GENERALIZED ROUGH APPROXIMATIONS IN ŁΠ$\frac{1}{2}$

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Abstract. In this paper we will treat a generalization of inner and outer approximations of fuzzy sets, which we will call $\mathcal{R}$-inner and $\mathcal{R}$-outer approximations respectively ($\mathcal{R}$ being any finite set of rational numbers in $[0,1]$). In particular we will discuss the case of those fuzzy sets which are definable in the logic ŁΠ$\frac{1}{2}$ by means of step functions from the hypercube $[0,1]^k$ and taking value in an arbitrary (finite) subset of $[0,1]\cap\mathbb{Q}$. Then, we will show that if a fuzzy set is definable as truth table of a formula of ŁΠ$\frac{1}{2}$, then both its $\mathcal{R}$-inner and $\mathcal{R}$-outer approximation are definable as truth table of formulas of ŁΠ$\frac{1}{2}$. Finally, we will introduce a generalization of abstract approximation spaces and compare out approach with the notion of fuzzy rough set.
Keywords: Fuzzy sets, Rough approximations, Abstract approximation spaces, Fuzzy rough sets.

1. Introduction

Fuzzy sets were introduced by Zadeh in [29] in order to generalize crisp (i.e. classical, Boolean) sets as follows.

Definition 1.1 ([29])**.** *Let X be a set of objects, called the* universe of discourse*. A* fuzzy set *on X is any mapping $f : X \to [0,1]$. In the sequel for each $a \in [0,1]$, we will denote by* **a** *the fuzzy set on X constantly equal to a.*

From the mathematical point of view, fuzzy sets introduce a theory for the treatment of vagueness.
On the other hand, rough sets were conceived by Pawlak in order to handle uncertainty in so called information systems [20, 21]. Several mathematical approches and generalizations have been done (for an overview see [23]) in the following years. To the scope of the present paper it is important to consider the rough approximation of a fuzzy set through a pair of crisp (classical, Boolean) sets [1, 4].

Definition 1.2. *Let f be a fuzzy set on X. The rough approximation of f is the pair of crisp sets $(\nu(f), \mu(f))$ defined pointwise as*

$$\nu(f)(x) := \begin{cases} 1 & \text{if} \quad f(x) = 1 \\ 0 & \text{otherwise} \end{cases} \qquad \mu(f)(x) := \begin{cases} 1 & \text{if} \quad f(x) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

It is obvious that $\nu(f) \leq f \leq \mu(f)$, i.e., in the rough-theory language ν is a lower approximation and μ an upper approximation of f. The main drawback of this approximation is that it looses all the semantic of a fuzzy set, reducing it to two, possibly very different, Boolean sets.

Remark 1.3. *For the sake of completeness, let us recall that there exist other approaches to the approximation of fuzzy sets. For instance, α-cuts* [9]*, level fuzzy*

sets [24] *and shadowed sets* [22, 3]. *All these methods are characterized by a modification of the original fuzzy set with a raising and/or lowering of the original value towards the values* $\{0,1\}$ *(α-cuts and level fuzzy sets) or* $\{0,1/2,1\}$ *(shadowed sets). Thus, as it will be clear later on, our work can be seen as a generalization of these methods.*
A further and different approach is given by fuzzy rough sets which will be discussed in the last section.

The logical counterpart of the mathematical theory of fuzzy sets is fuzzy logic. In the last years many fuzzy logical systems have been introduced and developed (see [11, 12, 14] just to quote some examples). Besides a solid and formal logical framework, those fuzzy logics also define wide classes of fuzzy sets. In other words, Boolean functions (and therefore crisp sets) stand to classical logics as (a class of) fuzzy sets to (a) fuzzy logic. Clearly, a fuzzy logic can define a more and more general class of fuzzy sets depending on its expressive power. For instance, the McNaughton theorem (see [18]) was stated for Łukasiewicz logic (see [14]) and it reads as follows.

Theorem 1.4 ([18])**.** *A function* $f : [0,1]^k \to [0,1]$ *is a truth table of a Łukasiewicz formula iff it is a McNaughton function, i.e., it is a continuous, piecewise linear function such that each piece has integer coefficients.*

In a fuzzy set theoretical setting, the previous theorem means that formulas of Łukasiewicz logic implicitly define those fuzzy sets being McNaughton functions.
Henceforth we will use the following notation. Let $\mathcal{L}$ be any logic whose standard truth value set is the real unit interval $[0,1]$. Then, a fuzzy sets $g : [0,1]^k \to [0,1]$ is an $\mathcal{L}$*-fuzzy set* if there is a formula φ in the language of $\mathcal{L}$ such that g is a truth table of φ. In such a case we will also write $g = \mathcal{F}_\varphi$ to denote the dependency on φ.
As said before, increasing the expressive power of a logic, we obtain for free a bigger class of fuzzy sets which are representable in the extended logic. From this point of view the most powerful fuzzy logics are ŁΠ and Ł$\Pi\frac{1}{2}$. Those logics were introduced by Esteva, Godo and Montagna in [12] and then improved and simplified in the axiomatization by Cintula in [6, 7]. Roughly speaking (we will give a formal definition of this logic in the following section), ŁΠ is obtained by joining together Łukasiewicz logic and product logic (see [14]), adding three further axioms making the relation between the two logics clear. Ł$\Pi\frac{1}{2}$ is obtained from ŁΠ by adding a truth constant $\overline{\frac{1}{2}}$ together with the axiom $\overline{\frac{1}{2}} \equiv \neg\overline{\frac{1}{2}}$.
As for the functional representation of Ł$\Pi\frac{1}{2}$, Montagna and Panti proved in [19] the following:

Theorem 1.5 ([19])**.** *A function* $f : [0,1]^k \to [0,1]$ *is a truth table of a formula of* Ł$\Pi\frac{1}{2}$ *iff there is a finite partition of* $[0,1]^k$ *such that each block of the partition is a* $\mathbb{Q}$*-semialgebraic set, and* f *restricted to each block is a fraction of two polynomials with rational coefficients.*

Ł$\Pi\frac{1}{2}$, besides a logic allowing to deal with a wide class of fuzzy sets, is a powerful setting also to deal with the generalized approximation of Ł$\Pi\frac{1}{2}$-fuzzy sets. Those approximations intend to generalize the usual rough approximation previously discussed. In Section 4, we will use Ł$\Pi\frac{1}{2}$ in order to define what we will call the $\mathcal{R}$*-inner approximation* and the $\mathcal{R}$*-outer approximation* of a fuzzy

set. The idea is the following: consider a fuzzy set $g : [0,1]^k \to [0,1]$ and let $\mathcal{R} = \{r_1, \ldots, r_n\}$ be a fixed finite set of rational numbers in $[0,1]$ such that $0 < r_1 < r_2 < \ldots < r_{n-1} < r_n < 1$. Then, roughly speaking, once defined the usual pointwise partial order: $g_1 \leq g_2$ iff $\forall \underline{x} \in [0,1]^k$, $g_1(\underline{x}) \leq g_2(\underline{x})$, we will define the $\mathcal{R}$-inner approximation of g to be that (unique) function $\chi_g : [0,1]^k \to \{0, r_1, \ldots, r_n, 1\}$ which is maximal (w.r.t. the above defined ordering) among all the rational valued step functions $\widehat{\chi} : [0,1]^k \to \{0, r_1, \ldots, r_n, 1\}$ approximating g from below. Analogously, the $\mathcal{R}$-outer approximation of g, will be that (unique) function $\tau_g : [0,1]^k \to \{0, r_1, \ldots, r_n, 1\}$, which is minimal (w.r.t. $\leq$) among all those rational valued step functions $\widehat{\tau} : [0,1]^k \to \{0, r_1, \ldots, r_n, 1\}$ approximating g from above.
In the following sections we will show that, if g is an $Ł\Pi\frac{1}{2}$-fuzzy set (and hence there is an $Ł\Pi\frac{1}{2}$-formula φ such that $g = \mathcal{F}_\varphi$), then also its $\mathcal{R}$-inner approximation and its $\mathcal{R}$-outer approximation are $Ł\Pi\frac{1}{2}$-fuzzy sets, that is there are two $Ł\Pi\frac{1}{2}$-formulas ψ_1 and ψ_2 such that (w.r.t. the above notation) $\chi_g = \mathcal{F}_{\psi_1}$, and $\tau_g = \mathcal{F}_{\psi_2}$.

2. The $Ł\Pi$ and $Ł\Pi\frac{1}{2}$ logics

The language of $Ł\Pi$ is constituted by a denumerable class of propositional variables $\{x_1, x_2 \ldots\}$, and the truth constant $\overline{0}$. Connectives are $\to$ (Łukasiewicz implication), and $\&_\Pi$ and $\to_\Pi$ (product strong conjunction and product implication respectively). Formulas are defined as usual: $\overline{0}$ is a formula, each propositional variable is a formula, and if φ and ψ are formulas, then so are $\varphi \to \psi$, $\varphi \&_\Pi \psi$, and $\varphi \to_\Pi \psi$. The language of $Ł\Pi\frac{1}{2}$ is obtained by adding the further constant $\overline{\frac{1}{2}}$.
The standard semantic of $\to$, $\&_\Pi$ and $\to_\Pi$ is as follows:

$$\begin{array}{ll} \varphi \to \psi & \text{interpretation: } x \Rightarrow y = \min\{1, 1 - x + y\} \\ \varphi \&_\Pi \psi & \text{interpretation: } x \odot_\Pi y = xy \\ \varphi \to_\Pi \psi & \text{interpretation: } x \Rightarrow_\Pi y = \begin{cases} 1 & \text{if } x \leq y \\ \frac{y}{x} & \text{otherwise} \end{cases} \end{array}$$

Further definable connectives are as follows, where the connective name, its definition and its standard interpretation are listed together:

Connective	Definition	Interpretation
$\neg\varphi$	$\varphi \to \overline{0}$	$\neg x = 1 - x$
$\neg_\Pi \varphi$	$\varphi \to_\Pi \overline{0}$	$\neg_\Pi x = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise.} \end{cases}$
$\Delta\varphi$	$\neg_\Pi \neg \varphi$	$\Delta(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{otherwise} \end{cases}$
$\nabla\varphi$	$\neg\Delta\neg\varphi$	$\nabla(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$
$\varphi \& \psi$	$\neg(\varphi \to \neg\psi)$	$x \odot y = \max\{0, x + y - 1\}$
$\varphi \oplus \psi$	$\neg(\neg\varphi \odot \neg\psi)$	$x \oplus y = \min\{1, x + y\}$
$\varphi \ominus \psi$	$\neg(\varphi \to \psi)$	$\max\{0, x - y\}$
$\varphi \wedge \psi$	$\varphi \& (\varphi \to \psi)$	$x \wedge y = \min\{x, y\}$
$\varphi \vee \psi$	$(\varphi \to \psi) \to \psi$	$x \vee y = \max\{x, y\}$
$\varphi \to_G \psi$	$\Delta(\varphi \to \psi) \vee \psi$	$x \Rightarrow_G y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$

There are many (equivalent) axiomatic systems for ŁΠ and ŁΠ$\frac{1}{2}$. We decided to cite the one from [7].

Definition 2.1 ([7]). *The following are the axioms of* ŁΠ*:*

(Ł): *All the axioms of Lukasiewicz logic (see* [14]*),*
(Π): *All the axioms of product logic (see* [14]*),*
(ŁΔ): $\Delta(\varphi \to \psi) \to (\varphi \to_\Pi \psi)$,
(ΠΔ): $\Delta(\varphi \to_\Pi \psi) \to (\varphi \to \psi)$,
(*Dist*): $\varphi \&_\Pi (\gamma \ominus \psi) \equiv (\varphi \&_\Pi \gamma) \ominus (\varphi \&_\Pi \psi)$.

Deduction rules are modus ponens *and* Δ-necessitation*.*

The logic ŁΠ$\frac{1}{2}$ *results from* ŁΠ *plus the further axiom for* $\overline{\frac{1}{2}}$*:*

(*Half*): $\overline{\frac{1}{2}} \equiv \neg\overline{\frac{1}{2}}$.

$\vdash_{Ł\Pi}$ *and* $\vdash_{Ł\Pi\frac{1}{2}}$ *will denote the relations of* logical consequence *for* ŁΠ *and* ŁΠ$\frac{1}{2}$ *respectively.*

Now we will introduce the algebraic counterpart of ŁΠ and ŁΠ$\frac{1}{2}$ logics, namely the classes of ŁΠ and ŁΠ$\frac{1}{2}$-algebras. Once again we will adopt the definition given in [7].

Definition 2.2 ([7]). *An* ŁΠ*-algebra is an algebraic structure* $\mathcal{A} = \langle A, \oplus, \neg, \Rightarrow_\Pi, \odot_\Pi, 0, 1\rangle$ *such that:*

- $\langle A, \oplus, \neg, 0, 1\rangle$ *is an MV-algebra (see* [14]*),*
- $\langle A, \Rightarrow_\Pi, \odot_\Pi, \wedge, \vee, 0, 1\rangle$ *is a* Π*-algebra (see* [14]*),*
- $x \odot_\Pi (y \ominus z) = (x \odot_\Pi y) \ominus (x \odot_\Pi z)$ *holds for each* $x, y, z \in A$
- $\Delta(x \Rightarrow y) \Rightarrow (x \Rightarrow_\Pi y) = 1$ *(where* $x \Rightarrow y$ *stands for* $\neg x \oplus y$*) holds for each* $x, y \in A$.

An ŁΠ$\frac{1}{2}$*-algebra is an algebraic structure* $\mathcal{A} = \langle A, \oplus, \neg, \Rightarrow_\Pi, \odot_\Pi, \frac{1}{2}, 0, 1\rangle$ *such that:*

- $\langle A, \oplus, \neg, \Rightarrow_\Pi, \odot_\Pi, 0, 1\rangle$ *is an* ŁΠ*-algebra,*
- $\frac{1}{2} = \neg\frac{1}{2}$ *holds.*

If $\mathcal{A}$ is an ŁΠ-algebra, an $\mathcal{A}$-*evaluation* is a map e from the propositional variable of the language of ŁΠ into A. An evaluation e can be extended to all the ŁΠ-formulas by the following conditions:

(−) $e(\varphi \to \psi) = \neg e(\varphi) \oplus e(\psi)$

(−) $e(\varphi \&_\Pi \psi) = e(\varphi) \odot_\Pi e(\psi)$,

(−) $e(\varphi \to_\Pi \psi) = e(\varphi) \Rightarrow_\Pi e(\psi)$

If $\mathcal{A}$ is an ŁΠ$\frac{1}{2}$-algebra then we need also to add the following:

(−) $e(\overline{\frac{1}{2}}) = \frac{1}{2}$.

The notion of *model* is defined as usual and $\models_{Ł\Pi}$ and $\models_{Ł\Pi\frac{1}{2}}$ will denote the semantic consequence relations for ŁΠ and ŁΠ$\frac{1}{2}$ respectively.

To give a complete description of the variety of ŁΠ$\frac{1}{2}$-algebras is not in the goals of this paper, anyway the following are the main examples of ŁΠ$\frac{1}{2}$-algebras which will turn out to be crucial for the understanding of the rest of this paper.

(1) The standard ŁΠ$\frac{1}{2}$-algebra is the system

$$[0,1]_{Ł\Pi\frac{1}{2}} = \langle [0,1], \oplus, \neg, \odot_\Pi, \Rightarrow_\Pi, \tfrac{1}{2}, 0, 1\rangle,$$

where, for each $a, b \in [0,1]$, $a \oplus b = \min\{1, a+b\}$, $\neg a = 1 - a$, $a \odot_\Pi b = ab$, and
$a \Rightarrow_\Pi b = \begin{cases} 1 & \text{if } a \leq b \\ \frac{b}{a} & \text{Otherwise.} \end{cases}$
The importance of the algebra above introduced lies in the fact that the whole class of $Ł\Pi\frac{1}{2}$-algebras is generated, as a variety, by the standard $Ł\Pi\frac{1}{2}$-algebra $[0,1]_{Ł\Pi\frac{1}{2}}$. In other words an equation $\varphi = \psi$ holds in every $Ł\Pi\frac{1}{2}$-algebra iff $\varphi = \psi$ holds in $[0,1]_{Ł\Pi\frac{1}{2}}$.

(2) Recalling Theorem 1.5, let $F(k)$ be the class of all the truth-tables of $Ł\Pi\frac{1}{2}$-formulas with at most k variables. Then the algebra

$$\mathcal{F}(k) = \langle F(k), \oplus, \neg, \odot_\Pi, \Rightarrow_\Pi, \mathbf{\tfrac{1}{2}}, \mathbf{0}, \mathbf{1}\rangle,$$

where the operations are defined by a pointwise application of the operation defined as in the above example (1), and $\mathbf{\frac{1}{2}}$, $\mathbf{0}$ and $\mathbf{1}$ respectively denote the functions constantly equal to $\frac{1}{2}$, 0 and 1, is an $Ł\Pi\frac{1}{2}$-algebra. Actually $\mathcal{F}(k)$ is the free $Ł\Pi\frac{1}{2}$-algebra over k generators.

In $Ł\Pi\frac{1}{2}$ propositional constants for rational values in $[0,1]$ are definable. In particular, passing through the functional representation theorem for $Ł\Pi\frac{1}{2}$ (Theorem 1.5), Cintula proved the following theorem in [7].

Theorem 2.3 ([7])**.** *Let $r \in [0,1]$ be a rational number. Then there is a formula ϕ_r of $Ł\Pi\frac{1}{2}$ such that $e(\phi_r) = r$ for any $[0,1]_{Ł\Pi\frac{1}{2}}$-evaluation e.*

Following the tradition, for each $r \in [0,1] \cap \mathbb{Q}$, we will use to denote ϕ_r by $\overline{r}$.

The algebra $\mathcal{F}(k)$ will turn out to be very useful in what follows. Indeed each element of $F(k)$ is an $Ł\Pi\frac{1}{2}$-fuzzy subset of the k-cube, hence $\mathcal{F}(k)$ describes all the possible fuzzy subset $\mathcal{F}_\varphi$ of $[0,1]^k$, φ being any $Ł\Pi\frac{1}{2}$-formula defined by using k propositional variables.

Example 2.4. *Let us consider the formula of three variable $\psi = (x \wedge y) \rightarrow z$. This formula induces the fuzzy set $\mathcal{F}_\psi : [0,1]^3 \rightarrow [0,1]$ such that, for any evaluation e mapping the variables x, y, z to $[0,1]$, $\mathcal{F}_\psi(e(x), e(y), e(z)) = min\{1, 1 - min\{e(x), e(y)\} + e(z)\}$. For the sake of simplicity, in the sequel we will avoid to explicitely state the evaluation e.*

3. $\mathcal{R}$-inner and $\mathcal{R}$-outer approximations

In the present section, starting from a finite set of rational numbers $\mathcal{R}$, we will define the $\mathcal{R}$-inner and the $\mathcal{R}$-outer approximation of a fuzzy set f. We will assume the reader to be familiar with the notion of *step-function* (if otherwise please see for instance [13]). As a notation, a step function ζ ranging on a finite set of rational number $\mathcal{R} = \{r_1, \ldots, r_n\}$, will be called an $\mathcal{R}$ *step-function* (or $\mathcal{R}$ s-f). For instance, the function $\zeta : \mathbb{R} \rightarrow \{0, 1/2, 1\}$ so defined: $\zeta(x) = 1$ if $x < 0$, $\zeta(0) = 1/2$ and $\zeta(x) = 0$ if $x > 0$ is an example of $\{0, 1/2, 1\}$ step-function.

Lemma 3.1. *Let $f : [0,1]^k \rightarrow [0,1]$ be a fuzzy subset of $[0,1]^k$, and let $\mathcal{R} = \{0, r_1, \ldots, r_n, 1\} \subset [0,1] \cap \mathbb{Q}$ be fixed. Then the set*

$$\mathcal{I}_f = \{\zeta : [0,1]^k \rightarrow \mathcal{R} \mid \zeta \textit{ is an } \mathcal{R} \textit{ s-f}, \zeta \leq f\}$$

has a maximum element, whereas the set

$$\mathcal{O}_f = \{\zeta : [0,1]^k \to \mathcal{R} \mid \zeta \textit{ is an } \mathcal{R} \textit{ s-f}, \zeta \geq f\}$$

has a minimum element.

Proof. We will prove it showing the maximum element of $\mathcal{I}_f$. The case relative to $\mathcal{O}_f$ is analogous and hence omitted. Without loss of generality we can assume $r_0 = 0 < r_1 < \ldots r_n < 1 = r_{n+1}$, hence let for $i = 0, \ldots, n$, $C_i = \{\underline{x} \in [0,1]^k \mid r_i \leq f(\underline{x}) < r_{i+1}\}$, and $C_{n+1} = \{\underline{x} \in [0,1]^k \mid f(\underline{x}) = 1\}$. Given that the domain of f is the whole k-cube $[0,1]^k$ it is clear that $C_0, C_1, \ldots, C_n, C_{n+1}$ provides a partition of $[0,1]^k$. Now define $\chi : [0,1]^k \to \mathcal{R}$ as: for each $i = 0, \ldots, n+1$, and for $\underline{x} \in C_i \subseteq [0,1]^k$,

$$\chi(\underline{x}) = r_i.$$

Notice that, in particular χ is constantly equal to 0 on C_0, while χ is constantly equal to 1 on C_{n+1}.
It is clear that $\chi \in \mathcal{I}_f$, moreover $\chi \geq \zeta$ for each $\zeta \in \mathcal{I}_f$. In fact if there is a $\zeta \neq \chi$, and a point $\underline{x} \in [0,1]^k$ where $\zeta(\underline{x}) > \chi(\underline{x}) = r_i$, then $\zeta(\underline{x}) \geq r_{i+1} > f(\underline{x})$ by the definition of the partition $\{C_i\}_{i=0,\ldots,n+1}$, and it leads to a contradiction because we assumed $\zeta \in \mathcal{I}_f$. Hence χ is the maximum element of $\mathcal{I}_f$. The proof is then complete. □

Definition 3.2. *Let $f : [0,1]^k \to [0,1]$ be a fuzzy subset of $[0,1]^k$, and let $\mathcal{R} = \{0, r_1, \ldots, r_n, 1\} \subset [0,1] \cap \mathbb{Q}$. Then we define:*

- *The $\mathcal{R}$-inner approximation of f to be $\chi = \max(\mathcal{I}_f)$.*
- *The $\mathcal{R}$-outer approximation of f to be $\tau = \min(\mathcal{O}_f)$.*

$\mathcal{I}_f$ and $\mathcal{O}_f$ being defined as in Lemma 3.1.

Remark 3.3. *Notice that there exist fuzzy subsets of $[0,1]^k$ whose range is not the whole real unit interval $[0,1]$, as for example $f : [0,1] \to [0,1/2]$ defined by $f(x) = \min\{x, 1-x\}$. In those cases it may happen that the rational values $r_1, \ldots, r_n$ we want to use for the definition of the $\mathcal{R}$-inner or the $\mathcal{R}$-outer approximations fall out of the range of f. This means that the above defined partition $C_0, \ldots, C_{n+1}$ may contain empty intervals as the following example shows:*
Consider the Ł$\Pi\frac{1}{2}$-formula $\varphi = p \wedge \neg p$ whose associated Ł$\Pi\frac{1}{2}$-term is the function $\mathcal{F}_\varphi = \min\{x, 1-x\}$ (for x ranging on $[0,1]$) having its maximum in $1/2$ (for $x = 1/2$). Let $r = 3/4$ the unique rational number we want to define the inner approximation with. Following the above lemma and definition, in order to get the $\left\{0, \frac{3}{4}, 1\right\}$-inner approximation of $\mathcal{F}_\varphi$, we have to define:

- $C_0 = \{x \in [0,1] \mid \mathcal{F}_\varphi(x) < 3/4\} = [0,1]$
- $C_1 = \{x \in [0,1] \mid 3/4 \leq \mathcal{F}_\varphi(x) < 1\} = C_2 = \{x \in [0,1] \mid \mathcal{F}_\varphi(x) = 1\} = \emptyset$.

Hence the $\left\{0, \frac{3}{4}, 1\right\}$-inner approximation of $\mathcal{F}_\varphi$ will be defined as: $\chi(x) = 0$ for all $x \in [0,1]$, that is the best $\left\{0, \frac{3}{4}, 1\right\}$-inner approximation we can describe for $\mathcal{F}_\varphi$ is the function constantly equal to 0.

The next lemma shows the notions of $\mathcal{R}$-inner and the $\mathcal{R}$-outer approximations of a fuzzy subset of $[0,1]^k$ to be dual notions.

Lemma 3.4. *Let $f : [0,1]^k \to [0,1]$ be a fuzzy subset of $[0,1]^k$, and let $\mathcal{R} = \{0, r_1, \ldots, r_n, 1\}$ be a finite set of rational numbers as above. Then $\chi : [0,1]^k \to \mathcal{R}$ is the $\mathcal{R}$-inner approximation of f iff $1-\chi$ is the $1-\mathcal{R}$-outer approximation of $1-f$, where $1-\mathcal{R} = \{0, 1-r_n, \ldots, 1-r_1, 1\}$.*

Proof. We will just prove the left-to-right direction, the other one easily follows by the involutive property of the function $1-x$ on $[0,1]$.
Let hence χ be the $\mathcal{R}$-inner approximation of f, and let $C_0, C_1, \ldots, C_n, C_{n+1}$ be the partition of $[0,1]^k$ defined as in Lemma 3.1. This means that for each $i = 0, \ldots, n+1$, and for all $\underline{x} \in [0,1]^k$,

$$1 - \chi(\underline{x}) = 1 - r_i.$$

In particular $1-\chi$ is constantly equal to 1 on C_0, whereas $1-\chi$ is constantly equal to 0 on C_{n+1}.
Hence $1-\chi$ is a $1-\mathcal{R}$ step-function and $1-\chi \geq f$. Moreover $1-\chi$ is minimum among all the $1-\mathcal{R}$ step functions $\zeta \geq f$. In fact is there is an $1-\mathcal{R}$ s-f and a point $\underline{x} \in [0,1]^k$ such that $\zeta(\underline{x}) < 1-\chi(\underline{x})$, then $1-\zeta$ will be a $\mathcal{R}$ s-f (it can be easily proved reasoning as above) and $1-\zeta(\underline{x}) > \chi(\underline{x})$. It is absurd χ being the $\mathcal{R}$-inner approximation of f. □

4. $\mathcal{R}$-inner and $\mathcal{R}$-outer approximations of ŁΠ$\frac{1}{2}$-fuzzy sets

Let now see more in detail how to define the $\mathcal{R}$-inner and the $\mathcal{R}$-outer approximations for functions being defined by formulas of ŁΠ$\frac{1}{2}$. The main result of this section says us how, starting from a finite set of rational numbers $\mathcal{R} = \{0, r_1, \ldots, r_n, 1\}$, one can define the ŁΠ$\frac{1}{2}$-connectives $\underset{\mathcal{R}}{\Delta}$ and $\underset{\mathcal{R}}{\nabla}$ satisfying the following property: If $\mathcal{F}_\varphi$ is an ŁΠ$\frac{1}{2}$-fuzzy set, then $\mathcal{F}_{\underset{\mathcal{R}}{\Delta}(\varphi)}$ and $\mathcal{F}_{\underset{\mathcal{R}}{\nabla}(\varphi)}$ are two ŁΠ$\frac{1}{2}$-fuzzy sets which respectively are the $\mathcal{R}$-inner, and the $\mathcal{R}$-outer approximation of $\mathcal{F}_\varphi$. Let start defining $\underset{\mathcal{R}}{\Delta}$ as:

$$\underset{\mathcal{R}}{\Delta}(\varphi) = \Delta\varphi \vee \left(\bigvee_{i=1}^{n} \overline{r_i} \&_\Pi \Delta(\overline{r_i} \to \varphi) \right).$$

Notice that, if $\mathcal{R}$ is empty what we obtain is: $\underset{\emptyset}{\Delta}(\varphi) = \Delta(\varphi)$. Hence, for every ŁΠ$\frac{1}{2}$-formula, and each $\underline{x}$ in the domain of $\mathcal{F}_\varphi$,

$$\mathcal{F}_{\underset{\emptyset}{\Delta}(\varphi)}(\underline{x}) = \begin{cases} 1 & \text{if } \mathcal{F}_\varphi(\underline{x}) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Thus it coincides with the first component of the rough approximation of $\mathcal{F}_\varphi$ (recall Definition 1.2).
Recalling Lemma 3.4 we are going now to define $\underset{\mathcal{R}}{\nabla}$ by duality. Hence, fixed $\mathcal{R} = \{0, r_1, \ldots, r_n, 1\} \subset [0,1] \cap \mathbb{Q}$ is the dual connective of $\underset{\mathcal{R}}{\Delta}$. Then, with respect to any ŁΠ$\frac{1}{2}$-formula φ, one has:

$$\underset{\mathcal{R}}{\nabla}(\varphi) = \neg\left(\underset{1-\mathcal{R}}{\Delta}(\neg\varphi) \right),$$

where $1-\mathcal{R}$ still denotes the set of rational numbers $\{0, 1-r_n, \ldots, 1-r_1, 1\}$. Hence

$$\underset{\mathcal{R}}{\nabla}(\varphi) = \neg\left(\Delta\neg\varphi \vee \left(\bigvee_{i=1}^{n} \neg\overline{r_i} \&_\Pi \Delta(\neg\overline{r_i} \to \neg\varphi) \right) \right) =$$

$$\nabla\varphi \wedge \left(\bigwedge_{i=1}^{n} \neg(\neg\overline{r_i} \&_\Pi \Delta(\varphi \to \overline{r_i})) \right).$$

If again $\mathcal{R}$ is empty, then $\underset{\emptyset}{\nabla}(\varphi) = \nabla(\varphi)$. Hence for every Ł$\Pi\frac{1}{2}$-formula φ, and every $\underline{x}$ in the domain of $\mathcal{F}_\varphi$, one has:

$$\mathcal{F}_{\underset{\emptyset}{\nabla}(\varphi)}(\underline{x}) = \begin{cases} 0 & \text{if } \mathcal{F}_\varphi(\underline{x}) = 0 \\ 1 & \text{otherwise.} \end{cases}$$

Thus $\mathcal{F}_{\underset{\emptyset}{\nabla}(\varphi)}$ is the second component of the rough approximation of $\mathcal{F}_\varphi$ (recall Definition 1.2).

Now we can prove the main result of this section. We are going to show that, given any Ł$\Pi\frac{1}{2}$-fuzzy set $\mathcal{F}_\varphi$, then the above defined connectives $\underset{\mathcal{R}}{\Delta}$ and $\underset{\mathcal{R}}{\nabla}$ actually define functions respectively being the $\mathcal{R}$-inner and $\mathcal{R}$-outer approximations of $\mathcal{F}_\varphi$. This also shows that for any Ł$\Pi\frac{1}{2}$-fuzzy set, both its $\mathcal{R}$-inner approximation and its $\mathcal{R}$-outer approximations still are Ł$\Pi\frac{1}{2}$-fuzzy sets.

Theorem 4.1. *Let $\varphi(x_1, \ldots, x_k)$ be an Ł$\Pi\frac{1}{2}$-formula, let $\mathcal{F}_\varphi : [0,1]^k \to [0,1]$ be the fuzzy subset of the k-cube defined by φ, and let $\mathcal{R} = \{0, r_1, \ldots, r_n, 1\} \subset [0,1] \cap \mathbb{Q}$. Then:*

(1) The function $\mathcal{F}_{\underset{\mathcal{R}}{\Delta}(\varphi)}$ is the $\mathcal{R}$-inner approximation of $\mathcal{F}_\varphi$.

(2) The function $\mathcal{F}_{\underset{\mathcal{R}}{\nabla}(\varphi)}$ is the $\mathcal{R}$-outer approximation of $\mathcal{F}_\varphi$.

Proof. (1) Recalling Definition 3.2, if we show that $\mathcal{F}_{\underset{\mathcal{R}}{\Delta}(\varphi)}$ behaves as the χ defined in the proof of Lemma 3.1, we are done. Let hence $r_0 = 0 < r_1 < \ldots < r_n < 1 = r_{n+1}$, and

- $C_0 = \{\underline{x} \in [0,1]^k \mid \mathcal{F}_\varphi(\underline{x}) < r_1\}$,
- for $i = 1, \ldots, n$, $C_i = \{\underline{x} \in [0,1]^k \mid r_i \leq \mathcal{F}_\varphi(\underline{x}) < r_{i+1}\}$,
- $C_{n+1} = \{\underline{x} \in [0,1]^k \mid \mathcal{F}_\varphi(\underline{x}) = 1\}$.

Now, for $\underline{x} \in [0,1]^k$ we have one of the following cases:

(i) If $\underline{x} \in C_0$, then $\mathcal{F}_{\underset{\mathcal{R}}{\Delta}(\varphi)}(\underline{x}) = \Delta\mathcal{F}_\varphi(\underline{x}) \vee (\bigvee_{i=1}^n r_i \odot_\Pi \Delta(r_i \Rightarrow \mathcal{F}_\varphi(\underline{x})))$, but $\Delta\mathcal{F}_\varphi(\underline{x}) = \Delta(r_i \Rightarrow \mathcal{F}_\varphi(\underline{x})) = 0$, hence $\mathcal{F}_{\underset{\mathcal{R}}{\Delta}(\varphi)}(\underline{x}) = 0$.

(ii) If $\underline{x} \in C_i$ (for some $i = 1, \ldots, n$), then $r_i \leq \mathcal{F}_{\underset{\mathcal{R}}{\Delta}(\varphi)}(\underline{x}) < r_{i+1} < 1$. This means that $\Delta\mathcal{F}_\varphi(\underline{x}) = 0$ and, for all $j \leq i$, $\Delta(r_j \Rightarrow \mathcal{F}_\varphi(\underline{x})) = 1$, while for $h > i$, $\Delta(r_h \Rightarrow \mathcal{F}_\varphi(\underline{x})) = 0$. Then $\mathcal{F}_{\underset{\mathcal{R}}{\Delta}(\varphi)(\underline{x})} = 0 \vee \left(\left(\bigvee_{j=1}^i r_j\right) \vee \left(\bigvee_{h=i+1}^n 0\right)\right) = r_i$.

(iii) If finally $\underline{x} \in C_{n+1}$, then $\mathcal{F}_\varphi(\underline{x}) = 1$. Thus $\Delta\mathcal{F}_\varphi(\underline{x}) = 1$. Hence $\mathcal{F}_{\underset{\mathcal{R}}{\Delta}(\varphi)}(\underline{x}) = 1$.

Then $\mathcal{F}_{\underset{\mathcal{R}}{\Delta}(\varphi)} = \chi$. Hence $\mathcal{F}_{\underset{\mathcal{R}}{\Delta}(\varphi)}$ is the $\mathcal{R}$-inner approximation of $\mathcal{F}_\varphi$.

(2) The claim easily follows by (1) and Lemma 3.4. $\square$

Henceforth, we will still denote by $\underset{\mathcal{R}}{\Delta}$ and $\underset{\mathcal{R}}{\nabla}$ the functionals mapping, for each $k \in \mathbb{N}$, $F(k)$ into $F(k)$ (recall that $F(k)$ denotes the domain of the free Ł$\Pi\frac{1}{2}$-algebra over k-generators) and associating to each Ł$\Pi\frac{1}{2}$-fuzzy set $\mathcal{F}_\varphi$ its $\mathcal{R}$-inner and, respectively, its $\mathcal{R}$-outer approximating function. That is

$$\underset{\mathcal{R}}{\Delta}(\mathcal{F}_\varphi) = \mathcal{F}_{\underset{\mathcal{R}}{\Delta}(\varphi)} \text{ and } \underset{\mathcal{R}}{\nabla}(\mathcal{F}_\varphi) = \mathcal{F}_{\underset{\mathcal{R}}{\nabla}(\varphi)}.$$

Example 4.2. *Let us consider the* $Ł\Pi\frac{1}{2}$*-formula* $\varphi(p) = 4.(p \ominus (p \odot_\Pi p))$ *(where for each* $n \in \mathbb{N}$ *and each* $Ł\Pi\frac{1}{2}$*-formula* φ*,* $n.\varphi$ *stands for* $\varphi \oplus \ldots \oplus \varphi$ *(n-times)) defining the* $Ł\Pi\frac{1}{2}$*-fuzzy set* $\mathcal{F}_\varphi(x) = 4(x - x^2)$*.*

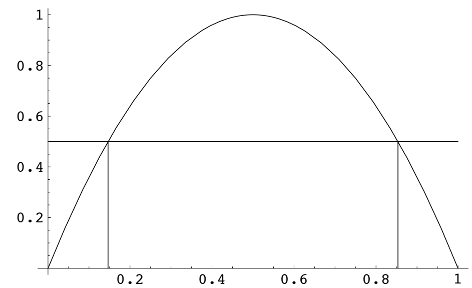

FIGURE 1. Fuzzy sets $\mathcal{F}_\varphi(x) = 4(x - x^2)$ and $\mathbf{\frac{1}{2}}$.

Thus, letting $\mathcal{R} = \{0, \frac{1}{2}, 1\}$*, we can compute the* $\mathcal{R}$*-inner approximation* $\underset{0,\frac{1}{2},1}{\Delta}\mathcal{F}_\varphi$*:*

$$\underset{0,\frac{1}{2},1}{\Delta}\mathcal{F}_\varphi(x) = \begin{cases} 0 & x \in C_0 = [0, \frac{\sqrt{2}-1}{2\sqrt{2}}) \cup (\frac{\sqrt{2}+1}{2\sqrt{2}}, 1] \\ \frac{1}{2} & x \in C_1 = [\frac{\sqrt{2}-1}{2\sqrt{2}}, \frac{1}{2}) \cup (\frac{1}{2}, \frac{\sqrt{2}+1}{2\sqrt{2}}] \\ 1 & x \in C_2 = \{\frac{1}{2}\}. \end{cases}$$

As can be seen in the last example, $\underset{0,\frac{1}{2},1}{\Delta}$ and $\underset{0,\frac{1}{2},1}{\nabla}$ give an approximation of a fuzzy set f through a three valued function:

$$\underset{0,\frac{1}{2},1}{\Delta}(f(x)) = \begin{cases} 0 & f(x) < \frac{1}{2} \\ \frac{1}{2} & f(x) \in [\frac{1}{2}, 1) \\ 1 & f(x) = 1 \end{cases} \qquad \underset{0,\frac{1}{2},1}{\nabla}(f(x)) = \begin{cases} 0 & f(x) = 0 \\ \frac{1}{2} & f(x) \in (0, \frac{1}{2}] \\ 1 & f(x) \in (\frac{1}{2}, 1] \end{cases}$$

Since shadowed sets [22, 3] act in a similar way it is natural to ask which is the relation among these different approximations.
First of all, let us recall the definition of shadowed sets.
Let $\alpha \in (0, \frac{1}{2})$ be a fixed value, the α–shadowed set of a fuzzy set f, denoted by $s_\alpha(f)$, is defined as

$$s_\alpha(f)(x) := \begin{cases} 0 & f(x) \leq \alpha \\ 1 & f(x) \geq 1 - \alpha \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

Evidently, a first difference between the two approaches is that shadowed sets depend on a parameter α, whereas generalized inner and outer approximations are uniquely defined (once set $\mathcal{R} = \{0, \frac{1}{2}, 1\}$).
Further, considering the values used to give the approximation, we have that among these three functions there exists the order relation: $\underset{0,\frac{1}{2},1}{\Delta} \leq s_\alpha \leq \underset{0,\frac{1}{2},1}{\nabla}$ as can be seen in detail in the following table:
We underline that, on the contrary, there does not exist an order relation between a fuzzy set and its related shadowed set, that is in general we cannot say that $f \leq s_\alpha(f)$ nor that $s_\alpha(f) \leq f$.

$f(x)$	$\underset{0,\frac{1}{2},1}{\Delta}$	$s_\alpha(x)$	$\underset{0,\frac{1}{2},1}{\nabla}$
0	0	0	0
$(0,\alpha]$	0	0	$\frac{1}{2}$
$(\alpha,\frac{1}{2})$	0	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$(\frac{1}{2},1-\alpha)$	$\frac{1}{2}$	$\frac{1}{2}$	1
$[1-\alpha,1)$	$\frac{1}{2}$	1	1
1	1	1	1

5. The rough approximation space of $\mathcal{R}$-inner and $\mathcal{R}$-outer approximations

Before analyzing the behaviour of the $\mathcal{R}$-inner and $\mathcal{R}$-outer approximation mappings, we recall some definitions and properties about abstract approximation spaces and topological operators. For a more detailed discussion, we refer to [2, 5].

Definition 5.1. *An abstract* approximation space *is a system* $\mathfrak{A} := \langle \Sigma, \mathbb{L}(\Sigma), \mathbb{U}(\Sigma)\rangle$*, where:*

(1) $\langle \Sigma, \wedge, \vee, 0, 1\rangle$ *is a bounded lattice. Elements from* Σ *are interpreted as concepts, data, etc., and are said to be* approximable *elements;*

(2) $\mathbb{L}(\Sigma)$ *and* $\mathbb{U}(\Sigma)$ *are bounded subposet of* Σ *(and thus* $0, 1 \in \mathbb{L}(\Sigma), \mathbb{U}(\Sigma)$*) consisting, respectively, of all available* lower (inner) *and* upper (outer) definable *elements;*

This system must satisfy the following axioms:

(Ax1): *For any approximable element* $a \in \Sigma$*, there exists one element* $i(a)$ *s.t.*

- $i(a)$ *is an inner definable element:* $i(a) \in \mathbb{L}(\Sigma)$*;*
- $i(a)$ *is an inner definable* lower *approximation of* a*:* $i(a) \le a$*;*
- $i(a)$ *is the best lower approximation of* a *by inner definable elements: let* $e \in \mathbb{L}(\Sigma)$ *be such that* $e \le a$*, then* $e \le i(a)$*.*

(Ax2): *For any approximable element* $a \in \Sigma$*, there exists one element* $o(a)$ *s.t.*

- $o(a)$ *is an outer definable element:* $o(a) \in \mathbb{U}(X)$*;*
- $o(a)$ *is an outer definable* upper *approximation of* a*:* $a \le o(a)$*;*
- $o(a)$ *is the best upper approximation of* a *by outer definable elements: let* $f \in \mathbb{U}(X)$ *be such that* $a \le f$*, then* $o(a) \le f$*.*

It turns out that abstract approximation spaces are equivalent, in a sense we are going to clarify, to lattices endowed with an interior and a closure operator.

Definition 5.2. *Let* $\langle \Sigma, \wedge, \vee, 0, 1\rangle$ *be a bounded lattice. An* interior *operator on* Σ *is a mapping* $^o : \Sigma \mapsto \Sigma$ *satisfying the following conditions for any* $a, b \in \Sigma$*:*

$(I1)$	$1^o = 1$	*(normalized)*
$(I2)$	$a^o \le a$	*(decreasing)*
$(I3)$	$a^o = a^{oo}$	*(idempotent)*
$(I4)$	$(a \wedge b)^o \le a^o \wedge b^o$	*(sub–multiplicative)*

Dually, a closure *operator on* Σ *is a mapping* $^* : \Sigma \mapsto \Sigma$ *satisfying for arbitrary* $a, b \in \Sigma$ *the following conditions:*

$(C1)$	$0^* = 0$	*(normalized)*
$(C2)$	$a \leq a^*$	*(increasing)*
$(C3)$	$a^* = a^{**}$	*(idempotent)*
$(C4)$	$a^* \vee b^* \leq (a \vee b)^*$	*(sub–additive)*

Topological (or Kuratowski) interior *and* closure *operations, are an interior and a closure operation in which condition (I4) and (C4) are substituted, respectively, by the following properties*

$(TI4)$	$a^o \wedge b^o = (a \wedge b)^o$	*(multiplicative)*
$(TC4)$	$a^* \vee b^* = (a \vee b)^*$	*(additive)*

Given an interior operator, we can define the subset of *open elements*, i.e., the collection of all elements which are equal to their interior: $\mathbb{O}(\Sigma) = \{a \in \Sigma : a = a^o\}$. Similarly, the set of *closed elements* is the collection of all elements which are equal to their closure: $\mathbb{C}(\Sigma) = \{a \in \Sigma : a = a^*\}$. Finally, the *clopen* elements are the ones which are at the same time open and closed.
The following proposition shows that rough approximation spaces and lattices with interior and closure are equivalent.

Proposition 5.3 ([2])**.** *First, suppose* $\mathcal{A} = \langle \Sigma, \mathbb{L}(\Sigma), \mathbb{U}(\Sigma) \rangle$ *is a rough approximation space and for arbitrary* $a \in \Sigma$ *let us define* $a^o := i(a)$ *and* $a^* := o(a)$*. Then,* $\mathcal{A}^{\blacktriangle} := \langle \Sigma, {}^o, {}^* \rangle$ *is a lattice equipped with an interior and a closure operation such that* $\mathbb{O}(\Sigma) = \mathbb{L}(\Sigma)$ *and* $\mathbb{C}(\Sigma) = \mathbb{U}(\Sigma)$*.*
Viceversa, suppose a lattice equipped with an interior and a closure operation $\mathcal{A} = \langle \Sigma, {}^o, {}^* \rangle$ *and let us define* $\mathbb{L}(\Sigma) := \mathbb{O}(\Sigma)$ *and* $\mathbb{U}(\Sigma) := \mathbb{C}(\Sigma)$*. Then,* $\mathcal{A}^{\blacktriangledown} := \langle \Sigma, \mathbb{L}(\Sigma), \mathbb{U}(\Sigma) \rangle$ *is a rough approximation space in which for arbitrary* a *it is* $i(a) = a^o$ *and* $o(a) = a^*$*.*
Finally, let $\mathcal{A} = \langle \Sigma, \mathbb{L}(\Sigma), \mathbb{U}(\Sigma) \rangle$ *be a rough approximation space. Then* $\mathcal{A}^{\blacktriangle\blacktriangledown} = \mathcal{A}$*. Dually, let* $\mathcal{A} = \langle \Sigma, {}^o, {}^* \rangle$ *be a lattice equipped with an interior and a closure operator. Then* $\mathcal{A}^{\blacktriangledown\blacktriangle} = \mathcal{A}$*.*

Now, for every finite $\mathcal{R} \subset [0,1] \cap \mathbb{Q}$, we prove that the $\mathcal{R}$-inner approximations and the $\mathcal{R}$-outer approximations introduced in the previous section have a topological behaviour and hence they give rise to a rough approximation space.
First of all recall that the free ŁΠ$\frac{1}{2}$-algebra over k generators $\mathcal{F}(k)$ consists in all the piecewise functions $\mathcal{F}_\varphi : [0,1]^k \to [0,1]$ such that all the $\mathcal{F}_\varphi$ pieces are fractions of continuous piecewise linear function, each piece having rational coefficients.

Proposition 5.4. *Let* $\underset{\mathcal{R}}{\Delta} : \mathcal{F}(k) \to \mathcal{F}(k)$ *be the* $\mathcal{R}$*-inner approximation map on* ŁΠ$\frac{1}{2}$ *fuzzy sets as defined in the previous section. Then,* $\underset{\mathcal{R}}{\Delta}$ *is an (additive) topological inner operator, that is, according to definition 5.2, the following hold:*

(a) $\underset{\mathcal{R}}{\Delta}(\mathcal{F}_{\overline{1}}) = \mathcal{F}_{\overline{1}}$,
(b) $\underset{\mathcal{R}}{\Delta}(\mathcal{F}_\varphi) \leq \mathcal{F}_\varphi$,
(c) $\underset{\mathcal{R}}{\Delta}(\mathcal{F}_\varphi) = \underset{\mathcal{R}}{\Delta}(\underset{\mathcal{R}}{\Delta}(\mathcal{F}_\varphi))$,

(d) $\underset{\mathcal{R}}{\Delta}(\mathcal{F}_{(\varphi_1\star\varphi_2)}) = \underset{\mathcal{R}}{\Delta}(\mathcal{F}_{\varphi_1})\hat{\star}\underset{\mathcal{R}}{\Delta}(\mathcal{F}_{\varphi_2})$*, for* $\star \in \{\wedge,\vee\}$*, and* $\hat{\star}$ *being the truth function associated to* $\star$*.*

Further $\underset{\mathcal{R}}{\Delta}$ *also satisfies the following property:*

(e) $\mathcal{F}_\varphi \odot \underset{\mathcal{R}}{\Delta}(\mathcal{F}_{\neg\varphi}) = \mathcal{F}_{\overline{0}}$*.*

Proof. Due to theorem 4.1, we have

$$\underset{\mathcal{R}}{\Delta}(\mathcal{F}_\varphi) = \max\{\zeta : [0,1]^k \to \mathcal{R} \mid \zeta \text{ is an } \mathcal{R} \text{ s-f}, \zeta \leq \mathcal{F}_\varphi\}. \tag{1}$$

So, points (a), (b) and (c) are trivially satisfied.
(d) We will prove the claim for $\star = \wedge$. Recall that in our notation

$$\underset{\mathcal{R}}{\Delta}(\mathcal{F}_{\varphi_1\wedge\varphi_2}) = \underset{\mathcal{R}}{\Delta}(\mathcal{F}_{\varphi_1} \wedge \mathcal{F}_{\varphi_2}),$$

and by (1) we have to prove that

$$\begin{gathered}\max\{\zeta : [0,1]^k \to \mathcal{R} \mid \zeta \text{ is an } \mathcal{R} \text{ s-f}, \zeta \leq \mathcal{F}_{\varphi_1} \wedge \mathcal{F}_{\varphi_2}\}\\ = \max\{\zeta : [0,1]^k \to \mathcal{R} \mid \zeta \text{ is an } \mathcal{R} \text{ s-f}, \zeta \leq \mathcal{F}_{\varphi_1}\}\wedge\\ \wedge \max\{\zeta : [0,1]^k \to \mathcal{R} \mid \zeta \text{ is an } \mathcal{R} \text{ s-f}, \zeta \leq \mathcal{F}_{\varphi_2}\}.\end{gathered}$$

This can be derived from the fact that $\forall x \in [0,1]^k$, it holds $\max_{\zeta(x)\in\mathcal{R}}\{\zeta(x) \leq \min\{\mathcal{F}_{\varphi_1}(x), \mathcal{F}_{\varphi_2}(x)\}\} = \min\{\max_{\zeta(x)\in\mathcal{R}}\{\zeta(x) \leq \mathcal{F}_{\varphi_1}(x)\}, \max_{\zeta(x)\in\mathcal{R}}\{\zeta(x) \leq \mathcal{F}_{\varphi_2}(x)\}\}$.
(e) By definition $\underset{\mathcal{R}}{\Delta}(\mathcal{F}_{\neg\varphi}) = \underset{\mathcal{R}}{\Delta}(\neg\mathcal{F}_\varphi)$. Then $\mathcal{F}_\varphi \odot \underset{\mathcal{R}}{\Delta}(\neg\mathcal{F}_\varphi)$. Now we know by (b) that $\underset{\mathcal{R}}{\Delta}(\neg\mathcal{F}_\varphi) \leq \neg\mathcal{F}_\varphi$ and $\mathcal{F}_\varphi \odot \neg\mathcal{F}_\varphi = \mathcal{F}_{\overline{0}}$. Therefore, a fortiori, $\mathcal{F}_\varphi \odot \underset{\mathcal{R}}{\Delta}(\mathcal{F}_{\neg\varphi}) = \mathcal{F}_{\overline{0}}$ given that $\odot$ is monotone. □

With respect to this interior operator, it is possible to define the collection of open elements as those fuzzy sets which coincide with their interior. In other words the subset $\mathbb{O}(k)$ of $F(k)$ defined as

$$\mathbb{O}(k) = \{\mathcal{F}_\varphi \in F(k) \mid \underset{\mathcal{R}}{\Delta}(\mathcal{F}_\varphi) = \mathcal{F}_\varphi\}$$

is the $F(k)$ subset of all its open elements.

In an analogous way it can be shown that, fixed a finite set of rational numbers $\mathcal{R} = \{0, r_1, \dots, r_n, 1\}$ for all $k \in \mathbb{N}$, the map $\underset{\mathcal{R}}{\nabla} : \mathcal{F}(k) \to \mathcal{F}(k)$ is a (multiplicative) topological closure operator, i.e., the following holds:

Proposition 5.5. *Let* $\underset{\mathcal{R}}{\nabla} : \mathcal{F}(k) \to \mathcal{F}(k)$ *be the* $\mathcal{R}$*-outer approximation map on* ŁΠ$\frac{1}{2}$ *fuzzy sets as defined in the previous section. Then,* $\underset{\mathcal{R}}{\nabla}$ *is a (multiplicative) topological outer operator, that is, according to definition 5.2, the following hold:*

(a) $\underset{\mathcal{R}}{\nabla}(\mathcal{F}_{\overline{0}}) = \mathcal{F}_{\overline{0}}$*,*

(b) $\underset{\mathcal{R}}{\nabla}(\mathcal{F}_\varphi) \geq \mathcal{F}_\varphi$*,*

(c) $\underset{\mathcal{R}}{\nabla}(\underset{\mathcal{R}}{\nabla}(\mathcal{F}_\varphi)) = \underset{\mathcal{R}}{\nabla}(\mathcal{F}_\varphi)$*,*

(d) $\underset{\mathcal{R}}{\nabla}(\mathcal{F}_{\varphi_1\star\varphi_2}) = \underset{\mathcal{R}}{\nabla}(\mathcal{F}_{\varphi_1})\hat{\star}\underset{\mathcal{R}}{\nabla}(\mathcal{F}_{\varphi_2})$*, for* $\star \in \{\wedge,\vee\}$*, and* $\hat{\star}$ *denoting the truth function associated to* $\star$*.*

Further $\underset{\mathcal{R}}{\nabla}$ *also fulfill the following:*

(e) $\mathcal{F}_\varphi \odot \underset{\mathcal{R}}{\nabla}(\mathcal{F}_{\neg\varphi}) = \mathcal{F}_{\overline{0}}$*.*

Proof. The steps allowing to prove the above proposition are analogous to those ones used in the proof of Proposition 5.4 and therefore omitted. □

Closed elements are then defined as those fuzzy sets which coincide with their closure. Let $\mathcal{F}(k)$ be the free ŁΠ$\frac{1}{2}$-algebra over k generators and $\underset{\mathcal{R}}{\nabla}$ an outer approximation map, then the subset $\mathbb{C}(k) \subseteq F(k)$ of *closed elements* is

$$\mathbb{C}(k) = \{\mathcal{F}_\varphi \in F(k) : \underset{\mathcal{R}}{\nabla}(\mathcal{F}_\varphi) = \mathcal{F}_\varphi\}.$$

It is easily seen that if a fuzzy set $\mathcal{F}_\varphi$ is open with respect to the interior operator relative to the set of rationals $\mathcal{R} = \{0 < r_1 < \ldots < r_n < 1\}$, then $\mathcal{F}_\varphi$ is also closed with respect to the outer operator based on the same set $\mathcal{R}$. Vice versa, any closed fuzzy set is also open. In conclusion, the set of clopen elements $\mathbb{CO}(k) := \mathbb{O}(k) \cap \mathbb{C}(k)$ coincides with both the set of closed and open elements.
Thanks to these two operators, it is possible (once fixed the set of rationals used to generate the approximations) to define the following *rough approximation space*:

- $\mathcal{F}(k)$ is the set of *approximable elements*;
- $\mathbb{CO}(k)$ is the set of *exact elements*;
- $\forall \mathcal{F}_\varphi \in F(k)$, $\underset{\mathcal{R}}{\Delta}(\mathcal{F}_\varphi)$ is the *lower approximation* of $\mathcal{F}_\varphi$ and $\underset{\mathcal{R}}{\nabla}(\mathcal{F}_\varphi)$ the *upper approximation* of $\mathcal{F}_\varphi$.

In this context a rough approximation of an ŁΠ$\frac{1}{2}$-fuzzy set $\mathcal{F}_\varphi$ is just the pair

$$r(\mathcal{F}_\varphi) = \langle \underset{\mathcal{R}}{\Delta}(\mathcal{F}_\varphi), \underset{\mathcal{R}}{\nabla}(\mathcal{F}_\varphi)\rangle \in \mathbb{CO}(k) \times \mathbb{CO}(k)$$

with $\underset{\mathcal{R}}{\Delta}(\mathcal{F}_\varphi) \leq \mathcal{F}_\varphi \leq \underset{\mathcal{R}}{\nabla}(\mathcal{F}_\varphi)$. Thus, $r(\mathcal{F}_\varphi)$ is the best approximation by exact (clopen) elements of $\mathcal{F}_\varphi$. That is, all the properties of axioms (Ax1) and (Ax2) in definition 5.1 are satisfied.
As a final remark, let us note that there exists a monotonicity property among approximations, as stated by the following proposition.

Proposition 5.6. *Let $\mathcal{R} = \{0, r_1, \ldots, r_n, 1\}$ and $\mathcal{S} = \{0, s_1, \ldots, s_m, 1\}$ two sets of rationals. If $\mathcal{R} \subseteq \mathcal{S}$ then, for all $k \in \mathbb{N}$, and $\forall \mathcal{F}_\varphi \in \mathcal{F}(k)$*

$$\underset{\mathcal{R}}{\Delta}(\mathcal{F}_\varphi) \leq \underset{\mathcal{S}}{\Delta}(\mathcal{F}_\varphi) \leq \mathcal{F}_\varphi \leq \underset{\mathcal{S}}{\nabla}(\mathcal{F}_\varphi) \leq \underset{\mathcal{R}}{\nabla}(\mathcal{F}_\varphi)$$

Proof. If $\mathcal{R} = \{0, r_1, \ldots r_n, 1\}$, then the interior operator $\underset{\mathcal{R}}{\Delta}(\mathcal{F}_\varphi)$ is defined as

$$\Delta\mathcal{F}_\varphi \vee \left(\bigvee_{i=0}^{n+1} r_i \odot_\Pi \Delta(r_i \Rightarrow \mathcal{F}_\varphi)\right).$$

Supposing that $\mathcal{S} = \mathcal{R} \cup \{s_{n+2}, \ldots, s_m\}$ then $\underset{\mathcal{S}}{\Delta}(\mathcal{F}_\varphi)$ is defined as

$$\Delta\mathcal{F}_\varphi \vee \left(\bigvee_{i=0}^{n+1} r_i \odot_\Pi \Delta(r_i \Rightarrow \mathcal{F}_\varphi)\right) \vee \left(\bigvee_{i=n+2}^{m} s_i \odot_\Pi \Delta(s_i \Rightarrow \mathcal{F}_\varphi)\right).$$

Thus, by trivial properties of $\vee$ we have that $\underset{\mathcal{R}}{\Delta}(\mathcal{F}_\varphi) \leq \underset{\mathcal{S}}{\Delta}(\mathcal{F}_\varphi)$.
Dually, for the closure operator we have:

$$\underset{\mathcal{R}}{\nabla}(\mathcal{F}_\varphi) = \nabla\mathcal{F}_\varphi \wedge \left(\bigwedge_{i=0}^{n+1} \neg(\neg r_i \odot_\Pi \Delta(\mathcal{F}_\varphi \Rightarrow r_i)\right).$$

and

$$\underset{\mathcal{S}}{\nabla}(\mathcal{F}_\varphi) = \nabla\mathcal{F}_\varphi \wedge \left(\bigwedge_{i=0}^{n+1} \neg(\neg r_i \odot_\Pi \Delta(\mathcal{F}_\varphi \Rightarrow r_i)\right) \wedge \left(\bigwedge_{i=n+2}^{m} \neg(\neg s_i \odot_\Pi \Delta(\mathcal{F}_\varphi \Rightarrow s_i)\right).$$

So, it easily follows that $\underset{\mathcal{S}}{\nabla}(\mathcal{F}_\varphi) \leq \underset{\mathcal{R}}{\nabla}(\mathcal{F}_\varphi)$. □

An increase in the number of rationals used to define an approximation means an augmented precision. So, as expected, proposition 5.6 says that to a better precision corresponds a better, i.e., closer to the original fuzzy set, approximation.

6. Relationship with Fuzzy Rough Sets

The relationship between fuzzy and rough sets has been analyzed by several authors and it brought to the definition of some hybrid model (see for instance [28]). Among them there are fuzzy rough sets which can be considered another way to approximate a fuzzy set through a pair of fuzzy or crisp sets [10, 27, 25, 15]. The approach is totally different from the one presented here, since it is based on an external similarity (sometimes equivalence) relation. Thus, we have not a pre-defined set of values which can be used to give the approximation. Instead, the approximation is obtained looking at the elements which are similar to some degree to the given fuzzy set. For the scope of the present paper, let us introduce the algebraic approach to fuzzy rough sets given in [26].
First, we need to recall some notions of fuzzy sets theory.

Definition 6.1 (cf [14])**.** *A* residuated lattice *is an algebra* $\mathcal{L} = \langle L, \wedge, \vee, *, \rightarrow, 0, 1\rangle$ *where:*

(1) $\langle L, \wedge, \vee, 0, 1\rangle$ *is a lattice with infimum* 0 *and supremum* 1.
(2) $\langle L, *, 0\rangle$ *is a commutative monoid.*
(3) $\mathcal{L}$ *satisfies the adjointness property: for all* $x, y, z \in L$, $x \leq y \rightarrow z$ *iff* $x * y \leq z$ *(*$\leq$ *being the induced lattice order).*

Definition 6.2. *Given a residuated lattice* $\mathcal{L} = \langle L, \wedge, \vee, *, \rightarrow, 0, 1\rangle$*, a* L-fuzzy similarity relation*, on a universe* X*, is defined as a binary fuzzy relation* $R : X \times X \mapsto L$ *satisfying*

(S1) $\forall x \in X\ R(x, x) = 1$ *(reflexive)*
(S2) $\forall x, y \in X\ R(x, y) = R(y, x)$ *(symmetric)*
(S3) $\forall x, y, z \in X\quad R(x, y) \geq R(x, z) * R(z, y)$ *(transitive)*

We can now define a fuzzy-rough approximation as follows.

Definition 6.3. *Given a residuated lattice* $\mathcal{L} = \langle L, \wedge, \vee, *, \rightarrow, 0, 1\rangle$*, a* fuzzy approximation space *is a pair* (X, R) *with* R *a* L*-fuzzy similarity relation on* X*. A* fuzzy rough approximation *for a fuzzy set* $f : X \mapsto L$ *is given by the pair of fuzzy sets* $\langle L_R(f), U_R(f)\rangle$ *defined pointwise as*

$$L_R(f)(x) := \inf_{y \in X}\{R(x, y) \rightarrow f(y)\} \quad \textit{Lower approximation} \tag{2}$$

$$U_R(f)(x) := \sup_{y \in X}\{R(x, y) * f(y)\} \quad \textit{Upper approximation} \tag{3}$$

Now, using some results of [26], it is possible to show that these approximation mappings have a topological behaviour.

Proposition 6.4. *Let (X, R) be a fuzzy approximation space and $\mathcal{L} = \langle L, \wedge, \vee, *, \rightarrow, 0, 1\rangle$ a complete residuated lattice. Then, the lower approximation $L_R : F(X) \mapsto F(X)$ is a topological inner operator:*

(L1) $L_R(\mathbf{1}) = \mathbf{1}$
(L2) $L_R(f) \leq f$
(L3) $L_R(f) = L_R(L_R(f))$
(L4) $L_R(f \wedge g) = L_R(f) \wedge L_R(g)$

On the other side, the upper approximation $U_R : F(X) \mapsto F(X)$ is a topological outer approximation:

(U1) $U_R(\mathbf{0}) = \mathbf{0}$
(U2) $f \leq U_R(f)$
(U3) $U_R(f) = U_R(U_R(f))$
(U4) $U_R(f \vee g) = U_R(f) \vee U_R(g)$

Proof. All these properties have been proved in [26]. Precisely: (L1),(L4),(U1) and (U4) in proposition 2; (L2) and (U2) in proposition 5; (L3) and (U3) in Corollary 2. □

Thus, since in $Ł\Pi\frac{1}{2}$, despite the three famous continuous t-norms (namely, $\wedge$, $\odot$, $\odot_\Pi$) as well as their corresponding residual implications, every continuous t-norm being a finite ordinal sum $\wedge, \odot$ and $\odot_\Pi$ and its corresponding residuum are definable in $Ł\Pi\frac{1}{2}$ (see the recent works [16, 17]), it is possible to define every fuzzy rough approximations (of course, after the definition of a fuzzy similarity relation R) made of a pair of inner-outer topological mappings.
So, from a topological-algebraic point of view, these approximations and the one defined in the previous sections have a similar behaviour. However, we remark that they are based on different ideas.
Of course, it would also be interesting to analyze all these approximations in practical situations, but such a study is clearly out of the scope of the present paper.

7. Conclusion and future work

A generalized notion of rough approximation of a fuzzy set has been investigated. The approximation is obtained according to a pre-defined set of rational values $\mathcal{R} = \{0, r_1, r_2, \ldots, r_n, 1\} \subset [0, 1]$ and is made of a lower approximation which gives the best approximation from below and an upper approximation which gives the best approximation from upward.
This rough approximation has been introduced in the context of $Ł\Pi\frac{1}{2}$ logic and we showed that if a fuzzy set is $Ł\Pi\frac{1}{2}$ definable then also its lower and upper approximations are.
Moreover, it has been proved that the lower and upper approximations are respectively a topological inner and outer operators on the collection of fuzzy sets $F(X)$ giving rise to a rough approximation space.
Finally, the notion of fuzzy rough sets is recalled and compared to our approach.
As a future work it would be interestering to describe a similar treatment for weaker logics than $Ł\Pi\frac{1}{2}$ and it will be useful (also from the application standpoint) to investigate how computationally hard is to find, given a fuzzy set f and a set of rational numbers $\mathcal{R}$, its $\mathcal{R}$-inner and its $\mathcal{R}$-outer approximations.

From the applicative point of view it is also worth to single out an evaluation criterion of the approximations, in order to understand which set of rational gives the best approximation.

References

[1] Cattaneo, G., Nisticò, G. Brouwer-Zadeh posets and three valued Łukasiewicz posets. *Fuzzy Sets and Systems* **33** (1989) 165–190

[2] Cattaneo, G. Abstract approximation spaces for rough theories. In Polkowski, L., Skowron, A., eds.: Rough Sets in Knowledge Discovery 1. Physica–Verlag, Heidelberg, New York (1998) 59–98

[3] Cattaneo, G., Ciucci, D. Shadowed sets and related algebraic structures. *Fundamenta Informaticae* **55**, 255–284, (2003)

[4] Cattaneo, G., Ciucci, D. Algebraic structures for rough sets. In Dubois, D., Gryzmala-Busse, J., Inuiguchi, M., Polkowski, L., eds.:*Fuzzy Rough Sets*. Volume 3135 of LNCS – Transactions on Rough Sets. Springer Verlag (2004) 218–264

[5] Cattaneo, G., Ciucci, D. Some methodological remarks about categorical equivalence in the abstract approach to roughness. Part I. Volume 4062 of LNCS. Springer Verlag (2006) 277–283

[6] Cintula P. An alternative approach to the ŁΠ logic, *Neural network world*, **11**, 561–572, 2001.

[7] Cintula P. Advances in the ŁΠ and Ł$\Pi\frac{1}{2}$ Logics, *Archive for Mathematical Logics*, **42**, 449–468, 2003.

[8] Ciucci D., Flaminio, T. Generalized Rough Approximation of Fuzzy Sets in Ł$\Pi\frac{1}{2}$, *Proc. ISFUROS'06, December 5-8, Santa Clara, Cuba*, Editorial Feijoo ISBN 959-250-308-7, 2006.

[9] Dubois, D., Prade, H. Fuzzy Sets and Systems. Theory and Applications. *Academic Press*, New York (1980)

[10] Dubois, D., Prade, H. Rough fuzzy sets and fuzzy rough sets, *Int.J. of General Systems*, **17**, 191–209, 1990.

[11] Esteva F., Godo L. Monoidal t-norm based logic: towards a logic for left-continuous t-norms. *Fuzzy Sets and Systems*, **124**, 271–288, 2001.

[12] Esteva F., Godo L., Montagna F. The ŁΠ and Ł$\Pi\frac{1}{2}$ logics: two complete fuzzy logics joining Łukasiewicz and product logic. *Archive for Mathematical Logic*, **40**, 39–67, 2001.

[13] Halmos P. R., *Measure Theory*. D. Van Nostrand, 1950.

[14] Hájek P. Metamathematics of Fuzzy Logic, *Kluwer*, 1998.

[15] Lin W.N., Yao J.T., Yao Y.Y. Rough Approximations under Level Fuzzy Sets. Volume 3066 of LNCS. Springer Verlag (2004) 78–83.

[16] Marchioni E., Montagna F., *A note in definability in* Ł$\Pi\frac{1}{2}$. In Proceedings of IPMU'06, 2006.

[17] Marchioni E., Montagna F., *On triangular norms and uninorms definable in* Ł$\Pi\frac{1}{2}$. International Journal of Approximate Reasoning. In print (2007).

[18] McNaughton R. A theorem about infinite valued sentential logic, *Journal of Symbolic Logic*,**16**, 1–13, 1951.

[19] Montagna F., Panti G. Adding structure to MV-algebras, *Journal of Pure and Applied Algebra*, **164**, 356–387, 2001.

[20] Pawlak, Z. Information systems - theoretical foundations. *Information Systems* **6** (1981) 205–218

[21] Pawlak, Z. Rough sets. *Int. J. Inform. Comput. Sci.* **11** (1982) 341–356

[22] Pedrycz, W. Shadowed sets: Representing and processing fuzzy sets. *IEEE Transaction on Systems, Man and Cybernetics - PART B: Cybernetics*, **28** (1998) 103–109

[23] Polkowski, L. Rough Sets. Mathematical Foundations. *Physica Verlag*, Heidelberg (2002)

[24] Radecki, T. Level fuzzy sets. *Journal of Cybernetics* **7** (1977) 189–198

[25] Radzikowska A.M., Kerre E.E., A comparative study of fuzzy rough sets, *Fuzzy Sets and Systems*, **126**, 137–155, 2002.

[26] Radzikowska A.M., Kerre E.E., Fuzzy Rough Sets Based on Residuated Lattices, *LNCS Transactions on Rough Sets 2*, **3135**, 278–296, 2004.

[27] Yao, Y.Y. Combination of rough and fuzzy sets based on α-level sets. In Lin, T.Y., Cercone, N., eds.: Rough Sets and Data Mining. Kluwer Academic press, Boston, 1997, 301–321

[28] Yao, Y.Y. A comparative study of fuzzy sets and rough sets. *Journal of Information Sciences* **109**, 227–242, 1998.

[29] Zadeh L. A. Fuzzy Sets, *Inform. and Control*, **8**, 1965, 338–353.

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