Approximating the Scattering Coefficients for a Non-Rayleigh Obstacle by Boundary Defect Minimization

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Abstract: The coefficients which appear in the representation of the far-field scattered by a two-dimensional, perfectly electrically conducting obstacle, depend on the normal derivative of the scattered field on the obstacle boundary, $\partial_N v|_\Gamma$. A family of functions, $\mathcal{W}$, deduced from the minimization of a boundary defect, is shown to be linearly independent and complete. As a consequence the approximation of $\partial_N v|_\Gamma$ results from a well posed algebraic problem. The approximation error is estimated from an *a priori* bound on the equation error in terms of the eigenvalues of a boundary integral operator. Rayleigh's hypothesis is nowhere required of the obstacle. These results justify some obstacle inversion methods which originated from heuristic arguments

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1. Scattering Coefficients and Complete Families

Families of functions which are linearly independent and complete, named *complete families* for short and generally denoted by $\mathcal{F}$, play a role in the exact or approximate representation of solutions to partial differential equations. The direct and inverse obstacle problems, both in acoustics and electromagnetics, have benefitted from the properties of $\mathcal{F}$s.

One class of obstacle scattering problems is characterized by a fixed incident wavenumber and by the assumption that the scattered wave has the same frequency, which makes sense whenever frequency conversion processes (e.g., fluorescence) are neglected.
Further specifications which apply to this presentation are: the problem is two-dimensional; the obstacle is non penetrable and perfectly electrically conducting (*PEC*); both the incident and scattered waves are complex scalar solutions to the Helmholtz equation.

Let $\Omega \subset \mathbf{R^2}$ be a star-shaped, bounded domain which contains the origin and has a smooth boundary, $\partial\Omega = \Gamma$. Let $D_R[\mathbf{0}]$ be the disk centered at the origin and of radius R circumscribed to Ω and let u denote the incident scalar plane wave and v the scattered wave complying with $(u + v)|_\Gamma = 0$ and the Sommerfeld radiation condition. This situation occurs when e.g., a *PEC* right cylinder of cross-section Ω and indefinite height is illuminated by a vertically polarized electromagnetic wave at fixed frequency.

Let $\lambda \equiv \{l, \wp\}$ denote the pair of indices {order, parity} such that $0 \le l$, $\wp = 0, 1$. The families of outgoing waves, $\mathcal{F}_1 = \{v_\lambda\}$, and that of their normal derivatives on Γ, $\mathcal{F}_2 = \{\partial_N v_\lambda\}$, where $\partial_N(.)$ is the outward normal derivative on Γ, are known to be unconditionally complete in $L^2(\Gamma)$ i.e. their completeness holds for any k^2. Instead, the family of real wave functions $\{u_\lambda\}$ is complete in $L^2(\Gamma)$ provided $k^2 \notin \sigma[-\Delta_D]$ i.e., the wavenumber squared is not an eigenvalue of the interior DIRICHLET-LAPLACE operator. (Similarly, the interior NEUMANN-LAPLACE operator will be denoted by $-\Delta_N$).

In general terms, *expansion coefficients* are those which appear in an exact (usually a series) or approximate (usually a sum) representation of a wave field by means of a suitable $\mathcal{F}$. For instance, the scattered field v can be represented in terms of $\mathcal{F}_1$ by the series $v = \sum_\lambda f_\lambda v_\lambda$, where $\{f_\lambda\}$ are the *far-field scattering coefficients*, derived from application of the HELMHOLTZ formula and given by:

$$f_\lambda = -(\mathrm{i}/4) \left\langle u | \partial_N (u + v) \right\rangle. \tag{1}$$

Here $\langle . | . \rangle$ denotes the inner product in $L^2(\Gamma)$.

This representation of v is known to converge uniformly outside $D_R[\mathbf{0}]$ i.e., $\mathcal{F}_1$ is a RIESZ basis of a space of functions supported in $\mathbf{R}^2 \backslash D_R[\mathbf{0}]$ [1]. If the above series converges up to Γ as well, then RAYLEIGH's hypothesis is said to hold or the obstacle to be in the RAYLEIGH class [2, 3, 4]. A general $\mathcal{F}$, however, does neither give rise to a SCHAUDER basis, nor to a RIESZ basis in $L^2(\Gamma)$.

In the past this author addressed inverse obstacle problems at the analytical and numerical levels. Algorithms were developed, including *forward propagation* [5], which relied on representations of the scattered waves on the obstacle boundary and in the far zone. Expansion coefficients and approximations thereof involved a variety of $\mathcal{F}$s.

This work addresses an approximation scheme of f_λ, for which error bounds can be eventually supplied.

Let L denote the order of approximation and let λ span the set $\Lambda[L] := \{0 \le l \le L, \wp = 0, 1\}$. The function to be approximated is the normal derivative on Γ of the scattered wave. Namely, $\partial_N v|_\Gamma$ is replaced by the sum:

$$(\partial_N v)_2^{(L)} = \sum_{\mu \in \Lambda[L]} c_\mu^{(L)} \partial_N v_\mu. \tag{2}$$

Here the subscript 2 refers to $\mathcal{F}_2$ and $\{c_\mu^{(L)} | \mu \in \Lambda[L]\}$ is a set of expansion coefficients to be determined as explained below. The corresponding far-field coefficients of order L are $\{p_\lambda^{(L)}\}$, where:

$$p_\lambda^{(L)} = -(i/4) \left\langle u_\lambda | \partial_N u + (\partial_N v)_2^{(L)} \right\rangle. \tag{3}$$

2. The Boundary Defect

In general, it is the choice of $\{c_\mu^{(L)}\}$ which qualifies the approximation scheme. Herewith, the determination of $\{c_\mu^{(L)}\}$ starts with defining a *boundary defect*, $\mathcal{D}^{(L)}$, of order L and seeking for a minimizer thereof. To this end let $\mathbf{x}, \mathbf{y} \in \Gamma$, denote by $\Phi[\mathbf{x}, \mathbf{y}]$ the fundamental solution $(= (\mathrm{i}/4) H_0^{(1)}[k|\mathbf{x} - \mathbf{y}|])$ to the HELMHOLTZ equation in $\mathbf{R}^2$, where $H_0^{(1)}[.]$ is the HANKEL function of the 1st kind and order 0, and define:

$$\mathcal{D}^{(L)}[\mathbf{x}] := (1/2)\left(\partial_{N[\mathbf{x}]}v\right)_2^{(L)}[\mathbf{x}] + \int_\Gamma \partial_{N[\mathbf{x}]}\Phi[\mathbf{x},\mathbf{y}]\left(\partial_{N[\mathbf{y}]}v\right)_2^{(L)}[\mathbf{y}]\mathrm{d}\Gamma_{\mathbf{y}} -$$

$$-(1/2)\left(\partial_{N[\mathbf{x}]}u\right)[\mathbf{x}] + \int_\Gamma \partial_{N[\mathbf{x}]}\Phi[\mathbf{x},\mathbf{y}]\left(\partial_{N[\mathbf{y}]}u\right)[\mathbf{y}]\mathrm{d}\Gamma_{\mathbf{y}}. \tag{4}$$

The quantity to be minimized will be $\mathcal{B}_2^{(L)} := (1/2)\|\mathcal{D}^{(L)}\|^2_{L^2(\Gamma)}$. The last two terms in $\mathcal{D}^{(L)}[\mathbf{x}]$ which depend on $u[.]$ give rise to the function $g[.]$, which plays the role of a 'known term':

$$g[\mathbf{x}] = (1/2)\left(\partial_{N[\mathbf{x}]}u\right)[\mathbf{x}] - (\mathrm{i}/4)\int_\Gamma \partial_{N[\mathbf{y}]}u[\mathbf{y}]\partial_{N[\mathbf{x}]}H_0^{(1)}[kR]\mathrm{d}\Gamma[\mathbf{y}]. \tag{5}$$

The first two terms, which contain approximations to $\partial_{N[.]}v[.]$ are re-arranged as:

$$\sum_{\lambda\in\Lambda[L]} c_\lambda^{(L)}\left((1/2)\left(\partial_{N[\mathbf{x}]}v_\lambda\right)[\mathbf{x}] + \int_\Gamma \partial_{N[\mathbf{x}]}\Phi[\mathbf{x},\mathbf{y}]\left(\partial_{N[\mathbf{y}]}v_\lambda\right)[\mathbf{y}]\mathrm{d}\Gamma_{\mathbf{y}}\right) := \sum_{\lambda\in\Lambda[L]} c_\lambda^{(L)} w_\lambda[\mathbf{x}]. \tag{6}$$

The functions of index λ between parentheses on the left side have thus been conveniently denoted by $w_\lambda[.]$: they are the members of the new family, $\mathcal{W}$, the properties of which shall be investigated.

The necessary condition for $\mathcal{B}_2^{(L)}$ to be stationary i.e., the set of normal equations, is:

$$\int_\Gamma \mathcal{D}^{(L)*}w_\lambda \mathrm{d}\Gamma = 0, \forall\lambda\in\Lambda[L]. \tag{7}$$

3. Properties of $\mathcal{W}$

THEOREM 1. The family $\mathcal{W}$ is linearly independent and complete in $L^2(\Gamma)$ provided $k^2 \notin (\sigma[-\Delta_D]\cup\sigma[-\Delta_N])$.

Proof (Linear independence). One has to show the implication:

$$\{w^{(L)}[\mathbf{x}] := \sum_{\lambda\in\Lambda[L]} b_\lambda w_\lambda[\mathbf{x}] = 0, \forall L\} \Rightarrow \{b_\lambda = 0 | \lambda\in\Lambda[L]\}. \tag{8}$$

By letting $\phi^{(L)} := \sum_{\lambda\in\Lambda[L]} b_\lambda \partial_{N[\mathbf{x}]}v_\lambda[.]$, the left side of the implication becomes:

$$(1/2)\phi^{(L)}[\mathbf{x}] + \int_\Gamma \partial_{N[\mathbf{x}]}\Phi[\mathbf{x},\mathbf{y}]\phi^{(L)}[\mathbf{y}]\mathrm{d}\Gamma_{\mathbf{y}} = 0, \forall\mathbf{x}\in\Gamma \tag{9}$$

i.e., the integral equation for the single layer potential $\psi[\mathbf{x}] := \int_\Gamma \Phi[\mathbf{x},\mathbf{y}]\phi^{(L)}[\mathbf{y}]\mathrm{d}\Gamma_{\mathbf{y}}$, which solves the interior homogeneous NEUMANN boundary value problem for the HELMHOLTZ equation. If

$k^2 \notin \sigma[-\Delta_N]$ then $\psi^{(L)} \equiv 0$ on $\bar{\Omega}$. By the linear independence of $\partial_N v_\lambda[.]$ in $L^2(\Gamma)$ the claim follows.
(Completeness). One has to show that the only $h[.] \in L^2(\Gamma)$ which complies with the 'orthogonality' condition $0 = \int_\Gamma h^*[\mathbf{x}] w_\lambda[\mathbf{x}] \mathrm{d}\Gamma_\mathbf{x}, \forall \lambda$ is $g \equiv 0$ on Γ. To this end one realizes that w_λ is the interior normal derivative (denoted by $\partial^{(-)}_{N[.]}[.]$) of the single layer potential with density $\partial_{N[.]} v_\lambda[.]$. Next, one lets D_0 denote an open disk centered at the origin and strictly contained in Ω and introduces $\Re v_\lambda[\mathbf{z}] := u_\lambda[\mathbf{z}]$, where $z \in D_0$. Multiplication by $u_\lambda[\mathbf{z}]$, summation over λ and application of the ERDELY expansion turn the orthogonality condition into a property of the function $\omega[.]$ defined by:

$$\omega[\mathbf{z}] := \int_\Gamma h^*[\mathbf{x}] \partial^{(-)}_{N[\mathbf{x}]} \left(\int_\Gamma \Phi[\mathbf{x},\mathbf{y}] \partial_{N[\mathbf{y}]} \Phi[\mathbf{y},\mathbf{z}] \mathrm{d}\Gamma_\mathbf{y} \right) \mathrm{d}\Gamma_\mathbf{x} \tag{10}$$

namely $\omega[\mathbf{z}] = 0, \forall [\mathbf{z}] \in D_0$, hence, by analytic continuation and continuity, $\omega^{(-)}[\mathbf{z}]|_\Gamma = 0$ and $\partial^{(-)}_N \omega[\mathbf{z}]|_\Gamma = 0$.
By interchanging the orders of integration, the function $\omega[.]$ can be represented by:

$$\omega[\mathbf{z}] = (1/2) \int_\Gamma h^*[\mathbf{x}] \partial_{N[\mathbf{x}]} \Phi[\mathbf{x},\mathbf{z}] \mathrm{d}\Gamma_\mathbf{x} + \int_\Gamma \partial_{N[\mathbf{y}]} \Phi[\mathbf{y},\mathbf{z}] \left(\int_\Gamma h^*[\mathbf{x}] \partial_{N[\mathbf{x}]} \Phi[\mathbf{x},\mathbf{y}] \mathrm{d}\Gamma_\mathbf{x} \right) \mathrm{d}\Gamma_\mathbf{y}, \tag{11}$$

which vanishes $\forall \mathbf{z} \in \Omega$. Now by passing to the limit i.e., letting $\mathbf{z} \to \mathbf{s} \in \Gamma$, one obtains an integral equation (THM. 2.3 of [6]) which characterizes $h^*[.]$:

$$0 = -(1/4) h^*[\mathbf{s}] + \int_\Gamma \partial_{N[\mathbf{y}]} \Phi[\mathbf{y},\mathbf{s}] \left(\int_\Gamma h^*[\mathbf{x}] \partial_{N[\mathbf{x}]} \Phi[\mathbf{x},\mathbf{y}] \mathrm{d}\Gamma_\mathbf{x} \right) \mathrm{d}\Gamma_\mathbf{y}. \tag{12}$$

The latter, by introducing the integral operator $(\mathbf{K}(.))[.]$ such that $(\mathbf{K}\psi)[\mathbf{x}] := 2 \int_\Gamma \partial_{N[\mathbf{y}]} \Phi[\mathbf{x},\mathbf{y}] \psi[\mathbf{y}] \mathrm{d}\Gamma_\mathbf{y}$, becomes:

$$(\mathbf{1} + \mathbf{K})\left[(\mathbf{1} - \mathbf{K})[h^*] \right] = 0, \forall \mathbf{s} \in \Gamma. \tag{13}$$

Since $k^2 \notin \sigma[-\Delta_N] \Rightarrow \mathrm{Ker}(\mathbf{1} + \mathbf{K}) = \{0\}$ and $k^2 \notin \sigma[-\Delta_D] \Rightarrow \mathrm{Ker}(\mathbf{1} - \mathbf{K}) = \{0\}$ (THMS. 3.17 and 3.22 of [6]), one concludes $h = 0_{L^2(\Gamma)}$. QED.

Let $\mathbf{W}^{(L)}$ denote the GRAMian matrix of $\mathcal{W}$ i.e.,

$$\mathbf{W}^{(L)} = [\langle w_\lambda | w_\mu \rangle]. \tag{14}$$

and define the vector of card$[\Lambda[L]]$ 'known terms' by $\mathbf{g}^{(L)} = [\langle g | w_\mu \rangle]$. Similarly denote by $c^{(L)}$ the vector, the elements of which are orderly taken from the set $\{c^{(L)}_\mu | \mu \in \Lambda[L]\}$.

THEOREM 2. The vector $c^{(L)}$ is the solution of the well-posed algebraic system:

$$\mathbf{W}^{(L)} \cdot c^{(L)} = \mathbf{g}^{(L)}. \tag{15}$$

Proof. The algebraic system comes from recasting the normal equations (EQ. 7) in matrix form.

Since $\mathcal{W}$ is linearly independent, then $\exists \left(\mathbf{W}^{(L)}\right)^{-1}$. Since $\mathcal{W}$ is complete, then L can be arbitrary.

QED

4. An Error Bound

The boundary integral equation for the (exact) scattered wave is well-known to read:

$$(1/2)\partial_{N[\mathbf{x}]}v + \partial_{N[\mathbf{x}]}\left(\int_{\Gamma}\Phi[\mathbf{x},\mathbf{y}]\partial_{N[\mathbf{y}]}v\mathrm{d}\Gamma_{\mathbf{y}}\right) = (1/2)\partial_{N[\mathbf{x}]}u - \partial_{N[\mathbf{x}]}\left(\int_{\Gamma}\Phi[\mathbf{x},\mathbf{y}]\partial_{N[\mathbf{y}]}u\mathrm{d}\Gamma_{\mathbf{y}}\right). \quad (16)$$

It is conveniently denoted by $(\mathbf{1}+\mathbf{K}')\partial_{N[\mathbf{x}]}v = (\mathbf{1}-\mathbf{K}')\partial_{N[\mathbf{x}]}u$. Let the equation error $\eta^{(L)}$ be defined by:

$$\eta(L) := (1/2)\partial_{N[\mathbf{x}]}v - (1/2)\left(\partial_{N[\mathbf{x}]}v\right)_2^{(L)} + \partial_{N[\mathbf{x}]}\left(\int_{\Gamma}\Phi[\mathbf{x},\mathbf{y}]\left(\partial_{N[\mathbf{x}]}v - \left(\partial_{N[\mathbf{x}]}v\right)_2^{(L)}\right)\mathrm{d}\Gamma_{\mathbf{y}}\right). \quad (17)$$

PROPOSITION 3. Let $\eta^{(L)}$ comply with:

$$\left\|\eta^{(L)}\right\|^2_{L^2(\Gamma)} < \epsilon^2 \quad (18)$$

and let λ_1 be the eigenvalue of $\mathbf{K}'$ which satisfies:

$$|1+\lambda_1| = \mathrm{argmin}_{\lambda\in\sigma[\mathbf{K}']}|1+\lambda| \quad (19)$$

then:

$$\left\|\partial_{N[\mathbf{x}]}v - (\partial_{N[\mathbf{x}]}v)_2^{(L)}\right\|^2_{L^2(\Gamma)} < \frac{4\epsilon^2}{|1+\lambda_1|^2}. \quad (20)$$

Proof. In terms of $\eta^{(L)}$ the boundary integral equation for the approximate normal derivative of the scattered wave reads:

$$(\mathbf{1}+\mathbf{K}')\left(\partial_{N[\mathbf{x}]}v\right)_2^{(L)} = (\mathbf{1}-\mathbf{K}')\partial_{N[\mathbf{x}]}u - 2\eta^{(L)}. \quad (21)$$

If one defines $\psi^{(L)} := \partial_{N[\mathbf{x}]}v - \left(\partial_{N[\mathbf{x}]}v\right)_2^{(L)}$, one has $(\mathbf{1}+\mathbf{K}')\psi^{(L)} = 2\eta^{(L)}$ and the inequalities:

$$|1+\lambda_1|\,\|\psi^{(L)}\| \leq \|(\mathbf{1}+\mathbf{K}')\psi^{(L)}\| = 2\|\eta^{(L)}\| \leq 2\epsilon. \quad (22)$$

QED.

5. Applications

There is an interplay between the approximation of a scattered wave function and that of expansion coefficients. Both eventually relate to solving boundary integral equations. A given approximation scheme is usually characterized by some convergence estimate. With reference to the already mentioned *forward propagation* scheme, the index-wise convergence of $p_\lambda^{(L)}$ of Eq. 3 to f_λ as $L \to \infty$ was proved in [5] under restrictive conditions, which applied to obstacles in the Rayleigh class. Instead, Ineq. 20, which immediately extends to $|f_\lambda - p_\lambda^{(L)}|$, has been derived without invoking the Rayleigh hypothesis. Therefore it represents a step towards fully justifying some obstacle inversion algorithms, including those in [7, 8], which originated from heuristic arguments.

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