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Abstract

Several authors have shown the ability of the variance gamma model to correct some biases of the Black-Scholes model. The variance gamma distribution has two additional parameters that allow to capture the skewness and kurtosis observed in financial data. However its density has not got a simple form formula and this implies numerical issues for historical estimation and option pricing.

This paper investigates the possibility of approximating the variance gamma distribution to a finite mixture of normals. Therefore, we apply this result to derive a simple historical estimation procedure by means of the Expectation Maximization algorithm and we obtain a simple formula to price a European call option.

Keywords: Variance Gamma distribution; Finite Mixture; EM algorithm; Option pricing

JEL classification codes: C00; C63; C65; G12; G13

1 Introduction

This paper investigates the possibility of approximating the variance gamma distribution with a finite mixture of normals.

The symmetric variance gamma model was first introduced by Madan and Seneta [27] in 1990. In 1998 Madan, Carr and Chang [8] proposed a three parameters version of the model, which allows to control over both skewness and kurtosis of the returns distribution.

The variance gamma distribution belongs to the class of normal variance mean mixture and corresponds to a gamma mixing density. Other cases of normal variance mean mixture are the normal inverse gaussian (Barndorff-Nielsen [4]), the

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hyperbolic and the generalized hyperbolic distribution (see Barndorff-Nielsen [3]).
For the historical estimation of the parameter values Madan and Seneta [27] employed the maximum likelihood (ML) method. Seneta [31] and Seneta and Tjetjep [32] discussed about fitting the variance gamma distribution by method of moments, while Finlay and Seneta [16] showed how to apply the generalized method of moments introduced by Hansen [20].
For option pricing purposes, Madan et al. [8] derived a semi-analytical formula for the price of a European call option while Carr and Madan [9] proposed to use the Fast Fourier Transform.

Another way to capture the skewness and kurtosis of the return distribution is through the finite normal mixture model. The normal mixture model has different economic interpretations such as market periods with different levels of volatility (see Bertholon et al. [6]) or heterogeneous groups of market participants (for example, "bullish" and "bearish" investors could behave differently see Haas et al. [19]).
After the first attempt made by Pearson in 1894, the problem of parameter estimation became easier since the introduction of the Expectation Maximization (EM) algorithm by Dempster et al. [12], based on the ML method (see McLachlan and Peel [28] for a complete survey of estimation problem in finite normal mixture models). The EM algorithm has also been used in the normal variance mean mixture (see Protassov [29] for the multivariate generalized hyperbolic distribution, Kostas [24] for the variance gamma distribution and Karlis [23] for the normal inverse gaussian distribution).
In 1990 Ritchey [30] introduced normal mixtures for option pricing; the resulting formula is a convex combination of Black-Scholes ones. See Bertholon [6] for recent results.

Empirical studies have shown the better performance for option pricing of the variance gamma (see Lam et al. [25]) and the normal mixture models (see Ritchey [30] and Gou [18]) compared to the Black-Scholes one. However the variance gamma density of returns has not got a simple form formula; this implies numerical issues for historical estimation and option pricing.

Following the idea of approximating the variance gamma distribution with a finite mixture of normals (see Bellini and Mercuri [5]), we derive a simple historical estimation procedure and an analytical formula for a European call option price.

The outline of the paper is as follows. In Section 2 we briefly review the variance gamma and the finite normal mixture models, and we show how the latter can be used to approximate the former. In Section 3 we provide a historical estimation procedure by means of the EM algorithm, and we apply it on different time series of the SPX index closing prices. In Section 4 we derive a simple formula to price European call options, and we compare our results with the prices obtained using Monte Carlo approach and the semi-analytical formula proposed by Madan et al. [8].

2 Variance gamma and finite mixture models

Variance gamma is a continuous probability distribution defined as a normal variance-mean mixture where the mixing density is a gamma distribution, that is:

$$Y := \mu_0 + \mu V + \sigma\sqrt{V}Z \tag{1}$$

where μ_0, $\mu \in R$, $\sigma \in [0, +\infty)$, $Z \sim N(0,1)$ and $V \sim \Gamma(\alpha, \beta)$ is independent from Z. The variance gamma density function can be written as follows:

$$f(y) = \int_0^\infty \frac{1}{\sigma\sqrt{2\pi v}} \exp\left(-\frac{(y-\mu_0-\mu v)^2}{2\sigma^2 v}\right) \frac{\beta^\alpha v^{\alpha-1}\exp(-\beta v)}{\Gamma(\alpha)} dv. \tag{2}$$

It has finite moments of all orders which can be easily calculated.
Given the scale property of gamma distribution, all the parameters in (1) are not identifiable. In literature, the common choice [27] to overcome this problem is to fix $\alpha = \frac{1}{\nu}$ and $\beta = \frac{1}{\nu}$ for the gamma mixing density. In this case, the σ, μ and ν parameters control respectively variability, skewness and kurtosis of the distribution, while μ_0 is a position parameter.
Without loss of generality, in this paper we choose $\beta = 1$, and the variance gamma density function assumes the following form:

$$\begin{aligned} f(y) &= \int_0^\infty \frac{1}{\sigma\sqrt{2\pi v}} \exp\left(-\frac{(y-\mu_0-\mu v)^2}{2\sigma^2 v}\right) \frac{v^{\alpha-1}\exp(-v)}{\Gamma(\alpha)} dv \\ &= \frac{\sqrt{2}\left(\frac{|y-\mu_0|}{\sqrt{\mu^2+2\sigma^2}}\right)^{a-0.5}}{\sqrt{\pi}\sigma\Gamma(a)} \exp\left(\frac{(y-\mu_0)\mu}{\sigma^2}\right) K_{a-0.5}\left(\frac{|y-\mu_0|\sqrt{\mu^2+2\sigma^2}}{\sigma^2}\right) \end{aligned} \tag{3}$$

where $K_p(u)$ is the modified Bessel function of the second type. This choice is equivalent to that proposed in [27] by imposing $\nu = \frac{1}{a}$, $\tilde{\sigma} = \sigma\sqrt{a}$ and $\tilde{\mu} = \mu a$.

Another flexible class of distributions is the finite normal mixture one. A random variable Y is a finite normal mixture if its density is a convex combination of normal densities:

$$f(y) = \sum_{i=1}^{n} \varphi(y; \mu_i, \sigma_i) p_i, \tag{4}$$

where $\varphi(y; \mu_i, \sigma_i)$ is the normal density function with mean μ_i and standard deviation σ_i; p_i are the weights satisfying the conditions:

$$p_i \geq 0 \quad \wedge \quad \sum_{i=1}^{n} p_i = 1.$$

As the variance gamma distribution, the finite normal mixture admits all finite moments.

In this section we show how to approximate the variance gamma density with a finite mixture of normals. In order to achieve our goal we use the Gauss-Laguerre quadrature (see [1]). With this method we have the following approximation:

$$\int_0^{+\infty} f(u)\, e^{-u} du \simeq \sum_{i=1}^{n} w(u_i)\, f(u_i) \tag{5}$$

where u_i are the $i^{th}-$ zeros of the Laguerre polynomial $L_n(u_i)$ and the weights $w(u_i)$ are:

$$w(u_i) = \frac{u_i}{(n+1)^2 L_{n+1}^2(u_i)}.$$

We begin approximating the gamma function using the result (5) and obtain:

$$\begin{aligned}\Gamma(a) &= \int_0^{+\infty} e^{-t} t^{a-1} dt \\ &\simeq \sum_{i=1}^{n} w(u_i)\, u_i^{a-1},\end{aligned} \tag{6}$$

Using the same approach, we evaluate the integral in (2) and use the result (6) to obtain the approximated formula of the variance gamma density:

$$f(y) = \sum_{i=1}^{n} \varphi\left(y; \mu_0 + \mu u_i, \sigma\sqrt{u_i}\right) p(u_i, a). \tag{7}$$

Defining

$$p(u_i, a) = \frac{w(u_i)\, u_i^{a-1}}{\sum_{i=1}^{n} w(u_i)\, u_i^{a-1}}, \tag{8}$$

we recognise that we have approximated the variance gamma density (a continuous mixture) with a finite normal mixture. The mixing variable U, discrete and with a finite support, is the following

$$U := \begin{cases} u_1 & p(u_1, a) \\ ... & ... \\ u_n & p(u_n, a) \end{cases}. \tag{9}$$

Moreover, for n increasing, a remains the only parameters to be estimated.
We report in Fig. 1, 2 and 3 the behaviour of our approximation for different values of the parameters μ, σ and a, for $n = 15$ and $n = 50$.

Insert here Fig. 1, 2 and 3

In our exercises, the choice of $n = 50$ seems the most appropriate; for $n = 15$ we observe some irregularities only for $\mu = -0.5$ that corresponds to strong negative skewness of log-returns distribution.

3 Historical estimation procedure

In this Section we use the previous result to provide a simple historical estimation procedure for the vector $\theta = (\sigma, \mu, \mu_0, a)$ by means of the EM algorithm. We compare our estimation procedure (EM-approx) with two methods: the Method of Moments (MoM) and the Maximum Likelihood Estimation (MLE).

3.1 Method of Moments

The key idea of the MoM is to match the theoretical moments with their sample counterparts. In our model the mean, variance, skewness and kurtosis are:

$$\begin{cases} E(y) = \mu_0 + \mu a \\ Var(y) = a\left(\mu^2 + \sigma^2\right) \\ skew(y) = \frac{\left(2\mu^2+3\sigma^2\right)\mu}{\sqrt{a}\sqrt{\left(\mu^2+\sigma^2\right)^3}} \\ k(y) = 3\left\{1 + \frac{2\mu^4+\sigma^4+4\sigma^2\mu^2}{a\left(\mu^2+\sigma^2\right)^2}\right\}. \end{cases}$$

Following [31], we neglect the terms μ^2, μ^3 and μ^4 since they do not have a big influence on the system solution when the value of μ is close to zero (as usually observed in real data).

3.2 Maximum Likelihood Estimation

The MLE procedure is based on the maximization of the log-likelihood function that, in the variance gamma model, is:

$$\begin{aligned} L(\mu_0, \mu, \sigma, a) = {} & \frac{T}{2}\log\left(\frac{2}{\pi}\right) + \sum_{t=1}^{T}\frac{(y-\mu_0)\mu}{\sigma^2} - \sum_{t=1}^{T}\log\left(\Gamma(a)\sigma\right) \\ & + \sum_{t=1}^{T}\log\left(K_{a-0.5}\left(\frac{\sqrt{2\sigma^2+\mu^2}\,|y-\mu_0|}{\sigma^2}\right)\right) + \\ & + \sum_{t=1}^{T}\left(a-\frac{1}{2}\right)\left[\log\left(|y-\mu_0| - \frac{1}{2}\log\left(2\sigma^2+\mu^2\right)\right)\right] \end{aligned} \tag{10}$$

where $K_p(u)$ is the modified Bessel function of the second type.
As observed in [27], the direct optimization of (10) is computationally expensive also in the symmetric case. Moreover, the results are strongly influenced by the initial values of the parameters. In order to overcome this problem a common choice is to initialize θ to the MoM results.

3.3 EM-based approach

Using our approximated formula (7), the log-likelihood function (10) becomes:

$$L(\theta) = \sum_{t=1}^{T}\ln\left(\sum_{i=1}^{n}\varphi\left(y_t; \mu_0 + \mu u_i, \sigma\sqrt{u_i}\right)p\left(u_i, a\right)\right). \tag{11}$$

Even in this case, the direct optimization of (11) is not trivial. For this reason, we provide a simple recursive procedure for the historical estimation using the EM algorithm.
To implement the EM algorithm we consider the complete log-likelihood function:

$$l^*(y,u,\theta) = \sum_{t=1}^{T} \ln\left(\varphi(y_t, \mu_0 + \mu u_t, \theta\sqrt{u_t}) p(u_t, a)\right).$$

Starting from an initial vector θ_0, the EM iterates two steps:

- **Expectation-step** (E-step henceforth) computes the expected value of l^* with respect to the mixing variable, given the observed data and the vector θ_{h-1} .
- **Maximization-step** (M-step henceforth) maximizes the quantity obtained in E-step with respect to θ_h.

Under fairly mild regularity conditions, the EM algorithm converges to a local maximum of the mixture likelihood function (see [12] and [35]).
First we compute the conditional distribution of the variable U given the observed data Y; applying the Bayes' theorem, we obtain:

$$P(u_i|y_t,\theta) = \frac{\varphi\left(y_t; \mu_0 + \mu u_i, \sigma\sqrt{u_i}\right) p\left(u_i, a\right)}{\sum_{i=1}^{n} \varphi\left(y_t; \mu_0 + \mu u_i, \sigma_h\sqrt{u_i}\right) p\left(u_i, a\right)}.$$

The conditional expectation of the complete log-likelihood function l^*, given the current data and current parameters, is computed as:

$$\begin{aligned}\text{E-step} &= \sum_{i=1}^{N} l^*(y, u_i, \theta_h) P(u_i|y_t, \theta_{h-1}) \\ &= \sum_{i=1}^{N}\sum_{t=1}^{T} \left[\ln(P(y_t, u_i|\theta_h))\right] P(u_i|y_t, \theta_{h-1})\end{aligned}$$

where

$$P(y_t, u_i|\theta_h) = \varphi(y_t, \mu_0 + \mu u_t, \theta\sqrt{u_t}) p(u_t, a).$$

The M-step involves the following optimization problem:

$$\max_{\mu_{0,h},\mu_h,\sigma_h,a_h} \sum_{i=1}^{n}\sum_{t=1}^{T} P(u_i|y_t,\theta_{h-1}) \ln\left(p\left(u_i\left(t\right), a_h\right) \varphi\left(y_t; \mu_{0,h} + \mu_h u_i, \sigma_h\sqrt{u_i}\right)\right),$$

which can be splitted as follows:

$$\max_{a_h} \sum_{i=1}^{n} \sum_{t=1}^{T} P(u_i|y_t, \theta_{h-1}) \ln \left(p\left(u_i\left(t\right), a_h\right)\right) \tag{12}$$

$$\max_{\mu_{0,h}, \mu_h, \sigma_h} \sum_{i=1}^{n} \sum_{t=1}^{T} P(u_i|y_t, \theta_{h-1}) \ln \left(\varphi\left(y_t; r + \mu_h u_i, \sigma_h \sqrt{u_i}\right)\right). \tag{13}$$

Once applied the first-order optimality conditions to (12), we obtain the following equation:

$$\sum_{i=1}^{n} \sum_{t=1}^{T} P(u_i|y_t, \theta_{h-1}) \left[\ln\left(u_i\right) - \frac{\sum_{i=1}^{n} w\left(u_i\right) u_i^{a_h - 1} \ln\left(u_i\right)}{\sum_{i=1}^{n} w\left(u_i\right) u_i^{a_h - 1}} \right] = 0,$$

which can be re-written in the form:

$$\sum_{t=1}^{T} \sum_{i=1}^{n} P(u_i|y_t, \theta_{h-1}) \ln\left(u_i\right) = T \frac{\sum_{i=1}^{n} w\left(u_i\right) u_i^{a_h - 1} \ln\left(u_i\right)}{\sum_{i=1}^{n} w\left(u_i\right) u_i^{a_h - 1}}. \tag{14}$$

The term in the right side of the equation (14) can be approximated by the digamma function $\psi\left(a_h\right)$ (see [1])

$$\frac{\sum_{t=1}^{T} \sum_{i=1}^{n} P(u_i|y_t, \theta_{h-1}) \ln\left(u_i\right)}{T} \tilde{=} \psi\left(a_h\right).$$

Applying the first-order optimality condition to (13), we obtain the following system:

$$\begin{cases} \sum_{i=1}^{n} \sum_{t=1}^{T} \frac{P(u_i|y_t,\theta_{h-1})}{(\sigma_h \sqrt{u_i})^2} \left(y_t - \mu_{0,h} - \mu_h u_i\right) = 0 \\ \sum_{i=1}^{n} \sum_{t=1}^{T} P(u_i|y_t, \theta_{h-1}) \left(y_t - \mu_{0,h} - \mu_h u_i\right) = 0 \\ \sum_{i=1}^{n} \sum_{t=1}^{T} P(u_i|y_t, \theta_{h-1}) \left[\left(\frac{y_t - \mu_{0,h} - \mu_h u_i}{\sqrt{u_i}} \right)^2 - \sigma_h^2 \right] = 0. \end{cases}$$

The parameters are given by

$$\begin{cases} \mu_{0,h} = \frac{\left(\sum_{i=1}^{n} \sum_{t=1}^{T} \frac{P(u_i|y_t,\theta_{h-1})}{u_i} y_t - \frac{T \sum_{t=1}^{T} y_t}{\sum_{t=1}^{T} \sum_{i=1}^{n} P(u_i|y_t,\theta_{h-1}) u_i} \right)}{\left(\sum_{i=1}^{n} \sum_{t=1}^{T} \frac{P(u_i|y_t,\theta_{h-1})}{u_i} - \frac{T^2}{\sum_{t=1}^{T} \sum_{i=1}^{n} P(u_i|y_t,\theta_{h-1}) u_i} \right)} \\ \mu_h = \frac{\sum_{t=1}^{T} y_t - \mu_{0,h} T}{\sum_{t=1}^{T} \sum_{i=1}^{n} P(u_i|y_t,\theta_{h-1}) u_i} \\ \sigma_h = \sqrt{\frac{\sum_{i=1}^{n} \sum_{t=1}^{T} P(u_i|y_t,\theta_{h-1}) \frac{(y_t - r - \mu_h u_i)^2}{u_i}}{T}}. \end{cases}$$

We can write

$$\begin{cases} \sum_{i=1}^{n} \frac{P(u_i|y_t,\theta_{h-1})}{u_i} = E\left(U^{-1}\left|y_t\theta_{h-1}\right.\right) \\ \sum_{i=1}^{n} P(u_i|y_t,\theta_{h-1})u_i = E\left(U\left|y_t\theta_{h-1}\right.\right) \\ \sum_{i=1}^{n} P(u_i|y_t,\theta_{h-1})\ln\left(u_i\right) = E\left(\ln\left(U\right)\left|y_t\theta_{h-1}\right.\right) \end{cases}$$

where U is the random variable defined in (9), and we have:

$$\begin{cases} \psi\left(a_h\right) = \frac{\sum_{t=1}^{T} E(\ln(U)|y_t\theta_{h-1})}{T} \\ \mu_{0,h} = \frac{\left(\sum_{t=1}^{T} E\left(U^{-1}|y_t\theta_{h-1}\right)y_t - \frac{T\sum_{t=1}^{T} y_t}{\sum_{t=1}^{T} E(U|y_t\theta_{h-1})}\right)}{\left(\sum_{t=1}^{T} E(U^{-1}|y_t\theta_{h-1}) - \frac{T^2}{\sum_{t=1}^{T} E(U|y_t\theta_{h-1})}\right)} \\ \mu_h = \frac{\sum_{t=1}^{T} y_t - \mu_{0,h}T}{\sum_{t=1}^{T} E(U|y_t\theta_{h-1})} \\ \sigma_h = \sqrt{\frac{\sum_{t=1}^{T}(y_t-\mu_0)^2 E(U^{-1}|y_t\theta_{h-1}) + \mu_h^2 \sum_{t=1}^{T} E(U|y_t\theta_{h-1}) - 2\mu_h \sum_{t=1}^{T}(y_t-\mu_0)}{T}} \end{cases}.$$

We conclude this section with an investigation of our procedure performances on real data. We provide a comparison with MoM and MLE estimates. We consider three datasets of the SPX daily closing prices. In table 1, we report the sample size and the sample estimates of mean, variance, skewness and kurtosis.

Insert here table 1

All procedures are implemented in Matlab. The optimization routine is performed using the Fmincon Matlab's function. For MLE, as starting points, we use the estimates obtained by MoM.

The estimation results are presented in table 2; in Figure 2 we compare the empirical density with those obtained by the considered procedures.

Insert table 2 and Figure 2

To measure the goodness of fit, we use the root mean square error (RMSE) and the mean absolute error (MAE) between the empirical and the estimated cumulative distribution functions:

$$MAE = \sum \left|F_{emp}\left(x\right) - F_{est}\left(x\right)\right|$$
$$RMSE = \sqrt{\sum \left(F_{emp}\left(x\right) - F_{est}\left(x\right)\right)^2}.$$

The best fit is achieved by the MLE procedure followed by the EM-approx. We remark that the MLE seems to be more sensible to the starting points than EM-approx. In order to investigate this fact, for the third considered time series, we repeat the estimation for different initial values of the parameters. In our study the choice of θ_0 seems not to affect the EM-approx results, as can be noticed in table 3

Insert table 3

4 Option Pricing

In Sections 2 and 3 we have studied how to approximate the variance gamma with a finite mixture of normals, and we have obtained a recursive system for historical estimation by means of the EM algorithm. In this Section we provide a discrete version of the semi-analytical formula proposed by Madan et al. [8]. The resulting prices will be compared with the ones obtained using Monte Carlo simulation and the original formula in [8] (MCC formula henceforth).

Using the same specification seen in [8], under the risk neutral measure, the underlying asset price follows an exponential variance gamma process:

$$\begin{aligned} S_T &= S_0 \exp\left(Y_T\right) \\ Y_T &= rT + X_T + gT \\ X_T &\sim VG\left(\mu, \sigma, \frac{T}{\nu}, \frac{1}{\nu}\right) \\ g &= \frac{1}{\nu}\ln\left(1 - \mu\nu - \frac{\sigma^2\nu}{2}\right). \end{aligned} \tag{15}$$

The density function of the random variable X_T can be approximated as follows:

$$\begin{aligned} f\left(x_T\right) &= \sum_{i=1}^{n} \varphi\left(x_T; \mu\nu u_i, \sigma\sqrt{\nu u_i}\right) p\left(u_i, \frac{T}{\nu}\right) \\ p\left(u_i, \frac{T}{\nu}\right) &= \frac{u_i^{\frac{T}{\nu}-1} w\left(u_i\right)}{\sum_{i=1}^{n} w\left(u_i\right) u_i^{\frac{T}{\nu}-1}} \end{aligned}$$

To ensure the martingale condition of the discounted price, we substitute the parameter g with

$$g_1 = -\frac{1}{T}\ln\left(\sum_{i=1}^{n} p\left(u_i, \frac{T}{\nu}\right) e^{\mu\nu u_i + \frac{\sigma^2}{2}\nu u_i}\right).$$

We are able to compute the expected value of the discounted final pay-off:

$$\begin{aligned} C_0 &= e^{-rT} E^Q\left[\left(S_T - K\right)^+\right] \\ &= e^{-rT} S_0 E^Q\left[\left(\exp\left(rT + X_T + g_1 T\right) - \frac{K}{S_0}\right)^+\right]. \end{aligned}$$

Defining $d = -\left(\ln\left(\frac{S_0}{K}\right) + \left(r + g_1\right)T\right)$, we have

$$C_0 = e^{-rT} S_0 \int_d^{+\infty} \left(\exp\left(rT + x_T + g_1 T\right) - \frac{K}{S_0} \right) \sum_{i=1}^{n} \varphi\left(x_T; \mu\nu u_i, \sigma\sqrt{\nu u_i}\right) p\left(u_i, \frac{T}{\nu}\right) dx_T =$$

$$= e^{-rT} S_0 \sum_{i=1}^{n} \left[\int_d^{+\infty} \frac{\left(\exp\left(rT + x_T + g_1 T\right) - \frac{K}{S_0} \right)}{\sqrt{2\pi\sigma^2 \nu u_i}} \exp\left(-\frac{\left(x_T - \mu\nu u_i\right)^2}{2\sigma^2 \nu u_i} \right) dx_T \right] p\left(u_i, \frac{T}{\nu}\right).$$

Substituting

$$\begin{aligned} z &= \frac{\left(x_T - \mu\nu u_i\right)}{\sigma\sqrt{\nu u_i}} \\ x_T &= z\sigma\sqrt{\nu u_i} + \mu\nu u_i \\ d_2\left(u_i\right) &= \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + g_1\right) T + \mu\nu u_i}{\sigma\sqrt{\nu u_i}}, \end{aligned}$$

we obtain

$$C_0 = e^{-rT} S_0 \sum_{i=1}^{n} \left[\int_{-d_2}^{+\infty} \frac{\left(\exp\left(rT + z\sigma\sqrt{\nu u_i} + \mu\nu u_i + g_1 T\right) - \frac{K}{S_0} \right)}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz \right] p\left(u_i, \frac{T}{\nu}\right) =$$

$$= S_0 \sum_{i=1}^{n} \left[\exp\left(\mu\nu u_i + g_1 T + \frac{\sigma^2 \nu u_i}{2} \right) \int_{-d_2(u_i)}^{+\infty} \frac{\left(\exp\left(-\frac{1}{2} \left(z - \sigma\sqrt{\nu u_i} \right)^2 \right) \right)}{\sqrt{2\pi}} dz \right] p\left(u_i, \frac{T}{\nu}\right) +$$

$$- K e^{-rT} \sum_{i=1}^{n} \left[\int_{-d_2(u_i)}^{+\infty} \frac{\exp\left(-\frac{z^2}{2}\right)}{\sqrt{2\pi}} dz \right] p\left(u_i, \frac{T}{\nu}\right).$$

Imposing

$$\begin{aligned} z_1 &= z - \sigma\sqrt{\nu u_i} \\ d_1(u_i) &= \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + g_1\right) T + \left(\sigma^2 + \mu\right) \nu u_i}{\sigma\sqrt{\nu u_i}} \\ d_2\left(u_i\right) &= d_1(u_i) - \sigma\sqrt{\nu u_i}, \end{aligned}$$

we get

$$\begin{aligned} C_0 = S_0 \sum_{i=1}^{n} &\left[\exp\left(\mu\nu u_i + g_1 T + \frac{\sigma^2 \nu u_i}{2} \right) N\left(d_1\left(u_i\right)\right) p\left(u_i, \frac{T}{\nu}\right) \right] \\ &- K e^{-rT} \sum_{i=1}^{n} N\left(d_2\left(u_i\right)\right) p\left(u_i, \frac{T}{\nu}\right), \end{aligned}$$

where

$$p\left(u_i, \frac{T}{\nu}\right) = \frac{u_i^{\frac{T}{\nu}-1} w\left(u_i\right)}{\sum_{i=1}^{n} w\left(u_i\right) u_i^{\frac{T}{\nu}-1}}$$

and $N(\cdot)$ is the normal distribution function.

In order to assess the accuracy of the approximated formula, we compute European call option prices using the approximated formula with mixtures of 20, 25 and 30 components. Then, we compare the results with Monte Carlo prices, obtained by means of 50000 resimulations, and the prices given by the MCC formula, for different maturities (15, 30 and 60 days) and for different levels of moneyness ($K = 100$ and $S = [90\ 95\ 100\ 105\ 110]$). The parameters chosen for the simulation are $\mu_0 = 0,\ \nu = 1,\ \sigma = 0.0113,\ \mu = -0.000478$.
We see that the prices obtained by our approach with 30 components are tipically close to MCC prices and, in every considered case, belong to the 95% confidence interval computed according to the methodology of Boyle [7]. The results are reported in table 4.

Insert here table 4

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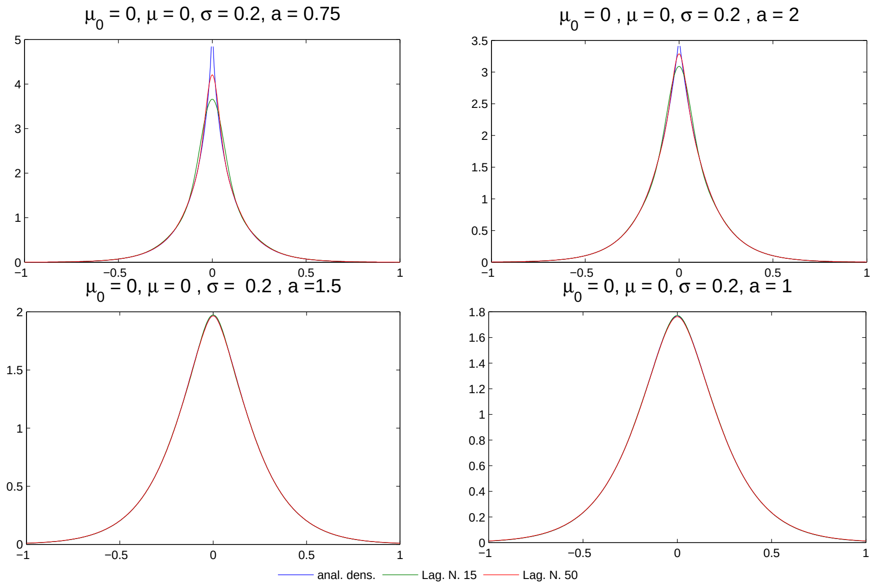

Figure 1: Comparison between analytical and approximated densities for different values of a

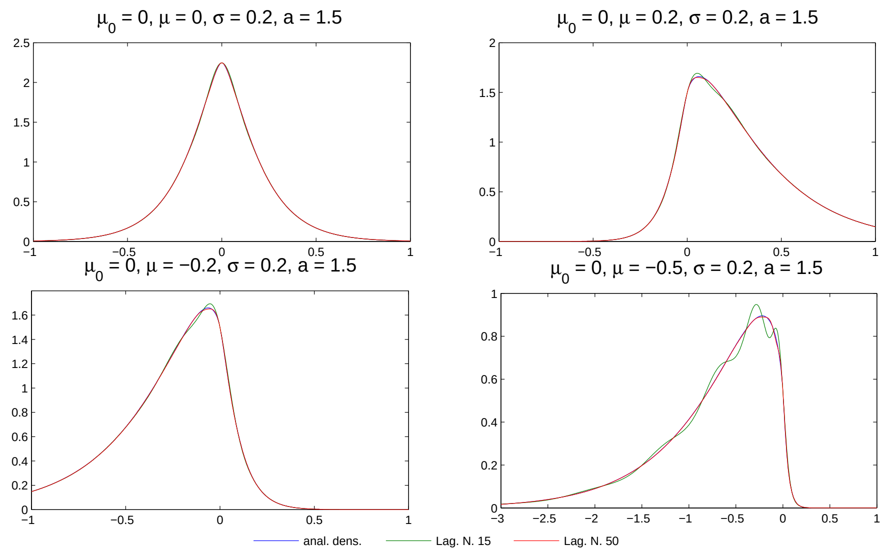

Figure 2: Comparison between analytical and approximated densities for different values of μ

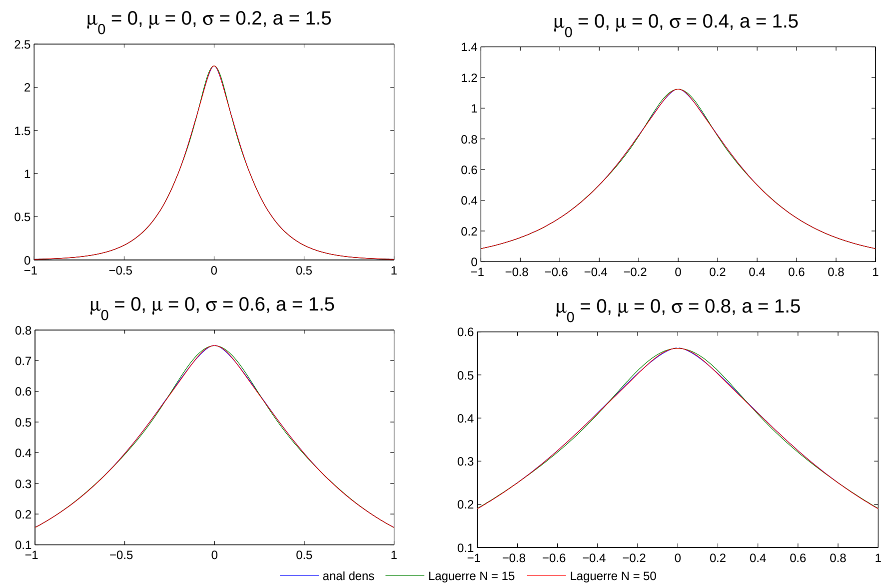

Figure 3: Comparison between analytical and approximated densities for different values of σ

Table 1: Sample size, mean, variance, skewness and kurtosis of the SPX index

Dataset	size	mean	var	skew	kurt
02/01/94-30/09/96	696	5.62485e-04	4.05e-05	-4.9e-01	5.30
31/12/01-30/09/04	693	-4.28e-05	1.53e-04	2.69e-01	4.80
31/12/07-30/09/10	694	-3.64e-04	3.94e-04	-1.41e-01	8.20

Table 2: Comparison between historical estimation procedures

I Dataset	μ_0	μ	σ	a	$RMSE$	MAE	$Logl$
MoM	2.06e-03	-1.07e-03	5.28e-03	1.40	1.07e-02	6.29e-03	2559.28
MLE	9.38e-04	-2.41e-04	4.97e-03	1.56	6.38e-03	4.93e-03	2563.35
EM approx	7.44e-04	-1.63e-04	6.22e-03	1.12	8.10e-03	6.22e-03	2561.30
II Dataset	μ_0	μ	σ	a	$RMSE$	MAE	$Logl$
MoM	-1.89e-03	1.11e-03	9.60e-03	1.67	9.97e-03	6.64e-03	2078.60
MLE	4.34e-04	-3.02e-04	9.86e-03	1.58	6.12e-03	4.28e-03	2081.60
EM approx	5.35e-04	-5.21e-04	1.24e-02	1.13	9.27e-03	7.42e-03	2079.33
III Dataset	μ_0	μ	σ	a	$RMSE$	MAE	$Logl$
MoM	1.76e-04	-9.32e-04	2.61e-02	5.78e-01	9.57e-03	5.35e-03	1816.67
MLE	1.40e-03	-2.22e-03	2.16e-02	7.92e-01	4.46e-03	3.03e-03	1827.69
EM approx	1.95e-03	-2.52e-03	1.97e-02	9.20e-01	5.56e-03	3.76e-03	1826.67

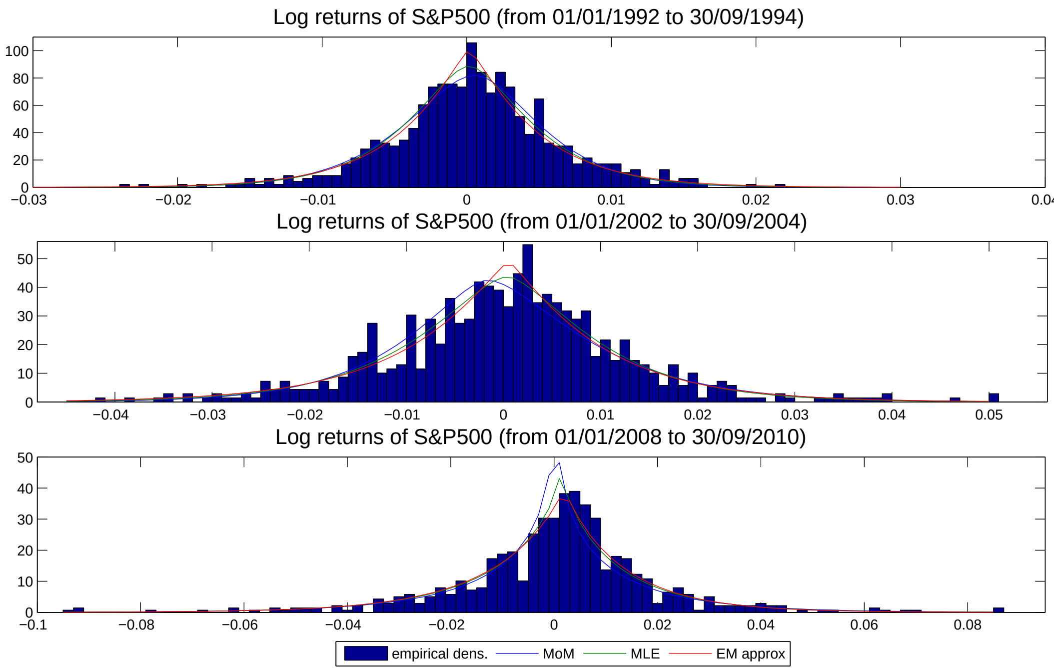

Figure 4: Comparison of the estimation procedures

Table 3: Estimated parameters by EM approx with different initial values

$\mu_0(0)$	$\mu(0)$	$\sigma(0)$	$a(0)$	μ_0	μ	σ	a
0	0	0.20	0.75	1.95e-03	-2.52e-03	1.97e-02	9.20e-01
0	0	0.20	1	1.95e-03	-2.52e-03	1.97e-02	9.20e-01
0	0	0.20	1.5	1.95e-03	-2.52e-03	1.97e-02	9.20e-01
0	0	0.40	1	1.95e-03	-2.52e-03	1.97e-02	9.20e-01
0	0	0.60	1.5	1.95e-03	-2.52e-03	1.97e-02	9.20e-01
0	-0.2	0.40	0.75	1.95e-03	-2.52e-03	1.97e-02	9.20e-01
0	-0.2	0.40	1	1.95e-03	-2.52e-03	1.97e-02	9.20e-01
0	-0.2	0.40	1.5	1.95e-03	-2.52e-03	1.97e-02	9.20e-01

Table 4: Option prices with the approximation approach, MCC formula and 95% confidence intervals.

S/K	T	**mix20**	**mix25**	**mix30**	**MCC**	**Monte Carlo**
0.9	15	0.01355	0.01355	0.01356	0.01608	(0.0132; 0.0158)
	30	0.10824	0.10824	0.10811	0.11570	(0.1080; 0.1172)
	60	0.45297	0.45989	0.45919	0.47149	(0.4472; 0.4802)
0.95	15	0.25191	0.25191	0.25201	0.26314	(0.2413; 0.2533)
	30	0.68302	0.68301	0.68255	0.69510	(0.6699; 0.6939)
	60	1.45292	1.46499	1.46373	1.47621	(1.4398; 1.5003)
1	15	1.73233	1.73233	1.73251	1.73358	(1.7061; 1.7385)
	30	2.45990	2.45989	2.45921	2.46167	(2.4606; 2.5073)
	60	3.47033	3.48551	3.48398	3.48802	(3.4288; 3.5236)
1.05	15	5.30627	5.30627	5.30639	5.29553	(5.2787; 5.3296)
	30	5.78105	5.78105	5.78059	5.77111	(5.7326; 5.7995)
	60	6.60654	6.62090	6.61959	6.61376	(6.5635; 6.6895)
1.1	15	10.03079	10.03079	10.03082	10.02694	(9.9957; 10.0544)
	30	10.18239	10.18239	10.18223	10.17375	(10.1361; 10.2162)
	60	10.63996	10.65059	10.64975	10.64007	(10.5778; 10.7295)