Embedding right-angled Artin groups into Brin-Thompson groups

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Abstract

We prove that every finitely-generated right-angled Artin group embeds into some Brin-Thompson group nV. It follows that any virtually special group can be embedded into some nV, a class that includes surface groups, all finitely-generated Coxeter groups, and many one-ended hyperbolic groups.

1. *Introduction*

The **Brin-Thompson groups** nV are a family of higher-dimensional generalizations of Thompson's group V, defined by Brin in [**4**]. In this paper we prove the following theorem.

Theorem 1. *For any finite simple graph* Γ*, there exists an* $n \geq 1$ *so that the right-angled Artin group* A_Γ *embeds into* nV*.*

Here A_Γ is the group with one generator for each vertex of Γ, where two generators commute if the corresponding vertices are connected by an edge (see [**5**]). Note that the only right-angled Artin groups that embed into Thompson's group V are direct products of free groups [**2**, **6**].

Combining our results with those in [**2**] gives us the following theorem.

Theorem 2. *For any finite simple graph* Γ*, there exists an* $n \geq 1$ *with the following properties:*

(i) *The restricted wreath product* $nV \wr A_\Gamma = \left(\bigoplus_{A_\Gamma} nV\right) \rtimes A_\Gamma$ *embeds into* nV*.*

(ii) *If* G *is any group that has a subgroup of finite index that embeds into* A_Γ*, then* G *embeds into* nV*.*

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Proof. Statement (1) follows from the fact that our embedding of A_Γ into nV is *demonstrative* in the sense of [**2**]. Statement (2) follows from the fact that $nV \wr H$ embeds into nV for any finite group H (since H is demonstrable for nV), together with the Kaloujnine-Krasner theorem [**13**]. □

Many different groups are known to embed (or virtually embed) into right-angled Artin groups, including the "virtually special" groups of Haglund and Wise [**9**]. It follows from Theorem 1 that all such groups embed into some nV. Here is a partial list of such groups:

(i) All finitely generated Coxeter groups [**10**].

(ii) Many word hyperbolic groups, including all hyperbolic surface groups [**15**], and all one-relator groups with torsion [**16**].

(iii) All graph braid groups [**8**].

(iv) All limit groups [**16**].

(v) Many 3-manifold groups, including the fundamental groups of all compact 3-manifolds that admit a Riemannian metric of nonpositive curvature [**14**], as well as all finite-volume hyperbolic 3-manifolds [**1**].

In addition, it follows from some recent work of Bridson [**3**] that there exists an $n \geq 1$ such that nV has unsolvable isomorphism and subgroup membership problems for its finitely presented subgroups, and also has a finitely presented subgroup with unsolvable conjugacy problem.

Our proof shows that A_Γ embeds into nV for $n = |V| + |E^c|$, where V is the set of vertices of Γ and E^c is the set of complementary edges, i.e. the set of all pairs of generators that do *not* commute. Kato has subsequently strengthened this result to $n = |E^c|$ [**11**]. Kato's bound is not sharp, and it remains an open problem to determine the smallest n for which a given right-angled Artin group embeds into nV.

2. *Background and Notation*

We will need to consider a certain generalization of the groups nV. Given finite alphabets $\Sigma_1, \ldots, \Sigma_n$, the corresponding **Cantor cube** is the product

$$X = \Sigma_1^\omega \times \cdots \times \Sigma_n^\omega$$

where Σ_i^ω denotes the space of all infinite strings of symbols from Σ_i. Given any tuple $(\alpha_1, \ldots, \alpha_n)$, where each α_i is a finite string over Σ_i, the corresponding **subcube** of X is the collection of all tuples $(x_1, \ldots, x_n) \in X$ for which each x_i has α_i as a prefix. There is a **canonical homeomorphism** between any two such subcubes given by prefix replacement, i.e.

$$(\alpha_1 \cdot x_1, \ldots, \alpha_n \cdot x_n) \;\mapsto\; (\beta_1 \cdot x_1, \ldots, \beta_n \cdot x_n).$$

where $\cdot$ denotes concatenation of strings.

A homeomorphism h of X is called a **rearrangement** if there exist partitions $D_1, \ldots, D_k$ and $R_1, \ldots, R_k$ of X into finitely many subcubes such that h maps each D_i to R_i by the canonical homeomorphism. The rearrangements of X form a group under composition, which we refer to as XV. The case where $X = (\{0,1\}^\omega)^n$ gives the Brin-Thompson group nV.

The following proposition is easy to prove using complete binary prefix codes.

PROPOSITION 3. *If $X = \Sigma_1^\omega \times \cdots \times \Sigma_n^\omega$ is any Cantor cube, then XV embeds into the Brin-Thompson group nV.* □

Next we need a version of the ping-pong lemma for actions of right-angled Artin groups. The following is a slightly modified version of the ping-pong lemma for right-angled Artin groups stated in [**7**] (also see [**12**]).

THEOREM 4 (Ping-Pong Lemma for Right-Angled Artin Groups). *Let A_Γ be a right-angled Artin group with generators $g_1, \ldots, g_n$ acting on a set X. Suppose that there exist subsets $\{S_i^+\}_{i=1}^n$ and $\{S_i^-\}_{i=1}^n$ of X, with $S_i = S_i^+ \cup S_i^-$, satisfying the following conditions:*

(i) *$g_i(S_i^+) \subseteq S_i^+$ and $g_i^{-1}(S_i^-) \subseteq S_i^-$ for all i.*

(ii) *If g_i and g_j commute (with $i \neq j$), then $g_i(S_j) = S_j$.*

(iii) *If g_i and g_j do not commute, then $g_i(S_j) \subseteq S_i^+$ and $g_i^{-1}(S_j) \subseteq S_i^-$.*

(iv) *There exists a point $x \in X - \bigcup_{i=1}^n S_i$ such that $g_i(x) \in S_i^+$ and $g_i^{-1}(x) \in S_i^-$ for all i.*

Then the action of A_Γ on X is faithful. □

Indeed, if U is any subset of $X - \bigcup_{i=1}^n S_i$ such that $g_i(U) \subseteq S_i^+$ and $g_i^{-1}(x) \in S_i^-$, then all of the sets $\{g(U) \mid g \in A_\Gamma\}$ are disjoint. In the case where X is a topological space and U is an open set, this means that the action of A_Γ on X is demonstrative in the sense of [**2**].

3. *Embedding Right-Angled Artin Groups*

Let A_Γ be a right-angled Artin group with generators $g_1, \ldots, g_n$. For convenience, we assume that none of the generators g_i lie in the center of A_Γ. For in this case $A_\Gamma \cong A'_{\Gamma'} \times \mathbb{Z}$ for some right-angled Artin group $A'_{\Gamma'}$ with fewer generators, and since $sV \times \mathbb{Z}$ embeds in sV, any embedding $A'_{\Gamma'} \to kV$ yields an embedding $A_\Gamma \to kV$.

Let P be the set of all pairs $\{i,j\}$ for which $g_i g_j \neq g_j g_i$, and note that each $i \in \{1, \ldots, n\}$ lies in at least one element of P. Let X be the following Cantor cube:

$$X \;=\; \prod_{i=1}^{n} \{0,1\}^\omega \;\times\; \prod_{\{i,j\}\in P} \{i,j,\emptyset\}^\omega .$$

We claim that A_Γ embeds into XV, and hence embeds into kV for $k = n + |P|$.

We begin by establishing some notation:

(i) For each point $x \in X$, we will denote its components by $\{x_i\}_{i\in\{1,\ldots,n\}}$ and $\{x_{ij}\}_{\{i,j\}\in P}$.

(ii) Given any $i \in \{1, \ldots, n\}$ and $\alpha \in \{0,1\}^*$, let $C_i(\alpha)$ be the subcube consisting of all $x \in X$ for which x_i begins with α. Let $L_{i,\alpha} \colon X \to C_i(\alpha)$ be the canonical homeomorphism, i.e. the map that prepends α to x_i.

(iii) For each $i \in \{1, \ldots, n\}$, let P_i be the set of all j for which $\{i,j\} \in P$, and let S_i be the subcube consisting of all $x \in X$ such that x_{ij} begins with i for all $j \in P_i$. Let $F_i \colon X \to S_i$ be the canonical homeomorphism, i.e. the map that prepends i to x_{ij} for each $j \in P_i$.

(iv) Let $S_{ii} = F_i(S_i) = F_i^2(X)$, i.e. the subcube consisting of all $x \in X$ such that x_{ij} begins with ii for each $j \in P_i$.

Now, for each $i \in \{1, \ldots, n\}$, define a homeomorphism $h_i\colon X \to X$ as follows:

(i) h_i maps $X - S_i$ to $(S_i - S_{ii}) \cap C_i(10)$ via $L_{i,10} \circ F_i$.

(ii) h_i is the identity on S_{ii}.

(iii) h_i maps $(S_i - S_{ii}) \cap C_i(1)$ to $(S_i - S_{ii}) \cap C_i(11)$ via $L_{i,1}$.

(iv) h_i maps $(S_i - S_{ii}) \cap C_i(01)$ to $X - S_i$ via $F_i^{-1} \circ L_{i,01}^{-1}$.

(v) h_i maps $(S_i - S_{ii}) \cap C_i(00)$ to $(S_i - S_{ii}) \cap C_i(0)$ via $L_{i,0}^{-1}$.

Note that the five domain pieces form a partition of X, and each is the union of finitely many subcubes. Similarly, the five range pieces form a partition of X, and each is the union of finitely many subcubes. Since each of the maps is a restriction of a canonical homeomorphism, it follows that h_i is an element of XV.

Note that for each $i, j \in \{1, \ldots, n\}$, if g_i and g_j commute, then so do h_i and h_j. Thus we can define a homomorphism $\Phi\colon A_\Gamma \to XV$ by $\Phi(g_i) = h_i$ for each i.

Proposition 5. *The homomorphism Φ is injective.*

Proof. For each i, let $S_i^+ = S_i \cap C_i(1)$, and let $S_i^- = S_i \cap C_i(0)$. These two sets form a partition of S_i, with

$$h_i(S_i^+) = S_i \cap C_i(11) \subseteq S_i^+ \qquad \text{and} \qquad h_i^{-1}(S_i^-) = S_i \cap C_i(00) \subseteq S_i^-.$$

Now suppose we are given two generators g_i and g_j. If g_i and g_j commute, then clearly $h_i(S_j) = S_j$. If g_i and g_j do not commute, then $S_j \subseteq X - S_i$, and therefore $h_i(S_j) \subseteq S_i^+$ and $h_i^{-1}(S_j) \subseteq S_i^-$.

Finally, let x be an point in X such that x_{ij} starts with $\emptyset$ for all $\{i, j\} \in P$. Then $x \in X - S_i$ for all i, so $h_i(x) \in S_i^+$ and $h_i^{-1}(x) \in S_i^-$. The homomorphism Φ is thus injective by Theorem 4. $\square$

This proves Theorem 1. Further, as observed at the end of Section 2, this embedding is demonstrative in the sense of [**2**].

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