NONLINEAR SCHRÖDINGER EQUATIONS WITH SYMMETRIC MULTI-POLAR POTENTIALS

VERONICA FELLI AND SUSANNA TERRACINI

1. Introduction

Schrödinger equations with Hardy-type singular potentials have been the object of a quite large interest in the recent literature, see e.g. [1, 7, 8, 12, 16, 19, 21, 26, 27, 29]. The singularity of inverse square potentials $V(x) \sim \lambda|x|^{-2}$ is critical both from the mathematical and the physical point of view. As it does not belong to the Kato's class, it cannot be regarded as a lower order perturbation of the laplacian but strongly influences the properties of the associated Schrödinger operator. Moreover, from the point of view of nonrelativistic quantum mechanics, among potentials of type $V(x) \sim \lambda|x|^{-\alpha}$, the inverse square case represent a *transition threshold*: for $\lambda < 0$ and $\alpha > 2$ (attractively singular potential), the energy is not lower-bounded and a particle near the origin in the presence a potential of this type "falls" to the center, whereas if $\alpha < 2$ the discrete spectrum has a lower bound (see [22]). Moreover inverse square singular potentials arise in many fields, such as quantum mechanics, nuclear physics, molecular physics, and quantum cosmology; we refer to [18] for further discussion and motivation.

The case of multi-polar Hardy-type potentials was considered in [14, 11]. In particular in [14] the authors studied the ground states of the following class of nonlinear elliptic equations with a critical power-nonlinearity and a potential exhibiting multiple inverse square singularities:

$$\begin{cases} -\Delta v - \displaystyle\sum_{i=1}^{k} \frac{\lambda_i}{|x-a_i|^2}\, v = v^{2^*-1}, \\ v > 0 \quad \text{in } \mathbb{R}^N \setminus \{a_1, \dots, a_k\}. \end{cases} \tag{1}$$

For Schrödinger operators $-\Delta + V$ the potential term V describes the interactions of the quantum particles with the environment. Hence, multi-singular inverse-square potentials are associated with the interaction of particles with a finite number of electric dipoles. The mathematical interest in this problem rests in its criticality, for the exponent of the nonlinearity as well as the singularities share the same order of homogeneity with the laplacian.

The analysis carried out in [14] highlighted how the existence of solutions to (1) heavily depends on the strength and the location of the singularities. For the scaling properties of the problem, the mutual interaction among the poles actually depends only on the shape of their configuration. When the poles form a symmetric structure, it is natural to wonder how the symmetry affects such mutual interaction. The present paper means to study this aspect from the point of view of the existence of solutions inheriting the same symmetry properties as the set of singularities. More

Supported by Italy MIUR, national project "Variational Methods and Nonlinear Differential Equations".

2000 *Mathematics Subject Classification.* 35J60, 35J20, 35B33.

precisely we deal with a class of nonlinear elliptic equations on $\mathbb{R}^N$, $N \geq 3$, involving a critical power-nonlinearity as well as a Hardy-type potential which is singular on sets exhibiting some simple kind of symmetry.

Let us start by considering a potential featuring multiple inverse square singularities located on the vertices of k-side regular concentric polygons. Let us write $\mathbb{R}^N = \mathbb{R}^2 \times \mathbb{R}^{N-2}$. For $k \in \mathbb{N}$, we consider the group $\mathbb{Z}_k \times \mathbb{SO}(N-2)$ acting on $\mathcal{D}^{1,2}(\mathbb{R}^N)$ as $u(y', z) \to v(y', z) = u(e^{2\pi\sqrt{-1}/k} y', Tz)$, T being any rotation of $\mathbb{R}^{N-2}$.

Let us consider m regular polygons (with k sides) centered at the origin and lying on the plane $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^N$. Let us denote as a_i^ℓ, $i = 1, 2, \dots, k$, the vertices of the ℓ-th polygon, $\ell = 1, 2, \dots, m$. Since the polygons are regular, we have $a_i^\ell = e^{2\pi\sqrt{-1}/k} a_{i-1}^\ell$. We look for $\mathbb{Z}_k \times \mathbb{SO}(N-2)$-invariant solutions to the following equation

$$\begin{cases} -\Delta v - \dfrac{\lambda_0}{|x|^2} v - \displaystyle\sum_{\ell=1}^{m} \sum_{i=1}^{k} \dfrac{\lambda_\ell}{|x - a_i^\ell|^2}\, v = v^{2^*-1}, \\ v > 0 \quad \text{in } \mathbb{R}^N \setminus \{0, a_i^\ell : \ 1 \leq \ell \leq m, \ 1 \leq i \leq k\}, \end{cases} \tag{2}$$

where $2^* = \frac{2N}{N-2}$, i.e. for solutions belonging to the space

$$\mathcal{D}_k^{1,2}(\mathbb{R}^N) = \{u(z, y) \in \mathcal{D}^{1,2}(\mathbb{R}^N) : \ u(e^{2\pi\sqrt{-1}/k} z, y) = u(z, |y|)\}.$$

Here $\mathcal{D}^{1,2}(\mathbb{R}^N)$ denotes the closure of the space $\mathcal{D}(\mathbb{R}^N)$ of smooth functions with compact support with respect to the Dirichlet norm

$$\|u\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right)^{1/2}.$$

Let us denote as

$$r_\ell = |a_1^\ell| = |a_2^\ell| = \cdots = |a_k^\ell|, \quad \text{for } \ell = 1, 2, \dots, m,$$

the radius of the ℓ-polygon and as

$$\Lambda_\ell = k\lambda_\ell, \quad \ell = 1, \dots, m,$$

the total mass of poles located on the ℓ-th polygon. The Rayleigh quotient associated with problem (2) in $\mathcal{D}_k^{1,2}(\mathbb{R}^N)$ is

$$S_k(\lambda_0, \Lambda_1, \dots, \Lambda_m) = \inf_{\substack{u \in \mathcal{D}_k^{1,2}(\mathbb{R}^N) \\ u \not\equiv 0}} \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla u|^2 dx - \int_{\mathbb{R}^N} \left(\frac{\lambda_0}{|x|^2} + \sum_{\ell=1}^{m} \sum_{i=1}^{k} \frac{\Lambda_\ell}{k|x - a_i^\ell|^2} \right) u^2(x) \, dx}{\left(\displaystyle\int_{\mathbb{R}^N} |u|^{2^*} dx \right)^{2/2^*}}. \tag{3}$$

It is well known that minimizers of (3) solve equation (2) up to a Lagrange multiplier. Theorems 1.4 and 1.5 give sufficient conditions for the attainability of $S_k(\lambda_0, \Lambda_1, \dots, \Lambda_m)$ for large values of k. Letting $k \to \infty$, the Schrödinger operator converges, in the sense of distributions, to the operator associated with a continuous distribution of mass on concentric circles. We stress that the convergence of the potentials does not hold in the natural way, i.e. in $L^p_{\rm loc}(\mathbb{R}^N)$ for any $p \leq \frac{N}{2}$, because of the singularity. To formulate the limiting problem, for any $r > 0$, we denote as

$$S_r := \{(x, 0) \in \mathbb{R}^2 \times \mathbb{R}^{N-2} : \ |x| = r\}$$

the circle of radius r lying on the plane $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^N$ and consider the distribution $\delta_{S_r} \in \mathcal{D}'(\mathbb{R}^N)$ supported in S_r and defined by

$$_{\mathcal{D}'(\mathbb{R}^N)}\Big\langle \delta_{S_r}, \varphi \Big\rangle_{\mathcal{D}(\mathbb{R}^N)} := \fint_{S_r} \varphi(x)\, d\sigma(x) = \frac{1}{2\pi r}\int_{S_r} \varphi(x)\, d\sigma(x) \quad \text{for any } \varphi \in \mathcal{D}(\mathbb{R}^N),$$

where $d\sigma$ is the line element on S_r. We look for solutions to the following equation

$$\begin{cases} -\Delta v - \dfrac{\lambda_0}{|x|^2} v - \displaystyle\sum_{\ell=1}^{m} \Lambda_\ell \Big(\delta_{S_{r_\ell}} * \frac{1}{|x|^2}\Big) v = v^{2^*-1}, \\ v > 0 \quad \text{in } \mathbb{R}^N \setminus \Big\{0, \bigcup_{1\le \ell \le m} S_{r_\ell}\Big\}, \end{cases} \tag{4}$$

which are invariant by the action of the group $\mathbb{SO}(2) \times \mathbb{SO}(N-2)$. To this purpose, the natural space to set the problem is

$$\mathcal{D}^{1,2}_{\rm circ}(\mathbb{R}^N) = \{u(z,y) \in \mathcal{D}^{1,2}(\mathbb{R}^N) : \ u(z,y) = u(|z|,|y|)\},$$

and to consider the associated Rayleigh quotient

$$\begin{aligned} & S_{\rm circ}(\lambda_0, \Lambda_1, \dots, \Lambda_m) \\ & = \inf_{\substack{u \in \mathcal{D}^{1,2}_{\rm circ}(\mathbb{R}^N) \\ u \not\equiv 0}} \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla u|^2\, dx - \int_{\mathbb{R}^N} \bigg(\frac{\lambda_0}{|y|^2} + \sum_{\ell=1}^{m} \Lambda_\ell \fint_{S_{r_\ell}} \frac{d\sigma(x)}{|x-y|^2}\bigg) u^2(y)\, dy}{\displaystyle\bigg(\int_{\mathbb{R}^N} |u|^{2^*} dx\bigg)^{2/2^*}}. \end{aligned} \tag{5}$$

The following theorem contains a Hardy type inequality for potentials which are singular at circles.

Theorem 1.1. *Let $N \ge 3$ and $r > 0$. For any $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ the map $y \mapsto u(y) \int_{S_r} \frac{d\sigma(x)}{|x-y|^2}$ belongs to $L^2(\mathbb{R}^N)$ and*

$$\Big(\frac{N-2}{2}\Big)^2 \int_{\mathbb{R}^N} |u(y)|^2 \bigg(\fint_{S_r} \frac{d\sigma(x)}{|x-y|^2}\bigg) dy \le \int_{\mathbb{R}^N} |\nabla u(y)|^2\, dy. \tag{6}$$

Moreover the constant $\big(\frac{N-2}{2}\big)^2$ is optimal and not attained.

Hardy type inequalities involving singularities at smooth compact boundaryless manifolds have been considered by several authors, see [9, 17] and references therein. In the aforementioned papers, the potentials taken into account are of the type $|\text{dist}(x,\Sigma)|^{-2}$, where $\text{dist}(x,\Sigma)$ denotes the distance from a smooth compact manifold Σ. We point out that such kind of potentials are quite different from the ones we are considering. Indeed an explicit computation yields

$$V^r(y) := \fint_{S_r} \frac{d\sigma(x)}{|x-y|^2} = \frac{1}{\sqrt{(r^2+|y|^2)^2 - 4r^2|y'|^2}} \quad \text{for all } y = (y', z) \in \mathbb{R}^N = \mathbb{R}^2 \times \mathbb{R}^{N-2}.$$

Hence

$$V^r(y) \sim \frac{1}{|y|^2} \quad \text{as} \quad |y| \to +\infty$$

whereas

$$V^r(y) = \frac{1}{\sqrt{r^2+|y|^2-2r|y'|}}\,\frac{1}{\sqrt{r^2+|y|^2+2r|y'|}} \sim \frac{1}{2r\big||y|-r\big|} \quad \text{as} \quad \text{dist}(y, S_r) \to 0.$$

Hence the singularity at the circle of V^r is weaker than the inverse square distance potential considered in [9, 17], but has the same behavior at ∞. We also remark that V^r is "regular" in the sense of the classification of singular potentials given in [18].

Arguing as in [14, Proposition 1.1], it is easy to verify that solvability of equations (2) and (4) requires the positivity of the associated quadratic forms. Let us consider for example the quadratic form associated with potentials singular on circles, i.e.

$$Q^{\rm circ}_{\lambda_0,\Lambda_1,\dots,\Lambda_m} = \int_{\mathbb{R}^N} |\nabla u|^2\,dx - \int_{\mathbb{R}^N}\bigg(\frac{\lambda_0}{|y|^2} + \sum_{\ell=1}^m \Lambda_\ell \fint_{S_{r_\ell}} \frac{d\sigma(x)}{|x-y|^2}\bigg) u^2(y)\,dy.$$

From (6) and Sobolev's inequality, it follows that

$$Q^{\rm circ}_{\lambda_0,\Lambda_1,\dots,\Lambda_m} \geq \bigg[1 - \frac{4}{(N-2)^2}\big(\lambda_0^+ + \textstyle\sum_{\ell=1}^m \Lambda_\ell^+\big)\bigg]\int_{\mathbb{R}^N} |\nabla u|^2\,dx,$$

where $t^+ := \max\{t, 0\}$ denotes the positive part. Hence $Q^{\rm circ}_{\lambda_0,\Lambda_1,\dots,\Lambda_m}$ is positive definite whenever

$$\lambda_0^+ + \sum_{\ell=1}^m \Lambda_\ell^+ < \frac{(N-2)^2}{4}, \tag{7}$$

see [14, Proposition 1.2] for further discussion on the positivity of the quadratic form. Condition (7) also ensures the positivity of the quadratic form associated to problem (2).

The attainability of $S_k(\lambda_0, \Lambda_1, \dots, \Lambda_m)$ and $S_{\rm circ}(\lambda_0, \Lambda_1, \dots, \Lambda_m)$ requires a delicate balance between the contribution of positive and negative masses. In particular, if $N \geq 4$ and all the masses have the same sign, $S_k(\lambda_0, \Lambda_1, \dots, \Lambda_m)$ and $S_{\rm circ}(\lambda_0, \Lambda_1, \dots, \Lambda_m)$ are never achieved.

Theorem 1.2. *Let $N \geq 4$, $\lambda_0, \Lambda_1, \dots, \Lambda_m \in \mathbb{R}$, $r_1, r_2, \dots, r_m \in \mathbb{R}^+$ satisfy (7). If*

$$\begin{aligned} &\textit{either} \quad \text{(i)} \quad \Lambda_\ell < 0 && \textit{for all } \ell = 1, \dots, m \\ &\textit{or} \quad \text{(ii)} \quad \lambda_0 > 0,\ \Lambda_\ell > 0 && \textit{for all } \ell = 1, \dots, m \end{aligned}$$

then neither $S_k(\lambda_0, \Lambda_1, \dots, \Lambda_m)$ nor $S_{\rm circ}(\lambda_0, \Lambda_1, \dots, \Lambda_m)$ are attained.

The analysis we are going to carry out in the present paper will highlight that, from the point of view of minimization of the Rayleigh quotient, spreading mass all over a continuum is more convenient than localization of mass at isolated points.

Theorem 1.3. *Let $N \geq 4$, $\lambda_0, \Lambda_1, \dots, \Lambda_m \in \mathbb{R}$, $r_1, r_2, \dots, r_m \in \mathbb{R}^+$ satisfy (7) and*

$$\sum_{\ell=1}^m \Lambda_\ell \leq 0, \qquad \lambda_0 < \frac{(N-2)^2}{4}, \tag{8}$$

and

$$\begin{cases} \displaystyle\sum_{\ell=1}^m \frac{\Lambda_\ell}{|r_\ell|^2} > 0, & \textit{if} \quad \lambda_0 \leq \dfrac{N(N-4)}{4}, \\ \displaystyle\sum_{\ell=1}^m \frac{\Lambda_\ell}{|r_\ell|^{\sqrt{(N-2)^2-4\lambda_0}}} > 0, & \textit{if} \quad \dfrac{N(N-4)}{4} < \lambda_0 < \dfrac{(N-2)^2}{4}. \end{cases} \tag{9}$$

Then the infimum in (5) is achieved. In particular equation (4) admits a solution which is $\mathbb{SO}(2) \times \mathbb{SO}(N-2)$-invariant.

As problem (5) is the limit of (3), when $k \to \infty$, we expect the assumptions of Theorem 1.3 to ensure the existence of solutions to (3) provided k is sufficiently large. Indeed the theorem below states that (8) and (9) are sufficient conditions on radii and masses of the polygons for the infimum in (3) to be achieved when k is large.

Theorem 1.4. *Let $N \geq 4$, $\lambda_0, \Lambda_1, \dots, \Lambda_m \in \mathbb{R}$, $r_1, r_2, \dots, r_m \in \mathbb{R}^+$ satisfy (7),* (8) *and* (9). *For any $\ell = 1, \dots, m$ and $k \in \mathbb{N}$ let $\{a_i^\ell\}_{i=1,\dots,k}$ be the vertices of a regular k-side polygon centered at* 0 *of radius r_ℓ and let $\lambda_\ell = \Lambda_\ell / k$. Then if k is sufficiently large, the infimum in (3) is achieved. In particular equation (2) admits a solution which is $\mathbb{Z}_k \times \mathbb{SO}(N-2)$-invariant.*

When $N > 4$, it is possible to estimate how large k must be in order to obtain the above existence result. This is the content of the following theorem.

Theorem 1.5. *Assume that $N > 4$, $\lambda_0^+ + k \sum_{j=1}^m \lambda_j^+ < \frac{(N-2)^2}{4}$,*

$$\sum_{\ell=1}^{m} \lambda_\ell \leq 0, \tag{10}$$

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m \leq \frac{N(N-4)}{4}, \quad \lambda_0 < \frac{(N-2)^2}{4}, \tag{11}$$

$$\begin{cases} \displaystyle\sum_{\ell=1}^{m} \frac{\lambda_\ell}{|r_\ell|^2} > 0, & \text{if} \quad \lambda_0 \leq \dfrac{N(N-4)}{4}, \\ \displaystyle\sum_{\ell=1}^{m} \frac{\lambda_\ell}{|r_\ell| \sqrt{(N-2)^2 - 4\lambda_0}} > 0, & \text{if} \quad \dfrac{N(N-4)}{4} < \lambda_0 < \dfrac{(N-2)^2}{4}, \end{cases} \tag{12}$$

$$\frac{\lambda_0}{|r_m|^2} + \lambda_m \sum_{i=1}^{k-1} \frac{1}{4r_m^2 \left| \sin \frac{i\pi}{k} \right|^2} + \sum_{\ell=1}^{m-1} \lambda_\ell \sum_{i=1}^{k} \frac{1}{r_m^2 + r_\ell^2 - 2r_m r_\ell \cos\left(\frac{2\pi i}{k} + \Theta_{j\ell}\right)} > 0, \tag{13}$$

where $\Theta_{j\ell}$ denoted the minimum angle formed by vectors a_i^j and a_s^ℓ, (see figure 4). Then the infimum in (3) is achieved. In particular equation (2) admits a solution.

This paper is organized as follows. In section 2 we recall some known facts about the single-polar problem and study the behavior of any solution (radial and non radial) to the one pole-equation near the singularities 0 and ∞. In section 3 we prove the Hardy type inequality for potentials which are singular at circles stated in Theorem 1.1. Section 4 contains an analysis of possible reasons for lack of compactness of minimizing sequences of problems (3) and (5) and a local Palais-Smale condition below some critical thresholds. In section 5 we provide some interaction estimates which are needed in section 6 to compare the concentration levels of minimization sequences and consequently to prove Theorem 1.3. Section 7 contains the study of behavior of energy levels of minimizing sequences as $k \to \infty$ which is needed in section 8 to prove Theorem 1.4. Last section is devoted to the proof of Theorem 1.5.

Notation. We list below some notation used throughout the paper.

- $B(a, r)$ denotes the ball $\{x \in \mathbb{R}^N : |x| < r\}$ in $\mathbb{R}^N$ with center at a and radius r.
- $S_r = \{(x, 0) \in \mathbb{R}^2 \times \mathbb{R}^{N-2} : |x| = r\}$ denotes the circle of radius r in the plane $\mathbb{R}^2 \times \{0\}$.
- δ_x denotes the Dirac mass located at point $x \in \mathbb{R}^N$.
- $\mathcal{D}(\mathbb{R}^N)$ is the space of smooth functions with compact support in $\mathbb{R}^N$.

- $\mathcal{D}^{1,2}(\mathbb{R}^N)$ is the closure of $\mathcal{D}(\mathbb{R}^N)$ with respect to the Dirichlet norm $(\int_{\mathbb{R}^N} |\nabla u|^2\,dx)^{1/2}$.
- $C_0(\mathbb{R}^N)$ denotes the closure of continuous functions with compact support in $\mathbb{R}^N$ with respect to the uniform norm.
- $\mathrm{dist}(x, A)$ denotes the distance of the point $x \in \mathbb{R}^N$ from the set $A \subset \mathbb{R}^N$.
- $\|\cdot\|_p$ denotes the norm in the Lebesgue space $L^p(\mathbb{R}^N)$.
- $A \triangle B$ denotes the symmetric difference of sets A and B, i.e. $A\triangle B = (A \setminus B) \cup (B \setminus A)$.

2. The problem with one singularity

For any $\lambda < (N-2)^2/4$, the problem with one singularity

$$\begin{cases} -\Delta u = \dfrac{\lambda}{|x|^2} u + u^{2^*-1}, \quad x \in \mathbb{R}^N, \\ u > 0 \text{ in } \mathbb{R}^N \setminus \{0\}, \text{ and } u \in \mathcal{D}^{1,2}(\mathbb{R}^N), \end{cases} \tag{14}$$

admits a family of positive solutions given by

$$w_\mu^\lambda(x) = \mu^{-\frac{N-2}{2}} w_1^\lambda\Big(\frac{x}{\mu}\Big), \ \mu > 0, \tag{15}$$

where we denote

$$w_1^\lambda(x) = \frac{\big(N(N-2)\nu_\lambda^2\big)^{\frac{N-2}{4}}}{\big(|x|^{1-\nu_\lambda}(1+|x|^{2\nu_\lambda})\big)^{\frac{N-2}{2}}}, \quad \text{and} \quad \nu_\lambda = \bigg(1 - \frac{4\lambda}{(N-2)^2}\bigg)^{1/2}.$$

Moreover, when $0 \le \lambda < (N-2)^2/4$, all $w_\mu^\lambda(x)$ minimize the associated Rayleigh quotient and the minimum can be computed as:

$$S(\lambda) := \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{Q_\lambda(u)}{\big(\int_{\mathbb{R}^N} |u|^{2^*} dx\big)^{2/2^*}} = \frac{Q_\lambda(w_\mu^\lambda)}{\big(\int_{\mathbb{R}^N} |w_\mu^\lambda|^{2^*} dx\big)^{2/2^*}} = \bigg(1 - \frac{4\lambda}{(N-2)^2}\bigg)^{\frac{N-1}{N}} S, \tag{16}$$

where we denoted the quadratic form $Q_\lambda(u) = \int_{\mathbb{R}^N} |\nabla u|^2 dx - \lambda \int_{\mathbb{R}^N} \frac{u^2}{|x|^2} dx$, see [29], and S is the best constant in the Sobolev inequality

$$S\|u\|^2_{L^{2^*}(\mathbb{R}^N)} \le \|u\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)}.$$

As minimizers of problem (16), we consider

$$z_\mu^\lambda(x) = \frac{w_\mu^\lambda(x)}{\big(\int_{\mathbb{R}^N} |w_\mu^\lambda|^{2^*} dx\big)^{1/2^*}} = \alpha_{\lambda,N}\, \mu^{-\frac{N-2}{2}} \bigg(\Big|\frac{x}{\mu}\Big|^{1-\nu_\lambda} + \Big|\frac{x}{\mu}\Big|^{1+\nu_\lambda}\bigg)^{-\frac{N-2}{2}} \tag{17}$$

where $\alpha_{\lambda,N} = \big(N(N-2)\nu_\lambda^2\big)^{\frac{N-2}{4}} \|w_1^\lambda\|^{-1}_{L^{2^*}}$ is a positive constant depending only on λ and N, so that for $0 \le \lambda < (N-2)^2/4$

$$S(\lambda) = Q_\lambda(z_\mu^\lambda) \quad \text{for all } \mu > 0.$$

For $-\infty < \lambda < (N-2)^2/4$, we also set

$$S_k(\lambda) := \inf_{u \in \mathcal{D}_k^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{Q_\lambda(u)}{\big(\int_{\mathbb{R}^N} |u|^{2^*} dx\big)^{2/2^*}}. \tag{18}$$

We note that $S(\lambda) \le S_k(\lambda)$, and equality holds whenever $\lambda \ge 0$. Moreover the following result has been proved in [29].

Lemma 2.1 (see [29], Lemma 6.1)**.** *Let $N \geq 4$. If $S_k(\lambda) < k^{2/N} S$ then $S_k(\lambda)$ is achieved.*

In [29] it is proved that if $\lambda \in (0, (N-2)^2/4)$ then all solutions to (14) are of the form (15) while if $\lambda \ll 0$ then also non radial solutions to (14) can exist. The behavior of any solution (radial and non radial) to problem (14) near the singularities 0 and ∞ is described by the following theorem.

Theorem 2.2. *If $\lambda < (N-2)^2/4$ and $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ is a solution to problem (14), then there exist positive constant $\kappa_0(u)$ and $\kappa_\infty(u)$ depending on u such that*

$$u(x) = |x|^{-\frac{N-2}{2}(1-\nu_\lambda)}\big[\kappa_0(u) + O(|x|^\alpha)\big], \quad \text{as } x \to 0, \tag{19}$$

$$u(x) = |x|^{-\frac{N-2}{2}(1+\nu_\lambda)}\big[\kappa_\infty(u) + O(|x|^{-\alpha})\big], \quad \text{as } |x| \to +\infty, \tag{20}$$

for some $\alpha \in (0,1)$.

Remark 2.3. *Putting together (19–20) we deduce that there exists a positive constant $\kappa(u)$ depending on u such that*

$$\frac{1}{\kappa(u)}\, w_1^\lambda(x) \leq u(x) \leq \kappa(u)\, w_1^\lambda(x). \tag{21}$$

Proof of Theorem 2.2. Set

$$a_\lambda = \frac{(N-2)\big(1-\nu_\lambda\big)}{2}, \quad v(x) = |x|^{a_\lambda} u(x).$$

Then the function v belongs to $\mathcal{D}^{1,2}_{a_\lambda}(\mathbb{R}^N)$ where $\mathcal{D}^{1,2}_{a_\lambda}(\mathbb{R}^N)$ denotes the space obtained by completion of $\mathcal{D}(\mathbb{R}^N)$ with respect to the weighted Dirichlet norm

$$\|v\|_{\mathcal{D}^{1,2}_{a_\lambda}(\mathbb{R}^N)} := \bigg(\int_{\mathbb{R}^N} |x|^{-2a_\lambda} |\nabla v|^2\, dx\bigg)^{1/2}.$$

Moreover v solves equation

$$-\operatorname{div}\big(|x|^{-2a_\lambda}\nabla v\big) = \frac{v^{2^*-1}}{|x|^{2^* a_\lambda}}. \tag{22}$$

From [13, Theorem 1.2], it follows that v is Hölder continuous; in particular expansion (19) holds for $\kappa_0(u) = v(0)$ and some $\alpha \in (0,1)$. Moreover $v(0)$ is strictly positive in view of Harnack's inequality for degenerate operators proved in [20], see also [10]; we mention that weights of type $|x|^{-2a}$ with $a < \frac{N-2}{2}$ belong to the class of quasi-conformal weights considered in [20].

To deduce (20), we perform the change of variable

$$\tilde{v}(x) = |x|^{a_\lambda-(N-2)} u\bigg(\frac{x}{|x|^2}\bigg),$$

and observe that the transformed function $\tilde{v}$ solves equation (22). Hence [13, Theorem 1.2] yields that $\tilde{v}$ is Hölder continuous and admits the following expansion

$$\tilde{v}(x) = \tilde{v}(0) + O(|x|^\alpha), \quad \text{as } x \to 0,$$

for some $\alpha \in (0,1)$, where $\tilde{v}(0) > 0$ in view of Harnack's inequality in [20]. Coming back to u we obtain that u satisfies (20) with $\kappa_\infty(u) = \tilde{v}(0) > 0$. ∎

3. Hardy's inequality with singularity on a circle

We prove now the Hardy type inequality for potentials which are singular at circles stated in Theorem 1.1.

Proof of Theorem 1.1. Let us consider the minimization problem

$$I(S_r) := \inf_{\substack{u\in\mathcal{D}^{1,2}(\mathbb{R}^N)\\ u\not\equiv 0}} \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla u(y)|^2\,dy}{\displaystyle\int_{\mathbb{R}^N} |u(y)|^2\bigg(\fint_{S_r}\frac{d\sigma(x)}{|x-y|^2}\bigg)\,dy} = \inf_{\substack{u\in\mathcal{D}(\mathbb{R}^N\setminus\{0\})\\ u\not\equiv 0}} \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla u(y)|^2\,dy}{\displaystyle\int_{\mathbb{R}^N} |u(y)|^2\bigg(\fint_{S_r}\frac{d\sigma(x)}{|x-y|^2}\bigg)\,dy},$$

where the last equality is due to density of $\mathcal{D}(\mathbb{R}^N\setminus\{0\})$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ (see e.g. [5, Lemma 2.1]). An easy calculation shows that for any $u\in\mathcal{D}^{1,2}(\mathbb{R}^N)$

$$\frac{\displaystyle\int_{\mathbb{R}^N} |\nabla u(y)|^2\,dy}{\displaystyle\int_{\mathbb{R}^N} |u(y)|^2\bigg(\fint_{S_r}\frac{d\sigma(x)}{|x-y|^2}\bigg)\,dy} = \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla v(y)|^2\,dy}{\displaystyle\int_{\mathbb{R}^N} |v(y)|^2\bigg(\fint_{S_1}\frac{d\sigma(x)}{|x-y|^2}\bigg)\,dy}$$

where $v(y)=u(ry)$. Therefore

$$I(S_r) = I(S_1)\quad\text{for any } r>0. \tag{23}$$

In view of (23), it is enough to prove the theorem for $r=1$. The proof consists in three steps.

Step 1: Inequality (6) i.e. $\boldsymbol{I(S_1)\geq\big(\frac{N-2}{2}\big)^2}$.

For any $u\in\mathcal{D}^{1,2}(\mathbb{R}^N)$, $u\geq 0$ a.e., we consider the Schwarz symmetrization u^* of u defined as

$$u^*(x) := \inf\big\{t>0:\ \big|\{y\in\mathbb{R}^N:\ u(y)>t\}\big|\leq\omega_N|x|^N\big\} \tag{24}$$

where $|\cdot|$ denotes the Lebesgue measure of $\mathbb{R}^N$ and ω_N is the volume of the standard unit N-ball. From [32, Theorem 21.8], it follows that for any $x\in S_1$

$$\int_{\mathbb{R}^N}\frac{|u(y)|^2}{|x-y|^2}\,dy \leq \int_{\mathbb{R}^N}|u^*(y)|^2\left[\left(\frac{1}{|x-y|}\right)^*\right]^2.$$

Since $\big(\frac{1}{|x-y|}\big)^* = \frac{1}{|y|}$, we deduce

$$\int_{\mathbb{R}^N}\frac{|u(y)|^2}{|x-y|^2}\,dy \leq \int_{\mathbb{R}^N}\frac{|u^*(y)|^2}{|y|^2}\,dy. \tag{25}$$

Moreover by Polya-Szego inequality

$$\int_{\mathbb{R}^N}|\nabla u^*|^2 \leq \int_{\mathbb{R}^N}|\nabla u|^2. \tag{26}$$

From (25–26) and the classical Hardy's inequality, it follows that, for any $u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\}$, $u \geq 0$ a.e.,

$$\frac{\displaystyle\int_{\mathbb{R}^N} |\nabla u(y)|^2\, dy}{\displaystyle\int_{\mathbb{R}^N} |u(y)|^2 \Big(\fint_{S_1} \frac{d\sigma(x)}{|x-y|^2} \Big)\, dy} = \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla u(y)|^2\, dy}{\displaystyle\fint_{S_1} \Big(\int_{\mathbb{R}^N} |u(y)|^2 \frac{dy}{|x-y|^2} \Big)\, d\sigma(x)}$$

$$\geq \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla u^*(y)|^2\, dy}{\displaystyle\int_{\mathbb{R}^N} \frac{|u^*(y)|^2}{|y|^2}\, dy} \geq \Big(\frac{N-2}{2}\Big)^2.$$

Due to evenness of the quotient we are minimizing, to compute $I(S_1)$ it is enough to take the infimum over positive functions. Hence passing to the infimum in the above inequality, we obtain $I(S_1) \geq \big(\frac{N-2}{2}\big)^2$.

Step 2: Optimality of the constant, i.e. $\boldsymbol{I(S_1) = \big(\frac{N-2}{2}\big)^2}$.

We fix $u \in \mathcal{D}(\mathbb{R}^N \setminus \{0\})$ and let $0 < r < R$ be such that $\operatorname{supp} u \subset \{x \in \mathbb{R}^N : \ r < |x| < R\}$. For any $0 < \lambda \ll 1$, we set $\tilde{u}_\lambda(x) = u(\lambda x)$. Hence we have

$$I(S_1) \leq \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla \tilde{u}_\lambda(y)|^2\, dy}{\displaystyle\int_{\mathbb{R}^N} |\tilde{u}_\lambda(y)|^2 \Big(\fint_{S_1} \frac{d\sigma(x)}{|x-y|^2} \Big)\, dy} = \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla u(y)|^2\, dy}{\displaystyle\int_{\mathbb{R}^N} |u(y)|^2 \Big(\fint_{S_1} \frac{d\sigma(x)}{|\lambda x-y|^2} \Big)\, dy}. \tag{27}$$

Since $\frac{1}{|\lambda x - y|^2} \leq \frac{4}{r^2}$ for all $y \in \operatorname{supp} u$, $x \in S_1$, and $0 < \lambda < \frac{r}{2}$, by Dominated Convergence Theorem we deduce that $\int_{\mathbb{R}^N} |u(y)|^2 \big(\fint_{S_1} \frac{d\sigma(x)}{|\lambda x - y|^2} \big)\, dy$ converges to $\int_{\mathbb{R}^N} \frac{|u(y)|^2}{|y|^2}\, dy$ as $\lambda \to 0$, hence passing to the limit in (27) we obtain

$$I(S_1) \leq \frac{\int_{\mathbb{R}^N} |\nabla u(y)|^2\, dy}{\int_{\mathbb{R}^N} |y|^{-2} |u(y)|^2\, dy} \quad \text{for any } u \in u \in \mathcal{D}(\mathbb{R}^N \setminus \{0\}).$$

By density of $\mathcal{D}(\mathbb{R}^N \setminus \{0\})$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ we deduce

$$I(S_1) \leq \inf_{\mathcal{D}^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u(y)|^2\, dy}{\int_{\mathbb{R}^N} |y|^{-2} |u(y)|^2\, dy} = \Big(\frac{N-2}{2}\Big)^2$$

where the last equality follows from the optimality of the constant $\big(\frac{N-2}{2}\big)^2$ in the classical Hardy inequality (see [19, Lemma 2.1]. Collecting the above inequality with the one proved in Step 1, we find $I(S_1) = \big(\frac{N-2}{2}\big)^2$.

Step 3: The infimum $\boldsymbol{I(S_1)}$ is not attained.

Arguing by contradiction, assume that the infimum $I(S_1)$ is achieved by some $\bar{u} \in \mathcal{D}^{1,2}(\mathbb{R}^N)$. We can assume that $\bar{u} \geq 0$ (otherwise we consider $|\bar{u}|$ which is also a minimizer by evenness of the quotient). Hence from (25) and (26)

$$\Big(\frac{N-2}{2}\Big)^2=\frac{\displaystyle\int_{\mathbb{R}^N}|\nabla\bar u(y)|^2\,dy}{\displaystyle\fint_{S_1}\bigg(\int_{\mathbb{R}^N}|\bar u(y)|^2\frac{dy}{|x-y|^2}\bigg)\,d\sigma(x)}\geq\frac{\displaystyle\int_{\mathbb{R}^N}|\nabla\bar u^*(y)|^2\,dy}{\displaystyle\int_{\mathbb{R}^N}\frac{|\bar u^*(y)|^2}{|y|^2}\,dy}\geq\Big(\frac{N-2}{2}\Big)^2.$$

Therefore the above inequalities are indeed equalities and this implies that the infimum

$$\Big(\frac{N-2}{2}\Big)^2=\inf_{u\in\mathcal{D}^{1,2}(\mathbb{R}^N)\setminus\{0\}}\frac{\displaystyle\int_{\mathbb{R}^N}|\nabla u(y)|^2\,dy}{\displaystyle\int_{\mathbb{R}^N}|y|^{-2}|u(y)|^2\,dy}\tag{28}$$

which yields the best constant in the classical Hardy inequality, is achieved by $\bar u^*$. Since it is known that the infimum in (28) cannot be attained (see [29, Remark 1.2]), we reach a contradiction. ■

4. The Palais-Smale condition under $\mathbb{Z}_k\times\mathbb{SO}(N-2)$ and $\mathbb{SO}(2)\times\mathbb{SO}(N-2)$-invariance

Let us define the functional $J_k:\mathcal{D}^{1,2}(\mathbb{R}^N)\to\mathbb{R}$ associated to equation (2) as

$$\begin{aligned}J_k(u)=&\frac12\int_{\mathbb{R}^N}|\nabla u|^2dx-\frac{\lambda_0}{2}\int_{\mathbb{R}^N}\frac{u^2(x)}{|x|^2}\,dx\\&-\sum_{\ell=1}^m\sum_{i=1}^k\frac{\lambda_\ell}{2}\int_{\mathbb{R}^N}\frac{u^2(x)}{|x-a_i^\ell|^2}\,dx-\frac{S_k(\lambda_0,\Lambda_1,\dots,\Lambda_m)}{2^*}\int_{\mathbb{R}^N}|u|^{2^*}\,dx.\end{aligned}$$

The choice of location of the singularities ensures that J_k is $\mathbb{Z}_k\times\mathbb{SO}(N-2)$-invariant. Since $\mathbb{Z}_k\times\mathbb{SO}(N-2)$ acts by isometries on $\mathcal{D}^{1,2}(\mathbb{R}^N)$, we can apply the *Principle of Symmetric Criticality* by Palais [25] to deduce that the critical points of J_k restricted to $\mathcal{D}^{1,2}_k(\mathbb{R}^N)$ are also critical points of J_k in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. Therefore, if u is a critical point of J_k in $\mathcal{D}^{1,2}_k(\mathbb{R}^N)$, $u>0$ outside singularities, then $v=S_k(\lambda_0,\Lambda_1,\dots,\Lambda_\ell)^{1/(2^*-2)}u$ is a solution to equation (2).

The following theorem provides a local Palais-Smale condition for J_k restricted to $\mathcal{D}^{1,2}_k(\mathbb{R}^N)$ below some critical threshold. We emphasize that the invariance of the problem by the action of a subgroup of orthogonal transformation allows to recover some compactness, in the sense that concentration points of invariant functions must be located in some symmetric way, thus reducing the possibility of loss of compactness. The restriction on dimension $N\geq4$ is required to avoid the presence of possible concentration points on $\{0\}\times\mathbb{R}^{N-2}$. Indeed when $N=3$, $\mathbb{SO}(N-2)=\mathbb{SO}(1)$ is a discrete group, making thus possible concentration at points on the axis $\{0\}\times\mathbb{R}$.

We mention that the *Concentration-Compactness* method under the action of $\mathbb{Z}_k\times\mathbb{SO}(N-2)$ was used by several authors to find k-bump solutions with prescribed symmetry for different classes of nonlinear elliptic equations: nonlinear Schrödinger equation in [30], nonlinear elliptic equations in symmetric domains in [4], nonlinear elliptic equations of Caffarelli-Kohn-Nirenberg type in [6], and elliptic equations with Hardy potential and critical growth in [29].

Theorem 4.1. *Assume $N\geq4$ and $\lambda_0^++k\sum_{j=1}^m\lambda_j^+<\frac{(N-2)^2}{4}$. Let $\{u_n\}_{n\in\mathbb{N}}\subset\mathcal{D}^{1,2}_k(\mathbb{R}^N)$ be a Palais-Smale sequence for J_k restricted to $\mathcal{D}^{1,2}_k(\mathbb{R}^N)$, namely*

$$\lim_{n\to\infty}J_k(u_n)=c<\infty\ \text{ in }\mathbb{R}\quad\text{and}\quad\lim_{n\to\infty}J_k'(u_n)=0\ \text{ in the dual space }(\mathcal{D}^{1,2}_k(\mathbb{R}^N))^\star.$$

If

$$c < \frac{S_k(\lambda_0, \Lambda_1, \dots, \Lambda_m)^{1-\frac{N}{2}}}{N} \min\Big\{k^{\frac{2}{N}} S, k^{\frac{2}{N}} S(\lambda_1), \dots, k^{\frac{2}{N}} S(\lambda_m), S_k(\lambda_0), S_k\Big(\lambda_0 + k \sum_{\ell=1}^{m} \lambda_\ell\Big)\Big\}^{\frac{N}{2}},$$

then $\{u_n\}_{n\in\mathbb{N}}$ *has a subsequence strongly converging in* $\mathcal{D}^{1,2}_k(\mathbb{R}^N)$*.*

The proof of the above theorem is based on a classical *Concentration Compactness* Argument (see [23, 24]) combined with a careful analysis of how the symmetric location of singularities influences possible concentration of minimizing sequences. For the sake of brevity, we omit the proof, which can be found with all details in the preprint version of the present paper [15].
The functional $J : \mathcal{D}^{1,2}(\mathbb{R}^N) \to \mathbb{R}$ associated to equation (4) is

$$\begin{aligned} J(u) =& \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{\lambda_0}{2}\int_{\mathbb{R}^N} \frac{u^2(x)}{|x|^2}\, dx \\ & - \sum_{\ell=1}^{m} \frac{\Lambda_\ell}{2} \int_{\mathbb{R}^N} \bigg(\fint_{S_{r_\ell}} \frac{u^2(y)}{|x-y|^2}\, d\sigma(x)\bigg)\, dy - \frac{S_{\rm circ}(\lambda_0, \Lambda_1, \dots, \Lambda_m)}{2^*} \int_{\mathbb{R}^N} |u|^{2^*} dx. \end{aligned}$$

The functional J is $\mathbb{SO}(2) \times \mathbb{SO}(N-2)$-invariant. Since $\mathbb{SO}(2) \times \mathbb{SO}(N-2)$ acts by isometries on $\mathcal{D}^{1,2}(\mathbb{R}^N)$, the *Principle of Symmetric Criticality* by Palais [25] implies that the critical points of J restricted to $\mathcal{D}^{1,2}_{\rm circ}(\mathbb{R}^N)$ are also critical points of J in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. Therefore, any critical point of J in $\mathcal{D}^{1,2}_{\rm circ}(\mathbb{R}^N)$ provides a solution to equation (4).

The following theorem is the analogous of Theorem 4.1 for J restricted to $\mathcal{D}^{1,2}_{\rm circ}(\mathbb{R}^N)$. However, the fact that the singularities are spread over circles instead of being concentrated at atoms reduces the possibility of lack of compactness. Indeed, according to P.L. Lions *Concentration-Compactness Principle*, possible forms of "non compactness" which can cause failure of Palais-Smale condition are loss of mass at infinity and concentration at an most countable set of points. When considering J restricted to $\mathcal{D}^{1,2}_{\rm circ}(\mathbb{R}^N)$, it turns out that if $\bar{x}$ is a concentration point of a Palais-Smale sequence, then all points of the orbit $\mathcal{O}(\bar{x}) = \{\tau\, \bar{x} : \ \tau \in \mathbb{SO}(2) \times \mathbb{SO}(N-2)\}$ must be concentration points. On the other hand, when $N \geq 4$ both groups $\mathbb{SO}(2)$ and $\mathbb{SO}(N-2)$ are continuous, hence the only point $\bar{x}$ for which $\mathcal{O}(\bar{x})$ is at most countable is the origin. Hence concentration can occur only at 0 and at ∞.

We mention that action of this type of groups was considered in [2] to find nonradial solutions to a Euclidean scalar field equation. We refer to [31, §1.5] for a discussion on the relation between symmetry and compactness in variational problems.

Let us define

$$S_{\rm circ}(\lambda_0) = \inf_{\substack{u \in \mathcal{D}^{1,2}_{\rm circ}(\mathbb{R}^N) \\ u \not\equiv 0}} \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla u|^2\, dx - \lambda_0 \int_{\mathbb{R}^N} \frac{u^2(x)}{|x|^2}\, dx}{\displaystyle\bigg(\int_{\mathbb{R}^N} |u|^{2^*} dx\bigg)^{2/2^*}}. \tag{29}$$

The following theorem provides a threshold up to which J satisfies Palais-Smale condition. For a detailed proof we remind to [15].

Theorem 4.2. *Assume $N \geq 4$ and $\lambda_0^+ + \sum_{j=1}^m \Lambda_j^+ < \frac{(N-2)^2}{4}$. Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{D}^{1,2}_{\rm circ}(\mathbb{R}^N)$ be a Palais-Smale sequence for J restricted to $\mathcal{D}^{1,2}_{\rm circ}(\mathbb{R}^N)$, namely*

$$\lim_{n\to\infty} J(u_n) = c < \infty \text{ in } \mathbb{R} \quad \text{and} \quad \lim_{n\to\infty} J'(u_n) = 0 \text{ in the dual space } (\mathcal{D}^{1,2}_{\rm circ}(\mathbb{R}^N))^\star.$$

If

$$c < \frac{S_{\rm circ}(\lambda_0,\Lambda_1,\dots,\Lambda_m)^{1-\frac{N}{2}}}{N} \min\Big\{S_{\rm circ}(\lambda_0), S_{\rm circ}\Big(\lambda_0 + \sum_{\ell=1}^m \Lambda_\ell\Big)\Big\}^{\frac{N}{2}}, \tag{30}$$

then $\{u_n\}_{n\in\mathbb{N}}$ has a converging subsequence in $\mathcal{D}^{1,2}_{\rm circ}(\mathbb{R}^N)$.

5. Interaction estimates

For any $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, let us consider the family of functions obtained from u by dilation, i.e.

$$u_\mu(x) = \mu^{-\frac{N-2}{2}} u(x/\mu), \quad \mu > 0. \tag{31}$$

The following lemma describes the behavior of $\int |x+\xi|^{-2}|u^\lambda_\mu|^2$ as $\mu \to 0$ for any solution u^λ of equation (14). We mention that estimates below were obtained in [14] for radial solutions to (14) (i.e. for functions $w^{(\lambda)}_\mu$ in (15)).

Lemma 5.1. *Let $u^\lambda \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ be a solution to (14). For any $\xi \in \mathbb{R}^N$ there holds*

$$\int_{\mathbb{R}^N} \frac{|u^\lambda_\mu|^2}{|x+\xi|^2}\,dx = \begin{cases} \frac{\mu^2}{|\xi|^2}\int_{\mathbb{R}^N}|u^\lambda|^2\,dx + o\left(\mu^2\right) & \text{if } \lambda < \frac{N(N-4)}{4}, \\ \kappa_\infty(u^\lambda)^2\,\frac{\mu^2|\ln\mu|}{|\xi|^2} + o(\mu^2|\ln\mu|) & \text{if } \lambda = \frac{N(N-4)}{4}, \\ \kappa_\infty(u^\lambda)^2\,\beta_{\lambda,N}\,\frac{\mu^{\sqrt{(N-2)^2-4\lambda}}}{|\xi|^{\sqrt{(N-2)^2-4\lambda}}} + o\big(\mu^{\sqrt{(N-2)^2-4\lambda}}\big) & \text{if } \lambda > \frac{N(N-4)}{4}, \end{cases}$$

as $\mu \to 0$, where

$$\beta_{\lambda,N} = \int_{\mathbb{R}^N} \frac{dx}{|x|^2|x-e_1|^{N-2+\sqrt{(N-2)^2-4\lambda}}}, \quad e_1 = (1,0,\dots,0) \in \mathbb{R}^N.$$

Proof. We have that

$$\int_{\mathbb{R}^N}\frac{|u^\lambda_\mu|^2}{|x+\xi|^2}\,dx = \mu^2\int_{\mathbb{R}^N}\frac{|u^\lambda|^2}{|\mu x+\xi|^2}\,dx = \mu^2\int_{|x|<\frac{|\xi|}{2\mu}}\frac{|u^\lambda|^2}{|\mu x+\xi|^2}\,dx + \mu^2\int_{|x|>\frac{|\xi|}{2\mu}}\frac{|u^\lambda|^2}{|\mu x+\xi|^2}\,dx. \tag{32}$$

For $\lambda < \frac{N(N-4)}{4}$, from (21) we have that $u^\lambda \in L^2(\mathbb{R}^N)$. From (21) and [14, Proof of Lemma 3.4] we deduce

$$\begin{aligned} &\bigg|\int_{|x|<\frac{|\xi|}{2\mu}}|u^\lambda(x)|^2\bigg[\frac{1}{|\mu x+\xi|^2}-\frac{1}{|\xi|^2}\bigg]\,dx\bigg| \\ &\qquad \leq \kappa(u^\lambda)^2\int_{|x|<\frac{|\xi|}{2\mu}}|w^\lambda_1(x)|^2\bigg|\frac{1}{|\mu x+\xi|^2}-\frac{1}{|\xi|^2}\bigg|\,dx = o(1) \end{aligned}$$

as $\mu \to 0$ and hence, since $u^\lambda \in L^2(\mathbb{R}^N)$,

$$\int_{|x|<\frac{|\xi|}{2\mu}} |u^\lambda(x)|^2 \frac{dx}{|\mu x+\xi|^2} = \frac{1}{|\xi|^2}\int_{|x|<\frac{|\xi|}{2\mu}} |u^\lambda(x)|^2\,dx + o(1) = \frac{1}{|\xi|^2}\int_{\mathbb{R}^N} |u^\lambda(x)|^2\,dx + o(1). \tag{33}$$

On the other hand, from [14, Proof of Lemma 3.4] we have

$$\begin{aligned}\mu^2\int_{|x|>\frac{|\xi|}{2\mu}} \frac{|u^\lambda|^2}{|\mu x+\xi|^2}\,dx &\le \kappa(u^\lambda)^2\mu^{2-N}\int_{|x-\xi|\ge\frac{|\xi|}{2}} \left|w_1^\lambda\Big(\frac{x-\xi}{\mu}\Big)\right|^2\frac{dx}{|x|^2}\\ &= O\big(\mu^{\nu_\lambda(N-2)}\big) = o(\mu^2).\end{aligned} \tag{34}$$

From (32), (33), and (34) we deduce that

$$\int_{\mathbb{R}^N} \frac{|u^\lambda|^2}{|x+\xi|^2}\,dx = \frac{\mu^2}{|\xi|^2}\int_{\mathbb{R}^N}|u^\lambda(x)|^2\,dx + o(\mu^2).$$

For $\lambda = \frac{N(N-4)}{4}$, from (21) and [14, Proof of Lemma 3.4] we deduce that

$$\int_{|x|<\frac{|\xi|}{2\mu}} |u^\lambda(x)|^2\frac{dx}{|\mu x+\xi|^2} = \frac{1}{|\xi|^2}\int_{|x|<\frac{|\xi|}{2\mu}}|u^\lambda(x)|^2\,dx + O(1). \tag{35}$$

On the other hand, from (20) we obtain

$$\begin{aligned}\int_{|x|<\frac{|\xi|}{2\mu}} |u^\lambda(x)|^2\,dx &= \int_{1<|x|<\frac{|\xi|}{2\mu}} |u^\lambda(x)|^2\,dx + \int_{|x|<1}|u^\lambda(x)|^2\,dx\\ &= \kappa_\infty(u^\lambda)^2\int_{1<|x|<\frac{|\xi|}{2\mu}}|x|^{-N}\,dx + O\bigg(\int_{1<|x|<\frac{|\xi|}{2\mu}}|x|^{-N-2\alpha}\,dx\bigg) + O(1)\\ &= \kappa_\infty(u^\lambda)^2|\ln\mu| + O(1).\end{aligned} \tag{36}$$

Arguing as above (see (34)), we obtain

$$\mu^2\int_{|x|>\frac{|\xi|}{2\mu}}\frac{|u^\lambda|^2}{|\mu x+\xi|^2}\,dx = O(\mu^2). \tag{37}$$

Gathering (32), (35), (36) and (37) we deduce that

$$\int_{\mathbb{R}^N}\frac{|u_\mu^\lambda|^2}{|x+\xi|^2}\,dx = \kappa_\infty(u^\lambda)^2\,\frac{\mu^2|\ln\mu|}{|\xi|^2} + o(\mu^2|\ln\mu|).$$

For $\lambda > \frac{N(N-4)}{4}$, in view of (20) we have that

$$\int_{\mathbb{R}^N}\frac{|u_\mu^\lambda|^2}{|x+\xi|^2}\,dx = \mu^{\nu_\lambda(N-2)}\bigg[\kappa_\infty^2(u^\lambda)\int_{\mathbb{R}^N}\frac{1}{|x|^2|x-\xi|^{(N-2)(1+\nu_\lambda)}} + o(1)\bigg]\,dx \tag{38}$$

As observed in [14, Proof of Lemma 3.4], the function

$$\varphi(\xi) := \int_{\mathbb{R}^N}\frac{dx}{|x|^2|x-\xi|^{(N-2)(1+\nu_\lambda)}}$$

can be written as

$$\varphi(\xi) = |\xi|^{-\sqrt{(N-2)^2-4\lambda}}\varphi(\xi/|\xi|) = |\xi|^{-\sqrt{(N-2)^2-4\lambda}}\varphi(e_1). \tag{39}$$

(38) and (39) yield the required estimate for $\lambda > \frac{N(N-4)}{4}$. ■

Let us now study the interaction between two minimizers of (16), i.e. functions z_μ^λ in (17), centered at different points as $\mu \to 0$. To this aim we note that a direct calculation yields

$$z_1^\lambda(x) = |x|^{-\frac{N-2}{2}(1+\nu_\lambda)}\big[\alpha_{\lambda,N} + O(|x|^{-\alpha})\big], \tag{40}$$

$$\nabla z_1^\lambda(x) = |x|^{-\frac{N+2}{2}-\nu_\lambda\frac{N-2}{2}}x\big[-\alpha_{\lambda,N}\tfrac{N-2}{2}(1+\nu_\lambda) + O(|x|^{-\alpha})\big], \tag{41}$$

for all $0 < \alpha \le 2\nu_\lambda$. From (17), (40), and (41), it is easy to deduce the following result.

Lemma 5.2. *For any $\lambda \in \big(0, (N-2)^2/4\big)$ and $\xi, \zeta \in \mathbb{R}^N$, $\xi \neq 0$, there holds*

$$\int_{\mathbb{R}^N} \frac{z_\mu^\lambda(x) z_\mu^\lambda(x+\xi)}{|x+\zeta|^2}\,dx = \mu^{\sqrt{(N-2)^2-4\lambda}}\bigg[\alpha_{\lambda,N}^2\int_{\mathbb{R}^N}\frac{dx}{|x|^{(1+\nu_\lambda)\frac{N-2}{2}}|x+\xi|^{(1+\nu_\lambda)\frac{N-2}{2}}|x+\zeta|^2} + o(1)\bigg]$$

and

$$\int_{\mathbb{R}^N} \nabla z_\mu^\lambda(x)\cdot\nabla z_\mu^\lambda(x+\xi)\,dx = \mu^{\sqrt{(N-2)^2-4\lambda}}\Big[\alpha_{\lambda,N}^2\tfrac{(N-2)^2}{4}(1+\nu_\lambda)^2\gamma_{\lambda,N}|\xi|^{-\sqrt{(N-2)^2-4\lambda}} + o(1)\Big]$$

as $\mu \to 0$, where

$$\gamma_{\lambda,N} = \int_{\mathbb{R}^N}\frac{x\cdot(x+e_1)\,dx}{|x|^{\frac{N+2}{2}+\nu_\lambda\frac{N-2}{2}}|x-e_1|^{\frac{N+2}{2}+\nu_\lambda\frac{N-2}{2}}}, \quad e_1 = (1,0,\dots,0)\in\mathbb{R}^N.$$

6. Comparison between concentration levels for J and proof of Theorem 1.3

In order to compare the level $S_{\rm circ}(\lambda_0, \Lambda_1, \dots, \Lambda_m)$ with the level $S_{\rm circ}(\lambda_0)$ of possible concentration at 0, we need the following lemma, which states that the infimum in (29) is achieved if $N \ge 4$. Such a result does not come unexpected, since it can be seen as the analogue of Lemma 2.1 when $k = \infty$; indeed when k becomes larger and larger, assumption $S_k(\lambda) < k^{2/N}S$ of Lemma 2.1 is weakened till it is no more needed in the limiting problem corresponding to singularities spread over circles.

Lemma 6.1. *For any $\lambda_0 \in (-\infty, (N-2)^2/4)$ and $N \ge 4$, the infimum in (29) is achieved.*

Being the proof of the above lemma a quite standard application of P.L. Lions' *Concentration-Compactness Principle*, we omit it and refer to [15] for details.
We now provide a sufficient condition for the infimum in (5) to stay below the level $S_{\rm circ}(\lambda_0)$, at which possible concentration at 0 can occur.

Lemma 6.2. *Let $\lambda_0, \Lambda_1, \dots, \Lambda_m \in \mathbb{R}$, $r_1, r_2, \dots, r_m \in \mathbb{R}^+$ satisfy (9). Then*

$$S_{\rm circ}(\lambda_0, \Lambda_1, \dots, \Lambda_m) < S_{\rm circ}(\lambda_0).$$

Proof. From Lemma 6.1, we have that $S_{\rm circ}(\lambda_0)$ is attained by some $u^{\lambda_0} \in \mathcal{D}^{1,2}_{\rm circ}(\mathbb{R}^N)$. By homogeneity of the Rayleigh quotient, we can assume $\int |u^{\lambda_0}|^{2^*} = 1$. Moreover, the function $v^{\lambda_0} = S_{\rm circ}(\lambda_0)^{1/(2^*-2)}|u^{\lambda_0}|$ is a nonnegative solution to (14), hence we can apply Lemma 5.1 to study the behavior of $\int_{\mathbb{R}^N}\frac{|u_\mu^{\lambda_0}|^2}{|x+\xi|^2}\,dx$ as $\mu \to 0$, where $u_\mu^{\lambda_0}$ are defined in (31). Hence for some positive constant $\tilde\kappa$

$$\int_{\mathbb{R}^N}\bigg(\fint_{S_{r_\ell}}\frac{|u_\mu^{\lambda_0}(y)|^2}{|x-y|^2}\,d\sigma(x)\bigg)\,dy$$

$$=\begin{cases}\frac{\mu^2}{r_\ell^2}\int_{\mathbb{R}^N}|u_1^{\lambda_0}|^2\,dx+o\left(\mu^2\right) & \text{if } \lambda_0<\frac{N(N-4)}{4},\\ \tilde\kappa^2\,\frac{\mu^2|\ln\mu|}{r_\ell^2}+o(\mu^2|\ln\mu|) & \text{if } \lambda_0=\frac{N(N-4)}{4},\\ \tilde\kappa^2\,\beta_{\lambda_0,N}\,\mu^{\sqrt{(N-2)^2-4\lambda_0}}|r_\ell|^{-\sqrt{(N-2)^2-4\lambda_0}}+o\big(\mu^{\sqrt{(N-2)^2-4\lambda_0}}\big) & \text{if } \lambda_0>\frac{N(N-4)}{4},\end{cases}$$

as $\mu\to0$, where $\beta_{\lambda_0,N}$ is defined in Lemma 5.1. Therefore

$$\begin{aligned}&S_{\rm circ}(\lambda_0,\Lambda_1,\dots,\Lambda_m)\\&\leq\int_{\mathbb{R}^N}|\nabla u_\mu^{\lambda_0}|^2\,dy-\lambda_0\int_{\mathbb{R}^N}\frac{|u_\mu^{\lambda_0}(y)|^2}{|y|^2}\,dy-\sum_{\ell=1}^m\Lambda_\ell\int_{\mathbb{R}^N}\bigg(\fint_{S_{r_\ell}}\frac{|u_\mu^{\lambda_0}(y)|^2}{|x-y|^2}\,d\sigma(x)\bigg)\,dy\\&=S_{\rm circ}(\lambda_0)-\begin{cases}\mu^2\big(\int_{\mathbb{R}^N}|u_1^{\lambda_0}|^2\big)\bigg(\sum_{\ell=1}^m\frac{\Lambda_\ell}{r_\ell^2}+o(1)\bigg) & \text{if } \lambda_0<\frac{N(N-4)}{4},\\ \mu^2|\ln\mu|\tilde\kappa^2\bigg(\sum_{\ell=1}^m\frac{\Lambda_\ell}{r_\ell^2}+o(1)\bigg) & \text{if } \lambda_0=\frac{N(N-4)}{4},\\ \mu^{\sqrt{(N-2)^2-4\lambda_0}}\tilde\kappa^2\,\beta_{\lambda_0,N}\bigg(\sum_{\ell=1}^m\frac{\Lambda_\ell}{|r_\ell|^{\sqrt{(N-2)^2-4\lambda_0}}}+o(1)\bigg) & \text{if } \lambda_0>\frac{N(N-4)}{4},\end{cases}\end{aligned}\tag{42}$$

as $\mu\to0$. Taking μ sufficiently small, assumption (9) yields $S_{\rm circ}(\lambda_0,\Lambda_1,\dots,\Lambda_m)<S_{\rm circ}(\lambda_0)$. ■

Proof of Theorem 1.3. Let $\{u_n\}_n\subset\mathcal{D}^{1,2}_{\rm circ}(\mathbb{R}^N)$ be a minimizing sequence for (5). From the homogeneity of the quotient there is no restriction requiring $\|u_n\|_{L^{2^*}(\mathbb{R}^N)}=1$. Moreover from Ekeland's variational principle we can assume that $\{u_n\}_n\subset\mathcal{D}^{1,2}_{\rm circ}(\mathbb{R}^N)$ is a Palais-Smale sequence, more precisely $J'(u_n)\to0$ in $(\mathcal{D}^{1,2}_{\rm circ}(\mathbb{R}^N))^\star$ and $J(u_n)\to\frac1N S_{\rm circ}(\lambda_0\Lambda_1,\dots,\Lambda_m)$. From assumption (8) we deduce that

$$S_{\rm circ}(\lambda_0)\leq S_{\rm circ}\Big(\lambda_0+\sum_{\ell=1}^m\Lambda_\ell\Big).\tag{43}$$

From Lemma 6.2 and (43), it follows that the level of the minimizing Palais-Smale sequence satisfies assumption (30). Hence from Theorem 4.2, $\{u_n\}_{n\in\mathbb{N}}$ has a subsequence strongly converging to some $u_0\in\mathcal{D}^{1,2}_{\rm circ}(\mathbb{R}^N)$ such that $J(u_0)=\frac1N S_{\rm circ}(\lambda_0,\Lambda_1,\dots,\Lambda_m)$. Hence u_0 achieves the infimum in (5). Since J is even, also $|u_0|$ is a minimizer in (5) and then $v_0=S_{\rm circ}(\lambda_0,\Lambda_1,\dots,\Lambda_m)^{1/(2^*-2)}|u_0|$ is a nonnegative solution to equation (4). The maximum principle implies the positivity outside singular circles of such a solution. ■

7. Limit of $S_k(\lambda)$ as $k \to \infty$.

Since $\mathcal{D}^{1,2}_{\rm circ}(\mathbb{R}^N) \subset \mathcal{D}^{1,2}_k(\mathbb{R}^N)$, for any $\lambda \in (-\infty, (n-2)^2/4)$, there holds

$$0 < S(\lambda) \leq S_k(\lambda) \leq S_{\rm circ}(\lambda). \tag{44}$$

From (44) and Lemma 2.1, it follows easily the following result.

Lemma 7.1. *Let $\lambda \in (-\infty, (n-2)^2/4)$ and $N \geq 4$. Then there exists $\bar{k} = \bar{k}(\lambda, N)$ such that $S_k(\lambda)$ is achieved for all $k \geq \bar{k}$.*

Let us now study the limit of $S_k(\lambda)$ as $k \to \infty$. Theorem 7.3 provides convergence of $S_k(\lambda)$ to $S_{\rm circ}(\lambda)$. To prove it we will need the following proposition.

Proposition 7.2. *Let $\lambda \in (-\infty, (n-2)^2/4)$ and let $\{w_k\}_k$ be a sequence in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ such that $w_k \in \mathcal{D}^{1,2}_k(\mathbb{R}^N)$,*

$$\int_{\mathbb{R}^N} |w_k|^{2^*} = 1, \quad Q_\lambda(w_k) = S_k(\lambda),$$

and w_k converges weakly to 0 in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ (at least along a subsequence). Then, for any $r > 0$ and $\varepsilon \in (-r, r)$, there exists ρ such that $0 < |\rho| < |\varepsilon|$ and , for a subsequence,

$$\text{either} \quad \int_{B(0,r+\rho)} |\nabla w_k|^2 \to 0, \quad \int_{B(0,r+\rho)} |w_k|^{2^*} \to 0, \quad \text{and} \int_{B(0,r+\rho)} \frac{|w_k|^2}{|x|^2} \to 0,$$

$$\text{or} \quad \int_{\mathbb{R}^N\setminus B(0,r+\rho)} |\nabla w_k|^2 \to 0, \quad \int_{\mathbb{R}^N\setminus B(0,r+\rho)} |w_k|^{2^*} \to 0, \quad \text{and} \int_{\mathbb{R}^N\setminus B(0,r+\rho)} \frac{|w_k|^2}{|x|^2} \to 0.$$

An analogous result is proved in [29] for minimizing sequences of quotient (16). Since the proof of Proposition 7.2 is similar, we omit it. The reader can find a detailed proof in [15].

Theorem 7.3. *Let $\lambda \in (-\infty, (n-2)^2/4)$ and $N \geq 4$. Then $\lim_{k\to+\infty} S_k(\lambda) = S_{\rm circ}(\lambda)$.*

Proof. From Lemma 7.1 we know that, for k sufficiently large, $S_k(\lambda)$ is achieved, hence there exists some $u_k \in \mathcal{D}^{1,2}_k(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} |u_k|^{2^*} = 1 \quad \text{and} \quad Q_\lambda(u_k) = S_k(\lambda).$$

From the uniform bound of $S_k(\lambda)$ (see (44)) and equivalence of Q_λ to $\mathcal{D}^{1,2}(\mathbb{R}^N)$-norm, it follows that $\{u_k\}_k$ is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. Let us set

$$\tilde{u}_k(x) = R_k^{-\frac{N-2}{2}} u_k\Big(\frac{x}{R_k}\Big) \quad \text{and} \quad v_k(x) = (S_k(\lambda))^{\frac{1}{2^*-2}} \tilde{u}_k(x) \tag{45}$$

where R_k is chosen such that

$$\int_{B(0,R_k)} \bigg[|\nabla u_k(x)|^2 - \lambda \frac{|u_k(x)|^2}{|x|^2}\bigg] dx = \int_{\mathbb{R}^N\setminus B(0,R_k)} \bigg[|\nabla u_k(x)|^2 - \lambda \frac{|u_k(x)|^2}{|x|^2}\bigg] dx = \frac{1}{2} S_k(\lambda).$$

Invariance by scaling yields

$$\int_{\mathbb{R}^N} |\tilde{u}_k|^{2^*} = 1, \quad Q_\lambda(\tilde{u}_k) = S_k(\lambda), \tag{46}$$

$$\int_{\mathbb{R}^N} |v_k|^{2^*} = (S_k(\lambda))^{\frac{N}{2}}, \quad Q_\lambda(v_k) = (S_k(\lambda))^{\frac{N}{2}}, \tag{47}$$

and

$$\int_{B(0,1)} \left[|\nabla v_k(x)|^2 - \lambda \frac{|v_k(x)|^2}{|x|^2}\right] dx = \int_{\mathbb{R}^N\setminus B(0,1)} \left[\nabla v_k(x)|^2 - \lambda \frac{|v_k(x)|^2}{|x|^2}\right], dx = \frac{1}{2}(S_k(\lambda))^{\frac{N}{2}}. \tag{48}$$

Invariance by scaling also implies that $\{\tilde u_k\}_k$ is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, hence there exists a subsequence (still denoted as $\{\tilde u_k\}_k$) weakly converging to some $\tilde u_0$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$.

Claim 1. We claim that $\tilde u_0 \not\equiv 0$. Assume by contradiction that $\tilde u_0 \equiv 0$. Using Proposition 7.2 for sequence $\tilde u_k$ with $r = 1$ and $\varepsilon = \pm\frac{1}{4}$ and taking into account (48), (44), and (45), we deduce that there exist $\rho^+ \in (0, 1/4)$ and $\rho^- \in (-1/4, 0)$ such that, up to a subsequence,

$$\int_{B(0,1+\rho^-)} |\nabla v_k|^2 \to 0, \quad \int_{B(0,1+\rho^-)} |v_k|^{2^*} \to 0, \quad \text{and} \int_{B(0,1+\rho^-)} \frac{|v_k|^2}{|x|^2} \to 0, \tag{49}$$

$$\int_{\mathbb{R}^N\setminus B(0,1+\rho^+)} |\nabla v_k|^2 \to 0, \quad \int_{\mathbb{R}^N\setminus B(0,1+\rho^+)} |v_k|^{2^*} \to 0, \quad \text{and} \int_{\mathbb{R}^N\setminus B(0,1+\rho^+)} \frac{|v_k|^2}{|x|^2} \to 0. \tag{50}$$

Note that weak convergence of $\tilde u_k \rightharpoonup 0$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, (44), and (45), imply weak convergence of $v_k \rightharpoonup 0$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. Let η be a smooth radial cut off function such that $0 \le \eta \le 1$, $\eta(x) \equiv 1$ for $1 + \rho^- \le |x| \le 1 + \rho^+$ and $\eta(x) \equiv 0$ for $|x| \notin [3/4, 5/4]$. Set $\tilde v_k := \eta v_k$. Clearly $\tilde v_k \in \mathcal{D}^{1,2}_k(\mathbb{R}^N)$. By choice of η and (49–50) we have

$$Q_\lambda(\tilde v_k) = Q_\lambda(v_k) + o(1), \quad \|\tilde v_k - v_k\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)} = o(1), \quad \int_{\mathbb{R}^N} |\tilde v_k|^{2^*} = \int_{\mathbb{R}^N} |v_k|^{2^*} + o(1).$$

Let us define

$$f(u) := \frac{1}{2}\int_{B(0,5/4)\setminus B(0,3/4)} |\nabla u|^2 - \frac{1}{2^*}\int_{B(0,5/4)\setminus B(0,3/4)} |u|^{2^*}, \quad u \in H^1_0(B(0,5/4) \setminus B(0,3/4)).$$

From (49–50), (44), and (47), it is easy to verify that

$$f'(\tilde v_k) \to 0 \quad \text{in } (\mathcal{D}^{1,2}(\mathbb{R}^N))^\star \quad \text{and} \quad f(\tilde v_k) = \frac{1}{N}(S_k(\lambda))^{N/2} + o(1) \le \frac{1}{N}(S_{\rm circ}(\lambda))^{N/2} + o(1),$$

i.e. $\tilde v_k$ is a Palais-Smale sequence for f in $H^1_0(B(0,5/4)\setminus B(0,3/4))$. From Struwe's representation lemma for diverging Palais-Smale sequences [28, Theorem III.3.1], we deduce the existence of an integer $M \in \mathbb{N}$, M sequences of points $\{x^i_k\}_k \subset B(0,5/4) \setminus B(0,3/4)$ and M sequences of radii $\{R^i_k\}_k$, $i = 1, \dots, M$, such that $\lim_k R^i_k = +\infty$ and

$$\tilde v_k(x) = \sum_{i=1}^{M} (R^i_k)^{\frac{N-2}{2}} \tilde v_0(R^i_k(x - x^i_k)) + \mathcal{R}_k(x) \quad \text{where } \mathcal{R}_k \to 0 \text{ in } \mathcal{D}^{1,2}(\mathbb{R}^N), \tag{51}$$

$\tilde v_0 = w^0_1$ and w^0_1 is the Talenti-Aubin function in (15). Let us consider the sequence $\{x^1_k\}_k$; up to subsequence we can assume that it converges to some point $x^1 \subset \overline{B(0,5/4) \setminus B(0,3/4)}$. Let us write $x^1 = (z_1, y_1) \in \mathbb{R}^2 \times \mathbb{R}^{N-2}$ where $z_1 = |z_1|e^{\bar\theta\sqrt{-1}}$.

Let us first assume that $|z_1| \ne 0$. We fix $J \in \mathbb{N}$ and for any $i = 1, 2, \dots, J$ we set

$$S_i = \left\{(z, y) = (|z|e^{\theta\sqrt{-1}}, y) \in \mathbb{R}^2 \times \mathbb{R}^{N-2} : \ \bar\theta - \frac{(2i-1)\pi}{J} < \theta < \bar\theta + \frac{(2i+1)\pi}{J}\right\}.$$

Note that $x^1 \in S_1$ and there exists $\delta = \delta(J) > 0$ such that

$$B(x^1, \delta) \subset \left\{ (|z|e^{\theta\sqrt{-1}}, y) \in \mathbb{R}^2 \times \mathbb{R}^{N-2} : \ \bar{\theta} - \frac{(2i-1)\pi}{2J} < \theta < \bar{\theta} + \frac{(2i+1)\pi}{2J} \right\}.$$

Choose $\bar{k} = \bar{k}(\delta)$ such that for all $k \geq \bar{k}$

$$x_k^1 = (z_k^1, y_k^1) \in B\Big(x^1, \frac{\delta}{2}\Big) \quad \text{and} \quad (R_k^1)^{-1} < \frac{\delta}{2}.$$

Moreover, if $\bar{k}$ is chosen sufficiently large, for each $i = 1, 2, \dots, J$ it is possible to find $\tau_k^i \in \mathbb{Z}_k$ such that $(\tau_k^i z_1, y_1)$ stays in the middle half of S_i, i.e.

$$(\tau_k^i z_1, y_1) \in \left\{ (z, y) = (|z|e^{\theta\sqrt{-1}}, y) \in \mathbb{R}^2 \times \mathbb{R}^{N-2} : \ \bar{\theta} - \frac{(2i-1)\pi}{2J} < \theta < \bar{\theta} + \frac{(2i+1)\pi}{2J} \right\}.$$

Hence

$$B\big((\tau_k^i z_1, y_1), \delta\big) \subset S_i, \quad (\tau_k^i z_k^1, y_k^1) \in B\Big((\tau_k^i z_1, y_1), \frac{\delta}{2}\Big)$$

and consequently

$$B\Big((\tau_k^i z_k^1, y_k^1), \frac{\delta}{2}\Big) \subset S_i$$

which yields

$$B\big((\tau_k^i z_k^1, y_k^1), (R_k^i)^{-1}\big) \subset S_i.$$

In particular the J balls $B\big((\tau_k^i z_k^1, y_k^1), (R_k^i)^{-1}\big)$ are disjoint, hence, by symmetry properties of $\tilde{v}_k$ we have that

$$(S_k(\lambda))^{N/2} + o(1) = \int_{\mathbb{R}^N} |\tilde{v}_k|^{2^*} \geq \sum_{i=1}^{J} \int_{B((\tau_k^i z_k^1, y_k^1), (R_k^i)^{-1})} |\tilde{v}_k|^{2^*} = \sum_{i=1}^{J} \int_{B(x_k^1, (R_k^i)^{-1})} |\tilde{v}_k|^{2^*}.$$

On the other hand from (51) we have, for k large,

$$\begin{aligned} &\bigg(\int_{B(x_k^1, (R_k^i)^{-1})} |\tilde{v}_k|^{2^*} \bigg)^{\frac{1}{2^*}} \\ &\geq \bigg(\int_{B(x_k^1, (R_k^i)^{-1})} (R_k^1)^N \big| \tilde{v}_0(R_k^1(x - x_k^1)) \big|^{2^*} \bigg)^{\frac{1}{2^*}} - \bigg(\int_{B(x_k^1, (R_k^i)^{-1})} |\mathcal{R}_k|^{2^*} \bigg)^{\frac{1}{2^*}} \geq \frac{1}{2} \bigg(\int_{B(0,1)} \tilde{v}_0^{2^*} \bigg)^{\frac{1}{2^*}}. \end{aligned}$$

Therefore

$$(S_k(\lambda))^{N/2} + o(1) \geq \frac{J}{2^{2^*}} \int_{B(0,1)} \tilde{v}_0^{2^*}$$

and, in view of (44)

$$(S_{\text{circ}}(\lambda))^{N/2} \geq \frac{J}{2^{2^*}} \int_{B(0,1)} \tilde{v}_0^{2^*}.$$

Letting $J \to +\infty$, we find a contradiction. Claim 1 is thereby proved in the case $|z_1| \neq 0$. The case $|z_1| = 0$ can be treated exploiting the radial symmetry of functions $\tilde{u}_k$ in the last $N-2$ variables with a similar argument (even simpler due to the stronger symmetry).

Claim 2. We claim that $\tilde u_0 \in \mathcal{D}^{1,2}_{\rm circ}(\mathbb{R}^N)$. We first note that $\tilde u_k$ satisfy the equation $-\Delta \tilde u_k - \lambda \frac{\tilde u_k}{|x|^2} = S_k(\lambda)\tilde u_k^{2^*-1}$. From (44), we can assume that $S_k(\lambda) \to L \in (0,+\infty)$ at least for a subsequence. Hence, due to weak convergence of $\tilde u_k \rightharpoonup \tilde u_0$, we can pass to the limit in the equation to find that $\tilde u_0$ satisfies the equation $-\Delta \tilde u_0 - \lambda \frac{\tilde u_0}{|x|^2} = L\tilde u_0^{2^*-1}$. By classical regularity theory for elliptic equations, we deduce that $\tilde u_0$ is a smooth function outside the origin.

Let $R>0$. Assume that there exist $(z_1,y),(z_2,y) \in B(0,R)\cap(\mathbb{R}^2\times\mathbb{R}^{N-2})$, $|z_1|=|z_2|$, such that $\tilde u_0(z_1,y) \neq \tilde u_0(z_2,y)$. Then there exist $\delta>0$ such that $\tilde u_0(x)\neq \tilde u_0(y)$ for any $x\in B((z_1,y),\delta)$, $y \in B((z_2,y),\delta)$. Let $0<\varepsilon<\frac12|B(0,\delta)|$. Since, up to a subsequence, $\tilde u_k \to \tilde u_0$ a.e. in $B(0,R)$, by the Severini-Egorov Theorem, there exists a measurable set $\Omega\subset B(0,R)$ such that $|\Omega|<\varepsilon$ and $\tilde u_k \to \tilde u_0$ uniformly in $B(0,R)\setminus\Omega$. Hence for k large, $\tilde u_k(x)\neq \tilde u_k(y)$ for any $x \in B((z_1,y),\delta)\setminus\Omega$, $y\in B((z_2,y),\delta)\setminus\Omega$. On the other hand, if k is large enough, there exists $\tau_k \in \mathbb{Z}_k$ such that

$$\big|\tau_k(B((z_1,y),\delta))\triangle B((z_2,y),\delta)\big| < \varepsilon,$$

where $\triangle$ denotes the symmetric difference of sets. Hence

$$\Big|\big(\tau_k(B((z_1,y),\delta))\cap B((z_2,y),\delta)\big)\setminus\Omega\Big| > |B(0,\delta)| - 2\varepsilon > 0.$$

In particular the set $\big(\tau_k(B((z_1,y),\delta))\cap B((z_2,y),\delta)\big)\setminus\Omega$ has non-zero measure. If $(z,y)\in \big(\tau_k(B((z_1,y),\delta))\cap B((z_2,y),\delta)\big)\setminus\Omega$, then $z=\tau_k\tilde z$ with $(\tilde z,y)\in B((z_1,y),\delta)$ and by symmetry of $\tilde u_k$, $\tilde u_k(z,y)=\tilde u_k(\tilde z,y)$, thus giving a contradiction. Hence $\tilde u_0$ is invariant by the $\mathbb{SO}(2)$-action on the first two variables on $B(0,R)$ for any R. Invariance by the $\mathbb{SO}(N-2)$-action on the last $(N-2)$ variables follows easily from pointwise convergence. Then we conclude that $\tilde u_0 \in \mathcal{D}^{1,2}_{\rm circ}(\mathbb{R}^N)$.

Hence we have proved that, up to a subsequence, $\tilde u_k \rightharpoonup \tilde u_0$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, with $\tilde u_0 \in \mathcal{D}^{1,2}_{\rm circ}(\mathbb{R}^N)\setminus\{0\}$. Weak convegence yields

$$Q_\lambda(\tilde u_k) = Q_\lambda(\tilde u_0) + Q_\lambda(\tilde u_k - \tilde u_0) + o(1) \tag{52}$$

while Brezis-Lieb Lemma implies

$$\|\tilde u_k\|_{2^*}^{2^*} = \|\tilde u_0\|_{2^*}^{2^*} + \|\tilde u_k - \tilde u_0\|_{2^*}^{2^*} + o(1). \tag{53}$$

From (52), (53), (46), and (18) we have

$$S_k(\lambda) \le \frac{Q_\lambda(\tilde u_0)}{\|\tilde u_0\|_{2^*}^2} \le S_k(\lambda)\frac{Q_\lambda(\tilde u_k) - Q_\lambda(\tilde u_k-\tilde u_0)+o(1)}{\big(Q_\lambda(\tilde u_k)^{2^*/2} - Q_\lambda(\tilde u_k-\tilde u_0)^{2^*/2}+o(1)\big)^{2/2^*}}.$$

Hence

$$\frac{Q_\lambda(\tilde u_k) - Q_\lambda(\tilde u_k-\tilde u_0)+o(1)}{\big(Q_\lambda(\tilde u_k)^{2^*/2} - Q_\lambda(\tilde u_k-\tilde u_0)^{2^*/2}+o(1)\big)^{2/2^*}} \ge 1.$$

Since $Q_\lambda(\tilde u_k)$ stay bounded away from 0, this is possible only when $Q_\lambda(\tilde u_k-\tilde u_0)\to 0$. Since $Q_\lambda^{1/2}$ is an equivalent norm, we deduce that $\tilde u_k \to \tilde u_0$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. In particular $\|\tilde u_0\|_{2^*} = \lim_k \|\tilde u_k\|_{2^*} = 1$. Hence, by weakly lower semi-continuity of Q_λ, (46), and (44)

$$S_{\rm circ}(\lambda) \le \frac{Q_\lambda(\tilde u_0)}{\|\tilde u_0\|_{2^*}^2} \le \liminf_k Q_\lambda(\tilde u_k) = \liminf_k S_k(\lambda) \le \limsup_k S_k(\lambda) \le S_{\rm circ}(\lambda).$$

Therefore all the above inequalities are indeed equalities. We have thus proved that along a subsequence, $S_k(\lambda)$ converges to $S_{\rm circ}(\lambda)$. The Uryson's property yields convergence of the entire sequence. ■

8. Proof of Theorem 1.4

The proof of Theorem 1.4 is based on Theorem 7.3 and the following lemma.

Lemma 8.1. $\limsup_{k\to+\infty} S_k(\lambda_0,\Lambda_1,\dots,\Lambda_m)\le S_{\rm circ}(\lambda_0,\Lambda_1,\dots,\Lambda_m)$.

Proof. Let $\varepsilon>0$. Then from (5) and density of $\mathcal{D}(\mathbb{R}^N\setminus\{0\})\cap\mathcal{D}^{1,2}_{\rm circ}(\mathbb{R}^N)$ in $\mathcal{D}^{1,2}_{\rm circ}(\mathbb{R}^N)$, there exists $u\in\mathcal{D}(\mathbb{R}^N\setminus\{0\})\cap\mathcal{D}^{1,2}_{\rm circ}(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N}|u|^{2^*}=1$ and

(54)
$$\int_{\mathbb{R}^N}|\nabla u|^2\,dx-\lambda_0\int_{\mathbb{R}^N}\frac{u^2(x)}{|x|^2}\,dx-\sum_{\ell=1}^m\Lambda_\ell\int_{\mathbb{R}^N}\bigg(\fint_{S_{r_\ell}}\frac{u^2(y)}{|x-y|^2}\,d\sigma(x)\bigg)\,dy<S_{\rm circ}(\lambda_0,\Lambda_1,\dots,\Lambda_m)+\varepsilon.$$

For any $\ell=1,\dots,m$, set

$$f_\ell(x):=\int_{\mathbb{R}^N}\frac{|u(y+x)|^2}{|y|^2}\,dy,\qquad x\in S_{r_\ell}.$$

It is easy to check that $f_\ell\in C^0(S_{r_\ell})$; indeed if $x_n\in S_{r_\ell}$ converge to $x\in S_{r_\ell}$, by the Dominated Convergence Theorem we conclude that $\lim_n f_\ell(x_n)=f_\ell(x)$. Hence the Riemann sum

$$\frac1k\sum_{i=1}^k f_\ell(a_i^\ell)=\frac1k\sum_{i=1}^k\int_{\mathbb{R}^N}\frac{|u(y)|^2}{|y-a_i^\ell|^2}\,dy$$

converges to the integral

$$\frac{1}{2\pi r_\ell}\int_{S_{r_\ell}}f_\ell(x)\,d\sigma(x)=\int_{\mathbb{R}^N}\bigg(\fint_{S_{r_\ell}}\frac{u^2(y)}{|x-y|^2}\,d\sigma(x)\bigg)\,dy.$$

Hence there exists $\bar k$ such that for all $k\ge\bar k$

(55)
$$\begin{aligned}\int_{\mathbb{R}^N}|\nabla u|^2\,dx-\lambda_0\int_{\mathbb{R}^N}\frac{u^2(x)}{|x|^2}\,dx-\sum_{\ell=1}^m\sum_{i=1}^k\frac{\Lambda_\ell}{k}\int_{\mathbb{R}^N}\frac{u^2(x)}{|x-a_i^\ell|^2}\,dx-\varepsilon\\ \le\int_{\mathbb{R}^N}|\nabla u|^2\,dx-\lambda_0\int_{\mathbb{R}^N}\frac{u^2(x)}{|x|^2}\,dx-\sum_{\ell=1}^m\Lambda_\ell\int_{\mathbb{R}^N}\bigg(\fint_{S_{r_\ell}}\frac{u^2(y)}{|x-y|^2}\,d\sigma(x)\bigg)\,dy.\end{aligned}$$

From (54–55), we deduce that

$$S_k(\lambda_0,\Lambda_1,\dots,\Lambda_m)-\varepsilon<S_{\rm circ}(\lambda_0,\Lambda_1,\dots,\Lambda_m)+\varepsilon.$$

Taking lim sup as $k\to+\infty$, since ε is arbitrary we reach the conclusion. ■

Proof of Theorem 1.4. As in the proof of Theorem 1.3, we can find a minimizing sequence $\{u_n\}_n\subset\mathcal{D}^{1,2}_k(\mathbb{R}^N)$ for (3) with the Palais-Smale property. Under assumption (9), Lemma 6.2 yields

(56)
$$S_{\rm circ}(\lambda_0,\Lambda_1,\dots,\Lambda_m)<S_{\rm circ}(\lambda_0),$$

while (8) implies

(57)
$$S_k(\lambda_0)\le S_k\Big(\lambda_0+k\sum_{\ell=1}^k\lambda_\ell\Big).$$

Let $0 < \varepsilon < S_{\rm circ}(\lambda_0) - S_{\rm circ}(\lambda_0, \Lambda_1, \dots, \Lambda_m)$. From Lemma 8.1, there exists $k_1 = k_1(\varepsilon)$ such that for all $k \geq k_1$

$$S_k(\lambda_0, \Lambda_1, \dots, \Lambda_m) < S_{\rm circ}(\lambda_0, \Lambda_1, \dots, \Lambda_m) + \varepsilon. \tag{58}$$

From Theorem 7.3 and (56), there exists k_2 such that for all $k \geq k_2$

$$S_{\rm circ}(\lambda_0, \Lambda_1, \dots, \Lambda_m) + \varepsilon < S_k(\lambda_0) \leq S_{\rm circ}(\lambda_0).$$

Let k_3 be such that for all $k \geq k_3$

$$S_{\rm circ}(\lambda_0, \Lambda_1, \dots, \Lambda_m) + \varepsilon \leq \min\Big\{k^{\frac{2}{N}} S, k^{\frac{2}{N}} S(\lambda_1), \dots, k^{\frac{2}{N}} S(\lambda_m)\Big\}. \tag{59}$$

From (57–59), we conclude that for all $k \geq \max\{k_1, k_2, k_3\}$

$$S_k(\lambda_0, \Lambda_1, \dots, \Lambda_m) < \min\Big\{k^{\frac{2}{N}} S, k^{\frac{2}{N}} S(\lambda_1), \dots, k^{\frac{2}{N}} S(\lambda_m), S_k(\lambda_0), S_k\Big(\lambda_0 + k \sum\nolimits_{\ell=1}^m \lambda_\ell\Big)\Big\}.$$

From above and the Palais-Smale condition proved in Theorem 4.1, we deduce that $\{u_n\}_{n\in\mathbb{N}}$ has a subsequence strongly converging to some $u_0 \in \mathcal{D}_k^{1,2}(\mathbb{R}^N)$ such that $J_k(u_0) = \frac{1}{N} S_k(\lambda_0, \Lambda_1, \dots, \Lambda_m)$. Hence u_0 achieves the infimum in (3). Since J_k is even, also $|u_0|$ is a minimizer in (3) and then $v_0 = S_k(\lambda_0, \Lambda_1, \dots, \Lambda_m)^{1/(2^*-2)}|u_0|$ is a nonnegative solution to equation (2). The maximum principle implies the positivity outside singularities of such a solution. ■

9. Proof of Theorem 1.5

We now provide a sufficient condition for the infimum in (3) to stay below the level $k^{2/N} S(\lambda_j)$, in correspondence of which possible concentration at singular points located at the j-th polygon can occur. We denote by $\Theta_{j\ell}$ the minimum angle formed by vectors a_i^j and a_s^ℓ, see figure below.

Figure 4 (The angle $\Theta_{j\ell}$.)

The following lemma can be proved by standard trigonometry calculus.

Lemma 9.1. *For any $i, s = 1, 2, \dots, k$, and $j, \ell = 1, 2, \dots, m$, there holds*

$$|a_i^j - a_s^j| = 2r_j \Big| \sin \frac{(s-i)\pi}{k} \Big|,$$
$$|a_i^j - a_s^\ell|^2 = r_j^2 + r_\ell^2 - 2r_j r_\ell \cos\Big(\frac{2\pi(i-s)}{k} + \Theta_{j\ell}\Big).$$

Lemma 9.2. *Let* $j \in \{1,2,\dots,m\}$. *If*

$$0 < \lambda_j \le \frac{N(N-4)}{4}$$

and

$$\frac{\lambda_0}{|r_j|^2} + \lambda_j \sum_{i=1}^{k-1} \frac{1}{4r_j^2\big|\sin\frac{i\pi}{k}\big|^2} + \sum_{\substack{\ell=1\\ \ell\neq j}}^{m} \lambda_\ell \sum_{i=1}^{k} \frac{1}{r_j^2 + r_\ell^2 - 2r_j r_\ell \cos\Big(\frac{2\pi i}{k} + \Theta_{j\ell}\Big)} > 0, \tag{60}$$

then

$$S_k(\lambda_0, \Lambda_1, \dots, \Lambda_m) < k^{2/N} S(\lambda_j). \tag{61}$$

Proof. Let $z(x) = \sum_{i=1}^k z_\mu^{\lambda_j}(x - a_i^j) \in \mathcal{D}_k^{1,2}(\mathbb{R}^N)$. Then

$$S_k(\lambda_0, \Lambda_1, \dots, \Lambda_m) \le \frac{Q_{\lambda_0,\lambda_1,\dots,\lambda_m}(z)}{\Big(\int_{\mathbb{R}^N} |z|^{2^*}dx\Big)^{2/2^*}},$$

where $Q_{\lambda_0,\lambda_1,\dots,\lambda_m}$ denotes the quadratic form defined by

$$Q_{\lambda_0,\lambda_1,\dots,\lambda_m}(u) = \int_{\mathbb{R}^N} |\nabla u|^2 dx - \int_{\mathbb{R}^N} \frac{\lambda_0}{|x|^2} u^2(x)\,dx - \sum_{\ell=1}^m \sum_{i=1}^k \lambda_\ell \int_{\mathbb{R}^N} \frac{u^2(x)}{|x - a_i^\ell|^2}\,dx.$$

Note that $\big(\int_{\mathbb{R}^N} |z|^{2^*}dx\big)^{2/2^*} \ge k^{2/2^*}$. Moreover, from (16), Lemmas 5.1 and 5.2 we find that

$$\begin{aligned}
Q_{\lambda_0,\lambda_1,\dots,\lambda_m}(z) &= k\int_{\mathbb{R}^N} |\nabla z_\mu^{\lambda_j}|^2\,dx - k\lambda_j \int_{\mathbb{R}^N} \frac{|z_1^{\lambda_j}|^2}{|x|^2}\,dx - \lambda_0 \sum_{i=1}^k \int_{\mathbb{R}^N} \frac{|z_\mu^{\lambda_j}|^2}{|x + a_i^j|^2}\,dx \\
&- \sum_{\ell=1}^m \sum_{\substack{i=1,s=1\\ (i,\ell)\neq(s,j)}}^k \int_{\mathbb{R}^N} \frac{\lambda_\ell |z_\mu^{\lambda_j}|^2}{|x + a_s^j - a_i^\ell|^2}\,dx + \sum_{\substack{i=1,s=1\\ i\neq s}}^k \int_{\mathbb{R}^N} \nabla z_\mu^{\lambda_j}(x - a_i^j)\cdot \nabla z_\mu^{\lambda_j}(x - a_s^j)\,dx \\
&- \sum_{\ell=1}^m \sum_{\substack{i=1,s=1,t=1\\ s\neq t}}^k \int_{\mathbb{R}^N} \frac{\lambda_\ell z_\mu^{\lambda_j}(x - a_s^j) z_\mu^{\lambda_j}(x - a_t^j)}{|x - a_i^\ell|^2}\,dx - \lambda_0 \sum_{\substack{i=1,s=1\\ i\neq s}}^k \int_{\mathbb{R}^N} \frac{z_\mu^{\lambda_j}(x - a_i^j) z_\mu^{\lambda_j}(x - a_s^j)}{|x|^2}\,dx \\
&= \begin{cases} kS(\lambda_j) - \mu^2\Big(\int_{\mathbb{R}^N} |z_1^{\lambda_j}|^2\Big)\Big[\frac{k\lambda_0}{|r_j|^2} + \sum_{\ell=1}^m \sum_{\substack{i=1,s=1\\ (i,\ell)\neq(s,j)}}^k \frac{\lambda_\ell}{|a_i^\ell - a_s^j|^2} + o(1)\Big] \text{if } \lambda_j < \frac{N(N-4)}{4}, \\[2ex] kS(\lambda_j) - \mu^2 |\ln\mu| \alpha_{\lambda_j,N}^2 \Big[\frac{k\lambda_0}{|r_j|^2} + \sum_{\ell=1}^m \sum_{\substack{i=1,s=1\\ (i,\ell)\neq(s,j)}}^k \frac{\lambda_\ell}{|a_i^\ell - a_s^j|^2} + o(1)\Big] \text{if } \lambda_j = \frac{N(N-4)}{4}. \end{cases}
\end{aligned}$$

Therefore (61) holds provided

$$\frac{k\lambda_0}{|r_j|^2} + \sum_{\ell=1}^m \sum_{\substack{i=1,s=1\\ (i,\ell)\neq(s,j)}}^k \frac{\lambda_\ell}{|a_i^\ell - a_s^j|^2} > 0. \tag{62}$$

It is easy to verify that assumption (60) and Lemma 9.1 imply (62). ■

Lemma 9.3. *For any $j, \ell = 1, \dots, m$, $j \neq \ell$, there holds*

$$\lim_{k\to\infty} \frac{1}{\Lambda_j}\left[\lambda_j \sum_{i=1}^{k-1} \frac{1}{4r_j^2 \big|\sin\frac{i\pi}{k}\big|^2}\right] = +\infty, \tag{63}$$

$$\lambda_\ell \sum_{i=1}^{k} \frac{1}{r_j^2 + r_\ell^2 - 2r_j r_\ell \cos\left(\frac{2\pi i}{k} + \Theta_{j\ell}\right)} = \Lambda_\ell\, O(1) \quad \textit{as } k \to +\infty. \tag{64}$$

Proof. A direct calculation yields

$$\frac{\lambda_j}{\Lambda_j}\sum_{i=1}^{k-1} \frac{1}{4r_j^2 \big|\sin\frac{i\pi}{k}\big|^2} \geq \frac{1}{k}\int_1^{k/2} \frac{ds}{4r_j^2\big|\sin\frac{s\pi}{k}\big|^2} \geq \frac{1}{\pi}\int_{\pi/k}^{\pi/2} \frac{dt}{4r_j^2|\sin t|^2} \underset{k\to+\infty}{\longrightarrow} +\infty.$$

On the other hand

$$\begin{aligned}\frac{\lambda_\ell}{\Lambda_\ell}\sum_{i=1}^{k} \frac{1}{r_j^2 + r_\ell^2 - 2r_j r_\ell \cos\left(\frac{2\pi i}{k} + \Theta_{j\ell}\right)} &\leq \frac{1}{k}\int_0^k \frac{ds}{r_j^2 + r_\ell^2 - 2r_j r_\ell \cos\left(\frac{2\pi s}{k} + \Theta_{j\ell}\right)} \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{dt}{r_j^2 + r_\ell^2 - 2r_j r_\ell \cos(t + \Theta_{j\ell})} \in \mathbb{R}^+,\end{aligned}$$

thus proving (64). ■

Remark 9.4. *Lemma 9.3 implies that if we fix $\lambda_0, \Lambda_1, \dots, \Lambda_m$ and let $k \to +\infty$, then the quantity in formula (60) tends to $+\infty$. Hence condition (60) is satisfied for k sufficiently large.*

Let us now compare levels $S_k(\lambda_0, \Lambda_1, \dots, \Lambda_m)$ and $S_k(\lambda_0)$ which is related to concentration at the origin. Two cases can occur:

$$(i) \qquad S_k(\lambda_0) \geq k^{2/N} S,$$

$$(ii) \qquad S_k(\lambda_0) < k^{2/N} S.$$

In case (i), since $S = S(0)$ and $\lambda \mapsto S(\lambda)$ is a nonincreasing function, to exclude that the infimum in (18) stays above $S_k(\lambda_0)$ it is enough to compare $S_k(\lambda_0, \Lambda_1, \dots, \Lambda_m)$ with $k^{2/N} S(\lambda_j)$ where $\lambda_j = \max\{\lambda_\ell\}_{1\leq \ell\leq m}$, as we have done in Lemma 9.2.

The study of case (ii) is based on Lemma 2.1. Indeed, using Lemma 2.1 and estimates of Lemma 5.1, we can prove the following lemma.

Lemma 9.5. *If $N \geq 4$, $S_k(\lambda_0) < k^{2/N} S$, and one of the following assumptions is satisfied*

$$\lambda_0 \leq \frac{N(N-4)}{4} \quad \textit{and} \quad \sum_{\ell=1}^{m} \frac{\lambda_\ell}{|r_\ell|^2} > 0, \tag{65}$$

$$\frac{N(N-4)}{4} < \lambda_0 < \frac{(N-2)^2}{4} \quad \textit{and} \quad \sum_{\ell=1}^{m} \frac{\lambda_\ell}{|r_\ell|^{\sqrt{(N-2)^2 - 4\lambda_0}}} > 0, \tag{66}$$

then

$$S_k(\lambda_0, \Lambda_1, \dots, \Lambda_m) < S_k(\lambda_0). \tag{67}$$

Proof. From Lemma 2.1, we have that $S_k(\lambda_0)$ is attained by some $u^{\lambda_0} \in \mathcal{D}_k^{1,2}(\mathbb{R}^N)$. By homogeneity of the Rayleigh quotient, we can assume $\int |u^{\lambda_0}|^{2^*} = 1$. Furthermore, the function $v^{\lambda_0} = S_k(\lambda_0, \Lambda_1, \dots, \Lambda_m)^{1/(2^*-2)}|u^{\lambda_0}|$ is a nonnegative solution to (14), hence we can apply Lemma 5.1 to study the behavior of $\int_{\mathbb{R}^N} \frac{|u_\mu^{\lambda_0}|^2}{|x+\xi|^2}\,dx$ as $\mu \to 0$. Hence we obtain that there exists some positive constant κ_0 such that

$$
\begin{aligned}
&S_k(\lambda_0, \Lambda_1, \dots, \Lambda_m)\\
&\le \int_{\mathbb{R}^N} |\nabla u_\mu^{\lambda_0}(x)|^2\,dx - \lambda_0 \int_{\mathbb{R}^N} \frac{|u_\mu^{\lambda_0}(x)|^2}{|x|^2}\,dx - \sum_{\ell=1}^m \sum_{i=1}^k \lambda_\ell \int_{\mathbb{R}^N} \frac{|u_\mu^{\lambda_0}(x)|^2}{|x-a_i^\ell|^2}\,dx\\
&= S_k(\lambda_0) - \begin{cases}
\mu^2 \int_{\mathbb{R}^N} |u^{\lambda_0}|^2 \left(\displaystyle\sum_{\ell=1}^m \sum_{i=1}^k \frac{\lambda_\ell}{|r_\ell|^2} + o(1) \right) & \text{if } \lambda_0 < \frac{N(N-4)}{4}\\[2ex]
\kappa_0^2 \mu^2 |\ln \mu| \left(\displaystyle\sum_{\ell=1}^m \sum_{i=1}^k \frac{\lambda_\ell}{|r_\ell|^2} + o(1) \right) & \text{if } \lambda_0 = \frac{N(N-4)}{4}\\[2ex]
\kappa_0^2\, \beta_{\lambda,N}\, \mu^{\sqrt{(N-2)^2-4\lambda}} \left(\displaystyle\sum_{\ell=1}^m \sum_{i=1}^k \frac{\lambda_\ell}{|r_\ell|^{\sqrt{(N-2)^2-4\lambda_0}}} + o(1) \right) & \text{if } \lambda_0 > \frac{N(N-4)}{4}.
\end{cases}
\end{aligned}
$$

Taking μ sufficiently small we obtain that either assumption (65) or (66) yield (67). ■

Proof of Theorem 1.5. As in the proof of Theorem 1.3, we can find a minimizing sequence $\{u_n\}_{n\in\mathbb{N}}$ which has the Palais-Smale property, more precisely $J_k'(u_n) \to 0$ in $(\mathcal{D}_k^{1,2}(\mathbb{R}^N))^\star$ and $J_k(u_n) \to \frac{1}{N} S_k(\lambda_0 \Lambda_1, \dots, \Lambda_m)$. Assumption (10) yields

$$S_k\Big(\lambda_0 + k \sum_{\ell=1}^k \lambda_\ell\Big) \ge S_k(\lambda_0).$$

Note also that (11) and (12) imply that $N > 4$ and $\lambda_m > 0$. Two cases can occur. If $S_k(\lambda_0) < k^{\frac{2}{N}} S$, then Lemma 9.5 and assumption (12) yield

$$S_k(\lambda_0, \Lambda_1, \dots, \Lambda_m) < S_k(\lambda_0).$$

If $S_k(\lambda_0) \ge k^{2/N} S$, from monotonicity we have $S_k(\lambda_0) \ge k^{2/N} S > k^{2/N} S(\lambda_m)$. In both cases from Lemma 9.2 and (13) we deduce $k^{2/N} S(\lambda_m) > S_k(\lambda_0, \Lambda_1, \dots, \Lambda_m)$. Hence we obtain that

$$S_k(\lambda_0, \Lambda_1, \dots, \Lambda_m) < \min\Big\{ k^{\frac{2}{N}} S, k^{\frac{2}{N}} S(\lambda_1), \dots, k^{\frac{2}{N}} S(\lambda_m), S_k(\lambda_0), S_k\Big(\lambda_0 + k \sum\nolimits_{\ell=1}^m \lambda_\ell\Big)\Big\}.$$

From above and the Palais-Smale condition proved in Theorem 4.1, we deduce that $\{u_n\}_{n\in\mathbb{N}}$ has a subsequence strongly converging to some $u_0 \in \mathcal{D}_k^{1,2}(\mathbb{R}^N)$ which achieves the infimum in (3). Moreover $v_0 = S_k(\lambda_0 \Lambda_1, \dots, \Lambda_m)^{\frac{1}{2^*-2}} |u_0|$ is a solution to (2). ■

Remark 9.6. *Theorem 1.5 contains an alternative proof to Theorem 1.4 in the case $N > 4$, as it follows easily gathering Theorem 1.5 and Remark 9.4. Note that the assumption $N > 4$ is needed to ensure that (11) and (12) hold. However, with respect to Theorem 1.4, it contains a more precise information on how k must be large in order to solve the problem.*

10. Proof of Theorem 1.2

Proof of Theorem 1.2. Assume first (i). Let $\varepsilon > 0$. Then from (3) and density of $\mathcal{D}(\mathbb{R}^N \setminus \{0\}) \cap \mathcal{D}^{1,2}_k(\mathbb{R}^N)$ in $\mathcal{D}^{1,2}_k(\mathbb{R}^N)$, there exists $u \in \mathcal{D}(\mathbb{R}^N \setminus \{0\}) \cap \mathcal{D}^{1,2}_k(\mathbb{R}^N)$ such that $Q_{\lambda_0}(u) \leq S_k(\lambda_0) + \varepsilon$. Let

$$u_\mu(x) = \mu^{-\frac{N-2}{2}} u(x/\mu), \quad \mu > 0.$$

By Dominated Convergence Theorem it is easy to verify that

$$\lim_{\mu \to 0} \int_{\mathbb{R}^N} \frac{|u_\mu(x)|^2}{|x - a_i^\ell|^2} = 0,$$

hence

$$S_k(\lambda_0, \Lambda_1, \dots, \Lambda_m) \leq \frac{\int_{\mathbb{R}^N} |\nabla u_\mu|^2 dx - \lambda_0 \int_{\mathbb{R}^N} \frac{u_\mu^2(x)}{|x|^2} - \int_{\mathbb{R}^N} \sum_{\ell=1}^m \frac{\Lambda_\ell}{k} \sum_{i=1}^k \frac{u_\mu^2(x)}{|x-a_i^\ell|^2} dx}{\big(\int_{\mathbb{R}^N} |u_\mu|^{2^*} dx\big)^{2/2^*}}$$

$$= Q_{\lambda_0}(u) + o(1) \leq S_k(\lambda_0) + \varepsilon + o(1) \quad \text{as } \mu \to 0.$$

Letting $\mu \to 0$ we have that $S_k(\lambda_0, \Lambda_1, \dots, \Lambda_m) \leq S_k(\lambda_0) + \varepsilon$ for all $\varepsilon > 0$. Hence

$$S_k(\lambda_0, \Lambda_1, \dots, \Lambda_m) \leq S_k(\lambda_0).$$

Assume by contradiction that $S_k(\lambda_0, \Lambda_1, \dots, \Lambda_m)$ is attained by $\bar u \in \mathcal{D}^{1,2}_k(\mathbb{R}^N) \setminus \{0\}$, then

$$S_k(\lambda_0, \Lambda_1, \dots, \Lambda_m) = \frac{\int_{\mathbb{R}^N} |\nabla \bar u|^2 dx - \lambda_0 \int_{\mathbb{R}^N} \frac{\bar u^2(x)}{|x|^2} - \int_{\mathbb{R}^N} \sum_{\ell=1}^m \frac{\Lambda_\ell}{k} \sum_{i=1}^k \frac{\bar u^2(x)}{|x-a_i^\ell|^2} dx}{\big(\int_{\mathbb{R}^N} |\bar u|^{2^*} dx\big)^{2/2^*}}$$

$$> \frac{\int_{\mathbb{R}^N} |\nabla \bar u|^2 \, dx - \int_{\mathbb{R}^N} \frac{\lambda_0}{|y|^2} \bar u^2(y) \, dy}{\big(\int_{\mathbb{R}^N} |\bar u|^{2^*} dx\big)^{2/2^*}} \geq S_k(\lambda_0),$$

giving rise to a contradiction. The proof of non-attainability of $S_{\rm circ}(\lambda_0, \Lambda_1, \dots, \Lambda_m)$ is analogous and is based on (42).

Assume now that (ii) holds. Then for all $u \in \mathcal{D}^{1,2}_{\rm circ}(\mathbb{R}^N)$, $u \geq 0$, denoting by u^* the Schwarz symmetrization of u (see (24)), from (25) and (26) it follows that

$$\frac{Q^{\rm circ}_{\lambda_0, \Lambda_1, \dots, \Lambda_m}(u)}{\big(\int_{\mathbb{R}^N} |u|^{2^*} dx\big)^{2/2^*}} \geq \frac{\int_{\mathbb{R}^N} |\nabla u^*|^2 \, dx - \big(\lambda_0 + \sum_{\ell=1}^m \Lambda_\ell\big) \int_{\mathbb{R}^N} \frac{|u^*(y)|^2}{|y|^2} \, dy}{\big(\int_{\mathbb{R}^N} |u^*|^{2^*} dx\big)^{2/2^*}}$$

$$\geq S_{\rm circ}\Big(\lambda_0 + \sum_{\ell=1}^m \Lambda_\ell\Big) = S\Big(\lambda_0 + \sum_{\ell=1}^m \Lambda_\ell\Big).$$

Hence $S_{\rm circ}(\lambda_0, \Lambda_1, \dots, \Lambda_m) \geq S_{\rm circ}\big(\lambda_0 + \sum_{\ell=1}^m \Lambda_\ell\big)$. On the other hand, setting $\Lambda = \lambda_0 + \sum_{\ell=1}^m \Lambda_\ell$, and using [14, Corollary 3.2] we obtain

$$S_{\rm circ}(\lambda_0, \Lambda_1, \dots, \Lambda_m) \leq Q^{\rm circ}_{\lambda_0, \Lambda_1, \dots, \Lambda_m}(z_\mu^\Lambda) = Q_\Lambda(z_\mu^\Lambda) + o(1) = S(\Lambda) + o(1)$$

$$= S_{\rm circ}\Big(\lambda_0 + \sum_{\ell=1}^m \Lambda_\ell\Big) + o(1) \quad \text{as } \mu \to \infty.$$

Then

$$S_{\mathrm{circ}}(\lambda_0, \Lambda_1, \dots, \Lambda_m) = S_{\mathrm{circ}}\Big(\lambda_0 + \sum_{\ell=1}^{m} \Lambda_\ell\Big). \tag{68}$$

If $S_{\mathrm{circ}}(\lambda_0, \Lambda_1, \dots, \Lambda_m)$ was attained by some $\bar{u} \in \mathcal{D}^{1,2}_{\mathrm{circ}}(\mathbb{R}^N) \setminus \{0\}$, then

$$S_{\mathrm{circ}}(\lambda_0, \Lambda_1, \dots, \Lambda_m) = \frac{Q^{\mathrm{circ}}_{\lambda_0,\Lambda_1,\dots,\Lambda_m}(\bar{u})}{\big(\int_{\mathbb{R}^N} |\bar{u}|^{2^*} dx\big)^{2/2^*}} \geq \frac{Q_\Lambda(\bar{u}^*)}{\big(\int_{\mathbb{R}^N} |\bar{u}^*|^{2^*} dx\big)^{2/2^*}} \geq S_{\mathrm{circ}}(\Lambda).$$

Due to (68), all above inequalities are indeed equalities; in particular $Q^{\mathrm{circ}}_{\lambda_0,\Lambda_1,\dots,\Lambda_m}(\bar{u}) = Q_\Lambda(\bar{u}^*)$ which (taking into account Polya-Szego inequality and (25)) yields

$$\begin{aligned} 0 &\leq \int_{\mathbb{R}^N} |\nabla \bar{u}|^2 dx - \int_{\mathbb{R}^N} |\nabla \bar{u}^*|^2 dx \\ &= \int_{\mathbb{R}^N} \bigg(\frac{\lambda_0}{|y|^2} + \sum_{\ell=1}^{m} \Lambda_\ell \fint_{S_{r_\ell}} \frac{d\sigma(x)}{|x-y|^2} \bigg) \bar{u}^2(y)\, dy - \Lambda \int_{\mathbb{R}^N} \frac{|\bar{u}^*(y)|^2}{|y|^2}\, dy \leq 0. \end{aligned}$$

Then $\int_{\mathbb{R}^N} |\nabla \bar{u}|^2 dx = \int_{\mathbb{R}^N} |\nabla \bar{u}^*|^2 dx$. From [3], it follows that $\bar{u}$ must be spherically symmetric with respect to some point. Since $\bar{u}$ is a solution to equation (4) (up to some Lagrange's multiplier), the potential in equation (4) must be spherically symmetric, thus giving rise to a contradiction. The proof of non-attainability of $S_k(\lambda_0, \Lambda_1, \dots, \Lambda_m)$ is contained in [14, Theorem 1.3]. ■

References

[1] B. Abdellaoui, V. Felli, I. Peral, *Existence and multiplicity for perturbations of an equation involving Hardy inequality and critical Sobolev exponent in the whole* $\mathbb{R}^N$, Adv. Differential Equations, 9 (2004), 481–508.

[2] T. Bartsch, M. Willem, *Infinitely many nonradial solutions of a Euclidean scalar field equation,* J. Funct. Anal., 117 (1993), no. 2, 447–460.

[3] J. E. Brothers, W. P. Ziemer, *Minimal rearrangements of Sobolev functions,* 15th winter school in abstract analysis (Srní, 1987), Acta Univ. Carolin. Math. Phys, 28 (1987), no. 2, 13–24.

[4] F. Catrina, Z.-Q. Wang, *Nonlinear elliptic equations on expanding symmetric domains,* J. Differential Equations, 156 (1999), no. 1, 153–181.

[5] F. Catrina, Z.-Q. Wang, *On the Caffarelli-Kohn-Nirenberg inequalities: sharp constants, existence (and nonexistence), and symmetry of extremal functions*, Comm. Pure Appl. Math., 54 (2001), no. 2, 229–258.

[6] F. Catrina, Z.-Q. Wang, *Positive bound states having prescribed symmetry for a class of nonlinear elliptic equations in* $\mathbb{R}^N$*,* Ann. Inst. H. Poincaré Anal. Non Linéaire, 18 (2001), no. 2, 157–178.

[7] M. Conti, S. Crotti, D. Pardo, *On the existence of positive solutions for a class of singular elliptic equations,* Adv. Differential Equations, 3 (1998), no. 1, 111–132.

[8] M. Conti, S. Terracini, G. Verzini, *Nodal solutions to a class of nonstandard superlinear equations on* $\mathbb{R}^N$*,* Adv. Differential Equations, 7 (2002), no. 3, 297–318.

[9] J. Dávila, L. Dupaigne, *Hardy-type inequalities,* J. Eur. Math. Soc. (JEMS), 6 (2004), no. 3, 335–365.

[10] V. De Cicco, M. A. Vivaldi, *Harnack inequalities for Fuchsian type weighted elliptic equations,* Comm. Partial Differential Equations, 21 (1996), no. 9-10, 1321–1347.

[11] T. Duyckaerts, *Opérateur de Schrödinger avec potentiel singulier multipolaire,* Bulletin de la SMF, to appear.

[12] H. Egnell, *Elliptic boundary value problems with singular coefficients and critical nonlinearities,* Indiana Univ. Math., J. 38 (1989), no. 2, 235–251.

[13] V. Felli, M. Schneider, *A note on regularity of solutions to degenerate elliptic equations of Caffarelli-Kohn-Nirenberg type,* Adv. Nonlinear Stud., 3 (2003), no. 4, 431–443.

[14] V. Felli, S. Terracini, *Elliptic equations with multi-singular inverse-square potentials and critical nonlinearity,* Preprint 2004.

[15] V. Felli, S. Terracini, *Nonlinear Schrödinger equations with symmetric multi-polar potentials,* Preprint, available on arXiv:math.AP/0506504, 24 Jun 2005.
[16] A. Ferrero, F. Gazzola, *Existence of solutions for singular critical growth semilinear elliptic equations,* J. Differential Equations, 177 (2001), no. 2, 494–522.
[17] S. Filippas, V. G. Maz'ya, A. Tertikas, *Sharp Hardy-Sobolev inequalities,* C. R. Math. Acad. Sci. Paris, 339 (2004), no. 7, 483–486.
[18] W. M. Frank, D. J. Land, R. M. Spector, *Singular potentials,* Rev. Modern Phys., 43 (1971), no. 1, 36–98.
[19] J. García Azorero, I. Peral, *Hardy Inequalities and some critical elliptic and parabolic problems,* J. Diff. Equations, 144 (1998), no. 2, 441–476.
[20] C. E. Gutiérrez, *Harnack's inequality for degenerate Schrödinger operators,* Trans. Amer. Math. Soc., 312 (1989), no. 1, 403–419.
[21] E. Jannelli, *The role played by space dimension in elliptic critical problems,* J. Differential Equations, 156 (1999), no. 2, 407–426.
[22] L. D. Landau, E. M. Lifshitz, *Quantum mechanics,* Pergamon Press Ltd., London-Paris, 1965.
[23] P. L. Lions, *The concentration-compactness principle in the calculus of variations. The limit case, part 1,* Rev. Matemática Iberoamericana, 1 (1985), no. 1, 145–201.
[24] P. L. Lions, *The concentration-compactness principle in the calculus of variations. The limit case, part 2,* Rev. Matemática Iberoamericana, 1 (1985), no. 2, 45–121.
[25] R. S. Palais, *The principle of symmetric criticality,* Comm. Math. Phys., 69 (1979), no. 1, 19–30.
[26] D. Ruiz, M. Willem, *Elliptic problems with critical exponents and Hardy potentials,* J. Differential Equations, 190 (2003), no. 2, 524–538.
[27] D. Smets, *Nonlinear Schrödinger equations with Hardy potential and critical nonlinearities,* Trans. AMS, to appear.
[28] M. Struwe, *Variational Methods and Applications to Nonlinear Partial Differential Equations and Hamiltonian systems,* Springer-Verlag, Berlin/New York (1990).
[29] S. Terracini, *On positive entire solutions to a class of equations with singular coefficient and critical exponent,* Adv. Diff. Equa., 1 (1996), no. 2, 241–264.
[30] Z.-Q. Wang, *Existence and symmetry of multi-bump solutions for nonlinear Schrdinger equations,* J. Differential Equations, 159 (1999), no. 1, 102–137.
[31] M. Willem, *Minimax theorems*, Progress in Nonlinear Differential Equations and their Applications, 24. Birkhuser Boston, Inc., Boston, MA, 1996.
[32] M. Willem, *Analyse fonctionnelle élémentaire*, Cassini Éditeurs. Paris, 2003.

Università di Milano Bicocca, Dipartimento di Matematica e Applicazioni, Via Cozzi 53, 20125 Milano.
E-mail address: veronica.felli@unimib.it, susanna.terracini@unimib.it.