Feature Extraction by Fractional Order Differentiation

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The spectrum enhancement algorithm has been used for some time for separating structure from texture and extracting features from images (including optical scattering patterns) for automatic classification and recognition. The relevant definition and properties of spectrum enhancement and the relation of the latter to fractional differentiation are outlined below.

Let $Q\Omega$ denote a square of sidelength L and T the surface of the torus obtained by glueing the opposite sides of $Q\Omega$ together. Assume the grayscale image is modeled by a function $Qg[\boldsymbol{x}]$, $\boldsymbol{x} \equiv \{x_1, x_2\} \in Q\Omega$, which is continuous on T. Next let $Q\Omega$ be discretized by a square grid of steplength ℓ. Let $\boldsymbol{u} \equiv \{u_1, u_2\}$ be the spatial frequency vector. Then the discrete FOURIER transform $G[\boldsymbol{u}]$ of $Qg[.]$ is supported in the square $0 \leq |u_1|, |u_2| \leq u_{max} = L/2\,\ell - 1$ cycles/image. Let $\boldsymbol{u}$ be represented in polar coordinates $\boldsymbol{u} \equiv \{u, \theta\}$. Denote by $|G[\mathrm{u}]|^2$ the power spectral density. Let Θ denote an arc symmetric with respect to either axis ($\boldsymbol{u}_1$ or $\boldsymbol{u}_2$) and let the normalized, arc-averaged spectral density profile be the function $s[.]$ of $u = |\boldsymbol{u}|$ defined in $0 \leq u \leq u_{max}$ (cycles/image) according to

$$s[u] = \frac{1}{|\Theta|} \int_{\Theta} 10\mathrm{Log}_{10} \left[\frac{|G[u]|^2}{|G[0]|^2} \right] u \, d\theta, \tag{1}$$

where $|\Theta|$ is the length of Θ and obviously $|G\,[0]|^2 \neq 0$ for any non-degenerate image. Let $m[u]$ be a model spectral density such that

$$m^{(p)}[u] = 0, \quad 0 \leq u \leq 1; \quad m^{(p)}[u] = -10\mathrm{Log}_{10}[u^p], \quad u \geq 1 \quad \text{cycles/image}, \tag{2}$$

where p (>0) is the model exponent. Then, the $m^{(p)}[.]$-enhanced spectrum $h[u]$ is defined by

$$h[u] = s[u] - m^{(p)}[u], \quad 1 \leq u \leq u_{\max}. \tag{3}$$

Intuitively, the function $h^{(p)}[.]$ represents deviations of $s[.]$ from the model $m^{(p)}[.]$. The values of L, u_{max}, $|\Theta|$, p are determined by the intended application.

Assume the image is not degenerate. Then the following properties can be shown to hold.

a) If p satisfies $p/2 = N$ (>0), integer, then the tempered distribution $H^{(p)}[\boldsymbol{u}]$ defined by

$$H^{(p)}\,[u] = |u|^p \, \frac{|G\,[u]|^2}{|a_{0,0}|^2} + \delta\,[u]\,, \tag{4}$$

has the following representation in terms of FOURIER transforms (F) of derivatives of $Qg[.]$:

$$H^{(p)}[u] = \frac{1}{|a_{0,0}|^2} \sum_{n=0}^{N} \binom{N}{n} \left| F\left[\frac{\partial^N Qg}{\partial^{(N-n)} x_1 \partial^n x_2} \right] \right|^2 + \delta\,[u]\,. \tag{5}$$

b) If $p/2$ is not an integer, then fractional derivatives and anti-derivatives of $Qg[.]$ of net order $p/2$ appear in the representation of $H^{(p)}[\boldsymbol{u}]$ and the sum in Eq. (5) is replaced by a binomial series.

c) In either case, if all FOURIER coefficients satisfy $|a_{l,m}|^2 \geq \varepsilon > 0$ the relation between $H^{(p)}[.]$ of Eq. (4) and the enhanced spectrum $h^{(p)}[.]$ of Eq. (3) is

$$h[u] = \frac{10}{|\Theta|} \int_{\Theta} \mathrm{Log}_{10}\,[H[u]]\, u \, d\vartheta. \tag{6}$$