Morphological Characterization of Two-dimensional Random Media and Patterns by Fractional Differentiation

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Let $\Omega \subset \mathbf{R}^2$ denote a square of sidelength $\frac{L}{2}$, $\mathbf{x} \equiv \{x_1, x_2\} \in \Omega$ and g denote a scalar function of $\mathbf{x}$ representing a property of the random medium or, more generally, a pattern. Let reflection operators with respect to the coordinate axis (*flip*, *flop*) be applied to Ω and give rise to the square $\mathcal{Q}\Omega$. Denote by $\mathcal{Q}g$ the corresponding function supported in $\mathcal{Q}\Omega$. Let $\mathbf{u} \equiv \{u_1, u_2\}$ be the spatial frequency vector and $|G[\mathbf{u}]|^2$ the (distribution-valued) power spectral density of $\mathcal{Q}g$. The spectrum enhancement (*SE*) algorithm, which has been introduced before [e.g., 1] consists of suitable transformations carried out on the function $H^{(p)}[\mathbf{u}] := |\mathbf{u}|^{2\beta} \frac{|G[\mathbf{u}]|^2}{|a_{0,0}|^2} + \delta[\mathbf{u}]$, where $\delta$is the DIRAC measure, $a_{0,0}$ appears in the FOURIER transform at the origin, $\mathcal{F}(\mathcal{Q}g)[\mathbf{0}] = a_{0,0}\delta[\mathbf{u}]$ and $\beta \in \mathbf{R}^+$ is the *enhancement order* such that $\beta = 2p$. The intepretation of $H^{(p)}$ when $\beta \in \mathbf{N}$ has already been given [2] and shall not be repeated here. The emphasis herewith is on $\beta \notin \mathbf{N}$. Let $\Phi_0 \subset \mathcal{S}$ denote the LIZORKIN space of functions with vanishing moments $\Phi_0 := \{\omega_1 | \int_{-\infty}^{+\infty} x_1^{k_1} \omega_1[x_1] \mathrm{d}x_1 = 0,\ \forall k_1 = 0, 1, 2, ...\}$. Introduce the space of test functions $\Phi := \{\omega | \omega \in \Phi_0 \times \Phi_0\ ;\ \omega[x_1, x_2] = \omega_1[x_1]\omega_2[x_2]\}$ and denote by $\Phi' (\supset \mathcal{S}')$ its dual. By extending to two dimensions the properties stated in Ch. 4 of Ref. 3, a double integral of ω of fractional orders α_1, α_2 is defined by

$$\left(I_{2,+}^{\alpha_2} I_{1,+}^{\alpha_1} \omega\right)[x_1, x_2] := \frac{1}{\Gamma[\alpha_1]\Gamma[\alpha_2]} \int_{-\infty}^{x_2} \frac{\omega_2[y_2]}{(x_2 - y_2)^{1-\alpha_2}} \mathrm{d}y_2 \int_{-\infty}^{x_1} \frac{\omega_1[y_1]}{(x_1 - y_1)^{1-\alpha_1}} \mathrm{d}y_1.$$

Fractional derivatives $\left(\mathcal{D}_{2,+}^{\alpha_2} \mathcal{D}_{1,+}^{\alpha_1} \omega\right)[x_1, x_2]$ are defined in accordance. At this point the main result of the paper can be stated.

THM. Let $\mathcal{Q}g \in \Phi'$, $\beta \in \mathbf{R}^+$, $\beta \notin \mathbf{N}$, $p = 2\beta$, $\gamma = 0, 1, 2, ...$ Assume, without loss of generality, $|u_1| > |u_2|$ and let $\frac{\partial^\beta \mathcal{Q}g}{\partial^{(\beta-\gamma)} x_1 \partial^\gamma x_2} = \mathcal{D}_{1,+}^{(\beta-\gamma)} \mathcal{D}_{2,+}^{\gamma} \mathcal{Q}g$ if $\beta > \gamma$ or $= I_{1,+}^{(\gamma-\beta)} \mathcal{D}_{2,+}^{\gamma} \mathcal{Q}g$ if $\beta < \gamma$. Then

$$H^{(p)}[\mathbf{u}] = \frac{1}{|a_{0,0}|^2} \sum_{\gamma=0}^{\infty} \binom{\beta}{\gamma} \left| \left(\mathcal{F} \left[\frac{\partial^\beta \mathcal{Q}g}{\partial^{(\beta-\gamma)} x_1 \partial^\gamma x_2} \right] \right) [\mathbf{u}] \right|^2 + \delta[\mathbf{u}].$$

In other words *SE* of order p amounts to evaluating derivatives and integrals of fractional order of the pattern $\mathcal{Q}g$, FOURIER -transforming and forming a binomial series. This result contributes to the justification of *SE* as a method for extracting morphological descriptors from 2-dimensional images of random media with the aim of automatic classification and characterization.

REFERENCES

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