$

$$\begin{cases} -\left(\frac{N-2}{2}\right)^2 < \mu_1(h_j) < -\left(\frac{N-2}{2}\right)^2 + 1 \quad \textit{and} \\ \displaystyle\sum_{\substack{i=1\\ i\neq j}}^{k}\int_{\mathbb{R}^N}\frac{h_i\left(\frac{x}{|x|}\right)\left[\psi_1^{h_j}\left(\frac{x+a_i-a_j}{|x+a_i-a_j|}\right)\right]^2}{|x|^2|x+a_i-a_j|^{2(\sigma_{h_j}+N-2)}} > 0. \end{cases} \tag{39}$$

Then $S(h_1,\dots,h_k) < S(h_j)$.

Proof. Since h_j satisfies (9), by Theorem 1.3 there exists $\phi_{h_j} \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, $\phi_{h_j} \geq 0$, $\phi_{h_j} \not\equiv 0$, attaining $S(h_j)$, i.e., satisfying (21) and (22) with $h = h_j$. Set $z_\mu(x) = \phi_\mu^{h_j}(x - a_j)$. Then

$$
\begin{aligned}
&S(h_1, \dots, h_k)\\
&\quad\leq \frac{\int_{\mathbb{R}^N} |\nabla z_\mu(x)|^2\,dx - \int_{\mathbb{R}^N} \frac{h_j\left(\frac{x-a_j}{|x-a_j|}\right)}{|x-a_j|^2}\, z_\mu^2(x)\,dx}{\|z_\mu\|^2_{L^{2^*}(\mathbb{R}^N)}}
- \frac{\sum_{i\neq j} \int_{\mathbb{R}^N} \frac{h_i\left(\frac{x-a_i}{|x-a_i|}\right)}{|x-a_i|^2}\, z_\mu^2(x)\,dx}{\|z_\mu\|^2_{L^{2^*}(\mathbb{R}^N)}}\\
&\quad= \frac{\int_{\mathbb{R}^N} |\nabla \phi_{h_j}(x)|^2\,dx - \int_{\mathbb{R}^N} \frac{h_j\left(\frac{x}{|x|}\right)}{|x|^2}\, \phi_{h_j}^2(x)\,dx}{\|\phi_{h_j}\|^2_{L^{2^*}(\mathbb{R}^N)}}
\frac{-\sum_{i\neq j} \int_{\mathbb{R}^N} \frac{h_i\left(\frac{x-(a_i-a_j)}{|x-(a_i-a_j)|}\right)}{|x-(a_i-a_j)|^2}\, (\phi_\mu^{h_j}(x))^2\,dx}{\|\phi_{h_j}\|^2_{L^{2^*}(\mathbb{R}^N)}}.
\end{aligned}
$$

From the above and Lemmas 3.1, 3.2 and 3.3, we deduce the following estimate:

$$
\begin{aligned}
(40)\quad & S(h_1, \dots, h_k) \leq S(h_j) - \|\phi_{h_j}\|^{-2}_{L^{2^*}(\mathbb{R}^N)}\\
&\times \begin{cases}
\mu^2\left(\int_{\mathbb{R}^N} \phi_{h_j}^2(x)\right)\left(\sum_{i\neq j} \frac{h_i\left(\frac{a_j-a_i}{|a_j-a_i|}\right)}{|a_j-a_i|^2} + o(1)\right) & \text{if } \mu_1(h_j) > -\left(\frac{N-2}{2}\right)^2 + 1\\
\mu^2\left(\int_{|x|<\frac{1}{\mu}} \phi_{h_j}^2(x)\right)\left(\sum_{i\neq j} \frac{h_i\left(\frac{a_j-a_i}{|a_j-a_i|}\right)}{|a_j-a_i|^2} + o(1)\right) & \text{if } \mu_1(h_j) = -\left(\frac{N-2}{2}\right)^2 + 1\\
\mu^{2\sigma_{h_j}+N-2}(c_\infty^{h_j})^2\left(\sum_{i\neq j} \int_{\mathbb{R}^N} \frac{h_i\left(\frac{x}{|x|}\right)\left[\psi_1^{h_j}\left(\frac{x+a_i-a_j}{|x+a_i-a_j|}\right)\right]^2}{|x|^2|x+a_i-a_j|^{2(\sigma_{h_j}+N-2)}} + o(1)\right) & \text{if } \mu_1(h_j) < -\left(\frac{N-2}{2}\right)^2 + 1
\end{cases}
\end{aligned}
$$

as $\mu \to 0^+$. Taking μ small enough in (40), we obtain from (38) and (39) that $S(h_1, \dots, h_k) < S(h_j)$. □

Remark 3.5. For $-\left(\frac{N-2}{2}\right)^2 < \mu_1(h_j) < -\left(\frac{N-2}{2}\right)^2 + 1$, assumption (39) can be rewritten as

$$\int_{\mathbb{R}^N} \left(\sum_{i \neq j} \frac{(h_i \circ A_{ij})\left(\frac{x}{|x|}\right)}{|a_i - a_j|^{2(\sigma_{h_j}+N-2)}} \left[(\psi_1^{h_j} \circ A_{ij}) \left(\frac{x+e_1}{|x+e_1|} \right) \right]^2 \right) \frac{dx}{|x|^2 |x+e_1|^{2(\sigma_{h_j}+N-2)}} > 0,$$

where $A_{ij} \in O(N)$ are such that $A_{ij} e_1 = \frac{a_i - a_j}{|a_i - a_j|}$.

4 The Palais–Smale condition and proof of Theorem 1.4

If $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, $u > 0$ a.e. in $\mathbb{R}^N$, is a critical point of the functional $J : \mathcal{D}^{1,2}(\mathbb{R}^N) \to \mathbb{R}$,

$$\begin{aligned} J(v) =& \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 dx - \frac{1}{2} \sum_{i=1}^k \int_{\mathbb{R}^N} \frac{h_i\left(\frac{x-a_i}{|x-a_i|}\right)}{|x-a_i|^2} v^2(x)\, dx \\ & - \frac{S(h_1, h_2, \dots, h_k)}{2^*} \int_{\mathbb{R}^N} |v|^{2^*} dx, \end{aligned} \tag{41}$$

then $w = S(h_1, h_2, \dots, h_k)^{1/(2^*-2)} u$ is a solution to equation (1) (weakly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and classically in $\mathbb{R}^N \setminus \{a_1, \dots, a_k\}$). From now on, for any $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, $J'(u) \in (\mathcal{D}^{1,2}(\mathbb{R}^N))^\star$ will denote the Fréchet derivative of J at u and $\langle \cdot, \cdot \rangle$ will remain as the duality product between $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and its dual space $(\mathcal{D}^{1,2}(\mathbb{R}^N))^\star$.

The Concentration-Compactness analysis of the behavior of Palais–Smale sequences provides the following local compactness result.

Theorem 4.1. *Let* (7) *hold and* $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{D}^{1,2}(\mathbb{R}^N)$ *be a Palais–Smale sequence for* J*, namely*

$$\lim_{n\to\infty} J(u_n) = c < \infty \text{ in } \mathbb{R} \quad \text{and} \quad \lim_{n\to\infty} J'(u_n) = 0 \text{ in the dual space } (\mathcal{D}^{1,2}(\mathbb{R}^N))^\star.$$

If

$$c < \frac{1}{N} S(h_1, h_2, \dots, h_k)^{1-\frac{N}{2}} \left(\min \left\{ S, S(h_1), \dots, S(h_k), S\left(\sum\nolimits_{j=1}^k h_j \right) \right\} \right)^{N/2}, \tag{42}$$

then $\{u_n\}_{n\in\mathbb{N}}$ *admits a subsequence strongly converging in* $\mathcal{D}^{1,2}(\mathbb{R}^N)$.

Proof. Let $\{u_n\}_{n\in\mathbb{N}}$ be a Palais–Smale sequence for J at level c. Then from (7) there exists some positive constant c_1 such that

$$\begin{aligned} c_1 \|u_n\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)} \leq \mathcal{Q}(u_n) &= N J(u_n) - \frac{N-2}{2} \langle J'(u_n), u_n \rangle \\ &= Nc + o(\|u_n\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}) + o(1) \end{aligned}$$

as $n \to +\infty$, hence $\{u_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. Then there exists $u_0 \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ such that, up to a subsequence still denoted by $\{u_n\}_{n\in\mathbb{N}}$, $u_n \rightharpoonup u_0$ weakly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, $u_n \to u_0$ a.e. in $\mathbb{R}^N$, and $u_n \to u_0$ in $L^\alpha_{\rm loc}(\mathbb{R}^N)$ for any $\alpha \in [1, 2^*)$. The *Concentration Compactness Principle* of P. L. Lion, (see [17] and [18]) ensures that, for an at most countable set $\mathcal{J}$, some points $x_j \in \mathbb{R}^N \setminus \{a_1, \dots, a_k\}$, some real numbers μ_{x_j}, ν_{x_j}, $j \in \mathcal{J}$, and $\mu_{a_i}, \nu_{a_i}, \gamma_i$, $i = 1, \dots, k$, the following convergences hold in the sense of measures up to a subsequence

$$|\nabla u_n|^2 \rightharpoonup d\mu \geq |\nabla u_0|^2 + \sum_{i=1}^k \mu_{a_i}\delta_{a_i} + \sum_{j\in\mathcal{J}} \mu_{x_j}\delta_{x_j}, \tag{43}$$

$$|u_n|^{2^*} \rightharpoonup d\nu = |u_0|^{2^*} + \sum_{i=1}^k \nu_{a_i}\delta_{a_i} + \sum_{j\in\mathcal{J}} \nu_{x_j}\delta_{x_j}, \tag{44}$$

$$h_i\Big(\frac{x-a_i}{|x-a_i|}\Big)\frac{u_n^2}{|x-a_i|^2} \rightharpoonup d\gamma_{a_i} = h_i\Big(\frac{x-a_i}{|x-a_i|}\Big)\frac{u_0^2}{|x-a_i|^2} + \gamma_i\delta_{a_i}, \quad \text{for any } i = 1, \dots, k. \tag{45}$$

Note that we can choose μ_{a_i} and μ_{x_j} such that $\mu_{a_i} = d\mu(\{a_i\})$, and $\mu_{x_j} = d\mu(\{x_j\})$. From Sobolev's inequality it follows that

$$S\nu_{x_j}^{2/2^*} \leq \mu_{x_j} \text{ for all } j \in \mathcal{J} \quad \text{and} \quad S\nu_{a_i}^{2/2^*} \leq \mu_{a_i} \text{ for all } i = 1, \dots, k. \tag{46}$$

The concentration at infinity of the sequence can be evaluated by the following quantities:

$$\nu_\infty = \lim_{R\to\infty}\limsup_{n\to\infty}\int_{|x|>R}|u_n(x)|^{2^*}dx, \quad \mu_\infty = \lim_{R\to\infty}\limsup_{n\to\infty}\int_{|x|>R}|\nabla u_n(x)|^2dx$$

and

$$\gamma_\infty = \lim_{R\to\infty}\limsup_{n\to\infty}\int_{|x|>R}\bigg(\sum_{i=1}^k h_i\big(\tfrac{x}{|x|}\big)\bigg)\frac{u_n^2(x)}{|x|^2}\,dx.$$

Testing $J'(u_n)$ with $u_n\phi_j^\varepsilon$, for some smooth cut-off function ϕ_j^ε centered at x_j and supported in $B(x_j, \varepsilon)$, and letting $n \to \infty$ and $\varepsilon \to 0$, we obtain $\mu_{x_j} \leq S(h_1, h_2, \dots, h_k)\nu_{x_j}$ which, together with (46), implies that

$$\mathcal{J} \text{ is finite, and for } j \in \mathcal{J} \text{ either } \nu_{x_j} = 0 \text{ or } \nu_{x_j} \geq \bigg(\frac{S}{S(h_1, h_2, \dots, h_k)}\bigg)^{N/2}. \tag{47}$$

To analyze concentration at singularities, for each $i = 1, 2, \dots, k$ we consider a smooth cut-off function ψ_i^ε satisfying $0 \leq \psi_i^\varepsilon(x) \leq 1$,

$$\psi_i^\varepsilon(x) = 1 \quad \text{if } |x-a_i| \leq \frac{\varepsilon}{2}, \quad \psi_i^\varepsilon(x) = 0 \quad \text{if } |x-a_i| \geq \varepsilon, \quad \text{and}$$
$$|\nabla\psi_i^\varepsilon(x)| \leq \frac{4}{\varepsilon} \quad \text{for all } x \in \mathbb{R}^N.$$

From (8), it follows that

$$\frac{\int_{\mathbb{R}^N} |\nabla(u_n\psi_i^\varepsilon)|^2\,dx - \int_{\mathbb{R}^N} h_i\Big(\frac{x-a_i}{|x-a_i|}\Big)\frac{|\psi_i^\varepsilon|^2 u_n^2}{|x-a_i|^2}\,dx}{\Big(\int_{\mathbb{R}^N} |\psi_i^\varepsilon u_n|^{2^*}\,dx\Big)^{2/2^*}} \geq S(h_i)$$

and hence

(48)

$$\int_{\mathbb{R}^N} |\psi_i^\varepsilon|^2 |\nabla u_n|^2 dx + \int_{\mathbb{R}^N} u_n^2 |\nabla\psi_i^\varepsilon|^2\,dx + 2\int_{\mathbb{R}^N} u_n\psi_i^\varepsilon \nabla u_n\cdot\nabla\psi_i^\varepsilon dx$$
$$\geq \int_{\mathbb{R}^N} h_i\Big(\frac{x-a_i}{|x-a_i|}\Big)\frac{|\psi_i^\varepsilon|^2 u_n^2}{|x-a_i|^2}\,dx + S(h_i)\Big(\int_{\mathbb{R}^N} |\psi_i^\varepsilon u_n|^{2^*} dx\Big)^{2/2^*}.$$

It is easy to verify that

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\Big[\int_{\mathbb{R}^N} u_n^2|\nabla\psi_i^\varepsilon|^2\,dx + 2\int_{\mathbb{R}^N} u_n\psi_i^\varepsilon\nabla u_n\cdot\nabla\psi_i^\varepsilon\,dx\Big] = 0;$$

then from (48) and (43)–(45) we deduce that

$$\mu_{a_i} \geq \gamma_i + S(h_i)\nu_{a_i}^{2/2^*}. \tag{49}$$

Testing $J'(u_n)$ with $u_n\psi_i^\varepsilon$ and letting $n\to+\infty$ and $\varepsilon\to 0$, we obtain

$$\mu_{a_i} - \gamma_i \leq S(h_1, h_2, \dots, h_k)\nu_{a_i}. \tag{50}$$

From (49) and (50) we conclude that, for each $i = 1, 2, \dots, k$,

$$\text{either}\quad \nu_{a_i} = 0 \quad\text{or}\quad \nu_{a_i} \geq \Big(\frac{S(h_i)}{S(h_1,h_2,\dots,h_k)}\Big)^{N/2}. \tag{51}$$

To study the possibility of concentration at ∞, we consider a regular cut-off function ψ_R such that

$$0\leq\psi_R(x)\leq 1 \text{ for all } x\in\mathbb{R}^N,\ \psi_R(x) = \begin{cases}1, & \text{if } |x|>2R,\\ 0, & \text{if } |x|<R,\end{cases} \quad\text{and}$$
$$|\nabla\psi_R(x)| \leq \frac{2}{R} \text{ for all } x\in\mathbb{R}^N.$$

From (8) we obtain

$$\frac{\int_{\mathbb{R}^N} |\nabla(u_n\psi_R)|^2 dx - \int_{\mathbb{R}^N}\Big(\sum_{i=1}^k h_i\big(\frac{x}{|x|}\big)\Big)\frac{\psi_R^2 u_n^2}{|x|^2}dx}{\Big(\int_{\mathbb{R}^N}|\psi_R u_n|^{2^*}dx\Big)^{2/2^*}} \geq S\Big(\sum_{i=1}^k h_i\Big)$$

and, consequently,

$$\int_{\mathbb{R}^N} \psi_R^2 |\nabla u_n|^2 dx + \int_{\mathbb{R}^N} u_n^2 |\nabla \psi_R|^2 dx + 2\int_{\mathbb{R}^N} u_n \psi_R \nabla u_n \cdot \nabla \psi_R \, dx \geq \int_{\mathbb{R}^N} \Big(\textstyle\sum_{i=1}^k h_i\big(\frac{x}{|x|}\big)\Big) \frac{\psi_R^2 u_n^2}{|x|^2} dx + S\Big(\sum_{i=1}^k h_i\Big)\Big(\int_{\mathbb{R}^N} |\psi_R u_n|^{2^*} dx\Big)^{2/2^*}. \tag{52}$$

It is easy to verify that

$$\lim_{R\to\infty} \limsup_{n\to\infty} \left\{ \int_{\mathbb{R}^N} u_n^2 |\nabla \psi_R|^2 dx + 2\int_{\mathbb{R}^N} u_n \psi_R \nabla u_n \cdot \nabla \psi_R \, dx \right\} = 0.$$

Then from (52) we infer

$$\mu_\infty - \gamma_\infty \geq S\Big(\textstyle\sum_{i=1}^k h_i\Big) \nu_\infty^{2/2^*}. \tag{53}$$

Testing $J'(u_n)$ with $u_n \psi_R$ we obtain

(54)

$$\begin{aligned} 0 &= \lim_{n\to\infty} \langle J'(u_n), u_n \psi_R \rangle \\ &= \lim_{n\to\infty} \Bigg[\int_{\mathbb{R}^N} |\nabla u_n|^2 \psi_R + \int_{\mathbb{R}^N} u_n \nabla u_n \cdot \nabla \psi_R \\ &\qquad - \sum_{i=1}^k \int_{\mathbb{R}^N} h_i\Big(\frac{x-a_i}{|x-a_i|}\Big) \frac{\psi_R u_n^2}{|x-a_i|^2} dx - S(h_1, h_2, \ldots, h_k) \int_{\mathbb{R}^N} \psi_R |u_n|^{2^*} \Bigg]. \end{aligned}$$

If $|x| \geq R$ with R sufficiently large, then

$$\begin{aligned} &\left| \frac{h_i\big(\frac{x-a_i}{|x-a_i|}\big)}{|x-a_i|^2} - \frac{h_i\big(\frac{x}{|x|}\big)}{|x|^2} \right| \\ &\qquad \leq \left| \frac{h_i\big(\frac{x-a_i}{|x-a_i|}\big)}{|x-a_i|^2} - \frac{h_i\big(\frac{x-a_i}{|x-a_i|}\big)}{|x|^2} \right| + \frac{1}{|x|^2} \left| h_i\Big(\frac{x-a_i}{|x-a_i|}\Big) - h_i\Big(\frac{x}{|x|}\Big) \right| \\ &\qquad \leq \|h_i\|_{L^\infty(\mathbb{S}^{N-1})} \frac{|2a_i \cdot x - |a_i|^2|}{|x-a_i|^2 |x|^2} + \frac{\text{const}}{|x|^2} \left| \frac{x-a_i}{|x-a_i|} - \frac{x}{|x|} \right| \\ &\qquad \leq \|h_i\|_{L^\infty(\mathbb{S}^{N-1})} \frac{2|a_i||x| + |a_i|^2}{|x-a_i|^2 |x|^2} + \frac{\sqrt{2}\,\text{const}}{|x|^2} \sqrt{\frac{|x|\big(|x-a_i| - |x|\big) + a_i \cdot x}{|x-a_i||x|}} \\ &\qquad \leq \frac{\text{const}}{|x|^{5/2}}. \end{aligned}$$

Since, by Hölder's inequality,

$$\int_{\mathbb{R}^N} \frac{u_n^2 \psi_R}{|x|^{5/2}} dx \leq \left(\int_{|x|>R} u_n^{2^*} \right)^{2/2^*} \left(\int_{|x|>R} |x|^{-\frac{5}{4}N} \right)^{2/N} = O(R^{-1/2})$$

as $R \to +\infty$ uniformly with respect to n, we deduce that

$$\sum_{i=1}^{k} \int_{\mathbb{R}^N} h_i\Big(\frac{x-a_i}{|x-a_i|}\Big)\frac{\psi_R u_n^2}{|x-a_i|^2}dx = \int_{\mathbb{R}^N} \frac{\sum_{i=1}^{k} h_i\big(\frac{x}{|x|}\big)}{|x|^2}\psi_R u_n^2\, dx + O(R^{-1/2})$$

as $R \to +\infty$ uniformly with respect to n, hence

$$\lim_{R\to+\infty} \limsup_{n\to\infty} \int_{\mathbb{R}^N} \sum_{i=1}^{k} h_i\Big(\frac{x-a_i}{|x-a_i|}\Big)\frac{\psi_R u_n^2}{|x-a_i|^2}dx = \gamma_\infty. \tag{55}$$

Passing to lim-sup as $n \to \infty$ and limits as $R \to \infty$ in (54) and using (55), we obtain

$$\mu_\infty - \gamma_\infty = S(h_1, h_2, \dots, h_k)\nu_\infty. \tag{56}$$

From (53) and (56) we conclude that

$$\text{either} \quad \nu_\infty = 0 \quad \text{or} \quad \nu_\infty \geq \left(\frac{S(\sum_{i=1}^{k} h_i)}{S(h_1, h_2, \dots, h_k)}\right)^{N/2}. \tag{57}$$

In conclusion, we obtain

$$\begin{aligned} c &= J(u_n) - \frac{1}{2}\langle J'(u_n), u_n\rangle + o(1) = \frac{1}{N}S(h_1, h_2, \dots, h_k)\int_{\mathbb{R}^N} |u_n|^{2^*}dx + o(1) \\ &= \frac{S(h_1, h_2, \dots, h_k)}{N}\bigg\{\int_{\mathbb{R}^N} |u_0|^{2^*}dx + \sum_{i=1}^{k}\nu_{a_i} + \nu_\infty + \sum_{j\in\mathcal{J}}\nu_{x_j}\bigg\}. \end{aligned} \tag{58}$$

From (42), (58), (47), (51), and (57), we deduce that $\nu_{x_j} = 0$ for any $j \in \mathcal{J}$, $\nu_{a_i} = 0$ for any $i = 1, \dots, k$, and $\nu_\infty = 0$. Then, up to a subsequence, $u_n \to u_0$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. □

The Palais–Smale condition recovered in Theorem 4.1 and the interaction estimates proved in Proposition 3.4 are the key tools in proving Theorem 1.4.

Proof of Theorem 1.4. Let $\{u_n\}_n \subset \mathcal{D}^{1,2}(\mathbb{R}^N)$ be a minimizing sequence for (3). From the homogeneity of the quotient, we can require without restriction that $\|u_n\|_{L^{2^*}(\mathbb{R}^N)} = 1$, while from Ekeland's variational principle we can assume that the sequence satisfies the Palais–Smale property, i.e., for any $v \in \mathcal{D}^{1,2}(\mathbb{R}^N)$

$$\begin{aligned} \int_{\mathbb{R}^N} \nabla u_n(x)\cdot\nabla v(x)\, dx - \sum_{i=1}^{k}\int_{\mathbb{R}^N} \frac{h_i\big(\frac{x-a_i}{|x-a_i|}\big)u_n(x)}{|x-a_i|^2}v(x)\, dx \\ - S(h_1, h_2, \dots, h_k)\int_{\mathbb{R}^N} |u_n(x)|^{2^*-2}u_n(x)v(x)\, dx = o\big(\|v\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}\big). \end{aligned}$$

Hence $J'(u_n) \to 0$ in $(\mathcal{D}^{1,2}(\mathbb{R}^N))^\star$ and

$$J(u_n) \to \Big(\frac{1}{2} - \frac{1}{2^*}\Big) S(h_1, h_2, \dots, h_k) = \frac{1}{N} S(h_1, h_2, \dots, h_k).$$

From assumption (13) and Proposition 3.4, we infer that

$$S(h_1, h_2, \dots, h_k) < S(h_k). \tag{59}$$

From assumptions (11) and (14) we have

$$S(h_k) \le S(h_i) \quad \text{for all} \quad i = 1, \dots, k-1, \quad \text{and} \quad S(h_k) \le S\Big(\textstyle\sum_{i=1}^k h_i\Big), \tag{60}$$

while from assumption (12) and Theorem 1.3,

$$S(h_k) < S. \tag{61}$$

Gathering (59), (60), and (61), we finally have

$$S(h_1, h_2, \dots, h_k) < \min\Big\{S, S(h_1), \dots, S(h_k), S\Big(\textstyle\sum_{i=1}^k h_i\Big)\Big\}$$

and hence

$$\frac{1}{N} S(h_1, h_2, \dots, h_k) < \frac{1}{N} S(h_1, h_2, \dots, h_k)^{1-\frac{N}{2}} \left(\min\Big\{S, S(h_1), \dots, S(h_k), S\Big(\sum_{i=1}^k h_i\Big)\Big\}\right)^{N/2}.$$

From Theorem 4.1 we deduce that $\{u_n\}_{n\in\mathbb{N}}$ has a subsequence strongly converging to some $u_0 \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, which satisfies $J(u_0) = \frac{1}{N} S(h_1, h_2, \dots, h_k)$. In particular, u_0 achieves the infimum in (3). Since $J(u_0) = J(|u_0|)$, we have that also $|u_0|$ is a minimizer in (3) and then $v_0 = S(h_1, h_2, \dots, h_k)^{1/(2^*-2)}|u_0|$ is a nonnegative solution to equation (1). The maximum principle in $\mathbb{R}^N \setminus \{a_1, \dots, a_k\}$ implies that $v_0 > 0$ in $\mathbb{R}^N \setminus \{a_1, \dots, a_k\}$. □

Let us now consider the case of singularities generated by electric dipoles. In order to prove Corollary 1.5, we first need to establish the following monotonicity property of ground state levels with respect to the dipole magnitudes.

Lemma 4.2. *If $\Lambda_N^{-1} > \lambda_1 \ge \lambda_2 > 0$, $\mathbf{d}_1, \mathbf{d}_2 \in \mathbb{R}^N$ with $|\mathbf{d}_1| = |\mathbf{d}_2| = 1$, and $h_i(\theta) = \lambda_i \theta \cdot \mathbf{d}_i$ for $i = 1, 2$, then $S(h_2) \ge S(h_1)$.*

Proof. We first note that, by rotation invariance, for any $\lambda > 0$, $S(\lambda\theta \cdot \mathbf{d})$ does not depend on the unit vector $\mathbf{d}$, hence $S(h_2) = S(\tilde{h}_2)$ where $\tilde{h}_2(\theta) = \lambda_2\theta \cdot \mathbf{d}_1$.

From Theorem 1.3, there exists $w \in \mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\}$ such that

$$\frac{\int_{\mathbb{R}^N} \left[|\nabla w(x)|^2 - \frac{\lambda_2 x\cdot \mathbf{d}_1}{|x|^3} w^2(x)\right] dx}{\left(\int_{\mathbb{R}^N} |w(x)|^{2^*} dx\right)^{2/2^*}} = S(\tilde{h}_2). \tag{62}$$

We claim that the quotient on the left-hand side decreases after passing to polarization with respect to the half-space $H_{\mathbf{d}_1} := \{x \in \mathbb{R}^N : x \cdot \mathbf{d}_1 \geq 0\}$. We denote by $\sigma_{\mathbf{d}_1} : \mathbb{R}^N \to \mathbb{R}^N$ the reflection with respect to the boundary of $H_{\mathbf{d}_1}$, i.e., $\sigma_{\mathbf{d}_1}(x) = x - 2(x \cdot \mathbf{d}_1)\mathbf{d}_1$. The polarization of any measurable nonnegative function u with respect to $H_{\mathbf{d}_1}$ is defined as

$$u_{\mathbf{d}_1}(x) := \begin{cases} \max\{u(x), u(\sigma_{\mathbf{d}_1}(x))\}, & \text{if } x \in H_{\mathbf{d}_1}, \\ \min\{u(x), u(\sigma_{\mathbf{d}_1}(x))\}, & \text{if } x \in \mathbb{R}^N \setminus H_{\mathbf{d}_1}. \end{cases}$$

From well-known properties of polarization, we have

$$\|\nabla |w|_{\mathbf{d}_1}\|_{L^2(\mathbb{R}^N)} = \|\nabla w\|_{L^2(\mathbb{R}^N)} \quad \text{and} \quad \||w|_{\mathbf{d}_1}\|_{L^{2^*}(\mathbb{R}^N)} = \|w\|_{L^{2^*}(\mathbb{R}^N)}; \tag{63}$$

see [22, Propositions 22.2 and 22.5]. Moreover,

$$\begin{aligned} &\int_{\mathbb{R}^N} \frac{x\cdot \mathbf{d}_1}{|x|^3}\left(|w|^2_{\mathbf{d}_1} - w^2\right) dx \\ &\quad = \int_{H_{\mathbf{d}_1}} \frac{x\cdot \mathbf{d}_1}{|x|^3}\left(|w|^2_{\mathbf{d}_1} - |w|^2\right) dx + \int_{\mathbb{R}^N\setminus H_{\mathbf{d}_1}} \frac{x\cdot \mathbf{d}_1}{|x|^3}\left(|w|^2_{\mathbf{d}_1} - |w|^2\right) dx \geq 0 \end{aligned} \tag{64}$$

and, by the change of variables $x = \sigma_{\mathbf{d}_1}(y)$,

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{x\cdot \mathbf{d}_1}{|x|^3} |w|^2_{\mathbf{d}_1}(x)\, dx &= \int_{H_{\mathbf{d}_1}} \frac{x\cdot \mathbf{d}_1}{|x|^3} |w|^2_{\mathbf{d}_1}(x)\, dx + \int_{\mathbb{R}^N\setminus H_{\mathbf{d}_1}} \frac{x\cdot \mathbf{d}_1}{|x|^3} |w|^2_{\mathbf{d}_1}(x)\, dx \\ &= \int_{H_{\mathbf{d}_1}} \frac{x\cdot \mathbf{d}_1}{|x|^3} |w|^2_{\mathbf{d}_1}(x)\, dx - \int_{H_{\mathbf{d}_1}} \frac{y\cdot \mathbf{d}_1}{|y|^3} |w|^2_{\mathbf{d}_1}(\sigma_{\mathbf{d}_1}(y))\, dy \\ &= \int_{H_{\mathbf{d}_1}} \frac{x\cdot \mathbf{d}_1}{|x|^3} \left(|w|^2_{\mathbf{d}_1}(x) - |w|^2_{\mathbf{d}_1}(\sigma_{\mathbf{d}_1}(x))\right) dx \geq 0. \end{aligned} \tag{65}$$

From (62)–(65), we obtain

$$\begin{aligned} S(h_2) = S(\tilde{h}_2) &\geq \frac{\int_{\mathbb{R}^N} \left[|\nabla |w|_{\mathbf{d}_1}(x)|^2 - \frac{\lambda_2 x\cdot \mathbf{d}_1}{|x|^3} |w|^2_{\mathbf{d}_1}(x)\right] dx}{\left(\int_{\mathbb{R}^N} ||w|_{\mathbf{d}_1}(x)|^{2^*} dx\right)^{2/2^*}} \\ &\geq \frac{\int_{\mathbb{R}^N} \left[|\nabla |w|_{\mathbf{d}_1}(x)|^2 - \frac{\lambda_1 x\cdot \mathbf{d}_1}{|x|^3} |w|^2_{\mathbf{d}_1}(x)\right] dx}{\left(\int_{\mathbb{R}^N} ||w|_{\mathbf{d}_1}(x)|^{2^*} dx\right)^{2/2^*}} \geq S(h_1), \end{aligned}$$

thus proving the stated inequality. □

Proof of Corollary 1.5. Theorem 1.4 applies with $h_i(\theta) = \lambda_i \theta \cdot \mathbf{d}_i$. Indeed, (11) follows from Lemma 4.2, (13) from (15), and (14) from (16) and Lemma 4.2. □

5 The problem on bounded domains

In this section we discuss the existence of ground state solutions to (20) by analyzing the associated minimization problem (19) on a bounded smooth domain $\Omega \subset \mathbb{R}^N$, $N \geq 3$, containing points $a_1, \dots, a_k$. The corresponding functional is given by (66)

$$J_\Omega(v) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{1}{2}\sum_{i=1}^{k}\int_{\mathbb{R}^N} \frac{h_i\big(\frac{x-a_i}{|x-a_i|}\big)}{|x-a_i|^2} v^2(x)\,dx - \frac{S(h_1,h_2,\dots,h_k)}{2^*}\int_\Omega |v|^{2^*}dx.$$

By boundedness of the domain, minimizing sequences of (19) cannot lose mass at infinity. Hence, arguing as in Theorem 4.1, the following local Palais–Smale condition can be obtained.

Theorem 5.1. *Assume that* (18) *holds. Let* $\{u_n\}_{n\in\mathbb{N}} \subset H^1_0(\Omega)$ *be a Palais–Smale sequence for* J_Ω, *namely* $\lim_{n\to\infty} J_\Omega(u_n) = c$ *in* $\mathbb{R}$ *and* $\lim_{n\to\infty} J'_\Omega(u_n) = 0$ *in the dual space* $(H^1_0(\Omega))^\star$. *If*

$$c < c^*_\Omega = \frac{1}{N} S_\Omega(h_1,h_2,\dots,h_k)^{1-\frac{N}{2}} \min\Big\{S, S(h_1),\dots,S(h_k)\Big\}^{N/2},$$

then $\{u_n\}_{n\in\mathbb{N}}$ *has a converging subsequence.*

In a bounded domain, the comparison between ground state levels of dipole-type and multi-dipole-type problems is more delicate and requires an analysis of the concentration behavior of cut-off test functions. With this aim we need, besides the asymptotic behavior of functions ϕ_h at infinity, also the behavior of their gradient, which we are going to deduce from Green's representation formula and the following property of differentiability of *Newtonian potentials*.

Lemma 5.2. *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded smooth domain,* $g \in L^p(\Omega)$, *for every* $p \in [1,2)$, *and let* u *be the Newtonian potential of* g, *i.e.,*

$$u(x) = \frac{1}{N(2-N)\omega_N}\int_\Omega \frac{g(y)}{|x-y|^{N-2}}\,dy.$$

Then $u \in W^{1,q}(\mathbb{R}^N)$ *for all* $q \in \big(\frac{N}{N-2}, \frac{2N}{N-2}\big)$ *and the weak derivatives of* u *are given by*

$$\frac{\partial u}{\partial x_i}(x) = \frac{1}{N\omega_N}\int_\Omega \frac{g(y)(x_i-y_i)}{|x-y|^N}\,dy, \quad i = 1,\dots,N.$$

Proof. The proof can be obtained by approximation from [14, Lemma 4.1, p. 54] using the L^p inequalities for singular Riesz potentials proved in [20, Theorem 1, p. 119]. We refer to [9, Lemma A.1] for a detailed proof in the case $g \in L^2(\Omega)$, which can be followed step by step to yield Lemma 5.2. □

The above lemma and Green's representation formula yields the following estimate on the behavior of solutions ϕ_h as $|x| \to +\infty$.

Lemma 5.3. *For* $h \in C^1(\mathbb{S}^N)$ *verifying* $\mu_1(h) \geq -\left(\frac{N-2}{2}\right)^2 + 1$ *and* (9)*, let* $\phi_h \in \mathcal{D}^{1,2}(\mathbb{R}^N)$*,* $\phi_h \geq 0$*,* $\phi_h \not\equiv 0$*, be as in* (21) *and* (22)*. Then, for every* $\varepsilon > 0$*,*

$$|\nabla \phi_h(x)| = \begin{cases} O\big(|x|^{-\sigma_h - N+1}\big), & \text{if } \mu_1(h) < N-1, \\ O\big(|x|^{-N+\varepsilon}\big), & \text{if } \mu_1(h) \geq N-1, \end{cases} \quad \text{as } |x| \to +\infty. \tag{67}$$

Proof. Let $w_h(x) := |x|^{-(N-2)}\phi_h(x/|x|^2)$ be the Kelvin transform of ϕ_h. Then w_h solves

$$-\Delta w_h = g \quad \text{in } \mathbb{R}^N,$$

where

$$g(x) = \frac{h(x/|x|)}{|x|^2} w_h(x) + w_h^{2^*-1}(x).$$

Moreover, a direct calculation yields the following relation between the gradients of ϕ_h and of its Kelvin transform:

$$\nabla \phi_h(x) = |x|^{-N} \nabla w_h\Big(\frac{x}{|x|^2}\Big) - 2x|x|^{-N-2} x \cdot \nabla w_h\Big(\frac{x}{|x|^2}\Big) - (N-2)|x|^{-N} w_h\Big(\frac{x}{|x|^2}\Big)x. \tag{68}$$

From (23), $w_h(x) = O(|x|^{\sigma_h})$ as $x \to 0$, hence $g(x) = O(|x|^{\sigma_h - 2})$ as $x \to 0$. Therefore, from $\mu_1(h) \geq -\left(\frac{N-2}{2}\right)^2 + 1$, it follows that $g \in L^p(B(0,1))$ for every $p \in [1,2)$.

Green's representation formula yields

$$\begin{aligned} w_h(x) = & \frac{1}{N(N-2)\omega_N} \bigg[\int_{B(0,1)} \frac{g(y)}{|x-y|^{N-2}}\, dy + \int_{\partial B(0,1)} \frac{1}{|x-y|^{N-2}} \frac{\partial w_h}{\partial \nu}\, dS(y) \bigg] \\ & + \frac{1}{N\omega_N} \int_{\partial B(0,1)} \frac{w_h(y)}{|x-y|^N} (y-x)\cdot \nu(y)\, dS(y), \quad x \in B(0,1), \end{aligned} \tag{69}$$

where ω_N denotes the volume of the unit ball in $\mathbb{R}^N$, ν is the unit outward normal to $\partial B(0,1)$, and dS indicates the $(N-1)$-dimensional area element in $\partial B(0,1)$. It is easy to verify that the functions

$$x \mapsto \int_{\partial B(0,1)} \frac{1}{|x-y|^{N-2}} \frac{\partial w_h}{\partial \nu}\, dS(y), \quad x \mapsto \int_{\partial B(0,1)} \frac{w_h(y)}{|x-y|^N} (y-x)\cdot \nu(y)\, dS(y)$$

are of class $C^1(B(0,1))$. From Lemma 5.2, we have

$$\nabla\bigg(\frac{1}{N(N-2)\omega_N}\int_{B(0,1)}\frac{g(y)}{|x-y|^{N-2}}\,dy\bigg)=-\frac{1}{N\omega_N}\int_{B(0,1)}\frac{x-y}{|x-y|^{N}}\,g(y)\,dy,$$

and hence

$$\bigg|\nabla\bigg(\frac{1}{N(N-2)\omega_N}\int_{B(0,1)}\frac{g(y)}{|x-y|^{N-2}}\,dy\bigg)\bigg|\le \text{const}\int_{B(0,1)}\frac{|y|^{\sigma_h-2}}{|x-y|^{N-1}}\,dy. \tag{70}$$

If $\mu_1(h)<N-1$, i.e., $\sigma_h<1$, then

$$\bigg|\nabla\bigg(\frac{1}{N(N-2)\omega_N}\int_{B(0,1)}\frac{g(y)}{|x-y|^{N-2}}\,dy\bigg)\bigg|\le \text{const}\,f(x),$$

where

$$f(x)=\int_{\mathbb{R}^N}\frac{|y|^{\sigma_h-2}}{|x-y|^{N-1}}\,dy.$$

An easy scaling argument shows that $f(\alpha x)=\alpha^{\sigma_h-1}f(x)$ for all $\alpha>0$, hence $f(x)=|x|^{\sigma_h-1}f(e_1)$, where $e_1=(1,0,\dots,0)\in\mathbb{R}^N$. Then, if $\mu_1(h)<N-1$,

$$\bigg|\nabla\bigg(\frac{1}{N(N-2)\omega_N}\int_{B(0,1)}\frac{g(y)}{|x-y|^{N-2}}\,dy\bigg)\bigg|\le \text{const}\,|x|^{\sigma_h-1}. \tag{71}$$

If $\mu_1(h)\ge N-1$, i.e., $\sigma_h\ge 1$, we fix $0<\varepsilon<N-1$ and note that, from (70),

$$\bigg|\nabla\bigg(\frac{1}{N(N-2)\omega_N}\int_{B(0,1)}\frac{g(y)}{|x-y|^{N-2}}\,dy\bigg)\bigg|\le \text{const}\,k_\varepsilon(x),$$

where

$$k_\varepsilon(x)=\int_{\mathbb{R}^N}\frac{1}{|y|^{1+\varepsilon}|y-x|^{N-1}}\,dy.$$

An easy scaling argument shows that $k_\varepsilon(\alpha x)=\alpha^{-\varepsilon}k_\varepsilon(x)$ for all $\alpha>0$, hence $k_\varepsilon(x)=|x|^{-\varepsilon}k_\varepsilon(e_1)$. Then, if $\mu_1(h)\ge N-1$,

$$\bigg|\nabla\bigg(\frac{1}{N(N-2)\omega_N}\int_{B(0,1)}\frac{g(y)}{|x-y|^{N-2}}\,dy\bigg)\bigg|\le C(\varepsilon)|x|^{-\varepsilon}, \tag{72}$$

for some positive constant $C(\varepsilon)$ depending on ε (and also on N, h, and w_h). Representation (69), regularity of the boundary terms, and estimates (71) and (72) yield

$$\nabla w_h(x)=\begin{cases}O\big(|x|^{\sigma_h-1}\big), & \text{if } \mu_1(h)<N-1,\\ O\big(|x|^{-\varepsilon}\big), & \text{if } \mu_1(h)\ge N-1,\end{cases}\qquad \text{as } x\to 0. \tag{73}$$

Estimate (67) then follows from (73) and (68). □

Lemma 5.4. *Let* $j \in \{1, 2, \ldots, k\}$. *Then*

(74) $$S_\Omega(h_1, \ldots, h_k) \le S(h_j) + O(\mu^{2\sigma_{h_j}+N-2}) - \begin{cases} \mu^2 \|\phi_{h_j}\|^{-2}_{L^{2^*}(\mathbb{R}^N)} \big(\int_{\mathbb{R}^N} \phi^2_{h_j}(x)\big) \left(\sum_{i \neq j} \frac{h_i\left(\frac{a_j-a_i}{|a_j-a_i|}\right)}{|a_j-a_i|^2} + o(1) \right) \\ \qquad \text{if } \mu_1(h_j) > -\big(\frac{N-2}{2}\big)^2 + 1, \\ \mu^2 \|\phi_{h_j}\|^{-2}_{L^{2^*}(\mathbb{R}^N)} \big(\int_{|x|<\frac{1}{\mu}} \phi^2_{h_j}(x)\big) \left(\sum_{i \neq j} \frac{h_i\left(\frac{a_j-a_i}{|a_j-a_i|}\right)}{|a_j-a_i|^2} + o(1) \right) \\ \qquad \text{if } \mu_1(h_j) = -\big(\frac{N-2}{2}\big)^2 + 1, \end{cases}$$

as $\mu \to 0^+$.

Proof. Let ω be an open set such that $\overline{\omega} \subset \Omega$ and $a_j \in \omega$, and let $\psi \in C^\infty_c(\mathbb{R}^N)$ be a smooth cut-off function such that $0 \le \psi(x) \le 1$, $\psi \equiv 0$ in $\mathbb{R}^N \setminus \Omega$, $\psi \equiv 1$ in ω. Then $\psi(x)\phi^{h_j}_\mu(x - a_j) \in H^1_0(\Omega)$.

Let $0 < \varepsilon < (N-2)/2$. We claim that, as $\mu \to 0^+$, the following estimates hold:

(75) $$\int_{\mathbb{R}^N} |\nabla(\psi(x)\phi^{h_j}_\mu(x-a_j))|^2\,dx = \int_{\mathbb{R}^N} |\nabla\phi_{h_j}(x)|^2\,dx + O(\mu^{2\sigma_{h_j}+N-2}) + O\big(\mu^{N-2\varepsilon}\big),$$

(76) $$\int_{\mathbb{R}^N} \frac{h_j\big(\frac{x-a_j}{|x-a_j|}\big)}{|x-a_j|^2} |\psi(x)\phi^{h_j}_\mu(x-a_j)|^2\,dx = \int_{\mathbb{R}^N} \frac{h_j\big(\frac{x}{|x|}\big)}{|x|^2} |\phi_{h_j}(x)|^2\,dx + O(\mu^{2\sigma_{h_j}+N-2})$$

(77) $$\int_{\mathbb{R}^N} \frac{h_i\big(\frac{x-a_i}{|x-a_i|}\big)}{|x-a_i|^2} |\psi(x)\phi^{h_j}_\mu(x-a_j)|^2\,dx = \int_{\mathbb{R}^N} \frac{h_i\big(\frac{x+a_j-a_i}{|x+a_j-a_i|}\big)}{|x+a_j-a_i|^2} |\phi^{h_j}_\mu(x)|^2\,dx + O(\mu^{2\sigma_{h_j}+N-2}),$$

(78) $$\left(\int_{\mathbb{R}^N} |\psi(x)\phi^{h_j}_\mu(x-a_j)|^{2^*}\,dx\right)^{2/2^*} = \left(\int_{\mathbb{R}^N} |\phi_{h_j}(x)|^{2^*}\,dx\right)^{2/2^*} + O(\mu^{2\sigma_{h_j}+N-2}).$$

Let us prove (75). We have

$$\begin{aligned}(79)\qquad &\int_{\mathbb{R}^N} |\nabla(\psi(x)\phi_\mu^{h_j}(x-a_j))|^2\,dx \\ &= \int_{\mathbb{R}^N} \psi^2(x)|\nabla\phi_\mu^{h_j}(x-a_j)|^2\,dx + \int_{\mathbb{R}^N} |\phi_\mu^{h_j}(x-a_j)|^2|\nabla\psi(x)|^2\,dx \\ &\quad + 2\int_{\mathbb{R}^N} \psi(x)\phi_\mu^{h_j}(x-a_j)\nabla\psi(x)\cdot\nabla\phi_\mu^{h_j}(x-a_j)\,dx.\end{aligned}$$

In view of (67) we have

(80)

$$\begin{aligned}\bigg|\int_{\mathbb{R}^N} \psi^2(x)|\nabla\phi_\mu^{h_j}(x-a_j)|^2\,dx - \int_{\mathbb{R}^N} |\nabla\phi_\mu^{h_j}(x-a_j)|^2\,dx\bigg| \\ = \int_{\mu^{-1}((\mathbb{R}^N\setminus\omega)-a_j)} (1-\psi^2(\mu y+a_j))|\nabla\phi_{h_j}(y)|^2\,dy \\ = \begin{cases} O\big(\mu^{N-2+2\sigma_{h_j}}\big), & \text{if } \mu_1(h_j) < N-1, \\ O\big(\mu^{N-2\varepsilon}\big), & \text{if } \mu_1(h_j) \geq N-1,\end{cases}\end{aligned}$$

and

$$\begin{aligned}(81)\qquad \int_{\mathbb{R}^N} |\phi_\mu^{h_j}(x-a_j)|^2|\nabla\psi(x)|^2\,dx &\leq \operatorname{const}\mu^2 \int_{\mu^{-1}((\Omega\setminus\omega)-a_j)} |\phi_{h_j}(y)|^2\,dy \\ &\leq \operatorname{const}\mu^2 \int_{\mu^{-1}r}^{\mu^{-1}R} s^{2(-\sigma_{h_j}-N+2)+N-1}\,ds \\ &= O(\mu^{2\sigma_{h_j}+N-2}),\end{aligned}$$

where $r = \operatorname{dist}(a_j, \mathbb{R}^N\setminus\omega)$ and $R > 0$ is such that $\Omega \subset B(a_j, R)$. Similarly,

$$(82)\quad \int_{\mathbb{R}^N} \psi(x)\phi_\mu^{h_j}(x-a_j)\nabla\psi(x)\cdot\nabla\phi_\mu^{h_j}(x-a_j)\,dx = O(\mu^{2\sigma_{h_j}+N-2}) + O\big(\mu^{N-2\varepsilon}\big).$$

Estimate (75) follows from (80)–(82). The proof of (76)–(78) is analogous and is based on (24). From

$$\begin{aligned}&S_\Omega(h_1,\dots,h_k) \\ &\leq \frac{\int_{\mathbb{R}^N}|\nabla(\psi(x)\phi_\mu^{h_j}(x-a_j))|^2\,dx - \int_{\mathbb{R}^N}\frac{h_j\left(\frac{x-a_j}{|x-a_j|}\right)}{|x-a_j|^2}|\psi(x)\phi_\mu^{h_j}(x-a_j)|^2\,dx}{\left(\int_{\mathbb{R}^N}|\psi(x)\phi_\mu^{h_j}(x-a_j)|^{2^*}\,dx\right)^{2/2^*}} \\ &\quad - \sum_{i\neq j}\frac{\int_{\mathbb{R}^N}\frac{h_i\left(\frac{x-a_i}{|x-a_i|}\right)}{|x-a_i|^2}|\psi(x)\phi_\mu^{h_j}(x-a_j)|^2\,dx}{\left(\int_{\mathbb{R}^N}|\psi(x)\phi_\mu^{h_j}(x-a_j)|^{2^*}\,dx\right)^{2/2^*}},\end{aligned}$$

Lemmas 3.1 and 3.2, and (75)–(78), it follows that

$$S_\Omega(h_1,\dots,h_k) \le S(h_j) + O(\mu^{2\sigma_{h_j}+N-2}) + O\big(\mu^{N-2\varepsilon}\big) - \begin{cases} \mu^2 \|\phi_{h_j}\|^{-2}_{L^{2^*}(\mathbb{R}^N)} \big(\int_{\mathbb{R}^N} \phi^2_{h_j}(x)\big) \left(\sum_{i\neq j} \frac{h_i\left(\frac{a_j-a_i}{|a_j-a_i|}\right)}{|a_j-a_i|^2} + o(1)\right) \\ \qquad \text{if } \mu_1(h_j) > -\left(\frac{N-2}{2}\right)^2 + 1, \\ \mu^2 \|\phi_{h_j}\|^{-2}_{L^{2^*}(\mathbb{R}^N)} \big(\int_{|x|<\frac{1}{\mu}} \phi^2_{h_j}(x)\big) \left(\sum_{i\neq j} \frac{h_i\left(\frac{a_j-a_i}{|a_j-a_i|}\right)}{|a_j-a_i|^2} + o(1)\right) \\ \qquad \text{if } \mu_1(h_j) = -\left(\frac{N-2}{2}\right)^2 + 1, \end{cases}$$

as $\mu \to 0^+$. Since $0 < \varepsilon < (N-2)/2$, we have $O\big(\mu^{N-2\varepsilon}\big) = o(\mu^2)$, thus implying the validity of (74). □

Corollary 5.5. *Let* $j \in \{1,2,\dots,k\}$ *such that* $\mu_1(h_j) \ge -\left(\frac{N-2}{2}\right)^2 + 1$. *If*

$$\sum_{i\neq j} \frac{h_i\left(\frac{a_j-a_i}{|a_j-a_i|}\right)}{|a_j-a_i|^2} > 0,$$

then

$$S_\Omega(h_1,\dots,h_k) < S(h_j).$$

Proof. It follows directly from Lemma 5.4 after noting that if

$$\mu_1(h_j) > -\left(\frac{N-2}{2}\right)^2 + 1,$$

then $2\sigma_{h_j} + N - 2 > 2$ and hence $O(\mu^{2\sigma_{h_j}+N-2}) = o(\mu^2)$ as $\mu \to 0^+$, while if $\mu_1(h_j) = -\left(\frac{N-2}{2}\right)^2 + 1$, then $2\sigma_{h_j} + N - 2 = 2$ and hence $O(\mu^{2\sigma_{h_j}+N-2}) = o\big(\mu^2 \int_{|x|<1/\mu} \phi^2_{h_j}(x)\big)$. Taking μ sufficiently small, we obtain $S_\Omega(h_1,\dots,h_k) < S(h_j)$. □

Proof of Theorem 1.6. This follows from Theorem 5.1 and Corollary 5.5, arguing as in the proof of Theorem 1.4. □

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