Option Pricing in Garch models

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Abstract

We review some classical Garch option pricing models in the unifying framework of conditional Esscher transform, in the spirit of Siu et al. ([8]). We propose a new model with Gamma innovations that admits a semianalytical recursive procedure for option pricing, similar to Heston and Nandi ([5]) and Christoffersen, Heston and Jacobs ([2]). We derive the risk neutral distribution of the innovations and the dynamics of the volatility using conditional Esscher transform; we compute the recursive relationships for the characteristic function of the underlying at maturity. We compare the calibration performances of the Gamma, the HN and CHJ models on closing prices of SP500 options.

Keywords: *Option Pricing, Garch models, Esscher transform, Gamma innovations, Semianalytical valuation.*

1 Introduction

Garch models have become increasingly popular in financial econometrics, either from a theoretical point of view or from a more applied one. Despite their simplicity, it is usually believed that they can capture some of the stylized facts characteristic of financial time series, such as volatility clustering and the presence of heavy tails in the unconditional distribution. Although their actual degree of realism is still under debate, they have firmly established both in the academy and in the industry. Usually, the main use of these models is to try to forecast future volatilities for Risk Management purposes; in this paper we focus on Option Pricing issues.

Since we are dealing with discrete time models with an infinity of states of the world, the model is typically incomplete; that is, there is not a unique price obtained by no-arbitrage arguments only. In other words, we face the well known problem of selecting a proper equivalent martingale measure for derivatives pricing.

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The first criterion proposed in Garch literature was Duan's (1995) "local risk neutral valuation relationship" (see [3]) that was motivated by equilibrium arguments under a specific choice of the utility function of the representative agent. The main drawback of the original Duan's model was that the risk neutral distribution of the underline is not available in analytic form, so option prices can only be computed by means of MonteCarlo simulation. A major step forward has been achieved with the Heston and Nandi (2000) model ([5]) since at the price of a slightly more complicated dynamic of the volatility, a recursive procedure for the computation of the characteristic function of the distribution of the underline at maturity was available. This opened the way for a semianalytical two-step pricing: in the first step the characteristic function of the underline at maturity is determined iteratively, in the second it is inverted numerically in order to compute option prices. This improves greatly the efficiency and stability of calibration procedures with respect to MonteCarlo methods. More recently an alternative model based on Inverse Gaussian innovations that admits a similar semianalytical valuation procedure has been proposed by Christoffersen, Heston and Jacobs (2004) (see [2]).

Up to this point however the "change of measure" problem was solved on a case by case basis, until in the remarkable paper ([8]) Siu, Tong and Yang pointed out that Duan's and Heston and Nandi's (HN henceforth) change of measure are actually a special case of the so-called "conditional Esscher transform". It is well known that Esscher transform is a very old actuarial tool that was first applied to option pricing in the seminal paper [4]; later the notion of "conditional Esscher transform" was developed for discrete time models (see for example [1] or [7]).

In this paper we propose an alternative model, based as the Siu et al. model ([8]) on Gamma innovations, but with a dynamic of the volatility similar to the CHJ model. After finding the equivalent martingale measure by the conditional Esscher transform approach (eq. 57), we are able to write the risk neutral dynamics of the volatility (eq. 58) and to compute the risk neutral parameters as a function of the initial parameters (eq. 60). Then, we show how a similar recursive procedure can be implemented and we obtain the recursive equations that are quite similar, although slightly simpler, to HN and CHJ models (eq. 62).

We conclude with a numerical comparison of the goodness of calibration of HN, CHJ and Gamma models on real option data; although the results are quite preliminary, they indicate that the Gamma model performs surely not worse than the other two and hence it deserves further investigations. The paper is structured as follows: in section 1 we review some basic facts about Esscher transform in Garch models, in section 2 we deal with the problem of computation of characteristic functions, in section 3 we introduce the Gamma model, while in section 4 we compare numerical results and calibrations of the three models.

2 Esscher transform in Garch models

The aim of this section is to review some of the existing Garch option pricing models and to present them in the unifying framework of the conditional Esscher transform, following the same approach of [8]. Let us fix some basic notations. A filtered complete probability space (Ω, F, F_t, P) is given with $t = 0, 1, ..., N$ and $F_N = F$. The stock price process is of the form

$$S_t = S_{t-1} \exp(X_t) \tag{1}$$

where X_t is adapted to the filtration F_t and represents the logarithmic return of the stock. The dynamic of the riskless security is specified by

$$B_t = B_{t-1} e^{r_t} \tag{2}$$

where r_t is a previsible process.

In order to price derivatives written on the stock S, it is necessary to find an equivalent martingale measure Q that is a measure Q that satisfies

$$E_Q[S_t | F_{t-1}] = S_{t-1} e^{r_t}$$

or equivalently

$$E_Q[\exp(X_t) | F_{t-1}] = e^{r_t} \tag{3}$$

For the class of models under consideration, this can be accomplished by means of the so-called *conditional Esscher transform*. We recall that the Esscher transform is a classical actuarial tool (see for example [10]) that has been applied to option pricing since the fundamental paper of Gerber and Shiu [4]. More recently, the notion of conditional Esscher transform has been proposed by Buhlmann et al. [1] and applied either in a theoretical framework (see for example [7]) or in an applied framework, especially in the field of discrete time models (see for example [8], [9]).

We adopt the same formalism as in [1] and [8]. Suppose that

$$E[\exp(\theta X_k) | F_{k-1}] < +\infty$$

for each $k = 1, ..., N$.
We denote with

$$M_{X_k | F_{k-1}}(\theta) = E[\exp(\theta X_k) | F_{k-1}] \tag{4}$$

the conditional moment generating function of X_k (m.g.f. henceforth) and define

$$\Lambda_t = \prod_{k=1}^{t} \frac{\exp(\theta_k X_k)}{M_{X_k | F_{k-1}}(\theta_k)}, \text{ for } t = 1, ..., N, \ \Lambda_0 = 1. \tag{5}$$

where θ_k is a previsible process.

We have that

$$
\begin{aligned}
E[\Lambda_t|F_{t-1}] &= \prod_{k=1}^{t-1} \frac{\exp(\theta_k X_k)}{M_{X_k|F_{k-1}}(\theta_k)} E[\frac{\exp(\theta_t X_t)}{M_{X_t|F_{t-1}}(\theta_t)}|F_{t-1}] = \\
&= \prod_{k=1}^{t-1} \frac{\exp(\theta_k X_k)}{M_{X_k|F_{k-1}}(\theta_k)} = \Lambda_t
\end{aligned}
$$

that is, Λ_t is a (positive) martingale with respect to the filtration F_k. Hence it defines a local change of measure with Radon-Nikodym density

$$\frac{dQ}{dP}|F_{t-1} = \Lambda_{t-1} \tag{6}$$

The martingale condition (3) for the discounted price process is satisfied if

$$E[\frac{\exp(\theta_t X_t)}{M_{X_t|F_{t-1}}(\theta_t)} \exp(X_t)|F_{t-1}] = e^{r_t}$$

that is if

$$\frac{M_{X_t|F_{t-1}}(\theta_t^* + 1)}{M_{X_t|F_{t-1}}(\theta_t^*)} = e^{r_t} \tag{7}$$

that is usually called the *conditional Esscher equation* or simply *Esscher equation.* Its solution determines the process θ_t that in turn determines the change of measure Λ_t through definition (5).

Moreover, the conditional m.g.f. of X under Q is given by

$$M^Q_{X_t|F_{t-1}}(\theta) = \frac{M_{X_t|F_{t-1}}(\theta_t^* + \theta)}{M_{X_t|F_{t-1}}(\theta_t^*)} \tag{8}$$

One of the main advantages of the Esscher approach is that often the conditional m.g.f. of the logreturns under Q can be determined explicitly from the previous equation.

Various models for the logreturns process X_t have been proposed in literature. We review them in the following subsections.

2.1 Models with normal innovations

2.1.1 The Duan model (1995)

The original Duan model [3] assumed that

$$
\begin{cases}
X_t = r + \lambda\sqrt{h_t} - \frac{h_t}{2} + \varepsilon_t \\
\varepsilon_t|F_{t-1} \sim N(0, h_t) \\
h_t = \alpha_0 + \alpha_1\varepsilon_{t-1}^2 + \beta_1 h_{t-1}
\end{cases}
\tag{9}
$$

that is ε_t follows a Garch (1,1) process with normal innovations.

In order to transform the discounted price process into a martingale, that is in order to satisfy equation (3), Duan originally proposed a change of measure that he called LRNVR ("locally risk neutralizing valuation relationship") that should satisfy the following assumptions:

- Q is equivalent to P
- $X_t|F_{t-1}$ has a normal distribution under Q
- $E_Q[\exp(X_t)|F_{t-1}] = e^r$
- $Var_Q[X_t|F_{t-1}] = Var_P[X_t|F_{t-1}]$

Adopting the Esscher approach, we have that the conditional m.g.f. is given by

$$M_{X_t|F_{t-1}}(\theta) = E[\exp(\theta X_t)|F_{t-1}] = \exp[\theta(r + \lambda\sqrt{h_t} - \frac{h_t}{2}) + \frac{h_t\theta^2}{2}] \tag{10}$$

and that the solution of the Esscher equation (7) is

$$\theta_t^* = -\frac{\lambda}{\sqrt{h_t}} \tag{11}$$

The conditional m.g.f. of X_t under Q is given by

$$M^Q_{X_t|F_{t-1}}(\theta) = \exp(\theta(r - \frac{h_t}{2}) + \frac{\theta^2}{2}h_t) \tag{12}$$

hence under Q we have that $X_t|F_{t-1}$ has still a normal distribution with

$$\begin{array}{l} E[X_t|F_{t-1}] = r - \frac{h_t}{2} \\ Var[X_t|F_{t-1}] = h_t \end{array} \tag{13}$$

It follows that all the four requirements of the Duan LRNVR are satisfied by the Esscher change of measure.
We note that

$$E_Q[\varepsilon_t|F_{t-1}] = -\lambda\sqrt{h_t}$$

hence introducing the process

$$\varepsilon'_t = \varepsilon_t + \lambda\sqrt{h_t}$$

we have that the dynamic of X_t under Q can be represented by

$$\begin{cases} X_t = r - \frac{h_t}{2} + \varepsilon'_t \\ \varepsilon'_t|F_{t-1} \sim N(0, h_t) \\ h_t = \alpha_0 + \alpha_1(\varepsilon'_{t-1} - \lambda\sqrt{h_{t-1}})^2 + \beta_1 h_{t-1} \end{cases} \tag{14}$$

Remark 1 *This model can be easily generalized to a generic risk premium $g(h_t)$ as follows:*

$$X_t = r + g(h_t) - \frac{h_t}{2} + \varepsilon_t$$

Repeating the calculations, we obtain that the Esscher parameter is given by

$$\theta_t^* = -\frac{g(h_t)}{h_t}$$

The conditional distribution of X_t under Q is still normal with mean $r - \frac{h_t}{2}$ and variance h_t, while the dynamics of the variance becomes

$$h_t = \alpha_0 + \alpha_1(\varepsilon'_{t-1} - g(h_{t-1}))^2 + \beta_1 h_{t-1}$$

2.1.2 The Heston and Nandi Model (2000)

Heston and Nandi [5] start with a different specification of the process under the P measure:

$$\begin{cases} X_t = r + \lambda h_t + \varepsilon_t \\ \varepsilon_t | F_{t-1} \sim N(0, h_t) \\ h_t = \alpha_0 + \alpha_1 \frac{(\varepsilon_{t-1} - \gamma h_{t-1})^2}{h_{t-1}} + \beta_1 h_{t-1} \end{cases} \tag{15}$$

The innovations are normal as in the Duan model, but the recursive equation for the variance is different. The other difference is that the risk premium is now linear in the variance. The Esscher change of measure follows the same lines as in the Duan model, since it depends only on the conditional distribution of the innovations, that is normal in both cases.
The conditional m.g.f. is given by

$$M_{X_t|F_{t-1}}(\theta) = E[\exp(\theta X_t)|F_{t-1}] = \exp[\theta(r + \lambda h_t) + \frac{\theta^2 h_t}{2}] \tag{16}$$

the Esscher equation (7) has explicit solution

$$\theta_t^* = -\lambda - \frac{1}{2} \tag{17}$$

and the m.g.f. of X_t under Q is given by

$$M^Q_{X_t|F_{t-1}}(\theta) = \exp[\theta(r - \frac{h_t}{2}) + \frac{\theta^2 h_t}{2}] \tag{18}$$

As in the Duan model, we have that $X_t|F_{t-1}$ has a normal distribution with

$$\begin{aligned} E[X_t|F_{t-1}] = r - \tfrac{h_t}{2} \\ Var[X_t|F_{t-1}] = h_t \end{aligned} \tag{19}$$

Since the conditional mean of the innovations under Q is given by

$$E_Q[\varepsilon_t|F_{t-1}] = -\frac{h_t}{2} - \lambda h_t$$

in order to specify the dynamic of X_t we introduce the process

$$\varepsilon'_t = \varepsilon_t + \lambda h_t + \frac{h_t}{2}$$

hence we have

$$\begin{cases} X_t = r - \frac{h_t}{2} + \varepsilon'_t \\ \varepsilon'_t|F_{t-1} \sim N(0, h_t) \\ h_t = \alpha_0 + \frac{\alpha_1(\varepsilon'_{t-1} - (\lambda + \frac{1}{2} + \gamma)h_{t-1})^2}{h_{t-1}} + \beta_1 h_{t-1} \end{cases} \tag{20}$$

2.2 Models with non-normal innovations

2.2.1 The Siu, Tong and Yang model (2004)

As previously remarked, the Esscher change of measure works very well if the m.g.f. of the innovations has a simple expression and the equation (7) can be solved explicitly. In the paper [8] Siu, Tong and Yang proposed to model the innovations with a *shifted gamma* distribution.

We recall that $Y \sim Ga(a, b)$ if its density is of the form

$$f_Y(x) = \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx}, \text{ with } a > 0,\ b > 0,\ x > 0 \tag{21}$$

We have that

$$E[e^{\theta Y}] = \left(\frac{b-\theta}{b}\right)^{-a} \tag{22}$$

and $E[Y] = \frac{a}{b}$, $Var[Y] = \frac{a}{b^2}$.
The proposed model is the following:

$$\begin{cases} X_t = r + \lambda\sqrt{h_t} - \frac{h_t}{2} + \varepsilon_t \\ \varepsilon_t|F_{t-1} = \sqrt{h_t}(\frac{Y_t - \frac{a}{b}}{\sqrt{\frac{a}{b^2}}}) \\ Y_t|F_{t-1} \sim Ga(a, b) \\ h_t = \alpha_0 + \alpha_1\varepsilon^2_{t-1} + \beta_1 h_{t-1} \end{cases} \tag{23}$$

the only difference with respect to the Duan model is that now the innovations

are shifted gamma.
The conditional m.g.f. can be easily computed as follows:

$$M_{X_t|F_{t-1}}(\theta) = E[\exp(\theta X_t)|F_{t-1}] = \exp[\theta(r+\lambda\sqrt{h_t}-\frac{h_t}{2}-\sqrt{ah_t})]\left(\frac{1}{1-\theta\sqrt{\frac{h_t}{a}}}\right)^a \tag{24}$$

The conditional Esscher equation (7) has now the explicit solution

$$\theta_t^* = \sqrt{\frac{a}{h_t}} - [1-\exp[(\frac{\lambda\sqrt{h_t}-\frac{h_t}{2}-\sqrt{ah_t}}{a})]]^{-1} \tag{25}$$

The conditional m.g.f. of X_t under Q is given by (8):

$$M^Q_{X_t|F_{t-1}}(\theta) = \exp[\theta(r+\lambda\sqrt{h_t}-\frac{h_t}{2}-\sqrt{ah_t})]\left(\frac{1}{1-\theta[1-\exp[(\frac{\lambda\sqrt{h_t}-\frac{h_t}{2}-\sqrt{ah_t}}{a})]]}\right)^a \tag{26}$$

by posing

$$b_t = [1-\exp[(\frac{\lambda\sqrt{h_t}-\frac{h_t}{2}-\sqrt{ah_t}}{a})]]^{-1} \tag{27}$$

we see that $X_t|F_{t-1}$ has still a shifted gamma distribution under Q with

$$\begin{array}{l} E[X_t|F_{t-1}] = r+\lambda\sqrt{h_t}-\frac{h_t}{2}-\sqrt{ah_t}+\frac{a}{b_t} \\ Var[X_t|F_{t-1}] = \frac{a}{b_t^2} \end{array} \tag{28}$$

It follows that also $\varepsilon_t|F_{t-1}$ has a shifted gamma distribution with

$$\begin{array}{l} E[\varepsilon_t|F_{t-1}] = -\sqrt{ah_t}+\frac{a}{b_t} \\ Var[\varepsilon_t|F_{t-1}] = \frac{a}{b_t^2} \end{array} \tag{29}$$

hence the dynamic of the model can be written as

$$\begin{cases} X_t = r+\lambda\sqrt{h_t}-\frac{h_t}{2}-\sqrt{ah_t}+\varepsilon'_t \\ \varepsilon'_t|F_{t-1} \sim Ga(a,b_t) \\ b_t = [1-\exp[(\frac{\lambda\sqrt{h_t}-\frac{h_t}{2}-\sqrt{ah_t}}{a})]]^{-1} \\ h_t = \alpha_0+\alpha_1(\varepsilon'_{t-1}-\sqrt{ah_{t-1}})^2+\beta_1 h_{t-1} \end{cases} \tag{30}$$

Remark 2 *The innovations under P are given by $\varepsilon_t|F_{t-1} = \sqrt{h_t}(\frac{Y_t-\frac{a}{b}}{\sqrt{\frac{a}{b^2}}})$ and are centered and depend on the scale parameter $\sqrt{h_t}$. The innovations under Q are given by $\varepsilon'_t|F_{t-1} \sim Ga(a,b_t)$ and are not centered, so in this case rescaled* *innovations are not identically distributed, contrary to what happens in the normal models. The evolution of the dynamic parameter under Q affects the shape of the distribution of the innovations in a more complicated fashion than just by scaling.*

Remark 3 *We have that*

$$\lim_{a\to+\infty} r + \lambda\sqrt{h_t} - \frac{h_t}{2} - \sqrt{ah_t} + a[1 - \exp[(\frac{\lambda\sqrt{h_t} - \frac{h_t}{2} - \sqrt{ah_t}}{a})]] = r - \frac{h_t}{2}$$

and

$$\lim_{a\to+\infty} a[1 - \exp[(\frac{\lambda\sqrt{h_t} - \frac{h_t}{2} - \sqrt{ah_t}}{a})]]^2 = h_t$$

that is, the Duan model can be recovered as a limiting case for $a \to +\infty$.

2.2.2 The Christoffersen, Heston and Jacobs model (2004)

Christoffersen, Heston and Jacobs [2] proposed a model with conditional *rescaled inverse gaussian innovations.*

We recall that Y has an inverse gaussian distribution $IG(\delta)$ if its density is of the form

$$f_Y(x) = \frac{\delta}{\sqrt{2\pi x^3}} \exp(-\frac{1}{2}(\sqrt{x} - \frac{\delta}{\sqrt{x}})^2) \text{ with } \delta > 0,\, x > 0. \tag{31}$$

the cumulative distribution function can be expressed as

$$F(t) = \int_{-\infty}^{t} f_Y(x)dx = e^{2\delta} N\left(-\sqrt{t} - \frac{\delta}{\sqrt{t}}\right) + N\left(\sqrt{t} - \frac{\delta}{\sqrt{t}}\right) \tag{32}$$

where $N(x)$ is the c.d.f. of a standard normal. See for example [6] for more properties of the IG distribution.

We have the following formula for a generalization of the m.g.f.:

$$E[\exp(\theta y + \frac{\phi}{y})] = \frac{\delta}{\sqrt{(\delta^2 - 2\phi)}} \exp(\delta - \sqrt{(\delta^2 - 2\phi)(1 - 2\theta)}) \tag{33}$$

the m.g.f is then obtained for $\phi = 0$

$$g(\theta) = E[\exp(\theta y)] = \exp(\delta(1 - \sqrt{1 - 2\theta})) \tag{34}$$

and the moments are given by

$$E[X] = g'(0) = \delta,\, E[X^2] = g''(0) = \delta^2 + \delta,\, Var[X] = \delta. \tag{35}$$

The proposed model is the following:

$$\begin{cases} X_t = r + \lambda h_t + \varepsilon_t \\ \varepsilon_t | F_{t-1} = \eta Y_t \\ Y_t \sim IG(\delta_t) \text{ with } \delta_t = \frac{h_t}{\eta^2} \\ h_t = \alpha_0 + \beta_1 h_{t-1} + \alpha_1 \varepsilon_{t-1} + \frac{\gamma h_{t-1}^2}{\varepsilon_{t-1}} \end{cases} \tag{36}$$

Remark 4 *Under P, the innovations have mean $E[\varepsilon_t|F_{t-1}] = \frac{h_t}{\eta}$ and variance $Var[\varepsilon_t|F_{t-1}] = h_t$. The α_1 term in the volatility dynamic equation is not squared, that is a very important difference with Heston-Nandi. As in the Siu. model, the normalized innovations $\frac{\varepsilon_t}{\sqrt{h_t}}$ are not identically distributed.*

The conditional m.g.f. is given by

$$M_{X_t|F_{t-1}}(\theta) = E[\exp(\theta X_t)|F_{t-1}] = \exp(\theta(r + \lambda h_t) + \frac{h_t}{\eta^2}(1 - \sqrt{1 - 2\eta\theta})) \tag{37}$$

the Esscher equation is

$$\exp(\lambda h_t + \frac{h_t}{\eta^2}(\sqrt{1 - 2\eta\theta} - \sqrt{1 - 2\eta(\theta + 1)}) = 1 \tag{38}$$

that gives the Esscher parameter

$$\theta^* = \frac{1}{2\eta}\left(1 - \eta - \frac{1}{\eta^2\lambda^2} - \frac{1}{4}\eta^4\lambda^2\right) \tag{39}$$

the m.g.f. under Q is

$$M^Q_{X_t|F_{t-1}}(\theta) = \exp(\theta(r + \lambda h_t) + \frac{h_t}{\eta^2}(\sqrt{1 - 2\eta\theta^*} - \sqrt{1 - 2\eta(\theta + \theta^*)}) \tag{40}$$

with some computations, we see that the following representation holds:

$$X_t|F_{t-1} = r + \lambda h_t + \eta' Y_t \tag{41}$$

with

$$\eta' = \frac{\eta}{[\frac{1}{\lambda\eta} + \frac{\eta^2\lambda}{2}]^2} = \frac{4\lambda^2\eta^3}{\left(\eta^3\lambda^2 + 2\right)^2} \tag{42}$$

and with $Y_t \sim IG(\delta_t)$ and

$$\delta_t = \frac{h_t}{\eta^2}[\frac{1}{\lambda\eta} + \frac{\eta^2\lambda}{2}] = \frac{h_t}{\eta^2}\sqrt{\frac{\eta}{\eta'}} \tag{43}$$

It follows that $X_t|F_{t-1}$ has still a rescaled $IG(\delta)$ distribution with

$$
\begin{array}{l}
E[X_t|F_{t-1}] = r + \lambda h_t + \frac{\lambda h_t}{2(\eta^3\lambda^2+2)} \\
Var[X_t|F_{t-1}] = \frac{h_t}{[\frac{1}{\lambda\eta}+\frac{\eta^2\lambda}{2}]^3} = h_t(\frac{\eta'}{\eta})^{\frac{3}{2}}
\end{array}
\tag{44}
$$

The dynamic under Q is given by

$$
\begin{cases}
X_t = r + \lambda h_t + \varepsilon_t \\
\varepsilon_t|F_{t-1} = \eta' Y_t, \text{ with } \eta' = \frac{\eta}{[\frac{1}{\lambda\eta}+\frac{\eta^2\lambda}{2}]^2} \\
Y_t \sim IG(\delta_t) \text{ with } \delta_t = \frac{h_t}{\eta^2}[\frac{1}{\lambda\eta} + \frac{\eta^2\lambda}{2}] = \frac{h_t}{\eta^2}\sqrt{\frac{\eta}{\eta'}} \\
h_t = \alpha_0 + \beta_1 h_{t-1} + \alpha_1\varepsilon_{t-1} + \frac{\gamma h_{t-1}^2}{\varepsilon_{t-1}}
\end{cases}
\tag{45}
$$

This set of equations completely specify the model under Q however, since in this model the conditional variance under Q is a linear function of the conditional variance under P, it is possible to rewrite the model in a simpler form.

If we pose

$$
\begin{cases}
\eta' = \frac{\eta}{[\frac{1}{\lambda\eta}+\frac{\eta^2\lambda}{2}]^2}, \quad \alpha_1' = (\frac{\eta'}{\eta})^{\frac{3}{2}}\alpha_1 \\
h_t' = h_t(\frac{\eta'}{\eta})^{\frac{3}{2}}, \qquad \gamma' = (\frac{\eta}{\eta'})^{\frac{3}{2}}\gamma \\
\alpha_0' = \alpha_0(\frac{\eta'}{\eta})^{\frac{3}{2}}, \qquad \lambda' = \lambda(\frac{\eta}{\eta'})^{\frac{3}{2}}
\end{cases}
\tag{46}
$$

the model becomes

$$
\begin{cases}
X_t = r + \lambda h_t + \varepsilon_t \\
\varepsilon_t|F_{t-1} = \eta' Y_t, \text{ with } \eta' = \frac{\eta}{[\frac{1}{\lambda\eta}+\frac{\eta^2\lambda}{2}]^2} \\
Y_t \sim IG(\delta_t') \text{ with } \delta_t' = \frac{h_t'}{\eta'^2} \\
h_t' = \alpha_0' + \beta_1 h_{t-1}' + \alpha_1'\varepsilon_{t-1} + \frac{\gamma' h_{t-1}'^2}{\varepsilon_{t-1}}
\end{cases}
\tag{47}
$$

3 The computation of option prices

In the preceding section we saw how using conditional Esscher transform it is possible to derive the risk-neutral distribution of the innovations and their recursive dynamics. The problem for option pricing is that in general only the distribution $X_{t+1}|F_t$ is known in an analytic form, while the distribution of the underlying at maturity T, that is $X_T|F_t$, does not admit such an explicit representation. For this reason in ([3]) and ([8]), among others, the valuation of options and the comparison with Black-Scholes prices has been based on MonteCarlo simulations.

A major breaktrough was due to Heston and Nandi ([5]), that proved that in their model it is possible to compute the m.g.f. of the distribution of $X_T|F_t$ by means of a *recursive procedure*, that can be easily implemented numerically. Option prices can then be recovered by numerical inversion of the Fourier transform.

After the Heston and Nandi paper, a search for similar models that admit this kind of *semianalytic valuation* has begun. In this section we review the recursive relationships for the HN and the CHJ models, while in the next section we propose a simple model with gamma innovations that has similar properties. For the ease of comparison we slightly modify the notations with respect to the preceding section, putting now

$$X_t = \ln S_t$$

Let

$$f(t, T, \theta) = E[\exp(\theta X_T)|F_t] = E[S_T^\theta|F_t] \tag{48}$$

be the conditional m.g.f of X_T evaluated at time t. We assume that the m.g.f. is log-linear in the two state variables X_t and h_{t+1}, that is

$$f(t, T, \theta) = \exp(\theta X_t + A(t, T, \theta) + B(t, T, \theta)h_{t+1}) \tag{49}$$

By the law of iterated expectations we have

$$E[f(t+1, T, \theta)|F_t] = f(t, T, \theta) \tag{50}$$

In the Heston and Nandi case, the first term can be explicitly computed (with some lengthy calculations, see the Appendix) using the relation

$$E[\exp(a(z+b)^2)] = \exp(-\frac{1}{2}\ln(1-2a) + \frac{ab^2}{1-2a}) \tag{51}$$

valid for $z \sim N(0, 1)$.

Substituting in equation (50) we get

$$\begin{cases} A(t, T, \theta) = \theta r + A(t+1, T, \theta) - \frac{1}{2}\ln(1 - 2\alpha_1 B(t+1, T, \theta)) + \alpha_0 B(t+1, T, \theta) \\ B(t, T, \theta) = \theta(\lambda+\gamma) - \frac{\gamma^2}{2} + \beta_1 B(t+1, T, \theta) + \frac{\frac{1}{2}(\theta-\gamma)^2}{1-2\alpha_1 B(t+1,T,\theta)} \end{cases} \tag{52}$$

In the Christoffersen et al. case, the integral in the first term of (50) can be easily calculated using the relation (33). After some calculations (see Appendix) we get the following recursive relations:

$$\begin{cases} A(t, T, \theta) = -\frac{1}{2}\ln[1 - 2\gamma\eta^4 B(t+1, T, \theta)] + \theta r + A(t+1, T, \theta) + \alpha_0 B(t+1, T, \theta) \\ B(t, T, \theta) = \beta_1 B(t+1, T, \theta) + \theta\lambda + \frac{1}{\eta^2} + \\ -\frac{1}{\eta^2}\sqrt{[1 - 2\gamma\eta^4 B(t+1, T, \theta)] \cdot [1 - 2\theta\eta - 2\alpha_1 B(t+1, T, \theta))]} \end{cases} \tag{53}$$

Both recursions, togheter with the terminal conditions

$$\begin{cases} A(T,T,\theta) = 0 \\ B(T,T,\theta) = 0 \end{cases} \tag{54}$$

can be used for the determination of the m.g.f. of $X_T|F_t$.

4 A simple model with Gamma innovations

In this section we propose a simple model with Gamma innovations, as in the Siu model, but with a different dynamic of the volatility, that is a special case of the CHJ model. We will see that also in this case the Esscher change of measure can be explicitly computed, and the m.g.f. of the underlying at maturity can be obtained through a recursive procedure.
The model is given by:

$$\begin{cases} X_t = r + \lambda h_t + \varepsilon_t \\ \varepsilon_t|F_{t-1} = -\frac{b}{\sqrt{a}} Y_t \\ Y_t \sim Ga(ah_t, b) \text{ with} \\ h_t = \alpha_0 + \beta_1 h_{t-1} + \alpha_1 \varepsilon_t \end{cases} \tag{55}$$

The dynamic of the volatility is a special case of CHJ's dynamic with $\gamma = 0$. The innovations have $E[\varepsilon_t] = -\sqrt{a} h_t$ and $Var[\varepsilon_t] = h_t$. The conditional m.g.f. is given by

$$M_{X_k|F_{k-1}}(\theta) = E[\exp(\theta X_k)|F_{k-1}] = \exp(\theta(r + \lambda h_t))(1 + \frac{\theta}{\sqrt{a}})^{-ah_t}$$

the conditional Esscher equation becomes

$$\frac{\exp((\theta^* + 1)(r + \lambda h_t))(1 + \frac{(\theta^*+1)}{\sqrt{a}})^{-ah_t}}{\exp(\theta^*(r + \lambda h_t))(1 + \frac{\theta}{\sqrt{a}})^{-ah_t}} = e^r \tag{56}$$

that gives

$$\theta^* = \frac{1}{\exp(\frac{\lambda}{a}) - 1} - \sqrt{a} \tag{57}$$

the conditional m.g.f. under Q is given by

$$M^Q_{X_t|F_{t-1}}(\theta) = \exp(\theta(r + \lambda h_t)) \left(1 + \frac{\theta}{\frac{1}{\exp(\frac{\lambda}{a})-1}}\right)^{-ah_t}$$

It follows that under Q

$$\begin{cases} X_t = r + \lambda h_t + \varepsilon_t \\ \varepsilon_t | F_{t-1} = -\frac{b}{\sqrt{a}} Y_t \\ Y_t \sim Ga(ah_t, \frac{b}{\sqrt{a}[\exp(\frac{\lambda}{a})-1]}) \text{ with} \\ h_t = \alpha_0 + \beta_1 h_{t-1} + \alpha_1 \varepsilon_t \end{cases} \tag{58}$$

Hence $X_t|F_{t-1}$ has still a shifted gamma distribution with

$$\begin{array}{l} E[X_t|F_{t-1}] = r + \lambda h_t - ah_t[\exp(\frac{\lambda}{a}) - 1] \\ Var[X_t|F_{t-1}] = ah_t[\exp(\frac{\lambda}{a}) - 1]^2 \end{array} \tag{59}$$

By posing

$$\begin{cases} b' = \frac{b}{\sqrt{a}[\exp(\frac{\lambda}{a})-1]}, & \beta'_1 = \beta_1 \\ h'_t = ah_t[\exp(\frac{\lambda}{a}) - 1]^2, & \alpha'_1 = \alpha_1 a[\exp(\frac{\lambda}{a}) - 1]^2 \\ a' = \frac{1}{[\exp(\frac{\lambda}{a})-1]^2}, & \lambda' = \frac{\lambda}{a[\exp(\frac{\lambda}{a})-1]^2} \\ \alpha'_0 = \alpha_0 a[\exp(\frac{\lambda}{a}) - 1]^2 \end{cases} \tag{60}$$

we see that the model can be rewritten as

$$\begin{cases} X_t = r + \lambda' h'_t + \varepsilon_t \\ \varepsilon_t | F_{t-1} = -\frac{b'}{\sqrt{a'}} Y_t \\ Y_t \sim Ga(a' h'_t, b') \text{ with} \\ h'_t = \alpha'_0 + \beta_1 h'_{t-1} + \alpha'_1 \varepsilon_t \end{cases} \tag{61}$$

The recursive relations can be computed along the preceding lines by assuming a log-linear form for the m.g.f. of $\ln S_T$

$$f(t, T, \theta) = \exp(\theta X_t + A(t, T, \theta) + B(t, T, \theta) h_{t+1})$$

here we have

$$\begin{array}{l} E[f(t+1, T, \theta)|F_t] = \exp[\theta(X_t + r + \lambda h_{t+1}) + A(t+1, T, \theta) + \\ + B(t+1, T, \theta)(\alpha_0 + \beta_1 h_{t+1})] \cdot E[\exp(\varepsilon_{t+1}(\theta + \alpha_1 B(t+1, T, \theta))|F_t] \end{array}$$

hence

$$\begin{array}{l} E[f(t+1, T, \theta)|F_t] = \exp[\theta(X_t + r + \lambda h_{t+1}) + A(t+1, T, \theta) + \\ + B(t+1, T, \theta)(\alpha_0 + \beta_1 h_{t+1})] \cdot \left(1 + \frac{\theta + \alpha_1 B(t+1, T, \theta)}{\sqrt{a}}\right)^{-ah_{t+1}} \end{array}$$

from equation (50) we obtain the desired recursion as follows:

$$\begin{cases} A(t,T,\theta) = \theta r + A(t+1,T,\theta) + \alpha_0 B(t+1,T,\theta) \\ B(t,T,\theta) = \theta\lambda + \beta_1 B(t+1,T,\theta) - a\ln\left(1 + \frac{\theta+\alpha_1 B(t+1,T,\theta)}{\sqrt{a}}\right) \end{cases} \tag{62}$$

From a numerical point of view this dynamic is quite simpler than the corresponding one in the HN and CHJ's models, since the first equation is linear. This is very important property from a practical point of view, since the iteration must be run for different discretized values of θ in order to reconstruct the m.g.f. of $\ln S_T$. So this is perhaps the simplest nontrivial Garch model that admits a semianalytic valuation procedure similar to HN. In the next section we will investigate its empirical performance versus the other two models.

5 Calibrations and comparisons

The aim of this section is to investigate the ability of the Gamma model to calibrate real options data and to compare it with HN and CHJ models. Altough the analysis carried out is quite preliminar, it already seems to indicate that the model is worth more extensive testing.

First of all, in order to check our numerical algorithms and to assess the efficiency of the semianalytical method, we compare MonteCarlo option prices and semianalytical prices on simulated data. To this end, we simulate model 58 with the following (risk neutral) parameters: $\alpha_0 = 7.14 \cdot 10^{-5}$, $\alpha_1 = 4.95 \cdot 10^{-6}$, $\beta_1 = 0.63$, $a = 8.08 \cdot 10^3$, $\lambda = 1, 10 \cdot 10^2$.
The chosen values are quite typical and have been obtained as a result of preliminary calibrations on the italian market..
We then compare MonteCarlo option prices (that is, discounted averages of the payoff in the N simulations) with the prices obtained by inverting the characteristic function of the distribution of S_T given by the recursive equations 62. The results for different strikes K, for different matirities T and for different number of simulations N are reported in the following tables:

Insert Table A around here

We see in general a good accordance between the two methods; for the longest maturity (T=90 days) we see that the semianalytical price tends to be slightly lower than the MonteCarlo price. The phenomenon is probably due to the increasing imprecision of the recursive method for longer maturities; also the issues related to the numerical inversion of the Fourier transform might be investigated furtherly. For the sake of comparison, we report in the following tables a similar numerical experiment with HN and CHJ models.

Insert Table B and Table C about here (63)

Having checked the validity of the numerical procedures on simulated data, we move to a comparison of the goodness of calibration of real option prices.

The dataset consists of 807 daily closing european call options prices on S&P500. Formally, we identify each price by the triplet (t, K, T) where t is the quotation date (varying from May 21, 2002 through June 16, 2002), K is the strike price (21 equally speced values ranging from 1000 to 1100) and T is the maturity (in our case "June", "July" and "August"). In the considered period, the S&P500 index has oscillated between the minimum value of 1010 and the maximum value of 1095. On this dataset we calibrated the HN, CHJ and Gamma models by the simple criterion of mean squared error minimization. That is, if we denote with $Call_{obs}(t, K, T)$ the observed option price and with $Call_{theo}(t, K, T)$ the theoretical price, we minimize

$$MSQ = \frac{1}{N} \sum_{t} \sum_{K} \sum_{T} \left[Call_{obs}(t, K, T) - Call_{theo}(t, K, T)\right]^2$$

It is well known that different criteria can give different calibrations' results; the main advantage of MSQ minimization is its simplicity, while a natural alternative could be the minimization of relative pricing error. Due to the preliminary nature of this calibration experiment we decided to follow the simplest approach. All minimizations (and inversion of characteristic functions) have been carried out numerically in Matlab environment.
In the following table we report the calibrated parameters for three models, considering the whole dataset of 807 prices:

Par.	HN	CHJ	Gamma
α_0'	$4.23 \cdot 10^{-5}$	$4.75 \cdot 10^{-6}$	$5.28 \cdot 10^{-10}$
α_1'	$2.80 \cdot 10^{-5}$	$2.69 \cdot 10^{-7}$	$5.30 \cdot 10^{-4}$
β_1'	$4.68 \cdot 10^{-1}$	$1.27 \cdot 10^{-6}$	$8.35 \cdot 10^{-1}$
γ'	4.67	0.500	–
a'	–	–	$9.50 \cdot 10^{4}$
η'	–	$7.64 \cdot 10^{-4}$	–
$\sqrt{MSQ}$	$4.30 \cdot 10^{-2}$	$4.04 \cdot 10^{-2}$	$3.95 \cdot 10^{-2}$

(64)

We see that the performance of the gamma model is comparable with the performance of HN and CHJ models, and actually has a smaller total mean squared error. In order to further analyze and compare the pricing errors, we define the average MSQ for each day t as

$$MSQ(t, T) = \frac{1}{N_K} \sum_{K} \left[Call_{obs}(t, K, T) - Call_{theo}(t, K, T)\right]^2$$

and the average MSQ for each strike K as

$$MSQ(K, T) = \frac{1}{N_t} \sum_{K} \left[Call_{obs}(t, K, T) - Call_{theo}(t, K, T)\right]^2$$

where $N_K = 21$ is the number of strikes and $N_t = 16$ are the number of strike prices and the number of days in the dataset.

In the following figure we plot the functions $MSQ\,(t, T)$ and $MSQ\,(K, T)$ for the three different models, for the two maturities June and July (we discarded August since we had quite few quotations):

Insert Fig.1 about here

Comparing the pictures, we see that for options with maturity July, the Gamma models outperforms the others uniformly with respect to the strike and with respect to the day. It is interesting to note that the time dependence of the average daily error is similar for the three models, with common upward and downward movements. For options with maturity June, the situation is less clear: we have that the Gamma model performs much worse for in the money and at the money options. This might be due to the fact that these options actually had a very short life to maturity (one week on average). Further analysis and comparisons are required, however it seems to us that the Gamma model performs at least not worse than HN and CHJ models.

6 Appendix

6.1 Tables and figures

$T = 60$ Gamma

N	K=0.95	K=0.975	K=1	K=1.025	K=1.05
500	0,07451	0,058689	0,04503	0,03345	0,02417
1000	0,07913	0,062982	0,04888	0,03697	0,02706
1500	0,07758	0,061114	0,04668	0,03475	0,02509
Semian. Price	0,07689	0,060556	0,04644	0,03462	0,02506

$T = 90$ Gamma

N	K=0.95	K=0.975	K=1	K=1.025	K=1.05
500	0,09129	0,075337	0,06113	0,04901	0,03859
1000	0,08972	0,07404	0,05996	0,04791	0,03796
1500	0,08941	0,073621	0,05962	0,04766	0,03760
Semian. Price	0,08762	0,071948	0,05811	0,04614	0,03600

TABLE A: Comparison between MonteCarlo and semianalytical prices on simulated data in the Gamma model

$T = 60$ Heston and Nandi

N	K=0.95	K=0.975	K=1	K=1.025	K=1.05
500	0,06947	0,051425	0,03606	0,02339	0,01412
1000	0,07165	0,053422	0,03780	0,02516	0,01547
1500	0,07017	0,052045	0,03642	0,02385	0,01447
Semian. Price	0,06911	0,050971	0,03548	0,02304	0,01378

$T = 90$ Heston and Nandi

N	K=0.95	K=0.975	K=1	K=1.025	K=1.05
500	0,07591	0,05886	0,04445	0,032359	0,02220
1000	0,07793	0,06066	0,04550	0,032525	0,02202
1500	0,07604	0,05895	0,04398	0,031314	0,02131
Semian. Price	0,07752	0,06003	0,04477	0,031963	0,02171

TABLE B: Comparison between MonteCarlo and semianalytical prices on simulated data in the Heston and Nandi model

$T = 60$	Christoffersen, Heston and Jacobs				
N	K=0.95	K=0.975	K=1	K=1.025	K=1.05
500	0,06472	0,045844	0,02986	0,01727	0,08656
1000	0,06383	0,044717	0,02891	0,01669	0,00840
1500	0,06467	0,045388	0,02915	0,01689	0,00885
Semian. Price	0,06419	0,044936	0,02888	0,01670	0,00852

$T = 90$	Christoffersen, Heston and Jacobs				
N	K=0.95	K=0.975	K=1	K=1.025	K=1.05
500	0,07162	0,053412	0,03743	0,02439	0,01487
1000	0,07184	0,053552	0,03772	0,02488	0,01544
1500	0,07153	0,052787	0,03657	0,02346	0,01376
Semian. Price	0,07104	0,052535	0,03674	0,02407	0,01464

TABLE C: Comparison between MonteCarlo and semianalytical prices on simulated data in the Christofferesen, Heston and Jacobs model

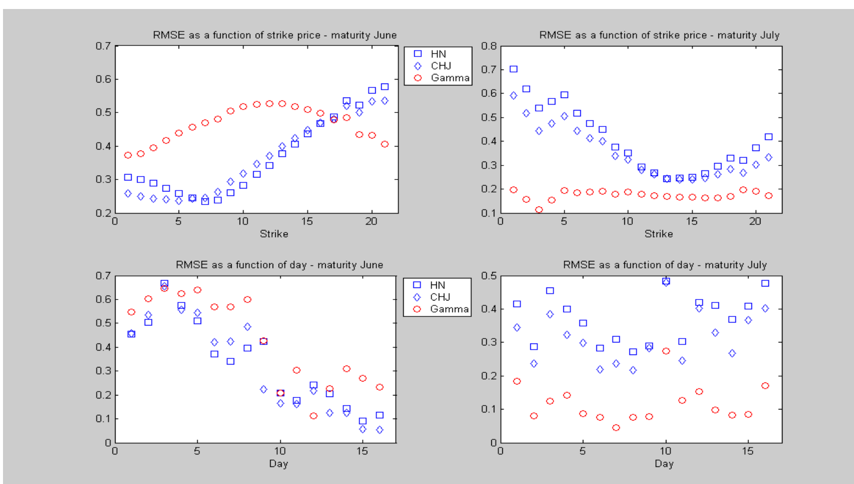

Fig. 1: Dependence of the Root Mean Square Error (RMSE) on the strike price (upper part) and on the day (lower part) for options with maturity June (left part) and July (right part).

6.2 Derivation of the recursive relations for the m.g.f.

We start by computing

$$E[f(t+1,T,\theta)|F_t] = f(t,T,\theta) \tag{65}$$

in the Heston case, where

$$f(t,T,\theta) = \exp(\theta X_t + A(t,T,\theta) + B(t,T,\theta)h_{t+1}) \tag{66}$$

We have that

$$E[f(t+1,T,\theta)|F_t] = \exp(\theta(X_t + r + \lambda h_{t+1}) + A(t+1,T,\theta)+$$
$$B(t+1,T,\theta)(\alpha_0 + \beta_1 h_{t+1})) \cdot E[\exp(\theta\sqrt{h_{t+1}}z_{t+1} + \alpha_1 B(t+1,T,\theta)(z_{t+1} - \gamma\sqrt{h_{t+1}})^2|F_t]$$

with some straightforward algebraic calculations we find

$$E[\exp(\theta\sqrt{h_{t+1}}z_{t+1} + \alpha_1 B(t+1,T,\theta)(z_{t+1} - \gamma\sqrt{h_{t+1}})^2|F_t]$$
$$= \exp(\alpha_1 B(t+1,T,\theta)(\gamma^2 h_{t+1} - \tfrac{h_{t+1}}{4}(\tfrac{\theta}{\alpha_1 B(t+1,T,\theta)} - 2\gamma)^2)\cdot$$
$$E[\exp(\alpha_1 B(t+1,T,\theta)[z_{t+1} + \tfrac{\sqrt{h_{t+1}}}{2}(\tfrac{\theta}{\alpha_1 B(t+1,T,\theta)} - 2\gamma)]^2)|F_t]$$

using the relation

$$E[\exp(a(z+b)^2)] = \exp(-\frac{1}{2}\ln(1-2a) + \frac{ab^2}{1-2a}) \tag{67}$$

valid for $z \sim N(0,1)$ we get

$$E[\exp(\alpha_1 B(t+1,T,\theta)[z_{t+1} + \tfrac{\sqrt{h_{t+1}}}{2}(\tfrac{\theta}{\alpha_1 B(t+1,T,\theta)} - 2\gamma)]^2)|F_t]$$
$$= \exp(-\tfrac{1}{2}\ln(1-2\alpha_1 B(t+1,T,\theta)) + \frac{\alpha_1 B(t+1,T,\theta)\frac{h_{t+1}}{4}(\frac{\theta}{\alpha_1 B(t+1,T,\theta)}-2\gamma)^2}{1-2\alpha_1 B(t+1,T,\theta)})$$

hence

$$E[f(t+1,T,\theta)|F_t] = \exp(\theta(X_t + r + \lambda h_{t+1}) + A(t+1,T,\theta)+$$
$$+B(t+1,T,\theta)(\alpha_0 + \beta_1 h_{t+1}) + \alpha_1 B(t+1,T,\theta)(\gamma^2 h_{t+1} - \tfrac{h_{t+1}}{4}(\tfrac{\theta}{\alpha_1 B(t+1,T,\theta)} - 2\gamma)^2)+$$
$$-\tfrac{1}{2}\ln(1-2\alpha_1 B(t+1,T,\theta)) + \frac{\alpha_1 B(t+1,T,\theta)\frac{h_{t+1}}{4}(\frac{\theta}{\alpha_1 B(t+1,T,\theta)}-2\gamma)^2}{1-2\alpha_1 B(t+1,T,\theta)})$$

From equation (50) this should coincide with 66; equating the terms we get (52).
In the Christoffersen's case we have

$$E[f(t+1,T,\theta)|F_t] = \exp[\theta(X_t + r + \lambda h_{t+1}) + A(t+1,T,\theta) + B(t+1,T,\theta)(\alpha_0 + \beta_1 h_{t+1})]\cdot$$
$$E[\exp(\varepsilon_{t+1}(\theta\eta + \alpha_1 B(t+1,T,\theta)) + \tfrac{B(t+1,T,\theta)\gamma h_{t+1}^2}{\varepsilon_{t+1}}))|F_t]$$

that can be easily calculated using the relation (33) since only terms of the form ε_{t+1} and $\frac{1}{\varepsilon_{t+1}}$ are present in the exponential. Remembering that $\delta_t = \frac{h_t}{\eta^2}$ we get

$$E[\exp(\varepsilon_{t+1}(\theta\eta + \alpha_1 B(t+1,T,\theta)) + \tfrac{B(t+1,T,\theta)\gamma h_{t+1}^2}{\varepsilon_{t+1}}))|F_t]$$
$$= \frac{\frac{1}{\eta^2}}{\sqrt{\frac{1}{\eta^4} - 2B(t+1,T,\theta)\gamma}}\exp(\tfrac{h_{t+1}}{\eta^2} - h_{t+1}\sqrt{[\tfrac{1}{\eta^4} - 2B(t+1,T,\theta)\gamma]\cdot[1 - 2(\theta\eta + \alpha_1 B(t+1,T,\theta))]})$$

from which the recursive relations (53) can be easily recovered.

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