

BALANCE LAWS WITH INTEGRABLE UNBOUNDED SOURCES*

GRAZIANO GUERRA[†], FRANCESCA MARCELLINI[†], AND VERONIKA SCHLEPER[‡]

Abstract. We consider the Cauchy problem for an $n \times n$ strictly hyperbolic system of balance laws $u_t + f(u)_x = g(x, u)$, $x \in \mathbb{R}$, $t > 0$, $\|g(x, \cdot)\|_{\mathbf{C}^2} \leq \tilde{M}(x) \in \mathbf{L}^1$, endowed with the initial data $u(0, \cdot) = u_o \in \mathbf{L}^1 \cap \mathbf{BV}(\mathbb{R}; \mathbb{R}^n)$. Each characteristic field is assumed to be genuinely nonlinear or linearly degenerate and nonresonant with the source, i.e., $|\lambda_i(u)| \geq c > 0$ for all $i \in \{1, \dots, n\}$. Assuming that the \mathbf{L}^1 norms of $\|g(x, \cdot)\|_{\mathbf{C}^1}$ and $\|u_o\|_{\mathbf{BV}(\mathbb{R})}$ are small enough, we prove the existence and uniqueness of global entropy solutions of bounded total variation extending the result in [D. Amadori, L. Gosse, and G. Guerra, *Arch. Ration. Mech. Anal.*, 162 (2002), pp. 327–366] to unbounded (in \mathbf{L}^∞) sources. Furthermore, we apply this result to the fluid flow in a pipe with discontinuous cross sectional area, showing existence and uniqueness of the underlying semigroup.

Key words. hyperbolic balance laws, unbounded sources, pipes with discontinuous cross sections

AMS subject classifications. 35L65, 35L45, 35L60

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1. Introduction. The recent literature offers several results on the properties of gas flows on networks. For instance, in [5, 6, 7, 9] the well posedness is established for the gas flow at a junction of n pipes with constant diameters. The equations governing the gas flow in a pipe with a smooth varying cross section $a(x)$ are given by (see, for instance, [12])

$$\begin{cases} \frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = -\frac{a'(x)}{a(x)} q, \\ \frac{\partial q}{\partial t} + \frac{\partial(\frac{q^2}{\rho} + p)}{\partial x} = -\frac{a'(x)}{a(x)} \frac{q^2}{\rho}, \\ \frac{\partial e}{\partial t} + \frac{\partial(\frac{q}{\rho}(e+p))}{\partial x} = -\frac{a'(x)}{a(x)} \left(\frac{q}{\rho}(e+p)\right). \end{cases}$$

The well posedness of this system is covered in [1], where an attractive unified approach to the existence and uniqueness theory for quasi-linear strictly hyperbolic systems of balance laws is proposed. The case of discontinuous cross sections is considered in the literature inserting a junction with suitable coupling conditions at the junction; see, for example, [5, 6, 10]. One way to obtain coupling conditions at the point of discontinuity of the cross section a is to take the limit of a sequence of Lipschitz continuous cross sections a^ε converging to a in \mathbf{L}^1 (for a different approach see, for instance, [8]). Unfortunately the results in [1] require \mathbf{L}^∞ bounds on the source term, and well posedness is proved on a domain depending on this \mathbf{L}^∞ bound. Since in the previous equations the source term contains the derivative of the cross sectional area one cannot hope to take the limit $a^\varepsilon \rightarrow a$. Indeed when a is discontinuous, the \mathbf{L}^∞ norm of $(a^\varepsilon)'$ goes to infinity. Therefore the purpose of this paper is to establish

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the result in [1] without requiring the \mathbf{L}^∞ bound. More precisely, we consider the Cauchy problem for the following $n \times n$ system of equations:

$$(1) \quad u_t + f(u)_x = g(x, u), \quad x \in \mathbb{R}, t > 0,$$

endowed with a (suitably small) initial datum

$$(2) \quad u(0, x) = u_o(x), \quad x \in \mathbb{R},$$

belonging to $\mathbf{L}^1 \cap \mathbf{BV}(\mathbb{R}; \mathbb{R}^n)$, the space of integrable functions with bounded total variation (Tot.Var.) in the sense of [13]. Here $u(t, x) \in \mathbb{R}^n$ is the vector of unknowns, and $f : \Omega \rightarrow \mathbb{R}^n$ denotes the fluxes, i.e., a smooth function defined on Ω which is an open neighborhood of the origin in \mathbb{R}^n . The system (1) is supposed to be strictly hyperbolic, with each characteristic field either genuinely nonlinear or linearly degenerate in the sense of Lax [11]. We recall that if zero does not belong to the domain Ω of definition of f , as in the case of gas dynamics away from vacuum, then a simple translation of the density vector u leads the problem back to the case $0 \in \Omega$. Concerning the source term g , we assume that it satisfies the following Caratheodory-type conditions:

(P₁) $g : \mathbb{R} \times \Omega \rightarrow \mathbb{R}^n$ is measurable with respect to (w.r.t.) x , for any $u \in \Omega$, and is \mathbf{C}^2 w.r.t. u , for any $x \in \mathbb{R}$;

(P₂) there exists an \mathbf{L}^1 function $\tilde{M}(x)$ such that $\|g(x, \cdot)\|_{\mathbf{C}^2} \leq \tilde{M}(x)$;

(P₃) there exists a function $\omega \in \mathbf{L}^1(\mathbb{R})$ such that $\|g(x, \cdot)\|_{\mathbf{C}^1} \leq \omega(x)$.

Remark 1. Note that the \mathbf{L}^1 norm of $\tilde{M}(x)$ does not have to be small but only bounded differently from $\omega(x)$, whose norm has to be small (see Theorem 1 below). Furthermore, condition (P₂) replaces the \mathbf{L}^∞ bound of the \mathbf{C}^2 norm of g in [1]. Finally observe that we do not require any \mathbf{L}^∞ bound on ω . On the other hand we will need the following observation: if we define

$$(3) \quad \tilde{\varepsilon}_h = \sup_{x \in \mathbb{R}} \int_0^h \omega(x + s) ds,$$

by absolute continuity one has $\tilde{\varepsilon}_h \rightarrow 0$ as $h \rightarrow 0$.

Moreover, we assume that a *nonresonance* condition holds; that is, the characteristic speeds of the system (1) are bounded away from zero:

$$(4) \quad \hat{\lambda} \geq |\lambda_i(u)| \geq c > 0 \quad \forall u \in \Omega, i \in \{1, \dots, n\}$$

for some $\hat{\lambda} > c > 0$.

Before stating the main theorem of this paper we need to recall the Riemann problem for the homogeneous system associated with (1):

$$(5) \quad u_t + f(u)_x = 0, \quad u(0, x) = \begin{cases} u_\ell & \text{if } x < x_o, \\ u_r & \text{if } x > x_o, \end{cases}$$

and we need the following two definitions.

DEFINITION 1. Given a BV function $u = u(x)$ and a point $\xi \in \mathbb{R}$, we denote by $U_{(u; \xi)}^\sharp$ the solution of the homogeneous Riemann problem (5) with data

$$(6) \quad u_\ell = \lim_{x \rightarrow \xi^-} u(x), \quad u_r = \lim_{x \rightarrow \xi^+} u(x), \quad x_o = \xi.$$

DEFINITION 2. Given a BV function $u = u(x)$ and a point $\xi \in \mathbb{R}$, we define $U_{(u;\xi)}^b$ as the solution of the linear hyperbolic Cauchy problem with constant coefficients

$$(7) \quad w_t + \tilde{A}w_x = \tilde{g}(x), \quad w(0, x) = u(x),$$

with $\tilde{A} = \nabla f(u(\xi))$, $\tilde{g}(x) = g(x, u(\xi))$.

THEOREM 1. Assume (P₁)–(P₃) and (4). If the norm of ω in $\mathbf{L}^1(\mathbb{R})$ is sufficiently small, there exist a constant $L > 0$, a closed domain \mathcal{D} of integrable functions with small total variation, and a unique semigroup $P : [0, +\infty) \times \mathcal{D} \rightarrow \mathcal{D}$ satisfying the following:

- (i) $P_0 u = u$, $P_{t+s} u = P_t \circ P_s u$ for all $u \in \mathcal{D}$ and $t, s \geq 0$;
- (ii) $\|P_s u - P_t v\|_{\mathbf{L}^1(\mathbb{R})} \leq L(|s - t| + \|u - v\|_{\mathbf{L}^1(\mathbb{R})})$ for all $u, v \in \mathcal{D}$ and $t, s \geq 0$;
- (iii) for all $u_o \in \mathcal{D}$ the function $u(t, \cdot) = P_t u_o$ is a weak entropy solution of the Cauchy problem (1)–(2) and for all $\tau \geq 0$ satisfies the following integral estimates:

(a) For every ξ , one has

$$(8) \quad \lim_{\theta \rightarrow 0} \frac{1}{\theta} \int_{\xi - \theta\lambda}^{\xi + \theta\lambda} \left| u(\tau + \theta, x) - U_{(u(\tau);\xi)}^b(\theta, x) \right| dx = 0.$$

(b) There exists a constant C such that, for every $a < \xi < b$ and $0 < \theta < \frac{b-a}{2\lambda}$, one has

$$(9) \quad \frac{1}{\theta} \int_{a+\theta\lambda}^{b-\theta\lambda} \left| u(\tau + \theta, x) - U_{(u(\tau);\xi)}^b(\theta, x) \right| dx \leq C \left[\text{Tot.Var.} \{u(\tau); (a, b)\} + \int_a^b \omega(x) dx \right]^2.$$

Conversely let $u : [0, T] \rightarrow \mathcal{D}$ be Lipschitz continuous as a map with values in $\mathbf{L}^1(\mathbb{R}, \mathbb{R}^n)$ and assume that $u(t, x)$ satisfies the integral conditions (a), (b) for almost all $\tau > 0$. Then $u(t, \cdot)$ coincides with a trajectory of the semigroup P .

The proof of this theorem is postponed to sections 3 and 4, where existence and uniqueness are proved. Before these technical details, we state the application of the above result to gas flow in section 2. Here we apply Theorem 1 to establish the existence and uniqueness of the semigroup related to pipes with discontinuous cross sections. Furthermore, we show that our approach yields the same semigroup as the approach followed in [7] in the special case of two connected pipes. The technical details of section 2 can be found at the end of the paper in section 5.

2. Application to gas dynamics. Theorem 1 provides an existence and uniqueness result for pipes with Lipschitz continuous cross section where the equations governing the gas flow are given by

$$(10) \quad \begin{cases} \frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = -\frac{a'(x)}{a(x)} q, \\ \frac{\partial q}{\partial t} + \frac{\partial(\frac{q^2}{\rho} + p)}{\partial x} = -\frac{a'(x)}{a(x)} \frac{q^2}{\rho}, \\ \frac{\partial e}{\partial t} + \frac{\partial(\frac{q}{\rho}(e+p))}{\partial x} = -\frac{a'(x)}{a(x)} \left(\frac{q}{\rho}(e+p) \right). \end{cases}$$

Here, as usual, ρ denotes the mass density, q is the linear momentum, e is the energy density, a is the area of the cross section of the pipe, and p is the pressure

which is related to the conserved quantities $u = (\rho, q, e)$ by the equations of state. In most situations, when two pipes of different size have to be connected, the length l of the adaptor is small compared to the length of the pipes. Therefore it is convenient to model these connections as pipes with a jump in the cross sectional area. These discontinuous cross sections, however, do not fulfil the requirements of Theorem 1. Nevertheless, we can use this theorem to derive the existence of solutions to the discontinuous problem by a limit procedure. To this end, we approximate the discontinuous function

$$(11) \quad a(x) = \begin{cases} a^-, & x < 0, \\ a^+, & x > 0, \end{cases}$$

by a sequence $a_l \in C^{0,1}(\mathbb{R}, \mathbb{R}^+)$ with the following properties:

$$(12) \quad a_l(x) = \begin{cases} a^-, & x < -\frac{l}{2}, \\ \varphi_l(x), & x \in [-\frac{l}{2}, \frac{l}{2}], \\ a^+, & x > \frac{l}{2}, \end{cases}$$

where φ_l is any smooth monotone function which connects the two strictly positive constants a^-, a^+ . One possible choice of the approximations a_l as well as the discontinuous pipe with cross section a are shown in Figure 1.

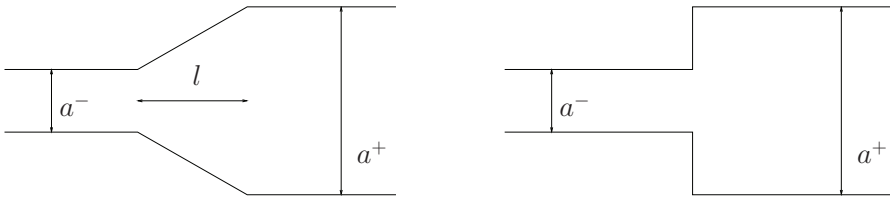


FIG. 1. Illustration of approximated and discontinuous cross sectional area.

With the positions $A_l(x) = \ln a_l(x)$ and $u = (\rho, q, e)$, we can write (10) in the following form:

$$(13) \quad u_t + f(u)_x = A'_l(x)g(u),$$

with the obvious definitions of the flux $f(u)$ and the source $g(u)$. Observe that $\|A'_l\|_{\mathbf{L}^1} = |A^+ - A^-| = |\ln a^+ - \ln a^-|$; hence, the smallness of $|a^+ - a^-|$ (away from zero) implies the smallness of $\|A'_l\|_{\mathbf{L}^1}$.

Let $\Phi(a, \bar{u})$ be the solution to the Cauchy problem

$$(14) \quad \begin{cases} \frac{d}{da}u(a) = [D_u f(u(a))]^{-1} g(u(a)), \\ u(0) = \bar{u}, \end{cases}$$

and define

$$(15) \quad \Psi(u^-, u^+) = u^+ - \Phi(A^+ - A^-, u^-).$$

DEFINITION 3. A solution to the Riemann problem with a junction in $x = 0$ described by the function Ψ

$$\begin{cases} u_t + f(u)_x = 0 & \text{for } x \neq 0, \\ u(0, x) = \begin{cases} u^\ell & \text{for } x < 0, \\ u_r & \text{for } x > 0, \end{cases} \end{cases}$$

$$(16) \quad \Psi(u(t, 0-), u(t, 0+)) = 0$$

is a function $u : [0, +\infty) \times \mathbb{R} \rightarrow \mathbb{R}^3$ such that the following hold:

- (i) the function $(t, x) \rightarrow u(t, x)$ is self-similar and in $x > 0$ coincides with a fan of Lax entropic waves with positive velocity outgoing from $(t, x) = (0, 0)$ while in $x < 0$ coincides with a fan of Lax entropic waves of negative velocity outgoing from $(t, x) = (0, 0)$;
- (ii) the traces $u(t, 0-), u(t, 0+)$ satisfy (16) for all $t > 0$.

With the help of Theorem 1 and the techniques used in its proof, we are now able to derive the following theorem (see also [8] for a similar result in the 2×2 case obtained with different methods).

THEOREM 2. *Let \bar{u} be a state satisfying (4) for the system (10). If $|a^+ - a^-|$ is sufficiently small, the semigroups $P^l : [0, +\infty) \times \mathcal{D}_l \rightarrow \mathcal{D}_l$, with $\mathcal{D}_l \subset \bar{u} + \mathbf{L}^1(\mathbb{R}, \mathbb{R}^3)$, related to the smooth section a_l converge to a unique semigroup P as $l \rightarrow 0$.*

Moreover, let U^\flat be defined for all ξ as in Definition 2 (here the hyperbolic flux f is the gas dynamic flux (10) and $\tilde{g} = 0$). Let U^\sharp be defined for all $\xi \neq 0$ as in Definition 1 and in the point $\xi = 0$ as the solution of the Riemann problem defined in Definition 3. Then the limit semigroup satisfies and is uniquely identified by the integral estimates (8), (9).

The proof is postponed to section 5.

Observe that the same theorem holds for the 2×2 isentropic system (see section 5)

$$(17) \quad \begin{cases} \frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = -\frac{a'(x)}{a(x)} q, \\ \frac{\partial q}{\partial t} + \frac{\partial(\frac{q^2}{\rho} + p)}{\partial x} = -\frac{a'(x)}{a(x)} \frac{q^2}{\rho}. \end{cases}$$

Also this system can be written in the form (13) with the positions $A_l(x) = \ln a_l(x)$, $u = (\rho, q)$ and the obvious definitions of $f(u)$ and $g(u)$.

In [7], 2×2 homogeneous conservation laws at a junction are considered for given admissible junction conditions. In the 2×2 case, the function Ψ in (15) depends only on four real variables:

$$(18) \quad \Psi((\rho^-, q^-), (\rho^+, q^+)) = (\rho^+, q^+) - \Phi(A^+ - A^-, (\rho^-, q^-)).$$

If $|A^+ - A^-|$ is sufficiently small, Ψ fulfils the determinant condition in [7, Proposition 2.2], that is,

$$(19) \quad \det \left[D_v \Psi(v, w)|_{(v,w)=(\bar{u}, \bar{u})} r_1(\bar{u}), D_w \Psi(v, w)|_{(v,w)=(\bar{u}, \bar{u})} r_2(\bar{u}) \right] \neq 0,$$

where $\bar{u} = (\bar{\rho}, \bar{q})$ is any subsonic state and $r_1(u), r_2(u)$ are the two right eigenvectors of the 2×2 matrix $Df(u)$.

Condition (16) is now given by

$$(20) \quad (\rho^+, q^+) - \Phi(A^+ - A^-, (\rho^-, q^-)) = 0.$$

Therefore we can apply the result in [7, Theorem 3.2] for the case of a junction with only two pipes with different cross sections and the function Ψ given by (20). This result can be stated in the following way.

THEOREM 3. *Given a subsonic state $\bar{u} = (\bar{\rho}, \bar{q})$, there exist a constant $L > 0$, a closed domain $\mathcal{D} \subset \bar{u} + \mathbf{L}^1(\mathbb{R}, \mathbb{R}^2)$ of functions with small total variation, and a unique semigroup $S : [0, +\infty) \times \mathcal{D} \rightarrow \mathcal{D}$ satisfying the following:*

- (i) $S_0u = u, S_{t+s}u = S_t \circ S_s u$ for all $u \in \mathcal{D}$ and $t, s \geq 0$;
- (ii) $\|S_s u - S_t v\|_{\mathbf{L}^1(\mathbb{R})} \leq L(|s - t| + \|u - v\|_{\mathbf{L}^1(\mathbb{R})})$ for all $u, v \in \mathcal{D}$ and $t, s \geq 0$;
- (iii) for all $u_o \in \mathcal{D}$ the function $u(t, \cdot) = S_t u_o \doteq (\rho, q)(t, \cdot)$ is a weak entropy solution to

$$(21) \quad \begin{cases} \frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0, \\ \frac{\partial q}{\partial t} + \frac{\partial(\frac{x^2}{2} + p)}{\partial x} = 0 \end{cases} \quad \text{for } x \neq 0,$$

equipped with the condition at the junction in $x = 0$

$$(22) \quad \Psi((\rho, q)(t, 0-); (\rho, q)(t, 0+)) = 0 \text{ for almost every } t > 0,$$

where Ψ is given by (20);

- (iv) if $u \in \mathcal{D}$ is piecewise constant, then for $t > 0$ sufficiently small, $S_t u$ coincides with the juxtaposition of the solutions to homogeneous Riemann problems centered at the points of jumps of u in $x \neq 0$ and the solution of the Riemann problem at the junction as defined in Definition 3 for the point $x = 0$.

The following proposition, whose proof can be found in section 5, implies that the semigroup obtained in [7] (and recalled in Theorem 3) and our limit semigroup coincide.

PROPOSITION 1. *The semigroup defined in [7] with the junction condition given by (20) satisfies the integral estimates (8), (9) with U^\sharp for the point $\xi = 0$ substituted by the solution of the Riemann problem described in Definition 3.*

Remark 2. Note that Proposition 1 justifies the coupling condition (20) as well as the condition used in [10] to study the Riemann problem for the gas flow through a nozzle.

3. Existence of BV entropy solutions. Throughout the next two sections, we follow the structure of [1]. We recall some definitions and notation in [1], and also the results which do not depend on the \mathbf{L}^∞ boundedness of the source term. We will prove only the results which in [1] depend on the \mathbf{L}^∞ bound using our weaker hypotheses.

3.1. The nonhomogeneous Riemann solver. Consider the stationary equations associated with (1), namely the system of ordinary differential equations

$$(23) \quad f(v(x))_x = g(x, v(x)).$$

For any $x_o \in \mathbb{R}, v \in \Omega$, consider the initial data

$$(24) \quad v(x_o) = v.$$

As in [1], we introduce a suitable approximation of the solutions to (23), (24). Thanks to (4), the map $u \mapsto f(u)$ is invertible inside some neighborhood of the origin; in this neighborhood, for small $h > 0$, we can define

$$(25) \quad \Phi_h(x_o, u) \doteq f^{-1} \left[f(u) + \int_0^h g(x_o + s, u) ds \right].$$

This map gives an approximation of the flow of (23) in the sense that

$$(26) \quad f(\Phi_h(x_o, u)) - f(u) = \int_0^h g(x_o + s, u) ds.$$

Throughout the paper we will use the Landau notation $\mathcal{O}(1)$ to indicate any function whose absolute value remains uniformly bounded, the bound depending only on f and $\|\tilde{M}\|_{\mathbf{L}^1}$.

LEMMA 1. *The function $\Phi_h(x_o, u)$ defined in (25) satisfies the following uniform (w.r.t. $x_o \in \mathbb{R}$ and to u in a suitable neighborhood of the origin) estimates:*

$$(27) \quad \begin{aligned} \|\Phi_h(x_o, \cdot)\|_{\mathbf{C}^2} &\leq \mathcal{O}(1), & \limsup_{h \rightarrow 0} \sup_{x_o \in \mathbb{R}} |\Phi_h(x_o, u) - u| &= 0, \\ \lim_{h \rightarrow 0} \|Id - D_u \Phi_h(x_o, u)\| &= 0. \end{aligned}$$

Proof. The Lipschitz continuity of f^{-1} and (3) imply

$$\begin{aligned} |\Phi_h(x_o, u) - u| &= |\Phi_h(x_o, u) - f^{-1}(f(u))| \leq \mathcal{O}(1) \left| \int_0^h g(x_o + s, u) ds \right| \\ &\leq \mathcal{O}(1) \left| \int_0^h \omega(x_o + s) ds \right| \leq \mathcal{O}(1) \tilde{\varepsilon}_h \xrightarrow{h \rightarrow 0} 0. \end{aligned}$$

Next we compute

$$\begin{aligned} D_u \Phi_h(x_o, u) &= Df^{-1} \left[f(u) + \int_0^h g(x_o + s, u) ds \right] \\ &\quad \cdot \left(Df(u) + \int_0^h D_u g(x_o + s, u) ds \right), \end{aligned}$$

which together with the identity $u = f^{-1}(f(u))$ implies

$$\begin{aligned} \|D_u \Phi_h(x_o, u) - Id\| &= \|D_u \Phi_h(x_o, u) - Df^{-1}(f(u))\| \\ &\leq \left\| Df^{-1} \left[f(u) + \int_0^h g(x_o + s, u) ds \right] - Df^{-1}(f(u)) \right\| \\ &\quad \cdot \left(\|Df(u)\| + \int_0^h \|D_u g(x_o + s, u)\| ds \right) \\ &\quad + \|Df^{-1}(f(u))\| \cdot \int_0^h \|D_u g(x_o + s, u)\| ds \\ &\leq \mathcal{O}(1) \tilde{\varepsilon}_h \xrightarrow{h \rightarrow 0} 0. \end{aligned}$$

Finally, denoting with D_i the partial derivative w.r.t. the i component of the state vector and by $\Phi_{h,\ell}$ the ℓ component of the vector Φ_h , we derive

$$\begin{aligned} D_i D_j \Phi_{h,\ell}(x_o, u) &= \sum_{k,k'} \left(D_k D_{k'} f_\ell^{-1} \left(f(u) + \int_0^h g(x_o + s, u) ds \right) \right. \\ &\quad \cdot \left(D_i f_k(u) + \int_0^h D_i g_k(x_o + s, u) ds \right) \\ &\quad \left. \cdot \left(D_j f_{k'}(u) + \int_0^h D_j g_{k'}(x_o + s, u) ds \right) \right) \end{aligned}$$

$$\begin{aligned}
 & + \sum_k D_k f_\ell^{-1} \left(f(u) + \int_0^h g(x_o + s, u) ds \right) \\
 & \cdot \left(D_j D_i f_k(u) + \int_0^h D_j D_i g_k(x_o + s, u) ds \right)
 \end{aligned}$$

so that

$$\|D^2 \Phi_h(x_o, u)\| \leq \mathcal{O}(1) \left(1 + \int_0^h \tilde{M}(x_o + s) ds \right) \leq \mathcal{O}(1) (1 + \|\tilde{M}\|_{L^1}) \leq \mathcal{O}(1). \quad \square$$

For any $x_o \in \mathbb{R}$ we consider the system (1), endowed with a Riemann initial datum:

$$(28) \quad u(0, x) = \begin{cases} u_\ell & \text{if } x < x_o, \\ u_r & \text{if } x > x_o. \end{cases}$$

If the two states u_ℓ, u_r are sufficiently close, let Ψ be the unique entropic homogeneous Riemann solver given by the map

$$u_r = \Psi(\boldsymbol{\sigma})(u_\ell) = \psi_n(\sigma_n) \circ \dots \circ \psi_1(\sigma_1)(u_\ell),$$

where $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n)$ denotes the (signed) wave strength vector in \mathbb{R}^n [11]. Here $\psi_j, j = 1, \dots, n$, is the shock–rarefaction curve of the j th family, parametrized as in [4] and related to the homogeneous system of conservation laws

$$(29) \quad u_t + f(u)_x = 0.$$

Observe that, due to (4), all the simple waves appearing in the solution of (29), (28) propagate with *nonzero* speed.

To take into account the effects of the source term, we consider a stationary discontinuity across the line $x = x_o$, that is, a wave whose speed is equal to 0, the so-called *zero-wave*. Now, given $h > 0$, we say that the particular Riemann solution

$$(30) \quad u(t, x) = \begin{cases} u_\ell & \text{if } x < x_o, \\ u_r & \text{if } x > x_o \end{cases} \quad \forall t \geq 0$$

is admissible if and only if $u_r = \Phi_h(x_o, u_\ell)$, where Φ_h is the map defined in (25). Roughly speaking, we require u_ℓ, u_r to be (approximately) connected by a solution of the stationary equations (23).

DEFINITION 4. *Let p be the number of waves with negative speed. Given $h > 0$ suitably small, $x_o \in \mathbb{R}$, we say that $u(t, x)$ is an h -Riemann solver for (1), (4), (28) if the following conditions hold:*

- (a) *there exist two states u^-, u^+ which satisfy $u^+ = \Phi_h(x_o, u^-)$;*
- (b) *on the set $\{t \geq 0, x < x_o\}$, $u(t, x)$ coincides with the solution to the homogeneous Riemann problem (29) with initial values u_ℓ, u^- and, on the set $\{t \geq 0, x > x_o\}$, with the solution to the homogeneous Riemann problem with initial values u^+, u_r ;*
- (c) *the Riemann problem between u_ℓ and u^- is solved only by waves with negative speed (i.e., of the families $1, \dots, p$);*
- (d) *the Riemann problem between u^+ and u_r is solved only by waves with positive speed (i.e., of the families $p + 1, \dots, n$).*

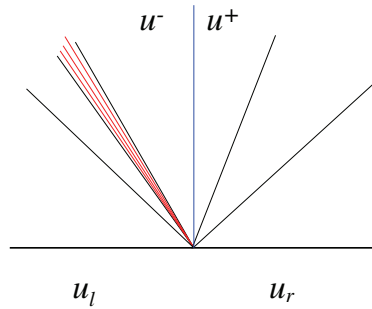


FIG. 2. Wave structure in an h -Riemann solver.

LEMMA 2. Let $x_o \in \mathbb{R}$ and let u, u_1, u_2 be three states in a suitable neighborhood of the origin. For h suitably small, one has

$$(31) \quad |\Phi_h(x_o, u) - u| = \mathcal{O}(1) \int_0^h \omega(x_o + s) ds,$$

$$(32) \quad |\Phi_h(x_o, u_2) - \Phi_h(x_o, u_1) - (u_2 - u_1)| = \mathcal{O}(1)|u_2 - u_1| \int_{x_o}^{x_o+h} \omega(s) ds.$$

The proof can be found in [1, Lemma 1].

LEMMA 3. For any $M > 0$ there exist $\delta'_1, h'_1 > 0$, depending only on M and the homogeneous system (29), such that for all maps $\phi \in \mathbf{C}^2(\mathbb{R}^n, \mathbb{R}^n)$ satisfying

$$\|\phi\|_{\mathbf{C}^2} \leq M, \quad |\phi(u) - u| \leq h'_1, \quad \|I - D\phi(u)\| \leq h'_1$$

and for all $u_\ell \in B(0, \delta'_1), u_r \in B(\phi(0), \delta'_1)$ there exist $n + 2$ states w_0, \dots, w_{n+1} and n wave sizes $\sigma_1, \dots, \sigma_n$, depending smoothly on u_ℓ, u_r which satisfy, with the previous notation, the following:

- (i) $w_0 = u_\ell, w_{n+1} = u_r$;
- (ii) $w_i = \Psi_i(\sigma_i)(w_{i-1}), i = 1, \dots, p$;
- (iii) $w_{p+1} = \phi(w_p)$;
- (iv) $w_{i+1} = \Psi_i(\sigma_i)(w_i), i = p + 1, \dots, n$.

Here p is the number of waves with negative velocity as in Definition 4.

The proof can be found in [1, Lemma 3].

The next lemma establishes existence and uniqueness for the h -Riemann solvers (see Figure 2).

LEMMA 4. There exist $\delta_1, h_1 > 0$ such that for any $x_o \in \mathbb{R}, h \in [0, h_1], u_\ell, u_r \in B(0, \delta_1)$ there exists a unique h -Riemann solver in the sense of Definition 4.

Proof. By Lemma 1, if $h_1 > 0$ is chosen sufficiently small, then for any $h \in [0, h_1], x_o \in \mathbb{R}$ the map $u \mapsto \Phi_h(x_o, u)$ meets the hypotheses of Lemma 3. Finally taking h_1 eventually smaller we can obtain that there exists $\delta_1 > 0$ such that $B(0, \delta_1) \subset B(0, \delta'_1) \cap B(\Phi_h(x_o, 0), \delta'_1)$ for any $h \in [0, h_1]$. \square

In what follows, E stands for the implicit function given by Lemmas 3 and 4:

$$\sigma \doteq E(h, u_\ell, u_r; x_o),$$

which plays the role of a wave-size vector. We recall that, by Lemma 3, E is a \mathbf{C}^2 function w.r.t. the variables u_ℓ, u_r and its \mathbf{C}^2 norm is bounded by a constant independent of h and x_o .

In contrast with the homogeneous case, the wave-size σ in the h -Riemann solver is not equivalent to the jump size $|u_\ell - u_r|$; an additional term appears, coming from the “Dirac source term” (see the special case $u_\ell = u_r$).

LEMMA 5. *Let δ_1, h_1 be the constants in Lemma 4. For $u_\ell, u_r \in B(0, \delta_1)$, $h \in [0, h_1]$, set $\sigma = E[h, u_\ell, u_r; x_o]$. Then it holds that*

$$(33) \quad \begin{aligned} |u_\ell - u_r| &= \mathcal{O}(1) \left(|\sigma| + \int_0^h \omega(x_o + s) ds \right), \\ |\sigma| &= \mathcal{O}(1) \left(|u_\ell - u_r| + \int_0^h \omega(x_o + s) ds \right). \end{aligned}$$

The proof can be found in [1, Lemma 4].

3.2. Existence of a Lipschitz semigroup of BV entropy solutions. Note that as shown in [1] we can identify the sizes of the zero-waves with the quantity

$$(34) \quad \sigma = \int_0^h \omega(jh + s) ds.$$

With this definition all the Glimm interaction estimates continue to hold with constants that depend only on f and $\|\tilde{M}\|_{\mathbf{L}^1}$; therefore the entire wave front tracking algorithm can be carried out obtaining the existence of ϵ, h -approximate solutions. Here we first give a precise definition of ϵ, h -approximate solutions and then outline the algorithm which allows us to obtain them. The detailed proof that the algorithm can be carried out for all times $t \geq 0$ is in [1].

DEFINITION 5. *Given $\epsilon, h > 0$, we say that a continuous map*

$$u^{\epsilon, h} : [0, +\infty) \rightarrow \mathbf{L}^1_{loc}(\mathbb{R}, \mathbb{R}^n)$$

is an ϵ, h -approximate solution of (1)–(2) if the following hold:

- *As a function of two variables, $u^{\epsilon, h}$ is piecewise constant with discontinuities occurring along finitely many straight lines in the (t, x) plane. Only finitely many wave-front interactions occur, each involving exactly two wave-fronts, and jumps can be of four types: shocks (or contact discontinuities), rarefaction waves, nonphysical waves, and zero-waves: $\mathcal{J} = \mathcal{S} \cup \mathcal{R} \cup \mathcal{NP} \cup \mathcal{Z}$.*
- *Along each shock (or contact discontinuity) $x_\alpha = x_\alpha(t)$, $\alpha \in \mathcal{S}$, the values of $u^- = u^{\epsilon, h}(t, x_\alpha^-)$ and $u^+ = u^{\epsilon, h}(t, x_\alpha^+)$ are related by $u^+ = \psi_{k_\alpha}(\sigma_\alpha)(u^-)$ for some $k_\alpha \in \{1, \dots, n\}$ and some wave-strength σ_α . If the k_α th family is genuinely nonlinear, then the Lax entropy admissibility condition $\sigma_\alpha < 0$ also holds. Moreover, one has*

$$|\dot{x}_\alpha - \lambda_{k_\alpha}(u^+, u^-)| \leq \epsilon,$$

where $\lambda_{k_\alpha}(u^+, u^-)$ is the speed of the shock front (or contact discontinuity) prescribed by the classical Rankine–Hugoniot conditions.

- *Along each rarefaction front $x_\alpha = x_\alpha(t)$, $\alpha \in \mathcal{R}$, one has $u^+ = \psi_{k_\alpha}(\sigma_\alpha)(u^-)$, $0 < \sigma_\alpha \leq \epsilon$, for some genuinely nonlinear family k_α . Moreover, we have $|\dot{x}_\alpha - \lambda_{k_\alpha}(u^+)| \leq \epsilon$.*
- *All nonphysical fronts $x = x_\alpha(t)$, $\alpha \in \mathcal{NP}$, travel at the same speed $\dot{x}_\alpha = \hat{\lambda} > \sup_{u, i} |\lambda_i(u)|$. Their total strength remains uniformly small, namely,*

$$\sum_{\alpha \in \mathcal{NP}} |u^{\epsilon, h}(t, x_{\alpha+}) - u^{\epsilon, h}(t, x_{\alpha-})| \leq \epsilon \quad \forall t > 0.$$

- The zero-waves are located at every point $x = jh, j \in (-\frac{1}{h\epsilon}, \frac{1}{h\epsilon}) \cap \mathbb{Z}$. Along a zero-wave located at $x_\alpha = j_\alpha h, \alpha \in \mathbb{Z}$, the values $u^- = u^{\epsilon,h}(t, x_\alpha^-)$ and $u^+ = u^{\epsilon,h}(t, x_\alpha^+)$ satisfy $u^+ = \Phi_h(x_\alpha, u^-)$ for all $t > 0$ except at the interaction points.
 - The total variation in space $\text{Tot.Var. } u^{\epsilon,h}(t, \cdot)$ is uniformly bounded for all $t \geq 0$. The total variation in time $\text{Tot.Var. } \{u^{\epsilon,h}(\cdot, x); [0, +\infty)\}$ is uniformly bounded for $x \neq jh, j \in \mathbb{Z}$.
- Finally, we require that $\|u^{\epsilon,h}(0, \cdot) - u_o\|_{L^1(\mathbb{R})} \leq \epsilon$.

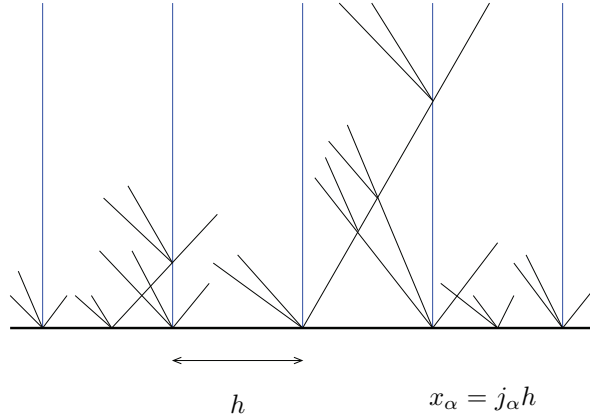


FIG. 3. Illustration of the wave front tracking algorithm.

Outline of the wave front tracking algorithm (see Figure 3). For notational convenience, we shall drop hereafter the ϵ, h superscripts, as there will be no ambiguity. Now, given any small value for these parameters, let us build up such an approximate solution.

- In order for our approximate solutions to be piecewise constant, we need to discretize the rarefactions; following [4], for a fixed small parameter δ , each rarefaction of size σ , when it is created for the first time, is divided into $m = \lceil \frac{\sigma}{\delta} \rceil + 1$ wave-fronts, each one with size $\sigma/m \leq \delta$.
- Given the initial data u_o , we can define a piecewise constant approximation $u(0, \cdot)$ satisfying the requirements of Definition 5; moreover, it is possible to guarantee that

$$\text{Tot.Var. } u(0, \cdot) \leq \text{Tot.Var. } u_o.$$

Then, $u(t, x)$ is constructed, for small t , by applying the h -Riemann solver at every point $x = jh$ with $j \in (-\frac{1}{h\epsilon}, \frac{1}{h\epsilon}) \cap \mathbb{Z}$ (even if in that point $u(0, \cdot)$ does not have a discontinuity), and by solving the remaining discontinuities in $u(0, \cdot)$ using a classical homogeneous Riemann solver for (29).

- At every interaction point, a new Riemann problem arises. Notice that because of their fixed speed, two nonphysical fronts cannot interact with each other; neither can the zero-waves. Moreover, by a slight modification of the speed of some waves (only among shocks, contact discontinuities, and rarefactions), it is possible to achieve the property that not more than two wave-fronts interact at a point.

After an interaction time, the number of wave-fronts may well increase. In order to prevent this number from becoming infinite in finite time, a particular treatment has

been proposed for waves whose strength is below a certain threshold value ρ by means of a *simplified Riemann solver* [4]. We shall use a similar trick in our construction, in the same spirit as in the homogeneous case.

Suppose that two wave-fronts of strengths σ, σ' interact at a given point (t, x) . If $x \neq x_\alpha$, for any $\alpha \in \mathcal{Z}$, we use the classical accurate or simplified homogeneous Riemann solver as in [4]. Assuming now that $x = x_\alpha = j_\alpha h$ with $\alpha \in \mathcal{Z}$, different situations can occur:

- If the wave approaching the zero-wave is physical and it holds that $|\sigma\sigma'| \geq \rho$, we use the (accurate) h -Riemann solver.
- If the wave incoming to the zero-wave is physical and it holds that $|\sigma\sigma'| < \rho$, we use a simplified one. Assume that the wave-front on the right is a zero-wave (the other case being similar): let $u_\ell, u_m = \psi_j(\sigma)(u_\ell), u_r = \Phi_h(x, u_m)$ be the states before the interaction.

We define the auxiliary states

$$\tilde{u}_m = \Phi_h(x, u_\ell), \quad \tilde{u}_r = \psi_j(\sigma)(\tilde{u}_m).$$

Then three fronts propagate after interaction: the zero-wave (u_ℓ, \tilde{u}_m) , the physical front $(\tilde{u}_m, \tilde{u}_r)$, and the nonphysical front (\tilde{u}_r, u_r) .

- Suppose now that the wave on the left belongs to \mathcal{NP} . Again we use a simplified solver: let $u_\ell, u_m, u_r = \Phi_h(x, u_m)$ be the states before the interaction, and define the new state $\tilde{u}_\ell = \Phi_h(x, u_\ell)$.

After the interaction time, only two fronts propagate: the zero-wave (u_ℓ, \tilde{u}_ℓ) and the nonphysical front (\tilde{u}_ℓ, u_r) .

Keeping $h > 0$ fixed, we are about to first let ϵ tend to zero. Hence we shall drop the superscript h for notational clarity.

THEOREM 4. *Let u^ϵ be a family of ϵ, h -approximate solutions of (1)–(2). There exists a subsequence u^{ϵ_i} converging as $i \rightarrow +\infty$ in $\mathbf{L}^1_{loc}((0, +\infty) \times \mathbb{R})$ to a function u which satisfies for any $\varphi \in \mathbf{C}^1_c((0, +\infty) \times \mathbb{R})$*

$$(35) \quad \int_0^\infty \int_{\mathbb{R}} [u\varphi_t + f(u)\varphi_x] dxdt + \int_0^\infty \sum_{j \in \mathbb{Z}} \varphi(t, jh) \left(\int_0^h g[jh + s, u(t, jh-)] ds \right) dt = 0.$$

Moreover, $\text{Tot.Var. } u(t, \cdot)$ is uniformly bounded and u satisfies the Lipschitz property

$$(36) \quad \int_{\mathbb{R}} |u(t', x) - u(t'', x)| dx \leq C'|t' - t''|, \quad t', t'' \geq 0.$$

Now we are in position to prove [1, Theorem 4] with our weaker hypotheses. As in [1] we can apply Helly’s compactness theorem to get a subsequence u^{h_i} converging to some function u in \mathbf{L}^1_{loc} whose total variation in space is uniformly bounded for all $t \geq 0$. Moreover, working as in [2, Proposition 5.1], one can prove that $u^{h_i}(t, \cdot)$ converges in \mathbf{L}^1 to $u(t, \cdot)$ for all $t \geq 0$.

THEOREM 5. *Let u^{h_i} be a subsequence of solutions to (35) with uniformly bounded total variation converging as $i \rightarrow +\infty$ in \mathbf{L}^1 to some function u . Then u is a weak solution to the Cauchy problem (1)–(2).*

We omit the proofs of Theorem 4 and 5 since they are very similar to the proofs of [1, Theorem 3 and 4]. We observe only that, in those proofs, the computations which rely on the \mathbf{L}^∞ bound on the source term have to be substituted by the following estimates.

- Concerning the proof of Theorem 4,

$$\begin{aligned} & \int_0^h |g(jh + s, u^\epsilon(t, jh-)) - g(jh + s, u(t, jh-))| \, ds \\ & \leq \int_0^h \|g(jh + s, \cdot)\|_{\mathbf{C}^1} \cdot |u^\epsilon(t, jh-) - u(t, jh-)| \, ds \\ & \leq \tilde{\epsilon}_h \cdot |u^\epsilon(t, jh-) - u(t, jh-)|. \end{aligned}$$

- Concerning the proof of Theorem 5,

$$\int_0^h |g(jh + s, u^h(t, jh-))| \, ds \leq \int_0^h \|g(jh + s, \cdot)\|_{\mathbf{C}^1} \, ds \leq \tilde{\epsilon}_h$$

and

$$\begin{aligned} & \int_0^h |g(jh + s, u^h(t, jh-)) - g(jh + s, u(t, jh + s))| \, ds \\ & \leq \int_0^h \|g(jh + s, \cdot)\|_{\mathbf{C}^1} \cdot |u^h(t, jh-) - u(t, jh + s)| \, ds \\ & \leq \tilde{\epsilon}_h \cdot \text{Tot.Var.} \{u^h(t, \cdot), [(j - 1)h, (j + 1)h]\} \\ & \quad + \int_{jh}^{(j+1)h} \omega(x) |u^h(t, x) - u(t, x)|. \end{aligned}$$

We observe that all the computations done in [1, section 4] rely on the source g only through the amplitude of the zero-waves and on the interaction estimates. Therefore the following two theorems still hold in the more general setting.

THEOREM 6. *If $\|\omega\|_{\mathbf{L}^1(\mathbb{R})}$ is sufficiently small, then for any (small) $h > 0$ there exist a nonempty closed domain \mathcal{D}_h and a unique uniformly Lipschitz semigroup $P^h : [0, +\infty) \times \mathcal{D}_h \rightarrow \mathcal{D}_h$ whose trajectories $u(t, \cdot) = P_t^h u_o$ solve (35) and are obtained as the limit of any sequence of ϵ, h -approximate solutions as ϵ tends to zero with fixed h . In particular the semigroups P^h satisfy for any $u_o, v_o \in \mathcal{D}_h, t, s \geq 0$*

$$(37) \quad P_0^h u_o = u_o, \quad P_t^h \circ P_s^h u_o = P_{s+t}^h u_o,$$

$$(38) \quad \|P_t^h u_o - P_s^h v_o\|_{\mathbf{L}^1(\mathbb{R})} \leq L \left[\|u_o - v_o\|_{\mathbf{L}^1(\mathbb{R})} + |t - s| \right]$$

for some $L > 0$, independent of h .

THEOREM 7. *If $\|\omega\|_{\mathbf{L}^1(\mathbb{R})}$ is sufficiently small, there exist a constant $L > 0$, a nonempty closed domain \mathcal{D} of integrable functions with small total variation, and a semigroup $P : [0, +\infty) \times \mathcal{D} \rightarrow \mathcal{D}$ with the following properties:*

- (i) $P_0 u = u$ for all $u \in \mathcal{D}$; $P_{t+s} u = P_t \circ P_s u$ for all $u \in \mathcal{D}, t, s \geq 0$.
- (ii) $\|P_s u - P_t v\|_{\mathbf{L}^1(\mathbb{R})} \leq L(|s - t| + \|u - v\|_{\mathbf{L}^1(\mathbb{R})})$ for all $u, v \in \mathcal{D}, t, s \geq 0$.
- (iii) For all $u_o \in \mathcal{D}$, the function $u(t, \cdot) = P_t u_o$ is a weak entropy solution of system (1).
- (iv) For all $h > 0$ small enough, $\mathcal{D} \subset \mathcal{D}_h$.
- (v) There exists a sequence of semigroups P^{h_i} such that $P_t^{h_i} u$ converges in \mathbf{L}^1 to $P_t u$ as $i \rightarrow +\infty$ for any $u \in \mathcal{D}$.

Remark 3. Looking at [1, (4.6)] and the proof of [1, Theorem 7] one realizes that the invariant domains \mathcal{D}_h and \mathcal{D} depend on the particular source term $g(x, u)$. On the other hand estimate [1, (4.4)] shows that all these domains contain all integrable functions with sufficiently small total variation. Since the bounds $\mathcal{O}(1)$ in Lemma 5 depend only on f and $\|\tilde{M}\|_{\mathbf{L}^1}$, also the constant C_1 in [1, (4.4)] depends only on f and $\|\tilde{M}\|_{\mathbf{L}^1}$. Therefore there exists $\tilde{\delta} > 0$ depending only on f and $\|\tilde{M}\|_{\mathbf{L}^1}$ such that \mathcal{D}_h and \mathcal{D} contain all integrable functions $u(x)$ with $\text{Tot.Var.}\{u\} \leq \tilde{\delta}$.

4. Uniqueness of BV entropy solutions. The proof of uniqueness in [1] strongly depends on the boundedness of the source; therefore we have to consider it in a more careful way.

4.1. Some preliminary results. As in [1] we shall make use of the following technical lemmas whose proofs can be found in [4].

LEMMA 6. *Let (a, b) a (possibly unbounded) open interval, and let $\hat{\lambda}$ be an upper bound for all wave speeds. If $\bar{u}, \bar{v} \in \mathcal{D}_h$, then for all $t \geq 0$ and $h > 0$ one has*

$$(39) \quad \int_{a+\hat{\lambda}t}^{b-\hat{\lambda}t} |(P_t^h \bar{u})(x) - (P_t^h \bar{v})(x)| dx \leq L \int_a^b |\bar{u}(x) - \bar{v}(x)| dx.$$

Remark 4. Observe that in the ϵ, h -approximate solutions, the waves do not travel faster than $\hat{\lambda}$; therefore the values of the function $P_t^h \bar{v}$ in the interval $(a + \hat{\lambda}t, b - \hat{\lambda}t)$ depend only on the values that \bar{v} assumes in the interval (a, b) . Therefore estimate (39) is only a localization of (38); in particular the Lipschitz constant L is the same and depends on g only through $\|\tilde{M}\|_{\mathbf{L}^1}$.

LEMMA 7. *Given any interval $I_0 = [a, b]$, define the interval of determinacy*

$$(40) \quad I_t = [a + \hat{\lambda}t, b - \hat{\lambda}t], \quad t < \frac{b-a}{2\hat{\lambda}}.$$

For every Lipschitz continuous map $w : [0, T] \mapsto \mathcal{D}_h$ and $h > 0$,

$$(41) \quad \begin{aligned} & \|w(t) - P_t^h w(0)\|_{\mathbf{L}^1(I_t)} \\ & \leq L \int_0^t \left\{ \liminf_{\eta \rightarrow 0} \frac{\|w(s+\eta) - P_\eta^h w(s)\|_{\mathbf{L}^1(I_{s+\eta})}}{\eta} \right\} ds. \end{aligned}$$

Remark 5. Lemmas 6 and 7 hold also substituting P^h with the operator P . In this case we obviously have to substitute the domains \mathcal{D}_h with the domain \mathcal{D} of Theorem 7.

Now let u_ℓ, u_r be two nearby states and $\lambda < \hat{\lambda}$; we consider the function

$$(42) \quad v(t, x) = \begin{cases} u_\ell & \text{if } x < \lambda t + x_o, \\ u_r & \text{if } x \geq \lambda t + x_o. \end{cases}$$

LEMMA 8. *Call $w(t, x)$ the self-similar solution given by the standard homogeneous Riemann solver with the Riemann data (28).*

(i) *In the general case, one has*

$$(43) \quad \frac{1}{t} \int_{-\infty}^{+\infty} |v(t, x) - w(t, x)| dx = \mathcal{O}(1)|u_\ell - u_r|.$$

(ii) Assuming the additional relations $u_r = R_i(\sigma)(u_\ell)$ and $\lambda = \lambda_i(u_r)$ for some $\sigma > 0$, $i = 1, \dots, n$, one has the sharper estimate

$$(44) \quad \frac{1}{t} \int_{-\infty}^{+\infty} |v(t, x) - w(t, x)| dx = \mathcal{O}(1)\sigma^2.$$

(iii) Let $u^* \in \Omega$ and call $\lambda_1^* < \dots < \lambda_n^*$ the eigenvalues of the matrix $A^* = \nabla f(u^*)$. If for some i it holds that $A^*(u_r - u_\ell) = \lambda_i^*(u_r - u_\ell)$ and $\lambda = \lambda_i^*$ in (42), then one has

$$(45) \quad \frac{1}{t} \int_{-\infty}^{+\infty} |v(t, x) - w(t, x)| dx = \mathcal{O}(1)|u_\ell - u_r| \left(|u_\ell - u^*| + |u^* - u_r| \right).$$

The proof can be found in [4, Lemma 9.1].

We now prove the next result, which is directly related to our h -Riemann solver.

LEMMA 9. Call $w(t, x)$ the self-similar solution given by the h -Riemann solver in x_o with the Riemann data (28).

(i) In the general case one has

$$(46) \quad \frac{1}{t} \int_{-\infty}^{+\infty} |v(t, x) - w(t, x)| dx = \mathcal{O}(1) \left(|u_\ell - u_r| + \int_0^h \omega(x_o + s) ds \right).$$

(ii) Assuming the additional relation

$$u_r = u_\ell + [\nabla f]^{-1}(u^*) \int_0^h g(x_o + s, u^*) ds$$

with $\lambda = 0$ in (42), one has the sharper estimate

$$(47) \quad \begin{aligned} & \frac{1}{t} \int_{-\infty}^{+\infty} |v(t, x) - w(t, x)| dx \\ &= \mathcal{O}(1) \left(\int_0^h \omega(x_o + s) ds + |u_\ell - u^*| \right) \cdot \int_0^h \omega(x_o + s) ds. \end{aligned}$$

Proof. Suppose $\lambda \geq 0$ (the other case being similar) and compute

$$\begin{aligned} & \frac{1}{t} \int_{-\infty}^{+\infty} |v(t, x) - w(t, x)| dx \\ &= \frac{1}{t} \int_{-\lambda t}^0 |u_\ell - w(t, x)| dx + \frac{1}{t} \int_0^{\lambda t} |u_\ell - w(t, x)| dx + \frac{1}{t} \int_{\lambda t}^{\lambda t} |u_r - w(t, x)| dx \\ &= \mathcal{O}(1) \sum_{\iota=1}^p |\sigma_\iota| + \frac{1}{t} \int_0^{\lambda t} |u_\ell - w(t, x)| dx + \mathcal{O}(1) \sum_{\iota=p+1}^n |\sigma_\iota| \\ &= \mathcal{O}(1)|\sigma| + \frac{1}{t} \int_0^{\lambda t} |u_\ell - w(t, x)| dx, \end{aligned}$$

where $\sigma = E[h, u_\ell, u_r; x_o]$. Since $w(t, x)$ is the solution to the h -Riemann problem, Lemma 5 implies $|u_\ell - w(t, x)| = \mathcal{O}(1)(|\sigma| + \int_0^h \omega(x_o + s) ds)$. Therefore again using Lemma 5 we get (i).

Let us now prove (ii). Setting $\lambda = 0$ in the previous computation gives

$$\frac{1}{t} \int_{-\infty}^{+\infty} |v(t, x) - w(t, x)| dx = \mathcal{O}(1)|\sigma|.$$

This leads to

$$\begin{aligned} |\sigma| &= \left| E[h, u_\ell, u_r; x_o] - E[h, u_\ell, \Phi_h(x_o, u_\ell); x_o] \right| \\ &= \mathcal{O}(1) |u_r - \Phi_h(x_o, u_\ell)|. \end{aligned}$$

To estimate this last term, we define $b(y, u) = f^{-1}(f(u) + y)$ and compute for some y_1, y_2

$$\begin{aligned} \left| u_\ell + [\nabla f]^{-1}(u^*)y_1 - b(y_2, u_\ell) \right| &\leq \mathcal{O}(1)|y_1| \cdot |u^* - u_\ell| + \mathcal{O}(1)|y_1 - y_2| \\ &\quad + \left| u_\ell + [\nabla f]^{-1}(u_\ell)y_2 - b(y_2, u_\ell) \right|. \end{aligned}$$

The function $z(y_2) = u_\ell + [\nabla f]^{-1}(u_\ell)y_2 - b(y_2, u_\ell)$ satisfies $z(0) = 0, D_{y_2}z(0) = 0$; hence we have the estimate

$$\left| u_\ell + [\nabla f]^{-1}(u^*)y_1 - b(y_2, u_\ell) \right| \leq \mathcal{O}(1) \left[|y_1| \cdot |u^* - u_\ell| + |y_1 - y_2| + |y_2|^2 \right].$$

If in this last expression we substitute

$$y_1 = \int_0^h g(x_o + s, u^*) ds, \quad y_2 = \int_0^h g(x_o + s, u_\ell) ds,$$

then we get

$$|u_r - \Phi_h(x_o, u_\ell)| = \mathcal{O}(1) \left(\int_0^h \omega(x_o + s) ds + |u_\ell - u^*| \right) \int_0^h \omega(x_o + s) ds,$$

which proves (47). \square

4.2. Characterization of the trajectories of the limit semigroup. In this section we are about to give necessary and sufficient conditions for a function $u(t, \cdot) \in \mathcal{D}$ to coincide with a semigroup's trajectory. To this end, we prove the uniqueness of the semigroup P and the convergence of all the sequence of semigroups P^h towards P as $h \rightarrow 0$.

Given a BV function $u = u(x)$ and a point $\xi \in \mathbb{R}$ let $w = U_{(u;\xi)}^b$ and \tilde{g} be the functions defined in Definition 2. We will need the following approximations of $U_{(u;\xi)}^b$. Let \bar{v} be a piecewise constant function. We will call w^h the solution of the following Cauchy problem:

$$(w^h)_t + \tilde{A}(w^h)_x = \sum_{j \in \mathbb{Z}} \delta(x - jh) \int_0^h \tilde{g}(jh + s) ds, \quad w^h(0, x) = \bar{v}(x).$$

Define $u^* \doteq u(\xi)$ and let $\lambda_i = \lambda_i(u^*), r_i = r_i(u^*),$ and $l_i = l_i(u^*)$ be, respectively, the i th eigenvalue and the i th right/left eigenvectors of the matrix \tilde{A} (see Definition 2).

As in [1], w and w^h have the following explicit representation:

$$(48) \quad \begin{aligned} w(t, x) &= \sum_{i=1}^n \left\{ \langle l_i, u(x - \lambda_i t) \rangle + \frac{1}{\lambda_i} \int_{x - \lambda_i t}^x \langle l_i, \tilde{g}(x') \rangle dx' \right\} r_i, \\ w^h(t, x) &= \sum_{i=1}^n \left\{ \langle l_i, \bar{v}(x - \lambda_i t) \rangle + \frac{1}{\lambda_i} \langle l_i, G^h(t, x) \rangle \right\} r_i, \end{aligned}$$

where the function $G^h(t, x) = \sum_{i=1}^n G_i^h(t, x)r_i$ is defined by

$$(49) \quad G_i^h(t, x) = \begin{cases} \sum_{j: jh \in (x - \lambda_i t, x)} \int_0^h \langle l_i, \tilde{g}(jh + s) \rangle ds & \text{if } \lambda_i > 0, \\ - \sum_{j: jh \in (x, x - \lambda_i t)} \int_0^h \langle l_i, \tilde{g}(jh + s) \rangle ds & \text{if } \lambda_i < 0. \end{cases}$$

Using (3) we can compute

$$(50) \quad \left| G_i^h(t, x) - \int_{x - \lambda_i t}^x \langle l_i, \tilde{g}(x') \rangle dx \right| = \mathcal{O}(1)\tilde{\varepsilon}_h.$$

Hence, for any $a, b \in \mathbb{R}$ with $a < b$, we have the error estimate

$$(51) \quad \int_{a + \lambda t}^{b - \lambda t} |w(t, x) - w^h(t, x)| dx \leq \mathcal{O}(1) \left[\int_a^b |u(x) - \bar{v}(x)| dx + (b - a)\tilde{\varepsilon}_h \right].$$

From (48), (49), it is easy to see that $w^h(t, x)$ is piecewise constant with discontinuities occurring along finitely many lines on compact sets in the (t, x) plane for $t \geq 0$. Only finitely many wave front interactions occur in a compact set, and jumps can be of two types: contact discontinuities or zero-waves. The zero-waves are located at the points $jh, j \in \mathbb{Z}$ and satisfy

$$(52) \quad w^h(t, jh+) - w^h(t, jh-) = [\nabla f]^{-1}(u^*) \int_{jh}^{(j+1)h} \tilde{g}(jh + s) ds.$$

Conversely a contact discontinuity of the i th family located at the point $x_\alpha(t)$ satisfies $\dot{x}_\alpha(t) = \lambda_i(u^*)$ and

$$(53) \quad w^h(t, x_\alpha(t)+) - w^h(t, x_\alpha(t)-) = \sigma r_i(u^*)$$

for some $\sigma \in \mathbb{R}$.

Now, we apply a technique introduced in [3] to state and prove the uniqueness result in our more general setting.

THEOREM 8. *Let $P : \mathcal{D} \times [0, +\infty) \rightarrow \mathcal{D}$ be the semigroup of Theorem 7, let $\hat{\lambda}$ be an upper bound for all wave speeds (see (4)), and let U^\sharp and U^\flat be the functions defined in Definitions 1 and 2. Then every trajectory $u(t, \cdot) = P_t u_0, u_0 \in \mathcal{D}$, satisfies the integral estimates (8) and (9) at every $\tau \geq 0$.*

Conversely let $u : [0, T] \rightarrow \mathcal{D}$ be Lipschitz continuous as a map with values in $\mathbf{L}^1(\mathbb{R}, \mathbb{R}^n)$ and assume that the conditions (8), (9) hold at almost every time τ . Then $u(t, \cdot)$ coincides with a trajectory of the semigroup P .

Remark 6. The difference w.r.t. the result in [1] is the presence of the integral in the right-hand side of formula (9). If ω is in \mathbf{L}^∞ , the integral can be bounded by $\mathcal{O}(1)(b - a)$ and we recover the estimates in [1]. Note also that the quantity

$$\mu((a, b)) = \text{Tot.Var.} \{u(\tau); (a, b)\} + \int_a^b \omega(x) dx$$

is a uniformly bounded finite measure and this is what is needed for proving the sufficiency part of the above theorem.

Proof of Theorem 8.

Part 1: Necessity. Given a semigroup trajectory $u(t, \cdot) = P_t \bar{u}$, $\bar{u} \in \mathcal{D}$ we now show that the conditions (8), (9) hold for every $\tau \geq 0$.

As in [1] we use the following notation. For fixed $h, \theta, \varepsilon > 0$ we define $J_t = J_t^- \cup J_t^o \cup J_t^+$ with

$$\begin{aligned} J_t^- &= \left(\xi - (2\theta - t + \tau) \hat{\lambda}, \xi - (t - \tau) \hat{\lambda} \right); \\ J_t^o &= \left[\xi - (t - \tau) \hat{\lambda}, \xi + (t - \tau) \hat{\lambda} \right]; \\ J_t^+ &= \left(\xi + (t - \tau) \hat{\lambda}, \xi + (2\theta - t + \tau) \hat{\lambda} \right). \end{aligned} \tag{54}$$

Let $U_{(u(\tau); \xi)}^{\sharp, \varepsilon}(\theta, x)$ be the piecewise constant function obtained from $U_{(u(\tau); \xi)}^\sharp(\theta, x)$ dividing the centered rarefaction waves in equal parts and replacing them by rarefaction fans containing wave-fronts whose strength is less than ε . Observe that

$$\frac{1}{t} \int_{-\infty}^{+\infty} \left| U_{(u(\tau); \xi)}^{\sharp, \varepsilon}(\theta, x) - U_{(u(\tau); \xi)}^\sharp(\theta, x) \right| dx = \mathcal{O}(1)\varepsilon. \tag{55}$$

Applying estimate (41) to the function $U_{(u(\tau); \xi)}^{\sharp, \varepsilon}$ we obtain

$$\begin{aligned} & \int_{J_{\tau+\theta}} \left| U_{(u(\tau); \xi)}^{\sharp, \varepsilon}(\theta, x) - \left(P_\theta^h U_{(u(\tau); \xi)}^{\sharp, \varepsilon}(0) \right)(x) \right| dx \\ & \leq L \int_\tau^{\tau+\theta} \liminf_{\eta \rightarrow 0} \frac{\left\| U_{(u(\tau); \xi)}^{\sharp, \varepsilon}(t - \tau + \eta) - P_\eta^h U_{(u(\tau); \xi)}^{\sharp, \varepsilon}(t - \tau) \right\|_{\mathbf{L}^1(J_{t+\eta})}}{\eta} dt. \end{aligned} \tag{56}$$

The discontinuities of $U_{(u(\tau); \xi)}^{\sharp, \varepsilon}$ do not cross the Dirac comb for almost all times $t \in (\tau, \tau + \theta)$. Therefore we compute for such a time t

$$\begin{aligned} & \frac{1}{\eta} \int_{J_{t+\eta}} \left| U_{(u(\tau); \xi)}^{\sharp, \varepsilon}(t - \tau + \eta, x) - \left(P_\eta^h U_{(u(\tau); \xi)}^{\sharp, \varepsilon}(t - \tau) \right)(x) \right| dx \\ & = \frac{1}{\eta} \int_{J_{t+\eta}^- \cup J_{t+\eta}^o \cup J_{t+\eta}^+} \left| U_{(u(\tau); \xi)}^{\sharp, \varepsilon}(t - \tau + \eta, x) - \left(P_\eta^h U_{(u(\tau); \xi)}^{\sharp, \varepsilon}(t - \tau) \right)(x) \right| dx. \end{aligned} \tag{57}$$

Define \mathcal{W}_t as the set of points in which $U_{(u(\tau); \xi)}^{\sharp, \varepsilon}(t - \tau)$ has a discontinuity, while \mathcal{Z}_h is the set of points in which the zero-waves are located. If η is sufficiently small, the solutions of the Riemann problems arising at the discontinuities of $U_{(u(\tau); \xi)}^{\sharp, \varepsilon}(t - \tau)$ do

not interact; therefore

$$\begin{aligned} & \frac{1}{\eta} \int_{J_{t+\eta}^o} \left| U_{(u(\tau);\xi)}^{\sharp,\varepsilon}(t-\tau+\eta,x) - \left(P_{\eta}^h U_{(u(\tau);\xi)}^{\sharp,\varepsilon}(t-\tau) \right) (x) \right| dx \\ &= \left(\sum_{x \in J_t^o \cap \mathcal{W}_t} + \sum_{x \in J_t^o \cap \mathcal{Z}_h} \right) \\ & \frac{1}{\eta} \int_{x-\hat{\lambda}\eta}^{x+\hat{\lambda}\eta} \left| U_{(u(\tau);\xi)}^{\sharp,\varepsilon}(t-\tau+\eta,y) - \left(P_{\eta}^h U_{(u(\tau);\xi)}^{\sharp,\varepsilon}(t-\tau) \right) (y) \right| dy. \end{aligned}$$

Note that the shocks are solved exactly both in $U_{(u(\tau);\xi)}^{\sharp,\varepsilon}$ and in $P^h U_{(u(\tau);\xi)}^{\sharp,\varepsilon}$; therefore they make no contribution in the summation. To estimate the approximate rarefactions we use the estimate (44); hence

$$\begin{aligned} & \sum_{x \in J_t^o \cap \mathcal{W}_t} \frac{1}{\eta} \int_{x-\hat{\lambda}\eta}^{x+\hat{\lambda}\eta} \left| U_{(u(\tau);\xi)}^{\sharp,\varepsilon}(t-\tau+\eta,x) - \left(P_{\eta}^h U_{(u(\tau);\xi)}^{\sharp,\varepsilon}(t-\tau) \right) (x) \right| dx \\ (58) \quad & \leq \mathcal{O}(1) \sum_{\substack{x \in J_t^o \cap \mathcal{W}_t \\ \text{rarefaction}}} |\sigma|^2 \leq \mathcal{O}(1)\varepsilon \text{Tot.Var.} \left\{ U_{(u(\tau);\xi)}^{\sharp,\varepsilon}(t-\tau); J_t^0 \right\} \\ & \leq \mathcal{O}(1)\varepsilon |u(\tau, \xi+) - u(\tau, \xi-)|. \end{aligned}$$

Concerning the zero-waves, recall that t is chosen such that $U_{(u(\tau);\xi)}^{\sharp,\varepsilon}$ is constant there, and P^h is the exact solution of an h -Riemann problem; hence we can apply (46) with $u_{\ell} = u_r$ and obtain

$$\begin{aligned} & \sum_{x \in J_t^o \cap \mathcal{Z}_h} \frac{1}{\eta} \int_{x-\hat{\lambda}\eta}^{x+\hat{\lambda}\eta} \left| U_{(u(\tau);\xi)}^{\sharp,\varepsilon}(t-\tau+\eta,x) - \left(P_{\eta}^h U_{(u(\tau);\xi)}^{\sharp,\varepsilon}(t-\tau) \right) (x) \right| dx \\ (59) \quad & \leq \mathcal{O}(1) \sum_{jh \in J_t^o} \int_0^h \omega(jh+s) ds \leq \mathcal{O}(1) \left(\int_{J_t^o} \omega(x) dx + \tilde{\varepsilon}_h \right). \end{aligned}$$

Finally using (59) and (58) we get in the end

$$\begin{aligned} (60) \quad & \frac{1}{\eta} \int_{J_{t+\eta}^o} \left| U_{(u(\tau);\xi)}^{\sharp,\varepsilon}(t-\tau+\eta,x) - \left(P_{\eta}^h U_{(u(\tau);\xi)}^{\sharp,\varepsilon}(t-\tau) \right) (x) \right| dx \\ & = \mathcal{O}(1) \left\{ \int_{J_t^o} \omega(x) dx + \tilde{\varepsilon}_h + \varepsilon \right\}. \end{aligned}$$

Moreover, following the same steps as before and using (43) and (46) with $u_{\ell} = u_r$ we get

$$\begin{aligned} (61) \quad & \frac{1}{\eta} \int_{J_{t+\eta}^+} \left| U_{(u(\tau);\xi)}^{\sharp,\varepsilon}(t-\tau+\eta,x) - \left(P_{\eta}^h U_{(u(\tau);\xi)}^{\sharp,\varepsilon}(t-\tau) \right) (x) \right| dx \\ & = \mathcal{O}(1) \left\{ \int_{J_t^+} \omega(x) dx + \tilde{\varepsilon}_h \right\}. \end{aligned}$$

Note that here there is no total variation of $U_{(u(\tau);\xi)}^{\sharp,\varepsilon}$ since in J_t^+ it is constant. A similar estimate holds for the interval $J_{t+\eta}^-$. Putting together (57), (60), (61), one has

$$\begin{aligned} \frac{1}{\eta} \int_{J_{t+\eta}} \left| U_{(u(\tau);\xi)}^{\sharp,\varepsilon}(t - \tau + \eta, x) - \left(P_{\eta}^h U_{(u(\tau);\xi)}^{\sharp,\varepsilon}(t - \tau) \right) (x) \right| dx \\ = \mathcal{O}(1) \left(\int_{J_{\tau}} \omega(x) dx + \tilde{\varepsilon}_h + \varepsilon \right). \end{aligned}$$

Hence, setting $\tilde{v} = U_{(u(\tau);\xi)}^{\sharp,\varepsilon}(0) = U_{(u(\tau);\xi)}^{\sharp}(0)$ by (56), we have

$$(62) \quad \int_{J_{\tau+\theta}} \left| U_{(u(\tau);\xi)}^{\sharp,\varepsilon}(\theta, x) - (P_{\theta}^h \tilde{v})(x) \right| dx = \mathcal{O}(1)\theta \left(\int_{J_{\tau}} \omega(x) dx + \tilde{\varepsilon}_h + \varepsilon \right).$$

Finally we take the sequence P^{h_i} converging to P . Using (39) we have

$$(63) \quad \begin{aligned} \frac{1}{\theta} \left\| P_{\theta}^{h_i} u(\tau) - P_{\theta}^{h_i} \tilde{v} \right\|_{\mathbf{L}^1(J_{\tau+\theta})} &\leq \frac{1}{\theta} L \|u(\tau) - \tilde{v}\|_{\mathbf{L}^1(J_{\tau})} \\ &= \frac{L}{\theta} \int_{\xi-2\hat{\lambda}\theta}^{\xi+2\hat{\lambda}\theta} |u(\tau, x) - \tilde{v}(x)| dx \\ &\doteq \bar{\varepsilon}_{\theta}, \end{aligned}$$

where $\bar{\varepsilon}_{\theta}$ tends to zero as θ tends to zero due to the fact that $u(\tau)$ has a right and left limit at any point: for any given $\epsilon > 0$, if θ is sufficiently small, $|u(\tau, x) - \tilde{v}(x)| = |u(\tau, x) - u(\tau, \xi-)| \leq \epsilon$ for $x \in (\xi - 2\hat{\lambda}\theta, \xi)$.

Therefore by (55), (62) we derive

$$\begin{aligned} \frac{1}{\theta} \int_{\xi-\theta\hat{\lambda}}^{\xi+\theta\hat{\lambda}} \left| u(\tau + \theta, x) - U_{(u(\tau);\xi)}^{\sharp}(\theta, x) \right| dx \\ = \frac{\left\| P_{\theta} u(\tau) - P_{\theta}^{h_i} u(\tau) \right\|_{\mathbf{L}^1(\mathbb{R})}}{\theta} + \bar{\varepsilon}_{\theta} + \mathcal{O}(1) \left[\int_{J_{\tau}} \omega(x) dx + \tilde{\varepsilon}_{h_i} \right]. \end{aligned}$$

The left-hand side of the previous estimate does not depend on ε and h_i ; hence

$$\frac{1}{\theta} \int_{\xi-\theta\hat{\lambda}}^{\xi+\theta\hat{\lambda}} \left| u(\tau + \theta, x) - U_{(u(\tau);\xi)}^{\sharp}(\theta, x) \right| dx = \mathcal{O}(1) \int_{J_{\tau}} \omega(x) dx + \bar{\varepsilon}_{\theta}.$$

Note that the intervals J_{τ} depend on θ (see 54). So taking the limit as $\theta \rightarrow 0$ in the previous estimate yields (8).

To prove (9) let $\theta > 0$ and let a point (τ, ξ) be given together with an open interval (a, b) containing ξ . Fix $\varepsilon > 0$ and choose a piecewise constant function $\bar{v} \in \mathcal{D}$ satisfying $\bar{v}(\xi) = u(\tau, \xi)$ together with

$$(64) \quad \int_a^b |\bar{v}(x) - u(\tau, x)| dx \leq \varepsilon, \quad \text{Tot.Var. } \{\bar{v}; (a, b)\} \leq \text{Tot.Var. } \{u(\tau); (a, b)\}.$$

Now let w^h be defined by (48) ($u^* = \bar{v}(\xi) = u(\tau, \xi)$). From (51), (64) we have the estimate

$$(65) \quad \int_{a+\theta\hat{\lambda}}^{b-\theta\hat{\lambda}} \left| U_{(u(\tau);\xi)}^{\flat}(\theta, x) - w^h(\theta, x) \right| dx \leq \mathcal{O}(1) (\varepsilon + \tilde{\varepsilon}_h(b-a)).$$

Using (40), (41) we get

$$(66) \quad \int_{a+\theta\hat{\lambda}}^{b-\theta\hat{\lambda}} |w^h(\theta, x) - (P_\theta^h w^h(0))(x)| dx \leq L \int_\tau^{\tau+\theta} \liminf_{\eta \rightarrow 0} \frac{\|w^h(t-\tau+\eta) - P_\eta^h w^h(t-\tau)\|_{\mathbf{L}^1(\tilde{I}_{t+\eta})}}{\eta} dt,$$

where we have defined $\tilde{I}_{t+\eta} = I_{t-\tau+\eta}$. Let $t \in (\tau, \tau+\theta)$ be a time for which there is no interaction in w^h ; in particular, discontinuities which travel with a nonzero velocity do not cross the Dirac comb (this happens for almost all t). We observe that by the explicit formula (48)

$$(67) \quad \text{Tot.Var.} \{w^h(t-\tau); \tilde{I}_t\} = \mathcal{O}(1) \left(\text{Tot.Var.} \{\bar{v}; (a, b)\} + \int_a^b \omega(x) dx + \tilde{\varepsilon}_h \right),$$

$$(68) \quad |w^h(t-\tau, x) - \bar{v}(\xi)| = \mathcal{O}(1) \left(\text{Tot.Var.} \{\bar{v}; (a, b)\} + \int_a^b \omega(x) dx + \tilde{\varepsilon}_h \right).$$

As before for η sufficiently small we can split homogeneous and zero-waves

$$(69) \quad \frac{1}{\eta} \int_{\tilde{I}_{t+\eta}} |w^h(t-\tau+\eta, x) - (P_\eta^h w^h(t-\tau))(x)| dx = \left(\sum_{x \in \tilde{I}_t \cap \mathcal{W}_t} + \sum_{x \in \tilde{I}_t \cap \mathcal{Z}_h} \right) \frac{1}{\eta} \int_{x-\hat{\lambda}\eta}^{x+\hat{\lambda}\eta} |w^h(t-\tau+\eta, x) - (P_\eta^h w^h(t-\tau))(x)| dx.$$

The homogeneous waves in w^h satisfy (53), with $\bar{v}(\xi)$ in place of u^* ; hence we can apply (45) which together with (67), (68) leads to

$$\begin{aligned} & \sum_{x \in \tilde{I}_t \cap \mathcal{W}_t} \frac{1}{\eta} \int_{x-\hat{\lambda}\eta}^{x+\hat{\lambda}\eta} |w^h(t-\tau+\eta, x) - (P_\eta^h w^h(t-\tau))(x)| dx \\ & \leq \mathcal{O}(1) \sum_{x \in \tilde{I}_t \cap \mathcal{W}_t} |\Delta w^h(t-\tau, x)| \left(\text{Tot.Var.} \{\bar{v}; (a, b)\} + \int_a^b \omega(x) dx + \tilde{\varepsilon}_h \right) \\ & \leq \mathcal{O}(1) \text{Tot.Var.} \{w^h(t-\tau), \tilde{I}_t\} \left(\text{Tot.Var.} \{\bar{v}; (a, b)\} + \int_a^b \omega(x) dx + \tilde{\varepsilon}_h \right) \\ & \leq \mathcal{O}(1) \left(\text{Tot.Var.} \{\bar{v}; (a, b)\} + \int_a^b \omega(x) dx + \tilde{\varepsilon}_h \right)^2, \end{aligned}$$

where $\Delta w^h(t-\tau, x)$ denotes the jump of $w^h(t-\tau)$ at x .

The zero-waves in w^h satisfy (52); hence we can apply (47) which together with

(68) leads to

$$\begin{aligned} & \sum_{x \in \tilde{I}_t \cap \mathcal{Z}_h} \frac{1}{\eta} \int_{x-\hat{\lambda}\eta}^{x+\hat{\lambda}\eta} |w^h(t-\tau+\eta, x) - (P_\eta^h w^h(t-\tau))(x)| dx \\ & \leq \mathcal{O}(1) \sum_{x \in \tilde{I}_t \cap \mathcal{Z}_h} \int_0^h \omega(x+s) ds \cdot \left(\text{Tot.Var.} \{ \bar{v}; (a, b) \} + \int_a^b \omega(x) dx + \tilde{\varepsilon}_h \right) \\ & \leq \mathcal{O}(1) \left(\int_{\tilde{I}_t} \omega(x) dx + \tilde{\varepsilon}_h \right) \left(\text{Tot.Var.} \{ \bar{v}; (a, b) \} + \int_a^b \omega(x) dx + \tilde{\varepsilon}_h \right) \\ & \leq \mathcal{O}(1) \left(\text{Tot.Var.} \{ \bar{v}; (a, b) \} + \int_a^b \omega(x) dx + \tilde{\varepsilon}_h \right)^2. \end{aligned}$$

Now let P^{h_i} be the subsequence converging to P . Since $w^h(0) = \bar{v}$ using (65), (66), (64), and the last estimates we get

$$\begin{aligned} & \frac{1}{\theta} \int_{a+\theta\hat{\lambda}}^{b-\theta\hat{\lambda}} |u(\tau+\theta, x) - U_{(u(\tau); \xi)}^b(\theta, x)| dx \\ & \leq \frac{\|P_\theta u(\tau) - P_\theta^{h_i} u(\tau)\|_{\mathbf{L}^1(\mathbb{R})}}{\theta} + L \frac{\|u(\tau) - \bar{v}\|_{\mathbf{L}^1(\mathbb{R})}}{\theta} \\ & \quad + \mathcal{O}(1) \left\{ \frac{\varepsilon + \tilde{\varepsilon}_{h_i} \cdot (b-a)}{\theta} + \left(\text{Tot.Var.} \{ \bar{v}; (a, b) \} + \int_a^b \omega(x) dx + \tilde{\varepsilon}_{h_i} \right)^2 \right\}. \end{aligned}$$

So for $\varepsilon, h_i \rightarrow 0$ we obtain the desired inequality.

Part 2: Sufficiency. By Remark 5 we can apply (41) to P , and hence the proof for the homogeneous case presented in [4], which relies on the property recalled in Remark 6, can be followed exactly for our case; hence it will not be repeated here. \square

Proof of Theorem 1. It is now a direct consequence of Theorems 7 and 8. \square

5. Proofs related to section 2. Consider the equation

$$u_t + f(u)_x = a'g(u)$$

for some $a \in \mathbf{BV}$. Equation (10) is comprised in this setting with the substitution $a \mapsto \ln a$. For this kind of equation we consider the exact stationary solutions besides the approximated ones considered in (25). Therefore call $\Phi(a, \bar{u})$ the solution of the following Cauchy problem:

$$(70) \quad \begin{cases} \frac{d}{da} u(a) = [D_u f(u(a))]^{-1} g(u(a)), \\ u(0) = \bar{u}. \end{cases}$$

If a is sufficiently small, the map $u \mapsto \Phi(a, u)$ satisfies Lemma 3. We call a -Riemann problem the Cauchy problem

$$(71) \quad \begin{cases} u_t + f(u)_x = a'g(u), \\ (a, u)(0, x) = \begin{cases} (a^-, u_l) & \text{if } x < 0, \\ (a^+, u_r) & \text{if } x > 0. \end{cases} \end{cases}$$

We define its solution as the function described in Definition 4 using the map $\Phi(a^+ - a^-, u^-)$ instead of the Φ_h used there (which also coincides with the solution in Definition 3 with Ψ given by (15) and the substitution $a^\pm \rightarrow A^\pm$). This solution exists if $|a^+ - a^-|$ is sufficiently small because of Lemma 3. Observe that if $a^+ = a^-$, the a -Riemann solver coincides with the usual homogeneous Riemann solver.

DEFINITION 6. *Given a function $u \in \mathbf{BV}$ and two states a^-, a^+ , we define $\bar{U}_u^\sharp(t, x)$ as the solution of the a -Riemann solver (71) with $u_l = u(0^-)$ and $u_r = u(0^+)$.*

Proof of Theorem 2. Since $\|a'_l\|_{\mathbf{L}^1} = |a^+ - a^-|$, hypothesis (P₂) is satisfied uniformly w.r.t. l ; moreover, the smallness of $|a^+ - a^-|$ ensures that the \mathbf{L}^1 norm of ω in (P₃) is small. Therefore the hypotheses of Theorem 1 are satisfied uniformly w.r.t. l .

Let P^l be the semigroup related to the smooth section a_l . By Remark 3, if $\text{Tot.Var. } \{u\}$ is sufficiently small, u belongs to the domain of P^l for every $l > 0$. Since the total variation of $P^l_t u$ is uniformly bounded for a fixed initial data u , Helly’s theorem guarantees that there is a converging subsequence $P^{l_i}_t u$. By a diagonal argument one can show that there is a converging subsequence of semigroups converging to a limit semigroup P defined on an invariant domain (see [1, Proof of Theorem 7]).

For the uniqueness we are left to prove the integral estimate (8) in the origin with U^\sharp substituted by \bar{U}^\sharp .

Therefore we have to show that the quantity

$$(72) \quad \frac{1}{\theta} \int_{-\theta\hat{\lambda}}^{+\theta\hat{\lambda}} \left| u(\tau + \theta, x) - \bar{U}_{u(\tau)}^\sharp(\theta, x) \right| dx$$

converges to zero as θ tends to zero. We will estimate (72) in several steps. First define $\bar{v} = \bar{U}_{u(\tau)}^\sharp(0, x)$ and compute

$$(73) \quad \frac{1}{\theta} \int_{-\theta\hat{\lambda}}^{+\theta\hat{\lambda}} |(P_\theta u(\tau))(x) - (P_\theta \bar{v})(x)(\theta, x)| dx \leq \bar{\epsilon}_\theta$$

as in (63). Then we consider the approximating sequence P^{l_i} corresponding to the source term a_{l_i} and the semigroups $P^{l_i, h}$ which converge to P^{l_i} in the sense of Theorem 7. Hence we have

$$\lim_{i \rightarrow \infty} \lim_{h \rightarrow 0} \frac{1}{\theta} \int_{-\theta\hat{\lambda}}^{+\theta\hat{\lambda}} |(P_\theta^{l_i, h} \bar{v})(x) - (P_\theta \bar{v})(x)| dx = 0.$$

For notational convenience we skip the subscript i in l_i . As in (55) we approximate rarefactions in $\bar{U}_{u(\tau)}^\sharp$ introducing the function $\bar{U}_{u(\tau)}^{\sharp, \epsilon}$. Then we define (see Figure 4)

$$\bar{U}_{u(\tau)}^{\sharp, \epsilon, l, h}(t - \tau, x) = \begin{cases} \bar{U}_{u(\tau)}^{\sharp, \epsilon}(t - \tau, x + \frac{1}{2}) & \text{for } x < -l/2, \\ \tilde{U}(x) & \text{for } -l/2 \leq x \leq l/2, \\ \bar{U}_{u(\tau)}^{\sharp, \epsilon}(t - \tau, x - \frac{1}{2}) & \text{for } x > l/2, \end{cases}$$

where $\tilde{U}(x)$ is piecewise constant with jumps in the points jh satisfying $\tilde{U}(jh+) = \Phi_h(jh, \tilde{U}(jh-))$. Furthermore, $\tilde{U}(-l/2-) = \bar{U}_{(u; \tau)}^{\sharp, \epsilon}(t - \tau, 0^-)$ and Φ_h is defined as in (25) using the source term $g(x, u) = a'_l(x)g(u)$. Observe that the jump between $\tilde{U}(l/2-)$ and $\bar{U}_{u(\tau)}^{\sharp, \epsilon, l, h}(t - \tau, l/2+)$ does not satisfy any jump condition, but as $\tilde{U}(x)$ is an “Euler” approximation of the ordinary differential equation $f(u)_x = a'_l g(u)$, this

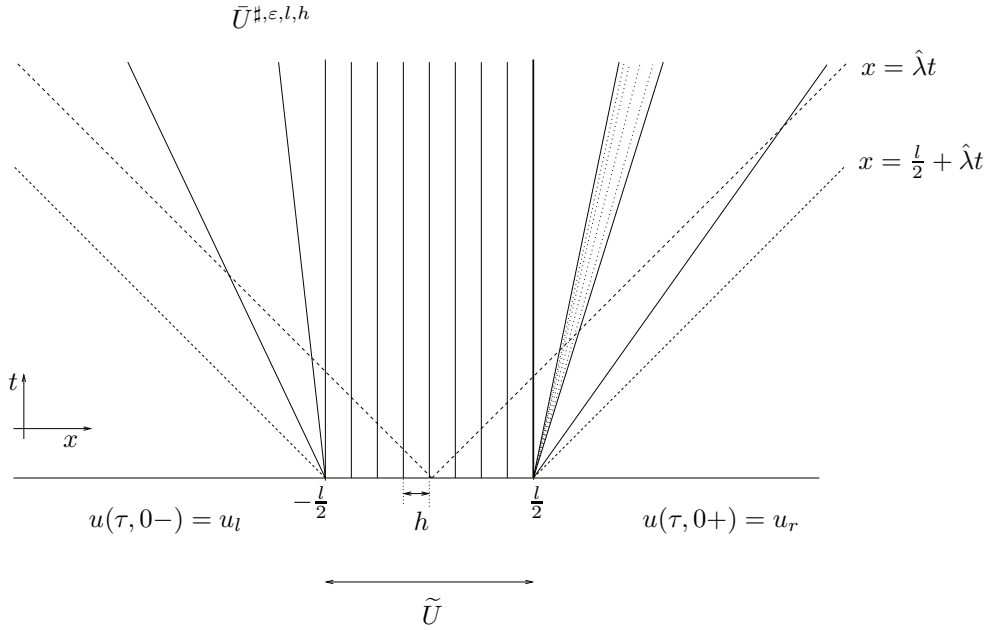


FIG. 4. Illustration of $\bar{U}_h^{\#, \epsilon, l, h}$ in the (t, x) plane.

jump is of order $\tilde{\epsilon}_h$. Since $\bar{U}_{u(\tau)}^{\#, \epsilon}$ and $\bar{U}_{u(\tau)}^{\#, \epsilon, l, h}$ have uniformly bounded total variation we have the estimate

$$\frac{1}{\theta} \int_{-\theta \hat{\lambda}}^{+\theta \hat{\lambda}} \left| \bar{U}_{u(\tau)}^{\#, \epsilon}(\theta, x) - \bar{U}_{u(\tau)}^{\#, \epsilon, l, h}(\theta, x) \right| dx \leq \mathcal{O}(1) \frac{l}{\theta},$$

the bound $\mathcal{O}(1)$ not depending on h . We apply Lemma 7 on the remaining term

$$\begin{aligned} & \frac{1}{\theta} \int_{-\theta \hat{\lambda}}^{+\theta \hat{\lambda}} \left| (P_\theta^{l, h} \bar{v})(x) - \bar{U}_{u(\tau)}^{\#, \epsilon, l, h}(\theta, x) \right| dx \\ & \leq L \int_\tau^{\tau + \theta} \liminf_{\eta \rightarrow 0} \frac{\| \bar{U}_{u(\tau)}^{\#, \epsilon, l, h}(t - \tau + \eta) - P_\eta^{l, h} \bar{U}_{u(\tau)}^{\#, \epsilon, l, h}(t - \tau) \|_{\mathbf{L}^1(J_{t+\eta})}}{\eta}. \end{aligned}$$

To estimate this last term we proceed as before. Observe that $P^{l, h}$ does not have zero-waves outside the interval $[-\frac{l}{2} - h, \frac{l}{2} + h]$ since outside the interval $[-\frac{l}{2}, \frac{l}{2}]$ the function a'_l is identically zero. If η is small enough, the waves in $P_\eta^{l, h} \bar{U}_{u(\tau)}^{\#, \epsilon, l, h}(t - \tau)$ do not interact; therefore the computation of the \mathbf{L}^1 norm in the previous integral, as before, can be split in a summation on the points in which there are zero-waves in $P^{l, h}$ or jumps in $\bar{U}_{u(\tau)}^{\#, \epsilon, l, h}(t - \tau)$. Observe that the jumps of $\bar{U}_{u(\tau)}^{\#, \epsilon, l, h}(t - \tau + \eta)$ in the interval $(-\frac{l}{2}, +\frac{l}{2})$ are defined exactly as the zero-waves in $P^{l, h}$, so we have no contribution to the summation from this interval. Outside the interval $[-\frac{l}{2} - h, \frac{l}{2} + h]$, P^h coincides with the homogeneous semigroup; hence we have only the second order contribution from the approximate rarefactions in $\bar{U}_{u(\tau)}^{\#, \epsilon, l, h}(t - \tau)$ as in (58). Furthermore, we might have a zero-wave in the interval $[-\frac{l}{2} - h, -\frac{l}{2}]$ and a discontinuity of $\bar{U}_{u(\tau)}^{\#, \epsilon, l, h}$ in the point $x = \frac{l}{2}$ of order $\tilde{\epsilon}_h$. Using (46) for the zero-wave and (43) for the discontinuity

(since P^h is equal to the homogeneous semigroup in $x = \frac{l}{2}$), we get

$$\liminf_{\eta \rightarrow 0} \frac{\|\bar{U}_{u(\tau)}^{\#, \varepsilon, l, h}(t - \tau + \eta) - P_{\eta}^{l, h} \bar{U}_{u(\tau)}^{\#, \varepsilon, l, h}(t - \tau)\|_{\mathbf{L}^1(J_{t+\eta})}}{\eta} \leq \mathcal{O}(1) (\varepsilon + \tilde{\varepsilon}_h),$$

which completes the proof if we first let ε tend to zero, then h tend to zero, then l tend to zero, and finally θ tend to zero. As in the previous proof, the sufficiency part can be obtained following the proof for the homogeneous case presented in [4]. \square

Proof of Proposition 1. Call S the semigroup defined in [7]. The estimates for this semigroup outside the origin are equal to the ones for the standard Riemann semigroup; see [4]. Concerning the origin we first observe that the choice (20) implies that the solution to the Riemann problem in [7, Proposition 2.2] coincides with $\bar{U}_{u(\tau)}^{\#}$. We need to show that

$$(74) \quad \lim_{\theta \rightarrow 0} \frac{1}{\theta} \int_{-\theta\hat{\lambda}}^{+\theta\hat{\lambda}} \left| u(\tau + \theta, x) - \bar{U}_{u(\tau)}^{\#}(\theta, x) \right| dx = 0$$

with $u(t, x) = (S_t u_o)(x)$. As before, we first approximate $\bar{U}_{u(\tau)}^{\#}$ with $\bar{U}_{u(\tau)}^{\#, \varepsilon}$ and $u(\tau)$ with $\bar{U}_{u(\tau)}^{\#}(0) \doteq \bar{v}$; then we apply Lemma 7 (which holds also for the semigroup S) and compute

$$\begin{aligned} & \frac{1}{\theta} \int_{-\theta\hat{\lambda}}^{+\theta\hat{\lambda}} \left| (S_{\theta} \bar{v})(x) - \bar{U}_{u(\tau)}^{\#, \varepsilon}(\theta, x) \right| dx \\ & \leq L \frac{1}{\theta} \int_{\tau}^{\tau+\theta} \liminf_{\eta \rightarrow 0} \frac{\|\bar{U}_{u(\tau)}^{\#, \varepsilon}(t - \tau + \eta) - S_{\eta} \bar{U}_{u(\tau)}^{\#, \varepsilon}(t - \tau)\|_{\mathbf{L}^1(J_{t+\eta})}}{\eta}. \end{aligned}$$

The discontinuities of $\bar{U}_{u(\tau)}^{\#, \varepsilon}$ are solved by S_{η} with exact shock or rarefaction for $x \neq 0$ and with the a -Riemann solver in $x = 0$; therefore the only difference between $\bar{U}_{u(\tau)}^{\#, \varepsilon}(t - \tau + \eta)$ and $S_{\eta} \bar{U}_{u(\tau)}^{\#, \varepsilon}(t - \tau)$ is that the rarefactions are solved in an approximate way in the first function and in an exact way in the second. Recalling (44) we know that this error is of second order in the size of the rarefactions.

To show that (74) holds, proceed as in (58). \square

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