SCUOLA DI DOTTORATO
UNIVERSITÀ DEGLI STUDI DI MILANO-BICOCCA

Dipartimento di / Department of

Matematica e Applicazioni

Dottorato di Ricerca in / PhD program Matematica Pura e Applicata Ciclo / Cycle XXX

Optimal control of pure jump Markov processes with noise-free partial observation

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ANNO ACCADEMICO / ACADEMIC YEAR 2016-2017

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Lunga e difficile è la via della ricerca, ma alla base di tutto c'è l'amore.

— Vincenzo Tiberio (1869–1915)

Ricercatore e Ufficiale medico del Corpo Sanitario della Marina Militare.

Abstract

This thesis is concerned with an infinite horizon optimal control problem for a pure jump Markov process with noise-free partial observation. We are given a pair of stochastic processes, named unobserved or signal process and observed or data process. The signal process is a continuous-time pure jump Markov process, taking values in a complete and separable metric space, whose controlled rate transition measure is known. The observed process takes values in another complete and separable metric space and is of noise-free type. With this we mean that its values at each time t are given as a function of the corresponding values at time t of the unobserved process. We assume that this function is a deterministic and, without loss of generality, surjective map between the state spaces of the signal and data processes. The aim is to control the dynamics of the unobserved process, i.e. its controlled rate transition measure, through a control process, taking values in the set of Borel probability measures on a compact metric space, named set of control actions. We take as admissible controls for our problem all the processes of this kind that are also predictable with respect to the natural filtration of the data process. The control process is chosen in this class to minimize a discounted cost functional on infinite time horizon. The infimum of this cost functional among all admissible controls is the value function.

In order to study the value function a preliminary step is required. We need to recast our optimal control problem with partial observation into a problem with complete observation. This is done studying the filtering process, a measure-valued stochastic process providing at each time t the conditional law of the unobserved process given the available observations up to time t (represented by the natural filtration of the data process at time t). We show that the filtering process satisfies an explicit stochastic differential equation and we characterize it as a Piecewise Deterministic Markov Process, in the sense of Davis.

To treat the filtering process as a state variable, we study a separated optimal control problem. We introduce it as a discrete-time one and we show that it is equivalent to the original one, i.e. their respective value functions are linked by an explicit formula. We also show that admissible controls of the original problem and admissible policies of the separated one have a specific structure and there is a precise relationship between them.

Next, we characterize the value function of the separated control problem (hence, indirectly, the value function of the original control problem) as the unique fixed point

of a contraction mapping, acting from the space of bounded continuous function on the state space of the filtering process into itself. Therefore, we prove that the value function is bounded and continuous.

The special case of a signal process given by a finite-state Markov chain is also studied. In this setting, we show that the value function of the separated control problem is uniformly continuous on the state space of the filtering process and that it is the unique constrained viscosity solution (in the sense of Soner) of a Hamilton-Jacobi-Bellman equation. We also prove that an optimal ordinary control exists, i.e. a control process taking values in the set of control actions, and that this process is a piecewise open-loop control in the sense of Vermes.

KEYWORDS: stochastic optimal control; nonlinear filtering.

Sommario

La presente tesi tratta un problema di controllo ottimo su orizzonte temporale infinito per un processo di puro salto Markoviano e con osservazione parziale di tipo noise-free. È definita una coppia di processi stocastici, detti processo non osservato o segnale e processo osservato o dei dati. Il segnale è un processo di puro salto Markoviano a tempo continuo, a valori in uno spazio metrico completo e separabile, di cui è nota la misura controllata dei tassi di transizione. Il processo osservato prende valori in un ulteriore spazio metrico completo e separabile ed è di tipo noise-free. Con questa espressione si intende che i suoi valori a ogni tempo t sono funzione dei corrispondenti valori al tempo t del processo non osservato. Si fa l'ipotesi che tale funzione sia un'applicazione deterministica e, senza perdita di generalità, suriettiva tra gli spazi di stato dei processi non osservato e osservato. L'obiettivo è controllare la dinamica del processo non osservato, ossia la sua misura controllata dei tassi di transizione, attraverso un processo di controllo, il quale prende valori nell'insieme delle misure di probabilità di Borel su uno spazio metrico compatto, detto spazio delle azioni di controllo. I controlli ammissibili per il nostro problema sono i processi appena descritti che siano anche prevedibili rispetto alla filtrazione naturale del processo osservato. Il processo di controllo è scelto in questa classe al fine di minimizzare un funzionale costo con fattore di sconto su orizzonte temporale infinito. L'estremo inferiore di tale funzionale costo tra tutti i controlli ammissibili è la funzione valore.

Per studiare la funzione valore è necessario un passo preliminare. Il problema di controllo ottimo a osservazione parziale deve essere espresso come problema a osservazione completa. Ciò è possibile grazie allo studio del processo di filtraggio, un processo a valori in misure che fornisce a ogni istante t la legge condizionale del processo non osservato data l'osservazione disponibile fino al tempo t (rappresentata dalla filtrazione naturale del processo osservato al tempo t). Si dimostra che il processo di filtraggio soddisfa un'equazione differenziale stocastica esplicita e si caratterizza tale processo come Piecewise Deterministic Markov Process, nel senso di Davis.

Allo scopo di trattare il processo di filtraggio come variabile di stato, si studia un problema di controllo separato. Questo è definito come problema a tempo discreto e si mostra che è equivalente a quello originario, nel senso che le rispettive funzioni valore sono legate da una formula esplicita. Si dimostra, inoltre, che i controlli ammissibili per il problema originario e le strategie ammissibili di quello separato hanno una ben precisa struttura ed esiste una specifica relazione tra di essi.

Si caratterizza, quindi, la funzione valore del problema di controllo separato (dunque, indirettamente, la funzione valore del problema originario) come unico punto fisso di un operatore di contrazione, il quale agisce dallo spazio delle funzioni continue e limitate sullo spazio di stato del processo di filtraggio in sé. Di conseguenza, si dimostra che la funzione valore è continua e limitata.

Si studia anche il caso di un processo non osservato dato da una catena di Markov a stati finiti. In questo contesto, si mostra che la funzione valore del problema di controllo separato è uniformemente continua sullo spazio di stato del processo di filtraggio e che è l'unica soluzione viscosa vincolata (nel senso di Soner) di un'equazione di Hamilton-Jacobi-Bellman. Si dimostra, inoltre, che esiste un controllo ottimo ordinario, ossia un processo di controllo che prende valori nell'insieme delle azioni di controllo, e che tale processo è un piecewise open-loop control nel senso di Vermes.

PAROLE CHIAVE: stochastic optimal control; nonlinear filtering.

Contents

Notation and Abbreviations

Sets

- $\mathbb{N} := \{1, 2, \dots\}$ – the set of natural integers.
- $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.
- $\bar{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$.
- $\bar{\mathbb{N}}_0 := \mathbb{N}_0 \cup \{\infty\}$.
- $\mathbb{R}^d$, $d \in \mathbb{N}$ – the d-dimensional Euclidean space.
- $\mathcal{N}$ – the collection of null sets in a specified probability space.

Spaces

In what follows, E denotes a metric space.

- $\mathcal{B}(E)$ – the Borel σ-algebra on E.
- $\mathrm{B}_b(E)$ – the space of real-valued bounded Borel-measurable functions on E.
- $\mathrm{C}(E)$ – the space of real-valued continuous functions on E.
- $\mathrm{C}_b(E)$ – the space of bounded and continuous real-valued functions on E.
- $\mathrm{C}^k(E)$, $k \in \mathbb{N}$ – the space of k-times continuously differentiable real-valued functions on E.
- $\mathcal{M}(E)$ – the space of finite signed Borel measures on $\big(E, \mathcal{B}(E)\big)$.
- $\mathcal{M}_+(E)$ – the space of finite Borel measures on $\big(E, \mathcal{B}(E)\big)$.
- $\mathcal{P}(E)$ – the space of Borel probability measures on $\big(E, \mathcal{B}(E)\big)$.

Functions, measures and integrals

- $\mathbb{1}_C$ – the indicator function of a set C.
- 1 – the constant function equal to 1.
- δ_a – the Dirac probability measure concentrated at $a \in E$.
- $\operatorname{supp}(\mu)$ – the support of a measure $\mu \in \mathcal{M}(E)$.
- $|\mu|$ – the total variation measure corresponding to $\mu \in \mathcal{M}(E)$.
- $\mu \ll \nu$ – indicates that the measure $\mu \in \mathcal{M}(E)$ is absolutely continuous with respect to the measure $\nu \in \mathcal{M}(E)$.
- $\mu \circ f$ – the image measure of $\mu \in \mathcal{M}(E)$ by the measurable function f.
- $f\mu(C) := \int_C f(x)\, \mathrm{d}\mu(x)$, $C \in \mathcal{B}(E)$ – $f\mu$ is the finite signed Borel measure on $\big(E, \mathcal{B}(E)\big)$ with density $f \in \mathrm{B}_b(E)$ with respect to $\mu \in \mathcal{M}(E)$.
- $\mu(f) := \int_E f(x)\, \mathrm{d}\mu(x)$ – the integral of a measurable function $f\colon E \to \mathbb{R}$ with respect to a measure $\mu \in \mathcal{M}(E)$.
- $\mu(f;\, \cdot) := \int_E f(x, \cdot)\, \mu(\mathrm{d}x)$ – the integral of a real-valued measurable function f of two variables with respect to the first one and against a measure μ.

Operations on sets

- $A \vee B$ – the smallest σ–algebra generated by the union $A \cup B$.
- $\operatorname{int}(C)$ – the interior of a set C.
- $\operatorname{cl}(C)$ – the closure of a set C.
- $\overline{\operatorname{co}}(C)$ – the closed convex hull of a set C.
- $|C|$ – the cardinality of a set C.

Norms and pairings

- $\|\cdot\|_\infty$ – the supremum norm.
- $\|\cdot\|_{TV}$ – the total variation norm.
- $\langle \cdot\,, \cdot \rangle$ – the duality pairing between a Banach space and its topological dual.

Miscellanea

- $s \wedge t := \min\{s, t\}$, with $s, t \in \mathbb{R}$.
- $s \vee t := \max\{s, t\}$, with $s, t \in \mathbb{R}$.
- D – the gradient symbol.

Abbreviations

- a.e., a.s. – almost everywhere, almost surely.
- BSDE – Backward Stochastic Differential Equation.
- càdlàg – right-continuous with left-limits.
- HJB – Hamilton-Jacobi-Bellman.
- MPP – Marked Point Process.
- ODE – Ordinary Differential Equation.
- PDMP (or PDP) – Piecewise Deterministic (Markov) Process.
- RCM – Random Counting Measure.
- SDE – Stochastic Differential Equation.

Introduction

This thesis deals with an infinite horizon optimal control problem for a continuous-time homogeneous pure jump Markov process with partial and noise-free observation.

Optimal control problems have been widely studied in the literature and their analysis continues nowadays with various ramifications. Starting from the celebrated *brachistochrone problem*, solved by J. Bernoulli in 1696, optimal control problems have been formulated in various forms (e. g. Lagrange, Bolza and Mayer problems) and with different purposes in mind. Initially, they were studied in a deterministic setting in the context of calculus of variations and in connection with problems coming from mechanics, optics and geometry. After World War II, new problems coming for instance from aerospace sciences, industrial control, financial and economic models, gave a renewed impulse to the study of optimal control problems, also in the stochastic case. The fundamental result provided by Bellman in his *Dynamic Programming Principle*, gave the optimal control branch its own *raison d'être* and its own tools, putting these problem in a different perspective from the one of the calculus of variations.

As we said in the opening statement, this thesis studies a stochastic optimal control problem. However, the reader should never lose contact with the deterministic counterpart of these problems (on this subject see e. g. [37] and [23]). Our problem, as we will later see, shares some similarities with deterministic ones. In some cases, for instance in Section 3.4, we will use typical results of this kind of problems.

Stochastic optimal control problems have been widely studied in recent years, both from a methodological and a modeling point of view. There are mainly two philosophies to tackle these problems: the first one is represented by Bellman's Dynamic Programming Principle, leading to the study of *Hamilton-Jacobi-Bellman* equations (or HJB for short). These are nonlinear partial differential equations, or integro-differential equations as it will be in our case, satisfied by the value function associated to an optimal control problem. Since, in general, one cannot expect the value function to be smooth enough (i. e. continuously differentiable as many times as needed) in order to be a classical solution to the HJB equation, existence and uniqueness results for solutions to these equations are formulated in the viscosity sense. We recall that viscosity solutions were introduced by M.G. Crandall and P.-L. Lions (see e. g. [28, 8] and [38] for connections with optimal control problems). The second philosophy is the approach provided by *Pontryagin's maximum principle* and *Backward Stochastic Differential Equations* (or BSDEs for short). These equations, introduced in the general framework by E. Pardoux and S. Peng in their seminal paper [52] and studied earlier in the linear case by J.-M. Bismut and A. Bensoussan (see [14, 9]), provide a different approach to characterize the value function of the optimal control problem, also in cases where HJB techniques fail. In this thesis we will adopt the first philosophy. For detailed expositions on stochastic optimal control problems, the reader is referred to [13, 41] for the discrete-time case (the first book treats also the deterministic setting), [65, 53, 24] for the continuous-time case (the second book provides also a great deal of financial applications), [10] for optimal control problems with partial observation and, finally, [48, 35] for the infinite dimensional case.

We analyze a model described by a triple $(X, Y, \mathbf{u}) = (X_t, Y_t, u_t)_{t \geqslant 0}$ of stochastic processes, defined in some suitable probability space. This triple is composed by the *unobserved* or *signal process* X, the *observed process* Y and the *control process* $\mathbf{u}$. The main feature of the model analyzed in this thesis is the *noise-free* observation. With this terminology we mean that no external source of randomness is acting on the observed process. We can say, in other words, that the noise on the observation is degenerate. This situation has been studied in very specific settings, also under various assumptions on the processes X and Y. For instance, in the context of an unobserved diffusion process, research papers as [18, 29, 46, 60] partially deal with this feature and analyze the filtering problem, i. e. the probabilistic estimation of the unobserved state of X at a time $t \geqslant 0$ given the observations available up to time t through the process Y (we will later discuss in full detail this problem, being it a fundamental point of this thesis). The case of an unobserved diffusion process is treated also in [44], that is devoted entirely to non-linear filtering with noise-free observation (therein called perfect observation). We also mention the chapter discussing singular filtering in the book by Xiong [64, Ch. 11]. The case of an unobserved process given by a pure-jump $\mathbb{R}^d$–valued Markov process is studied in [19, 20, 21, 22], where the authors consider counting observations. These models, that cannot be analyzed with well established results, have received a sporadic treatment in the literature, despite their potential and useful connection with applications, such as queuing systems (see e. g. [3, 17]) and inventory models (see e. g. [11]). We point out that our problem is connected to Hidden Markov Models (see [34] for a comprehensive exposition on this subject).

We now discuss some aspects of our control problem and anticipate some results. We warn the reader that all the results stated in the Introduction are given without proof and not provided with all the precise definitions needed. This is done on purpose, in order to convey a global idea of the original results contained in this thesis, that will be discussed and proved in full detail in the following Chapters.

We study an optimal control problem for the triple $(X, Y, \mathbf{u})$ introduced above in the following setting. The unobserved process X is a continuous-time homogeneous pure jump Markov process, taking values in a complete and separable metric space I. We are given its *rate transition measure*. This kernel, denoted by λ, along with the initial distribution determines the law of the process X. In other words, its sojourn times and its post jump locations are random variables whose law can be expressed in terms of λ. The case of I being a finite set is also studied in this thesis (see Chapter 3). The process X reduces to a continuous-time homogeneous Markov chain, whose rate transition matrix Λ is given. This matrix is sometimes called Q-matrix (see e. g. [51]). Such a setting may be more familiar to the reader and we invite she/he to keep in mind this situation also when considering the general case of a pure jump unobserved process.

The observed process Y takes its values in another complete and separable metric space O and will be of noise-free type in the following sense. We are given a measurable function $h\colon I \to O$ and the observed process satisfies the equality

$$Y_t(\omega) = h\big(X_t(\omega)\big), \quad t \geqslant 0$$

for each ω in the sample space on which our processes are defined. Of course, we exclude from our analysis two cases where the problem is not of true partial observation nature, i. e. when h is one-to-one or constant. In the first case we would obtain a problem with complete observation, while in the second one the observation would give no information about the unobserved process X. We point out that these assumptions on the function h do not affect the results presented in this thesis. All of them remain valid even if we take h to be one-to-one or constant. Without loss of generality, we may take h to be surjective. The function h generates a partition of the set I through its level sets $h^{-1}(y)$ as y varies in the set O. This means that if at a time $t \geqslant 0$ the controller observes the state $Y_t = y$, for some $y \in O$, then she/he immediately knows that $X_t \in h^{-1}(y)$, almost surely. In other words, she/he knows (almost surely) to which level set of the function h the random variable X_t belongs, but she/he does not know which is the actual state occupied by the unobserved process at time t. Another point of view, equivalent to this one, is to see the observation as the set-valued process $\big(h^{-1}(Y_t)\big)_{t\geqslant 0}$.

The control process $\mathbf{u}$ takes values in the space of Borel probability measures on the *set of control actions* U. We assume that U is a compact metric space. Thus, the process $\mathbf{u}$ represents the action of a *relaxed control*. The reason for this formulation is just a technical one and related to the so called *Young topology*, that is used to gain an important compactness property (see e. g. [32]), as we shall later explain in full detail. We will also see that we are able to recover *ordinary controls*, i. e. control processes taking values in the set U, thanks to some approximation argument. This is important since relaxed controls are not easily implementable in practice and have little meaning in applications.

Control processes are required to be in the class $\mathcal{U}_{ad}$ of predictable processes with respect to the natural filtration generated by the observed process. Such a choice, that is quite standard in the literature, is motivated by two aspects: first of all, it is obvious that a controller has the opportunity to choose her/his actions based on quantities that are actually observable, hence on the observed process Y in our case; second, it is not to be *a priori* excluded a dependency of a control action taken at some time $t \geqslant 0$ on the past trajectory of the observed process up to time t.

The aim of our control problem is to drive the dynamics of the unobserved process X by manipulating its controlled rate transition measure. Such a control will be exerted by the control process $\mathbf{u}$ in order to minimize, for each possible initial law μ of X, the *cost functional*

$$J(\mu, \mathbf{u}) = \mathrm{E}_\mu^{\mathbf{u}}\left[\int_0^\infty e^{-\beta t} \int_U f(X_t, \mathrm{u})\, u_t(\mathrm{d}\mathrm{u})\, \mathrm{d}t\right].$$

Here f is the *cost function*, that we take to be bounded and uniformly continuous, and β is a positive *discount factor*. The expectation is taken under a specific probability measure $\mathrm{P}_\mu^{\mathbf{u}}$, depending on the initial law μ of the unobserved process X and on the chosen control process $\mathbf{u}$. Our control problem is, thus, formulated in a weak sense. The infimum among all possible controls in the class $\mathcal{U}_{ad}$ earlier introduced is the *value function* $V(\mu)$. The study of various properties of the value function is a central topic of this thesis.

Solving control problems with partial observation requires a two step procedure. The first step consists in providing a probabilistic estimate of the state of the unobserved process. This is done in Chapter 2 and, to simplify matters, we study this subject in the uncontrolled case. We are given the pair of unobserved/observed processes (X, Y), with Y of noise-free type as explained above, defined in a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The estimate that we are looking for is provided by the *filtering process* $\pi = (\pi_t)_{t\geqslant 0}$. This is a measure-valued process representing the conditional law at each time $t \geqslant 0$ of the random variable X_t given the σ-algebra $\mathcal{Y}_t$. Here $(\mathcal{Y}_t)_{t\geqslant 0}$ denotes the natural completed filtration of the process Y. Otherwise said, the filtering process satisfies

$$\pi_t(\varphi) := \int_I \varphi(x)\,\pi_t(\mathrm{d}x) = \mathbb{E}[\varphi(X_t) \mid \mathcal{Y}_t], \quad t \geqslant 0,\ \mathbb{P}\text{–a.s.}$$

for all bounded and measurable functions $\varphi\colon I \to \mathbb{R}$. The filtering process is a key tool to solve our control problem since it allows us to transform it from a partial information one to a complete information one, called the separated problem, as we will explain later. Given its central role in this thesis, it is fundamental to characterize it and study its properties.

The first result that we obtain is an explicit stochastic evolution equation satisfied, for each $\varphi\colon I \to \mathbb{R}$ bounded and measurable, by the real-valued process $\pi(\varphi)$.

Theorem. *Let $\varphi\colon I \to \mathbb{R}$ be a bounded and measurable function and define, for each fixed $y \in O$, the linear operator $\mathcal{A}_y$ as*

$$\mathcal{A}_y\varphi(x) := \int_I \big[\varphi(z) - \varphi(x)\big]\,\lambda(x, \mathrm{d}z) - \int_I \mathbb{1}_{h^{-1}(y)^c}(z)\varphi(z)\lambda(x, \mathrm{d}z).$$

Let us denote by $\mathbb{1}\colon I \to \mathbb{R}$ the function identically equal to 1.

The process $\pi(\varphi)$ satisfies for all $t \geqslant 0$ and $\mathbb{P}$–a.s. the following equation

$$\begin{aligned}\pi_t(\varphi) &= H_{Y_0}[\mu](\varphi)\\ &\quad + \int_0^t \int_I \mathcal{A}_{Y_{s-}}\,\varphi(x)\,\pi_{s-}(\mathrm{d}x)\,\mathrm{d}s - \int_0^t \pi_{s-}(\varphi)\int_I \mathcal{A}_{Y_{s-}}\,\mathbb{1}(x)\,\pi_{s-}(\mathrm{d}x)\,\mathrm{d}s\\ &\quad + \sum_{0<\tau_n\leqslant t}\Big\{H_{Y_{\tau_n}}[\Lambda(\pi_{\tau_n^-})](\varphi) - \pi_{\tau_n-}(\varphi)\Big\}\end{aligned}$$

where μ is the initial law of X, $(\tau_n)_{n\in\mathbb{N}}$ are the jump times of the process Y and H_y, $y \in O$, Λ, are suitably defined operators acting on finite measures on I, introduced in Section 2.1 and Section 2.2 respectively.

From the equation for the process $\pi(\varphi)$ we can derive the equation satisfied by the measure-valued filtering process π.

Theorem. *For each fixed $y \in O$ let $\mathcal{B}_y$ the operator defined for all finite signed Borel measures ν on I as*

$$\mathcal{B}_y\nu(\mathrm{d}z) := \mathbb{1}_{h^{-1}(y)}(z)\int_I \lambda(x, \mathrm{d}z)\,\nu(\mathrm{d}x) - \lambda(z)\nu(\mathrm{d}z)$$

where $\lambda(z) := \lambda(z, I)$. The filtering process $\pi = (\pi_t)_{t\geqslant 0}$ satisfies for all $t \geqslant 0$ and

$\mathbb{P}$–a.s. the following SDE

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\pi_t = \mathcal{B}_{Y_t}\pi_t - \pi_t\,\mathcal{B}_{Y_t}\pi_t(I), & t \in [\tau_n, \tau_{n+1}),\, n \in \mathbb{N}_0 \\ \pi_0 = H_{Y_0}[\mu] \\ \pi_{\tau_n} = H_{Y_{\tau_n}}[\Lambda(\pi_{\tau_n^-})], & n \in \mathbb{N} \end{cases}$$

where μ is the initial law of X, $(\tau_n)_{n\in\mathbb{N}}$ are the jump times of the process Y and Λ, H_y, $y \in O$, are suitably defined operators acting on finite measures on I.

A peculiarity of the filtering process is that it takes values on a proper subset of the set of Borel probability measures on I. This subset, denoted by Δ_e, is called *effective simplex* and is the disjoint union of all the families Δ_y of probability measures concentrated on the level sets $h^{-1}(y)$, as y varies in the set O. This feature is due to the fact, noticed earlier, that if at some time $t \geqslant 0$ we observe the value $Y_t = y$ for some $y \in O$, then we know that, $\mathbb{P}$–a.s., $X_t \in h^{-1}(y)$. This implies that the filtering process at time t must be a random probability measure on I that is $\mathbb{P}$–a.s. concentrated on the level set $h^{-1}(y)$. Expressing this fact with our notation, in this situation we have that, $\mathbb{P}$–a.s., $\pi_t \in \Delta_y \subset \Delta_e$.

At least heuristically, the equation satisfied by the filtering process indicates that π is a *Piecewise Deterministic Markov Process*, or PDMP for short. This class of processes, introduced by M.H.A. Davis (see e. g. the monograph [32]), has been widely studied in the literature, also in connection with optimal control problems, and has important applications. The reason is that such class of processes mostly covers models that are not of diffusive type. In fact, their behavior (as the name suggests) is deterministic between some random jump times. In this time window they follow a deterministic flow, associated to a vector field, satisfying an ordinary differential equation (ODE). The occurrence of a random jump time, governed by a *rate function*, makes the process restart after this random time in a new position, determined by a *transition probability*. The distribution of a PDMP is, thus, completely characterized by the *characteristic triple*, formed by the flow (or, equivalently, the vector field), the rate function and the transition probability.

Our intuition is confirmed by the following theorem, that characterizes the filtering process as a PDP and provides its characteristic triple.

Theorem. *For each fixed $y \in O$ let $\mathcal{B}_y$ the operator defined for all finite signed Borel measures ν on I as*

$$\mathcal{B}_y\nu(\mathrm{d}z) \coloneqq \mathbb{1}_{h^{-1}(y)}(z) \int_I \lambda(x, \mathrm{d}z)\,\nu(\mathrm{d}x) - \lambda(z)\nu(\mathrm{d}z)$$

where $\lambda(z) \coloneqq \lambda(z, I)$.

For every initial law $\nu \in \Delta_e$ of the unobserved process X, the filtering process is a Piecewise Deterministic Process *with starting point ν and with respect to the following characteristic triple (F, r, R)*

$$F(\nu) \coloneqq \mathcal{B}_y\nu - \nu\,\mathcal{B}_y\nu(I), \quad \nu \in \Delta_y.$$

$$r(\nu) \coloneqq -\mathcal{B}_y\nu(I) = \int_I \lambda\big(x, h^{-1}(y)^c\big)\,\nu(\mathrm{d}x), \quad \nu \in \Delta_y.$$

$$R(\nu, D) \coloneqq \int_O \mathbb{1}_D\big(H_\upsilon[\Lambda(\nu)]\big)\,\rho(\nu, \mathrm{d}\upsilon), \quad \nu \in \Delta_y,\ D \in \mathcal{B}(\Delta_e)$$

where Λ is a suitably defined operator acting on finite measures on I and ρ is a transition probability defined for all $\nu \in \Delta_y$ and all Borel subsets B of O as

$$\rho(\nu, B) := \begin{cases} \dfrac{1}{r(\nu)} \displaystyle\int_I \lambda\big(x, h^{-1}(B \setminus \{y\})\big)\, \nu(\mathrm{d}x), & \text{if } r(\nu) > 0 \\ q_y(B), & \text{if } r(\nu) = 0 \end{cases}$$

where $(q_y)_{y \in O}$ is a family of probability measures, each concentrated on the level set $h^{-1}(y)$, $y \in O$, whose exact values are irrelevant.

As said earlier, the importance of characterizing the filtering process resides in the fact that it allows to transform our optimal control problem with partial observation into a complete observation one, where the state variable is the filtering process in place of the unobserved process X. This follows from an easy computation performed on the cost functional J and involving nothing more than conditional expectations and Fubini-Tonelli theorem. However, proving the equivalence between the problem with partial observation and the one with complete observation is not an easy task. In fact, to solve our optimal control problem we need to reformulate it as a control problem with complete observation for a PDP, called the *separated problem*, for various reasons that will be thoroughly discussed. More specifically, the separated problem will be a discrete-time one related to a specific Markov decision model formulated in terms of a PDP. The reduction of PDP optimal control problems to discrete-time Markov decision processes is exploited e. g. in [1, 26, 30, 32]. This reformulation produces the separated control problem that we prove to be equivalent to the original one and that allows to study its value function. The equivalence between the original and the separated control problems can be summarized by the explicit equality linking the corresponding value functions, denoted respectively by V and v.

Theorem. *For all initial laws μ of the unobserved process X we have that*

$$V(\mu) = \int_O v(H_y[\mu])\, \mu \circ h^{-1}(\mathrm{d}y)$$

where $\mu \circ h^{-1}$ is the image measure defined as $\mu \circ h^{-1}(B) := \mu\big(h^{-1}(B)\big)$ for all Borel subsets B of O and H_y, $y \in O$, are suitably defined operators acting on finite measures on I.

This equality allows to study the value function v of the separated control problem to provide an indirect characterization of the value function V of the original control problem, that we are not able to analyze directly.

It is worth noticing that there is a significant difference between the approach to PDP optimal control problems presented in [32] and ours. In the book by Davis the class of control processes is represented by *piecewise open-loop controls*, a class of processes introduced by D. Vermes in [61] depending only on the time elapsed since the last jump and the position at the last jump time of the PDP. In our separated control problem, instead, we are forced to use a more general class of control policies depending on the past history of jump times and jump positions of the PDP. In fact, as we shall later see, it is only looking at this larger class that we can find a correspondence between controls for the original problem with partial observation, i. e. in the class $\mathcal{U}_{ad}$, and policies for the separated PDP control problem. In this sense, an approach closer to ours can be traced in [27]. However, in that paper the authors consider an optimal control problem for a PDP (with complete observation), where the control parameter acts only on the jump intensity and on the transition measure of the process but not on its deterministic flow.

The value function is characterized as the unique fixed point of a contraction mapping. This operator, denoted by $\mathcal{G}$, is defined for all real-valued bounded and continuous functions w on Δ_e as

$$\mathcal{G}w(\nu) := \inf_{\alpha \in A} \int_0^\infty e^{-\beta t} L(\phi_\nu^\alpha(t), \chi_\nu^\alpha(t), \alpha(t), w)\,\mathrm{d}t, \quad \nu \in \Delta_e,$$

where the infimum is taken in the set $A := \{\alpha\colon [0, +\infty) \to U, \text{ measurable}\}$ of all possible ordinary controls, instead of relaxed ones, and L is a real-valued function depending on all the quantities relevant to the separated control problem, i. e. the characteristic triple of the PDP, the cost function f and the control function $\alpha \in A$. The functions ϕ_ν^α, χ_ν^α and L are introduced in Sections 3.1 and 3.2 in the Markov chain setting and in Sections 4.1 and 4.2 in the jump Markov process case.

The operator $\mathcal{G}$ can be associated with a deterministic optimal control problem connected with the stochastic one. It is the problem that an agent must solve to optimize the dynamics of the PDP given by the filtering process between two consecutive jump times, i. e. when the filtering process moves along the deterministic flow of the PDP. We stress once more that in this deterministic problem the agent optimizes among all ordinary control functions, not relaxed ones.

We will prove that the operator $\mathcal{G}$ maps the space of real-valued bounded and continuous functions on the effective simplex into itself and we will show the following result.

Theorem. *Under suitable assumptions on the cost function f and the rate transition measure λ of the unobserved process X, the value function v of the separated optimal control problem is the unique fixed point of the operator $\mathcal{G}$ in the space of real-valued bounded and continuous functions on Δ_e.*

Our value function v is, thus, bounded and continuous.

We can provide a further characterization of this value function in the case where the set I, the state space of the unobserved process, is a finite set. In this case, covered in Chapter 3, the unobserved process X is a continuous-time homogeneous Markov chain, whose controlled rate transition matrix Λ is given. The space of probability measures on I can be identified with the canonical simplex on $\mathbb{R}^{|I|}$, where $|I|$ denotes the cardinality of the set I. Hence, the effective simplex Δ_e is a proper subset of this canonical simplex and it is a compact set. The filtering process can be viewed as a vector-valued process and, in particular, is regarded as a row vector. Also the cost function f is seen as a (column) vector-valued function $\mathbf{f}$ defined on the space of control actions U.

Before anticipating this characterization, we point out that this setting has been analyzed in two other works. The filtering problem for a continous-time Markov chain has been studied in [25]. In that paper, filtering equations are computed, the filtering process is characterized as a PDP and its local characteristics are written down explicitly. There the authors consider an application of those results to an optimal stopping problem.

An optimal control problem is, instead, studied in [63]. This PhD thesis analyzes a more general model than ours: alongside the processes X and Y with values in finite spaces, a further finite-state jump process appears, called *environmental*, influencing both the unobserved and the observed processes. Our function h is encoded in the specification of an *information structure*, i. e. a partition of the state space I. Although in some specific situations our problem can be described in the setting of [63], there are some differences, both at level of definitions and of techniques adopted. In our thesis, for instance, the initial state X_0 of the unobserved process is a random variable with law μ, not just a pre-specified deterministic state; this is a common feature of Markov chains models but it induces some non-trivial complications as we shall see, in particular in connection with the value function v of the separated control problem. In addition, as we anticipated earlier, we prove the equivalence between the original problem and the separated one and we provide a detailed description of the structure of admissible controls in both problems: this is required to make the results in [63] fully rigorous.

Moreover, alongside the characterization of the value function v as unique fixed point of the contraction mapping $\mathcal{G}$ mentioned earlier, we are able to prove the following result.

Theorem. *Under suitable assumptions on the cost function f and the rate transition measure λ of the unobserved process X, the value function v of the separated optimal control problem is the unique constrained viscosity solution of the following HJB equation*

$$H(\nu, \mathrm{D}v(\nu), v) + \beta v(\nu) = 0, \quad \nu \in \Delta_e$$

where the hamiltonian function H is defined for all $\nu \in \Delta_e$, all vectors $\mathbf{b} \in \mathbb{R}^{|I|}$ and all bounded and continuous functions $w \colon \Delta_e \to \mathbb{R}$ as

$$H(\nu, \mathbf{b}, w) := \sup_{u \in U} \left\{ -F(\nu, u)\mathbf{b} - \nu \mathbf{f}(u) - r(\nu, u) \int_{\Delta_e} \big[w(p) - w(\nu)\big] R(\nu, u; \mathrm{d}p) \right\}.$$

The concept of constrained viscosity solution was introduced by H.M. Soner in [58, 59]. This result allows to avoid using generalized gradient methods, as in [63], which require locally Lipschitz continuity of v and additional assumptions on the data of the problem. It is our opinion that the viscosity solutions approach deserves a detailed exposition, since this concept is extensively adopted in the literature to solve HJB equations associated to stochastic optimal control problems (see e. g. [8, 38]). Considering in particular PDP optimal control problems, this approach can be found in [31, 33]. We also mention, as recalled at the beginning of the Introduction, that a characterization of the value function v via BSDEs may be studied (in the PDP optimal control setting see e. g. [5, 6]).

The last result that we anticipate here, remaining in the Markov chain setting, is the existence of an optimal ordinary control.

Theorem. *For each initial law μ of the unobserved process X there exists an optimal ordinary control $\mathbf{u}^\star \in \mathcal{U}_{ad}$, i. e. an admissible control process $(u_t^\star)_{t \geqslant 0} \in \mathcal{U}_{ad}$ such that, for each time $t \geqslant 0$, $u_t^\star$ depends on the last jump time and position of the process Y prior to time t, takes values in the set U, and minimizes the cost functional J.*

We notice that this optimal control corresponds to a piecewise open-loop control in the sense of Vermes. Thus, in some sense this result closes the circle, ensuring the existence of control processes that are standard in PDP optimal control problems.

The thesis is organized as follows. In order to make it as self-contained as possible, the first Chapter is devoted to recall some results on Marked Point Processes (Section 1.1) and Piecewise Deterministic Markov Processes (Section 1.2).

Marked Point Processes are a useful tool to describe the control problem discussed above and play a central rôle in Chapter 2, where we obtain the explicit filtering equations anticipated earlier (Section 2.1) and we prove the characterization of the filtering process as a PDP (Section 2.2). In Section 2.3 we introduce the notation adopted in the Markov chain case, studied in Chapter 3, and in Section 2.4 we provide some remarks on the rôle of the function h in the filtering problem.

In Section 3.1 we introduce the setting of the optimal control problem and we prove a useful property concerning the transition kernel associated to the filtering process. In Section 3.2 we define the separated optimal control problem for the filtering process and we prove its equivalence with the original one. In Section 3.3 we prove the two characterizations of the value function associated to the separated control problem, as the unique fixed point of the contraction mapping $\mathcal{G}$ mentioned earlier and as the unique constrained viscosity solution of the HJB equation stated above. Finally, in Section 3.4 we prove the existence of an ordinary optimal control for our problem and in Section 3.5 we provide an example where we are able to solve explicitly our optimal control problem.

In Chapter 4, the final one, the optimal control problem for a continuous-time homogeneous pure jump Markov process is studied. The steps are almost the same of Chapter 3. In Section 4.1 the optimal control problem is introduced and its equivalence with the separated optimal control problem for the filtering process is proved in Section 4.2. The characterization of the value function associated to the separated control problem as the unique fixed point of the contraction mapping $\mathcal{G}$ is studied in Section 4.3. In Section 4.4 we make some comments on the rôle of the function h in the control problem.

We point out that all the proofs contained in this thesis are original. Results stated without proof are either generalizations (or slight modifications) of other proofs contained in this work or can be found elsewhere in the literature. In this case, we explicitly indicate references for the interested reader.

CHAPTER 1

Preliminaries

This Chapter is devoted to a synthetic and brief review of the main concepts regarding marked point processes and piecewise deterministic processes. The reader will encounter these kind of processes in the following Chapters and we will use various results, stated in this Chapter without proof in order to make this thesis as self contained as possible. For the reader's convenience, we will point out the precise reference of the presented results.

1.1 Marked point processes

In this Section we introduce *marked point processes*, or MPP for short. These processes have been deeply studied and characterized in the past years. The main references on this topic are [43, 17, 47, 42] and the interested reader is invited to consult them for detailed expositions on the subject. Here we present a summary of the main results on MPPs that we are going to use in the next Chapters. We mostly follow the discussions in [43] and [17] and we provide the precise reference to the results shown, for sake of clarity.

1.1.1 General results on marked point processes

Marked point processes are countable collections of couples of random variables, denoted by $(T_n, \xi_n)_{n \in \mathbb{N}}$ and defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The random variables $T_n \colon \Omega \to (0, +\infty]$, $n \in \mathbb{N}$ satisfy

$$\begin{aligned} T_n &\leqslant T_{n+1}, && \mathbb{P}\text{–a.s., } n \in \mathbb{N}. \\ T_n < +\infty &\Rightarrow T_n < T_{n+1}, && \mathbb{P}\text{–a.s., } n \in \mathbb{N}. \end{aligned}$$

They represent the time of occurrence of some random phenomenon. The collection $(T_n)_{n \in \mathbb{N}}$ is called *simple point process*, or just *point process*. The random variables $\xi_n \colon \Omega \to E$, $n \in \mathbb{N}$, take their values in a measurable space $(E, \mathcal{E})$, called the *mark space*, and represent a quantity related to each random time T_n. We denote by

$T_\infty(\omega) := \lim_{n\to\infty} T_n(\omega)$, $\omega \in \Omega$ the accumulation or explosion point of the MPP. If $T_\infty = +\infty$, $\mathbb{P}$–a.s., the MPP is said to be *non-explosive*. Notice that the accumulation point may be finite and, in that case, there is no point after T_∞.

A marked point process can be equivalently described via a *random counting measure*, or RCM for short. This is a random measure on $\big((0,+\infty)\times E, \mathcal{B}\big((0,+\infty)\big)\otimes\mathcal{E}\big)$ defined as

$$\mu(\omega, \mathrm{d}t\,\mathrm{d}x) := \sum_{n\in\mathbb{N}} \delta_{\big(T_n(\omega),\xi_n(\omega)\big)}(\mathrm{d}t\,\mathrm{d}x)\mathbb{1}_{T_n(\omega)<+\infty}, \quad \omega \in \Omega$$

where δ_a denotes the Dirac probability measure concentrated on the point a. With the expression *random measure* on some measurable space $(A, \mathcal{A})$ we mean a transition kernel from $(\Omega, \mathcal{F})$ to $(A, \mathcal{A})$. We will base our analysis of MPPs on their corresponding RCMs.

Another quantity associated to MPPs is the *counting process* $N_t := \mu\big((0,t]\times E\big)$, $t \geqslant 0$, that counts the number of jumps occurred up to time t. We can also define a family of counting processes parameterized by measurable sets in the σ-algebra $\mathcal{E}$, i. e. consider the processes

$$N_t(A) := \mu\big((0,t]\times A\big), \quad t \geqslant 0,\ A \in \mathcal{E}. \tag{1.1.1}$$

Notice that, in the case of a simple point process, the counting process $N = (N_t)_{t\geqslant 0}$ completely describes it and there is no need to consider its associated RCM.

For the rest of this Chapter, the following assumption will be in force.

Assumption 1.1.1. The mark space E is a Borel subset of a compact metric space (a.k.a. *Lusin space*). The σ-algebra $\mathcal{E}$ is the Borel σ-algebra on E, i. e. $\mathcal{E} = \mathcal{B}(E)$.

We will sometimes consider an extra point Δ that we add to the mark space E, defining $E_\Delta := E \cup \{\Delta\}$ with its Borel σ-algebra $\mathcal{E}_\Delta := \mathcal{B}(E_\Delta)$.

To study MPPs we will adopt a dynamic point of view and treat them as continuous time stochastic processes through their RCMs. From now on, we are given a filtration $\mathbb{F} := (\mathcal{F}_t)_{t\geqslant 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$, i. e. an increasing sequence of sub-σ-algebras of $\mathcal{F}$. We assume that the usual conditions of Dellacherie are satisfied, meaning that the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is complete and all the σ-algebras of $\mathbb{F}$ are augmented with the collection $\mathcal{N}$ of all $\mathbb{P}$–null sets of $\mathcal{F}$.

A marked point process $(T_n, \xi_n)_{n\in\mathbb{N}}$ is said to be $\mathbb{F}$–adapted if the sequence of $\big((0,+\infty]\times E\big)$–valued random variables $(T_n, \xi_n)_{n\in\mathbb{N}}$ is such that for all $n \in \mathbb{N}$

1. T_n is a $\mathbb{F}$–stopping time and $T_n \leqslant T_{n+1}$, $\mathbb{P}$–a.s.
2. ξ_n is $\mathcal{F}_{T_n}$–measurable.
3. If $T_n < +\infty$ then $T_n < T_{n+1}$, $\mathbb{P}$–a.s.

Clearly, we can always consider a marked point process $(T_n, \xi_n)_{n\in\mathbb{N}}$ as adapted to its natural completed filtration, that we denote by $\mathbb{G} := (\mathcal{G}_t)_{t\geqslant 0}$, where

$$\mathcal{G}_t := \sigma\big(\mu\big((0,s]\times A\big) \colon 0 \leqslant s \leqslant t,\ A \in \mathcal{E}\big) \vee \mathcal{N}, \quad t \geqslant 0. \tag{1.1.2}$$

It is possible to give an extremely careful description of the filtration $\mathbb{G}$.

Theorem 1.1.1 ([17, Appendix A2, Th. T30]). *Let $(T_n, \xi_n)_{n\in\mathbb{N}}$ be an E–marked point process defined on $(\Omega, \mathcal{F})$. We have that*

1. $\mathcal{G}_{T_n} = \sigma(T_1, \xi_1, \dots, T_n, \xi_n) \vee \mathcal{N}$, $n \in \mathbb{N}$.
2. $\mathcal{G}_{T_n^-} = \sigma(T_1, \xi_1, \dots, T_{n-1}, \xi_{n-1}, T_n) \vee \mathcal{N}$, $n \in \mathbb{N}$.
3. $\mathcal{G}_{T_\infty} = \sigma(T_1, \xi_, \dots) \vee \mathcal{N}$.

Let $\mathcal{P}(\mathbb{F})$ be the σ-algebra on $\Omega \times [0, +\infty)$ generated by the maps $(\omega, t) \mapsto Y_t(\omega)$ that are $\mathcal{F}_t$-measurable in ω and left-continuous in t. We recall that a real-valued process $X = (X_t)_{t \geqslant 0}$ is called $\mathbb{F}$–*predictable* if the map $(t, \omega) \mapsto X_t(\omega)$ is $\mathcal{P}(\mathbb{F})$–measurable. An important characterization of predictable processes is given in the following Theorem. First, we need to introduce the following assumption concerning the structure of the filtration $\mathbb{F}$.

Assumption 1.1.2 ([43, (A.1)]). We have that for all $t \geqslant 0$, $\mathcal{F}_t = \mathcal{F}_0 \vee \mathcal{G}_t = \sigma(\mathcal{F}_0 \cup \mathcal{G}_t)$, where $\mathbb{G} = (\mathcal{G}_t)_{t \geqslant 0}$ is the natural completed filtration of the MPP $(T_n, \xi_n)_{n \in \mathbb{N}}$ introduced in (1.1.2).

Theorem 1.1.2 ([17, Appendix A2, Th. T34] and [43, Lemma 3.3]). *Let $T_0(\omega) = 0, \omega \in \Omega$ and let Assumption 1.1.2 be in force. In order for the process $X = (X_t)_{t \geqslant 0}$ to be $\mathbb{F}$–predictable it is necessary and sufficient that X_0 is $\mathcal{F}_0$–measurable and it admits the representation*

$$X_t(\omega) = \sum_{n \in \mathbb{N}_0} f^{(n)}(t, \omega) \mathbb{1}_{T_n(\omega) < t \leqslant T_{n+1}(\omega)} + f^{(\infty)}(t, \omega) \mathbb{1}_{T_\infty(\omega) < t < +\infty},\ t > 0,\ \omega \in \Omega$$

where for each $n \in \bar{\mathbb{N}}_0$ the mapping $(t, \omega) \mapsto f^{(n)}(t, \omega)$ is $\mathcal{F}_{T_n} \otimes \mathcal{B}\big([0, +\infty)\big)$–measurable.

Remark 1.1.1. Thanks to Theorem 1.1.1 it is possible to write the functions $f^{(n)}$, $n \in \bar{\mathbb{N}}_0$ of Theorem 1.1.2 as

- $f^{(0)}(t, \omega) = f_0\big(t, X_0(\omega)\big)$, $t > 0$, $\omega \in \Omega$.
- $f^{(n)}(t, \omega) = f_n\big(t, X_0(\omega), \dots, T_n(\omega), \xi_n(\omega)\big)$, $t > 0$, $\omega \in \Omega$, $n \in \mathbb{N}$.
- $f^{(\infty)}(t, \omega) = f_\infty\big(t, X_0(\omega), T_1(\omega), \xi_1(\omega), \dots\big)$, $t > 0$, $\omega \in \Omega$.

where the functions f_n, $n \in \bar{\mathbb{N}}_0$ are suitably defined deterministic functions.

In what follows we also consider *stochastic* or *random fields* on E, i. e. stochastic processes depending on an additional parameter $x \in E$. Similarly to stochastic processes, a random field $Z = \big(Z_t(x)\big)_{t \geqslant 0,\, x \in E}$ is said to be $\mathbb{F}$–predictable if the map $(t, \omega, x) \mapsto Z_t(\omega, x)$ is $\mathcal{P}(\mathbb{F}) \otimes \mathcal{E}$–measurable.

The concept of predictability can be extended to random measures thanks to the following definition.

Definition 1.1.1 ([43]). A random measure η on $(E, \mathcal{E})$ is said to be $\mathbb{F}$–predictable if for all nonnegative $\mathbb{F}$–predictable random fields $Z = \big(Z_t(x)\big)_{t \geqslant 0,\, x \in E}$ we have that the real-valued process

$$\left(\int_{(0,t] \times E} Z_s(x)\, \eta(\mathrm{d}s\, \mathrm{d}x) \right)_{t \geqslant 0}$$

is $\mathbb{F}$–predictable.

Dual predictable projection of a RCM

Let $(T_n, \xi_n)_{n\in\mathbb{N}}$ be a $\mathbb{F}$–adapted marked point process with associated random counting measure μ. One of the main results concerning marked point processes is the existence of the dual predictable projection of the associated random counting measure.

Theorem 1.1.3 ([43, Th. 2.1]). *There exists one and only one (up to a modification on a $\mathbb{P}$–null set) $\mathbb{F}$–predictable random measure ν such that for each nonnegative $\mathbb{F}$–predictable random field $Z = \big(Z_t(x)\big)_{t\geqslant 0,\, x\in E}$ we have:*

$$\mathbb{E}\int_{(0,+\infty)\times E} Z_t(x)\,\mu(\mathrm{d}t\,\mathrm{d}x) = \mathbb{E}\int_{(0,+\infty)\times E} Z_t(x)\,\nu(\mathrm{d}t\,\mathrm{d}x).$$

The $\mathbb{F}$–predictable random measure ν is called the $\mathbb{F}$–dual predictable projection *of μ.*

Proposition 1.1.4 ([43, Prop. 2.3]). *One can choose a version of ν satisfying identically*

$$\begin{aligned} \nu\big(\{t\}\times E\big) &\leqslant 1, \quad t>0 \\ \nu\big([T_\infty,+\infty)\times E\big) &= 0. \end{aligned} \tag{1.1.3}$$

Theorem 1.1.3 is of fundamental importance because it paves the way for a martingale description of a MPP. In particular, we have the following result.

Proposition 1.1.5 ([43, (2.4), (2.5) and (2.6)]). *The $\mathbb{F}$–dual predictable projection ν of μ is characterized by any of these equivalent facts.*

1. *ν satisfies (1.1.3) and for each $A\in\mathcal{E}$ the process $\big(\nu\big((0,t]\times A\big)\big)_{t\geqslant 0}$ is the $\mathbb{F}$–dual predictable projection of $\big(\mu\big((0,t]\times A\big)\big)_{t\geqslant 0}$.*
2. *ν satisfies (1.1.3) and*
   - *(a) for all $A\in\mathcal{E}$ the process $\big(\nu\big((0,t]\times A\big)\big)_{t\geqslant 0}$ is $\mathbb{F}$–predictable,*
   - *(b) for all $A\in\mathcal{E}$ and all $\mathbb{F}$–stopping times T it holds*

     $$\mathbb{E}\big[\nu\big((0,T]\times A\big)\big] = \mathbb{E}\big[\mu\big((0,T]\times A\big)\big].$$
3. *ν satisfies (1.1.3) and*
   - *(a) for all $A\in\mathcal{E}$ the process $\big(\nu\big((0,t]\times A\big)\big)_{t\geqslant 0}$ is $\mathbb{F}$–predictable,*
   - *(b) for all $A\in\mathcal{E}$ and all $n\in\mathbb{N}$ the process*

     $$\big(\nu\big((0,t\wedge T_n]\times A\big) - \mu\big((0,t\wedge T_n]\times A\big)\big)_{t\geqslant 0}$$

     is a uniformly integrable $\mathbb{F}$–martingale. If $\mathbb{P}(T_\infty = +\infty) = 1$ one can replace (3b) by
   - *(b') for all $A\in\mathcal{E}$ the process*

     $$\big(\nu\big((0,t]\times A\big) - \mu\big((0,t]\times A\big)\big)_{t\geqslant 0}$$

     is a $\mathbb{F}$–local martingale.

The following result allows to give a complete description of a marked point process in terms of its *local characteristics*, by disintegrating the dual predictable projection ν as specified below (here Assumption 1.1.1 is fundamental).

Theorem 1.1.6 ([17, Ch. VIII, Th. T14]). *There exists*

1. *a unique (up to $\mathbb{P}$–indistinguishability) right-continuous $\mathbb{F}$–predictable increasing process $A = (A_t)_{t \geqslant 0}$ with $A_0 \equiv 0$,*
2. *a transition measure $\Phi_t(\omega, \mathrm{d}x)$ from $\big((0, +\infty) \times \Omega, \mathcal{P}(\mathbb{F})\big)$ into $(E, \mathcal{E})$, such that, for all $n \in \mathbb{N}$*

$$\mathbb{E} \int_{(0,T_n] \times E} Z_s(x)\, \mu(\mathrm{d}s\, \mathrm{d}x) = \mathbb{E} \int_{(0,T_n] \times E} Z_s(x)\, \Phi_s(\mathrm{d}x)\, \mathrm{d}A_s$$

for all nonnegative $\mathbb{F}$–predictable random fields $Z = \big(Z_t(x)\big)_{t \geqslant 0,\, x \in E}$.

*The pair $\big(A, \Phi_t(\mathrm{d}z)\big)$ gives the $(\mathbb{P}, \mathbb{F})$–*local characteristics *of the MPP $(T_n, \xi_n)_{n \in \mathbb{N}}$.*

The following result gives an important characterization of the probability kernel $\Phi_t(\mathrm{d}x)$.

Theorem 1.1.7 ([17, Ch. VIII, Th. T16]). *Let $(T_n, \xi_n)_{n \in \mathbb{N}}$ be a $\mathbb{F}$–adapted marked point process, with $\mathbb{F}$–local characteristics $\big(A, \Phi_t(\mathrm{d}x)\big)$. Under Assumption 1.1.2, the transition probability $\Phi_t(\mathrm{d}x)$ satisfies for all $n \in \mathbb{N}$*

$$\Phi_{T_n}(A) = \mathbb{P}\big[\xi_n \in A \mid \mathcal{F}_{T_n^-}\big], \quad \mathbb{P}\text{–a.s. on } \{T_n < +\infty\}.$$

It is possible in some cases to give an explicit form for the dual predictable projection ν. Let us define the sojourn times $(S_n)_{n \in \mathbb{N}}$ by

$$S_n = \begin{cases} T_n - T_{n-1}, & \text{on } \{T_{n-1} < +\infty\} \\ +\infty, & \text{on } \{T_{n-1} = +\infty\} \end{cases} \tag{1.1.4}$$

Let us denote by $G_n(\omega, \mathrm{d}t\, \mathrm{d}x)$, $n \in \mathbb{N}_0$ a regular version of the conditional law of (S_{n+1}, ξ_{n+1}) given $\mathcal{F}_{T_n}$ and let $H_n(\omega, \mathrm{d}t) := G_n(\omega, \mathrm{d}t \times E_\Delta)$, be the conditional law of S_{n+1} given $\mathcal{F}_{T_n}$. We point out that thanks to Assumption 1.1.1 the regular versions of these conditional laws always exist.

Proposition 1.1.8 ([43, Prop. 3.1]). *Under Assumption 1.1.2 the following formula defines a version of the $\mathbb{F}$–dual predictable projection of μ (which satisfies (1.1.3)).*

$$\nu(\mathrm{d}t\, \mathrm{d}x) = \sum_{n \in \mathbb{N}_0} \frac{G_n(\mathrm{d}t - T_n\, \mathrm{d}x)}{H_n([t - T_n, +\infty])} \mathbb{1}_{T_n < t \leqslant T_{n+1}}.$$

Another important question regarding dual predictable projections concerns the opposite question answered by Theorem 1.1.3. Suppose that we have a predictable random measure ν: under which conditions it is possible to construct a probability measure $\mathbb{P}$ on a suitably defined measurable sample space such that ν is the dual predictable projection of μ?

Assumption 1.1.3 ([43, (A.2)]). We have that $\Omega = \Omega' \times \Omega''$ where

- Ω' is the canonical space for MPPs, i. e. the set of all possible marked point processes $(T'_n, \xi'_n)_{n \in \mathbb{N}}$.
- $(\Omega'', \mathcal{F}'')$ is an arbitrary measurable space.

Moreover, we are given a MPP $(T_n, \xi_n)_{n\in\mathbb{N}}$ such that

$$(T_n, \xi_n)(\omega', \omega'') = (T'_n, \xi'_n)(\omega'), \quad \omega = (\omega', \omega'') \in \Omega,\, n \in \mathbb{N}$$

and a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geqslant 0}$ such that

$$\mathcal{F}_0 = \{\emptyset, \Omega'\} \otimes \mathcal{F}'', \qquad\qquad \mathcal{F}_t = \mathcal{F}_0 \vee \mathcal{G}_t$$

where $(\mathcal{G}_t)_{t\geqslant 0}$ is the natural completed filtration of μ defined in (1.1.2).

We denote by $\mathcal{F}_\infty := \bigvee_{t\geqslant 0} \mathcal{F}_t$. We have the following important result.

Theorem 1.1.9 ([43, Th. 3.6]). *Let Assumption 1.1.3 be in force. Let* P_0 *be a probability measure on* $(\Omega, \mathcal{F}_0)$ *and* ν *a predictable random measure satisfying (1.1.3). Then there exists a unique probability measure* P *on* $(\Omega, \mathcal{F}_\infty)$ *whose restriction to* $\mathcal{F}_0$ *is* P_0 *and for which* ν *is the predictable projection of* μ*.*

Martingale representation theorem

As it is known, martingale representation theorems are ubiquitous and fundamental in stochastic processes analysis. As everyone expects, such a theorem exists also for marked point processes. Its importance is never to be underestimated and in our case it will be central to solve the stochastic filtering problem in Chapter 2.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a fixed probability space and let $(T_n, \xi_n)_{n\in\mathbb{N}}$ a $\mathbb{F}$–adapted marked point process defined on it, with associated random counting measure μ. We have the following two results.

Proposition 1.1.10 ([43, Prop. 5.3]). *Let* $Z = \big(Z_t(x)\big)_{t\geqslant 0,\, x\in E}$ *be a real-valued* $\mathbb{F}$*–predictable random field satisfying*

$$\int_{(0,t]\times E} |Z_s(x)|\, \nu(\mathrm{d}s\, \mathrm{d}x) < +\infty, \quad t > 0,\, \mathbb{P}\text{–a.s. on } \{t < T_\infty\}.$$

Let $X = (X_t)_{t\geqslant 0}$ *be a right-continuous* $\mathbb{F}$*–adapted process, such that*

$$X_t = X_0 + \int_{(0,t]\times E} Z_s(x) \left[\mu(\mathrm{d}s\, \mathrm{d}x) - \nu(\mathrm{d}s\, \mathrm{d}x)\right], \quad t > 0,\, \mathbb{P}\text{–a.s. on } \{t < T_\infty\}.$$

Then there exists a sequence $(S_n)_{n\in\mathbb{N}}$ *of* $\mathbb{F}$*–stopping times increasing* $\mathbb{P}$*–a.s. towards* T_∞*, for which* $(X_{t\wedge S_n})_{t\geqslant 0}$ *is a uniformly integrable martingale for each* $n \in \mathbb{N}$*.*

Theorem 1.1.11 ([43, Th. 5.4]). *Let Assumption 1.1.2 be in force and let* $X = (X_t)_{t\geqslant 0}$ *be a right-continuous* $\mathbb{F}$*–adapted process. Then there exists a sequence* $(S_n)_{n\in\mathbb{N}}$ *of* $\mathbb{F}$*–stopping times increasing* $\mathbb{P}$*–a.s. towards* T_∞*, for which* $(X_{t\wedge S_n})_{t\geqslant 0}$ *is a uniformly integrable martingale for each* $n \in \mathbb{N}$*, if and only if there exists a real-valued* $\mathbb{F}$*–predictable random field* $Z = \big(Z_t(x)\big)_{t\geqslant 0,\, x\in E}$ *satisfying*

$$\int_{(0,t]\times E} |Z_s(x)|\, \nu(\mathrm{d}s\, \mathrm{d}x) < +\infty, \qquad t > 0,\, \mathbb{P}\text{–a.s. on } \{t < T_\infty\}.$$

$$X_t = X_0 + \int_{(0,t]\times E} Z_s(x) \left[\mu(\mathrm{d}s\, \mathrm{d}x) - \nu(\mathrm{d}s\, \mathrm{d}x)\right], \quad t > 0,\, \mathbb{P}\text{–a.s. on } \{t < T_\infty\}.$$

1.1.2 Stochastic intensities

Important examples of marked point processes are those admitting a *stochastic intensity*.

Let us start discussing the case of a $\mathbb{F}$–adapted simple point process $(T_n)_{n\in\mathbb{N}}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$. For sake of simplicity, we assume that it is $\mathbb{P}$–a.s. non-explosive. We denote by $N = (N_t)_{t\geqslant 0}$ its associated counting process.

Definition 1.1.2 ([17, Ch. II, Def. D7]). Let $\lambda = (\lambda_t)_{t\geqslant 0}$ be a $\mathbb{F}$–progressive nonnegative process such that for all $t \geqslant 0$

$$\int_0^t \lambda_s \, \mathrm{d}s < +\infty \quad \mathbb{P}\text{–a.s.}$$

If for all nonnegative $\mathbb{F}$–predictable processes $Z = (Z_t)_{t\geqslant 0}$ the equality

$$\mathbb{E}\int_0^\infty Z_s \, \mathrm{d}N_s = \mathbb{E}\int_0^\infty Z_s \lambda_s \, \mathrm{d}s$$

is verified, then we say that N *admits the* $\mathbb{F}$*-stochastic intensity* λ.

The stochastic intensity may fail to exist. However, we know from Theorem 1.1.3 that the dual predictable projection always does. In the case of a simple point process this can be identified with an increasing right-continuous $\mathbb{F}$–predictable process $A = (A_t)_{t\geqslant 0}$ with $A_0 \equiv 0$ (cfr. Theorem 1.1.6). In particular, we are granted the existence of a stochastic intensity if the measure $\mathrm{d}A$ is absolutely continuous with respect to the Lebesgue measure on $\big((0,+\infty), \mathcal{B}\big((0,+\infty)\big)\big)$, in the sense that there exists a $\mathbb{F}$–predictable nonnegative process $\lambda = (\lambda_t)_{t\geqslant 0}$ such that

$$A_t = \int_0^t \lambda_s \, \mathrm{d}s, \quad t \geqslant 0, \; \mathbb{P}\text{–a.s.}$$

We now present some explicit examples of stochastic intensities.

Example 1.1.1 (Homogeneous Poisson Process). Let $(T_n)_{n\in\mathbb{N}}$ be a $\mathbb{F}$–adapted point process, with associated counting process $N = (N_t)_{t\geqslant 0}$, and let $\lambda > 0$.

If for all $0 \leqslant s \leqslant t$ and all $u \in \mathbb{R}$ it holds

$$\mathbb{E}\big[e^{iu(N_t - N_s)} \mid \mathcal{F}_s\big] = \exp\big\{\lambda(t-s)(e^{iu} - 1)\big\}, \tag{1.1.5}$$

then N is called a $\mathbb{F}$–*homogeneous Poisson process* with intensity λ. In other words, its stochastic intensity is the deterministic constant process equal to λ.

Condition (1.1.5) implies that for all $0 \leqslant s \leqslant t$ the increments $N_t - N_s$ are $\mathbb{P}$–independent of $\mathcal{F}_s$ given $\mathcal{F}_0$. Moreover, it leads to the usual formula

$$\mathbb{P}(N_t - N_s = k \mid \mathcal{F}_s) = e^{-\lambda(t-s)} \frac{\big(\lambda(t-s)\big)^k}{k!}, \quad k \in \mathbb{N}.$$

A simple calculation using formula (1.1.5) shows that $\mathbb{E}N_t = \lambda t$. This allows us to interpret the intensity of the process N as the expected number of "events" that occur per unit time and identifying it with λ. This reasoning can be further generalized in order to consider a wider class of processes that are still related to the Poisson distribution, as shown in the following example.

Example 1.1.2 (Conditional Poisson Process)**.** Let $(T_n)_{n\in\mathbb{N}}$ be a $\mathbb{F}$–adapted point process, with associated counting process $N = (N_t)_{t\geqslant 0}$, and let $\lambda = (\lambda_t)_{t\geqslant 0}$ be a nonnegative $\mathbb{F}$–progressive process.

Suppose that the following conditions hold:

- λ_t is $\mathcal{F}_0$–measurable, for all $t \geqslant 0$,
- $\int_0^t \lambda_s \, \mathrm{d}s < +\infty$, $\mathbb{P}$–a.s., for all $t \geqslant 0$,
- $\mathbb{E}\big[e^{iu(N_t - N_s)} \mid \mathcal{F}_s\big] = \exp\Big\{(e^{iu} - 1) \int_s^t \lambda_r \, \mathrm{d}r\Big\}$, $\mathbb{P}$–a.s., for all $0 \leqslant s \leqslant t$.

Then N is called a $\mathbb{F}$-*conditional Poisson process*, or $\mathbb{F}$–Cox process, with stochastic intensity λ.

Allowing stochastic intensities to be $\mathbb{F}$–progressive processes has the effect that uniqueness (modulo indistinguishability) is lost. However, it is always possible to find a $\mathbb{F}$–predictable version of the stochastic intensity, so that uniqueness is restored, as the following Theorem shows.

Theorem 1.1.12 ([17, Ch. II, Th. T12 and T13])**.** *Let $(T_n)_{n\in\mathbb{N}}$ be a $\mathbb{F}$–adapted point process, with associated counting process $N = (N_t)_{t\geqslant 0}$. Suppose that it admits a $\mathbb{F}$–stochastic intensity $\lambda = (\lambda_t)_{t\geqslant 0}$. Then a $\mathbb{F}$–predictable version of λ exists. Moreover, if $\hat{\lambda} = (\hat{\lambda}_t)_{t\geqslant 0}$ and $\tilde{\lambda} = (\tilde{\lambda}_t)_{t\geqslant 0}$ are two $\mathbb{F}$–predictable intensities of N, then*

$$\hat{\lambda}_t(\omega) = \tilde{\lambda}_t(\omega) \quad \mathbb{P}(\mathrm{d}\omega)\mathrm{d}N_t(\omega)\text{–a.e.} \tag{1.1.6}$$

In particular, $\mathbb{P}$–a.s.,

$$\hat{\lambda}_{T_n} = \tilde{\lambda}_{T_n} \quad \text{on } \{T_n < \infty\}, \quad n \in \mathbb{N}, \tag{1.1.7a}$$

$$\hat{\lambda}_t(\omega) = \tilde{\lambda}_t(\omega) \quad \hat{\lambda}_t(\omega)\mathrm{d}t\text{– and } \tilde{\lambda}_t(\omega)\mathrm{d}t\text{–a.e.}, \quad \omega \in \Omega, \tag{1.1.7b}$$

$$\hat{\lambda}_{T_n} > 0 \quad \text{on } \{T_n < \infty\}, \quad n \in \mathbb{N}. \tag{1.1.7c}$$

Let us know discuss the existence of a stochastic intensity for a marked point process.

Definition 1.1.3 ([17, Ch. VIII, Def. D2])**.** Let $(T_n, \xi_n)_{n\in\mathbb{N}}$ be a $\mathbb{F}$–adapted marked point process, with associated random counting measure μ. Suppose that for each $A \in \mathcal{E}$, the counting process $\big(N_t(A)\big)_{t\geqslant 0}$ defined in (1.1.1) admits the $\mathbb{F}$–predictable intensity $\big(\lambda_t(A)\big)_{t\geqslant 0}$, where $\lambda_t(\omega, \mathrm{d}x)$ is a transition measure from $\big(\Omega \times [0, +\infty), \mathcal{F} \otimes \mathcal{B}\big([0, +\infty)\big)\big)$ into $(E, \mathcal{E})$. We say that μ admits the $\mathbb{F}$–intensity kernel $\lambda_t(\mathrm{d}x)$.

Also in this case, intensity kernels may fail to exist. Again, by Theorem 1.1.3 we know that the $\mathbb{F}$–dual predictable projection ν of μ always exists and we are granted the existence of the $\mathbb{F}$–intensity kernel whenever ν is absolutely continuous with respect to the Lebesgue measure on $\big((0, +\infty), \mathcal{B}\big((0, +\infty)\big)\big)$. This means that there exists a transition measure $\lambda_t(\omega, \mathrm{d}x)$ from $\big(\Omega \times [0, +\infty), \mathcal{F} \otimes \mathcal{B}\big([0, +\infty)\big)\big)$ into $(E, \mathcal{E})$ such that $(\lambda_t(A))_{t\geqslant 0}$ is a $\mathbb{F}$–predictable nonnegative process for all $A \in \mathcal{E}$ and

$$\nu\big((0, t] \times A\big) = \int_0^t \lambda_s(A) \, \mathrm{d}s, \quad t \geqslant 0, \; A \in \mathcal{E}, \; \mathbb{P}\text{–a.s.}$$

We recall that, since Assumption 1.1.1 is in force, the $\mathbb{F}$–dual predictable projection ν of μ is a random measure that can always be disintegrated to produce the $\mathbb{F}$–local characteristics of the MPP $(T_n, \xi_n)_{n\in\mathbb{N}}$. When a stochastic kernel exists, we obtain that the $\mathbb{F}$–local characteristics can be expressed as in the following definition.

Definition 1.1.4 ([17, Ch. VIII, Def. D5]). Let $(T_n, \xi_n)_{n\in\mathbb{N}}$ be a $\mathbb{F}$–adapted marked point process, with associated random counting measure μ. Suppose that it admits the $\mathbb{F}$–intensity kernel $\lambda_t(\mathrm{d}x)$. Then we have that

$$\lambda_t(\mathrm{d}x) = \lambda_t \Phi_t(\mathrm{d}x), \quad t \geqslant 0, \, \mathbb{P}\text{–a.s.}$$

where $\lambda = (\lambda_t)_{t\geqslant 0}$ is a nonnegative $\mathbb{F}$–predictable process and $\Phi_t(\omega, \mathrm{d}x)$ is a transition probability from $\big(\Omega \times [0, +\infty), \mathcal{F} \otimes \mathcal{B}\big([0, +\infty)\big)\big)$ into $(E, \mathcal{E})$. The pair $\big(\lambda, \Phi_t(\mathrm{d}x)\big)$ gives the $\mathbb{F}$*–local characteristics* of μ.

Under some conditions, we are able to obtain an explicit form of the $\mathbb{F}$–local characteristics $\big(\lambda, \Phi_t(\mathrm{d}x)\big)$ of a marked point process, similarly to what we saw in Proposition 1.1.8.

Theorem 1.1.13 ([17, Ch. VIII, Th. T7]). *Let Assumption 1.1.2 be in force and let $(T_n, \xi_n)_{n\in\mathbb{N}}$ be a $\mathbb{F}$–adapted marked point process, with associated random counting measure μ. Suppose that, for each $n \in \mathbb{N}_0$, there exists a regular conditional distribution of (S_{n+1}, ξ_{n+1}) given $\mathcal{F}_{T_n}$ of the form*

$$\mathbb{P}(S_{n+1} \in A, \xi_{n+1} \in C \mid \mathcal{F}_{T_n}) = \int_A g^{(n+1)}(s, C)\,\mathrm{d}s, \quad A \in \mathcal{B}\big([0, +\infty)\big), \, C \in \mathcal{E}$$

where $(S_n)_{n\in\mathbb{N}}$ are the sojourn times defined in (1.1.4) and, for each $n \in \mathbb{N}_0$, $g^{(n+1)}$ is a finite kernel from $\big(\Omega \times [0, \infty), \mathcal{F}_{T_n} \otimes \mathcal{B}\big([0, +\infty)\big)\big)$ into $(E, \mathcal{E})$, that is to say:

1. *$(\omega, s) \mapsto g^{(n+1)}(\omega, s, C)$ is $\mathcal{F}_{T_n} \otimes \mathcal{B}\big([0, +\infty)\big)$–measurable, for all $C \in \mathcal{E}$,*
2. *for all $(\omega, s) \in \Omega \times [0, \infty)$, $C \mapsto g^{(n+1)}(\omega, s, C)$ is a finite measure on $(E, \mathcal{E})$.*

Then μ admits the $\mathbb{F}$–local characteristics $\big(\lambda, \Phi_t(\mathrm{d}x)\big)$ defined by (set $T_0 \equiv 0$)

$$\lambda_t(C) = \frac{g^{(n+1)}(t - T_n, C)}{1 - \int_0^{t-T_n} g^{(n+1)}(s, E)\,\mathrm{d}s}, \, t \in (T_n, T_{n+1}], \, n \in \mathbb{N}_0 \tag{1.1.8a}$$

$$\lambda_t = \lambda_t(E), \quad t > 0 \tag{1.1.8b}$$

$$\Phi_t(C) = \frac{\lambda_t(C)}{\lambda_t(E)}, \quad t > 0, \, C \in \mathcal{E}. \tag{1.1.8c}$$

1.1.3 Filtering with marked point process observation

Stochastic filtering techniques address the issue of estimating the state at time t of a given dynamical stochastic system, based on the available information at the same time t. Historically and in the context of second-order stationary processes, two approaches have mainly been used:

- Frequency spectra analysis (Kolmogorov-Wiener).
- Time-domain analysis (Kalman).

Since we adopted a dynamical approach so far to describe and study marked point processes, we will use tools that are based on Kalman's innovations theory.

The basic datum of a filtering problem is a pair of stochastic processes: a *state process* and an *observed process*. The former is also said *unobserved* or *signal* process; we are interested in the estimation of its state or, more generally, of the state of a process that depends on it. To estimate its state we will use the information provided by the latter process, i. e. the observed one.

Having in mind this setting, we will proceed along this path:

1. Find the innovating representation of the state process and then project this representation on the natural filtration of the observed process.
2. Search for filtering formulas, expressed in terms of the innovations gain and of the innovating part, using the representation of martingales with respect to the observed filtration.
3. Use martingale calculus to identify the innovations gain.

The Innovating Structure of the Filter

Let $X = (X_t)_{t\geqslant 0}$ and $Y = (Y_t)_{t\geqslant 0}$ be two $(E, \mathcal{E})$ valued processes and let $Z = (Z_t)_{t\geqslant 0} = (\varphi(X_t))_{t\geqslant 0}$ be a real-valued process, with $\varphi\colon E \to \mathbb{R}$ a measurable function. We assume that X is the unobserved process, Y is the observed process and Z as the process that we aim to filter.

Let $\mathbb{X} = (\mathcal{X}_t)_{t\geqslant 0}$ and $\mathbb{Y} = (\mathcal{Y}_t)_{t\geqslant 0}$ be the natural completed filtrations of the processes X and Y respectively. With $\mathbb{F} = (\mathcal{F}_t)_{t\geqslant 0}$, where $\mathcal{F}_t = \mathcal{X}_t \vee \mathcal{Y}_t$, $t \geqslant 0$, we denote the *global filtration*.

In the sequel we suppose that the process Z satisfies the equation

$$Z_t = Z_0 + \int_0^t f_s \,\mathrm{d}s + M_t, \quad \mathbb{P}\text{–a.s., } t \geqslant 0 \tag{1.1.9}$$

where

1. $(f_t)_{t\geqslant 0}$ is a $\mathbb{F}$–progressive process such that
$$\int_0^t |f_s| \,\mathrm{d}s < +\infty \quad \mathbb{P}\text{–a.s., } t \geqslant 0,$$
2. $(M_t)_{t\geqslant 0}$ is a zero mean $\mathbb{F}$–local martingale.

Equation (1.1.9) is called the *semi-martingale representation* of Z. In most cases of practical interest, the existence of this representation can be directly exhibited as shown in the following examples.

Example 1.1.3 (Signal corrupted by a white noise[1])**.** Let X be the real-valued process

$$X_t = X_0 + \int_0^t S_u \,\mathrm{d}u + W_t, \quad t \geqslant 0$$

where

[1] For a background in stochastic processes driven by Wiener-processes, see [45].

- $(S_t)_{t\geqslant 0}$ is a $\mathbb{F}$–adapted process such that

$$\int_0^t |S_u| \,\mathrm{d}u < +\infty \quad \mathbb{P}\text{–a.s., } t \geqslant 0$$

- $(W_t)_{t\geqslant 0}$ is a $\mathbb{F}$-Wiener process.

Let $\varphi\colon \mathbb{R} \to \mathbb{R}$ be a twice continuously differentiable function and let $Z_t = \varphi(X_t)$, $t \geqslant 0$. Then, an application of Ito's formula yields

$$Z_t = Z_0 + \int_0^t \left(S_u \varphi'(X_u) + \frac{1}{2}\varphi''(X_u) \right) \mathrm{d}u + \int_0^t \varphi'(X_u)\,\mathrm{d}W_u, \quad t \geqslant 0 \tag{1.1.10}$$

where the last term in the sum is an Ito's integral. Formula (1.1.10) is a representation for the process X of type (1.1.9) with

$$f_t = S_t \varphi'(X_t) + \frac{1}{2}\varphi''(X_t), \quad t \geqslant 0$$
$$M_t = \int_0^t \varphi'(X_u)\,\mathrm{d}W_u, \quad t \geqslant 0.$$

Example 1.1.4 (Markov processes with a generator)**.** Let X be a E-valued homogeneous $\mathbb{F}$-Markov process with infinitesimal generator $\mathcal{L}$ of domain $\mathcal{D}(\mathcal{L})$. Then applying Dynkin's formula we obtain that for any $\varphi \in \mathcal{D}(\mathcal{L})$ it holds

$$\varphi(X_t) = \varphi(X_0) + \int_0^t \mathcal{L}\varphi(X_s)\,\mathrm{d}s + M_t, \quad t \geqslant 0 \tag{1.1.11}$$

where $(M_t)_{t\geqslant 0}$ is a $\mathbb{F}$–martingale. The representation (1.1.11) is clearly of the form (1.1.9) and will be used in Chapter 2.

As previously stated, the first step in the innovations method consists in projecting the semi-martingale representation given in (1.1.9) on the observed filtration $\mathbb{Y}$. This is the content of the following Theorem.

Theorem 1.1.14 (Projection of the State [17, Ch. IV, Th. T1])**.** *Let $Z = (Z_t)_{t\geqslant 0}$ be an integrable real-valued process with semi-martingale representation given by*

$$Z_t = Z_0 + \int_0^t f_s \,\mathrm{d}s + M_t, \quad t \geqslant 0$$

where

(i) *$(f_t)_{t\geqslant 0}$ is a $\mathbb{F}$–progressive process such that*

$$\mathbb{E}\left[\int_0^t |f_s|\,\mathrm{d}s\right] < +\infty, \quad t \geqslant 0$$

(ii) *$(M_t)_{t\geqslant 0}$ is a zero mean $\mathbb{F}$–martingale.*

Then

$$\mathbb{E}[Z_t \mid \mathcal{Y}_t] = \mathbb{E}[Z_0 \mid \mathcal{Y}_0] + \int_0^t \hat{f}_s \,\mathrm{d}s + \hat{M}_t, \quad t \geqslant 0 \tag{1.1.12}$$

where

- $(\hat{M}_t)_{t\geqslant 0}$ *is a zero mean* $\mathbb{Y}$*–martingale,*
- $(\hat{f}_t)_{t\geqslant 0}$ *is a* $\mathbb{Y}$*–progressive process defined by*

$$\mathbb{E}\left[\int_0^t C_s f_s \,\mathrm{d}s\right] = \mathbb{E}\left[\int_0^t C_s \hat{f}_s \,\mathrm{d}s\right], \quad t \geqslant 0 \tag{1.1.13}$$

for all nonnegative bounded $\mathbb{Y}$*–progressive processes* $(C_t)_{t\geqslant 0}$.

Remark 1.1.2. It can be shown that a process $(\hat{f}_t)_{t\geqslant 0}$ satisfying (1.1.13) always exists (see e. g. [17, Remark (β), p. 88]). A more concrete version of it can be obtained whenever we are granted the existence of a version of the conditional expectation of f_t given $\mathcal{Y}_t$ such that the mapping $(\omega, t) \mapsto \mathbb{E}[f_t \mid \mathcal{Y}_t](\omega)$ is $\mathbb{Y}$–progressively measurable, for all $t \geqslant 0$. Then the process $\big(\hat{f}_t(\omega)\big)_{t\geqslant 0} = \big(\mathbb{E}[f_t \mid \mathcal{Y}_t](\omega)\big)_{t\geqslant 0}$ clearly satisfies (1.1.13), as an application of the Fubini-Tonelli theorem shows.

Filtering Equations

We now assume that the observed process Y is a $\mathbb{F}$–adapted E–marked point process $(T_n, \xi_n)_{n\in\mathbb{N}}$, with associated random counting measure μ. Until the end of this discussion, Assumption 1.1.2 will be in force and we suppose that μ admits the $\mathbb{F}$-local characteristics $\big(\lambda, \Phi_t(\mathrm{d}x)\big)$ and the $\mathbb{Y}$-local characteristics $\big(\hat{\lambda}, \hat{\Phi}_t(\mathrm{d}x)\big)$. For technical reasons, Dellacherie's usual conditions stated in Section 1.1.1 are assumed to hold for the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and for all the filtrations here specified.

Let $Z = (Z_t)_{t\geqslant 0}$ be a real-valued process satisfying the conditions stated in Theorem 1.1.14. We add the following hypothesis.

Assumption 1.1.4. The semi-martingale representation of Z is such that

(H1) $M_t = M_t^d + M_t^c, t \geqslant 0$, where $(M_t^d)_{t\geqslant 0}$ is a $\mathbb{F}$–martingale of integrable variation over finite intervals and $(M_t^c)_{t\geqslant 0}$ is a continuous $\mathbb{F}$–martingale.

(H2) $(Z_t - M_t^c)_{t\geqslant 0}$ is a bounded process.

We are now in a position to state the main result of this Subsection. In fact, recalling the Martingale Representation Theorem 1.1.11, we can express in a more precise way the $\mathbb{Y}$–martingale $(\hat{M}_t)_{t\geqslant 0}$ that figures in (1.1.12).

Theorem 1.1.15 (Filtering Theorem [17, Ch. VIII, Th. T9])**.** *Let the conditions stated in Theorem 1.1.14 and Assumption 1.1.4 hold. Then for all* $t \geqslant 0$ *and* $\mathbb{P}$*–a.s.*

$$\hat{Z}_t = \mathbb{E}[Z_t \mid \mathcal{Y}_t] = \mathbb{E}[Z_0 \mid \mathcal{Y}_0] + \int_0^t \hat{f}_s \,\mathrm{d}s + \\ + \int_{(0,t]\times E} K_s(x)\big[\mu(\mathrm{d}s\,\mathrm{d}x) - \hat{\lambda}_s \hat{\Phi}_s(\mathrm{d}x)\,\mathrm{d}s\big]. \tag{1.1.14}$$

The random field $\big(K_t(x)\big)_{t\geqslant 0, x\in E}$ *is* $\mathbb{Y}$*–predictable and is defined* $\mathbb{P}(\mathrm{d}\omega)\mu(\mathrm{d}t\,\mathrm{d}x)$*–essentially uniquely by*

$$K_t(x) = \Psi_t^1(x) - \Psi_t^2(x) + \Psi_t^3(x), \quad t \geqslant 0,\, x \in E. \tag{1.1.15}$$

The $\mathbb{Y}$–predictable random fields $\big(\Psi^i_t(x)\big)_{t\geqslant 0,\, x\in E}$, $i = 1,2,3$, are $\mathbb{P}(\mathrm{d}\omega)\,\mu(\mathrm{d}t\,\mathrm{d}x)$–essentially uniquely defined by the following equalities, holding for all $t \geqslant 0$ and all bounded $\mathbb{Y}$–predictable random fields $\big(C_t(x)\big)_{t\geqslant 0, x\in E}$

$$\begin{aligned}
\mathbb{E}\bigg[\int_0^t \int_E \Psi^1_s(x) C_s(x)\,\hat{\lambda}_s(\mathrm{d}x)\,\mathrm{d}s\bigg] &= \mathbb{E}\bigg[\int_0^t \int_E Z_s C_s(x)\,\lambda_s(\mathrm{d}x)\,\mathrm{d}s\bigg] \\
\mathbb{E}\bigg[\int_0^t \int_E \Psi^2_s(x) C_s(x)\,\hat{\lambda}_s(\mathrm{d}x)\,\mathrm{d}s\bigg] &= \mathbb{E}\bigg[\int_0^t \int_E Z_s C_s(x)\,\hat{\lambda}_s(\mathrm{d}x)\,\mathrm{d}s\bigg], \qquad (1.1.16) \\
\mathbb{E}\bigg[\int_0^t \int_E \Psi^3_s(x) C_s(x)\,\hat{\lambda}_s(\mathrm{d}x)\,\mathrm{d}s\bigg] &= \mathbb{E}\bigg[\int_{(0,t]\times E} \big[Z_s - Z_{s-}\big] C_s(x)\,\mu(\mathrm{d}s\,\mathrm{d}x)\bigg].
\end{aligned}$$

Remark 1.1.3. The existence of the random fields $\big(\Psi^i_t(x)\big)_{t\geqslant 0,\, x\in E}$, $i = 1,2,3$ and, in turn, of the random field $\big(K_t(x)\big)_{t\geqslant 0,\, x\in E}$, is granted since they are Radon-Nikodym derivatives. In fact:

1. The map $(\omega, t, x) \mapsto \Psi^1_t(\omega, x)$ is the Radon-Nikodym derivative of the measure $\mu^1_1(\mathrm{d}\omega\,\mathrm{d}t\,\mathrm{d}x)$ with respect to the measure $\mu^1_2(\mathrm{d}\omega\,\mathrm{d}t\,\mathrm{d}x)$, where

$$\begin{aligned}
\mu^1_1(\mathrm{d}\omega\,\mathrm{d}t\,\mathrm{d}x) &= \mathbb{P}(\mathrm{d}\omega)\, Z_t(\omega)\,\lambda_t(\omega, \mathrm{d}x)\,\mathrm{d}t, \\
\mu^1_2(\mathrm{d}\omega\,\mathrm{d}t\,\mathrm{d}x) &= \mathbb{P}(\mathrm{d}\omega)\,\hat{\lambda}_t(\omega, \mathrm{d}x)\,\mathrm{d}t.
\end{aligned}$$

   Both measures are defined on $\big(\Omega \times (0,\infty) \times E, \mathcal{P}(\mathbb{Y}) \otimes \mathcal{E}\big)$. The first one is a signed measure, is σ-finite since Z is bounded and is absolutely continuous with respect to the second one. Moreover, being a Radon-Nikodym derivative, the random field $\big(\Psi^1_t(x)\big)_{t\geqslant 0,\, x\in E}$ is $\mathbb{Y}$–predictable.

2. The map $(\omega, t, x) \mapsto \Psi^2_t(\omega, x)$ is the Radon-Nikodym derivative of the measure $\mu^2_1(\mathrm{d}\omega\,\mathrm{d}t\,\mathrm{d}x)$ with respect to the measure $\mu^2_2(\mathrm{d}\omega\,\mathrm{d}t\,\mathrm{d}x)$, where

$$\begin{aligned}
\mu^2_1(\mathrm{d}\omega\,\mathrm{d}t\,\mathrm{d}x) &= \mathbb{P}(\mathrm{d}\omega)\, Z_t(\omega)\,\hat{\lambda}_t(\omega, \mathrm{d}x)\,\mathrm{d}t, \\
\mu^2_2(\mathrm{d}\omega\,\mathrm{d}t\,\mathrm{d}x) &= \mathbb{P}(\mathrm{d}\omega)\,\hat{\lambda}_t(\omega, \mathrm{d}x)\,\mathrm{d}t.
\end{aligned}$$

   Similar considerations to the ones made for the random field $\big(\Psi^1_t(x)\big)_{t\geqslant 0,\, x\in E}$ apply to this process.

3. The map $(\omega, t, x) \mapsto \Psi^3_t(\omega, x)$ is the Radon-Nikodym derivative of the measure $\mu^3_1(\mathrm{d}\omega\,\mathrm{d}t\,\mathrm{d}x)$ with respect to the measure $\mu^3_2(\mathrm{d}\omega\,\mathrm{d}t\,\mathrm{d}x)$, where

$$\begin{aligned}
\mu^3_1(\mathrm{d}\omega\,\mathrm{d}t\,\mathrm{d}x) &= \mathbb{P}(\mathrm{d}\omega)\,\mathrm{d}Z_t(\omega)\,\mu(\mathrm{d}t\,\mathrm{d}x), \\
\mu^3_2(\mathrm{d}\omega\,\mathrm{d}t\,\mathrm{d}x) &= \mathbb{P}(\mathrm{d}\omega)\,\hat{\lambda}_t(\omega, \mathrm{d}x)\,\mathrm{d}t.
\end{aligned}$$

   Both measures are defined on $\big(\Omega \times (0,\infty) \times E, \mathcal{P}(\mathbb{Y}) \otimes \mathcal{E}\big)$. The first one is a signed measure, is σ-finite since Z, hence $(|Z_t - Z_{t-}|)_{t\geqslant 0}$ is bounded and is absolutely continuous with respect to the second one, because on the space of definition of these measures, $\mathbb{P}(\mathrm{d}\omega)\,\hat{\lambda}_t(\omega, \mathrm{d}x)\,\mathrm{d}t = \mathbb{P}(\mathrm{d}\omega)\,\mu(\mathrm{d}t\,\mathrm{d}x)$. The $\mathbb{Y}$–predictability of the random field $\big(\Psi^3_t(x)\big)_{t\geqslant 0, x\in E}$ comes from the same arguments applied to the previous points.

Remark 1.1.4. We end this Subsection with a consideration very useful when applying the filtering formula given in (1.1.14). The random field $\big(\Psi^2_t(x)\big)_{t\geqslant 0,\, x\in E}$ is $\mathbb{P}$–a.s. equal to the process $(\hat{Z}_{t-})_{t\geqslant 0}$. This can be easily obtain using the Fubini-Tonelli Theorem in the second relation of (1.1.16).

1.2 Piecewise Deterministic Markov Processes

In this Section we discuss *piecewise deterministic Markov processes*, or PDMP for short. These processes have been introduced by M.H.A. Davis and we will refer to the monograph [32] to present some general facts about them. The interested reader may also consult [42]. PDMPs have received a lot of attention in the literature, since they can efficiently describe a wide range of non-diffusive stochastic models. They have also been studied in connection with optimal control problems, starting from the work by D. Vermes [61], in which the author introduces the concept of *piecewise open-loop control*.

Here we give a synthetic introduction to PDMPs, limiting our statements only to those that will be referred to in the next Chapters. We also simplify the setting of [32], for sake of clarity.

1.2.1 Construction of a PDMP

As the name suggests, piecewise deterministic Markov processes are Markovian processes that evolve deterministically between some random jump times.

To start constructing a PDMP, we specify first the state space. Let E^0 be an open subset of $\mathbb{R}^d$, $d \in \mathbb{N}$. We are given a locally Lipschitz continuous vector field $F\colon E^0 \to E^0$, determining a flow $\phi(t, x)$, where $t \geqslant 0$ is the time variable and $x \in E^0$ is the starting point of the PDMP. We will often write $\phi(t, x) = \phi_x(t)$.

Let us denote by $\partial E^0 \coloneqq \operatorname{cl} E^0 \setminus E^0$ the boundary of E^0 and define

$$t_\star(x) \coloneqq \inf\{t > 0 \colon \phi_x(t) \in \partial E^0\}, \quad x \in E^0$$

under the usual convention $\inf \emptyset \coloneqq +\infty$. It can be shown (see [32, Lemma 27.1]) that $t_\star\colon E^0 \to [0, +\infty]$ is a Borel-measurable function. We also make the following assumption, that excludes the possibility of "explosions" of the flow.

Assumption 1.2.1. Denote by $t_\infty(x)$ the explosion time of the flow $\phi_x(t)$, $x \in E$. We assume that if $t_\star(x) = +\infty$, then $t_\infty(x) = +\infty$.

Next, define the following two sets

$$\partial^\pm E^0 \coloneqq \{z \in \partial E^0 \colon z = \phi_x(\pm t), \text{ for some } x \in E^0 \text{ and } t > 0\}.$$

The set $\partial^+ E^0$ (resp. $\partial^- E^0$) is the set of points of the boundary of E^0 that can be reached forwards (resp. backwards) by the flow, starting from some point $x \in E^0$.

Finally, we set $E \coloneqq E^0 \cup (\partial^- E^0 \setminus \partial^+ E^0)$ and, to ease the notation, $\Gamma^\star = \partial^+ E^0$. The set E is the state space of our PDMP. We denote by $\mathcal{E} \coloneqq \mathcal{B}(E)$ its Borel σ-algebra. It is worth noticing that the measurable space $(E, \mathcal{E})$ is a Borel-space, i. e. a Borel subset of a complete metric space (cfr. [13]). The set $\Gamma^\star$ represents the boundary of our PDMP.

Remark 1.2.1. Choosing a state space for a PDMP as the one above is not the only available option. It is possible to choose E as a closed subset of $\mathbb{R}^d$, $d \in \mathbb{N}$ (as did, e. g. , in [39]), as a differentiable manifold (as said in [32, Par. 24]) or even an infinite dimensional set (as in [55]). We will see such different settings also in the following Chapters.

The vector field F determines the behavior of the PDMP between two consecutive jump times. The distribution of jump times is governed by a *jump rate function* $r\colon E \to$

$[0, +\infty)$. This is a measurable function and we assume that for each $x \in E$ there exists $\varepsilon(x) > 0$ such that the following integrability condition holds

$$\int_0^{\varepsilon(x)} r\big(\phi_x(t)\big)\,\mathrm{d}t < +\infty.$$

The post-jump location of the PDMP is given by a *transition probability* $R\colon E \cup \Gamma^\star \to \mathcal{P}(E)$, such that

1. for each $A \in \mathcal{E}$ the map $x \mapsto R(x, A)$ is measurable,
2. $R(x, \{x\}) = 0$.

Notice that this transition probability prescribes also jumps from the boundary $\Gamma^\star$ of our PDMP. The functions in the triple (F, r, R) give the *local characteristics* of the PDMP.

We are now ready to explicitly construct a PDMP, that we indicate by $X(\omega) = \big(X_t(\omega)\big)_{t \geqslant 0}$, $\omega \in \Omega$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space defined as follows.

- $\Omega := [0,1]^{\mathbb{N}}$.
- $\mathcal{F} := \mathcal{A}^{\otimes \mathbb{N}}$, where $\mathcal{A}$ is the Lebesgue σ-algebra on $[0,1]$.
- $\mathbb{P} := \lambda^{\otimes \mathbb{N}}$, where λ is the Lebesgue measure on $([0,1], \mathcal{A})$.

This is the canonical space for a countable sequence of independent random variables uniformly distributed on $[0,1]$, that we denote by $(U_n)_{n\in\mathbb{N}}$.

Let us fix a starting point $x \in E$ and define the survivor function of the first jump time T_1 of the PDMP as

$$G(t, x) := \mathbb{1}_{t < t_\star(x)} \exp\left\{-\int_0^t r\big(\phi_x(s)\big)\,\mathrm{d}s\right\}, \quad t \geqslant 0.$$

We set its generalized inverse as $\psi(u, x) := \inf\{t \geqslant 0\colon G(t, x) \leqslant u\}$ (again, under the assumption $\inf \emptyset := +\infty$) and define the random variables $S_1(\omega) := T_1(\omega) := \psi(U_1(\omega), x)$. It is clear that, by definition, $\mathbb{P}(T_1 > t) = G(t, x)$, $t \geqslant 0$. To define the post-jump location of the PDMP after the first jump time T_1, let $\Psi\colon [0,1] \times E \cup \Gamma^\star \to E$ be a measurable function such that $\lambda(\{u \in [0,1]\colon \Psi(u, z) \in A\}) = R(z, A)$, $z \in E \cup \Gamma^\star$, $A \in \mathcal{E}$, where we recall that λ is the Lebesgue measure on $([0,1], \mathcal{A})$. It can be shown that such a function exists (see [32, Par. 24]).

The sample path $X(\omega)$, $\omega \in \Omega$, of our PDMP up the first jump time T_1 is defined as follows

$$X_t(\omega) := \begin{cases} \phi_x(t), & t \geqslant 0, \text{ if } T_1(\omega) = +\infty \\ \phi_x(t), & t \in [0, T_1(\omega)), \text{ if } T_1(\omega) < +\infty \end{cases}.$$

If $\omega \in \Omega$ is such that $T_1(\omega) < +\infty$ we set $X_{T_1(\omega)}(\omega) := \Psi\big(U_2(\omega), \phi_x\big(T_1(\omega)\big)\big)$. Notice that, given the definition of the transition probability R, we have that our PDMP immediately jumps away from the boundary $\Gamma^\star$ into E whenever the flow reaches $\Gamma^\star$.

Now our PDMP restarts from the post-jump location following the same recipe. With this we mean that, for those $\omega \in \Omega$ such that $T_1(\omega) < +\infty$ we define $S_2(\omega) := \psi\big(U_3(\omega), X_{T_1(\omega)}(\omega)\big)$ and $T_2(\omega) := T_1(\omega) + S_2(\omega)$. Then, the sample path $X(\omega)$ of our PDMP up the second jump time $T_2(\omega)$ is given by

$$X_t(\omega) := \begin{cases} \phi_{X_{T_1(\omega)}(\omega)}\big(t - T_1(\omega)\big), & t \geqslant T_1(\omega), \text{ if } T_2(\omega) = +\infty \\ \phi_{X_{T_1(\omega)}(\omega)}\big(t - T_1(\omega)\big), & t \in [T_1(\omega), T_2(\omega)), \text{ if } T_2(\omega) < +\infty \end{cases}$$

and the post-jump location after $T_2(\omega)$, if $T_2(\omega) < +\infty$, is specified by $X_{T_2(\omega)}(\omega) := \Psi\big(U_4(\omega), \phi_{X_{T_1(\omega)}(\omega)}\big(T_2(\omega) - T_1(\omega)\big)\big)$. And so on.

Notice that with this construction we implicitly defined a map from Ω into the set of all possible marked point processes with mark space E, i. e.

$$\Omega \ni \omega \mapsto \{T_1(\omega), \xi_1(\omega), \dots, T_{\bar{k}-1}(\omega), \xi_{\bar{k}-1}(\omega), T_{\bar{k}}(\omega)\}$$

where $\bar{k} = \bar{k}(\omega) := \min\{k \in \bar{\mathbb{N}} \colon T_k(\omega) = +\infty\}$ and $\xi_n(\omega) := X_{T_n(\omega)}(\omega)$, $\mathbb{N} \ni n < \bar{k}$. Therefore, we can always associate a MPP to a PDMP that, together with the flow, completely specifies it.

On the MPP associated to our PDMP we make the following assumption.

Assumption 1.2.2. Let $N = (N_t)_{t \geqslant 0}$ denote the counting process associated to a PDMP, i. e.

$$N_t(\omega) := \sum_{n \in \mathbb{N}} \mathbb{1}_{T_n(\omega) \leqslant t}, \quad t \geqslant 0,\ \omega \in \Omega.$$

We assume that for every starting point $x \in E$ of the PDMP we have that $\mathbb{E} N_t < +\infty$ for all $t \geqslant 0$. In other words, we require that the simple point process $(T_n)_{n \in \mathbb{N}}$ is $\mathbb{P}$–a.s. non-explosive (cfr. Subsection 1.1.1).

Although this assumption is usually satisfied in applications, it is difficult to provide general conditions under which it holds. This is due to the complicated interaction between the three components (F, r, R) of the local characteristics of a PDMP and the geometry of the boundary, as the following simple example shows.

Example 1.2.1 ([32, Example 24.5])**.** Let us define

$$E = [0,1) \times [0,2], \quad \Gamma^\star = \{1\} \times [0,2],$$
$$F \equiv (1,0), \quad r \equiv 0, \quad R\big((x,y), A\big) = \delta_{(1-y/2, y/2)}(A),\ (x,y) \in \Gamma^\star.$$

Notice that the flow, starting from any point $z = (x,y) \in E$ is equal to $\phi_z(t) = (x+t, y)$. Suppose that the PDMP starts from $z = (0,1)$. Then we have that $T_1 = 1$, $T_2 = 1 + \frac{1}{2}$, $T_3 = 1 + \frac{3}{4}$, hence $T_\infty = \lim_{n \to \infty} T_n = 2$.

This example shows that explosion of the MPP associated to a PDMP is possible even when the jump rate function r is null, if the sequence $\big(t_\star(X_{T_n})\big)_{n \in \mathbb{N}}$ is not bounded away from zero.

A pair of useful condition guaranteeing that Assumption 1.2.2 is satisfied is given in the following Proposition.

Proposition 1.2.1 ([32, Prop. 24.6])**.** *Suppose that for all $x \in E$, $r(x) \leqslant c$ for some real constant $c > 0$. Then Assumption 1.2.2 holds if at least one of the following facts is satisfied.*

1. *There are no "active" boundaries, i. e. $t_\star(x) = +\infty$ for all $x \in E$.*
2. *For some $\varepsilon > 0$, $R(x, A_\varepsilon) = 1$ for all $x \in \Gamma^\star$, where $A_\varepsilon := \{x \in E \colon t_\star(x) \geqslant \varepsilon\}$.*

As a final comment to this Subsection, let us specify that to identify a process as a PDMP it is not necessary to trace step by step the construction previously shown. As we said earlier, a PDMP is completely determined by the flow and the law of its associated MPP, which in turn is given by the jump rate function and the transition probability. Hence, whatever $(\Omega, \mathcal{F}, \mathbb{P})$ is the probability space on which X is defined, one just need to prove that

- X is a (strong) Markov process on $(\Omega, \mathcal{F}, \mathbb{P})$ with respect to some filtration $(\mathcal{F}_t)_{t\geqslant 0}$.
- X has càdlàg paths, jumping at some random times $(T_n)_{n\in\mathbb{N}}$ and satisfying for all $n \in \mathbb{N}_0$ (set $T_0 \equiv 0$)

$$X_t = \phi_{X_{T_n}}(t - T_n), \quad t \in [T_n, T_{n+1}), \ \mathbb{P}\text{–a.s. on } \{T_n < +\infty\}.$$

$$\begin{aligned}\mathbb{P}(T_{n+1} - T_n > t, \, T_n < +\infty \mid \mathcal{F}_{T_n}) \\ = \mathbb{1}_{T_n<+\infty} \exp\left\{-\int_0^t r\big(\phi_{X_{T_n}}(s)\big)\,\mathrm{d}s\right\}, \quad t \geqslant 0.\end{aligned}$$

$$\begin{aligned}\mathbb{P}(X_{T_{n+1}} \in A, \, T_{n+1} < +\infty \mid \mathcal{F}_{T_{n+1}}) \\ = \mathbb{1}_{T_{n+1}<+\infty} R\big(\phi_{X_{T_n}}(T_{n+1} - T_n), A\big), \quad A \in \mathcal{E}.\end{aligned}$$

1.2.2 The strong Markov property

A fundamental property regarding stochastic processes is undoubtedly the strong Markov property. In this brief Subsection we will state a result confirming that the Piecewise Deterministic Process constructed in Subsection 1.2.1 satisfies it. However, to do so we need to see our process as defined on a suitable canonical space. We warn the reader that such a procedure will be extensively used in Chapters 3 and 4.

Let $\bar{\Omega} = \{\bar{\omega} \colon [0, +\infty) \to E, \text{ cádlág}\}$ denote the canonical space for E–valued piecewise deterministic processes. We define the coordinate mapping $\bar{X}_t(\bar{\omega}) = \bar{\omega}(t)$, for $\bar{\omega} \in \bar{\Omega}$, $t \geqslant 0$.

Let $(\bar{\mathcal{F}}_t^\circ)_{t\geqslant 0}$ denote the natural filtration of our PDP, i. e.

$$\bar{\mathcal{F}}_t^\circ \coloneqq \sigma(\bar{X}_s, 0 \leqslant s \leqslant t), \quad \bar{\mathcal{F}}^\circ \coloneqq \sigma(\bar{X}_s, s \geqslant 0).$$

Let X be the PDP constructed on $(\Omega, \mathcal{F}, \mathbb{P})$ in Subsection 1.2.1. Under Assumption 1.2.2, such a construction defines for each starting point $x \in E$ a measurable mapping $\psi_x \colon \Omega \to \bar{\Omega}$ such that $\bar{X}_t\big(\psi_x(\omega)\big) = X_t(\omega)$. We denote by $\bar{\mathrm{P}}_x \coloneqq \mathbb{P} \circ \psi_x^{-1}$, $x \in E$ the image measure of $\mathbb{P}$ under ψ_x. This provides a family of measures $(\bar{\mathrm{P}}_x)_{x\in E}$ on $(\bar{\Omega}, \bar{\mathcal{F}}^\circ)$ and our PDP can be also thought of as a Markov family defined on $\bar{\Omega}$, i. e. as $(\bar{\Omega}, \bar{\mathcal{F}}^\circ, (\bar{\mathcal{F}}_t^\circ)_{t\geqslant 0}, (\bar{X}_t)_{t\geqslant 0}, (\bar{\mathrm{P}}_x)_{x\in E})$.

For any probability measure Q on $(E, \mathcal{E})$ indicate by $\bar{\mathrm{P}}_Q$ the following probability measure on $(\bar{\Omega}, \bar{\mathcal{F}}^\circ)$

$$\bar{\mathrm{P}}_Q(A) \coloneqq \int_E \bar{\mathrm{P}}_x(A)\, Q(\mathrm{d}x), \quad A \in \mathcal{E}.$$

Finally, let $\bar{\mathcal{F}}^Q$ be the $\bar{\mathrm{P}}_Q$-completion of $\bar{\mathcal{F}}^\circ$ (we still denote by $\bar{\mathrm{P}}_Q$ the measure naturally extended to this new σ-algebra) and, indicating by $\bar{\mathcal{Z}}^Q$ the family of sets in $\bar{\mathcal{F}}^Q$ with zero $\bar{\mathrm{P}}_Q$-probability, define

$$\bar{\mathcal{F}}_t^Q \coloneqq \sigma(\bar{\mathcal{F}}_t^\circ \cup \bar{\mathcal{Z}}^Q), \quad \bar{\mathcal{F}}_t \coloneqq \bigcap_{Q\in\mathcal{P}(E)} \bar{\mathcal{F}}_t^Q, \quad t \geqslant 0.$$

$(\bar{\mathcal{F}}_t)_{t\geqslant 0}$ is called the *natural completed filtration* of $\bar{X}$. It satisfies the following important property.

Theorem 1.2.2 ([32, Th. 25.3]). *The natural completed filtration $(\bar{\mathcal{F}}_t)_{t\geqslant 0}$ of $\bar{X}$ is right continuous, i. e. for any $t > 0$ we have that*

$$\bar{\mathcal{F}}_t = \bar{\mathcal{F}}_{t+} \coloneqq \bigcap_{\varepsilon>0} \bar{\mathcal{F}}_{t+\varepsilon}.$$

We are now ready to state the main result of this Subsection. For any bounded and measurable function $\varphi\colon E \to \mathbb{R}$, define $P_t\varphi(x) \coloneqq \bar{\mathrm{E}}_x\varphi(\bar{X}_t)$, where $\bar{\mathrm{E}}_x$ denotes expectation with respect to the probability measure $\bar{\mathrm{P}}_x$.

Theorem 1.2.3 ([32, Th. 25.5]). *The process $\bar{X}$ is a homogeneous strong Markov process, i. e. for any $x \in E$, any $(\bar{\mathcal{F}}_t)_{t\geqslant 0}$–stopping time T and any bounded measurable function $\varphi\colon E \to \mathbb{R}$ we have that*

$$\bar{\mathrm{E}}_x[\varphi(\bar{X}_{T+s})\mathbb{1}_{T<+\infty} \mid \bar{\mathcal{F}}_T] = P_s\varphi(\bar{X}_T)\mathbb{1}_{T<+\infty}, \quad s \geqslant 0,\ \bar{\mathrm{P}}_x\text{–}a.s.$$

CHAPTER 2

The filtering problem

In this Chapter we study the filtering problem connected to the optimal control problem with partial observation that we intend to analyze in Chapters 3 and 4. To simplify matters, in this Chapter we introduce the setting and prove the results of the filtering problem in the uncontrolled case. We will see in the next Chapters that, thanks to some measurability properties of controls, we will be able to reproduce these results also in the controlled case (and exactly in the same way).

We briefly recall some basic aspects of stochastic filtering. Let us fix two complete and separable metric spaces I and O equipped with their respective Borel σ-algebras $\mathcal{I}$ and $\mathcal{O}$. The basic datum of a stochastic filtering problem consists in a couple of stochastic processes $(X, Y) = (X_t, Y_t)_{t \geqslant 0}$, defined on some complete probability space $(\Omega,\, \mathcal{F},\, \mathbb{P})$. The process X, called *unobserved* or *signal process*, takes values in the set I, while the process Y, called *observed* or *data process* takes values in the set O. The aim is to find a $\mathcal{P}(I)$-valued process $\pi = (\pi_t)_{t \geqslant 0}$ such that for all $t \geqslant 0$ and all $\varphi \in \mathrm{B}_b(I)$

$$\int_I \varphi(x)\pi_t(\mathrm{d}x) = \mathbb{E}\left[\varphi(X_t) \mid \mathcal{Y}_t\right], \quad \mathbb{P}-\text{a.s.} \tag{2.0.1}$$

where $(\mathcal{Y}_t)_{t \geqslant 0}$ is the natural completed filtration of the observed process Y. That is, we are looking for a probabilistic estimate of the unobserved state (or of a measurable function of it) given the observation provided by Y. For sake of brevity we will often write

$$\pi_t(\varphi) \coloneqq \int_I \varphi(x)\pi_t(\mathrm{d}x).$$

The key tool used in this Chapter is given by *Marked Point Processes*. Thanks to these processes, whose main properties are recalled in Section 1.1, we are able to use known results on filtering with point process observations and to deduce in Section 2.1 the filtering equation, i. e. the evolution equation satisfied by the filtering process π. With this result at our disposal, in Section 2.2 we characterize the filtering process as a *Piecewise Deterministic Markov Process* (see Section 1.2 for a recap on PDMPs), a fact that will be fundamental to study the optimal control problem in Chapters 3 and 4. In Section 2.3 we introduce the notation and recall the main results (already known from [25]) on the filtering problem for a Markov chain with noise-free observation. Finally, in Section 2.4 we provide some remarks on the rôle of the function h in the filtering problem.

Setting of the filtering problem

We now introduce the setting of the filtering problem discussed in this thesis. Let us fix a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The I-valued unobserved process defined on this space is a continuous time homogeneous pure jump Markov process. We are given the law μ of X_0. We can equivalently describe it (see e. g. [47]) by recording its jump times and jump locations, i. e. by defining for all $n \in \mathbb{N}_0$ the random variables $T_n \colon \Omega \to [0, +\infty]$ and $\xi_n \colon \Omega \to I$, where for each $n \in \mathbb{N}$

$$T_0(\omega) := 0 \qquad T_n(\omega) := \inf\{t > T_{n-1}(\omega) \colon X_t(\omega) \neq X_{T_{n-1}(\omega)}(\omega)\}. \quad (2.0.2)$$

$$\xi_0(\omega) := X_0(\omega) \quad \xi_n(\omega) := X_{T_n(\omega)}(\omega). \quad (2.0.3)$$

We denote by $\mathbb{X} := (\mathcal{X}_t)_{t \geqslant 0}$ the natural completed filtration associated to the unobserved process, where $\mathcal{X}_t := \sigma(X_s \colon 0 \leqslant s \leqslant t) \vee \mathcal{N}$ for each $t \geqslant 0$ and $\mathcal{N}$ is the collection of $\mathbb{P}$–null sets in $\mathcal{F}$. Its dynamics are described by a *rate transition measure* λ, i. e. a transition kernel from $(I, \mathcal{I})$ into itself such that for all $n \in \mathbb{N}_0$ and all $t \geqslant 0$

$$\mathbb{P}(T_{n+1} - T_n > t,\, \xi_{n+1} \in A \mid \mathcal{X}_{T_n}) = \frac{\lambda(\xi_n, A)}{\lambda(\xi_n, I)} e^{-\lambda(\xi_n, I)t}, \quad \mathbb{P}\text{–a.s.} \quad (2.0.4)$$

To have a more synthetic notation it is convenient to define the *jump rate function* $\lambda \colon I \to [0, +\infty)$ as

$$\lambda(x) := \lambda(x, I), \quad x \in I.$$

We point out that no confusion shall arise from the notation λ used both for the transition measure and the jump rate function. It will always be clear from the context to which object we will be referring to.

The following assumption will be in force throughout this section and ensures some important facts about the process X that we will recall later on.

Assumption 2.0.1. The jump rate function λ satisfies

$$\sup_{x \in I} \lambda(x) < +\infty. \quad (2.0.5)$$

Remark 2.0.1. It should be noted that the definition of rate transition measure given in (2.0.4) implies that $\lambda(x, \{x\}) = 0$ for all $x \in I$. This is evident by looking at the definition of the jump times appearing in (2.0.2).

We assume that the O-valued observed process is a function of the signal process via a given measurable function $h \colon I \to O$, i. e. it satisfies

$$Y_t(\omega) = h\big(X_t(\omega)\big), \quad \omega \in \Omega,\, t \geqslant 0. \quad (2.0.6)$$

This means that in our setting the observation is *noise-free*: the only source of randomness is represented by the unobserved process and no exogenous noise is acting on the observation. Of course, we exclude the cases where the function h is one to one or constant, being the first case trivial and the second one of no interest in the context of a filtering problem. We will provide some remarks on these special cases in Section

2.4. We assume, without loss of generality, that this function is surjective. It is straightforward to notice that also in this case we can equivalently describe the process Y by defining for all $n \in \mathbb{N}_0$ the random variables $\tau_n \colon \Omega \to [0, +\infty]$ and $\eta_n \colon \Omega \to O$, where for each $n \in \mathbb{N}$

$$\tau_0(\omega) := 0 \qquad \tau_n(\omega) := \inf\{t > \tau_{n-1}(\omega) \colon Y_t(\omega) \neq Y_{\tau_{n-1}(\omega)}(\omega)\}. \tag{2.0.7}$$

$$\eta_0(\omega) := Y_0(\omega) \qquad \eta_n(\omega) := Y_{\tau_n(\omega)}(\omega). \tag{2.0.8}$$

We denote by $\mathbb{Y} := (\mathcal{Y}_t)_{t \geqslant 0}$ the natural completed filtration associated to the observed process, where $\mathcal{Y}_t := \sigma(Y_s \colon 0 \leqslant s \leqslant t) \vee \mathcal{N}$ for each $t \geqslant 0$. Finally, we define the *explosion points* of the processes X and Y as the random variables

$$T_\infty(\omega) := \lim_{n \to \infty} T_n(\omega) \qquad \tau_\infty(\omega) := \lim_{n \to \infty} \tau_n(\omega). \tag{2.0.9}$$

2.1 The filtering equation

To tackle the noise-free filtering problem described previously, we will adopt an innovations approach (see Subsection 1.1.3 for more details), basing our analysis on the fact that we can represent (X, Y) as a pair of Marked Point Processes (or MPPs for short). These processes are countable collections of pairs of random variables, describing the occurrence of some random phenomena by recording the time of these events and a related mark. The main features of MPPs are recalled in Section 1.1 and in this Section we will use some results contained therein.

It is immediate to see that the pairs $(T_n, \xi_n)_{n \in \mathbb{N}}$ and $(\tau_n, \eta_n)_{n \in \mathbb{N}}$ are MPPs. Moreover, thanks to Assumption 2.0.1 they are $\mathbb{P}$–a.s. non-explosive, i. e. we have that $\mathbb{P}$-a.s.

$$T_\infty(\omega) = +\infty \qquad \tau_\infty(\omega) = +\infty. \tag{2.1.1}$$

Together with the initial conditions $\xi_0 = X_0$ and $\eta_0 = Y_0$, they describe completely the unobserved process X and the observed process Y respectively, so when speaking of the signal or data process we can equivalently use the jump process formulation or the MPP one. A third useful description of these two processes is possible. Let us define the following *random counting measures* (or RCMs for short)

$$n(\omega, \mathrm{d}t\,\mathrm{d}z) := \sum_{n \in \mathbb{N}} \delta_{\left(T_n(\omega),\, \xi_n(\omega)\right)}(\mathrm{d}t\,\mathrm{d}z) \mathbb{1}_{\{T_n(\omega) < +\infty\}}, \quad \omega \in \Omega \tag{2.1.2}$$

$$m(\omega, \mathrm{d}t\,\mathrm{d}y) := \sum_{n \in \mathbb{N}} \delta_{\left(\tau_n(\omega),\, \eta_n(\omega)\right)}(\mathrm{d}t\,\mathrm{d}y) \mathbb{1}_{\{\tau_n(\omega) < +\infty\}}, \quad \omega \in \Omega \tag{2.1.3}$$

where, for any arbitrary point a in some measurable space, δ_a denotes the Dirac probability measure concentrated at a. For each fixed $\omega \in \Omega$, n is a measure on $\big((0, +\infty) \times I, \mathcal{B}\big((0, +\infty)\big) \otimes \mathcal{I}\big)$ and is associated to the unobserved process X, while m is a measure on $\big((0, +\infty) \times O, \mathcal{B}\big((0, +\infty)\big) \otimes \mathcal{O}\big)$ and is associated to the observed process Y. Random counting measures are particularly useful in connection with their *dual predictable projections* (see e. g. [43, Th. 2.1] for more details), also called *compensators*.

It is a known fact that the $\mathbb{X}$–compensator of the RCM n is given by the predictable random measure $\lambda(X_{t-}(\omega), \mathrm{d}z)\,\mathrm{d}t$, since it is associated to a pure jump process with known rate transition measure. We will denote by $\tilde{n}$ the corresponding *compensated random measure*, i. e.

$$\tilde{n}(\omega, \mathrm{d}t\,\mathrm{d}z) := n(\omega, \mathrm{d}t\,\mathrm{d}z) - \lambda(X_{t-}(\omega),\, \mathrm{d}z)\,\mathrm{d}t. \tag{2.1.4}$$

It is important to recall that for all $A \in \mathcal{I}$ the compensated process $\tilde{n}\big((0,t] \times A\big)$ is a $\mathbb{X}$–martingale.

A preliminary step to the solution of our filtering problem consists in computing the $\mathbb{X}$– and $\mathbb{Y}$–compensators of the RCM m.

Lemma 2.1.1. *The $\mathbb{X}$– and $\mathbb{Y}$–dual predictable projections of the random counting measure m are respectively given by the predictable random measures $\mu_t(\omega, \mathrm{d}y)\,\mathrm{d}t$ and $\hat{\mu}_t(\omega, \mathrm{d}y)\,\mathrm{d}t$, where for all $B \in \mathcal{O}$ and $t > 0$*

$$\mu_t(\omega, B) := \lambda\big(X_{t-}(\omega),\, h^{-1}(B \setminus \{Y_{t-}(\omega)\})\big) \tag{2.1.5}$$

$$\hat{\mu}_t(\omega, B) := \int_I \lambda\big(x,\, h^{-1}(B \setminus \{Y_{t-}(\omega)\})\big)\, \pi_{t-}(\omega\,;\,\mathrm{d}x) \tag{2.1.6}$$

and $\pi = (\pi_t)_{t \geqslant 0}$ is the filtering process defined in (2.0.1).

Proof. Let $B \in \mathcal{O}$ and $t > 0$ be fixed. Then we can write

$$\begin{aligned} m\big(\omega, (0,t] \times B\big) &= \sum_{0<s\leqslant t} \mathbb{1}_{\{Y_s \neq Y_{s-},\, Y_s \in B\}}(\omega) \\ &= \sum_{0<s\leqslant t} \mathbb{1}_{\{X_s \in h^{-1}(Y_{s-})^c,\, X_s \in h^{-1}(B)\}}(\omega) \\ &= \int_{(0,t]\times I} \mathbb{1}_{h^{-1}(B\setminus\{Y_{s-}(\omega)\})}(z)\, n(\omega, \mathrm{d}s\,\mathrm{d}z). \end{aligned}$$

The field $\big(\mathbb{1}_{h^{-1}(B\setminus\{Y_{s-}(\omega)\})}(z)\big)_{s\in(0,t],\, z\in I}$ is bounded and $\mathbb{X}$–predictable and under Assumption 2.0.1 we have that

$$\mathbb{E}\int_{(0,t]\times I} \mathbb{1}_{h^{-1}(B\setminus\{Y_{s-}\})}(z)\, \lambda(X_{s-}, \mathrm{d}z)\,\mathrm{d}s = \mathbb{E}\int_{(0,t]} \lambda\big(X_{s-}, h^{-1}(B \setminus \{Y_{s-}\})\big)\,\mathrm{d}s < \infty.$$

Therefore, by Proposition 1.1.10

$$\int_{(0,t]\times I} \mathbb{1}_{h^{-1}(B\setminus\{Y_{s-}\})}(z)\, \tilde{n}(\mathrm{d}s\,\mathrm{d}z) = m\big((0,t]\times B\big) - \int_{(0,t]} \lambda(X_{s-},\, h^{-1}(B\setminus\{Y_{s-}\}))\,\mathrm{d}s$$

is a $\mathbb{X}$–local martingale. The first part of the claim follows from number 3 of Proposition 1.1.5 by noticing that

$$\left(\int_{(0,t]\times B} \mu_s(\mathrm{d}z)\,\mathrm{d}s\right)_{t\geqslant 0} = \left(\int_{(0,t]} \lambda(X_{s-},\, h^{-1}(B \setminus \{Y_{s-}\}))\,\mathrm{d}s\right)_{t\geqslant 0}$$

is a $\mathbb{X}$–predictable process for all $B \in \mathcal{O}$.

What we have just shown is that for all $\mathbb{X}$–predictable and non-negative random fields $C = (C_t(z))_{t\geqslant 0,\, z\in I}$ it holds

$$\mathbb{E}\int_{(0,+\infty)\times I} C_s(z) m(\mathrm{d}s\,\mathrm{d}z) = \mathbb{E}\int_{(0,+\infty)\times I} C_s(z)\mu_s(\mathrm{d}z)\,\mathrm{d}s.$$

To prove the second part of the claim we have to show that the same kind equality holds for all $\mathbb{Y}$–predictable and non-negative random fields C, with $\hat{\mu}$ replacing μ on the right hand side. Let a $\mathbb{Y}$–predictable and non-negative random field C be fixed and write

$$\begin{aligned}
&\mathbb{E}\int_{(0,+\infty)\times I} C_s(z)\mu_s(\mathrm{d}z)\,\mathrm{d}s = \mathbb{E}\int_{(0,+\infty)}\int_I C_s(z)\mathbb{1}_{h^{-1}(Y_{s-})^c}(z)\lambda(X_{s-},\,\mathrm{d}z)\,\mathrm{d}s \\
&= \int_{(0,+\infty)} \mathbb{E}\Big[\mathbb{E}\Big[\int_I C_s(z)\mathbb{1}_{h^{-1}(Y_{s-})^c}(z)\lambda(X_s,\,\mathrm{d}z)\mid \mathcal{Y}_s\Big]\Big]\,\mathrm{d}s \\
&= \int_{(0,+\infty)} \mathbb{E}\int_I\int_I C_s(z)\mathbb{1}_{h^{-1}(Y_{s-})^c}(z)\lambda(x,\,\mathrm{d}z)\,\pi_s(\mathrm{d}x)\,\mathrm{d}s \\
&= \mathbb{E}\int_{(0,+\infty)}\int_I\int_I C_s(z)\mathbb{1}_{h^{-1}(Y_{s-})^c}(z)\lambda(x,\,\mathrm{d}z)\,\pi_{s-}(\mathrm{d}x)\,\mathrm{d}s \\
&= \mathbb{E}\int_{(0,+\infty)\times I} C_s(z)\hat{\mu}_s(\mathrm{d}z)\,\mathrm{d}s
\end{aligned}$$

where the passages from the first to the second and from the third to the fourth line are justified by repeatedly using the Fubini-Tonelli theorem and the fact that $X_{s-} = X_s$ and $\pi_s = \pi_{s-}$, $\mathrm{d}s$–a.e., while the passage from the second to the third line is due to the freezing lemma.

Now, thanks to number 1 of Proposition 1.1.5 it suffices to take $C_s(z) = C_s\mathbb{1}_B(z)$, for any fixed $B \in \mathcal{O}$, with $(C_t)_{t\geqslant 0}$ a $\mathbb{Y}$–predictable process. Then we get that

$$\mathbb{E}\int_{(0,+\infty)} C_s m(\mathrm{d}s\times B) = \mathbb{E}\int_{(0,+\infty)} C_s\hat{\mu}_s(B)\,\mathrm{d}s$$

holds for all $\mathbb{Y}$–predictable processes $(C_t)_{t\geqslant 0}$, i. e. $\left(\int_{(0,t]} C_s\hat{\mu}_s(B)\,\mathrm{d}s\right)_{t\geqslant 0}$ is the $\mathbb{Y}$–dual predictable projection of $m\big((0,t]\times B\big)$ for all $B\in\mathcal{O}$. The claim follows, being clearly $\hat{\mu}_t(\omega,\mathrm{d}y)\,\mathrm{d}t$ a $\mathbb{Y}$–predictable random measure. □

Remark 2.1.1. To be more precise we should have defined the $\mathbb{X}$– and $\mathbb{Y}$–compensators of the RCM m as

$$\begin{aligned}
\mu_t(\omega,B)\,\mathrm{d}t &= \mathbb{1}_{\big(0,\tau_\infty(\omega)\big)}(t)\lambda\big(X_{t-}(\omega),\,h^{-1}(B\setminus\{Y_{t-}(\omega)\})\big)\,\mathrm{d}t \\
\hat{\mu}_t(\omega,B)\,\mathrm{d}t &= \mathbb{1}_{\big(0,\tau_\infty(\omega)\big)}(t)\int_I \lambda\big(x,\,h^{-1}(B\setminus\{Y_{t-}(\omega)\})\big)\,\pi_{t-}(\omega\,;\,\mathrm{d}x)\,\mathrm{d}t
\end{aligned}$$

so that the two random measures would have satisfied (1.1.3). However, these measures coincide $\mathbb{P}$–a.s. with the corresponding ones defined in (2.1.5)–(2.1.6), since the MPP $(\tau_n,\eta_n)_{n\in\mathbb{N}}$ is $\mathbb{P}$–a.s. non-explosive, thanks to Assumption 2.0.1. For this reason we will adopt the simpler notation of (2.1.5)–(2.1.6).

Another important step to solve the filtering problem is to represent the process to be filtered (in this case $\varphi(X_t)$, for some $\varphi\in\mathrm{B}_b(I)$) as a semimartingale and then use a martingale representation theorem to obtain an expression for the filtering process $\pi(\varphi)$.

Let us fix $\varphi\in\mathrm{B}_b(I)$. A semimartingale representation for $\varphi(X_t)$ is easily obtained by using Dynkin's formula (cfr. Example 1.1.4)

$$\varphi(X_t) = \varphi(X_0) + \int_0^t \mathcal{L}\varphi(X_s)\,\mathrm{d}s + M_t, \quad t\geqslant 0 \tag{2.1.7}$$

where $\mathcal{L}$ is the infinitesimal generator associated to the process X, i. e.

$$\mathcal{L}f(x) := \int_I \big[f(z) - f(x)\big]\lambda(x,\, \mathrm{d}z), \quad f \in \mathrm{B}_b(I) \tag{2.1.8}$$

and $(M_t)_{t \geqslant 0}$ is a $\mathbb{X}$–martingale whose expression is given by

$$M_t = \int_{(0,t] \times I} \big[\varphi(z) - \varphi(X_{s-})\big]\tilde{n}(\mathrm{d}s\, \mathrm{d}z), \quad t \geqslant 0. \tag{2.1.9}$$

In order to get the expression for the filtering process provided in the next Proposition, some assumptions have to be checked and this is done in the following Lemma.

Lemma 2.1.2. *Let $\varphi \in \mathrm{B}_b(I)$ be fixed. The next properties hold true:*

1. *The process $\big(\mathcal{L}\varphi(X_t)\big)_{t \geqslant 0}$ is $\mathbb{X}$–progressive and for all $t > 0$*
$$\mathbb{E}\int_0^t \Big|\mathcal{L}\varphi(X_s)\Big|\,\mathrm{d}s < \infty.$$
2. *The $\mathbb{X}$–martingale $(M_t)_{t \geqslant 0}$ is such that for all $t > 0$*
$$\mathbb{E}\int_{(0,t]} \Big|\mathrm{d}M_s\Big| < \infty.$$
3. *The process $\big(\varphi(X_t)\big)_{t \geqslant 0}$ is bounded.*

Proof. **Claim 1.** Clearly enough, the process $\big(\mathcal{L}\varphi(X_t)\big)_{t \geqslant 0}$ is $\mathbb{X}$–adapted and càdlàg, since X is. Hence it is $\mathbb{X}$–progressive. Moreover we have the following estimate, thanks to Assumption 2.0.1, holding for all $t > 0$.

$$\begin{aligned}
\mathbb{E}\int_0^t \Big|\mathcal{L}\varphi(X_s)\Big|\,\mathrm{d}s &= \mathbb{E}\int_0^t \Big|\int_I \big[\varphi(z) - \varphi(X_s)\big]\lambda(X_s,\, \mathrm{d}z)\Big|\,\mathrm{d}s \\
\leqslant \mathbb{E}\int_0^t \Big[\int_I \big|\varphi(z)\big|\lambda(X_s,\, \mathrm{d}z) &+ \int_I \big|\varphi(X_s)\big|\lambda(X_s,\, \mathrm{d}z)\Big]\,\mathrm{d}s \\
\leqslant 2\sup_{z \in I}|\varphi(z)|\,\mathbb{E}\int_0^t \lambda(X_s)\,\mathrm{d}s &\leqslant 2t\sup_{z \in I}\big[|\varphi(z)|\lambda(z)\big] < +\infty.
\end{aligned}$$

Claim 2. From (2.1.9) we have $\mathrm{d}M_t = \int_I \big[\varphi(z) - \varphi(X_{t-})\big]\tilde{n}(\mathrm{d}t\, \mathrm{d}z)$ for all $t > 0$, hence

$$\begin{aligned}
\mathbb{E}\int_{(0,t]} \Big|\mathrm{d}M_s\Big| &= \mathbb{E}\int_0^t \Big|\int_I \big[\varphi(z) - \varphi(X_{s-})\big]\tilde{n}(\mathrm{d}s\, \mathrm{d}z)\Big| \\
\leqslant \mathbb{E}\int_{(0,t]} \Big|\int_I \varphi(z)\tilde{n}(\mathrm{d}s\, \mathrm{d}z)\Big| &+ \Big|\int_I \varphi(X_s)\tilde{n}(\mathrm{d}s\, \mathrm{d}z)\Big|.
\end{aligned} \tag{2.1.10}$$

From the definition of the compensated measure $\tilde{n}$ given in (2.1.4) we get that

$$\begin{aligned}
\int_{(0,t]} \Big|\int_I \varphi(z)\tilde{n}(\mathrm{d}t\, \mathrm{d}z)\Big| &= \Big|\int_I \varphi(z)\big[n(\mathrm{d}t\, \mathrm{d}z) - \lambda(X_{t-}, \mathrm{d}z)\,\mathrm{d}t\big]\Big| \\
\leqslant \int_{(0,t]} \int_I |\varphi(z)| n(\mathrm{d}t\, \mathrm{d}z) &+ \int_I |\varphi(z)|\lambda(X_{t-}, \mathrm{d}z)\,\mathrm{d}t \\
\leqslant \sup_{z \in I}\big|\varphi(z)\big| \int_{(0,t]} &\big[n(\mathrm{d}t \times I) + \lambda(X_{t-})\,\mathrm{d}t\big] \\
\leqslant \sup_{z \in I}\big|\varphi(z)\big| \,\big[n\big((0,t] \times I\big) &+ \sup_{z \in I}\lambda(z)\big] < +\infty, \quad \mathbb{P}\text{–a.s.}
\end{aligned}$$

thanks to the fact that n is $\mathbb{P}$–a.s. non-explosive. A similar estimate is obtained analogously for the second summand of (2.1.10), whence the claim.
Claim 3. It is obvious, thanks to the boundedness of the function φ. □

Proposition 2.1.3. *Let $\varphi \in \mathrm{B}_b(I)$ be fixed. Then for all $t \geqslant 0$ we have that $\mathbb{P}$–a.s.*

$$\pi_t(\varphi) = \pi_0(\varphi) + \int_0^t \int_I \mathcal{L}\varphi(x)\, \pi_{s-}(\mathrm{d}x)\, \mathrm{d}s + \int_{(0,t]\times O} \left\{ \Psi_s(y) - \pi_{s-}(\varphi) \right\} \tilde{m}(\mathrm{d}s\, \mathrm{d}y) \tag{2.1.11}$$

where $\tilde{m}(\mathrm{d}s\,\mathrm{d}y) = m(\mathrm{d}s\,\mathrm{d}y) - \hat{\mu}_s(\mathrm{d}y)\,\mathrm{d}s$, $\big(\Psi_t(y)\big)_{t\geqslant 0,\, y\in O}$ is the $\mathbb{Y}$–predictable random field defined as

$$\Psi_t(\omega, y) \coloneqq \frac{\psi_t(\omega;\, \mathrm{d}\upsilon)}{\hat{\mu}_t(\omega;\, \mathrm{d}\upsilon)}(y) \tag{2.1.12}$$

and $\psi_t(\omega,\, \mathrm{d}\upsilon)$ is the $\mathbb{Y}$–predictable random measure given by

$$\psi_t(\omega, B) = \int_I \int_I \varphi(z) \mathbb{1}_{h^{-1}(B\setminus\{Y_{t-}(\omega)\})}(z) \lambda(x,\, \mathrm{d}z)\pi_{t-}(\omega;\, \mathrm{d}x), \quad B \in \mathcal{O}.$$

Proof. Thanks to Lemma 2.1.2 we can apply Theorem 1.1.15 and we can write

$$\pi_t(\varphi) = \pi_0(\varphi) + \int_0^t \hat{f}_s\, \mathrm{d}s + \hat{m}_t, \quad t \geqslant 0, \quad \mathbb{P}\text{–a.s.},$$

where $(\hat{f}_t)_{t\geqslant 0}$ is a $\mathbb{Y}$–progressive version of $\big(\mathbb{E}[\mathcal{L}\varphi(X_t) \mid \mathcal{Y}_t]\big)_{t\geqslant 0}$ (see Remark 1.1.2) and $(\hat{m}_t)_{t\geqslant 0}$ is the $\mathbb{Y}$–martingale given by

$$\hat{m}_t = \int_{(0,t]\times O} \{\Psi_s^1(y) - \Psi_s^2(y) + \Psi_s^3(y)\}\, \tilde{m}(\mathrm{d}s\, \mathrm{d}y), \quad t \geqslant 0$$

with the $\mathbb{Y}$–predictable fields $\big(\Psi_t^i(y)\big)_{t\geqslant 0,\, y\in O}$, $i = 1, 2, 3$ defined by the following equalities

$$\mathbb{E}\int_0^t \int_O C_s(y)\, \Psi_s^1(y)\, \hat{\mu}_s(\mathrm{d}y)\, ds = \mathbb{E}\int_0^t \int_O C_s(y)\, \varphi(X_s)\, \mu_s(\mathrm{d}y)\, \mathrm{d}s \tag{2.1.13}$$

$$\mathbb{E}\int_0^t \int_O C_s(y)\, \Psi_s^2(y)\, \hat{\mu}_s(\mathrm{d}y)\, \mathrm{d}s = \mathbb{E}\int_0^t \int_O C_s(y)\, \varphi(X_s)\, \hat{\mu}_s(\mathrm{d}y)\, \mathrm{d}s \tag{2.1.14}$$

$$\mathbb{E}\int_{(0,t]\times O} C_s(y)\, \Psi_s^3(y)\, \hat{\mu}_s(\mathrm{d}y)\, \mathrm{d}s = \mathbb{E}\int_{(0,t]\times O} C_s(y)\, [\varphi(X_s) - \varphi(X_{s-})]\, m(\mathrm{d}s\, \mathrm{d}y) \tag{2.1.15}$$

holding for all $t \geqslant 0$ and all non-negative $\mathbb{Y}$–predictable fields $\big(C_t(y)\big)_{t\geqslant 0,\, y\in O}$.

It is clear that $\hat{f}_t = \int_I \mathcal{L}\varphi(x)\pi_t(\mathrm{d}x)$, $t \geqslant 0$, as a straightforward computation shows. It is also immediate to notice that $\Psi_t^2(y) = \pi_{t-}(\varphi)$, $t > 0$ (see Remark 1.1.4).

We now proceed to compute $\Psi_t^3(y)$, $t \geqslant 0$, $y \in O$. We will see that it is not necessary to compute the term $\Psi_t^1(y)$. Let us elaborate the right hand side of (2.1.15)

$$\begin{aligned}
&\mathbb{E}\int_{(0,t]\times O} C_s(y)\, [\varphi(X_s) - \varphi(X_{s-})]\, m(\mathrm{d}s\, \mathrm{d}y) \\
&= \mathbb{E}\int_I C_s(h(z))\, [\varphi(z) - \varphi(X_{s-})]\, \mathbb{1}_{h^{-1}(Y_{s-})^c}(z)\, n(\mathrm{d}s\, \mathrm{d}z) \\
&= \mathbb{E}\int_0^t \int_I C_s(h(z))\, [\varphi(z) - \varphi(X_{s-})]\, \mathbb{1}_{h^{-1}(Y_{s-})^c}(z)\, \lambda(X_{s-},\, \mathrm{d}z)\, \mathrm{d}s
\end{aligned}$$

where the former passage is justified by the same reasoning as in the beginning of the proof of Lemma 2.1.1 and the latter one is due to the fact that $\lambda(X_{s-},\, \mathrm{d}z)\,\mathrm{d}s$ is the $\mathbb{X}$–compensator of $n(\mathrm{d}s\,\mathrm{d}z)$ and that the integrand is a $\mathbb{X}$–predictable process.

It is easy to check that the term

$$\mathbb{E}\int_0^t \int_I C_s(h(z))\,\varphi(X_{s-})\,\mathbb{1}_{h^{-1}(Y_{s-})^c}(z)\,\lambda(X_{s-},\, \mathrm{d}z)\,\mathrm{d}s$$

leads to the expression of the process $\big(\Psi_t^1(y)\big)_{t\geqslant 0,\, y\in O}$, obtainable by elaborating the right hand side of (2.1.13). Hence defining $\Psi_t(\omega, y) = \Psi_t^1(\omega, y) + \Psi_t^3(\omega, y)$, $\omega \in \Omega$, $t \geqslant 0$, $y \in O$, we are left to characterize the following equality, holding for all $t \geqslant 0$ and all non-negative $\mathbb{Y}$–predictable random fields $\big(C_t(y)\big)_{t\geqslant 0,\, y\in O}$.

$$\mathbb{E}\int_0^t \int_O C_s(y)\,\Psi_s(y)\,\hat{\mu}_s(\mathrm{d}y)\,\mathrm{d}s = \mathbb{E}\int_0^t \int_I C_s(h(z))\,\varphi(z)\,\mathbb{1}_{h^{-1}(Y_{s-})^c}(z)\,\lambda(X_{s-},\, \mathrm{d}z)\,\mathrm{d}s. \tag{2.1.16}$$

Applying the freezing lemma and the Fubini-Tonelli theorem to the right hand side of (2.1.16) and noticing that $X_s = X_{s-}$ and $\pi_s = \pi_{s-}$, $\mathrm{d}s$–a.e., we get that

$$\begin{aligned}
&\mathbb{E}\int_0^t \int_I C_s(h(z))\,\varphi(z)\,\mathbb{1}_{h^{-1}(Y_{s-})^c}(z)\,\lambda(X_{s-},\, \mathrm{d}z)\,\mathrm{d}s \\
&= \mathbb{E}\int_0^t \int_I \int_I C_s(h(z))\,\varphi(z)\,\mathbb{1}_{h^{-1}(Y_{s-})^c}(z)\,\lambda(x,\, \mathrm{d}z)\,\pi_{s-}(\mathrm{d}x)\,\mathrm{d}s \\
&= \mathbb{E}\int_0^t \int_O C_s(y)\,\psi_s(\mathrm{d}y)\,\mathrm{d}s = \mathbb{E}\int_0^t \int_O C_s(y)\,\Psi_s(y)\,\hat{\mu}_s(\mathrm{d}y)\,\mathrm{d}s.
\end{aligned}$$

Therefore, if we define the random field $\big(\Psi_t(y)\big)_{t\geqslant 0,\, y\in O}$ as in (2.1.12) we get (2.1.11). The random field $\big(\Psi_t(y)\big)_{t\geqslant 0,\, y\in O}$ is well defined since for all $t \geqslant 0$ and all $\omega \in \Omega$ we have that $\psi_t(\omega;\, \mathrm{d}y) \ll \hat{\mu}_t(\omega; \mathrm{d}y)$, as it is straightforward to verify. Its $\mathbb{Y}$–predictability follows from an easy generalization of [15, Exercise 6.10.72]. □

Before stating an explicit equation for the filtering process $\pi(\varphi)$, we need to define an operator, denoted by H. The notation adopted is due to [25], where the authors discuss the case of a signal process X given by a finite-state Markov chain. The aim is to characterize, for each $n \in \mathbb{N}$, the probability measure π_{τ_n}, i. e. the filtering process evaluated at each jump time τ_n of the observed process Y (we will use H also to characterize the initial value π_0). This will be done by identifying π_{τ_n} as a probability measure obtained via this operator and depending on the position Y_{τ_n} of the observed process at the n-th jump time and on a specific random measure. This measure will be determined by the values $\pi_{\tau_n -}$ and $Y_{\tau_n -}$ and by the rate transition measure λ.

Let us introduce some more notation. Let $\mathcal{M}(I)$, $\mathcal{M}_+(I)$, $\mathcal{P}(I)$ be the sets of (respectively) finite signed, finite, probability Borel measures on $(I, \mathcal{I})$. For all signed measures $\mu \in \mathcal{M}(I)$ let $\mu \circ h^{-1}$ denote the image measure of μ by h, i. e. the signed measure on $(O, \mathcal{O})$ defined as

$$\mu \circ h^{-1}(B) := \mu\big(h^{-1}(B)\big), \quad B \in \mathcal{O}.$$

In addition, for all $\varphi \in \mathrm{B}_b(I)$ let $\varphi\mu$ be the (signed) measure on $(I, \mathcal{I})$ defined as

$$\varphi\mu(A) := \int_A \varphi(z)\, \mu(\mathrm{d}z), \quad A \in \mathcal{I}.$$

In what follows, for fixed $\mu \in \mathcal{M}_+(I)$ and $\varphi \in \mathrm{B}_b(I)$, we will need to consider the Radon-Nikodym derivative of the signed measure $\varphi\mu \circ h^{-1}$ with respect to the measure $\mu \circ h^{-1}$. It is clear that this derivative is well defined, being the former measure absolutely continuous with respect to the latter.

Remark 2.1.2. The operator H presents strong analogies with the regular version of a conditional probability. Consider $(I, \mathcal{I}) =: (\Omega, \mathcal{F})$ as the measurable sample space; take $\mathbf{P} \in \mathcal{P}(I)$ and define the sub-σ-algebra $h^{-1}(\mathcal{O}) =: \mathcal{G}$. Then what we will denote by $H_{h(z)}[\mathbf{P}]$ (in this example, we should write $H_{h(\omega)}[\mathbf{P}]$) corresponds to a regular version of the conditional distribution $\mathbf{P}$ given $\mathcal{G}$, i. e. a function $(\mathbf{P} \mid \mathcal{G})\colon \mathcal{F} \times \Omega \to [0, 1]$ such that

- the map $\omega \mapsto (\mathbf{P} \mid \mathcal{G})(F, \omega)$ is a version of $\mathbf{P}(F \mid \mathcal{G})$ for all $F \in \mathcal{F}$
- the function $F \mapsto (\mathbf{P} \mid \mathcal{G})(F, \omega)$ is a probability measure on $(\Omega, \mathcal{F})$ for all $\omega \in \Omega$.

However, in the definition of the operator H we do not use as "parameter" space (i. e. the sample space) the set I, but the state space O of the observed process. Moreover, the operator H acts on the larger space $\mathcal{M}_+(I)$.

Let us begin the construction of the operator H with the following Lemma.

Lemma 2.1.4. *Suppose that I is a compact metric space and fix $\mu \in \mathcal{M}_+(I)$. For each $\varphi \in \mathrm{C}(I)$ take a version (i. e. any function in the equivalence class) of the Radon-Nikodym derivative of $\varphi\mu \circ h^{-1}$ with respect to $\mu \circ h^{-1}$ and define, for fixed $y \in O$, the functional $L_y\colon \mathrm{C}(I) \to \mathbb{R}$ as*

$$L_y(\varphi) := \frac{\varphi\mu \circ h^{-1}(\mathrm{d}\upsilon)}{\mu \circ h^{-1}(\mathrm{d}\upsilon)}(y), \quad \varphi \in \mathrm{C}(I).$$

If $y \in \operatorname{supp}(\mu \circ h^{-1})$ there exists a unique probability measure ρ_y on $(I, \mathcal{I})$ such that

$$L_y(\varphi) = \int_I \varphi(z)\, \rho_y(\mathrm{d}z), \quad \varphi \in \mathrm{C}(I).$$

Remark 2.1.3. The hypothesis that the point y belongs to the support of the measure $\mu \circ h^{-1}$ ensures that the functional L is not zero on the whole space $\mathrm{B}_b(I)$ (for instance, it takes value 1 on the function $\varphi = 1$, as will be proved). To see what happens if this is not the case, consider the example below.

Example 2.1.1. Let $\mu \in \mathcal{M}_+(I)$, $O = [0, 1]$ and a point $y \in O$, $y \notin \operatorname{supp}(\mu \circ h^{-1})$. Take $(A_n)_{n \in \mathbb{N}} \subset \mathcal{O}$ to be the sequence of open intervals given by

$$A_n := \left(y - \frac{1}{n}, y + \frac{1}{n}\right) \cap O, \quad n \in \mathbb{N}.$$

By definition of support and set inclusion, we have that there exists a natural number $\bar{n} \in \mathbb{N}$ such that $\mu \circ h^{-1}(A_n) = 0$ for all $n \geqslant \bar{n}$. Since $\varphi\mu \circ h^{-1}$ is absolutely continuous with respect to $\mu \circ h^{-1}$, we also have that $\varphi\mu \circ h^{-1}(A_n) = 0$ for all $n \geqslant \bar{n}$

and for all $\varphi \in \mathrm{B}_b(I)$. Therefore, thanks to [15, Theorem 5.8.8.] and under the usual convention $\frac{0}{0} := 0$ we have that

$$\frac{\varphi\mu \circ h^{-1}(\mathrm{d}\upsilon)}{\mu \circ h^{-1}(\mathrm{d}\upsilon)}(y) = \lim_{n\to\infty} \frac{\varphi\mu \circ h^{-1}(A_n)}{\mu \circ h^{-1}(A_n)} = 0$$

and we obtain $L(\varphi) = 0$ for all $\varphi \in \mathrm{B}_b(I)$.

Proof of Lemma 2.1.4. Fix $\mu \in \mathcal{M}_+(I)$ and $y \in O$ such that y is in the support of $\mu \circ h^{-1}$. Let $\mathcal{C}$ denote a countable dense subset of the set $\mathrm{C}(I)$ of continuous functions on I, containing the constant function equal to 1 (denoted by 1) and such that $\mathcal{C}$ is a vector space over $\mathbb{Q}$. The result will follow from an application of a slight modification of the Riesz Representation Theorem to the functional L_y (see [56, Par. 88] for further details). What we need to prove is that (remember: $y \in \operatorname{supp}(\mu \circ h^{-1})$):

1. L_y is a linear functional on $\mathcal{C}$ (as a vector space over $\mathbb{Q}$).
2. $L_y(\varphi) \leqslant L_y(\psi)$, whenever $\varphi \leqslant \psi$, $\varphi, \psi \in \mathcal{C}$.
3. $L_y(1) = 1$.

Claim 1. Let $\alpha, \beta \in \mathbb{Q}$ and $\varphi, \psi \in \mathcal{C}$ be fixed and let us define

$$g(\upsilon) := \frac{(\alpha\varphi + \beta\psi)\mu \circ h^{-1}(\mathrm{d}\upsilon)}{\mu \circ h^{-1}(\mathrm{d}\upsilon)}(\upsilon), \quad \upsilon \in O.$$

By definition of Radon-Nikodym derivative we have that for all $B \in \mathcal{O}$

$$\int_B g(\upsilon)\, \mu \circ h^{-1}(\mathrm{d}\upsilon) = (\alpha\varphi + \beta\psi)\mu \circ h^{-1}(B) = \alpha\varphi\mu \circ h^{-1}(B) + \beta\psi\mu \circ h^{-1}(B).$$

Therefore, setting $g_1 := \frac{\alpha\varphi\mu\circ h^{-1}(\mathrm{d}\upsilon)}{\mu\circ h^{-1}(\mathrm{d}\upsilon)}$ and $g_2 := \frac{\beta\psi\mu\circ h^{-1}(\mathrm{d}\upsilon)}{\mu\circ h^{-1}(\mathrm{d}\upsilon)}$ we get that $g = g_1 + g_2$ except on a $(\mu \circ h^{-1})$–null measure set C. On this set we may redefine, for instance, $g_1(\upsilon) := g(\upsilon)$ and $g_2(\upsilon) = 0$, $\upsilon \in C$ to have that $g = g_1 + g_2$ for all $\upsilon \in O$, whence $L_y(\alpha\varphi + \beta\psi) = \alpha L_y(\varphi) + \beta L_y(\psi)$.
Claim 2. By linearity, this is equivalent to prove that $L_y(\varphi) \geqslant 0$, for all $\varphi \in \mathcal{C}$, $\varphi \geqslant 0$. It is immediate to see that, for all $\varphi \geqslant 0$, we have that $\varphi\mu \circ h^{-1} \in \mathcal{M}_+(I)$, hence $g := \frac{\varphi\mu\circ h^{-1}(\mathrm{d}\upsilon)}{\mu\circ h^{-1}(\mathrm{d}\upsilon)} \geqslant 0$, $\mu \circ h^{-1}$–a.e.. Redefining g to be zero on the $\mu \circ h^{-1}$–null measure set $C \in \mathcal{O}$ where this does not happen, we get that $L_y(\varphi) \geqslant 0$.
Claim 3. If $\{y\}$ is an atom for μ then the result is obvious. Otherwise, we can consider μ to be atomless, without loss of generality.

Consider first the case $O = [0, 1]$. Define $(A_n)_{n\in\mathbb{N}} \subset \mathcal{O}$ to be the sequence of open intervals given by

$$A_n := \Big(y - \frac{1}{n}, y + \frac{1}{n}\Big) \cap O, \quad n \in \mathbb{N}.$$

Then, by definition of support, we have that for all $n \in \mathbb{N}$

$$\mu \circ h^{-1}(A_n) = \mu\big(h^{-1}(A_n)\big) > 0.$$

Therefore, thanks to [15, Theorem 5.8.8.] we have that

$$\frac{\mu \circ h^{-1}(\mathrm{d}\upsilon)}{\mu \circ h^{-1}(\mathrm{d}\upsilon)}(y) = \lim_{n\to\infty} \frac{\mu\big(h^{-1}(A_n)\big)}{\mu\big(h^{-1}(A_n)\big)} = 1.$$

The case of O being a complete and separable metric space is treated by reducing it to the previous case. This is possible since we are taking the σ-algebra $\mathcal{O}$ on O as the Borel one, hence we know that $\mathcal{O}$ is countably generated and countably separated. Then, by [15, Theorem 6.5.5.], there exists a measurable function $\psi\colon O \to [0,1]$ such that

$$\mathcal{O} = \{\psi^{-1}(B)\colon B \in \mathcal{B}\big([0,1]\big)\}$$

and, by [15, Theorem 6.5.7.], we know that this function is injective.

We are now in a position to apply the Riesz Representation Theorem to the functional L_y and say that there exists a unique probability measure ρ_y on $(I, \mathcal{I})$ such that

$$L_y(\varphi) = \int_I \varphi(z)\, \rho_y(\mathrm{d}z), \quad \varphi \in \mathcal{C}.$$

We get the same equality for all $\varphi \in \mathrm{C}(I)$ by uniform convergence. □

Proposition 2.1.5. *Let $\mu \in \mathcal{M}_+(I)$ be fixed. Then there exists a probability measure ρ_y on $(I, \mathcal{I})$ such that for all $\varphi \in \mathrm{B}_b(I)$ and $\mu \circ h^{-1}$–almost all $y \in O$ it holds*

$$\frac{\varphi\mu \circ h^{-1}(\mathrm{d}\upsilon)}{\mu \circ h^{-1}(\mathrm{d}\upsilon)}(y) = \int_I \varphi(z)\, \rho_y(\mathrm{d}z).$$

Moreover the set

$$A := \{z \in I\colon \rho_{h(z)}(G) = \mathbb{1}_G(z),\, \forall G \in \mathcal{H}\}, \quad \mathcal{H} := h^{-1}(\mathcal{O})$$

is such that $\mu(A^c) = 0$ and

$$\rho_y(h^{-1}(y)) = 1, \quad \mu \circ h^{-1}\text{–a.e.}$$

Proof. The scheme followed in this proof is the same used to prove the existence of a regular version of a conditional probability (see e. g. [56, Th. 89.1]). To start, let us prove the first claim in the case where I is a compact metric space. From Lemma 2.1.4 we know that there exists a unique probability measure ρ_y on $(I, \mathcal{I})$ such that

$$\frac{\varphi\mu \circ h^{-1}(\mathrm{d}\upsilon)}{\mu \circ h^{-1}(\mathrm{d}\upsilon)}(y) = \int_I \varphi(z)\, \rho_y(\mathrm{d}z), \quad \varphi \in \mathrm{C}(I)$$

for all $y \in \operatorname{supp}(\mu \circ h^{-1})$. Clearly the set $\operatorname{supp}(\mu \circ h^{-1})^c$ has $\mu \circ h^{-1}$–measure zero and on this set we can define $\rho_y = \nu_y$, where ν_y is an arbitrary but fixed probability measure on $(I, \mathcal{I})$, such that $\nu_y\big(h^{-1}(y)\big) = 1$. Then, we get that for $\mu \circ h^{-1}$–almost all $y \in O$

$$\frac{\varphi\mu \circ h^{-1}(\mathrm{d}\upsilon)}{\mu \circ h^{-1}(\mathrm{d}\upsilon)}(y) = \int_I \varphi(z)\, \rho_y(\mathrm{d}z), \quad \varphi \in \mathrm{C}(I).$$

By a monotone class argument, the same equality holds for all $\varphi \in \mathrm{B}_b(I)$.

To show that the second assertion holds, let us notice, first, that the σ-algebra $\mathcal{H}$ is countably generated. In fact, since O is a complete and separable metric space, its Borel σ-algebra $\mathcal{O}$ can be written as $\mathcal{O} = \sigma(C_1, C_2, \dots)$, for some countable collection $\mathcal{C} = (C_i)_{i\in\mathbb{N}}$ of subsets of O. By a standard fact from measure theory (see e. g. [15, Corollary 1.2.9]) we have that

$$\mathcal{H} = h^{-1}(\mathcal{O}) = h^{-1}\big(\sigma(\mathcal{C})\big) = \sigma\big(h^{-1}(\mathcal{C})\big)$$

hence the collection $h^{-1}(\mathcal{C}) = \big(h^{-1}(C_1), h^{-1}(C_2), \dots\big)$ forms a countable class generating $\mathcal{H}$.

Now, let $\mathcal{K}$ be the π-system formed by all finite intersections of sets in $h^{-1}(\mathcal{C})$ (notice that $\sigma(\mathcal{K}) = \mathcal{H}$) and define

$$A_1 \coloneqq \{z \in I\colon \rho_{h(z)}(K) = \mathbb{1}_K(z),\ \forall K \in \mathcal{K}\}.$$

Fix $K \in \mathcal{K}$. By definition of Radon-Nikodym derivative we have that for all $B \in \mathcal{O}$

$$\int_{h^{-1}(B)} \mathbb{1}_K(z)\, \mu(\mathrm{d}z) = \mathbb{1}_K \mu \circ h^{-1}(B) = \int_B \int_I \mathbb{1}_K(z)\, \rho_y(\mathrm{d}z)\, \mu \circ h^{-1}(\mathrm{d}y) = \int_B \rho_y(K)\, \mu \circ h^{-1}(\mathrm{d}y) = \int_{h^{-1}(B)} \rho_{h(z)}(K)\, \mu(\mathrm{d}z)$$

therefore $\rho_{h(z)}(K) = \mathbb{1}_K(z)$, μ–a.e. on $\mathcal{H}$. This equality holds for all $K \in \mathcal{K}$ and since this is a countable collection of sets, we get that $\mu(A_1^c) = 0$.

Next, for fixed $z \in A_1$, let $\mathcal{D} \coloneqq \{G \in \mathcal{H}\colon \rho_{h(z)}(G) = \mathbb{1}_G(z)\}$. Obviously $I \in \mathcal{D}$ and it is immediate to see that for any $A, B \in \mathcal{D}$ with $A \subset B$

$$\rho_{h(z)}(B \setminus A) = \rho_{h(z)}(B) - \rho_{h(z)}(A) = \mathbb{1}_B(z) - \mathbb{1}_A(z) = \mathbb{1}_{B\setminus A}(z).$$

In addition, for any sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{D}$, $A_n \uparrow A$

$$\rho_{h(z)}(A) = \lim_{n\to\infty} \rho_{h(z)}(A_n) = \lim_{n\to\infty} \mathbb{1}_{A_n}(z) = \mathbb{1}_A(z).$$

Hence, $\mathcal{D}$ is a d-system, clearly containing the π-system $\mathcal{K}$. Therefore, by Dynkin's $\pi - \lambda$ theorem we get that $\sigma(\mathcal{K}) = \mathcal{H} \subset \mathcal{D}$. This implies that the equality

$$\rho_{h(z)}(G) = \mathbb{1}_G(z), \quad \text{for all } G \in \mathcal{H}$$

holds for all $z \in A_1$, hence $\mu(A^c) = 0$.

Finally, fix $y \in O$. Clearly $h^{-1}(y) \in \mathcal{H}$ and from the previous discussion we have that for μ–almost all $z \in I$

$$\rho_{h(z)}(h^{-1}(y)) = \mathbb{1}_{h^{-1}(y)}(z) = \begin{cases} 1, & \text{if } z \in h^{-1}(y) \\ 0, & \text{if } z \notin h^{-1}(y) \end{cases}.$$

Notice, also, that

$$\rho_{h(z)}(h^{-1}(y)) = \begin{cases} \rho_y(h^{-1}(y)), & \text{if } z \in h^{-1}(y) \\ \rho_v(h^{-1}(y)), & \text{if } z \notin h^{-1}(y) \end{cases}$$

for some $v \in O$, $v \neq y$. Therefore, $\rho_y(h^{-1}(y)) = 1$ for $\mu \circ h^{-1}$–almost all $y \in O$.

To prove the claim in the case where I is a complete and separable metric space, it suffices to remember (see e. g. [4, Theorem A.7]) that I is homeomorfic to a Borel subset of some compact metric space J (in particular, $I \in \mathcal{B}(J)$). After extending the measure μ to $(J, \mathcal{B}(J))$ in the usual way, one can apply the result just shown to the measure space $(J, \mathcal{B}(J), \mu)$, considering $\mathcal{H} \coloneqq \sigma\big(h^{-1}(\mathcal{O})\big) \subset \mathcal{B}(J)$. To conclude, it is enough to set ρ_y, for each $y \in O$, to be the restriction to $h^{-1}(\mathcal{O})$ of the probability measure found with the above procedure. □

Following these results, we can give the definition of the operator H.

Definition 2.1.1 (Operator H)**.** For each $y \in O$ the operator $H_y \colon \mathcal{M}_+(I) \to \mathcal{P}(I)$ is given by

$$H_y[\mu] \coloneqq \begin{cases} \rho_y, & \text{if } y \in \operatorname{supp}(\mu \circ h^{-1}) \\ \nu_y, & \text{if } y \notin \operatorname{supp}(\mu \circ h^{-1}) \end{cases}$$

where ρ_y is the unique probability measure on $(I, \mathcal{I})$ satisfying

$$\frac{\varphi\mu \circ h^{-1}(\mathrm{d}y)}{\mu \circ h^{-1}(\mathrm{d}y)}(y) = \int_I \varphi(z)\, \rho_y(\mathrm{d}z), \quad \varphi \in \mathrm{B}_b(I)$$

and ν_y is an arbitrary probability measure on $(I, \mathcal{I})$, such that $\nu_y\big(h^{-1}(y)\big) = 1$.

Remark 2.1.4. A more explicit definition of the operator H can be obtained whenever this operator acts on a measure $\mu \in \mathcal{M}_+(I)$ such that $\mu \circ h^{-1}$ is discrete. It is clear that in this case we have that

$$H_y[\mu](\varphi) = \frac{\varphi\mu\big(h^{-1}(y)\big)}{\mu\big(h^{-1}(y)\big)} = \frac{1}{\mu\big(h^{-1}(y)\big)} \int_{h^{-1}(y)} \varphi(z)\, \mu(\mathrm{d}z), \quad y \in O$$

under the usual assumption $\frac{0}{0} \coloneqq 0$. Otherwise said, the probability measure $H_y[\mu]$ is

$$H_y[\mu](\mathrm{d}z) = \frac{1}{\mu\big(h^{-1}(y)\big)} \mathbb{1}_{h^{-1}(y)}(z)\, \mu(\mathrm{d}z), \quad y \in O.$$

As an example of such a setting, see [25], where both the state spaces I and O of the unobserved and observed processes are assumed to be finite sets.

This simplification can be interpreted in a Bayesian setting, by looking at this simple dominated model. We can view $(I, \mathcal{I})$ as the parameter space and $(O, \mathcal{O})$ as the data one. If we fix a *prior distribution* μ on $(I, \mathcal{I})$ of our unknown parameter X, such that $\mu \circ h^{-1}$ is discrete, then we can interpret $\mu \circ h^{-1}$ as the *likelihood*, i. e. the law on $(O, \mathcal{O})$ of the datum Y given X and, finally, we can see $H_Y[\mu]$ as the *posterior distribution* of X given Y.

Unfortunately, this setting cannot be generalized to encompass the range of possible cases covered by our model and, as it is known, the Bayesian framework fails in a non-dominated setting.

We are now ready to state the final version of the filtering equation, giving the dynamics of the process $\pi(\varphi)$.

Theorem 2.1.6 (Filtering equation)**.** *Let $\varphi \in \mathrm{B}_b(I)$ be fixed. Let us define, for each fixed $y \in O$, the linear operator $\mathcal{A}_y \colon \mathrm{B}_b(I) \to \mathrm{B}_b(I)$ as*

$$\mathcal{A}_y\varphi(x) \coloneqq \mathcal{L}\varphi(x) - \int_I \mathbb{1}_{h^{-1}(y)^c}(z)\varphi(z)\lambda(x, \mathrm{d}z), \quad x \in I \tag{2.1.17}$$

and let us denote by $\mathbf{1} \colon I \to \mathbb{R}$ the function identically equal to 1.

The process $\pi(\varphi)$ satisfies for all $t \geqslant 0$ and $\mathbb{P}$–a.s. the following equation

$$\begin{aligned}\pi_t(\varphi) &= H_{Y_0}[\mu](\varphi) \\ &+ \int_0^t \int_I \mathcal{A}_{Y_{s-}}\, \varphi(x)\, \pi_{s-}(\mathrm{d}x)\, \mathrm{d}s - \int_0^t \pi_{s-}(\varphi) \int_I \mathcal{A}_{Y_{s-}}\, \mathbf{1}(x)\, \pi_{s-}(\mathrm{d}x)\, \mathrm{d}s \\ &+ \sum_{0 < \tau_n \leqslant t} \Big\{ H_{Y_{\tau_n}}[\mu_n](\varphi) - \pi_{\tau_n -}(\varphi) \Big\},\end{aligned} \tag{2.1.18}$$

or, in differential form

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\pi_t(\varphi) = \pi_t(\mathcal{A}_{Y_{\tau_n}}\varphi) - \pi_t(\varphi)\pi_t(\mathcal{A}_{Y_{\tau_n}}1), & t \in [\tau_n, \tau_{n+1}),\ n \in \mathbb{N}_0 \\ \pi_{\tau_n}(\varphi) = H_{Y_{\tau_n}}[\mu_n](\varphi), & n \in \mathbb{N}_0 \end{cases} \tag{2.1.19}$$

where the (random) measures μ_n on $(I, \mathcal{I})$ are given by

$$\mu_n(\mathrm{d}z) \coloneqq \begin{cases} \mu(\mathrm{d}z), & n = 0 \\ \mathbb{1}_{h^{-1}(Y_{\tau_n-})^c}(z) \displaystyle\int_I \lambda(x, \mathrm{d}z)\, \pi_{\tau_n-}(\mathrm{d}x), & n \in \mathbb{N} \end{cases}. \tag{2.1.20}$$

Proof. To prove this theorem it suffices to elaborate the terms appearing in (2.1.11). We show that the integral form (2.1.18) holds true. The differential form (2.1.19) follows immediately.

Let us start by computing $\pi_0(\varphi) = \mathbb{E}[\varphi(X_0) \mid Y_0]$. By definition of conditional expectation the equality

$$\mathbb{E}[Z\pi_0(\varphi)] = \mathbb{E}[Z\varphi(X_0)]$$

holds for all bounded and $\sigma(Y_0)$ measurable random variables Z. Otherwise said, we have that

$$\mathbb{E}[g(Y_0)f(Y_0)] = \mathbb{E}[g(Y_0)\varphi(X_0)]$$

for all bounded and measurable functions $g\colon O \to \mathbb{R}$, where $f\colon O \to \mathbb{R}$ is a measurable function such that $f(Y_0) = \pi_0(\varphi)$, $\mathbb{P}$–a.s. (notice that $\pi_0(\varphi)$ is $\sigma(Y_0)$ measurable). Then we can write

$$\mathbb{E}[g(Y_0)f(Y_0)] = \int_I g\big(h(x)\big) f\big(h(x)\big)\, \mu(\mathrm{d}x) = \int_O g(y)f(y)\, \mu \circ h^{-1}(\mathrm{d}y)$$

on one hand. On the other hand

$$\mathbb{E}[g(Y_0)\varphi(X_0)] = \int_I g\big(h(x)\big)\varphi(x)\, \mu(\mathrm{d}x) = \int_O g(y)\, \varphi\mu \circ h^{-1}(\mathrm{d}y).$$

Therefore

$$\int_O g(y)f(y)\, \mu \circ h^{-1}(\mathrm{d}y) = \int_O g(y)\, \varphi\mu \circ h^{-1}(\mathrm{d}y)$$

for all bounded and measurable functions $g\colon O \to \mathbb{R}$, whence

$$f(y) = \frac{\varphi\mu \circ h^{-1}(\mathrm{d}y)}{\mu \circ h^{-1}(\mathrm{d}y)}(y) = \int_I \varphi(z)\, H_y[\mu](\mathrm{d}z) = H_y[\mu](\varphi), \quad y \in O$$

and finally $\pi_0(\varphi) = f(Y_0) = H_{Y_0}[\mu](\varphi)$, $\mathbb{P}$–a.s..

Let us now analyze the term

$$\int_{(0,t]\times O} \Big\{ \Psi_s(y) - \pi_{s-}(\varphi) \Big\} \big[m(\mathrm{d}s\, \mathrm{d}y) - \hat{\mu}_s(\mathrm{d}y)\, \mathrm{d}s \big]$$

appearing in (2.1.11).

From the definition of the field $\big(\Psi_t(y)\big)_{t\geqslant 0,\, y\in O}$ given in (2.1.12) and recalling that $\mathcal{L}\,1 = 0$, we easily get that $\mathbb{P}$–a.s.

$$\begin{aligned} &\int_0^t \int_O \Big\{ \Psi_s(y) - \pi_{s-}(\varphi) \Big\} \hat{\mu}_s(\mathrm{d}y)\, \mathrm{d}s \\ &= \int_0^t \int_I \Big[\int_I \mathbb{1}_{h^{-1}(Y_{s-})^c}(z)\varphi(z)\lambda(x, \mathrm{d}z) - \pi_{s-}(\varphi)\mathcal{A}_{Y_{s-}}\, 1(x) \Big] \pi_{s-}(\mathrm{d}x)\, \mathrm{d}s. \end{aligned}$$

Therefore $\mathbb{P}$–a.s.

$$\int_0^t \int_I \mathcal{L}\varphi(x)\, \pi_{s-}(\mathrm{d}x)\, \mathrm{d}s - \int_{(0,t]\times O} \Big\{ \Psi_s(y) - \pi_{s-}(\varphi) \Big\}\, \hat{\mu}_s(\mathrm{d}y)\, \mathrm{d}s$$
$$= \int_0^t \int_I \mathcal{A}_{Y_{s-}}\, \varphi(x)\, \pi_{s-}(\mathrm{d}x)\, \mathrm{d}s - \int_0^t \pi_{s-}(\varphi) \int_I \mathcal{A}_{Y_{s-}}\, \mathbb{1}(x)\, \pi_{s-}(\mathrm{d}x)\, \mathrm{d}s.$$

We are left to elaborate the term

$$\int_{(0,t]\times O} \Big\{ \Psi_s(y) - \pi_{s-}(\varphi) \Big\}\, m(\mathrm{d}s\, \mathrm{d}y) = \sum_{0<\tau_n\leqslant t} \Big\{ \Psi_{\tau_n}(Y_{\tau_n}) - \pi_{\tau_n -}(\varphi) \Big\}.$$

Let us recall that the $\mathbb{Y}$–predictable random field $\big(\Psi_t(y)\big)_{t\geqslant 0,\, y\in O}$ satisfies

$$\mathbb{E} \int_0^t \int_O C_s(y)\, \psi_s(\mathrm{d}y)\, \mathrm{d}s = \mathbb{E} \int_0^t \int_O C_s(y)\, \Psi_s(y)\, \hat{\mu}_s(\mathrm{d}y)\, \mathrm{d}s$$

for all non-negative $\mathbb{Y}$–predictable random fields $\big(C_t(y)\big)_{t\geqslant 0,\, y\in O}$. A simple computation involving just the Fubini-Tonelli theorem shows that we can rewrite the previous equation as

$$\mathbb{E} \int_0^t \int_O C_s(y)\, \varphi\nu_t \circ h^{-1}(\mathrm{d}y)\, \mathrm{d}s = \mathbb{E} \int_0^t \int_O C_s(y)\, \Psi_s(y)\, \nu_t \circ h^{-1}(\mathrm{d}y)\, \mathrm{d}s$$

where the $\mathbb{Y}$–predictable random measure $\nu_t(\omega;\, \mathrm{d}z)\, \mathrm{d}t$ is given by

$$\nu_t(\mathrm{d}z)\, \mathrm{d}t = \mathbb{1}_{h^{-1}(Y_{t-})^c}(z) \int_I \lambda(x, \mathrm{d}z)\, \pi_{t-}(\mathrm{d}x)\, \mathrm{d}t.$$

Therefore, we have that

$$\Psi_t(\omega, y) = H_y[\nu_t(\omega)](\varphi), \quad \nu_t(\omega) \circ h^{-1}(\mathrm{d}y)\, \mathrm{d}t\, \mathbb{P}(\mathrm{d}\omega)\text{–a.e.}$$

or, equivalently

$$\Psi_t(\omega, y) = H_y[\nu_t(\omega)](\varphi), \quad m(\omega, \mathrm{d}t\, \mathrm{d}y)\, \mathbb{P}(\mathrm{d}\omega)\text{–a.e.}$$

whence we deduce that $\Psi_{\tau_n}(Y_{\tau_n}) = H_{Y_{\tau_n}}[\mu_n](\varphi)$, $\mathbb{P}$–a.s, for all $n \in \mathbb{N}$. □

The differential form (2.1.19) of the filtering equation gives an important insight on the structure of the filtering process $\pi(\varphi)$. In fact, in each time interval $[\tau_n, \tau_{n+1})$, $n \in \mathbb{N}_0$, the filtering process satisfies $\mathbb{P}$–a.s. a deterministic differential equation (observe that since $Y_t = Y_{\tau_n}$ for all $t \in [\tau_n, \tau_{n+1})$, the operator A_{Y_t} is defined and fixed at each jump time τ_n). This will be a crucial fact in the characterization of π as a PDP.

The final and most important Theorem of this Section shows that, starting from the filtering equation (2.1.18) we can obtain an explicit equation for the measure-valued filtering process π. It provides the evolution equation satisfied by the filtering process π on the space $\mathcal{P}(I)$.

Theorem 2.1.7. *For each fixed $y \in O$ let $\mathcal{B}_y \colon \mathcal{M}(I) \to \mathcal{M}(I)$ be the operator*

$$\mathcal{B}_y\nu(\mathrm{d}z) := \mathbb{1}_{h^{-1}(y)}(z) \int_I \lambda(x, \mathrm{d}z)\, \nu(\mathrm{d}x) - \lambda(z)\nu(\mathrm{d}z), \quad \nu \in \mathcal{M}(I). \tag{2.1.21}$$

The filtering process $\pi = (\pi_t)_{t \geqslant 0}$ satisfies for all $t \geqslant 0$ and $\mathbb{P}$–a.s. the following SDE

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\pi_t = \mathcal{B}_{Y_t}\pi_t - \pi_t\,\mathcal{B}_{Y_t}\pi_t(I), & t \in [\tau_n, \tau_{n+1}),\, n \in \mathbb{N}_0 \\ \pi_{\tau_n} = H_{Y_{\tau_n}}[\mu_n], & n \in \mathbb{N}_0 \end{cases} \tag{2.1.22}$$

where the random measures μ_n, $n \in \mathbb{N}_0$ were defined in (2.1.20).

Proof. Let us notice first that from (2.1.19) we have that for all $n \in \mathbb{N}_0$ and $\mathbb{P}$–a.s.

$$\pi_{\tau_n}(\varphi) = \int_I \varphi(z)\, H_{Y_{\tau_n}}[\mu_n](\mathrm{d}z), \quad \mathbb{P}\text{–a.s.}, \quad \varphi \in \mathrm{B}_b(I).$$

Since we also have that $\pi_{\tau_n}(\varphi) = \int_I \varphi(z)\,\pi_{\tau_n}(\mathrm{d}z)$, $\mathbb{P}$–a.s., we get

$$\pi_{\tau_n} = H_{Y_{\tau_n}}[\mu_n], \quad \mathbb{P}\text{–a.s.},\ n \in \mathbb{N}_0.$$

We need to prove that the filtering process satisfies for each $n \in \mathbb{N}_0$ the ODE

$$\frac{\mathrm{d}}{\mathrm{d}t}\pi_t = \mathcal{B}_{Y_t}\pi_t - \pi_t\,\mathcal{B}_{Y_t}\pi_t(I), \quad t \in [\tau_n, \tau_{n+1}),\ \mathbb{P}\text{–a.s.}$$

It suffices to show that for each fixed $y \in O$ the operator $\mathcal{B}_y$ is the restriction to $\mathcal{M}(I)$ of the adjoint $\mathcal{A}_y^\star$ of the operator $\mathcal{A}_y$ introduced in Theorem 2.1.6. To see this, we have to recall that the dual space $\mathrm{B}_b(I)^\star$ of $\mathrm{B}_b(I)$ is isometrically isomorfic to the space $\mathrm{ba}(I)$ of bounded finitely additive regular measures defined on the algebra generated by open sets in I. Notice that $\mathcal{M}(I) \subset \mathrm{ba}(I)$.

Denote by $\langle \varphi, \nu \rangle := \int_I \varphi(x)\,\nu(\mathrm{d}x)$ the duality pairing between $\varphi \in \mathrm{B}_b(I)$ and $\nu \in \mathrm{B}_b(I)^\star$ and fix $n \in \mathbb{N}_0$. Then (2.1.19) can be written as

$$\langle \varphi, \dot{\pi}_t \rangle = \langle \mathcal{A}_{Y_t}\varphi, \pi_t \rangle - \langle \varphi, \pi_t \rangle \langle \mathcal{A}_{Y_t}1, \pi_t \rangle, \quad t \in [\tau_n, \tau_{n+1}),\ \mathbb{P}\text{–a.s.}\,.$$

The claim follows if we are able to show that

$$\langle \varphi, \dot{\pi}_t \rangle = \langle \varphi, \mathcal{A}_{Y_t}^\star \pi_t - \pi_t \mathcal{A}_{Y_t}^\star \pi_t(I) \rangle, \quad t \in [\tau_n, \tau_{n+1}),\ \mathbb{P}\text{–a.s.}$$

This fact follows from a repeated application of the Fubini-Tonelli theorem in the following chain of equalities, holding for all $\nu \in \mathcal{M}(I)$ and $\varphi \in \mathrm{B}_b(I)$.

$$\langle \mathcal{A}_y\varphi, \nu \rangle = \int_I \mathcal{L}\varphi(x)\,\nu(dx) - \int_I \int_I \mathbb{1}_{h^{-1}(y)^c}\varphi(z)\lambda(x, dz)\,\nu(dx) =$$
$$\int_I \varphi(z)\left\{\mathbb{1}_{h^{-1}(y)}(z) \int_I \lambda(x, dz)\,\nu(dx) - \lambda(z)\,\nu(dz)\right\} = \langle \varphi, \mathcal{B}_y\nu \rangle.$$

So, clearly $\mathcal{B}_y = \mathcal{A}_y^\star|_{\mathcal{M}(I)}$. □

2.2 The filtering process

In this Section we want to investigate the properties of the filtering process π. The core of this Section will be devoted to prove that this is a *Piecewise Deterministic Markov Process*, or PDMP for short. This class of processes, whose study has been started by by M.H.A. Davis (see [32] or [42]), has gained a lot of attention in applications, since it provides a framework to describe a vast range of phenomena whose behavior does not fit any diffusive model.

The main feature of PDMPs is that their dynamic, as the name suggests, is deterministic in specific time intervals, given by the occurrence of random jumps. In this time window the evolution of the process is governed by a *flow*, determined by a *vector field*, satisfying an ODE. The distribution of the time passing between two consecutive jump times is given by an exponential-like law. The position of the process after a jump, i. e. its post jump location, is provided by another specified probability measure. In Section 1.2 the reader can find a synthetic summary of the main results concerning PDMPs.

It is clear that, in our situation, the filtering process π appears to be a PDMP. The main task is to identify the characteristic triple that uniquely determines a PDMP, composed by the flow, the distribution of sojourn times and the law of the post jump locations.

Let us start by studying the flow. In particular, we are concerned with the well-posedness of the initial value problem described by equation (2.1.22) between two consecutive jump times. From now on we consider the set $\mathcal{M}(I)$ endowed with the total variation norm, indicated by $\|\cdot\|_{TV} := |\cdot|(I)$ (where $|\cdot|$ denotes the total variation measure). It is worth to recall that this norm is equivalent to the one defined as $\mathcal{M}(I) \ni \mu \mapsto \sup_{A\in\mathcal{I}} |\mu(A)|$. In particular, from the Hahn decomposition of signed measures, we have that

$$\|\mu\|_{TV} \leqslant 2 \sup_{A\in\mathcal{I}} |\mu(A)| \leqslant 2\|\mu\|_{TV}, \quad \mu \in \mathcal{M}(I). \tag{2.2.1}$$

We define, for each fixed $y \in O$, the set Δ_y as the family of probability measures on $(I, \mathcal{I})$ concentrated on $h^{-1}(y)$, i. e.

$$\Delta_y := \{\nu \in \mathcal{P}(I) \colon \nu\big(h^{-1}(y)^c\big) = 0\}, \quad y \in O. \tag{2.2.2}$$

This is a closed subset of $\mathcal{M}(I)$ since it can be written as the intersection of closed sets $\mathcal{P}(I)$ and $\mathcal{M}_y(I)$, where $\mathcal{M}_y(I) := \{\mu \in \mathcal{M}(I) \colon \mu\big(h^{-1}(y)\big) = 1\}$. In particular, $\mathcal{M}_y(I)$ is closed since the functional $\mu \mapsto \mu\big(h^{-1}(y)\big)$ is continuous on $\mathcal{M}(I)$ for all $y \in O$.

Let us define, for each fixed $y \in O$ the vector field $F_y \colon \mathcal{M}(I) \to \mathcal{M}(I)$

$$F_y(\nu) := \mathcal{B}_y\nu - \nu\mathcal{B}_y\nu(I), \quad \nu \in \mathcal{M}(I),\, y \in O \tag{2.2.3}$$

where $\mathcal{B}_y$ is the operator defined in (2.1.21). We already know that the solution to the following ODE

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} z_t = F_y(z_t), & t \geqslant 0 \\ z_0 = \rho, & \rho \in \Delta_y, \end{cases} \tag{2.2.4}$$

exists for each fixed $y \in O$ (for instance, consider the fact that the filtering process π satisfies it $\mathbb{P}$–a.s. in the time interval $[0, \tau_1)$ when $y = Y_0$). We want to investigate whether the solution to that equation is unique. In the following Lemma we prove that for each $y \in O$ the operator $\mathcal{B}_y$ is linear and continuous. This fact will be used in Proposition 2.2.2 to prove the local Lipschitz continuity of the vector field F_y.

Lemma 2.2.1. *Under Assumption 2.0.1, for each fixed $y \in O$ the operator $\mathcal{B}_y$ is linear and continuous on $\mathcal{M}(I)$.*

Proof. Fix $y \in O$. Linearity is obvious. To prove continuity, fix $A \in \mathcal{I}$ and $\mu, \nu \in \mathcal{M}(I)$. Then, we get

$$\begin{aligned}
&|\mathcal{B}_y\nu(A) - \mathcal{B}_y\mu(A)| \\
&= \left| \int_A \mathbb{1}_{h^{-1}(y)}(z) \int_I \lambda(x, \mathrm{d}z)\, [\nu - \mu](\mathrm{d}x) - \int_A \lambda(x)\, [\nu - \mu](\mathrm{d}x) \right| \\
&\leqslant \left| \int_I \lambda\big(x, A \cap h^{-1}(y)\big)\, [\nu - \mu](\mathrm{d}x) \right| + \left| \int_A \lambda(x)\, [\nu - \mu](\mathrm{d}x) \right| \\
&\leqslant 2 \int_I \lambda(x)\, |\nu - \mu|(\mathrm{d}x) \leqslant 2 \sup_{x \in I} \lambda(x) \|\nu - \mu\|_{TV}.
\end{aligned}$$

Since this equality holds for all $A \in \mathcal{I}$ we easily get

$$\|\mathcal{B}_y\nu - \mathcal{B}_y\mu\|_{TV} \leqslant 2 \sup_{A \in \mathcal{I}} |\mathcal{B}_y\nu(A) - \mathcal{B}_y\mu(A)| \leqslant 4 \sup_{x \in I} \lambda(x) \|\nu - \mu\|_{TV}$$

whence the continuity of the operator $\mathcal{B}_y$. □

Proposition 2.2.2. *Under Assumption 2.0.1, for each fixed $y \in O$, the map F_y is locally Lipschitz continuous on $\mathcal{M}(I)$.*

Proof. Fix $y \in O$ and $\mu, \nu \in \mathcal{M}(I)$. Then, recalling (2.2.1) and the result of Lemma 2.2.1, we have that

$$\begin{aligned}
&\|F_y(\nu) - F_y(\mu)\|_{TV} = \|\mathcal{B}_y\nu - \nu\mathcal{B}_y\nu(I) - \mathcal{B}_y\mu + \mu\mathcal{B}_y\mu(I)\|_{TV} \\
&\leqslant \|\mathcal{B}_y(\nu - \mu)\|_{TV} + \|\mathcal{B}_y\mu(I)[\nu - \mu]\|_{TV} + \|\nu[\mathcal{B}_y\nu(I) - \mathcal{B}_y\mu(I)]\|_{TV} \\
&\leqslant 4 \sup_{x \in I} \lambda(x) \|\nu - \mu\|_{TV} + |\mathcal{B}_y\mu(I)| \|\nu - \mu\|_{TV} + \|\nu\|_{TV} |\mathcal{B}_y\nu(I) - \mathcal{B}_y\mu(I)| \\
&\leqslant (4 + \|\mu\|_{TV}) \sup_{x \in I} \lambda(x) \|\nu - \mu\|_{TV} + \|\nu\|_{TV} \|\mathcal{B}_y(\nu - \mu)\|_{TV} \\
&\leqslant (4 + \|\mu\|_{TV} + 4\|\nu\|_{TV}) \sup_{x \in I} \lambda(x) \|\nu - \mu\|_{TV}
\end{aligned}$$

whence the result. Notice that the term $|\mathcal{B}_y\mu(I)| = \left| \int_I \lambda\big(x, h^{-1}(y)^c\big)\, \mu(\mathrm{d}x) \right|$ is easily majorized by $\|\mu\|_{TV} \sup_{x \in I} \lambda(x)$. □

Remark 2.2.1. It is worth noting that from the previous computations we deduce that the field F_y is Lipschitz continuous on $\mathcal{P}(I)$.

Theorem 2.2.3. *Under Assumption 2.0.1, for each fixed $y \in O$ the ODE (2.2.4) admits a unique global solution $z \in \mathrm{C}^1\big([0, +\infty); \Delta_y\big)$.*

Proof. Fix $y \in O$. The claim follows from [50, Th. 4]. To apply it we have to verify the following assumptions (we point out in square brackets the reference to the corresponding hypotheses of the cited work. We invite the interested reader to consult it for further details).

1. F_y is continuous from Δ_y into $\mathcal{M}(I)$ [Condition C1].
2. $\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \inf_{\nu \in \Delta_y} \|\mu + \varepsilon F_y(\mu) - \nu\|_{TV} = 0$ for all $\mu \in \Delta_y$ [Condition C2].
3. For all $K > 0$ there exists $C_K > 0$ such that for all $\mu \in \Delta_y$ with $\|\mu\|_{TV} \leqslant K$ it holds that $\|F_y(\mu)\|_{TV} \leqslant C_K$ [(i) of Theorem 4].

4. $\langle \mu - \nu, F_y(\mu) - F_y(\nu) \rangle_+ \leqslant C\|\mu - \nu\|_{TV}^2$ for all $\mu, \nu \in \Delta_y$ and some $C > 0$ [(3.11)]

where for all $\mu, \nu \in \mathcal{M}(I)$ we define $\langle \mu, \nu \rangle_+ := \sup_{\substack{\Phi \in \mathcal{M}(I)^\star \\ \Phi\mu = \|\mu\|_{TV}^2}} \Phi\nu$. The set $\mathcal{M}(I)^\star$ is the topological dual space of $\mathcal{M}(I)$.

Claim 1. In Proposition 2.2.2 we proved that the vector field F_y is locally Lipschitz continuous on $\mathcal{M}(I)$. In Remark 2.2.1 we also noted that it is Lipschitz continuous on $\mathcal{P}(I)$, hence we easily deduce its continuity on Δ_y.

Claim 2. Fix $\mu \in \Delta_y$. To get the claim, it suffices to prove that for $\varepsilon > 0$ small enough $\mu + \varepsilon F_y(\mu) \in \Delta_y$.

We prove, first that $\mu + \varepsilon F_y(\mu) \in \mathcal{M}_+(I)$ for $\varepsilon > 0$ small enough. For all fixed $A \in \mathcal{I}$ we have that

$$\begin{aligned}
\big[\mu + \varepsilon F_y(\mu)\big](A) &= \mu(A) + \varepsilon F_y(\mu;\, A) = \mu(A) + \varepsilon\big[\mathcal{B}_y\mu(A) - \mu(A)\mathcal{B}_y\mu(I)\big] \\
&= \mu(A) + \varepsilon\bigg[\int_I \lambda(x, A \cap h^{-1}(y))\, \mu(\mathrm{d}x) - \int_A \lambda(z)\, \mu(\mathrm{d}z) \\
&\qquad + \mu(A)\int_I \lambda(x, h^{-1}(y)^c)\, \mu(\mathrm{d}x)\bigg]
\end{aligned}$$

Recalling that Assumption 2.0.1 is in force, we have the obvious estimate

$$\int_A \lambda(z)\, \mu(\mathrm{d}z) \leqslant \sup_{x \in A} \lambda(x)\mu(A) \leqslant \sup_{x \in I} \lambda(x)\mu(A).$$

Hence, taking $\varepsilon < [\sup_{x \in I} \lambda(x)]^{-1}$ we get

$$\begin{aligned}
\big[\mu + \varepsilon F_y(\mu)\big](A) > \varepsilon\bigg[&\int_I \lambda(x, A \cap h^{-1}(y))\, \mu(\mathrm{d}x) \\
&+ \mu(A)\int_I \lambda(x, h^{-1}(y)^c)\, \mu(\mathrm{d}x)\bigg] \geqslant 0.
\end{aligned}$$

Now it remains to prove that $\mu + \varepsilon F_y(\mu)$ is a probability measure and that is concentrated on $h^{-1}(y)$. This can be easily shown, since it is immediately seen that $\big[\mu + \varepsilon F_y(\mu)\big](I) = 1$ and we have that

$$\begin{aligned}
\big[\mu + \varepsilon F_y(\mu)\big]\big(h^{-1}(y)\big) &= \mu\big(h^{-1}(y)\big) + \varepsilon F_y\big(\mu;\, h^{-1}(y)\big) \\
&= 1 + \varepsilon\bigg[\int_{h^{-1}(y)} \int_I \lambda(x, \mathrm{d}z)\, \mu(\mathrm{d}x) - \int_{h^{-1}(y)} \lambda(z)\, \mu(\mathrm{d}z) \\
&\qquad - \int_{h^{-1}(y)} \int_I \lambda(x, \mathrm{d}z)\, \mu(\mathrm{d}x) + \int_I \lambda(z)\, \mu(\mathrm{d}z)\bigg] = 1
\end{aligned}$$

thanks to the equality $\int_{h^{-1}(y)} \lambda(z)\, \mu(\mathrm{d}z) = \int_I \lambda(z)\, \mu(\mathrm{d}z)$, implied by the fact that $\mu \in \Delta_y$.

Claim 3. Fix $\mu \in \Delta_y$. The claim is easily proved thanks to the following estimate, holding for all $A \in \mathcal{I}$.

$$\begin{aligned}|F_y(\mu;\, A)| &= |\mathcal{B}_y\mu(A) - \mu(A)\mathcal{B}_y(I)| \\ &= \left| \int_I \lambda(x, A \cap h^{-1}(y))\, \mu(\mathrm{d}x) - \int_A \lambda(z)\, \mu(\mathrm{d}z) \right. \\ &\qquad \left. + \mu(A) \int_I \lambda(x, h^{-1}(y)^c)\, \mu(\mathrm{d}x) \right| \leqslant 3 \sup_{x \in I} \lambda(x).\end{aligned}$$

From this inequality, it follows that $\|F_y(\mu)\|_{TV} \leqslant 6 \sup_{x \in I} \lambda(x)$, whence the result.
Claim 4. Fix $\mu, \nu \in \Delta_y$ and take $\Phi \in \mathcal{M}(I)^\star$ such that $\Phi(\mu - \nu) = \|\mu - \nu\|_{TV} = \|\Phi\|_\star$, where $\|\cdot\|_\star$ denotes the norm in the dual space $\mathcal{M}(I)^\star$. Thanks to Proposition 2.2.2 (see also Remark 2.2.1) we have that

$$\Phi\big(F_y(\mu) - F_y(\nu)\big) \leqslant \|\Phi\|_\star \, \|F_y(\mu) - F_y(\nu)\|_{TV} \leqslant 9 \sup_{x \in I} \lambda(x) \|\mu - \nu\|_{TV}^2.$$

Since this estimate holds for all required Φ, we get the result taking the supremum. □

Remark 2.2.2. In what follows, we will denote the solution z by $\phi_{y,\rho}(\cdot)$, to stress the dependence on $y \in O$ and $\rho \in \Delta_y$. By standard results on ODEs, $(t, \rho) \mapsto \phi_{y,\rho}(t)$ is continuous for each $y \in O$ and it enjoys the flow property, i. e. $\phi_{y,\phi_y(s,\rho)}(t) = \phi_{y,\rho}(t + s)$, for $t, s \geqslant 0$. The function $y \mapsto \phi_{y,\cdot}(\cdot)$ is called the *flow* associated with the vector field F_y on Δ_y. To simplify the notation, it is convenient to define the set $\Delta_e = \bigcup_{y \in O} \Delta_y$, named the *effective simplex* to preserve the terminology used in [25]. Notice that the union is disjoint, as is immediate to prove from the definition given in (2.2.2). In this way we can define a *global flow* ϕ on Δ_e setting $\phi_\rho(t) = \phi_{y,\rho}(t)$, if $\rho \in \Delta_y$. For all fixed $t \geqslant 0$, $\rho \mapsto \phi_\rho(t)$ is a function mapping Δ_e into itself and leaving each set Δ_y invariant. Finally, we can associate to the global flow a *global vector field* $F\colon \Delta_e \to \Delta_e$ defined as

$$F(\nu) := F_y(\nu) = \mathcal{B}_y\nu - \nu\, \mathcal{B}_y\nu(I), \quad \nu \in \Delta_y. \tag{2.2.5}$$

The effective simplex bears this name because of its relationship with the canonical simplex on euclidean spaces. In fact, if we consider the state spaces I and O of the signal and observed processes as finite sets (we will do so in Chapter 3), then the effective simplex is made of pairwise disjoint faces of the canonical simplex on $\mathbb{R}^{|I|}$, where $|I|$ denotes the cardinality of the set I. The shape of these faces (points, segments, triangles, tetrahedra, etc . . .) depends on the function h. The evolution of the filtering process takes place only on parts of the boundary of the canonical simplex.

Before moving on to prove the characterization of the filtering process as a PDMP, let us precise that, as far as topology is concerned, the effective simplex will be regarded as a topological space (Δ_e, τ_e) under the relative topology τ_e. This is defined as

$$\tau_e := \{\Delta_e \cap U,\, U \in \tau_{TV}\}$$

where τ_{TV} is the topology on $\mathcal{M}(I)$ induced by the total variation norm. In this way, we can also consider the effective simplex as a measurable space, endowing it with the Borel σ-algebra $\mathcal{B}(\Delta_e)$.

In order to prove that the filtering process is a PDMP, it is convenient to put ourselves in a canonical setting for our filtering problem with respect to the unobserved process X. This construction will have a fundamental role in studying the optimal control problem, as done in Chapters 3 and 4.

Let us define Ω as the set

$$\Omega = \{\omega = (i_0, t_1, i_1, t_2, i_2, \ldots) \colon \\ i_0 \in I, i_n \in I, t_n \in (0, +\infty], t_n < +\infty \Rightarrow t_n < t_{n+1}, n \in \mathbb{N}\}.$$

For each $n \in \mathbb{N}$ we introduce the following random variables

$$T_0(\omega) = 0; \quad T_n(\omega) = t_n; \quad T_\infty(\omega) = \lim_{n\to\infty} T_n(\omega); \quad \xi_0(\omega) = i_0; \quad \xi_n(\omega) = i_n$$

and we define the random measure on $\big((0, +\infty) \times I,\, \mathcal{B}\big((0, +\infty)\big) \otimes \mathcal{I}\big)$

$$n(\omega, \mathrm{d}t\,\mathrm{d}z) = \sum_{n\in\mathbb{N}} \delta_{\left(T_n(\omega),\, \xi_n(\omega)\right)}(\mathrm{d}t\,\mathrm{d}z)\mathbb{1}_{\{T_n<+\infty\}}(\omega), \quad \omega \in \Omega$$

with associated natural filtration $\mathcal{N}_t = \sigma\big(n\big((0, t] \times A\big)\big),\, 0 \leqslant s \leqslant t,\, A \in \mathcal{I}\big)$. Finally, let us specify the σ-algebras

$$\mathcal{X}_0^\circ = \sigma(\xi_0); \qquad \mathcal{X}_t^\circ = \sigma(\mathcal{X}_0^\circ \cup \mathcal{N}_t); \qquad \mathcal{X}^\circ = \sigma\Big(\bigcup_{t\geqslant 0} \mathcal{X}_t\Big).$$

The unobserved process X is defined as

$$X_t(\omega) = \begin{cases} \xi_n(\omega), & t \in \big[T_n(\omega), T_{n+1}(\omega)\big),\, n \in \mathbb{N}_0,\, T_n(\omega) < +\infty \\ i_\infty, & t \in \big[T_\infty(\omega), +\infty),\, T_\infty(\omega) < +\infty \end{cases}$$

where $i_\infty \in I$ is an arbitrary state, that is irrelevant to specify. Next, we define the observed process Y and its natural filtration $(\mathcal{Y}_t^\circ)_{t\geqslant 0}$ as

$$Y_t(\omega) = h(X_t(\omega)),\, t \geqslant 0,\, \omega \in \Omega; \qquad \mathcal{Y}_t^\circ = \sigma\big(Y_s,\, 0 \leqslant s \leqslant t\big),\, t \geqslant 0.$$

It is clear that we can equivalently describe this process (as is the case for X) via a MPP $(\eta_n, \tau_n)_{n\in\mathbb{N}}$ together with the initial condition $\eta_0 = h(\xi_0) = Y_0$. Accordingly, the σ-algebras of the natural filtration of Y are the smallest σ-algebras generated by the union of $\sigma(\eta_0)$ and the σ-algebras of the natural filtration of the MPP $(\eta_n, \tau_n)_{n\in\mathbb{N}}$.

Remark 2.2.3. Notice that here we constructed the unobserved process X starting from its MPP counterpart $(T_n, \xi_n)_{n\in\mathbb{N}}$, whereas at the beginning of this Chapter we were given a pure jump Markov process and we associated its corresponding MPP. We could have done the same also here and the subsequent results would have remained the same. However, it is preferable to use this construction because this definition of the space Ω is the most natural to the MPP setting, upon which we heavily rely in all of this thesis.

Next, for every $\mu \in \mathcal{P}(I)$ let P_μ be the unique probability measure on $(\Omega, \mathcal{X}^\circ)$ such that X is a $(\mathcal{X}^\circ, \mathrm{P}_\mu)$–Markov process with state space I, initial law μ and generator $\mathcal{L}$, defined in (2.1.8). This means that for all $A \in \mathcal{I}$, all $s, t \geqslant 0$ and all $f \in \mathrm{B}_b(I)$ it holds

$$\mathrm{P}_\mu(X_0 \in A) = \mu(A), \quad \mathrm{P}_\mu\text{–a.s.}$$
$$\mathrm{E}_\mu\big[f(X_{t+s}) \mid \mathcal{X}_t^\circ\big] = e^{s\mathcal{L}} f(X_t), \quad \mathrm{P}_\mu\text{–a.s.}$$

It follows from Assumption 2.0.1 and by standard arguments that the point process n is P_μ-a.s. non-explosive, i. e. that $T_\infty = +\infty$, P_μ-a.s..

To conclude the previous construction, for a fixed probability measure μ on I we define

- $\mathcal{X}^\mu$ the P_μ-completion of $\mathcal{X}^\circ$ (P_μ is extended to $\mathcal{X}^\mu$ in the natural way).
- $\mathcal{Z}^\mu$ the family of elements of $\mathcal{X}^\mu$ with zero P_μ – probability.
- $\mathcal{Y}_t^\mu = \sigma(\mathcal{Y}_t^\circ, \mathcal{Z}^\mu)$, for $t \geqslant 0$.

$(\mathcal{Y}_t^\mu)_{t\geqslant 0}$ is called the *natural completed filtration* of Y.

In this canonical framework for the unobserved process X we can consider, for each fixed $\mu \in \mathcal{P}(I)$, the process $\pi^\mu = (\pi_t^\mu)_{t\geqslant 0}$ defined as the trajectory-wise unique solution to the SDE (2.1.22). This means that for each $\omega \in \Omega$ we define $\pi^\mu(\omega)$ to be the unique solution to

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\pi_t^\mu(\omega) = F_{Y_t(\omega)}\big(\pi_t^\mu(\omega)\big), \quad t \in [\tau_n(\omega), \tau_{n+1}(\omega)),\ n \in \mathbb{N}_0 \\ \pi_0^\mu(\omega) = H_{Y_0(\omega)}[\mu] \\ \pi_{\tau_n(\omega)}^\mu(\omega) = H_{Y_{\tau_n(\omega)}}\big[\Lambda\big(\pi_{\tau_n^-}^\mu(\omega)\big)\big],\ n \in \mathbb{N} \end{cases} \tag{2.2.6}$$

where $\Lambda\colon \Delta_e \to \mathcal{M}_+(I)$ is the function given by

$$\Lambda(\nu) := \mathbb{1}_{h^{-1}(y)^c}(z) \int_I \lambda(x, \mathrm{d}z)\,\nu(\mathrm{d}x), \quad \nu \in \Delta_y \tag{2.2.7}$$

and the quantity $\pi_{\tau_n^-(\omega)}(\omega)$ is defined as

$$\pi_{\tau_n^-(\omega)}(\omega) := \lim_{t\to\tau_n(\omega)^-} \pi_t(\omega), \quad \text{on } \{\omega \in \Omega\colon \tau_n(\omega) < +\infty\}$$

Thanks to Theorem 2.2.3, Equation (2.2.6) uniquely determines a $(\mathcal{Y}_t^\circ)_{t\geqslant 0}$–adapted cádlág and Δ_e–valued process. By Theorem 2.1.7, we deduce that it is a modification of the filtering process, i. e. for all $t \geqslant 0$ and all $A \in \mathcal{I}$ it holds

$$\pi_t^\mu(A) = \mathrm{P}_\mu(X_t \in A \mid \mathcal{Y}_t^\mu), \quad \mathrm{P}_\mu\text{–a.s.}$$

Since the filtering process is $(\mathcal{Y}_t^\mu)_{t\geqslant 0}$–adapted and the filtration $(\mathcal{Y}_t^\mu)_{t\geqslant 0}$ is right-continuous, we can choose (and we will, whenever needed) a $(\mathcal{Y}_t^\mu)_{t\geqslant 0}$–progressive version of the filtering process itself.

We are now ready to state the Markov property for the filtering process π^μ with respect to the natural completed filtration of the observed process Y, for each fixed $\mu \in \mathcal{P}(I)$. This is the content of Proposition 2.2.5, preceded by the useful technical Lemma 2.2.4. We omit their proof, being slight generalizations of [25, Proposition 3.3 and Proposition 3.4].

Lemma 2.2.4. *For fixed t $\geqslant 0$, let us denote by X_t^∞ the future trajectory of the process X starting at time t. For all $\mu \in \mathcal{P}(I)$, $t \geqslant 0$ and $C \in \mathcal{X}^\mu$, it holds*

$$\mathrm{P}_\mu(X_t^\infty \in C \mid \mathcal{Y}_t^\mu) = \mathrm{P}_{\pi_t^\mu}(C), \quad \mathrm{P}_\mu\text{–a.s.}$$

Proposition 2.2.5. *For fixed t $\geqslant 0$ consider the transition kernel p_t from $\big(\Delta_e,\, \mathcal{B}(\Delta_e)\big)$ into itself given by*

$$p_t(\nu, D) := \mathrm{P}_\nu(\pi_t^\nu \in D), \quad \nu \in \Delta_e,\ D \in \mathcal{B}(\Delta_e).$$

Then $(p_t)_{t\geqslant 0}$ is a Markov transition function on $\big(\Delta_e,\, \mathcal{B}(\Delta_e)\big)$. Moreover, for every fixed $\mu \in \mathcal{P}(I)$, the process π^μ in the probability space $(\Omega, \mathcal{X}^\mu, \mathrm{P}_\mu)$ is a Δ_e–valued Markov process with respect to $(\mathcal{Y}_t^\mu)_{t\geqslant 0}$, having transition function $(p_t)_{t\geqslant 0}$. Otherwise said, the following equality holds, for all $s, t \geqslant 0$ and all $D \in \mathcal{B}(\Delta_e)$

$$\mathrm{P}_\mu(\pi_{t+s}^\mu \in D \mid \mathcal{Y}_t^\mu) = p_s(\pi_t^\mu, D), \quad \mathrm{P}_\mu\text{–a.s.}$$

Having proved the Markov property for the filtering process π^μ, we can show that it is a PDMP by introducing the following three quantities that will be proved to form its *characteristic triple*.

1. The *vector field* F is the global vector field defined in Equation (2.2.5), i. e.

$$F(\nu) \coloneqq \mathcal{B}_y\nu - \nu\,\mathcal{B}_y\nu(I), \quad \nu \in \Delta_y. \tag{2.2.8}$$

   The global flow ϕ previously introduced is associated to this vector field (see Remark 2.2.2).

2. The *jump rate function* $r\colon \Delta_e \to [0, +\infty)$ is defined by

$$r(\nu) \coloneqq -\mathcal{B}_y\nu(I) = \int_I \lambda\big(x, h^{-1}(y)^c\big)\,\nu(\mathrm{d}x), \quad \nu \in \Delta_y. \tag{2.2.9}$$

3. The *transition probability* R from $\big(\Delta_e, \mathcal{B}(\Delta_e)\big)$ into itself is defined by

$$R(\nu, D) \coloneqq \int_O \mathbb{1}_D\big(H_\upsilon[\Lambda(\nu)]\big)\,\rho(\nu, \mathrm{d}\upsilon), \quad \nu \in \Delta_y,\, D \in \mathcal{B}(\Delta_e) \tag{2.2.10}$$

   where ρ is a transition probability from $\big(\Delta_e, \mathcal{B}(\Delta_e)\big)$ into $(O, \mathcal{O})$ defined for all $\nu \in \Delta_y$ and all $B \in \mathcal{O}$ as

$$\rho(\nu, B) \coloneqq \begin{cases} \dfrac{1}{r(\nu)} \displaystyle\int_I \lambda\big(x, h^{-1}(B \setminus \{y\})\big)\,\nu(\mathrm{d}x), & \text{if } r(\nu) > 0 \\ q_y(B), & \text{if } r(\nu) = 0 \end{cases} \tag{2.2.11}$$

   where $(q_y)_{y \in O}$ is a family of probability measures, each concentrated on the level set $h^{-1}(y)$, $y \in O$, whose exact values are irrelevant.

   Since for any given $\nu \in \Delta_y$ the probability $\rho(\nu, \cdot)$ is concentrated on the set $O \setminus \{y\}$, the probability $R(\nu, \cdot)$ is concentrated on $\Delta_e \setminus \Delta_y$, as we expected to be, given the structure of the filtering process.

Remark 2.2.4. Note that if $r(\nu) > 0$ then $r(\nu)\,\rho(\nu, \mathrm{d}y) = \Lambda(\rho) \circ h^{-1}(\mathrm{d}y)$.

In the following Proposition, which will be needed to characterize the filtering process as a PDMP, we show that the law of the observed process Y can be expressed via the filtering process itself. It is clear that since the process Y is piecewise constant, its law is uniquely determined by the finite dimensional distributions of the process $\{Y_0, \tau_1, Y_{\tau_1}, \dots\}$. These in turn are completely characterized by the law of Y_0, which is obvious, and by the following distributions

$$\begin{aligned} &\mathrm{P}_\mu(\tau_{n+1} - \tau_n > t,\, \tau_n < +\infty \mid \mathcal{Y}^\mu_{\tau_n}), && t \geqslant 0,\, n \in \mathbb{N}_0 \\ &\mathrm{P}_\mu(Y_{\tau_{n+1}} \in B,\, \tau_{n+1} < +\infty \mid \mathcal{Y}^\mu_{\tau_n}), && B \in \mathcal{O},\, n \in \mathbb{N}_0. \end{aligned}$$

Proposition 2.2.6. *For all fixed $\mu \in \mathcal{P}(I)$ the distributions of the sojourn times and the post jump locations of the observed process Y are given by the following equalities, holding P_μ–a.s. for all $t \geqslant 0$, $B \in \mathcal{O}$ and all $n \in \mathbb{N}_0$*

$$\mathrm{P}_\mu(\tau_{n+1} - \tau_n > t,\, \tau_n < +\infty \mid \mathcal{Y}^\mu_{\tau_n}) = \exp\left\{-\int_0^t r\big(\phi_{\pi^\mu_{\tau_n}}(s)\big)\,\mathrm{d}s\right\}\mathbb{1}_{\tau_n < +\infty} \tag{2.2.12}$$

$$\mathrm{P}_\mu(Y_{\tau_{n+1}} \in B,\, \tau_{n+1} < +\infty \mid \mathcal{Y}^\mu_{\tau_n}) = \rho\big(\phi_{\pi^\mu_{\tau_n}}(\tau^-_{n+1} - \tau_n), B\big)\mathbb{1}_{\tau_{n+1} < +\infty}. \tag{2.2.13}$$

Proof. Let us fix $n \in \mathbb{N}_0$. To start, we will look for an expression for the joint distribution of the jump times and post jump locations of the process Y, i. e. for all $T \geqslant 0$ and all $B \in \mathcal{O}$ the quantity

$$\mathrm{P}_\mu(\tau_{n+1} \leqslant T,\, Y_{\tau_{n+1}} \in B \mid \mathcal{Y}^\mu_{\tau_n}), \quad \text{on } \{\tau_n < +\infty\}.$$

Notice, first, that for all fixed $T \geqslant 0$ and $B \in \mathcal{O}$ we can write

$$Z_T(B) \coloneqq \mathbb{1}_{\tau_{n+1} \leqslant T} \mathbb{1}_{Y_{\tau_{n+1}} \in B} = m\big((0, T \wedge \tau_{n+1}] \times B\big) - m\big((0, T \wedge \tau_n] \times B\big)$$

where m is the random counting measure associated to Y and defined in (2.1.3). To see this, it suffices to observe that

$$m\big((0, T \wedge \tau_n] \times B\big) = \sum_{k=1}^{n} \mathbb{1}_{\tau_k \leqslant T} \mathbb{1}_{Y_{\tau_k} \in B}.$$

Clearly, $(Z_T(B))_{T \geqslant 0}$ is a $(\mathcal{Y}^\mu_T)_{T \geqslant 0}$–adapted point process and, thanks to Lemma 2.1.1, a straightforward computation (using once more the obvious facts $Y_{s-} = Y_s$ and $\pi_{s-} = \pi_s$, $\mathrm{d}s$–a.e.) shows that its $(\mathcal{Y}^\mu_T)_{T \geqslant 0}$–compensator is given by

$$\begin{aligned}
\zeta_T(B) &\coloneqq \int_{T \wedge \tau_n}^{T \wedge \tau_{n+1}} \int_I \lambda\big(x, h^{-1}(B \setminus \{Y_{s-}\})\big)\, \pi^\mu_{s-}(\mathrm{d}x)\, \mathrm{d}s \\
&= \int_0^T \mathbb{1}_{\tau_n \leqslant s < \tau_{n+1}} \int_I \lambda\big(x, h^{-1}(B \setminus \{Y_s\})\big)\, \pi^\mu_s(\mathrm{d}x)\, \mathrm{d}s.
\end{aligned}$$

Moreover, since for all $k \in \mathbb{N}$ the stopped process

$$\left(m\big((0, T \wedge \tau_k] \times B\big) - \int_{(0, T \wedge \tau_k]} \int_I \lambda\big(x, h^{-1}(B \setminus \{Y_{s-}\})\big)\, \pi^\mu_{s-}(\mathrm{d}x)\, \mathrm{d}s\right)_{T \geqslant 0}$$

is a uniformly integrable $(\mathcal{Y}^\mu_T)_{T \geqslant 0}$–martingale (cfr. number 3 of Proposition 1.1.5), the compensated process $(Z_T(B) - \zeta_T(B))_{T \geqslant 0}$ is a uniformly integrable $(\mathcal{Y}^\mu_T)_{T \geqslant 0}$–martingale. Hence by applying Doob's optional sampling theorem we get that for all $T \geqslant 0$

$$\mathrm{E}_\mu[Z_T(B) \mid \mathcal{Y}^\mu_{\tau_n}] = \mathrm{E}_\mu[\zeta_T \mid \mathcal{Y}^\mu_{\tau_n}], \quad \mathrm{P}_\mu\text{–a.s.}$$

or otherwise written

$$\begin{aligned}
&\mathrm{P}_\mu(\tau_{n+1} \leqslant T,\, Y_{\tau_{n+1}} \in B \mid \mathcal{Y}^\mu_{\tau_n}) = \\
&\qquad \mathrm{E}_\mu\left[\int_0^T \mathbb{1}_{\tau_n \leqslant s < \tau_{n+1}} \int_I \lambda\big(x, h^{-1}(B \setminus \{Y_s\})\big)\, \pi^\mu_s(\mathrm{d}x)\, \mathrm{d}s \;\middle|\; \mathcal{Y}^\mu_{\tau_n}\right], \quad \mathrm{P}_\mu\text{–a.s.}
\end{aligned}$$

Noting that for $\tau_n \leqslant s < \tau_{n+1}$ we have that $Y_s = Y_{\tau_n}$ and $\pi_s = \phi_{\pi^\mu_{\tau_n}}(s - \tau_n)$ we can write the previous equation as

$$\begin{aligned}
&\mathrm{P}_\mu(\tau_{n+1} \leqslant T,\, Y_{\tau_{n+1}} \in B \mid \mathcal{Y}^\mu_{\tau_n}) = \\
&\mathrm{E}_\mu\left[\int_0^T \int_I \mathbb{1}_{\tau_n \leqslant s < \tau_{n+1}} \lambda\big(x, h^{-1}(B \setminus \{Y_{\tau_n}\})\big)\, \phi_{\pi^\mu_{\tau_n}}(s - \tau_n;\, \mathrm{d}x)\, \mathrm{d}s \;\middle|\; \mathcal{Y}^\mu_{\tau_n}\right], \mathrm{P}_\mu\text{–a.s.}
\end{aligned} \tag{2.2.14}$$

Now let $t \geqslant 0$ be fixed. Since the random variable $t + \tau_n$ is $Y^\mu_{\tau_n}$–measurable, we immediately get from (2.2.14)

$$\begin{aligned}
&\mathrm{P}_\mu(\tau_{n+1} - \tau_n \leqslant t,\, Y_{\tau_{n+1}} \in B \mid \mathcal{Y}^\mu_{\tau_n}) = \\
&\mathrm{E}_\mu\left[\int_0^{t+\tau_n} \int_I \mathbb{1}_{\tau_n \leqslant s < \tau_{n+1}} \lambda\big(x, h^{-1}(B \setminus \{Y_{\tau_n}\})\big)\, \phi_{\pi^\mu_{\tau_n}}(s - \tau_n;\, \mathrm{d}x)\, \mathrm{d}s \mid \mathcal{Y}^\mu_{\tau_n}\right] = \\
&\quad \mathrm{E}_\mu\left[\int_0^{t} \int_I \mathbb{1}_{\tau_{n+1} - \tau_n > s} \lambda\big(x, h^{-1}(B \setminus \{Y_{\tau_n}\})\big)\, \phi_{\pi^\mu_{\tau_n}}(s;\, \mathrm{d}x)\, \mathrm{d}s \mid \mathcal{Y}^\mu_{\tau_n}\right],\ \mathrm{P}_\mu\text{–a.s.}
\end{aligned} \tag{2.2.15}$$

We need to exchange the conditional expectation and the time integral appearing in the last line of (2.2.15). Let us consider a regular version G^n of the conditional distribution $\mathrm{P}_\mu(\tau_{n+1} - \tau_n \in \cdot, Y_{\tau_n} \in \cdot \mid \mathcal{Y}^\mu_{\tau_n})$, which always exists in our setting. Define the function $g^n \colon \Omega \times [0, +\infty) \to [0, 1]$ as

$$g^n(\omega, t) \coloneqq G^n\big(\omega, (t, +\infty], O\big).$$

Clearly g^n enjoys the following properties

- $g^n(\omega, t) = \mathrm{P}_\mu(\tau_{n+1} - \tau_n > t \mid \mathcal{Y}^\mu_{\tau_n})$, P_μ–a.s.
- g^n is $\big(\mathcal{Y}^\mu_{\tau_n} \otimes \mathcal{B}([0, +\infty))\big)$–measurable.
- For all $\omega \in \Omega$ the map $t \mapsto g^n(\omega, t)$ is non-increasing and right-continuous.

Applying the Fubini-Tonelli theorem to the last line of (2.2.15) we get

$$\begin{aligned}
&\mathrm{P}_\mu(\tau_{n+1} - \tau_n \leqslant t,\, Y_{\tau_{n+1}} \in B \mid \mathcal{Y}^\mu_{\tau_n}) = G\big((0, t], B\big) \\
&\qquad \int_0^t g^n(s) \int_I \lambda\big(x, h^{-1}(B \setminus \{Y_{\tau_n}\})\big)\, \phi_{\pi^\mu_{\tau_n}}(s;\, \mathrm{d}x)\, \mathrm{d}s, \quad \mathrm{P}_\mu\text{–a.s.}
\end{aligned} \tag{2.2.16}$$

We are now ready to use Proposition 1.1.8 (or, equivalently, Theorem 1.1.13) and obtain that g^n satisfies

$$g^n(t) = 1 - \int_0^t g^n(s) \int_I \lambda\big(x, h^{-1}(Y_{\tau_n})^c\big)\, \phi_{\pi^\mu_{\tau_n}}(s;\, \mathrm{d}x)\, \mathrm{d}s, \quad t \in (0, \tau_{n+1} - \tau_n].$$

This equality implies that on $(0, \tau_{n+1} - \tau_n]$ the function g^n is absolutely continuous for each $\omega \in \Omega$ and solves ω-by-ω the following ODE

$$\begin{cases}
\dfrac{\mathrm{d}}{\mathrm{d}t} g^n(t) = -g^n(t) \displaystyle\int_I \lambda\big(x, h^{-1}(Y_{\tau_n})^c\big)\, \phi_{\pi^\mu_{\tau_n}}(t;\, \mathrm{d}x), \quad t \in (0, \tau_{n+1} - \tau_n] \\
g_n(0) = 1
\end{cases}$$

whose solution is clearly $g_n(t) = \exp\big\{-\int_0^t \int_I \lambda\big(x, h^{-1}(Y_{\tau_n})^c\big)\, \phi_{\pi^\mu_{\tau_n}}(s;\, \mathrm{d}x)\, \mathrm{d}s\big\}$ for $t \in (0, \tau_{n+1} - \tau_n]$. Therefore we get (2.2.12).

Finally, (2.2.13) follows from an immediate application of both Theorem 1.1.13 and Theorem 1.1.7. In fact, we have that on $\{\tau_{n+1} < +\infty\}$

$$\begin{aligned}
&\mathrm{P}_\mu(Y_{\tau_{n+1}} \in B \mid \mathcal{Y}^\mu_{\tau_n}) = \\
&= \frac{1}{r\big(\phi_{\pi^\mu_{\tau_n}}(\tau^-_{n+1} - \tau_n)\big)} \int_I \lambda\big(x, h^{-1}(B \setminus \{Y_{\tau_n}\})\big)\, \phi_{\pi^\mu_{\tau_n}}(\tau^-_{n+1} - \tau_n;\, \mathrm{d}x), \quad \mathrm{P}_\mu\text{–a.s.}
\end{aligned}$$

whence the desired equality. Notice that the fraction is well defined since, by Theorem 1.1.12, $r\big(\phi_{\pi^\mu_{\tau_n}}(\tau^-_{n+1} - \tau_n)\big) > 0$ on $\{\tau_{n+1} < +\infty\}$. □

We are now ready to prove the main Theorem of this Section, that is the characterization of the filtering process as a PDMP.

Theorem 2.2.7. *For every $\nu \in \Delta_e$ the filtering process $\pi^\nu = (\pi^\nu_t)_{t\geqslant 0}$ defined on the probability space $(\Omega, \mathcal{X}^\nu, \mathrm{P}_\nu)$ and taking values in Δ_e is a* Piecewise Deterministic Markov Process *with respect to the filtration $(\mathcal{Y}^\nu_t)_{t\geqslant 0}$ and the triple (F, r, R) defined in (2.2.8)–(2.2.10) and with starting point ν.*

More specifically, we have that π^ν is a $(\mathcal{Y}^\nu_t)_{t\geqslant 0}$–Markov process and the following equalities hold P_ν–a.s.

$$\pi^\nu_t = \phi_{\pi^\nu_{\tau_n}}(t - \tau_n), \quad t \in [\tau_n, \tau_{n+1}),\, n \in \mathbb{N}_0 \tag{2.2.17}$$

$$\mathrm{P}_\nu(\tau_{n+1} - \tau_n > t,\, \tau_n < +\infty \mid \mathcal{Y}^\nu_{\tau_n}) = \mathbb{1}_{\tau_n<+\infty} \exp\left\{-\int_0^t r\big(\phi_{\pi^\mu_{\tau_n}}(s)\big)\,\mathrm{d}s\right\}, \quad t \geqslant 0,\, n \in \mathbb{N}_0 \tag{2.2.18}$$

$$\mathrm{P}_\nu(\pi^\nu_{\tau_{n+1}} \in D,\, \tau_{n+1} < +\infty \mid \mathcal{Y}^\nu_{\tau^-_{n+1}}) = \mathbb{1}_{\tau_{n+1}<+\infty} R\big(\phi_{\pi^\nu_{\tau_n}}(\tau^-_{n+1} - \tau_n); D\big), \quad D \in \mathcal{B}(\Delta_e),\, n \in \mathbb{N}_0 \tag{2.2.19}$$

where, for each $n \in \mathbb{N}_0$, $\phi_{\pi^\nu_{\tau_n}}$ is the flow starting from $\pi^\nu_{\tau_n}$ and determined by the vector field F.

Proof. Fix $\nu \in \Delta_e$, hence $\nu \in \Delta_y$ for some $y \in O$. It is clear that $\mathrm{P}_\nu(Y_0 = y) = 1$ and that $H_y[\nu] = \nu$. Hence $\mathrm{P}_\nu(\pi^\nu_0 = \nu) = 1$, i. e. the filtering process π^ν starts from ν. The $(\mathcal{Y}^\nu_t)_{t\geqslant 0}$–Markov property for the process π^ν has already been proved in Proposition 2.2.5. The deterministic dynamic between consecutive jump times expressed by (2.2.17) easily follows from (2.2.6). Moreover (2.2.18) coincides with (2.2.12).

It remains to prove (2.2.19). From (2.2.6) we have that on $\{\tau_{n+1} < +\infty\}$ and for all $D \in \mathcal{B}(\Delta_e)$

$$\mathrm{P}_\nu\Big(\pi^\nu_{\tau_{n+1}} \in D \;\Big|\; \mathcal{Y}^\nu_{\tau^-_{n+1}}\Big) = \mathrm{P}_\nu\Big(H_{Y_{\tau_{n+1}}}\big[\Lambda(\pi^\nu_{\tau^-_{n+1}})\big] \in D \;\Big|\; \mathcal{Y}^\nu_{\tau^-_{n+1}}\Big).$$

Observing that $\Lambda(\pi^\nu_{\tau_{n+1}-}) = \Lambda\big(\phi_{\pi^\nu_{\tau_n}}(\tau^-_{n+1} - \tau_n)\big)$ is a $\mathcal{Y}^\nu_{\tau^-_{n+1}}$–measurable random variable (with values on Δ_e), an easy application of the freezing lemma to the last displayed equation entails that

$$\mathrm{P}_\nu\Big(\pi^\nu_{\tau_{n+1}} \in D \;\Big|\; \mathcal{Y}^\nu_{\tau^-_{n+1}}\Big) = \int_O \mathbb{1}_D\Big(H_\upsilon\big[\Lambda\big(\phi_{\pi^\nu_{\tau_n}}(\tau^-_{n+1} - \tau_n)\big)\big]\Big)\rho\big(\phi_{\pi^\nu_{\tau_n}}(\tau^-_{n+1} - \tau_n), \mathrm{d}\upsilon\big)$$

hence the desired result. □

Remark 2.2.5. In PDMP literature it is common to require that R is a Feller kernel, i. e. that for all $w \in \mathrm{C}_b(\Delta_e)$ the map $\rho \mapsto \int_{\Delta_e} w(p)\, R(\rho; \mathrm{d}p)$ is bounded and continuous on Δ_e. In our situation this fact may fail, since it may happen that the function r is null on some non-empty subset of Δ_e. However, we are able to show the following weaker form of the Feller property of R.

Proposition 2.2.8. *Let Assumption 2.0.1 hold. Then for every bounded and continuous function $w\colon \Delta_e \to \mathbb{R}$ the function $\rho \mapsto r(\rho)\int_{\Delta_e} w(p) R(\rho; \mathrm{d}p)$ is bounded and continuous on Δ_e.*

To prove this Proposition we will use the following Lemma.

Lemma 2.2.9. *Suppose that $(\mu_n)_{n\in\mathbb{N}} \subset \mathcal{M}_+(I)$ is a sequence converging in total variation to a measure $\mu \in \mathcal{M}_+(I)$, i. e. $\|\mu_n - \mu\|_{TV} \to 0$. Then, the sequence of $\mathcal{P}(I)$–valued functions on O given by $(H_y[\mu_n])_{n\in\mathbb{N}}$ converges in measure $\mu \circ h^{-1}$ to $H_y[\mu]$, i. e.*

$$\lim_{n\to\infty} \mu \circ h^{-1}(\{y \in O \colon \|H_y[\mu_n] - H_y[\mu]\|_{TV} > \delta\}) = 0, \quad \textit{for all } \delta > 0.$$

Proof. Fix $\delta > 0$ and a sequence $(\mu_n)_{n\in\mathbb{N}} \subset \mathcal{M}_+(I)$ such that $\|\mu_n - \mu\|_{TV} \to 0$ for some $\mu \in \mathcal{M}_+(I)$. Let us define the following sets

$$C_n^\delta \coloneqq \{y \in O \colon \sup_{A\in\mathcal{I}} \big|H_y[\mu_n](A) - H_y[\mu](A)\big| > \delta\}, \quad n \in \mathbb{N}$$
$$C_n^{\delta,A} \coloneqq \{y \in O \colon \big|H_y[\mu_n](A) - H_y[\mu](A)\big| > \delta\}, \quad n \in \mathbb{N},\ A \in \mathcal{I}.$$

We immediately notice that, since $\|\mu_n - \mu\|_{TV} \to 0$, we have

$$\|\mu_n \circ h^{-1} - \mu \circ h^{-1}\| \xrightarrow[n\to\infty]{} 0$$
$$\|\mathbb{1}_A\mu_n \circ h^{-1} - \mathbb{1}_A\mu \circ h^{-1}\| \xrightarrow[n\to\infty]{} 0, \quad \text{for all } A \in \mathcal{I}.$$

Recalling Definition 2.1.1 and thanks to [15, Exercise 3.10.36], we deduce that for all $A \in \mathcal{I}$ we have that $\lim_{n\to\infty} \mu \circ h^{-1}\big(C_n^{\delta,A} \cap \operatorname{supp}(\mu \circ h^{-1})\big) = 0$, whence $\mu \circ h^{-1}(C_n^{\delta,A}) \xrightarrow[n\to\infty]{} 0$, being $\operatorname{supp}(\mu \circ h^{-1})$ a set of full $\mu \circ h^{-1}$–measure.

Given (2.2.1), to prove our claim it suffices to show that $\mu \circ h^{-1}(C_n^\delta) \xrightarrow[n\to\infty]{} 0$ for all $\delta > 0$. To start, let us recall that since I is a separable metric space, there exists a countable base $\mathcal{U} = (U_j)_{j\in\mathbb{N}}$ for its topology. Let us define the set of all possible finite or countable unions of sets in $\mathcal{U}$, i. e.

$$\mathcal{B} \coloneqq \{B = \bigcup_{j\in J} U_j,\ U_j \in \mathcal{U},\ j \in J,\ |J| \leqslant |\mathbb{N}|\}.$$

Notice that $\mathcal{B}$ is a countable class.

We want to prove first that

$$\sup_{A\in\mathcal{I}} \big|H_y[\mu_n](A) - H_y[\mu](A)\big| = \sup_{B\in\mathcal{B}} \big|H_y[\mu_n](F) - H_y[\mu](F)\big| \tag{2.2.20}$$

It is clear that the term on the left is greater or equal to the one on the right. To prove the reverse inequality, fix $y \in O$ and $\varepsilon > 0$. Thanks to the fact that for all $y \in O$ each of the measures $H_y[\mu_n]$, $n \in \mathbb{N}$ and $H_y[\mu]$ is regular, we have that for each $A \in \mathcal{I}$ there exist open sets F^ε, $F_n^\varepsilon \in \mathcal{I}$, $n \in \mathbb{N}$ and closed sets G^ε, $G_n^\varepsilon \in \mathcal{I}$, $n \in \mathbb{N}$ such that $G^\varepsilon \subset A \subset F^\varepsilon$, $G_n^\varepsilon \subset A \subset F_n^\varepsilon$ for all $n \in \mathbb{N}$ and

$$H_y[\mu_n](F_n^\varepsilon) - \varepsilon \leqslant H_y[\mu_n](A) \leqslant H_y[\mu_n](G_n^\varepsilon) + \varepsilon$$
$$H_y[\mu](F^\varepsilon) - \varepsilon \leqslant H_y[\mu](A) \leqslant H_y[\mu](G^\varepsilon) + \varepsilon.$$

Moreover, there exist at most countable index sets I_n^ε and I^ε such that $F_n^\varepsilon = \bigcup_{i\in I_n^\varepsilon} U_i$ and $F^\varepsilon = \bigcup_{i\in I^\varepsilon} U_i$, where $U_i \in \mathcal{U}$, $i \in I_n^\varepsilon$, I^ε. Thanks to this, we have that Therefore, we get that

$$\begin{aligned}&H_y[\mu_n](A) - H_y[\mu](A) \leqslant H_y[\mu_n](G_n^\varepsilon) + \varepsilon - H_y[\mu](F_n^\varepsilon)\\ &\leqslant H_y[\mu_n](F_n^\varepsilon) - H_y[\mu](F_n^\varepsilon) + \varepsilon \leqslant H_y[\mu_n]\Bigg(\bigcup_{i\in I_n^\varepsilon} U_i\Bigg) - H_y[\mu]\Bigg(\bigcup_{i\in I_n^\varepsilon} U_i\Bigg) + \varepsilon\\ &\leqslant \sup_{B\in\mathcal{B}} H_y[\mu_n](B) - H_y[\mu](B) + \varepsilon.\end{aligned}$$

Similarly, exchanging the roles of $H_y[\mu_n](A)$ and $H_y[\mu](A)$ we get

$$H_y[\mu](A) - H_y[\mu_n](A) \leqslant \sup_{B\in\mathcal{B}} H_y[\mu_n](B) - H_y[\mu](B) + \varepsilon$$

whence

$$\big|H_y[\mu_n](A) - H_y[\mu](A)\big| \leqslant \sup_{B\in\mathcal{B}} H_y[\mu_n](B) - H_y[\mu](B) + \varepsilon.$$

Taking the supremum with respect to all $A \in \mathcal{I}$ in the l.h.s. and then letting $\varepsilon \to 0^+$ we get the desired inequality and from (2.2.20) we deduce that

$$C_n^\delta = \{y \in O\colon \sup_{A\in\mathcal{I}}\big|H_y[\mu_n](A) - H_y[\mu](A)\big| > \delta\}.$$

Now, let $y \in C_n^\delta$. Then, by definition of supremum, we have that there exists some $B \in \mathcal{B}$ such that $\big|H_y[\mu_n](B) - H_y[\mu](B)\big| > \delta$, hence $y \in \bigcup_{B\in\mathcal{B}} C_n^{\delta,B} = \bigcup_{\substack{i\in\mathbb{N}\\ B_i\in\mathcal{B}}} C_n^{\delta,B_i}$ (recall that $\mathcal{B}$ is a countable class). Therefore, for all $n \in \mathbb{N}$

$$\mu \circ h^{-1}(C_n^\delta) \leqslant \sum_{i\in\mathbb{N}} \mu \circ h^{-1}(C_n^{\delta,B_i})$$

whence

$$\begin{aligned}&\limsup_{n\to\infty} \mu \circ h^{-1}(C_n^\delta) \leqslant \limsup_{n\to\infty} \sum_{i\in\mathbb{N}} \mu \circ h^{-1}(C_n^{\delta,B_i})\\ &\leqslant \sum_{i\in\mathbb{N}} \limsup_{n\to\infty} \mu \circ h^{-1}(C_n^{\delta,B_i}) = \sum_{i\in\mathbb{N}} \lim_{n\to\infty} \mu \circ h^{-1}(C_n^{\delta,B_i}) = 0\end{aligned}$$

and our claim is, thus, proved. □

Proof of Proposition 2.2.8. Fix $w\colon \Delta_e \to \mathbb{R}$ bounded and continuous and $\rho \in \Delta_e$, hence $\rho \in \Delta_y$ for some $y \in O$. Let $(\rho_n)_{n\in\mathbb{N}}$ be a sequence such that $\|\rho_n - \rho\|_{TV} \to 0$ as $n \to \infty$. Notice that we can assume, without loss of generality that $\rho_n \in \Delta_y$ for all $n \in \mathbb{N}$.

We consider first the case where $r(\rho) > 0$. It is easy to show that the function $\rho \mapsto r(\rho)$ is continuous on Δ_e, hence $r(\rho_n) > 0$ apart from a finite number of indices $n \in \mathbb{N}$. We want to prove that

$$\left|r(\rho_n)\int_{\Delta_e} w(p)R(\rho_n;\, \mathrm{d}p) - r(\rho)\int_{\Delta_e} w(p)R(\rho;\, \mathrm{d}p)\right| \xrightarrow[n\to\infty]{} 0. \tag{2.2.21}$$

Recalling Remark 2.2.4, from (2.2.21) we obtain

$$\begin{aligned}
&\left| r(\rho_n) \int_{\Delta_e} w(p) R(\rho_n;\, \mathrm{d}p) - r(\rho) \int_{\Delta_e} w(p) R(\rho;\, \mathrm{d}p) \right| \\
&= \left| \int_O w\big(H_v[\Lambda(\rho_n)]\big) \Lambda(\rho_n) \circ h^{-1}(\mathrm{d}v) - \int_O w\big(H_v[\Lambda(\rho)]\big) \Lambda(\rho) \circ h^{-1}(\mathrm{d}v) \right| \\
&\leqslant \left| \int_O w\big(H_v[\Lambda(\rho_n)]\big) \Big[\Lambda(\rho_n) \circ h^{-1}(\mathrm{d}v) - \Lambda(\rho) \circ h^{-1}(\mathrm{d}v) \Big] \right| \\
&\quad + \int_O \Big| w\big(H_v[\Lambda(\rho_n)]\big) - w\big(H_v[\Lambda(\rho)]\big) \Big| \Lambda(\rho) \circ h^{-1}(\mathrm{d}v).
\end{aligned} \tag{2.2.22}$$

The first summand of the last line of (2.2.22) can be easily estimated thanks to the boundedness of w and Assumption 2.0.1 by

$$\begin{aligned}
&\left| \int_O w\big(H_v[\Lambda(\rho_n)]\big) \Big[\Lambda(\rho_n) \circ h^{-1}(\mathrm{d}v) - \Lambda(\rho) \circ h^{-1}(\mathrm{d}v) \Big] \right| \\
&\leqslant \sup_{p \in \Delta_e} |w(p)| \, \| \Lambda(\rho_n) \circ h^{-1} - \Lambda(\rho) \circ h^{-1} \|_{TV} \\
&\leqslant 2 \sup_{p \in \Delta_e} |w(p)| \sup_{x \in I} \lambda(x) \, \| \rho_n - \rho \|_{TV}.
\end{aligned}$$

Hence it vanishes as $n \to \infty$.

As for the second summand of the last line of (2.2.22) notice that

$$\| \Lambda(\rho_n) - \Lambda(\rho) \|_{TV} \leqslant 2 \sup_{x \in I} \lambda(x) \, \| \rho_n - \rho \|_{TV} \xrightarrow[n \to \infty]{} 0$$

thanks again to Assumption 2.0.1. Thanks to Lemma 2.2.9, taking also into account that w is continuous, the sequence $\Big(w\big(H_v[\Lambda(\rho_n)]\big) \Big)_{n \in \mathbb{N}}$ converges to $w\big(H_v[\Lambda(\rho)]\big)$ in $\Lambda(\rho) \circ h^{-1}$–measure. Applying the dominated convergence theorem (since $\Lambda(\rho) \circ h^{-1}$ is a finite measure, we can replace almost everywhere convergence of the sequence $\Big(w\big(H_v[\Lambda(\rho_n)]\big) \Big)_{n \in \mathbb{N}}$ with convergence in measure, see e. g. [15, Th. 2.8.5]) we get that

$$\int_O \Big| w\big(H_v[\Lambda(\rho_n)]\big) - w\big(H_v[\Lambda(\rho)]\big) \Big| \Lambda(\rho) \circ h^{-1}(\mathrm{d}v) \xrightarrow[n \to \infty]{} 0$$

as desired.

We conclude the proof considering the case $r(\rho) = 0$. By continuity of the function r, we have that $r(\rho_n) \to 0$ as $n \to \infty$. In case $r(\rho_n) = 0$ eventually, the claim is trivial. So, let us consider the case $r(\rho_n) \neq 0$ for all n big enough. We have to prove that

$$\left| r(\rho_n) \int_{\Delta_e} w(p) R(\rho_n;\, \mathrm{d}p) \right| \xrightarrow[n \to \infty]{} 0. \tag{2.2.23}$$

Notice that $0 = r(\rho) = -\mathcal{B}_y \rho(I) = \lambda(\rho) \circ h^{-1}(O)$, hence by boundedness of w

and Assumption 2.0.1 we get

$$\begin{aligned}
&\left|r(\rho_n)\int_{\Delta_e} w(p)R(\rho_n;\,\mathrm{d}p)\right| = \left|\int_O w\big(H_v[\Lambda(\rho_n)]\big)\Lambda(\rho_n)\circ h^{-1}(\mathrm{d}v)\right| \\
&\leqslant \sup_{p\in\Delta_e}|w(p)|\Lambda(\rho_n)\circ h^{-1}(O) = \sup_{p\in\Delta_e}|w(p)|\big[\Lambda(\rho_n)\circ h^{-1}(O)-\Lambda(\rho)\circ h^{-1}(O)\big] \\
&\leqslant \sup_{p\in\Delta_e}|w(p)|\,\|\Lambda(\rho_n)\circ h^{-1}-\Lambda(\rho)\circ h^{-1}\|_{TV} \\
&\leqslant 2\sup_{p\in\Delta_e}|w(p)|\,\sup_{x\in I}\lambda(x)\,\|\rho_n-\rho\|_{TV}
\end{aligned}$$

whence (2.2.23). □

At this point, many properties of the filtering process may be deduced from the fact that it is a PDMP, thanks to the widespread study of this class of processes. However, in the following Chapters we will need nothing more than what we have already proved about the PDMP nature of the filtering process.

To end this Section we introduce a canonical version of the filtering process. This construction is useful for various applications, e. g. optimal stopping and optimal switching. In Chapter 3 and 4 we will adopt a slightly different construction, to take into account the setting of the optimal control problem. Let us introduce the following objects.

- $\bar{\Omega} = \{\bar{\omega}\colon [0,+\infty)\to\Delta_e,\ \text{cádlág}\}$ denotes the canonical space for Δ_e – valued PDMPs. We define $\bar{\pi}_t(\bar{\omega}) = \bar{\omega}(t)$, for $\bar{\omega}\in\bar{\Omega}$, $t\geqslant 0$, and

$$\begin{aligned}
\bar{\tau}_0(\bar{\omega}) &= 0, \\
\bar{\tau}_n(\bar{\omega}) &= \inf\{t > \bar{\tau}_{n-1}(\bar{\omega}) \text{ s.t. } \bar{\pi}_t(\bar{\omega}) \neq \bar{\pi}_{t^-}(\bar{\omega})\}, \quad n\in\mathbb{N}, \\
\bar{\tau}_\infty(\bar{\omega}) &= \lim_{n\to\infty}\bar{\tau}_n(\bar{\omega}).
\end{aligned}$$

- The family of σ-algebras $(\bar{\mathcal{F}}_t^\circ)_{t\geqslant 0}$ given by

$$\bar{\mathcal{F}}_t^\circ = \sigma(\bar{\pi}_s, 0\leqslant s\leqslant t), \quad \bar{\mathcal{F}}^\circ = \sigma(\bar{\pi}_s, s\geqslant 0),$$

  is the natural filtration of the process $\bar{\pi} = (\bar{\pi}_t)_{t\geqslant 0}$.

- For every $\nu\in\Delta_e$ we denote by $\bar{\mathrm{P}}_\nu$ the probability measure on $(\bar{\Omega},\bar{\mathcal{F}}^\circ)$ such that the process $\bar{\pi}$ is a PDMP, starting from the point ν and with characteristic triple (F,r,R). We this, we mean that $\bar{\mathrm{P}}_\nu$–a.s.

$$\bar{\pi}_t = \phi_{\bar{\pi}_{\bar{\tau}_n}}(t-\bar{\tau}_n), \quad t\in[\bar{\tau}_n,\bar{\tau}_{n+1}),\ n\in\mathbb{N}_0. \tag{2.2.24}$$

$$\begin{aligned}
&\bar{\mathrm{P}}_\nu(\bar{\tau}_{n+1}-\bar{\tau}_n > t,\ \bar{\tau}_n < +\infty \mid \bar{\mathcal{F}}^\circ_{\bar{\tau}_n}) = \\
&\qquad \mathbb{1}_{\bar{\tau}_n<+\infty}\exp\left\{-\int_0^t r\big(\phi_{\bar{\pi}_{\bar{\tau}_n}}(t)\big)\,\mathrm{d}s\right\}, \quad t\geqslant 0,\ n\in\mathbb{N}_0.
\end{aligned} \tag{2.2.25}$$

$$\begin{aligned}
&\bar{\mathrm{P}}_\nu(\bar{\pi}_{\bar{\tau}_{n+1}}\in D,\ \bar{\tau}_{n+1} < +\infty \mid \bar{\mathcal{F}}^\circ_{\bar{\tau}_{n+1}^-}) = \\
&\qquad \mathbb{1}_{\bar{\tau}_{n+1}<+\infty}R(\phi_{\bar{\pi}_{\bar{\tau}_n}}(\bar{\tau}_{n+1}^- - \bar{\tau}_n); D), \quad D\in\mathcal{B}(\Delta_e),\ n\in\mathbb{N}_0.
\end{aligned} \tag{2.2.26}$$

  where, for each $n\in\mathbb{N}_0$, $\phi_{\bar{\pi}_{\bar{\tau}_n}}$ is the flow starting from $\bar{\pi}_{\bar{\tau}_n}$ and determined by the vector field F. We recall that this probability measure always exists by the canonical construction of a PDMP (see Section 1.2).

- For every $Q \in \mathcal{P}(\Delta_e)$ we define a probability $\bar{\mathrm{P}}_Q$ on $(\bar{\Omega}, \bar{\mathcal{F}}^\circ)$ by $\bar{\mathrm{P}}_Q(C) = \int_{\Delta_e} \bar{\mathrm{P}}_\nu(C)\, Q(\mathrm{d}\nu)$ for $C \in \bar{\mathcal{F}}^\circ$. This means that Q is the initial distribution of $\bar{\pi}$ under $\bar{\mathrm{P}}_Q$.
- Let $\bar{\mathcal{F}}^Q$ be the $\bar{\mathrm{P}}_Q$-completion of $\bar{\mathcal{F}}^\circ$. We still denote by $\bar{\mathrm{P}}_Q$ the measure naturally extended to this new σ-algebra. Let $\bar{\mathcal{Z}}^Q$ be the family of sets in $\bar{\mathcal{F}}^Q$ with zero $\bar{\mathrm{P}}_Q$-probability and define

$$\bar{\mathcal{F}}_t^Q = \sigma(\bar{\mathcal{F}}_t^\circ \cup \bar{\mathcal{Z}}^Q), \quad \bar{\mathcal{F}}_t = \bigcap_{Q \in \mathcal{P}(\Delta_e)} \bar{\mathcal{F}}_t^Q, \quad t \geqslant 0.$$

$(\bar{\mathcal{F}}_t)_{t \geqslant 0}$ is called the *natural completed filtration* of $\bar{\pi}$. By Theorem 1.2.2 it is right-continuous.

The PDMP $(\bar{\Omega}, \bar{\mathcal{F}}, (\bar{\mathcal{F}}_t)_{t\geqslant 0}, (\bar{\pi}_t)_{t\geqslant 0}, (\bar{\mathrm{P}}_\nu)_{\nu \in \Delta_e})$ constructed as above admits the characteristic triple (F, r, R).

The following Proposition, that we will revisit later in Chapters 3 and 4, provides for each initial distribution μ of the unobserved process X the law of the filtering process π^μ under P_μ. Its proof will be omitted, since it is a slight generalization of [25, Prop. 5.6.].

Proposition 2.2.10. *For every $\mu \in \mathcal{P}(I)$ the law of $\pi^\mu = (\pi_t^\mu)_{t \geqslant 0}$ is $\bar{P}_Q$ where Q is the Borel probability measure on Δ_e defined as*

$$Q \coloneqq \mu \circ h^{-1} \circ \mathcal{H}^{-1}, \quad \mathcal{H}(y) \coloneqq H_y[\mu].$$

Moreover, Q is concentrated on the set $\bigcup_{y \in O} \{H_y[\mu]\}$.

Remark 2.2.6. The set $\cup_{y \in O}\{H_y[\mu]\}$ is the image of the map $\mathcal{H}$ defined in Proposition 2.2.10. This map is clearly a bijection, with inverse given for all $\rho \in \Delta_e$ by $\mathcal{H}^{-1}(\rho) = \mathrm{proj}_{\mathrm{Y}}(\rho) \coloneqq y$, whenever $\rho \in \Delta_y$. This shows also that $\mathcal{H}$ and its inverse are both measurable, hence $\cup_{y \in O}\{H_y[\mu]\} \in \mathcal{B}(\Delta_e)$.

2.3 The Markov chain case

In this brief Section we introduce the notation that will be used in Chapter 3, where we assume that the state space I of the unobserved process is a finite set. The notation adopted is almost the same used in [25]. The purpose of this Section is to get the reader acquainted with this setting and also to give the chance to revisit the results obtained so far in this Chapter in a simpler situation, before tackling the optimal control problem studied in the next Chapters. Since any result that we are going to show is an adaptation of the corresponding ones earlier obtained, we will state them without proof.

Let the set I be finite. We put ourselves in the canonical framework described in Section 2.2. In particular, the definitions of the function h, the unobserved and observed processes X and Y remain unchanged. Notice that the state space O of the observed process is a finite set, too, and since we are considering h as a surjective function its cardinality is no greater than that of the set I. As done in the general case, for any fixed probability distribution μ on I we can find a probability measure P_μ on $(\Omega, \mathcal{X}^\circ)$ such that the unobserved process X is a finite-state $(\mathcal{X}^\circ, \mathrm{P}_\mu)$–Markov chain, with initial distribution μ and known rate transition matrix $\Lambda = (\lambda_{ij})_{i,j \in I}$. This is a real square matrix such that

1. $\lambda_{ij} \geqslant 0$ for all $i, j \in I, i \neq j$.
2. $\sum_{j \in I} \lambda_{ij} = 0$ for all $i \in I$.

It is common to define $\lambda_i \coloneqq -\lambda_{ii} = \sum_{\substack{j \in I \\ j \neq i}} \lambda_{ij}$ for each $i \in I$. Analogously to Assumption 2.0.1, we require that $\lambda_i < +\infty$ for all $i \in I$.

The filtering process is defined for all $i \in I$ as

$$\mathrm{P}_\mu(X_t = i \mid \mathcal{Y}_t^\mu), \quad t \geqslant 0.$$

Its state space $\mathcal{P}(I)$ can be naturally identified with the canonical simplex on $\mathbb{R}^{|I|}$, i. e.

$$\Delta \coloneqq \{\rho \in \mathbb{R}^{|I|} \colon \rho(i) \geqslant 0, \, \forall i = 1, \dots, |I|, \sum_{i=1}^{|I|} \rho(i) = 1\}.$$

However, as we already know, the actual values of π lie in the so called *effective simplex* $\Delta_e = \cup_{y \in O} \Delta_y$, where for each $y \in O$ the set Δ_y is the family of probability measures concentrated on $h^{-1}(y)$. The effective simplex is a proper subset of Δ (unless the function h is constant) and it is compact in the present finite dimensional setting.

We identify probability measures (or, more generally, finite measures) on I with row vectors on $\mathbb{R}^{|I|}$. We will also denote by $\mathbb{1}_{h^{-1}(y)}$ the column vector

$$[\mathbb{1}_{h^{-1}(y)}]_i = \begin{cases} 1, & i \in h^{-1}(y) \\ 0, & \text{otherwise} \end{cases}$$

with subscript i indicating the i-th component of a vector.

Before discussing the filtering equation, let us observe that the definition of the operator H is greatly simplified in this setting (see also Remark 2.1.4). In fact, we can write it as

$$H_y[\mu](i) = \begin{cases} 0, & \text{if } i \notin h^{-1}(y), \\ \frac{\mu(i)}{\mu \mathbb{1}_{h^{-1}(y)}}, & \text{if } i \in h^{-1}(y), \quad \mu \mathbb{1}_{h^{-1}(y)} \neq 0, \\ \nu_y, & \text{if } \mu \mathbb{1}_{h^{-1}(y)} = 0, \end{cases} \tag{2.3.1}$$

where ν_y is an arbitrary probability measure concentrated on $h^{-1}(y)$ whose exact values are irrelevant.

The analogue of Equation (2.1.22) satisfied by the filtering process can be deduced immediately from (2.1.19) considering the test functions $\varphi = \mathbb{1}_{\{i\}}$, $i \in I$. To further simplify the notation, let us introduce the revisited version of the vector field $F \colon \Delta_e \to \Delta_e$.

$$F_j(\nu) = \begin{cases} [\nu \Lambda]_j - (\nu \Lambda \mathbb{1}_{h^{-1}(y)}) \nu_j, & j \in h^{-1}(y) \\ 0, & \text{otherwise} \end{cases}, \quad \nu \in \Delta_y, \, y \in O. \tag{2.3.2}$$

Before introducing the filtering equation, we need to be sure that the ODE governed by the vector field F admits a unique solution.

Proposition 2.3.1 ([25, Prop. 2.1])**.** *For every $y \in O$, $\rho \in \Delta_y$ the differential equation*

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} z(t) = F\big(z(t)\big) & t \geqslant 0 \\ z(0) = \rho \end{cases} \tag{2.3.3}$$

has a unique global solution $z \colon [0, +\infty) \to \mathbb{R}^{|I|}$. Moreover $z(t) \in \Delta_y$ for all $t \geqslant 0$.

The filtering equation can be stated as follows.

Theorem 2.3.2 ([25, Th. 2.2]). *For all* $\omega \in \Omega$ *define* $\tau_0(\omega) \equiv 0$ *and for fixed* $\mu \in \Delta$ *the stochastic process* $\pi^\mu = (\pi_t^\mu)_{t\geqslant 0}$ *as the unique solution of the following system of ODEs*

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\pi_t^\mu(\omega) = F\big(\pi_t^\mu(\omega)\big), \quad t \in [\tau_n(\omega), \tau_{n+1}(\omega)),\, n \in \mathbb{N}_0 \\ \pi_0^\mu(\omega) = H_{Y_0(\omega)}[\mu], \\ \pi_{\tau_n}^\mu(\omega) = H_{Y_{\tau_n}(\omega)}[\pi_{\tau_n^-}^\mu(\omega)\Lambda],\, n \in \mathbb{N}. \end{cases} \tag{2.3.4}$$

where F *is the vector field defined in (2.3.2).*

Then, π^μ *is* $(\mathcal{Y}_t^\mu)_{t\geqslant 0}$ *- adapted and is a modification of the filtering process, i. e.*

$$\pi_t^\mu(i) = \mathrm{P}_\mu(X_t = i \mid \mathcal{Y}_t^\mu), \quad \mathrm{P}_\mu\text{–}a.s.,\, t \geqslant 0,\, i \in I.$$

To end this Section, we discuss the characterization of the filtering process π as a PDMP. The characteristic triple (F, r, R) is given by the vector field F introduced in (2.3.2) and by

- The jump rate function $r\colon \Delta_e \to [0, +\infty)$ defined as
$$r(\rho) \coloneqq -\rho\Lambda\mathbb{1}_{h^{-1}(y)}, \quad \rho \in \Delta_y. \tag{2.3.5}$$
- The transition probability R from $\big(\Delta_e, \mathcal{B}(\Delta_e)\big)$ into itself given for all $D \in \mathcal{B}(\Delta_e)$ by
$$R(\rho; D) \coloneqq \sum_{\upsilon\in O} \mathbb{1}_D\big(H_\upsilon[\rho\Lambda]\big)\, q(\rho, \upsilon), \quad \rho \in \Delta_y \tag{2.3.6}$$
with
$$q(\rho, \upsilon) \coloneqq \begin{cases} \dfrac{\rho\Lambda\mathbb{1}_{h^{-1}(\upsilon)}}{-\rho\Lambda\mathbb{1}_{h^{-1}(y)}}\mathbb{1}_{\upsilon\neq y}, & \text{if } \rho\Lambda\mathbb{1}_{h^{-1}(y)} \neq 0 \\ q_y(\upsilon), & \text{if } \rho\Lambda\mathbb{1}_{h^{-1}(y)} = 0 \end{cases}, \quad \rho \in \Delta_y$$
and where for each $y \in O$ we denote by $q_y = (q_y(\upsilon))_{\upsilon\in O}$ a probability measure concentrated on $O \setminus \{y\}$ whose exact values are irrelevant.

The following Theorem is the counterpart of Theorem 2.2.7 and characterizes the filtering process as a PDMP.

Theorem 2.3.3 ([25, Th. 5.4]). *For every* $\nu \in \Delta_e$ *the filtering process* $\pi^\nu = (\pi_t^\nu)_{t\geqslant 0}$ *defined on the probability space* $(\Omega, \mathcal{X}, \mathrm{P}_\nu)$ *and taking values in* Δ_e *is a controlled* Piecewise Deterministic Markov Process *with respect to the triple* (F, r, R) *defined in (2.3.2), (2.3.5), (2.3.6) and with starting point* ν.

More specifically, we have that P_ν*–a.s.*

$$\pi_t^\nu = \phi_{\pi_{\tau_n}^\nu}(t - \tau_n), \quad t \in [\tau_n, \tau_{n+1}),\, n \in \mathbb{N}_0 \tag{2.3.7}$$

$$\mathrm{P}_\nu(\tau_{n+1} - \tau_n > t,\, \tau_n < +\infty \mid \mathcal{Y}_{\tau_n}^\nu) = \mathbb{1}_{\tau_n<+\infty} \exp\Big\{-\int_0^t r\big(\phi_{\pi_{\tau_n}^\nu}(s)\big)\,\mathrm{d}s\Big\}, \quad t \geqslant 0,\, n \in \mathbb{N}_0 \tag{2.3.8}$$

$$\mathrm{P}_\nu(\pi^\nu_{\tau_{n+1}} \in D,\, \tau_{n+1} < +\infty \mid \mathcal{Y}^\nu_{\tau^-_{n+1}}) = \\ \mathbb{1}_{\tau_{n+1}<+\infty} R\big(\phi_{\pi^\nu_{\tau_n}}(\tau^-_{n+1} - \tau_n); D\big), \quad D \in \mathcal{B}(\Delta_e),\, n \in \mathbb{N}_0 \quad (2.3.9)$$

where, for each $n \in \mathbb{N}_0$, $\phi_{\pi^\nu_{\tau_n}}$ *is the flow starting from* $\pi^\nu_{\tau_n}$ *and determined by the vector field* F.

2.4 Some remarks on the observed process

In this brief Section we highlight what happens to the results proved in this Chapter if we allow the function h, providing the observed process as in (2.0.6), to be one-to-one or constant. We recall that these cases were excluded from our analysis since the control problem arising from them is not of true partial observation nature and the filtering problem is, in some sense, trivial. However, the assumption that h is neither one-to-one nor constant is not used in any of the proofs contained in this Chapter and, in fact, the reader may check that they still hold even in these cases, leading to the results shown below.

The case where h is one-to-one is associated to a control problem with complete observation. The state spaces of the processes X and Y are isomorphic and the filtering problem is trivial, in the sense that the filtering process possesses piecewise constant trajectories and takes values in the subset of $\mathcal{P}(I)$ given by Dirac probability measures. In fact, the σ-algebras $\mathcal{X}_t$ and $\mathcal{Y}_t$ coincide for all $t \geqslant 0$, up to $\mathbb{P}$–null sets. Therefore, we get that for all $\varphi \in \mathrm{B}_b(I)$

$$\pi_t(\varphi) = \mathbb{E}[\varphi(X_t) \mid \mathcal{Y}_t] = \mathbb{E}[\varphi(X_t) \mid \mathcal{X}_t] = \varphi(X_t), \quad t \geqslant 0,\ \mathbb{P}\text{–a.s.}$$

whence $\pi_t(\mathrm{d}x) = \delta_{X_t}(\mathrm{d}x)$, $t \geqslant 0$, $\mathbb{P}$–a.s.. Notice that the operator H, given in Definition 2.1.1, reduces to $H_y[\mu] = \delta_y$, $y \in O$, for all $\mu \in \mathcal{P}(I)$.

The filtering process π is still a PDMP although, more specifically, is a pure jump process. Its state space Δ_e, i. e. the effective simplex, can be identified with the state space I of the unobserved process X via Dirac probability measures

$$\Delta_e = \{\delta_x : x \in I\} \quad \leftrightarrow \quad I.$$

Its local characteristics are given for all $\nu \in \Delta_e$ and all $D \in \mathcal{B}(\Delta_e)$ by

$$F(\nu) = 0 \qquad r(\nu) = r(\delta_x) = \lambda(x) \qquad R(\nu, D) = R(\delta_x, D) = \frac{\lambda(x, A)}{\lambda(x)}$$

where $A \in \mathcal{B}(I)$ is the unique Borel set corresponding to $D \in \mathcal{B}(\Delta_e)$ and λ is the rate transition measure of the process X.

The case where h is constant is associated to a control problem with no information. The state space of the observed process Y reduces to a single point and the filtering problem is trivial, in the sense that the filtering process becomes a deterministic function of the time variable, taking values in $\mathcal{P}(I)$. In fact, the σ-algebras $\mathcal{Y}_t$ coincide with the trivial σ-algebra for all $t \geqslant 0$. Therefore, we get that for all $\varphi \in \mathrm{B}_b(I)$

$$\pi_t(\varphi) = \mathbb{E}[\varphi(X_t) \mid \mathcal{Y}_t] = \mathbb{E}[\varphi(X_t)], \quad t \geqslant 0.$$

The filtering process π coincides with the law of the unobserved process X, its state space is $\Delta_e = \mathcal{P}(I)$ and it satisfies the following evolution equation

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\pi_t = \mathcal{L}^\star \pi_t, & t \geqslant 0 \\ \pi_0 = \mu \in \mathcal{P}(I) \end{cases} \qquad (2.4.1)$$

where $\mathcal{L}^\star$ is the adjoint operator associated to the infinitesimal generator $\mathcal{L}$ of the unobserved process X, defined in (2.1.8). Since $\mathcal{L}$ is linear and bounded, the unique solution to (2.4.1) is given by $\pi_t = e^{t\mathcal{L}^\star}\mu$, $t \geqslant 0$, where $\left(e^{t\mathcal{L}^\star}\right)_{t\geqslant 0}$ is the strongly continuous semigroup associated to $\mathcal{L}^\star$. Notice that if we assume that the process X takes values on $\mathbb{R}^n$ and its law admits density with respect to the Lebesgue measure, then (2.4.1) can be rewritten as a PIDE for the density process of X, called the *Fokker-Planck* equation. Sometimes, also evolution equations as (2.4.1) in the space $\mathcal{P}(I)$ are called Fokker-Planck equations.

CHAPTER 3

Optimal control: the finite dimensional case

In this Chapter we specialize the setting of Chapter 2 to the case where the unobserved process $X = (X_t)_{t \geqslant 0}$ is a continuous-time homogeneous Markov chain and we study an optimal control problem on infinite time horizon with partial observation.

The aim of our control problem is to optimize the dynamics of the Markov chain X through the actions described by another stochastic process $\mathbf{u} = (u_t)_{t \geqslant 0}$, with values in the set of Borel probability measures $\mathcal{P}(U)$ on a measurable space $(U, \mathcal{U})$, the *space of control actions*. Thus, control actions are specified by *relaxed controls*. This process is chosen in a well defined class and is called *control process*. At any time the chosen control action shall be based on the information provided by the observed process $Y = (Y_t)_{t \geqslant 0}$, that will be of noise-free type as in the previous Chapter. The choice of the control process is done following a performance criterion that, in our setting, is the minimization of a discounted cost functional.

Throughout this Chapter we will assume that I and O are finite sets and that U is a compact metric space equipped with its Borel σ-algebra $\mathcal{U}$. Therefore $\mathcal{P}(U)$ is a compact metric space, too. As in Chapter 2, we are given a function $h \colon I \to O$ that gives the values of the observed process Y as a deterministic transformation of the values assumed by the unobserved process X. We consider this function to be surjective, without loss of generality. We remind that, in general, h can be constant or one-to-one, but we will exclude these cases in what follows. In the next Chapter, precisely in Section 4.4, we will make some comments on the rôle of the function h in the control problem.

We start our analysis in Section 3.1 by formulating our optimal control problem with partial observation in a canonical setting for the Markov chain X. We will see that, thanks to the filtering process (introduced in the uncontrolled case in Chapter 2), we are able to rewrite this control problem in an equivalent one with complete observation, where the new *state variable* is the filtering process itself, in place of the unobserved Markov chain X. However, we will need to reformulate our control problem, introducing a separated discrete-time control problem for the filtering process, seen as a PDP.

The separated control problem will be formulated in Section 3.2 in a canonical setting for the PDP given by the filtering process. We will prove that the original and the separated control problem are linked in various ways, culminating with a formula providing an equality involving the value function V of the original control problem and the value function v of the separated control problem.

At this point, as is common in the study of optimal control problems, we want to characterize the original value function V. To do so, it is convenient to focus on the value function v of the separated control problem. This is done in Section 3.3, where we obtain, first, that v is the unique fixed point of a certain contraction mapping and, second, that is the unique constrained viscosity solution (in Soner's sense, see [59]) of a *Hamilton-Jacobi-Bellman* integro-differential equation.

In Section 3.4 we provide an existence result of an ordinary optimal control. This is of great importance since from the preceding results we are sure only about the existence of a relaxed optimal control. Finally, in Section 3.5 we provide an example where we are able to solve explicitly our optimal control problem.

3.1 The Markov chain optimal control problem

The setting that we will adopt in this Section is fairly similar to that introduced in Section 2.3 and we advise the reader to give at least a quick glance to its contents, in order to get acquainted with the notation used here. Nonetheless, we provide here all the definitions and the results needed for our optimal control problem, since they must be modified with respect to Section 2.3 to take into account the control process.

The aim of this Section is to provide a canonical framework for a continuous time homogeneous Markov chain described by an initial law and a *controlled rate transition matrix* on I, sometimes called Q-matrix (see e. g. [51]). By this we mean that for each fixed $u \in U$ we have a real square matrix $\Lambda(u) = (\lambda_{ij}(u))_{i,j \in I}$ such that

1. $\lambda_{ij}(u) \geqslant 0$ for all $i, j \in I$, $i \neq j$.
2. $\sum_{j \in I} \lambda_{ij}(u) = 0$ for all $i \in I$.

It is quite common to write for each $i \in I$

$$\lambda_i(u) := -\lambda_{ii}(u) = \sum_{\substack{j \in I \\ j \neq i}} \lambda_{ij}(u).$$

On these matrix coefficients we introduce the following assumption

Assumption 3.1.1. For each $i, j \in I$ the map $u \mapsto \lambda_{ij}(u)$ is continuous (hence bounded and uniformly continuous). In particular, we have that

$$\sup_{u \in U} \lambda_i(u) < +\infty, \quad i \in I.$$

We are going now to build the probability space on which the processes X, Y, $\mathbf{u}$ are defined. The construction is very similar to that shown in Section 2.2. Let us define Ω as the set

$$\Omega = \{\omega = (i_0, t_1, i_1, t_2, i_2, \dots) \colon \\ i_0 \in I, i_n \in I, t_n \in (0, +\infty], t_n < +\infty \Rightarrow t_n < t_{n+1}, n \in \mathbb{N}\}.$$

For each $n \in \mathbb{N}$ we introduce the following random variables

$$T_n(\omega) = t_n; \qquad T_\infty(\omega) = \lim_{n\to\infty} T_n(\omega); \qquad \xi_0(\omega) = i_0; \qquad \xi_n(\omega) = i_n$$

and we define the random measure on $\big((0,+\infty] \times I,\, \mathcal{B}\big((0,+\infty]\big) \otimes \mathcal{I}\big)$

$$n(\omega, \mathrm{d}t\,\mathrm{d}z) = \sum_{n\in\mathbb{N}} \delta_{\big(T_n(\omega),\, \xi_n(\omega)\big)}(\mathrm{d}t\,\mathrm{d}z)\mathbb{1}_{\{T_n<+\infty\}}(\omega), \quad \omega \in \Omega$$

with associated natural filtration $\mathcal{N}_t = \sigma\big(n\big((0,t] \times \{i\}\big),\, 0 \leqslant s \leqslant t,\, i \in I\big)$. Finally, let us specify the σ-algebras

$$\mathcal{X}_0^\circ = \sigma(\xi_0); \qquad \mathcal{X}_t^\circ = \sigma(\mathcal{X}_0 \cup \mathcal{N}_t); \qquad \mathcal{X}^\circ = \sigma\Big(\bigcup_{t\geqslant 0} \mathcal{X}_t\Big).$$

The unobserved process X is defined as

$$X_t(\omega) = \begin{cases} \xi_0(\omega), & t \in \big[0, T_1(\omega)\big) \\ \xi_n(\omega), & t \in \big[T_n(\omega), T_{n+1}(\omega)\big),\, n \in \mathbb{N},\, T_n(\omega) < +\infty \\ i_\infty, & t \in \big[T_\infty(\omega), +\infty),\, T_\infty(\omega) < +\infty \end{cases}$$

where $i_\infty \in I$ is an arbitrary state, that is irrelevant to specify. Next, we define the observed process Y and its natural filtration $(\mathcal{Y}_t^\circ)_{t\geqslant 0}$ as

$$Y_t(\omega) = h(X_t(\omega)),\, t \geqslant 0,\, \omega \in \Omega; \qquad \mathcal{Y}_t^\circ = \sigma\big(Y_s,\, 0 \leqslant s \leqslant t\big),\, t \geqslant 0.$$

As we already pointed out in Section 2.2, we notice that we can equivalently describe this process via a MPP $(\eta_n, \tau_n)_{n\in\mathbb{N}}$ with initial condition $\eta_0 = h(\xi_0) = Y_0$. Each σ-algebra $\mathcal{Y}_t^\circ$, $t \geqslant 0$ is the smallest σ-algebra generated by the union of $\sigma(\eta_0)$ and the σ-algebra at time t of the natural filtration of the MPP $(\eta_n, \tau_n)_{n\in\mathbb{N}}$.

As said at the beginning of this Chapter, we need to consider control processes $\mathbf{u}$ that are based on the information brought by the observed process Y. More precisely, we will choose controls in the class of *admissible controls*, defined as the set

$$\mathcal{U}_{ad} = \Big\{\mathbf{u}\colon \Omega \times [0,+\infty) \to \mathcal{P}(U),\, (\mathcal{Y}_t^\circ)_{t\geqslant 0} - \text{predictable}\Big\}. \tag{3.1.1}$$

Remark 3.1.1. It must be pointed out that considering $\mathcal{P}(U)$ – valued processes (the so called *relaxed controls*), instead of ordinary U – valued processes, has considerable technical benefits that will be fully clear in Section 3.2. At this stage such a choice has almost no impact on the problem itself (except for a slightly more complicated notation), being both U and $\mathcal{P}(U)$ compact metric spaces. It is important to notice that any ordinary control is included in this formulation by considering its corresponding process in $\mathcal{U}_{ad}$ whose value at each time $t \geqslant 0$ and $\omega \in \Omega$ is given by a probability measure concentrated at a single point in U. Ordinary controls are far easier to understand and implement and we will later mention some technical results enabling us to use such controls to prove the main results of this Chapter. As a final note on this subject, we recall that the existence of an ordinary optimal control will be proved in Section 3.4.

Thanks to the peculiar structure of the natural filtration of Y we have a precise characterization of the class $\mathcal{U}_{ad}$ (see Theorem 1.1.2 and Remark 1.1.1 following it). A control process $\mathbf{u} \in \mathcal{U}_{ad}$ is completely determined by a sequence of Borel-measurable functions $(u_n)_{n\in\bar{\mathbb{N}}_0}$, with $u_n\colon [0,+\infty)\times O\times\big((0,+\infty)\times O\big)^n \to \mathcal{P}(U)$ for each $n\in\bar{\mathbb{N}}_0$ and we can write

$$\begin{aligned} u_t(\omega) &= u_0(t,Y_0(\omega))\mathbb{1}(0\leqslant t\leqslant \tau_1(\omega))+\\ &\sum_{n=1}^{\infty} u_n(t,Y_0(\omega),\tau_1(\omega),Y_{\tau_1}(\omega),\dots,\tau_n(\omega),Y_{\tau_n}(\omega))\mathbb{1}(\tau_n(\omega)<t\leqslant\tau_{n+1}(\omega))+\\ &\qquad u_\infty(t,Y_0(\omega),\tau_1(\omega),Y_{\tau_1}(\omega),\dots)\mathbb{1}(t>\tau_\infty(\omega)), \end{aligned} \tag{3.1.2}$$

where $\tau_\infty(\omega)=\lim_{n\to\infty}\tau_n(\omega)$. This kind of decomposition of a control process $\mathbf{u}\in\mathcal{U}_{ad}$ will be of fundamental importance throughout this Chapter and we will frequently switch between the notation $(u_t)_{t\geqslant 0}$ and $(u_n)_{n\in\bar{\mathbb{N}}_0}$. To simplify the notation, we will often use the more compact writing $u_n(\cdot)$ instead of $u_n(\cdot,Y_0(\omega),\dots,\tau_n(\omega),Y_{\tau_n}(\omega))$, for all $n\in\mathbb{N}_0$.

The dynamics of the unobserved process will be specified by the initial distribution μ, a probability measure on I, and by the following random measure depending on u

$$\nu^{\mathbf{u}}(\omega;\,\mathrm{d}t\times\{i\}) = \begin{cases} \mathbb{1}\big(t<T_\infty(\omega)\big)\displaystyle\int_U \lambda_{X_{t-}(\omega)i}(\mathrm{u})\,u_t(\omega\,;\mathrm{du})\,\mathrm{d}t, & \text{if } i\neq X_{t-}(\omega)\\ 0, & \text{if } i=X_{t-}(\omega)\end{cases} \tag{3.1.3}$$

for any $\omega\in\Omega$ and $\mathbf{u}\in\mathcal{U}_{ad}$. For sake of simplicity, we will drop ω in what follows.

Now set P_0 as the probability measure on $\big(\Omega,\mathcal{X}_0^\circ\big)$ such that $X_0=\xi_0$ has law μ. It is easy to see that the previously described setting is equivalent to that provided in Assumption 1.1.3. In fact, one can show that the random measure $\nu^{\mathbf{u}}$ is $(\mathcal{X}_t^\circ)_{t\geqslant 0}$–predictable and satisfies (1.1.3), i. e.

1. $\nu^{\mathbf{u}}\big(\{t\}\times I\big)\leqslant 1$,
2. $\nu^{\mathbf{u}}\big([T_\infty,+\infty)\times I\big)=0$.

Therefore, by Theorem 1.1.9, there exists a unique probability measure $\mathrm{P}_\mu^{\mathbf{u}}$ on $\big(\Omega,\mathcal{X}^\circ\big)$, such that $\mathrm{P}_\mu^{\mathbf{u}}|_{\mathcal{X}_0^\circ}=\mathrm{P}_0$ and $\nu^{\mathbf{u}}$ is the $\big(\mathrm{P}_\mu^{\mathbf{u}},\mathcal{X}_t^\circ\big)$–predictable projection of n. Once specified the control $\mathbf{u}\in\mathcal{U}_{ad}$ and consequently the probability measure $\mathrm{P}_\mu^{\mathbf{u}}$, it follows from Assumption 3.1.1 and by standard arguments that the point process n is $\mathrm{P}_\mu^{\mathbf{u}}$-a.s. non-explosive, i. e. that $T_\infty=+\infty$, $\mathrm{P}_\mu^{\mathbf{u}}$-a.s.. For this reason we will drop the term $\mathbb{1}(t<T_\infty)$ appearing in (3.1.3) and, since also $\tau_\infty=+\infty$ $\mathrm{P}_\mu^{\mathbf{u}}$-a.s., we will avoid specifying the function u_∞ in (3.1.2).

Finally, we define for each probability measure μ on I and $\mathbf{u}\in\mathcal{U}_{ad}$ the completions of the natural filtrations of the processes X and Y as follows.

- $\mathcal{X}^{\mu,\mathbf{u}}$ is the $\mathrm{P}_\mu^{\mathbf{u}}$-completion of $\mathcal{X}^\circ$ ($\mathrm{P}_\mu^{\mathbf{u}}$ is extended to $\mathcal{X}^{\mu,\mathbf{u}}$ in the natural way).
- $\mathcal{Z}^{\mu,\mathbf{u}}$ is the family of elements of $\mathcal{X}^{\mu,\mathbf{u}}$ with zero $\mathrm{P}_\mu^{\mathbf{u}}$ – probability.
- $\mathcal{Y}_t^{\mu,\mathbf{u}}:=\sigma(\mathcal{Y}_t^\circ,\mathcal{Z}^{\mu,\mathbf{u}})$, for $t\geqslant 0$.

$(\mathcal{Y}_t^{\mu,\mathbf{u}})_{t\geqslant 0}$ is called the *natural completed filtration* of Y.

As we anticipated earlier, we choose control actions in order to minimize, for all possible choices of the initial distribution μ of the process X, the following *cost functional*

$$J(\mu,\mathbf{u})=\mathrm{E}_\mu^{\mathbf{u}}\left[\int_0^\infty e^{-\beta t}\int_U f(X_t,\mathrm{u})\,u_t(\mathrm{du})\,\mathrm{d}t\right] \tag{3.1.4}$$

where f is called *cost function* and $\beta > 0$ is a fixed constant called *discount factor*. In other words, we want to characterize the *value function*

$$V(\mu) = \inf_{\mathbf{u} \in \mathcal{U}_{ad}} J(\mu, \mathbf{u}). \tag{3.1.5}$$

We make the following Assumption on the cost function f, ensuring that the functional J is well defined (and also bounded).

Assumption 3.1.2. The function $f\colon I \times U \to \mathbb{R}$ is continuous. Since U is compact and I finite, f is uniformly continuous and it holds that

$$\sup_{(i,u) \in I \times U} |f(i, u)| \leqslant C_f, \tag{3.1.6}$$

for some constant $C_f > 0$.

Since I is a finite set, we will denote by $\mathbf{f}(u)$ the column vector whose components are the values $f(i, u)$ as i varies in the set I, for each fixed $u \in U$.

We can transform the problem formulated above into a complete observation problem by means of the *filtering process*. In what follows we state some results on this process that in the uncontrolled case were already obtained in [25] and previously recalled in Section 2.3. Hence, they can be deduced in the present controlled setting as slight generalizations of those results or, alternatively, as generalizations of the corresponding statements obtained in Chapter 2. That considered, we will present these facts without proof.

The filtering process is defined for all $i \in I$ as

$$\mathrm{P}_\mu^{\mathbf{u}}(X_t = i \mid \mathcal{Y}_t^{\mu,\mathbf{u}}), \quad t \geqslant 0.$$

It takes values on the set $\mathcal{P}(I)$ of probability measures on I which, being I a finite set, can be naturally identified with the canonical simplex on $\mathbb{R}^{|I|}$, i. e.

$$\Delta \coloneqq \{\rho \in \mathbb{R}^{|I|} \colon \rho(i) \geqslant 0, \, \forall i = 1, \dots, |I|, \sum_{i=1}^{|I|} \rho(i) = 1\}.$$

As we already know the actual values of the filtering process lie in the so called *effective simplex* Δ_e, defined as $\Delta_e \coloneqq \cup_{y \in O} \Delta_y$; for each $y \in O$, Δ_y indicates the set of probability measures concentrated on $h^{-1}(y)$. It is important to point out that in the present setting we have also compactness of the effective simplex. Moreover, it is a proper subset of Δ unless the function h is constant.

It is worth noticing that the filtering process is a $(\mathcal{Y}_t^{\mu,\mathbf{u}})_{t \geqslant 0}$ – adapted process and since $(\mathcal{Y}_t^{\mu,\mathbf{u}})_{t \geqslant 0}$ is right continuous we can choose a $(\mathcal{Y}_t^{\mu,\mathbf{u}})_{t \geqslant 0}$ – progressive version. We will assume this whenever needed.

As did in Section 2.3, we will identify probability measures (or, more generally, finite measures) on I with row vectors on $\mathbb{R}^{|I|}$.

Let us now define on the effective simplex the vector field $F\colon \Delta_e \times U \to \Delta_e$ as

$$F_j(\nu, u) = \begin{cases} [\nu\Lambda(u)]_j - (\nu\Lambda(u)\mathbb{1}_{h^{-1}(y)})\nu_j, & j \in h^{-1}(y) \\ 0, & \text{otherwise} \end{cases}, \; \nu \in \Delta_y, \, y \in O, \, u \in U. \tag{3.1.7}$$

We recall that subscript j denotes the j-th component of a vector and $\mathbb{1}_{h^{-1}(y)}$ is the column vector

$$[\mathbb{1}_{h^{-1}(y)}]_i = \begin{cases} 1, & i \in h^{-1}(y) \\ 0, & \text{otherwise} \end{cases}, \quad y \in O.$$

It is clear that the map $u \mapsto F(\nu, u)$ is measurable for all $\nu \in \Delta_e$. Moreover, Assumption 3.1.1 implies that F is Lipschitz continuous in ν uniformly in u, i. e. there exists a constant $L_F > 0$ such that

$$\sup_{u \in U} |F(\nu, u) - F(\rho, u)| \leqslant L_F |\nu - \rho|, \quad \text{for all } \nu, \rho \in \Delta_y,\ y \in O. \tag{3.1.8}$$

Therefore, a generalization of Proposition 2.3.1 or Theorem 2.2.3 provides us with the following result.

Proposition 3.1.1. *For every $y \in O$, $\rho \in \Delta_y$ and all measurable $m\colon [0, +\infty) \to \mathcal{P}(U)$, the differential equation*

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} z(t) = \displaystyle\int_U F(z(t), u)\, m(t\,; \mathrm{d}u), & t \geqslant 0 \\ z(0) = \rho \end{cases} \tag{3.1.9}$$

has a unique global solution $z\colon [0, +\infty) \to \mathbb{R}^{|I|}$. Moreover $z(t) \in \Delta_y$ for all $t \geqslant 0$.

We will write $\phi^m_{y,\rho}(t)$ instead of $z(t)$ to stress the dependence of the solution to (3.1.9) on ρ and on the measurable function m. To ease up the notation a little, we also define $\phi^m_\rho(t) = \phi^m_{y,\rho}(t)$ if $\rho \in \Delta_y$ to denote the global flow associated to the vector field F.

As we already know, the filtering process solves a SDE, the *filtering equation*. This SDE can be written pathwise as a system of ODEs, that specify the deterministic behavior of the filtering process between two consecutive jump times of the process Y; its post-jump locations and its initial value are determined by the operator H, i. e. the collection of functions $H_y\colon \mathbb{R}^{|I|} \to \mathbb{R}^{|I|}$, as y varies in O, mapping row vectors into row vectors and defined for each $y \in O$ as

$$H_y[\mu](i) = \begin{cases} 0, & \text{if } i \notin h^{-1}(y), \\ \frac{\mu(i)}{\mu \mathbb{1}_{h^{-1}(y)}}, & \text{if } i \in h^{-1}(y), \quad \mu \mathbb{1}_{h^{-1}(y)} \neq 0, \\ \nu_y, & \text{if } \mu \mathbb{1}_{h^{-1}(y)} = 0, \end{cases} \tag{3.1.10}$$

where ν_y is an arbitrary probability measure supported on $h^{-1}(y)$ whose exact values are irrelevant. Whenever the process Y jumps, say to $y \in O$, the filtering process jumps to a specific state prescribed by the function H_y. This state belongs to the subset Δ_y of the effective simplex, necessarily different from the subset of Δ_e to which the pre-jump state belonged.

We are now ready to state the filtering equation, which extends to the case of controlled process X the corresponding Theorem 2.3.2 and Theorem 2.1.6.

Theorem 3.1.2 (Filtering equation). *For all $\omega \in \Omega$ define $\tau_0(\omega) \equiv 0$ and for fixed $\mathbf{u} \in \mathcal{U}_{ad}$ the stochastic process $\pi^{\mu,\mathbf{u}} = (\pi^{\mu,\mathbf{u}}_t)_{t \geqslant 0}$ as the unique solution of the following* *system of ODEs*

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\pi_t^{\mu,\mathbf{u}}(\omega) = \displaystyle\int_U F(\pi_t^{\mu,\mathbf{u}}(\omega), \mathfrak{u})\, u_t(\omega\,; \mathrm{d}\mathfrak{u}), \quad t \in [\tau_n(\omega), \tau_{n+1}(\omega)),\, n \in \mathbb{N}_0 \\ \pi_0^{\mu,\mathbf{u}}(\omega) = H_{Y_0(\omega)}[\mu], \\ \pi_{\tau_n}^{\mu,\mathbf{u}}(\omega) = H_{Y_{\tau_n(\omega)}}\left[\pi_{\tau_n^-(\omega)}^{\mu,\mathbf{u}}(\omega) \displaystyle\int_U \Lambda(\mathfrak{u})\, u_{\tau_n^-}(\omega\,; \mathrm{d}\mathfrak{u})\right],\, n \in \mathbb{N}. \end{cases} \tag{3.1.11}$$

where F is the vector field defined in (3.1.7).

Then, $\pi^{\mu,\mathbf{u}}$ is $(\mathcal{Y}_t^\circ)_{t\geqslant 0}$ - adapted and is a modification of the filtering process, i. e.

$$\pi_t^{\mu,\mathbf{u}}(i) = \mathrm{P}_\mu^{\mathbf{u}}(X_t = i \mid \mathcal{Y}_t^{\mu,\mathbf{u}}), \quad \mathrm{P}_\mu^{\mathbf{u}}\text{–a.s.},\, t \geqslant 0,\, i \in I.$$

Remark 3.1.2. Thanks to the structure of admissible controls shown in (3.1.2) we can write (3.1.11) as

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\pi_t^{\mu,\mathbf{u}} = \displaystyle\int_U F(\pi_t^{\mu,\mathbf{u}}, \mathfrak{u})\, u_n(t, Y_0, \dots, \tau_n, Y_{\tau_n}\,; \mathrm{d}\mathfrak{u}), \quad t \in [\tau_n, \tau_{n+1}),\, n \in \mathbb{N}_0 \\ \pi_0^{\mu,\mathbf{u}} = H_{Y_0}[\mu], \\ \pi_{\tau_n}^{\mu,\mathbf{u}} = H_{Y_{\tau_n}}\left[\pi_{\tau_n^-}^{\mu,\mathbf{u}} \displaystyle\int_U \Lambda(\mathfrak{u})\, u_{n-1}(\tau_n^-, Y_0, \dots, \tau_{n-1}, Y_{\tau_{n-1}}\,; \mathrm{d}\mathfrak{u})\right],\, n \in \mathbb{N}. \end{cases}$$

Also in the controlled case, we can clearly characterize the process π as a *Piecewise Deterministic Process* (PDP), generalizing the corresponding Theorem 2.3.3 and Theorem 2.2.7. Its characteristic triple (F, r, R) is given by the controlled vector field F defined in (3.1.7), a controlled jump rate function $r\colon \Delta_e \times U \to [0, +\infty)$ and a controlled stochastic kernel R, i. e. a probability transition kernel from $(\Delta_e \times U, \mathcal{B}(\Delta_e) \otimes \mathcal{U})$ to $(\Delta_e, \mathcal{B}(\Delta_e))$. We define the functions in this triple as

$$\begin{aligned} F_j(\nu, u) &= \begin{cases} [\nu\Lambda(u)]_j - (\nu\Lambda(u)\mathbb{1}_{h^{-1}(y)})\nu_j, & j \in h^{-1}(y) \\ 0, & \text{otherwise} \end{cases}, \quad \nu \in \Delta_y, y \in O. \\ r(\rho, u) &= -\rho\Lambda(u)\mathbb{1}_{h^{-1}(y)}, \quad \rho \in \Delta_y \\ R(\rho, u; D) &= \sum_{b \in O} \mathbb{1}_D\big(H_b[\rho\Lambda(u)]\big)\, q(\rho, u, b), \quad \rho \in \Delta_y \\ q(\rho, u, b) &= \begin{cases} \dfrac{\rho\Lambda(u)\mathbb{1}_{h^{-1}(b)}}{-\rho\Lambda(u)\mathbb{1}_{h^{-1}(y)}}\mathbb{1}_{b \neq y}, & \text{if } \rho\Lambda(u)\mathbb{1}_{h^{-1}(y)} \neq 0 \\ q_y(b), & \text{if } \rho\Lambda(u)\mathbb{1}_{h^{-1}(y)} = 0 \end{cases}, \quad \rho \in \Delta_y \end{aligned} \tag{3.1.12}$$

where for each $y \in O$ we denote by $q_y = (q_y(b))_{b \in O}$ a probability measure concentrated on $O \setminus \{y\}$ whose exact values are irrelevant. It is important to notice that under Assumption 3.1.1 r is Lipschitz continuous uniformly in u, i. e.

$$\sup_{u \in U} |r(\rho, u) - r(\vartheta, u)| \leqslant L_r|\rho - \vartheta|, \quad \text{for all } \rho, \vartheta \in \Delta_y,\, y \in O, \tag{3.1.13}$$

with Lipschitz constant given by $L_r = \sum_{i \in I} \sup_{u \in U} \lambda_i(u)$. We also have that for some $C_r > 0$

$$\sup_{(\rho,u) \in \Delta_e \times U} |r(\rho, u)| \leqslant C_r. \tag{3.1.14}$$

Theorem 3.1.3. *For every* $\nu \in \Delta_e$ *and all* $\mathbf{u} \in \mathcal{U}_{ad}$ *the filtering process* $\pi^{\nu,\mathbf{u}} = (\pi_t^{\nu,\mathbf{u}})_{t\geqslant 0}$ *defined on the probability space* $(\Omega, \mathcal{X}^\circ, \mathrm{P}_\nu^{\mathbf{u}})$ *and taking values in* Δ_e *is a controlled* Piecewise Deterministic Process *with respect to the triple* (F, r, R) *defined in (3.1.12) and with starting point* ν.

More specifically, we have that $\mathrm{P}_\nu^{\mathbf{u}}$*–a.s.*

$$\pi_t^{\nu,\mathbf{u}} = \phi^{u_n}_{\pi^{\nu,\mathbf{u}}_{\tau_n}}(t-\tau_n), \quad \text{on } \{\tau_n < +\infty\},\ t \in [\tau_n, \tau_{n+1}),\ n \in \mathbb{N}_0 \tag{3.1.15}$$

$$\mathrm{P}_\nu^{\mathbf{u}}(\tau_{n+1} - \tau_n > t,\ \tau_n < +\infty \mid \mathcal{Y}^{\nu,\mathbf{u}}_{\tau_n}) = \mathbb{1}_{\tau_n<+\infty} \exp\left\{-\int_0^t \int_U r\big(\phi^{u_n(\cdot\,+\tau_n)}_{\pi^{\nu,\mathbf{u}}_{\tau_n}}(s), \mathfrak{u}\big)\, u_n(s+\tau_n\,; \mathrm{d}\mathfrak{u})\, \mathrm{d}s\right\}, \quad t \geqslant 0 \tag{3.1.16}$$

$$\mathrm{P}_\nu^{\mathbf{u}}(\pi^{\nu,\mathbf{u}}_{\tau_{n+1}} \in D,\ \tau_{n+1} < +\infty \mid \mathcal{Y}^{\nu,\mathbf{u}}_{\tau^-_{n+1}}) = \mathbb{1}_{\tau_{n+1}<+\infty} \int_U R\big(\phi^{u_n(\cdot\,+\tau_n)}_{\pi^{\nu,\mathbf{u}}_{\tau_n}}(\tau^-_{n+1} - \tau_n), \mathfrak{u}; D\big)\, u_n(\tau^-_{n+1}\,; \mathrm{d}\mathfrak{u}), \quad D \in \mathcal{B}(\Delta_e) \tag{3.1.17}$$

where, for each $n \in \mathbb{N}_0$, $\phi^{u_n}_{\pi^{\nu,\mathbf{u}}_{\tau_n}}$ *is the flow starting from* $\pi^{\nu,\mathbf{u}}_{\tau_n}$ *and determined by the controlled vector field* F *under the action of the control function* $u_n(\cdot, Y_0, \dots, \tau_n, Y_{\tau_n})$.

Remark 3.1.3. We already pointed out in Section 2.2 the importance of characterizing the filtering process as a PDP. Concerning the situation described in this Chapter, an additional advantage of this characterization comes from the available works in the literature regarding optimal control problems involving PDPs. In these problems it is customary to define the class of admissible controls as *piecewise open-loop controls*. These control functions, first studied by Vermes in [61], depend at any time $t \geqslant 0$ on the position of the PDP at the last jump-time prior to t and on the time elapsed since the last jump.

In (3.1.1) we specified a different class of admissible controls, more suited to our problem and imposed by the fact that we are dealing with partial observation, hence equations (3.1.15), (3.1.16) and (3.1.17) are changed with respect to the standard formulation with piecewise open-loop controls. Another element in contrast with the usual definition of a PDP is the absence in our model of a boundary, since this will be enough for our purposes.

A common assumption in PDP optimal control problems is that the transition measure R is a Feller kernel. This fails to happen in our situation but, nonetheless, a weaker form of this property holds and it is stated in the following Proposition.

Proposition 3.1.4. *Let Assumption 3.1.1 hold. Then for every bounded and continuous function* $w\colon \Delta_e \to \mathbb{R}$ *the function* $\rho \mapsto r(\rho, u) \int_{\Delta_e} w(p) R(\rho, u; \mathrm{d}p)$ *is continuous on* Δ_e *uniformly in* $u \in U$.

Proof. Fix $\rho \in \Delta_e$, i.e. $\rho \in \Delta_a$ for some $a \in O$, and $u \in U$. Let $(\rho_n)_{n\in\mathbb{N}}$ be a sequence such that $\rho_n \to \rho$ as $n \to +\infty$. Without loss of generality we can assume that $\rho_n \in \Delta_a$ for all $n \in \mathbb{N}$.

Let us consider, first, the case where $r(\rho, u) > 0$. It is easy to see that the function $\rho \mapsto r(\rho, u)$ is continuous on Δ_e uniformly in $u \in U$, therefore $r(\rho_n, u) > 0$ apart from a finite number of indices $n \in \mathbb{N}$. We want to prove that

$$\left| r(\rho_n, u) \int_{\Delta_e} w(p) R(\rho_n, u; \mathrm{d}p) - r(\rho, u) \int_{\Delta_e} w(p) R(\rho, u; \mathrm{d}p) \right| \to 0, \quad \text{as } n \to +\infty,$$

uniformly with respect to $u \in U$, i. e.

$$\Big|\sum_{a\neq b\in O}\sum_{i\in h^{-1}(a)}\sum_{j\in h^{-1}(b)}\Big\{w(H_b[\rho_n\Lambda^u])\rho_i^n - w(H_b[\rho\Lambda^u])\rho_i\Big\}\lambda_{ij}(u)\Big| \to 0, \quad (3.1.18)$$

as $n \to +\infty$ uniformly in $u \in U$.

Let $B = \{b \in O, b \neq a,$ such that $\sum_{i\in h^{-1}(a)}\sum_{j\in h^{-1}(b)} \rho_i\lambda_{ij}(u) > 0\}$ and $B_0 = B^c \setminus \{a\}$. Clearly, reasoning as before, we have that $\sum_{i\in h^{-1}(a)}\sum_{j\in h^{-1}(b)} \rho_i^n\lambda_{ij}(u) > 0$, apart from a finite number of indices $n \in \mathbb{N}$, for each $b \in B$. We have also that $\sum_{i\in h^{-1}(a)}\sum_{j\in h^{-1}(b)} \rho_i^n\lambda_{ij}(u) \to 0$ for all $b \in B_0$ uniformly in u. Then from equation (3.1.18) we can get the estimate

$$\begin{aligned}
&\Big|\sum_{b\in B}\sum_{i\in h^{-1}(a)}\sum_{j\in h^{-1}(b)}\Big\{w(H_b[\rho_n\Lambda^u])\rho_i^n - w(H_b[\rho\Lambda^u])\rho_i\Big\}\lambda_{ij}(u)\\
&\qquad + \sum_{b\in B_0}\sum_{i\in h^{-1}(a)}\sum_{j\in h^{-1}(b)} w(H_b[\rho_n\Lambda^u])\rho_i^n\lambda_{ij}(u)\Big|\\
&\leqslant C_\Lambda\sum_{b\in B}\big|w(H_b[\rho_n\Lambda^u]) - w(H_b[\rho\Lambda^u])\big| + |B|C_\Lambda \sup_{p\in\Delta_e}|w(p)|\sum_{i\in h^{-1}(a)}\big|\rho_i^n - \rho_i\big|\\
&\qquad + \sup_{p\in\Delta_e}|w(p)|\sum_{b\in B_0}\sum_{i\in h^{-1}(a)}\sum_{j\in h^{-1}(b)}\rho_i^n\lambda_{ij}(u),
\end{aligned} \quad (3.1.19)$$

where $C_\Lambda = \max_{i\in h^{-1}(a)}\sup_{u\in U}\lambda_i(u)$ is finite thanks to Assumption 3.1.1.

It is clear that the second and the third summand of the last inequality tend to 0, as n goes to infinity, uniformly in u. The difficult task is to show that this is the case also for the first summand. Since the function w is continuous on Δ_e, there exists a modulus of continuity η_w such that

$$\sum_{b\in B}\big|w(H_b[\rho_n\Lambda^u]) - w(H_b[\rho\Lambda^u])\big| \leqslant \sum_{b\in B}\eta_w\big(\big|H_b[\rho_n\Lambda^u] - H_b[\rho\Lambda^u]\big|\big). \quad (3.1.20)$$

Therefore we can fix $b \in B$ and concentrate ourselves on the term $\big|H_b[\rho_n\Lambda^u] - H_b[\rho\Lambda^u]\big|$. We need to show that

$$\begin{aligned}
&\big|H_b[\rho_n\Lambda^u] - H_b[\rho\Lambda^u]\big| =\\
&\sum_{j\in h^{-1}(b)}\left|\frac{\sum_{i\in h^{-1}(a)}\rho_i\lambda_{ij}(u)}{\sum_{k\in h^{-1}(a)}\sum_{l\in h^{-1}(b)}\rho_k\lambda_{kl}(u)} - \frac{\sum_{i\in h^{-1}(a)}\rho_i^n\lambda_{ij}(u)}{\sum_{k\in h^{-1}(a)}\sum_{l\in h^{-1}(b)}\rho_k^n\lambda_{kl}(u)}\right|
\end{aligned} \quad (3.1.21)$$

tends to 0 as $n \to +\infty$ uniformly in u. Notice that defining

$$K := \frac{1}{\Big(\sum_{k\in h^{-1}(a)}\sum_{l\in h^{-1}(b)}\rho_k\lambda_{kl}(u)\Big)\Big(\sum_{k\in h^{-1}(a)}\sum_{l\in h^{-1}(b)}\rho_k^n\lambda_{kl}(u)\Big)}$$

we can rewrite $\left|H_b[\rho_n\Lambda^u] - H_b[\rho\Lambda^u]\right|$ as

$$\begin{aligned}\left|H_b[\rho_n\Lambda^u] - H_b[\rho\Lambda^u]\right| &= \\ K \sum_{j\in h^{-1}(b)} \Bigg| \sum_{i\in h^{-1}(a)} &\Big\{\rho_i\lambda_{ij}(u) \sum_{k\in h^{-1}(a)} \sum_{l\in h^{-1}(b)} \rho_k^n\lambda_{kl}(u) \\ &- \rho_i^n\lambda_{ij}(u) \sum_{k\in h^{-1}(a)} \sum_{l\in h^{-1}(b)} \rho_k\lambda_{kl}(u)\Big\}\Bigg| \end{aligned} \tag{3.1.22}$$

Let $A_0 = \{i \in h^{-1}(a) \text{ such that } \rho_i = 0\}$ and $A = h^{-1}(a) \setminus A_0$. It is obvious that $\sum_{k\in h^{-1}(a)} \sum_{l\in h^{-1}(b)} \rho_k\lambda_{kl}(u) = \sum_{k\in A} \sum_{l\in h^{-1}(b)} \rho_k\lambda_{kl}(u)$. Using such a decomposition of the set $h^{-1}(a)$ the sums appearing in (3.1.22) can be estimated in the following way

$$\begin{aligned}
&\sum_{j\in h^{-1}(b)} \Bigg| \sum_{i\in h^{-1}(a)} \Bigg\{\rho_i\lambda_{ij}(u) \sum_{k\in h^{-1}(a)} \sum_{l\in h^{-1}(b)} \rho_k^n\lambda_{kl}(u) \\
&\qquad - \rho_i^n\lambda_{ij}(u) \sum_{k\in h^{-1}(a)} \sum_{l\in h^{-1}(b)} \rho_k\lambda_{kl}(u)\Bigg\}\Bigg| \\
&= \sum_{j\in h^{-1}(b)} \Bigg| \sum_{i\in A} \rho_i\lambda_{ij}(u) \sum_{k\in h^{-1}(a)} \sum_{l\in h^{-1}(b)} \rho_k^n\lambda_{kl}(u) \\
&\qquad - \sum_{i\in h^{-1}(a)} \rho_i^n\lambda_{ij}(u) \sum_{k\in A} \sum_{l\in h^{-1}(b)} \rho_k\lambda_{kl}(u)\Bigg| \\
&\leqslant 2\sum_{i\in A} |\rho_i - \rho_i^n| \sum_{j\in h^{-1}(b)} \lambda_{ij}(u) \sum_{k\in h^{-1}(a)} \sum_{l\in h^{-1}(b)} \rho_k^n\lambda_{kl}(u) \\
&\qquad + 2\sum_{i\in A} \sum_{k\in A_0} \sum_{j,l\in h^{-1}(b)} \rho_i^n\rho_k^n\lambda_{ij}(u)\lambda_{kl}(u).
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}&\left|H_b[\rho_n\Lambda^u] - H_b[\rho\Lambda^u]\right| \\ &\quad\leqslant 2\frac{\sum_{i\in A} |\rho_i - \rho_i^n| \sum_{j\in h^{-1}(b)} \lambda_{ij}(u)}{\sum_{i\in A} \sum_{j\in h^{-1}(b)} \rho_i\lambda_{ij}(u)} + 2\frac{\sum_{i\in A} \sum_{k\in A_0} \sum_{j,l\in h^{-1}(b)} \rho_i^n\rho_k^n\lambda_{ij}(u)\lambda_{kl}(u)}{\sum_{i\in A} \sum_{k\in A} \rho_i\rho_k^n \sum_{j,l\in h^{-1}(b)} \lambda_{ij}(u)\lambda_{kl}(u)}.\end{aligned} \tag{3.1.23}$$

We are left to prove that the two terms appearing in (3.1.23) tend to zero uniformly in u. It suffices to rewrite them in a suitable way, exploiting the properties granted by the decomposition $h^{-1}(a) = A \cup A_0$. As for the first summand:

$$\frac{\sum\limits_{i\in A} |\rho_i - \rho_i^n| \sum\limits_{j\in h^{-1}(b)} \lambda_{ij}(u)}{\sum\limits_{i\in A}\sum\limits_{j\in h^{-1}(b)} \rho_i\lambda_{ij}(u)} =$$

$$\sum_{i\in A} \frac{|\rho_i - \rho_i^n| \sum\limits_{j\in h^{-1}(b)} \lambda_{ij}(u)}{\rho_i \sum\limits_{j\in h^{-1}(b)} \lambda_{ij}(u)} \underbrace{\frac{\rho_i \sum\limits_{j\in h^{-1}(b)} \lambda_{ij}(u)}{\sum\limits_{i\in A}\sum\limits_{j\in h^{-1}(b)} \rho_i\lambda_{ij}(u)}}_{\leqslant 1\ \forall u\in U} \leqslant \sum_{i\in A} \frac{|\rho_i - \rho_i^n|}{\rho_i} \to 0$$

uniformly in $u \in U$. Finally, for the second summand [1]:

$$\frac{\sum\limits_{i\in A}\sum\limits_{k\in A_0}\sum\limits_{j,l\in h^{-1}(b)} \rho_i^n\rho_k^n \lambda_{ij}(u)\lambda_{kl}(u)}{\sum\limits_{i\in A}\sum\limits_{k\in A} \rho_i\rho_k^n \sum\limits_{j,l\in h^{-1}(b)} \lambda_{ij}(u)\lambda_{kl}(u)} =$$

$$\sum_{i\in A}\sum_{k\in A_0} \frac{\rho_i^n\rho_k^n \sum\limits_{j,l\in h^{-1}(b)} \lambda_{ij}(u)\lambda_{kl}(u)}{\rho_i\rho_i^n \sum\limits_{j,l\in h^{-1}(b)} \lambda_{ij}(u)\lambda_{kl}(u)} \underbrace{\frac{\rho_i\rho_i^n \sum\limits_{j,l\in h^{-1}(b)} \lambda_{ij}(u)\lambda_{kl}(u)}{\sum\limits_{i\in A}\sum\limits_{k\in A} \rho_i\rho_k^n \sum\limits_{j,l\in h^{-1}(b)} \lambda_{ij}(u)\lambda_{kl}(u)}}_{\leqslant 1\ \forall u\in U}$$

$$\leqslant \sum_{i\in A}\sum_{k\in A_0} \frac{\rho_k^n}{\rho_i} \to 0 \text{ uniformly in } u \in U.$$

Combining the result just obtained with equations (3.1.19) and (3.1.20) we get the claim in the case $r(\rho, u) > 0$.

The case $r(\rho, u) = 0$ is much less cumbersome to analyze. Without loss of generality we can assume that the sequence $r(\rho_n, u) \neq 0$ starting from some index n on (the case in which the sequence is equal to 0 eventually is trivial). We have to prove that

$$\left| r(\rho_n, u) \int_{\Delta_e} w(p) R(\rho_n, u; \mathrm{d}p) \right| = \left| \sum_{a\neq b\in O} w(H_b[\rho_n\Lambda^u]) \sum_{i\in h^{-1}(a)} \sum_{j\in h^{-1}(b)} \rho_i^n \lambda_{ij}(u) \right|$$

tends to zero, as n tends to infinity, uniformly with respect to $u \in U$. Thanks to the boundedness of the function w we immediately get

$$\left| \sum_{a\neq b\in O} w(H_b[\rho_n\Lambda^u]) \sum_{i\in h^{-1}(a)} \sum_{j\in h^{-1}(b)} \rho_i^n \lambda_{ij}(u) \right| \leqslant \sup_{p\in\Delta_e} |w(p)| \sum_{a\neq b\in O} \sum_{i\in h^{-1}(a)} \sum_{j\in h^{-1}(b)} \rho_i^n \lambda_{ij}(u). \quad (3.1.24)$$

The properties of the matrix coefficients $\lambda_{ij}(u)$ ensure that

$$0 = \underbrace{\sum_{i\in h^{-1}(a)} \sum_{j\in h^{-1}(a)} \rho_i \lambda_{ij}(u)}_{r(\rho,u)=0} + \sum_{a\neq b\in O} \underbrace{\sum_{i\in h^{-1}(a)} \sum_{j\in h^{-1}(b)} \rho_i \lambda_{ij}(u)}_{\geqslant 0,\ \forall b\neq a}.$$

[1]Provided that $\sum\limits_{j,l\in h^{-1}(b)} \lambda_{ij}(u)\lambda_{kl}(u) \neq 0$ for all $i \in A$ and all $k \in A_0$. If this is not the case for some indices i, k, one can just exclude these indices from the sum appearing in the numerator of this term.

Therefore the terms $\sum_{i \in h^{-1}(a)} \sum_{j \in h^{-1}(b)} \rho_i \lambda_{ij}(u)$ are equal to zero for all $b \neq a$ and since $\rho_i^n \sum_{j \in h^{-1}(b)} \lambda_{ij}(u) \to \rho_i \sum_{j \in h^{-1}(b)} \lambda_{ij}(u) = 0$ for all $i \in h^{-1}(a)$ and all $b \neq a$ uniformly in u, we get the desired result from equation (3.1.24). □

We can now turn our attention back to the optimal control problem. We recall that the aim is to minimize the cost functional J defined in (3.1.4), i. e. to study the value function V defined in (3.1.5). Since the control processes $\mathbf{u}$ are $(\mathcal{Y}_t^\circ)_{t \geqslant 0}$ – predictable and we know that the filtering process $\pi^{\mu,\mathbf{u}}$ provides us with the conditional law of X_t given $\mathcal{Y}_t^{\mu,\mathbf{u}}$, for all $t \geqslant 0$, it is easy to show that

$$J(\mu, \mathbf{u}) = \mathrm{E}_\mu^{\mathbf{u}} \left[\int_0^\infty e^{-\beta t} \pi_t^{\mu,\mathbf{u}} \int_U \mathbf{f}(\mathrm{u})\, u_t(\mathrm{d}\mathrm{u})\, \mathrm{d}t \right]. \tag{3.1.25}$$

Evidently, this form of the functional J has the advantage of depending on completely observable processes, namely $\pi^{\mu,\mathbf{u}}$ and $\mathbf{u}$ (which in turn depend on Y), so that we have turned the optimal control problem for the Markov chain X into an optimal control problem for the PDP $\pi^{\mu,\mathbf{u}}$. Moreover, we can write J in a way that allows us to interpret our problem as a discrete-time control problem. Exploiting the structure of admissible controls $\mathbf{u} = (u_n)_{n \in \mathbb{N}_0} \in \mathcal{U}_{ad}$ and recalling that for each $n \in \mathbb{N}_0$, u_n is $\mathcal{Y}_{\tau_n}^\circ$–measurable, it is easy to see that

$$\begin{aligned}
J(\mu, \mathbf{u}) &= \mathrm{E}_\mu^{\mathbf{u}} \left[\sum_{n=0}^{+\infty} \int_{\tau_n}^{\tau_{n+1}} e^{-\beta t} \pi_t^{\mu,\mathbf{u}} \int_U \mathbf{f}(\mathrm{u})\, u_n(t\,; \mathrm{d}\mathrm{u})\, \mathrm{d}t \right] \\
=& \mathrm{E}_\mu^{\mathbf{u}} \left[\sum_{n=0}^{+\infty} e^{-\beta \tau_n} \int_0^{\tau_{n+1}-\tau_n} e^{-\beta t} \phi_{\pi_{\tau_n}^{\mu,\mathbf{u}}}^{u_n(\cdot\, + \tau_n)}(t) \int_U \mathbf{f}(\mathrm{u})\, u_n(t+\tau_n\,; \mathrm{d}\mathrm{u})\, \mathrm{d}t \right] \\
=& \sum_{n=0}^{+\infty} \mathrm{E}_\mu^{\mathbf{u}} \left[\mathrm{E}_\mu^{\mathbf{u}} \left[e^{-\beta \tau_n} \int_0^{\tau_{n+1}-\tau_n} e^{-\beta t} \phi_{\pi_{\tau_n}^{\mu,\mathbf{u}}}^{u_n(\cdot\, + \tau_n)}(t) \int_U \mathbf{f}(\mathrm{u}) u_n(t+\tau_n\,; \mathrm{d}\mathrm{u})\, \mathrm{d}t \mid \mathcal{Y}_{\tau_n}^{\mu,\mathbf{u}} \right] \right] \\
=& \mathrm{E}_\mu^{\mathbf{u}} \left[\sum_{n=0}^{+\infty} e^{-\beta \tau_n} \int_0^{+\infty} e^{-\beta t} \chi_{\pi_{\tau_n}^{\mu,\mathbf{u}}}^{u_n(\cdot\, + \tau_n)}(t) \phi_{\pi_{\tau_n}^{\mu,\mathbf{u}}}^{u_n(\cdot\, + \tau_n)}(t) \int_U \mathbf{f}(\mathrm{u})\, u_n(t+\tau_n\,; \mathrm{d}\mathrm{u})\, \mathrm{d}t \right] \\
=& \mathrm{E}_\mu^{\mathbf{u}} \left[\sum_{n=0}^{+\infty} e^{-\beta \tau_n} g\big(\pi_{\tau_n}^{\mu,\mathbf{u}}, u_n(\cdot + \tau_n, Y_0, \tau_1, Y_{\tau_1}, \dots, \tau_n, Y_{\tau_n})\big) \right]
\end{aligned} \tag{3.1.26}$$

where the function g (that will be defined precisely in Section 3.2) represents the double integral appearing in the fourth line and $\chi_{\pi_{\tau_n}^{\mu,\mathbf{u}}}^{u_n(\cdot\, + \tau_n)}$ is the survival distribution appearing in (3.1.16).

Unfortunately, the reformulated problem does not fit in the framework of a classical discrete-time optimal control problem (see e. g. [13]) for various reasons. For instance, the problem should be based only on the discrete-time process given by the pairs of jump times and jump locations of the filtering process $\pi^{\mu,\mathbf{u}}$ (notice that in (3.1.26) also the process Y appears) which, in turn, should not depend on the initial law of the process X and on the control trajectory $\mathbf{u}$. Moreover, the class of admissible controls $\mathcal{U}_{ad}$ is not adequate for a discrete-time problem, since its policies should be functions depending at each time step exclusively on the past trajectory of a discrete-time process (in this case, the one based on the filtering process, as explained above). It is immediate to see that this is not the case for (3.1.26), since each of the functions u_n depends on a continuous-time variable and on the positions of the process Y.

We will solve these issues by reformulating our original control problem into a discrete-time one based on the filtering process, introducing also a new class of controls strongly related to the family $\mathcal{U}_{ad}$.

3.2 The separated optimal control problem

In this Section we will reformulate the original optimal control problem into a discrete-time one based on the filtering process. This reformulation will fall in the framework of [13] (from which we will borrow some terminology), a fact that enables us to use known results to study the value function V defined in (3.1.5). We will prove that the original control problem and the separated one are deeply connected. In particular, we will show that the value function V can be indirectly characterized by its counterpart in the separated problem, that will be analyzed in detail in the next section.

Let us introduce the *action space*

$$\mathcal{M} = \{m\colon [0,+\infty) \to \mathcal{P}(U),\ \text{measurable}\} \tag{3.2.1}$$

whose elements are *relaxed controls*. It is known that this space endowed with the *Young topology* is compact (see e. g. [32]). As already pointed out in Remark 3.1.1, the set of *ordinary controls*

$$A = \{\alpha\colon [0,+\infty) \to U,\ \text{measurable}\}$$

can be identified as a subset of $\mathcal{M}$ via the function $t \mapsto \delta_{\alpha(t)}$, $\alpha \in A$, where δ_u denotes the Dirac probability measure concentrated at the point $u \in U$. As proved in [67, Lemma 1], this set becomes a *Borel space* when endowed with the coarsest σ-algebra such that the maps

$$\alpha \mapsto \int_0^{+\infty} e^{-t}\psi(t,\alpha(t))\,\mathrm{d}t$$

are measurable for all $\psi\colon [0,+\infty) \times U \to \mathbb{R}$, bounded and measurable. This is a fundamental fact to be used in the sequel. Finally, we define the class of *admissible policies* $\mathcal{A}_{ad}$ for the separated optimal control problem as

$$\mathcal{A}_{ad} = \{\mathbf{a} = (a_n)_{n\in\bar{\mathbb{N}}_0}, a_n\colon \Delta_e \times \big((0,+\infty)\times\Delta_e\big)^n \to \mathcal{M} \text{ measurable } \forall n \in \bar{\mathbb{N}}_0\}. \tag{3.2.2}$$

We are now ready to introduce the separated PDP optimal control problem. Since this separated control problem uses the filtering process as a state variable, we need to put ourselves in a canonical framework for this process. Notice that the construction is similar to that given at the end of Section 2.2. We define the following objects.

- $\bar{\Omega} = \{\bar{\omega}\colon [0,+\infty) \to \Delta_e,\ \text{cádlág}\}$ denotes the canonical space for Δ_e – valued PDPs. We define $\bar{\pi}_t(\bar{\omega}) = \bar{\omega}(t)$, for $\bar{\omega} \in \bar{\Omega}$, $t \geqslant 0$, and

$$\begin{aligned}
\bar{\tau}_0(\bar{\omega}) &= 0,\\
\bar{\tau}_n(\bar{\omega}) &= \inf\{t > \bar{\tau}_{n-1}(\bar{\omega}) \text{ s.t. } \bar{\pi}_t(\bar{\omega}) \neq \bar{\pi}_{t^-}(\bar{\omega})\}, \quad n \in \mathbb{N},\\
\bar{\tau}_\infty(\bar{\omega}) &= \lim_{n\to\infty} \bar{\tau}_n(\bar{\omega}).
\end{aligned}$$

- The family of σ-algebras $(\bar{\mathcal{F}}_t^\circ)_{t\geqslant 0}$ given by

$$\bar{\mathcal{F}}_t^\circ = \sigma(\bar\pi_s, 0\leqslant s\leqslant t), \quad \bar{\mathcal{F}}^\circ = \sigma(\bar\pi_s, s\geqslant 0),$$

is the natural filtration of the process $\bar\pi = (\bar\pi_t)_{t\geqslant 0}$.

- For every $\nu\in\Delta_e$ and all $\mathbf{a}\in\mathcal{A}_{ad}$ we denote by $\bar{\mathrm{P}}_\nu^{\mathbf{a}}$ the probability measure on $(\bar\Omega, \bar{\mathcal{F}}^\circ)$ such that the process $\bar\pi$ is a PDP, starting from the point ν and with characteristic triple (F, r, R). We this, we mean that $\bar{\mathrm{P}}_\nu^{\mathbf{a}}$–a.s.

$$\bar\pi_t = \phi^{a_n}_{\bar\pi_{\bar\tau_n}}(t-\bar\tau_n), \quad \text{on } \{\bar\tau_n < +\infty\},\ t\in[\bar\tau_n, \bar\tau_{n+1}),\ n\in\mathbb{N}_0. \tag{3.2.3}$$

$$\bar{\mathrm{P}}_\nu^{\mathbf{a}}(\bar\tau_{n+1}-\bar\tau_n > t,\ \bar\tau_n < +\infty \mid \bar{\mathcal{F}}^\circ_{\bar\tau_n}) = \mathbb{1}_{\bar\tau_n<+\infty}\exp\left\{-\int_0^t\int_U r(\phi^{a_n}_{\bar\pi_{\bar\tau_n}}(t), \mathfrak{u})\, a_n(s\,;\mathrm{d}\mathfrak{u})\,\mathrm{d}s\right\}, \quad t\geqslant 0. \tag{3.2.4}$$

$$\bar{\mathrm{P}}_\nu^{\mathbf{a}}(\bar\pi_{\bar\tau_{n+1}}\in D,\ \bar\tau_{n+1}<+\infty \mid \bar{\mathcal{F}}^\circ_{\bar\tau^-_{n+1}}) = \mathbb{1}_{\bar\tau_{n+1}<+\infty}\int_U R(\phi^{a_n}_{\bar\pi_{\bar\tau_n}}(\bar\tau^-_{n+1}-\bar\tau_n), \mathfrak{u}; D)\, a_n(\bar\tau^-_{n+1}-\bar\tau_n\,;\mathrm{d}\mathfrak{u}), \quad D\in\mathcal{B}(\Delta_e). \tag{3.2.5}$$

where we simplified the notation by indicating $a_n = a_n(\bar\pi_0,\dots,\bar\tau_n,\bar\pi_{\bar\tau_n})$ and, for each $n\in\mathbb{N}_0$, we denoted by $\phi^{a_n}_{\bar\pi_{\bar\tau_n}}$ the flow starting from $\bar\pi_{\bar\tau_n}$ and determined by the controlled vector field F under the action of the relaxed control $a_n(\bar\pi_0,\dots,\bar\tau_n,\bar\pi_{\bar\tau_n})$. We recall that the probability measure $\bar{\mathrm{P}}_\nu^{\mathbf{a}}$ always exists by the canonical construction of a PDP (see Section 1.2).

- For every $Q\in\mathcal{P}(\Delta_e)$ and every $\mathbf{a}\in\mathcal{A}_{ad}$ we define a probability $\bar{\mathrm{P}}_Q^{\mathbf{a}}$ on $(\bar\Omega,\bar{\mathcal{F}}^\circ)$ by $\bar{\mathrm{P}}_Q^{\mathbf{a}}(C) = \int_{\Delta_e}\bar{\mathrm{P}}_\nu^{\mathbf{a}}(C)Q(\mathrm{d}\nu)$ for $C\in\bar{\mathcal{F}}^\circ$. This means that Q is the initial distribution of $\bar\pi$ under $\bar{\mathrm{P}}_Q^{\mathbf{a}}$.

- Let $\bar{\mathcal{F}}^{Q,\mathbf{a}}$ be the $\bar{\mathrm{P}}_Q^{\mathbf{a}}$-completion of $\bar{\mathcal{F}}^\circ$. We still denote by $\bar{\mathrm{P}}_Q^{\mathbf{a}}$ the measure naturally extended to this new σ-algebra. Let $\bar{\mathcal{Z}}^{Q,\mathbf{a}}$ be the family of sets in $\bar{\mathcal{F}}^{Q,\mathbf{a}}$ with zero $\bar{\mathrm{P}}_Q^{\mathbf{a}}$-probability and define

$$\bar{\mathcal{F}}_t^{Q,\mathbf{a}} = \sigma(\bar{\mathcal{F}}_t^\circ\cup\bar{\mathcal{Z}}^{Q,\mathbf{a}}), \quad \bar{\mathcal{F}}_t = \bigcap_{\substack{Q\in\mathcal{P}(\Delta_e)\\ \mathbf{a}\in\mathcal{A}_{ad}}}\bar{\mathcal{F}}_t^{Q,\mathbf{a}}, \quad t\geqslant 0.$$

$(\bar{\mathcal{F}}_t)_{t\geqslant 0}$ is called the *natural completed filtration* of $\bar\pi$. By a slight generalization of Theorem 1.2.2 it is right-continuous.

The PDP $(\bar\Omega, \bar{\mathcal{F}}, (\bar{\mathcal{F}}_t)_{t\geqslant 0}, (\bar\pi_t)_{t\geqslant 0}, (\bar{\mathrm{P}}_\nu^{\mathbf{a}})^{\mathbf{a}\in\mathcal{A}_{ad}}_{\nu\in\Delta_e})$ constructed as above admits the characteristic triple (F, r, R). For sake of brevity, let us introduce the function χ^m_ρ, depending on $\rho\in\Delta_e$ and $m\in\mathcal{M}$, given by

$$\chi^m_\rho(t) = \exp\left\{-\int_0^t\int_U r(\phi^m_\rho(s), \mathfrak{u})\, m(s\,;\mathrm{d}\mathfrak{u})\,\mathrm{d}s\right\}, \quad t\geqslant 0. \tag{3.2.6}$$

In this way, we can write (3.2.4) as

$$\bar{\mathrm{P}}^{\mathbf{a}}_{\nu}(\bar{\tau}_{n+1} - \bar{\tau}_n > t \mid \bar{\mathcal{F}}^{\circ}_{\bar{\tau}_n}) = \chi^{a_n}_{\nu}(t), \quad t \geqslant 0, \text{ on } \{\bar{\tau}_n < +\infty\}.$$

It is worth noticing that χ^m_ρ solves the ODE

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} z(t) = -z(t) \displaystyle\int_U r(\phi^m_\rho(t), \mathrm{u})\, m(t\,; \mathrm{du}), \quad t \geqslant 0 \\ z(0) = 1 \end{cases} \tag{3.2.7}$$

We define the observed process $\bar{Y}$ on $\bar{\Omega}$ as follows. Let us introduce the (obviously measurable) function $\mathrm{proj}_{\mathrm{Y}} \colon \Delta_e \to O$ given by

$$\mathrm{proj}_{\mathrm{Y}}(p) = y, \quad \text{if } p \in \Delta_y, \text{ for some } y \in O$$

and set

$$\bar{Y}_t(\bar{\omega}) = \begin{cases} \mathrm{proj}_{\mathrm{Y}}(\bar{\pi}_0(\bar{\omega})), & t \in \big[0, \bar{\tau}_1(\bar{\omega})\big) \\ \mathrm{proj}_{\mathrm{Y}}(\bar{\pi}_{\bar{\tau}_n(\bar{\omega})}(\bar{\omega})), & t \in \big[\bar{\tau}_n(\bar{\omega}), \bar{\tau}_{n+1}(\bar{\omega})\big),\ n \in \mathbb{N},\ \bar{\tau}_n(\bar{\omega}) < +\infty \\ o_\infty, & t \in \big[\bar{\tau}_\infty(\bar{\omega}), +\infty),\ \bar{\tau}_\infty(\bar{\omega}) < +\infty \end{cases},$$

where $o_\infty \in O$ is an arbitrary state, that is irrelevant to specify. In fact, it is easy to prove by standard arguments that under Assumption 3.1.1 for each fixed $\nu \in \Delta_e$ and $\mathbf{a} \in \mathcal{A}_{ad}$ we have that $\bar{\tau}_\infty = +\infty$, $\bar{\mathrm{P}}^{\mathbf{a}}_\nu$–a.s., i. e. also in this framework the observed process is $\bar{\mathrm{P}}^{\mathbf{a}}_\nu$–a.s. non explosive.

Next, let us define the counterpart of the functional J, appearing in (3.1.25), as follows. Let $g \colon \Delta_e \times \mathcal{M} \to \mathbb{R}$ be the discrete-time one-stage cost function defined as

$$g(\nu, m) = \int_0^{+\infty} e^{-\beta t} \chi^m_\nu(t) \phi^m_\nu(t) \int_U \mathbf{f}(\mathrm{u})\, m(t\,; \mathrm{du})\, \mathrm{d}t. \tag{3.2.8}$$

For each $\nu \in \Delta_e$ and $\mathbf{a} \in \mathcal{A}_{ad}$ the cost functional $\bar{J}$ of the separated problem is defined in analogy with the last line of (3.1.26) as

$$\bar{J}(\nu, \mathbf{a}) = \bar{\mathrm{E}}^{\mathbf{a}}_\nu \left[\sum_{n=0}^{+\infty} e^{-\beta \bar{\tau}_n} g\big(\bar{\pi}_{\bar{\tau}_n}, a_n(\bar{\pi}_{\bar{\tau}_0}, \dots, \bar{\tau}_n, \bar{\pi}_{\bar{\tau}_n})\big) \right]. \tag{3.2.9}$$

Finally, we define the value function of the separated problem as

$$v(\nu) = \inf_{\mathbf{a} \in \mathcal{A}_{ad}} \bar{J}(\nu, \mathbf{a}). \tag{3.2.10}$$

It is now fundamental to establish a connection between the cost functionals given in (3.1.25) and (3.2.9). This link will be given by constructing corresponding admissible controls in $\mathcal{U}_{ad}$ and admissible policies in $\mathcal{A}_{ad}$.

Theorem 3.2.1. *Fix $\mu \in \Delta$ and let $Q \in \mathcal{P}(\Delta_e)$ the Borel probability measure on Δ_e concentrated at points $H_y[\mu] \in \Delta_e$, as y varies in the set O, defined as*

$$Q(D) = \sum_{y \in O} \mu(h^{-1}(y)) \delta_{H_y[\mu]}(D), \quad D \in \mathcal{B}(\Delta_e). \tag{3.2.11}$$

For all $\mu \in \Delta$ and all $\mathbf{u} \in \mathcal{U}_{ad}$ there exists an admissible policy $\mathbf{a} \in \mathcal{A}_{ad}$ such that the laws of $\pi^{\mu,\mathbf{u}}$ under $\mathrm{P}^{\mathbf{u}}_{\mu}$ and of $\bar{\pi}$ under $\bar{\mathrm{P}}^{\mathbf{a}}_{Q}$ are the same. Moreover, for such an admissible policy

$$J(\mu, \mathbf{u}) = \sum_{y \in O} \mu(h^{-1}(y)) \bar{J}(H_y[\mu], \mathbf{a}). \tag{3.2.12}$$

Viceversa, for all $\mu \in \Delta$ and all $\mathbf{a} = (a_n)_{n \in \mathbb{N}} \in \mathcal{A}_{ad}$ there exists an admissible control $\mathbf{u} \in \mathcal{U}_{ad}$ such that the same conclusions hold.

Proof. Let us prove the first part of the theorem. Let $\mathbf{u} \in \mathcal{U}_{ad}$ be fixed and for all $n \in \mathbb{N}_0$ let us define the functions $a_n \colon \Delta_e \times \big((0, +\infty) \times \Delta_e\big)^n \to \mathcal{M}$ as

$$a_n(\nu_0, \dots, s_n, \nu_n)(t\,; \mathrm{du}) = u_n\big(t + s_n, \mathrm{proj}_{\mathrm{Y}}(\nu_0), \dots, s_n, \mathrm{proj}_{\mathrm{Y}}(\nu_n)\,; \mathrm{du}\big)$$

for all possible sequences $(\nu_i)_{i=0}^n \subset \Delta_e$ and $(s_i)_{i=1}^n \subset (0, +\infty)$.

Thanks to the fact that $\mathrm{proj}_{\mathrm{Y}}$ is Borel-measurable and that $\mathcal{M}$ is a Borel space, we can apply [67, Lemma 3(i)] and it follows that each function a_n is measurable. Therefore we have that $\mathbf{a} = (a_n)_{n \in \mathbb{N}_0} \in \mathcal{A}_{ad}$.

The laws of $\pi^{\mu,\mathbf{u}}$ under $\mathrm{P}^{\mathbf{u}}_{\mu}$ and $\bar{\pi}$ under $\bar{\mathrm{P}}^{\mathbf{a}}_{Q}$ are determined respectively by the finite-dimensional distributions of the stochastic processes $\{\pi^{\mu,\mathbf{u}}_0, \tau_1, \pi^{\mu,\mathbf{u}}_{\tau_1}, \dots\}$ and $\{\bar{\pi}_0, \bar{\tau}_1, \bar{\pi}_{\bar{\tau}_1}, \dots\}$ and by the flows associated to the controlled vector fields $F^{\mathbf{u}}$ and $F^{\mathbf{a}}$. These laws, in turn, can be expressed via the initial distributions of $\pi^{\mu,\mathbf{u}}_0$ and $\bar{\pi}_0$ and the conditional distributions of the sojourn times and post-jump locations, i. e. for $t \geqslant 0$ and $D \in \mathcal{B}(\Delta_e)$ the quantities

$$\mathrm{P}^{\mathbf{u}}_{\mu}(\tau_n - \tau_{n-1} > t,\, \tau_{n-1} < +\infty \mid \pi^{\mu,\mathbf{u}}_0, \dots, \tau_{n-1}, \pi^{\mu,\mathbf{u}}_{\tau_{n-1}}); \tag{3.2.13}$$

$$\bar{\mathrm{P}}^{\mathbf{a}}_{Q}(\bar{\tau}_n - \bar{\tau}_{n-1} > t,\, \bar{\tau}_{n-1} < +\infty \mid \bar{\pi}_0, \dots, \bar{\tau}_{n-1}, \bar{\pi}_{\bar{\tau}_{n-1}}); \tag{3.2.14}$$

$$\mathrm{P}^{\mathbf{u}}_{\mu}(\pi^{\mu,\mathbf{u}}_{\tau_n} \in D,\, \tau_n < +\infty \mid \pi^{\mu,\mathbf{u}}_0, \dots, \pi^{\mu,\mathbf{u}}_{\tau_{n-1}}, \tau_n); \tag{3.2.15}$$

$$\bar{\mathrm{P}}^{\mathbf{a}}_{Q}(\bar{\pi}_{\bar{\tau}_n} \in D,\, \bar{\tau}_n < +\infty \mid \bar{\pi}_0, \dots, \bar{\pi}_{\bar{\tau}_{n-1}}, \bar{\tau}_n). \tag{3.2.16}$$

We will now prove that under the two different probability measures $\mathrm{P}^{\mathbf{u}}_{\mu}$ and $\bar{\mathrm{P}}^{\mathbf{a}}_{Q}$ the distributions (3.2.13) - (3.2.16) along with the initial laws of $\pi^{\mu,\mathbf{u}}_0$ and $\bar{\pi}_0$ are equal.
Initial distribution. Fix $D \in \mathcal{B}(\Delta_e)$. Then

$$\begin{aligned}\mathrm{P}^{\mathbf{u}}_{\mu}(\pi^{\mu,\mathbf{u}}_0 \in D) = \mathrm{P}^{\mathbf{u}}_{\mu}(H_{Y_0}[\mu] \in D) &= \sum_{y \in O} \mathrm{P}^{\mathbf{u}}_{\mu}(H_y[\mu] \in D, Y_0 = y)\\ &= \sum_{y \in O} \mathrm{P}^{\mathbf{u}}_{\mu}(Y_0 = y)\delta_{H_y[\mu]}(D) = \sum_{y \in O} \mu(h^{-1}(y))\delta_{H_y[\mu]}(D) = Q(D),\end{aligned}$$

since the events $\{H_y[\mu] \in D\}$ are either of probability zero or one with respect to $\mathrm{P}^{\mathbf{u}}_{\mu}$ for all $y \in O$. On the other side

$$\begin{aligned}\bar{\mathrm{P}}^{\mathbf{a}}_{Q}(\bar{\pi}_0 \in D) &= \int_{\Delta_e} \bar{\mathrm{P}}^{\mathbf{a}}_{\nu}(\bar{\pi}_0 \in D)\, Q(\mathrm{d}\nu) = \sum_{y \in O} \mu(h^{-1}(y)) \bar{\mathrm{P}}^{\mathbf{a}}_{H_y[\mu]}(\bar{\pi}_0 \in D)\\ &= \sum_{y \in O} \mu(h^{-1}(y))\delta_{H_y[\mu]}(D) = Q(D).\end{aligned}$$

Sojourn times. Let us analyze, first, the conditional law (3.2.13). Notice that since we are considering (3.2.13) on the set $\tau_{n-1} < +\infty$, $\pi^{\mu,\mathbf{u}}_{\tau_{n-1}}$ is well defined and the law of $\tau_n - \tau_{n-1}$ is not trivial. Fix $p_0, \dots, p_{n-1} \in \Delta_e$, where for each $i = 0, \dots, n-1$,

$p_i \in \Delta_{b_i}$ for some $b_0 \neq b_1 \neq \cdots \neq b_{n-1} \in O$; fix also $0 < s_1 < \cdots < s_{n-1} < +\infty$. Since a trajectory of the observed process Y uniquely determines a trajectory of the filtering process $\pi^{\mu,\mathbf{u}}$ and viceversa, we can immediately deduce that, up to $\mathrm{P}^{\mathbf{u}}_{\mu}$-null sets

$$\mathcal{Y}^{\mu,\mathbf{u}}_{\tau_{n-1}} = \sigma(\pi^{\mu,\mathbf{u}}_0, \dots, \tau_{n-1}, \pi^{\mu,\mathbf{u}}_{\tau_{n-1}}) \qquad \text{and} \qquad \mathcal{Y}^{\mu,\mathbf{u}}_{\tau_n^-} = \sigma(\pi^{\mu,\mathbf{u}}_0, \dots, \pi^{\mu,\mathbf{u}}_{\tau_{n-1}}, \tau_n).$$

From this fact and (3.1.16) we can write for $t \geqslant 0$

$$\begin{aligned}
&\mathrm{P}^{\mathbf{u}}_{\mu}(\tau_n - \tau_{n-1} > t,\, \tau_{n-1} < +\infty \mid \pi^{\mu,\mathbf{u}}_0 = p_0, \dots, \tau_{n-1} = s_{n-1}, \pi^{\mu,\mathbf{u}}_{\tau_{n-1}} = p_{n-1}) \\
=&\chi^{u_{n-1}}_{p_{n-1}}(t) = \exp\Big\{-\int_0^t \int_U r\big(\phi^{u_{n-1}}_{p_{n-1}}(s), \mathrm{u}\big)\, u_{n-1}(s + s_{n-1}\,; \mathrm{d}\mathrm{u})\, \mathrm{d}s\Big\}.
\end{aligned}$$

The function $u_{n-1} = u_{n-1}(\cdot + s_{n-1},\, b_0,\, \dots,\, s_{n-1},\, b_{n-1})$ can be clearly expressed as

$$u_{n-1}(\cdot + s_{n-1},\, \mathrm{proj}_{\mathrm{Y}}(p_0),\, \dots,\, s_{n-1},\, \mathrm{proj}_{\mathrm{Y}}(p_{n-1})).$$

Therefore, if we compare the previous computation with

$$\begin{aligned}
&\bar{\mathrm{P}}^{\mathbf{a}}_{Q}(\bar{\tau}_n - \bar{\tau}_{n-1} > t,\, \bar{\tau}_{n-1} < +\infty \mid \bar{\pi}_0 = p_0, \dots, \bar{\tau}_{n-1} = s_{n-1}, \bar{\pi}_{\tau_{n-1}} = p_{n-1}) \\
=&\chi^{a_{n-1}}_{p_{n-1}}(t) = \exp\Big\{-\int_0^t \int_U r\big(\phi^{a_{n-1}}_{p_{n-1}}(s), \mathrm{u}\big)\, a_{n-1}(p_0,\, \dots,\, s_{n-1},\, p_{n-1})(s\,; \mathrm{d}\mathrm{u})\, \mathrm{d}s\Big\},
\end{aligned}$$

we get the desired result, by definition of $\mathbf{a}$.

Post-jump locations. Continuing with the notation previously introduced (where we add only a new value s_n such that $0 < s_1 < \cdots < s_n < +\infty$), we can write (3.2.15) as

$$\begin{aligned}
&\mathrm{P}^{\mathbf{u}}_{\mu}(\pi^{\mu,\mathbf{u}}_{\tau_n} \in D,\, \tau_n < +\infty \mid \pi^{\mu,\mathbf{u}}_0 = p_0, \dots, \pi^{\mu,\mathbf{u}}_{\tau_{n-1}} = p_{n-1}, \tau_n = s_n) \\
&= \int_U R\big(\phi^{u_{n-1}}_{p_{n-1}}(s_n^- - s_{n-1}), \mathrm{u}\,; D\big)\, u_{n-1}(s_n^-,\, b_0,\, \dots,\, s_{n-1},\, b_{n-1}\,; \mathrm{d}\mathrm{u}).
\end{aligned}$$

On the other hand, we know from (3.2.4)

$$\begin{aligned}
&\bar{\mathrm{P}}^{\mathbf{a}}_{Q}(\bar{\pi}_{\tau_n} \in D,\, \bar{\tau}_n < +\infty \mid \bar{\pi}_0 = p_0, \dots, \bar{\pi}_{\tau_{n-1}} = p_{n-1}, \bar{\tau}_n = s_n) \\
&= \int_U R\big(\phi^{a_{n-1}}_{p_{n-1}}(s_n^- - s_{n-1}), \mathrm{u}\,; D\big)\, a_{n-1}(p_0,\, \dots,\, s_{n-1},\, p_{n-1})(s_n^- - s_{n-1}\,; \mathrm{d}\mathrm{u}).
\end{aligned}$$

Hence again by definition of $\mathbf{a}$ we get the equality of the conditional laws (3.2.15) and 3.2.16.

It remains to prove (3.2.12). Fix $\mu \in \Delta$ and $\mathbf{u} \in \mathcal{U}_{ad}$ with corresponding $\mathbf{a} \in \mathcal{A}_{ad}$ defined as above. Let us define the function $\Phi\colon \bar{\Omega} \to \mathbb{R}$ as

$$\begin{aligned}
\Phi(\bar{\omega}) &= \sum_{n=0}^{+\infty} e^{-\beta \bar{\tau}_n(\bar{\omega})} g\big(\bar{\pi}_{\bar{\tau}_n(\bar{\omega})}(\bar{\omega}), a_n(\dots, \bar{\tau}_n(\bar{\omega}), \bar{\pi}_{\tau_n(\bar{\omega})}(\bar{\omega}))\big) \\
&= \sum_{n=0}^{+\infty} e^{-\beta \bar{\tau}_n(\bar{\omega})} g\big(\bar{\pi}_{\bar{\tau}_n(\bar{\omega})}(\bar{\omega}), u_n(\cdot + \bar{\tau}_n(\bar{\omega}), \dots, \bar{\tau}_n(\bar{\omega}), \mathrm{proj}_{\mathrm{Y}}(\bar{\pi}_{\tau_n(\bar{\omega})}(\bar{\omega})))\big).
\end{aligned}$$

Thanks to Assumptions 3.1.1 and 3.1.2 this function is bounded. Since for each $n \in \mathbb{N}_0$ the functions a_n (equivalently u_n) are measurable it is also $\bar{\mathcal{F}}$-measurable.

Now, take $\bar\omega = \pi^{\mu,\mathbf{u}}(\omega)$, $\omega \in \Omega$. It is clear that for all $t \geqslant 0$ we have $\bar\pi_t(\bar\omega) = \bar\omega(t) = \pi_t^{\mu,\mathbf{u}}(\omega)$ and also, by definition of the jump times $(\bar\tau_n)_{n\in\mathbb{N}_0}$, that $\bar\tau_n(\bar\omega) = \tau_n(\omega)$, $\mathrm{P}^{\mathbf{u}}_\mu$-a.s.. Then, we get that $\mathrm{P}^{\mathbf{u}}_\mu$-a.s.

$$\begin{aligned}\Phi(\pi^{\mu,\mathbf{u}}(\omega)) &= \sum_{n=0}^{+\infty} e^{-\beta\tau_n(\omega)} g\big(\pi^{\mu,\mathbf{u}}_{\tau_n(\omega)}(\omega), u_n(\cdot + \tau_n(\omega)\big)\\ &= \sum_{n=0}^{+\infty} e^{-\beta\tau_n(\omega)} g\big(\pi^{\mu,\mathbf{u}}_{\tau_n(\omega)}(\omega), u_n(\cdot + \tau_n(\omega), \dots, \tau_n(\omega), Y_{\tau_n(\omega)}(\omega))\big),\end{aligned}$$

hence, comparing this result with (3.1.26) we obtain

$$\begin{aligned}J(\mu,\mathbf{u}) &= \int_\Omega \Phi(\pi^{\mu,\mathbf{u}}(\omega))\mathrm{P}^{\mathbf{u}}_\mu(\mathrm{d}\omega) = \int_{\bar\Omega} \Phi(\bar\omega)\bar{\mathrm{P}}^{\mathbf{a}}_Q(\mathrm{d}\bar\omega)\\ &= \sum_{y\in O} \mu(h^{-1}(y)) \int_{\bar\Omega} \Phi(\bar\omega)\bar{\mathrm{P}}^{\mathbf{a}}_{H_y[\mu]}(\mathrm{d}\bar\omega) = \sum_{y\in O} \mu(h^{-1}(y))\bar J(H_y[\mu],\mathbf{a}),\end{aligned}$$

by definition of the functional $\bar J$.

To prove the second part of the theorem, fix $\mu \in \Delta$ and $\mathbf{a} = (a_n)_{n\in\mathbb{N}} \in \mathcal{A}_{ad}$. Let us start by defining, for each possible sequence $b_0, b_1, \dots \in O$ and $s_1, \dots \in (0,+\infty)$ the following quantities by recursion for all $n \in \mathbb{N}$

$$\begin{aligned}&p_0 = p_0(b_0) = H_{b_0}[\mu]\\ &p_n = p_n(b_0, s_1, \dots, s_n, b_n) =\\ &\begin{cases} H_{b_n}\left[\phi^{a_{n-1}}_{p_{n-1}}(s_n^- - s_{n-1}) \displaystyle\int_U \Lambda(\mathfrak{u})\, a_{n-1}(s_n^- - s_{n-1}\,; \mathrm{d}\mathfrak{u})\right], & \text{if } s_1 < \dots < s_n\\ \rho, & \text{otherwise}\end{cases}\end{aligned}$$

Here $s_0 = 0$ and $\rho \in \Delta_e$ is an arbitrarily chosen value.

For all $n \in \mathbb{N}_0$ we define the functions $u_n \colon [0,+\infty) \times O \times \big((0,+\infty)\times O\big)^n \to \mathcal{P}(U)$ as

$$u_n(t, b_0, \dots, s_n, b_n\,; \mathrm{d}\mathfrak{u}) = \begin{cases} a_n(p_0, \dots, s_n, p_n)(t - s_n\,; \mathrm{d}\mathfrak{u}), & \text{if } t \geqslant s_n,\\ \mathfrak{u}, & \text{if } t < s_n,\end{cases}$$

where $\mathfrak{u} \in U$ is some fixed value that is irrelevant to specify. Thanks to the fact that each of the functions $(b_0, \dots, s_n, b_n) \mapsto p_n$ is Borel-measurable and that $\mathcal{M}$ is a Borel space, we can use [67, Lemma 3(ii)] to conclude that all the functions u_n are Borel-measurable and therefore $\mathbf{u} = (u_n)_{n\in\mathbb{N}_0} \in \mathcal{U}_{ad}$.

Similarly to what we did in the proof of the first part of the theorem, we need to characterize the laws of $\pi^{\mu,\mathbf{u}}$ under $\mathrm{P}^{\mathbf{u}}_\mu$ and $\bar\pi$ under $\mathrm{P}^{\mathbf{a}}_Q$. First of all, let us notice that we do not need to prove again that the initial distributions of the two processes are equal since they do not depend on the controls $\mathbf{u}$ and $\mathbf{a}$. Therefore, we need only to compare the conditional distributions

$$\begin{aligned}&\mathrm{P}^{\mathbf{u}}_\mu(\tau_n - \tau_{n-1} > t,\, \tau_{n-1} < +\infty \mid Y_0, \dots, \tau_{n-1}, Y_{\tau_{n-1}});\\ &\bar{\mathrm{P}}^{\mathbf{a}}_Q(\bar\tau_n - \bar\tau_{n-1} > t,\, \bar\tau_{n-1} < +\infty \mid \bar Y_0, \dots, \bar\tau_{n-1}, \bar Y_{\bar\tau_{n-1}});\\ &\mathrm{P}^{\mathbf{u}}_\mu(\pi^{\mu,\mathbf{u}}_{\tau_n} \in D,\, \tau_n < +\infty \mid Y_0, \dots, Y_{\tau_{n-1}}, \tau_n);\\ &\bar{\mathrm{P}}^{\mathbf{a}}_Q(\bar\pi_{\bar\tau_n} \in D,\, \bar\tau_n < +\infty \mid \bar Y_0, \dots, \bar Y_{\bar\tau_{n-1}}, \bar\tau_n),\end{aligned}$$

where $t > 0$ and $D \in \mathcal{B}(\Delta_e)$. This can be done in the same way as in the first part of the proof, this time using the definition of the control $\mathbf{u} \in \mathcal{U}_{ad}$ and the obvious fact that, up to $\bar{\mathrm{P}}^{\mathbf{a}}_{Q}$-null sets we have

$$\bar{\mathcal{F}}_{\bar{\tau}_{n-1}} = \sigma(\bar{Y}_0, \dots, \bar{\tau}_{n-1}, \bar{Y}_{\bar{\tau}_{n-1}}) \qquad \text{and} \qquad \bar{\mathcal{F}}_{\tau_n^-} = \sigma(\bar{Y}_0, \dots, \bar{Y}_{\bar{\tau}_{n-1}}, \bar{\tau}_n).$$

Finally, to prove (3.2.12) it suffices to define $\Phi \colon \bar{\Omega} \to \mathbb{R}$ as

$$\Phi(\bar{\omega}) = \sum_{n=0}^{+\infty} e^{-\beta \bar{\tau}_n(\bar{\omega})} g\big(\bar{\pi}_{\bar{\tau}_n(\bar{\omega})}(\bar{\omega}), u_n(\cdot + \bar{\tau}_n, \bar{Y}_0(\bar{\omega}), \dots, \bar{\tau}_n(\bar{\omega}), \bar{Y}_{\tau_n(\bar{\omega})}(\bar{\omega})\big).$$

and notice that $p_n(\bar{Y}_0, \dots, \bar{\tau}_n, \bar{Y}_{\bar{\tau}_n}) = \bar{\pi}_{\bar{\tau}_n}$, so that we can write

$$\Phi(\bar{\omega}) = \sum_{n=0}^{+\infty} e^{-\beta \bar{\tau}_n(\bar{\omega})} g\big(\bar{\pi}_{\bar{\tau}_n(\bar{\omega})}(\bar{\omega}), a_n(\bar{\pi}_0(\bar{\omega}), \dots, \bar{\tau}_n(\bar{\omega}), \bar{\pi}_{\tau_n(\bar{\omega})}(\bar{\omega})\big).$$

The desired equality follows from the same reasoning as in the first part of the proof. □

Remark 3.2.1. The proof of this theorem provides us with an explicit way to construct an admissible policy $\mathbf{a}$ given an admissible control $\mathbf{u}$ and viceversa. The case that most concerns us is to build an admissible control $\mathbf{u}$ when $\mathbf{a}$ is a stationary admissible policy. A policy $\mathbf{a} \in \mathcal{A}_{ad}$ is said to be *stationary* if it is of the form $\mathbf{a} = (a_0, a, a, \dots)$, where $a_0 \colon \Delta_e \to \mathcal{M}$ and $a \colon (0, +\infty) \times \Delta_e \to \mathcal{M}$. The function a_0 depends on the starting point of the filtering process while at each discrete time step the function a depends on the last jump time and jump location of the filtering process. In other words, this kind of admissible policy represents a piecewise open-loop control. Notice that here dependency on jump times (and not only on the time elapsed since the last one) must be taken into account. This is a generalization of the original definition by Vermes (cfr. [61]).

Having identified the original problem with the discrete-time PDP problem, we can concentrate our analysis on the latter one. What we are aiming at is to prove that v is the unique fixed point of the operator $\mathcal{T} \colon \mathrm{B}_b(\Delta_e) \to \mathrm{B}_b(\Delta_e)$ defined for all $\nu \in \Delta_e$ as

$$\begin{aligned}
\mathcal{T} w(\nu) &\coloneqq \inf_{m \in \mathcal{M}} \int_0^\infty \int_U e^{-\beta t} L(\phi_\nu^m(t), \chi_\nu^m(t), \mathrm{u}, w)\, m(t; \mathrm{du})\, \mathrm{d}t \\
&\coloneqq \inf_{m \in \mathcal{M}} \int_0^\infty \int_U e^{-\beta t} \chi_\nu^m(t) \Big[\phi_\nu^m(t) \mathbf{f}(\mathrm{u}) + \\
&\qquad r(\phi_\nu^m(t), \mathrm{u}) \int_{\Delta_e} w(p) R(\phi_\nu^m(t), \mathrm{u}; \mathrm{d}p) \Big]\, m(t; \mathrm{du})\, \mathrm{d}t. \qquad (3.2.17)
\end{aligned}$$

A first result in this direction is provided by the following lemma. In what follows we equip the space $\mathrm{B}_b(\Delta_e)$ with the usual sup-norm, denote by $\|\cdot\|_\infty$. In this way $\mathrm{B}_b(\Delta_e)$ becomes a Banach space.

Lemma 3.2.2. *Under Assumptions 3.1.1 and 3.1.2 the operator $\mathcal{T}$ is a contraction mapping on the space $\mathrm{B}_b(\Delta_e)$.*

Proof. Fix $w_1,\, w_2 \in \mathrm{B}_b(\Delta_e)$. For each $\nu \in \Delta_e$ we have that

$$\big|\mathcal{T}w_1(\nu) - \mathcal{T}w_2(\nu)\big| \leqslant \sup_{m\in\mathcal{M}} \bigg|\int_0^\infty \int_U e^{-\beta t}\big[L(\phi_\nu^m(t), \chi_\nu^m(t), \mathfrak{u}, w_1) - L(\phi_\nu^m(t), \chi_\nu^m(t), \mathfrak{u}, w_2)\big]\, m(t; \mathrm{d}\mathfrak{u})\, \mathrm{d}t\bigg|. \tag{3.2.18}$$

Observe that for all $m \in \mathcal{M},\ u \in U$ and $t \geqslant 0$

$$\big|L(\phi_\nu^m(t), \chi_\nu^m(t), u, w_1) - L(\phi_\nu^m(t), \chi_\nu^m(t), u, w_2)\big| \leqslant \|w_1 - w_2\|_\infty \chi_\nu^m(t) r(\phi_\nu^m(t), u)$$

whence

$$\bigg|\int_0^\infty \int_U e^{-\beta t}\big[L(\phi_\nu^m(t), \chi_\nu^m(t), \mathfrak{u}, w_1) - L(\phi_\nu^m(t), \chi_\nu^m(t), \mathfrak{u}, w_2)\big]\, m(t; \mathrm{d}\mathfrak{u})\, \mathrm{d}t\bigg|$$
$$\leqslant \|w_1 - w_2\|_\infty \int_0^\infty e^{-\beta t} \chi_\nu^m(t) \int_U r(\phi_\nu^m(t), u)\, m(t; \mathrm{d}\mathfrak{u})\, \mathrm{d}t \leqslant \|w_1 - w_2\|_\infty \frac{C_r}{\beta + C_r}. \tag{3.2.19}$$

The last inequality comes from the following estimate, that can be obtained integrating by parts and recalling that χ_ν^m satisfies (3.2.7)

$$\int_0^\infty e^{-\beta t} \chi_\nu^m(t) \int_U r(\phi_\nu^m(t), u)\, m(t; \mathrm{d}\mathfrak{u})\, \mathrm{d}t = 1 - \beta \int_0^\infty e^{-\beta t} \chi_\nu^m(t)\, \mathrm{d}t \leqslant \frac{C_r}{\beta + C_r}$$

where C_r is the constant appearing in (3.1.14). Since the estimate obtained in (3.2.19) does not depend on $m \in \mathcal{M}$ we can take the supremum on the l.h.s. with respect to $m \in \mathcal{M}$ and obtain from (3.2.18)

$$\big|\mathcal{T}w_1(\nu) - \mathcal{T}w_2(\nu)\big| \leqslant \|w_1 - w_2\|_\infty \frac{C_r}{\beta + C_r}, \quad \nu \in \Delta_e. \tag{3.2.20}$$

Finally, taking the supremum with respect to $\nu \in \Delta_e$ on the l.h.s. of (3.2.20) we get the result. □

At this point, we just need to show that v is a fixed point of $\mathcal{T}$. To do so, we need to build a *Markov Decision Process* (or MDP) so that we can resort to results connected with the so called *lower semicontinuous model* of [13], that ensure the existence of an optimal non-randomized stationary (Borel-)measurable policy $\mathbf{a} \in \mathcal{A}_{ad}$, in the same sense given in Remark 3.2.1 above. For sake of clarity, we briefly sketch here the construction of the MDP and recall what is meant by a lower semicontinuous model.

Let $\bar{Z} = (\bar{Z}_n)_{n\in\mathbb{N}_0}$ the discrete-time process defined on $(\bar{\Omega}, \bar{\mathcal{F}}^0)$ as

$$\bar{Z}_0 \coloneqq \bar{\pi}_{\bar{\tau}_0}, \qquad \bar{Z}_k \coloneqq (\bar{\tau}_k \wedge \bar{\tau},\, \bar{\pi}_{\bar{\tau}_k} \mathbb{1}_{\bar{\tau}_k < \bar{\tau}} + \delta \mathbb{1}_{\bar{\tau}_k \geqslant \bar{\tau}}), \quad k \in \mathbb{N}$$

where δ is a *cemetery point* outside Δ_e and $\bar{\tau}$ is an exponential random variable, independent of all the random variables $\bar{\tau}_1, \bar{\pi}_{\bar{\tau}_1}, \dots$, with rate parameter β, the discount factor of our control problem. The state space of the process $(\bar{Z}_n, n)_{n\in\mathbb{N}_0}$ is given by

$$S \coloneqq \big\{(z_0, 0),\, z_0 \in \Delta_e\big\} \bigcup_{k=1}^\infty \big\{(z_k, k),\, z_k \in (0, +\infty) \times \Delta_e \cup \{\delta\}\big\}.$$

The reason for considering in this definition the value of the current discrete-time step, is that in this way we are able to obtain a stationary MDP. Otherwise we should consider a *non-stationary* MDP (to use the terminology of [13]), i. e. a model with parameters (state space, action space, cost function, etc. . .) that can change at each time step.

As control space we choose the set $\mathcal{M}$ introduced in (3.2.1) and, for all $t \geqslant 0,\ D \in \mathcal{B}(\Delta_e),\ m \in \mathcal{M}$, we set as one-stage cost function $\bar{g}$ and transition kernel $\bar{q}$

$$\bar{g}\big((z_k, k), m\big) = \begin{cases} g(\rho, m), & \text{if } z_0 = \rho \in \Delta_e \text{ or } z_k = (s, \rho) \in (0, +\infty) \times \Delta_e,\ k \in \mathbb{N} \\ 0, & \text{if } k \in \mathbb{N}, \text{ with } z_k = (s, \delta),\ s > 0 \end{cases} \tag{3.2.21}$$

$$\bar{q}\big(\big\{(z_{k+1}, k+1) \colon z_{k+1} \in (0, t] \times D\big\} \mid (z_k, k), m\big) = \\ \begin{cases} \displaystyle\int_0^t q(\sigma, D \mid \rho, m)\,\mathrm{d}\sigma, & \text{if } k = 0, \text{ with } z_0 = \rho \in \Delta_e \\ \displaystyle\mathbb{1}_{t>s}\, e^{-\beta s} \int_0^{t-s} q(\sigma, D \mid \rho, m)\,\mathrm{d}\sigma, & \text{if } k \in \mathbb{N}, \text{ with} \\ & z_k = (s, \rho) \in (0, +\infty) \times \Delta_e \end{cases} \tag{3.2.22}$$

$$\bar{q}\big(\big\{(z_{k+1}, k+1) \colon z_{k+1} \in (0, t] \times \{\delta\}\big\} \mid (z_k, k), m\big) = \\ \begin{cases} \displaystyle\int_0^t \beta e^{-\beta\sigma} \chi_\rho^m(\sigma)\,\mathrm{d}\sigma, & \text{if } k = 0, \text{ with } z_0 = \rho \in \Delta_e \\ \displaystyle\mathbb{1}_{t>s}\left[1 - e^{-\beta s} + e^{-\beta s} \int_0^{t-s} \beta e^{-\beta\sigma} \chi_\rho^m(\sigma)\,\mathrm{d}\sigma\right], & \text{if } k \in \mathbb{N}, \text{ with} \\ & z_k = (s, \rho) \in (0, +\infty) \times \Delta_e \end{cases} \tag{3.2.23}$$

where the function g is given in (3.2.8) and we define for all $t \geqslant 0,\ D \in \mathcal{B}(\Delta_e)$, $\nu \in \Delta_e$ and $m \in \mathcal{M}$

$$q(t, D \mid \nu, m) \coloneqq e^{-\beta t} \chi_\nu^m(t) \int_U r(\phi_\nu^m(t), u) R(\phi_\nu^m(t), u; D)\, m(t; \mathrm{d}u). \tag{3.2.24}$$

The so called *k-th originating* cost associated with the MDP, denoted for each $k \in \mathbb{N}_0$ by $\mathcal{J}_{\mathbf{a}}(z_k, k)$, is the cost functional for the optimal control problem starting at the k-th stage from $(z_k, k) \in S$ with control policy $\mathbf{a} \in \mathcal{A}_{ad}$. A simple calculation shows that we can write it as follows

$$\mathcal{J}_{\mathbf{a}}(z_k, k) \coloneqq \begin{cases} \bar{J}(\nu, \mathbf{a}), & \text{if } z_0 = \nu \text{ or } z_k = (t, \nu),\ k \in \mathbb{N} \\ 0, & \text{if } k \in \mathbb{N}, \text{ with } z_k = (t, \delta) \end{cases}. \tag{3.2.25}$$

The optimal cost at $(z_k, k) \in S,\ k \in \mathbb{N}_0$ is given by

$$\mathcal{J}^\star(z_k, k) = \inf_{\mathbf{a} \in \mathcal{A}_{ad}} \mathcal{J}_{\mathbf{a}}(z_k, k). \tag{3.2.26}$$

Notice that $v(\nu) = \mathcal{J}^\star(z_k, k)$, whenever $z_0 = \nu$ or $z_k = (t, \nu),\ k \in \mathbb{N}$. From [13, Prop. 10.1] we know that each of the k-th originating optimal costs satisfies

$$\mathcal{J}^\star(z_k, k) = \inf_{m \in \mathcal{M}} \left\{ \bar{g}\big((z_k, k), m\big) + \int_{(0,+\infty) \times \Delta_e} \mathcal{J}^\star(z_{k+1}, k+1)\, \bar{q}(\mathrm{d}z_{k+1} \mid z_k, m) \right\} \tag{3.2.27}$$

and is a lower semianalytic function. Hence, given the previous discussion, we deduce that the value function v is lower semianalytic and satisfies the optimality equation [2]

$$v(\nu) := \inf_{m \in \mathcal{M}} \int_0^\infty \int_U e^{-\beta t} \chi_\nu^m(t) \Big[\phi_\nu^m(t) \mathbf{f}(\mathrm{u}) + r(\phi_\nu^m(t), \mathrm{u}) \int_{\Delta_e} v(p) R(\phi_\nu^m(t), \mathrm{u}; \mathrm{d}p) \Big] m(t; \mathrm{d}\mathrm{u}) \, \mathrm{d}t. \quad (3.2.28)$$

As we anticipated earlier, to deduce that v is the unique fixed point of the operator $\mathcal{T}$ we use results connected to the so called lower semicontinuous model of [13]. In our context, given the construction of the MDP above, this means that

a. The control space $\mathcal{M}$ is compact.

b. The transition kernel $\bar{q}$ is continuous on $S \times \mathcal{M}$.

c. The cost function $\bar{g}$ is lower semicontinuous and bounded below on $S \times \mathcal{M}$.

From the discussion at the beginning of this Section, we know that $\mathcal{M}$ is compact under the Young topology. The other points are proved in the following Lemma.

Lemma 3.2.3. *Under Assumptions 3.1.1 and 3.1.2 we have the following results.*

1. *The transition kernel $\bar{q}$ defined in (3.2.22) and (3.2.23) is continuous on $S \times \mathcal{M}$.*
2. *The cost function $\bar{g}$ defined in (3.2.21) is bounded and continuous on $S \times \mathcal{M}$.*

Proof. To start, let us observe that the maps $(\nu, m) \mapsto \phi_\nu^m$ and $(\nu, m) \mapsto \chi_\nu^m$ are continuous from $\Delta_e \times \mathcal{M}$ into $\mathrm{C}([0,T]; \Delta_e)$ and $\mathrm{C}([0,T]; \mathbb{R})$ respectively, for all $T > 0$. This follows from an application of Warga's Theorem (see [62, Proof of Th. V.6.1, p. 325]. The interested reader may also look at [32, Th. 44.11]).
Claim 1. It is clear that continuity of the transition kernel $\bar{q}$ on $S \times \mathcal{M}$ is implied by continuity on $\Delta_e \times \mathcal{M}$ of the kernel $q(t, \mathrm{d}p \mid \nu, m)\, \mathrm{d}t$, as in (3.2.24). By definition we have to check that for all functions $w \in \mathrm{C}_b\big([0, +\infty) \times \Delta_e\big)$ the map

$$\Delta_e \times \mathcal{M} \ni (\nu, m) \mapsto \int_0^{+\infty} \int_{\Delta_e} w(t, p)\, q(t, \mathrm{d}p \mid \nu, m) \, \mathrm{d}t \quad (3.2.29)$$

is continuous (clearly, we consider the product topology). Fix $w \in \mathrm{C}_b\big([0, +\infty) \times \Delta_e\big)$. Continuity with respect to the ν variable is easy to show, thanks to Proposition 3.1.4 and the fact, noticed at the beginning of the proof, that ϕ_ν^m and χ_ν^m are continuous in ν.

It remains to show continuity with respect to the m variable. Let us fix $\nu \in \Delta_e, m \in \mathcal{M}$ and consider a sequence $(m_n)_{n \in \mathbb{N}}$ converging to m with respect to the Young to-

[2] Observe that the integral with respect to the measure R appearing in (3.2.28) is well defined, being the integral of a lower semianalytic function against a Borel probability measure (see [13] for more details).

pology. We have that

$$
\begin{aligned}
&\left|\int_0^{+\infty}\int_{\Delta_e} w(t,p)\,q(t,\mathrm{d}p\mid\nu,m_n)\,\mathrm{d}t-\int_0^{+\infty}\int_{\Delta_e} w(t,p)\,q(t,\mathrm{d}p\mid\nu,m)\,\mathrm{d}t\right|\\
&\leqslant \int_0^{+\infty} e^{-\beta t}\big|\chi_\nu^{m_n}(t)-\chi_\nu^m(t)\big|\\
&\qquad\times\left|\int_{\Delta_e} w(t,p)\int_U r(\phi_\nu^{m_n}(t),u)R(\phi_\nu^{m_n}(t),u;\mathrm{d}p)\,m_n(t;\mathrm{d}u)\right|\mathrm{d}t\\
&+\int_0^{+\infty} e^{-\beta t}\chi_\nu^m(t)\int_U\bigg|r(\phi_\nu^{m_n}(t),u)\int_{\Delta_e} w(t,p)R(\phi_\nu^{m_n}(t),u;\mathrm{d}p)\\
&\qquad-r(\phi_\nu^m(t),u)\int_{\Delta_e} w(t,p)R(\phi_\nu^m(t),u;\mathrm{d}p)\,m_n(t;\mathrm{d}u)\bigg|\,\mathrm{d}t\\
&+\bigg|\int_0^{+\infty} e^{-\beta t}\chi_\nu^m(t)\\
&\qquad\times\int_U r(\phi_\nu^m(t),u)\int_{\Delta_e} w(t,p)R(\phi_\nu^m(t),u;\mathrm{d}p)\,\big[m_n(t;\mathrm{d}u)-m(t;\mathrm{d}u)\big]\,\mathrm{d}t\bigg|
\end{aligned}
\tag{3.2.30}
$$

We want to show that all of the three terms appearing in (3.2.30) tend to zero as $n\to\infty$. We can estimate the first summand by

$$
\begin{aligned}
&\int_0^{+\infty} e^{-\beta t}\big|\chi_\nu^{m_n}(t)-\chi_\nu^m(t)\big|\\
&\qquad\times\left|\int_{\Delta_e} w(t,p)\int_U r(\phi_\nu^{m_n}(t),u)R(\phi_\nu^{m_n}(t),u;\mathrm{d}p)\,m_n(t;\mathrm{d}u)\right|\mathrm{d}t\\
&\qquad\qquad\leqslant C_r\sup_{t\geqslant 0,\,p\in\Delta_e}\big|w(t,p)\big|\int_0^{+\infty} e^{-\beta t}\big|\chi_\nu^{m_n}(t)-\chi_\nu^m(t)\big|\mathrm{d}t\xrightarrow[n\to\infty]{}0
\end{aligned}
$$

where C_r is the constant given in (3.1.14). The conclusion is justified by an application of Lebesgue's dominated convergence theorem, that is possible thanks to the continuity of χ_ν^m with respect to m and the obvious fact $e^{-\beta t}\big|\chi_\nu^{m_n}(t)-\chi_\nu^m(t)\big|\leqslant 2e^{-\beta t}$.

As for the second summand

$$
\begin{aligned}
&\int_0^{+\infty} e^{-\beta t}\chi_\nu^m(t)\int_U\bigg|r(\phi_\nu^{m_n}(t),u)\int_{\Delta_e} w(t,p)R(\phi_\nu^{m_n}(t),u;\mathrm{d}p)\\
&\qquad-r(\phi_\nu^m(t),u)\int_{\Delta_e} w(t,p)R(\phi_\nu^m(t),u;\mathrm{d}p)\,m_n(t;\mathrm{d}u)\bigg|\,\mathrm{d}t\\
&\leqslant\int_0^{+\infty} e^{-\beta t}\sup_{u\in U}\bigg|r(\phi_\nu^{m_n}(t),u)\int_{\Delta_e} w(t,p)R(\phi_\nu^{m_n}(t),u;\mathrm{d}p)\\
&\qquad-r(\phi_\nu^m(t),u)\int_{\Delta_e} w(t,p)R(\phi_\nu^m(t),u;\mathrm{d}p)\bigg|\,\mathrm{d}t\xrightarrow[n\to\infty]{}0
\end{aligned}
$$

since the supremum converges to zero, as $n\to\infty$, thanks to Proposition 3.1.4, the continuity properties of ϕ_ν^m recalled at the beginning of the proof and the fact that the integrand is bounded by $2\,C_r e^{-\beta t}\sup_{t\geqslant 0,\,p\in\Delta_e}\big|w(t,p)\big|$.

Finally, to get the same conclusion for the third summand let us notice that the map

$$
[0,+\infty)\times U\ni(t,u)\mapsto e^{-\beta t}\chi_\nu^m(t)r(\phi_\nu^m(t),u)\int_{\Delta_e} w(t,p)R(\phi_\nu^m(t),u;\mathrm{d}p)
$$

is measurable in t, continuous in u and such that

$$\int_0^{+\infty} e^{-\beta t} \chi_\nu^m(t) \max_{u \in U} \left| r(\phi_\nu^m(t), u) \int_{\Delta_e} w(t,p) R(\phi_\nu^m(t), u; \mathrm{d}p) \right| \mathrm{d}t < +\infty$$

(continuity with respect to the u variable can be obtained following a reasoning similar to that of the proof of Proposition 3.1.4). Hence, we get our result by definition of Young topology.

Claim 2. Let us notice, first, the obvious fact that the properties we want to prove about the function $\bar{g}$ follow from the analogous properties of the function g defined in (3.2.8). It is clear that the function g is bounded thanks to Assumption 3.1.2. Keeping in mind the continuity properties of ϕ_ν^m and χ_ν^m recalled at the beginning of the proof, continuity of g with respect to the ν variable is easy to show. As for continuity with respect to control functions, fix $\nu \in \Delta_e$, $m \in \mathcal{M}$ and consider a sequence $(m_n)_{n \in \mathbb{N}}$ converging to m with respect to the Young topology. We have that

$$\begin{aligned} \left| g(\nu, m_n) - g(\nu, m) \right| &\leqslant C_f \int_0^{+\infty} e^{-\beta t} \left| \chi_\nu^{m_n}(t) - \chi_\nu^m(t) \right| \mathrm{d}t \\ + C_f \int_0^{+\infty} e^{-\beta t} \left| \phi_\nu^{m_n}(t) - \phi_\nu^m(t) \right| \mathrm{d}t \\ + \left| \int_0^{+\infty} \int_U e^{-\beta t} \chi_\nu^m(t) \phi_\nu^m(t) \mathbf{f}(u) \left[m_n(t; \mathrm{d}u) - m(t; \mathrm{d}u) \right] \mathrm{d}t \right| \end{aligned}$$

where C_f is the constant appearing in (3.1.6). Proceeding similarly to previous claim we get the result. □

Since the hypotheses of the lower semicontinuous model of [13] are verified, we are able to state (details on the proof can be found in e. g. [13, Corollary 9.17.2]) the following standard result on the existence of an optimal policy and regularity of the value function. Notice that we can apply it since our MDP can be equivalently formulated in the so called *positive case*, being the cost function $\bar{g}$ bounded, hence positive up to the addition of a suitable constant.

Proposition 3.2.4. *Under Assumptions 3.1.1 and 3.1.2 there exists an optimal policy* $\mathbf{a}^\star \in \mathcal{A}_{ad}$*, i. e. a policy such that*

$$v(\nu) = \bar{J}(\nu, \mathbf{a}^\star), \quad \textit{for all } \nu \in \Delta_e.$$

Moreover, this policy is stationary, the value function v is lower semicontinuous and it is the unique fixed point of the operator $\mathcal{T}$.

Remark 3.2.2. It is worth noticing that Assumption 3.1.1 reveals its fundamental role in the course of the proof of Proposition 3.2.4. In fact, it ensures the continuity of the function $u \mapsto r(\rho, u) \int_{\Delta_e} w(p) R(\rho, u; \mathrm{d}p)$ for all $\rho \in \Delta_e$ and all $w \in \mathrm{C}_b(\Delta_e)$, which is crucial to guarantee continuity of q with respect to $m \in \mathcal{M}$. However all the other results shown so far remain true even if we weaken Assumption 3.1.1 and just ask that the maps $u \mapsto \lambda_{ij}(u)$ are measurable for all $i, j \in I$ and that $\sup_{u \in U} \lambda_i(u) < +\infty$ for all $i \in I$.

Relaxed controls are difficult to interpret and implement in practice. Fortunately, we are able to show from Proposition 3.2.4 that v is also the unique fixed point of the operator $\mathcal{G} \colon \mathrm{B}_b(\Delta_e) \to \mathrm{B}_b(\Delta_e)$ given by

$$\mathcal{G}w(\nu) = \inf_{\alpha \in A} \int_0^{\infty} e^{-\beta t} L(\phi_\nu^\alpha(t), \chi_\nu^\alpha(t), \alpha(t), w) \, \mathrm{d}t, \quad \nu \in \Delta_e \tag{3.2.31}$$

where the infimum is taken among all possible ordinary controls instead of relaxed ones. It can be proved, similarly to Lemma 3.2.2, that under Assumptions 3.1.1 and 3.1.2 $\mathcal{G}$ is a contraction.

Theorem 3.2.5. *Under Assumptions 3.1.1 and 3.1.2 v is the unique fixed point of the operator $\mathcal{G}$.*

Proof. It is clear that $v = \mathcal{T}v \leqslant \mathcal{G}v$, so we just need to prove the reverse inequality. We previously saw that there exists a stationary optimal policy $\mathbf{a}^\star$ for the discrete time control problem. By [13, Corollary 9.12.1] this implies that the infimum in (3.2.17) is attained for each $\nu \in \Delta_e$ by some $m^\star \in \mathcal{M}$, with $m^\star = m^\star(\nu)$, and since the set A of ordinary controls is dense in $\mathcal{M}$ with respect to the Young topology (see e. g. [57, V, Th. 7]), we can construct a sequence $(\alpha_n)_{n\in\mathbb{N}} \subset A$ such that $\alpha_n \to m^\star$ as $n \to \infty$. Moreover we have that the function $\mathcal{J}(\nu, m) := \int_0^\infty \int_U e^{-\beta t} L(\phi_\nu^m(t), \chi_\nu^m(t), u, v)\, m(t; \mathrm{d}u)\, \mathrm{d}t$ is continuous in m for all $\nu \in \Delta_e$ (the computations are similar to those of Lemma **??**). Hence we get that for each fixed $\nu \in \Delta_e$

$$\mathcal{J}(\nu, \alpha_n) \to \mathcal{J}(\nu, m^\star) = \mathcal{T}v(\nu) = v(\nu).$$

Noticing that $\mathcal{G}v(\nu) \leqslant \mathcal{J}(\nu, \alpha_n)$ for all $n \in \mathbb{N}$, we get the result. □

We can finally provide the link between the two value functions V and v.

Theorem 3.2.6. *For all $\mu \in \Delta$ we have that*

$$V(\mu) = \sum_{y\in O} \mu(h^{-1}(y)) v(H_y[\mu]). \tag{3.2.32}$$

Proof. Recall that we know from Theorem 3.2.1 that for all $\mu \in \Delta$

$$J(\mu, \mathbf{u}) = \sum_{y\in O} \mu(h^{-1}(y)) \bar{J}(H_y[\mu], \mathbf{a})$$

where $\mathbf{u} \in \mathcal{U}_{ad}$ and $\mathbf{a} \in \mathcal{A}_{ad}$ are corresponding admissible controls and admissible policies.

Let now $\mu \in \Delta$ be fixed. It is obvious that $V(\mu) \geqslant \sum_{y\in O} \mu(h^{-1}(y)) v(H_y[\mu])$. In fact, since $\bar{J}(H_y[\mu], \mathbf{a}) \geqslant v(H_y[\mu])$ for all $\mathbf{a} \in \mathcal{A}_{ad}$ and all $y \in O$, we get that for all $\mathbf{u} \in \mathcal{U}_{ad}$

$$J(\mu, \mathbf{u}) \geqslant \sum_{y\in O} \mu(h^{-1}(y)) v(H_y[\mu])$$

and we get the desired inequality by taking the infimum on the left hand side with respect to all $\mathbf{u} \in \mathcal{U}_{ad}$.

The reverse inequality is easily obtained by taking an optimal policy $\mathbf{a}^\star \in \mathcal{A}_{ad}$ (whose existence is guaranteed by Proposition 3.2.4) and considering its corresponding admissible control $\mathbf{u}^\star \in \mathcal{U}_{ad}$. From Theorem 3.2.1 we immediately get that

$$V(\mu) \leqslant J(\mu, \mathbf{u}^\star) = \sum_{y\in O} \mu(h^{-1}(y)) \bar{J}(H_y[\mu], \mathbf{a}^\star) = \sum_{y\in O} \mu(h^{-1}(y)) v(H_y[\mu]). \quad \square$$

Theorem 3.2.6 gives us a way to go back and forth between the original control problem and the separated one. Moreover, we easily deduce that an admissible control $\mathbf{u} \in \mathcal{U}_{ad}$ is optimal if and only if its corresponding admissible policy $\mathbf{a} \in \mathcal{A}_{ad}$ is. In the next Section we will focus our attention on the analysis of the value function v, that will indirectly give informations about the original value function V.

3.3 Characterization of the value function

We will characterize the PDP value function v in two ways: first we will study a fixed point problem related to the operator $\mathcal{G}$. We already know that v is the unique fixed point of $\mathcal{G}$ as an operator acting on the space of bounded Borel-measurable functions on Δ_e into itself. What we will prove is that it is the unique fixed point of $\mathcal{G}$ as an operator acting on the space of continuous functions into itself. Once gained the continuity of v on Δ_e, hence its uniform continuity and boundedness, we will prove that it is also a *constrained viscosity solution* of a HJB equation.

3.3.1 The fixed point problem

Let us denote by $\mathrm{C}(\Delta_e)$ the space of continuous functions on Δ_e equipped with the usual sup norm. We recall that, since Δ_e is a compact subset of $\mathbb{R}^{|I|}$, it coincides with the space of bounded and uniformly continuous functions on Δ_e.

To prove continuity of v we need to show that $\mathcal{G}$ maps the space $\mathrm{C}(\Delta_e)$ into itself and that v is its unique fixed point in that space (recall that we already established that $\mathcal{G}$ is a contraction). We shall also need a version of the *Dynamic Programming Principle* suited to this problem, that we are going to prove.

Proposition 3.3.1 (Dynamic Programming Principle)**.** *For all functions $w \in \mathrm{B}_b(\Delta_e)$ and all $T > 0$ the function $\mathcal{G}w$ satisfies the following identity*

$$\mathcal{G}w(\nu) = \inf_{\alpha \in A} \left\{ \int_0^T e^{-\beta t} L(\phi_\nu^\alpha(t), \chi_\nu^\alpha(t), \alpha(t), w)\,\mathrm{d}t + e^{-\beta T} \chi_\nu^\alpha(T) \mathcal{G}w(\phi_\nu^\alpha(T)) \right\}. \tag{3.3.1}$$

Remark 3.3.1. It is worth noticing that taking $w = v$ we get the standard statement of the Dynamic Programming Principle.

Proof. Let $T > 0$, $w \in \mathrm{B}_b(\Delta_e)$ and $\nu \in \Delta_e$ be fixed and let us define $\tilde{w}(\nu)$ the right hand side of (3.3.1).

We will show first that $\mathcal{G}w(\nu) \leqslant \tilde{w}(\nu)$. Choose an arbitrary $\alpha \in A$ and define $\rho := \phi_\nu^\alpha(T)$. For some fixed $\varepsilon > 0$, let $\alpha^\varepsilon \in A$ be such that, in accordance with (3.2.31)

$$\mathcal{G}w(\rho) + \varepsilon \geqslant \int_0^\infty e^{-\beta t} L(\phi_\rho^{\alpha^\varepsilon}(t), \chi_\rho^{\alpha^\varepsilon}(t), \alpha^\varepsilon(t), w)\,\mathrm{d}t. \tag{3.3.2}$$

Next, define the function $\tilde{\alpha}\colon [0, +\infty) \to U$ as

$$\tilde{\alpha}(t) = \alpha(t)\mathbb{1}_{[0,T]}(t) + \alpha^\varepsilon(t - T)\mathbb{1}_{(T,+\infty)}(t).$$

It is clearly measurable, i. e. $\tilde{\alpha} \in A$, and it is straightforward to notice that

$$\mathcal{G}w(\nu) \leqslant \int_0^T e^{-\beta t} L(\phi_\nu^\alpha(t), \chi_\nu^\alpha(t), \alpha(t), w)\,\mathrm{d}t + \int_T^\infty e^{-\beta t} L(\phi_\nu^{\tilde{\alpha}}(t), \chi_\nu^{\tilde{\alpha}}(t), \tilde{\alpha}(t), w)\,\mathrm{d}t.$$

Thanks to the flow property of ϕ we have that the equality $\phi_\nu^{\tilde{\alpha}}(t) = \phi_\rho^{\alpha^\varepsilon}(t - T)$ holds, for all $t > T$. Moreover, it can be easily shown that $\chi_\nu^{\tilde{\alpha}}(t) = \chi_\nu^\alpha(T)\chi_\nu^{\alpha^\varepsilon}(t - T)$, for

$t > T$. With this in mind and performing a simple change of variables, we get that

$$\int_T^\infty e^{-\beta t} L(\phi_\nu^{\tilde\alpha}(t), \chi_\nu^{\tilde\alpha}(t), \tilde\alpha(t), w)\,\mathrm{d}t = \\ e^{-\beta T}\chi_\nu^\alpha(T)\int_0^\infty e^{-\beta t} L(\phi_\rho^{\alpha^\varepsilon}(t), \chi_\rho^{\alpha^\varepsilon}(t), \alpha^\varepsilon(t), w)\,\mathrm{d}t.$$

Therefore, we have from (3.3.2) that for all $\varepsilon > 0$

$$\mathcal{G}w(\nu) \leqslant \int_0^T e^{-\beta t} L(\phi_\nu^\alpha(t), \chi_\nu^\alpha(t), \alpha(t), w)\,\mathrm{d}t + e^{-\beta T}\chi_\nu^\alpha(T)\big[\mathcal{G}w(\rho) + \varepsilon\big]$$

Since α is arbitrary, we can take the limit as $\varepsilon \to 0^+$ and then the infimum on the set A to get that $\mathcal{G}w(\nu) \leqslant \tilde w(\nu)$. The reverse inequality is easily obtained with similar computations. □

We provide now an estimate that will be fundamental in proving the next Proposition.

Lemma 3.3.2. *Let $T > 0$ and $w \in \mathrm{C}(\Delta_e)$ be fixed and define for all $\nu \in \Delta_e$ and all $\alpha \in A$*

$$\mathcal{J}_{T,w}(\nu, \alpha) = \int_0^T e^{-\beta t} L(\phi_\nu^\alpha(t), \chi_\nu^\alpha(t), \alpha(t), w)\,\mathrm{d}t. \tag{3.3.3}$$

Then, under Assumptions 3.1.1 and 3.1.2, there exist constants $C, K_1, K_2 > 0$ and a modulus of continuity η [3] *such that for all $\alpha \in A$*

$$\big|\mathcal{J}_{T,w}(\nu, \alpha) - \mathcal{J}_{T,w}(\rho, \alpha)\big| \leqslant K_1|\nu - \rho| + K_2\eta(C|\nu - \rho|). \tag{3.3.4}$$

Proof. Let $\alpha \in A$ and $\nu \in \Delta_e$ be fixed. It is clear that $\nu \in \Delta_a$ for some $a \in O$. Let us consider a sequence $(\nu_k)_{k\in\mathbb{N}}$ such that $\nu_k \to \nu$ as $k \to +\infty$. Without loss of generality we can take $(\nu_k)_{k\in\mathbb{N}} \subset \Delta_a$.

First of all, we need an estimate for the term

$$\big|L(\phi_\nu^\alpha(t), \chi_\nu^\alpha(t), \alpha(t), w) - L(\phi_\rho^\alpha(t), \chi_\rho^\alpha(t), \alpha(t), w)\big|.$$

Thanks to the linearity of L in the second argument, it is easy to get that for all $t \in [0, T]$

$$\begin{aligned}&\big|L(\phi_\nu^\alpha(t), \chi_\nu^\alpha(t), \alpha(t), w) - L(\phi_\rho^\alpha(t), \chi_\rho^\alpha(t), \alpha(t), w)\big| \\ \leqslant&\big|\chi_\nu^\alpha(t) - \chi_\rho^\alpha(t)\big|\left|\phi_\nu^\alpha(t)\mathbf{f}(\alpha(t)) + r(\phi_\nu^\alpha(t), \alpha(t))\int_{\Delta_e} w(p)R(\phi_\nu^\alpha(t), \alpha(t); \mathrm{d}p)\right| \\ &+ \chi_\rho^\alpha(t)\left|\left[\phi_\nu^\alpha(t)\mathbf{f}(\alpha(t)) + r(\phi_\nu^\alpha(t), \alpha(t))\int_{\Delta_e} w(p)R(\phi_\nu^\alpha(t), \alpha(t); \mathrm{d}p)\right]\right. \\ &\left. - \left[\phi_\rho^\alpha(t)\mathbf{f}(\alpha(t)) + r(\phi_\rho^\alpha(t), \alpha(t))\int_{\Delta_e} w(p)R(\phi_\rho^\alpha(t), \alpha(t); \mathrm{d}p)\right]\right|.\end{aligned}$$

The first summand can be estimated observing that Assumptions 3.1.1, 3.1.2 entail that

$$\left|\phi_\nu^\alpha(t)\mathbf{f}(\alpha(t)) + r(\phi_\nu^\alpha(t), \alpha(t))\int_{\Delta_e} w(p)R(\phi_\nu^\alpha(t), \alpha(t); \mathrm{d}p)\right| \leqslant K,$$

[3] i. e. a continuous, nondecreasing, subadditive function $\eta\colon [0, +\infty) \to [0, +\infty)$ such that $\eta(t) \to 0$ as $t \downarrow 0$.

where $K > 0$ is a constant depending on C_f and C_r defined in (3.1.6) and (3.1.14) and on $\sup_{\vartheta \in \Delta_e} |w(\vartheta)|$. Moreover, by repeatedly applying Gronwall's Lemma, it can be shown that for all $t \in [0, T]$

$$\left|\chi_\nu^\alpha(t) - \chi_\rho^\alpha(t)\right| \leqslant \frac{L_r}{L_F}(e^{L_F T} - 1)e^{L_r T}|\nu - \rho|$$

where L_F is the constant defined in (3.1.8).

As for the second summand, notice that $\chi_\nu^\alpha(t) \leqslant 1$. In addition, Assumption 3.1.2 and Proposition 3.1.4 imply that there exists a modulus of continuity $\eta\colon [0, +\infty) \to [0, +\infty)$ such that

$$\sup_{u \in U}\bigg|\nu \mathbf{f}(u) + r(\nu, u)\int_{\Delta_e} w(p)R(\nu, u; \mathrm{d}p) - \rho \mathbf{f}(u) \\ - r(\rho, u)\int_{\Delta_e} w(p)R(\rho, u; \mathrm{d}p)\bigg| \leqslant \eta(|\nu - \rho|).$$

So we have that for all $t \in [0, T]$

$$\Bigg|\bigg[\phi_\nu^\alpha(t)\mathbf{f}(\alpha(t)) + r(\phi_\nu^\alpha(t), \alpha(t))\int_{\Delta_e} w(p)R(\phi_\nu^\alpha(t), \alpha(t); \mathrm{d}p)\bigg] \\ - \bigg[\phi_\nu^\alpha(t)\mathbf{f}(\alpha(t)) + r(\phi_\nu^\alpha(t), \alpha(t))\int_{\Delta_e} w(p)R(\phi_\nu^\alpha(t), \alpha(t); \mathrm{d}p)\bigg]\Bigg| \\ \leqslant \eta(|\phi_\nu^\alpha(t) - \phi_\rho^\alpha(t)|) \leqslant \eta(|\nu - \rho|e^{L_F T}),$$

where the last inequality follows from the fact that η is non decreasing and Gronwall's Lemma again.

Collecting all the computations made so far and defining $C = e^{L_F T}$ we get

$$\left|L(\phi_\nu^\alpha(t), \chi_\nu^\alpha(t), \alpha(t), w) - L(\phi_\rho^\alpha(t), \chi_\rho^\alpha(t), \alpha(t), w)\right| \leqslant \\ \frac{L_r}{L_F}(e^{L_F T} - 1)e^{L_r T}|\nu - \rho| + \eta(C|\nu - \rho|).$$

We are now in a position to prove our claim. It suffices to notice that

$$\begin{aligned} &\left|\mathcal{J}_{T,w}(\nu, \alpha) - \mathcal{J}_{T,w}(\rho, \alpha)\right| \\ &\quad\leqslant \int_0^T e^{-\beta t}\Big|L(\phi_\nu^\alpha(t), \chi_\nu^\alpha(t), \alpha(t), w) - L(\phi_\rho^\alpha(t), \chi_\rho^\alpha(t), \alpha(t), w)\Big|\,\mathrm{d}t \\ &\quad\leqslant \frac{e^{-\beta T} - 1}{\beta}\left[\frac{L_r}{L_F}(e^{L_F T} - 1)e^{L_r T}|\nu - \rho| + \eta(C|\nu - \rho|)\right] \end{aligned} \tag{3.3.5}$$

and define $K_1 = \frac{e^{-\beta T}-1}{\beta}\frac{L_r}{L_F}(e^{L_F T} - 1)e^{L_r T}$ and $K_2 = \frac{e^{-\beta T}-1}{\beta}$. □

Proposition 3.3.3. *Under Assumptions 3.1.1 and 3.1.2, for each function $w \in \mathrm{C}(\Delta_e)$ we have that $\mathcal{G}w \in \mathrm{C}(\Delta_e)$.*

Remark 3.3.2. In the literature we could only find [58, Theorem 3.3] as a result similar to this one. However, it is not directly applicable to our case. Therefore we provide a complete proof of this Proposition, adapting whenever necessary the arguments of the cited work.

Proof. To start, let us pick $\nu, \rho \in \Delta_a$, $a \in O$, such that for some $\delta > 0$ $|\nu - \rho| < \delta$. Let $\varepsilon > 0$, $T > 0$ be arbitrarily fixed and choose $\alpha^\varepsilon \in A$ such that

$$\mathcal{G}w(\rho) + \varepsilon \geqslant \int_0^T e^{-\beta t} L(\phi_\rho^{\alpha^\varepsilon}(t), \chi_\rho^{\alpha^\varepsilon}(t), \alpha^\varepsilon(t), w)\,\mathrm{d}t + e^{-\beta T}\chi_\rho^{\alpha^\varepsilon}(T)\mathcal{G}w(\phi_\rho^{\alpha^\varepsilon}(T)) \tag{3.3.6}$$

according to the Dynamic Programming Principle. We immediately get from (3.3.6)

$$\begin{aligned}
\mathcal{G}w(\nu) - \mathcal{G}w(\rho) &\leqslant \mathcal{J}_{T,w}(\nu, \alpha^\varepsilon) - \mathcal{J}_{T,w}(\rho, \alpha^\varepsilon) + \varepsilon \\
&\quad + e^{-\beta T}\big[\chi_\nu^{\alpha^\varepsilon}(T)\mathcal{G}w(\phi_\nu^{\alpha^\varepsilon}(T)) - \chi_\rho^{\alpha^\varepsilon}(T)\mathcal{G}w(\phi_\rho^{\alpha^\varepsilon}(T))\big] \\
&\leqslant \big|\mathcal{J}_{T,w}(\nu, \alpha^\varepsilon) - \mathcal{J}_{T,w}(\rho, \alpha^\varepsilon)\big| \\
&\quad + e^{-\beta T}\big|\chi_\nu^{\alpha^\varepsilon}(T) - \chi_\rho^{\alpha^\varepsilon}(T)\big| \sup_{\vartheta \in \Delta_e}\big|\mathcal{G}w(\vartheta)\big| \\
&\quad + e^{-\beta T}\big|\mathcal{G}w(\phi_\nu^{\alpha^\varepsilon}(T)) - \mathcal{G}w(\phi_\rho^{\alpha^\varepsilon}(T))\big| + \varepsilon
\end{aligned}$$

where $\mathcal{J}_{T,w}$ was defined in (3.3.3) and $\sup_{\vartheta \in \Delta_e}\big|\mathcal{G}w(\vartheta)\big| < +\infty$ since w is bounded and $\mathcal{G}$ maps bounded functions into bounded functions.

We need to provide an estimate for the terms appearing in the last lines of the previous equation. We know from Lemma 3.3.2 that

$$\big|\mathcal{J}_{T,w}(\nu, \alpha^\varepsilon) - \mathcal{J}_{T,w}(\rho, \alpha^\varepsilon)\big| \leqslant K_1\delta + K_2\eta(C\delta)$$

where $C, K_1, K_2 > 0$, η is a modulus of continuity and it is worth remarking that the estimate is independent of α^ε. In particular, $C = e^{L_F T}$. Applying Gronwall's Lemma one is able to obtain (see the proof of Lemma 3.3.2 for more details)

$$\big|\chi_\nu^{\alpha^\varepsilon}(T) - \chi_\rho^{\alpha^\varepsilon}(T)\big| \leqslant \frac{K_1}{K_2}\delta.$$

As for the term $\big|\mathcal{G}w(\phi_\nu^{\alpha^\varepsilon}(T)) - \mathcal{G}w(\phi_\rho^{\alpha^\varepsilon}(T))\big|$, let us define for $r > 0$

$$\zeta(r) = \sup_{\substack{\nu,\rho \in \Delta_e \\ |\nu - \rho| < r}} \big|\mathcal{G}w(\nu) - \mathcal{G}w(\rho)\big|$$

and set $\zeta(0) = \lim_{r \downarrow 0}\zeta(r)$. Since $|\phi_\nu^{\alpha^\varepsilon}(T) - \phi_\rho^{\alpha^\varepsilon}(T)| \leqslant C\delta$, we get that

$$\big|\mathcal{G}w(\phi_\nu^{\alpha^\varepsilon}(T)) - \mathcal{G}w(\phi_\rho^{\alpha^\varepsilon}(T))\big| \leqslant \zeta(C\delta).$$

Summarizing all the results obtained so far, we get that for all $\varepsilon > 0$ and all $\nu, \rho \in \Delta_a$, $a \in O$, with $|\nu - \rho| < \delta$,

$$\mathcal{G}w(\nu) - \mathcal{G}w(\rho) \leqslant K_1\delta + K_2\eta(C\delta) + e^{-\beta T}\sup_{\vartheta \in \Delta_e}\big|\mathcal{G}w(\vartheta)\big|\frac{K_1}{K_2}\delta + e^{-\beta T}\zeta(C\delta) + \varepsilon.$$

Thus, as $\varepsilon \to 0^+$ and defining $K_0 = K_1 + e^{-\beta T}\sup_{\vartheta \in \Delta_e}\big|\mathcal{G}w(\vartheta)\big|\frac{K_1}{K_2}$,

$$\zeta(\delta) \leqslant K_0\delta + K_2\eta(C\delta) + e^{-\beta T}\zeta(C\delta). \tag{3.3.7}$$

Now it is left to prove that ζ is a modulus of continuity for the function $\mathcal{G}w$ and to do so it suffices to show that $\zeta(0) = 0$. Let us choose $\delta = \frac{1}{C^{2n}}$, for some $n \in \mathbb{N}$. Since $C = e^{L_F T} > 1$, proving that $\zeta(0) = 0$ is equivalent to verify that $\lim_{n \to +\infty}\zeta(\frac{1}{C^{2n}}) =$

0, by definition of ζ in 0. Assuming, without loss of generality, that $Ce^{-\beta T} \neq 1$ and iterating the inequality shown in (3.3.7) we get

$$\begin{aligned}\zeta(0) &\leqslant \lim_{n\to+\infty}\left[\frac{K_0}{C^{2n}}\sum_{j=0}^{n-1}(Ce^{-\beta T})^j + K_2\eta\Big(\frac{1}{C^n}\Big)\sum_{j=0}^{n-1}(e^{-\beta T})^j + e^{-n\beta T}\zeta(1)\right]\\ &\leqslant \lim_{n\to+\infty}\left[\frac{K_0}{1-Ce^{-\beta T}}\left[\frac{1}{C^{2n}} - \frac{e^{-n\beta T}}{C^n}\right] + \frac{K_2}{1-e^{-\beta T}}\eta\Big(\frac{1}{C^n}\Big)[1-e^{-n\beta T}]\right] = 0,\end{aligned}$$

hence the desired result. □

We are now ready to state the first characterization of the value function v.

Theorem 3.3.4. *Under Assumptions 3.1.1 and 3.1.2 we have that v is the unique fixed point of the operator $\mathcal{G}$ in the space of continuous functions on Δ_e.*

Proof. The result follows by combining the fact that v is the unique fixed point of $\mathcal{G}$ in the space $\mathrm{B}_b(\Delta_e)$, the fact that the operator $\mathcal{G}\colon \mathrm{C}_b(\Delta_e) \to \mathrm{C}_b(\Delta_e)$ is a contraction mapping and, finally, Proposition 3.3.3. □

3.3.2 The HJB equation

Now we move to the second characterization of the value function v of the separated problem in the sense of viscosity solutions. Using standard arguments of control theory, the Dynamic Programming Principle stated in Proposition 3.3.1 admits a *local* version in the form the following *Hamilton-Jacobi-Bellman* equation

$$H(\nu, \mathrm{D}v(\nu), v) + \beta v(\nu) = 0, \quad \nu \in \Delta_e. \tag{3.3.8}$$

The function $H\colon \Delta_e \times \mathbb{R}^{|I|} \times \mathrm{C}(\Delta_e) \to \mathbb{R}$ is called the *hamiltonian* and is defined as

$$H(\nu, \mathbf{b}, w) := \sup_{u\in U}\left\{-F(\nu,u)\mathbf{b} - \nu\mathbf{f}(u) - r(\nu,u)\int_{\Delta_e}\big[w(p) - w(\nu)\big]R(\nu,u;\mathrm{d}p)\right\}. \tag{3.3.9}$$

The aim of this subsection to characterize the value function v as the unique *constrained viscosity solution* of the HJB equation (3.3.8). This concept has been developed by H. M. Soner. In [58] it is used to characterize the value function of a deterministic optimal control problem with state space constraint; in [59] the author extends this definition to study the solution to an integro-differential HJB, associated to an optimal control problem of a PDP with state space constraint.

This approach is particularly well suited to our problem, not only because of the similarities between our situation and the one studied in [59], but also because of the fact that the state space constraint is embedded in our formulation. In fact, the trajectories of the PDP $\bar{\pi}$ lie in the effective simplex Δ_e and may as well take values on the boundary of Δ_e. Despite these similarities we will not able to apply directly results of [59] to our problem. Some assumptions are not satisfied in our case, e. g. the hypothesis stated in (1.3) of that paper, and the proof of the main theorem relies on a slightly different (and somewhat more classical) version of the Dynamic Programming Principle. We will, then, provide a full proof of the following Theorem 3.3.6 adapting the arguments given in [59, Th. 1.1] as needed.

First, let us recall the definition of constrained viscosity solution. In what follows, whenever K is a subset of Δ_e, we will denote by $\bar{K}$ its relative closure and by $\operatorname{int} K$

its relative interior. It is understood that all statements referring to topological properties are with respect to the relative topology of Δ_e as a subset of $\mathbb{R}^{|I|}$ (the latter one equipped with the standard euclidean topology). The set $\mathrm{C}^1(K)$ will be the set of continuously differentiable real functions on K.

Definition 3.3.1. A uniformly continuous and bounded function $w\colon \bar{K} \to \mathbb{R}$ is called a

- *viscosity subsolution* of $H(\nu, \mathrm{D}w(\nu), w) + \beta w(\nu) = 0$ on K if

$$H(\rho, \mathrm{D}\psi(\rho), w) + \beta w(\rho) \leqslant 0$$

  whenever $\psi \in \mathrm{C}^1(N_\rho)$ and $(w - \psi)$ has a global maximum, relative to K, at $\rho \in K$, where N_ρ is a neighborhood of ρ.

- *viscosity supersolution* of $H(\nu, \mathrm{D}w(\nu), w) + \beta w(\nu) = 0$ on K if

$$H(\rho, \mathrm{D}\psi(\rho), w) + \beta w(\rho) \geqslant 0$$

  whenever $\psi \in \mathrm{C}^1(N_\rho)$ and $(w - \psi)$ has a global minimum, relative to K, at $\rho \in K$, where N_ρ is a neighborhood of ρ.

- *constrained viscosity solution* of $H(\nu, \mathrm{D}w(\nu), w) + \beta w(\nu) = 0$ on $\bar{K}$ if it is a subsolution on K and a supersolution on $\bar{K}$.

Remark 3.3.3. The fact that w is a viscosity supersolution on the closed set $\bar{K}$ of (3.3.8) automatically imposes a boundary condition. For more details, see the Remark following [58, Definition 2.1]

Before stating the main Theorem, we need the following lemma. We omit its proof for the reader's convenience. It can be found in [59, Lemma 2.1] (see also Remark 2.1 therein).

Lemma 3.3.5. *Let Assumption 3.1.1 hold. A function $w \in \mathrm{C}(\Delta_e)$ is a viscosity subsolution on $\operatorname{int}\Delta_e$ (resp. supersolution on Δ_e) of $H(\nu, \mathrm{D}w(\nu), w) + \beta w(\nu) = 0$ if and only if*

$$H(\rho, \mathrm{D}\psi(\rho), \psi) + \beta w(\rho) \leqslant (\textit{resp. } \geqslant)\, 0,$$

whenever $\psi \in \mathrm{C}^1(N_\rho) \cap \mathrm{C}_b(\Delta_e)$ and $(v - \psi)$ has a global maximum relative to Δ_e at $\rho \in \operatorname{int}\Delta_e$ (resp. minimum at $\rho \in \Delta_e$), where N_ρ is a neighborhood of ρ.

Theorem 3.3.6. *Under Assumptions 3.1.1 and 3.1.2, the value function v of the separated problem is the unique constrained viscosity solution of (3.3.8).*

Proof. Uniqueness follows easily from the very same argument given in [59, Th. 1.1]. In fact, the hypothesis labelled as (A1) is satisfied in our framework by each connected component of Δ_e and other hypotheses are invoked only to show that the functions

$$f_i(\nu, u) = \nu\mathbf{f}(u) + r(\nu, u) \int_{\Delta_e} \big[w_i(p) - w_i(\nu)\big] R(\nu, u; \mathrm{d}p), \quad \nu \in \Delta_e, u \in U, i = 1, 2,$$

are uniformly continuous in ν, uniformly with respect to u (here w_1, w_2 are two arbitrary constrained viscosity solutions of (3.3.8)). This is true in our setting because of Assumption 3.1.2 and Proposition 3.1.4. Therefore, one can follow the same reasoning to show uniqueness of the solution.

Let us now show that v is a viscosity subsolution on $\operatorname{int}\Delta_e$ of (3.3.8). It is easy to see that in Lemma 3.3.5 we can substitute $\psi \in \mathrm{C}^1(N_\rho) \cap \mathrm{C}_b(\Delta_e)$ by $\psi \in \mathrm{C}^1(\Delta_e)$ (see also [59, Remark 2.1]). So, let us fix $\psi \in \mathrm{C}^1(\Delta_e)$ and $\rho \in \operatorname{int}\Delta_e$ such that $(v-\psi)(\rho) = \max_{\nu\in\Delta_e}\{(v-\psi)(\nu)\} = 0$. Since $v \leqslant \psi$, from the DPP we get that for all $\alpha \in A$

$$v(\rho) = \psi(\rho) \leqslant \int_0^T e^{-\beta t} L(\phi_\rho^\alpha(t), \chi_\rho^\alpha(t), \alpha(t), \psi)\,\mathrm{d}t + e^{-\beta T}\chi_\rho^\alpha(T)\psi(\phi_\rho^\alpha(T)). \tag{3.3.10}$$

Differentiating $e^{-\beta t}\chi_\rho^\alpha(t)\psi(\phi_\rho^\alpha(t))$ we have

$$\begin{aligned}&\mathrm{d}\big(e^{-\beta t}\chi_\rho^\alpha(t)\psi(\phi_\rho^\alpha(t))\big) = e^{-\beta t}\chi_\rho^\alpha(t)\\ &\big\{-\beta\psi(\phi_\rho^\alpha(t)) - r(\phi_\rho^\alpha(t),\alpha(t))\psi(\phi_\rho^\alpha(t)) + F(\phi_\rho^\alpha(t),\alpha(t))\mathrm{D}\psi(\phi_\rho^\alpha(t))\big\}\mathrm{d}t.\end{aligned} \tag{3.3.11}$$

Integrating (3.3.11) in $[0,T]$ and substituting the result in (3.3.10) we obtain

$$\begin{aligned}&\int_0^T e^{-\beta t}\chi_\rho^\alpha(t)\bigg\{\beta\psi(\phi_\rho^\alpha(t)) - F(\phi_\rho^\alpha(t),\alpha(t))\mathrm{D}\psi(\phi_\rho^\alpha(t))\\ &-\phi_\rho^\alpha(t)\mathbf{f}(\alpha(t)) - r(\phi_\rho^\alpha(t),\alpha(t))\int_{\Delta_e}\big[\psi(p)-\psi(\phi_\rho^\alpha(t))\big]R(\phi_\rho^\alpha(t),\alpha(t);\mathrm{d}p)\bigg\}\mathrm{d}t \leqslant 0.\end{aligned} \tag{3.3.12}$$

By means of Assumption 3.1.2, Proposition 3.1.4 and the properties of the flow $\phi_\rho^\alpha(\cdot)$, we are able to obtain from the previous inequality the estimate

$$\begin{aligned}\frac{1}{T}\int_0^T &\bigg\{\beta\psi(\rho) - F(\rho,\alpha(t))\mathrm{D}\psi(\rho)\\ &- \rho\mathbf{f}(\alpha(t)) - r(\rho,\alpha(t))\int_{\Delta_e}\big[\psi(p)-\psi(\rho)\big]R(\rho,\alpha(t);\mathrm{d}p)\bigg\}\mathrm{d}t \leqslant h(T)\end{aligned}$$

where h is a continuous function such that $h(0) = 0$. Now, let $t_0 = \frac{\mathrm{dist}(\rho,\partial\Delta_e)}{C_F}$, where $C_F = \sup_{(\nu,u)\in\Delta_e\times U} F(\nu,u)$, so that on $[0,t_0)$ the flow never reaches the boundary of Δ_e. For each fixed $u \in U$ it is clearly possible to pick a control $\alpha \in A$ such that $\alpha(T) = u$, for all $T < t_0$. Using this strategy in the last inequality we get that for all $T \in [0,t_0)$ and all $u \in U$

$$\beta\psi(\rho) - F(\rho,u)\mathrm{D}\psi(\rho) - \rho\mathbf{f}(u) - r(\rho,u)\int_{\Delta_e}\big[\psi(p)-\psi(\rho)\big]R(\rho,u;\mathrm{d}p) \leqslant h(T).$$

Taking the limit as $T \to 0^+$ and the supremum with respect to all $u \in U$ we obtain the subsolution property.

Let us now show that v is a viscosity supersolution on Δ_e of (3.3.8). Let $\psi \in \mathrm{C}^1(\Delta_e)$ and $\rho \in \Delta_e$ such that $(v-\psi)(\rho) = \min_{\nu\in\Delta_e}\{(v-\psi)(\nu)\} = 0$. Since $v \geqslant \psi$, from the DPP we get that for all $T > 0$

$$v(\rho) = \psi(\rho) \geqslant \inf_{\alpha\in A}\bigg\{\int_0^T e^{-\beta t} L(\phi_\rho^\alpha(t), \chi_\rho^\alpha(t), \alpha(t), \psi)\,\mathrm{d}t + e^{-\beta T}\chi_\rho^\alpha(T)\psi(\phi_\rho^\alpha(T))\bigg\}. \tag{3.3.13}$$

For each $n \in \mathbb{N}$ consider $T = 1/n$ and pick a control $\alpha^n \in A$ such that

$$\psi(\rho) + \frac{1}{n^2} \geqslant \int_0^{1/n} e^{-\beta t} L(\phi_\rho^\alpha(t), \chi_\rho^\alpha(t), \alpha(t), \psi)\, \mathrm{d}t + e^{-\beta/n} \chi_\rho^\alpha\Big(\frac{1}{n}\Big) \psi\Big(\phi_\rho^\alpha\Big(\frac{1}{n}\Big)\Big).$$

With similar computations as before we are able to obtain

$$\begin{aligned} n \int_0^{1/n} \Big\{ & \beta\psi(\rho) - F(\rho, \alpha^n(t)) \mathrm{D}\psi(\rho) \\ & - \rho \mathbf{f}(\alpha^n(t)) - r(\rho, \alpha^n(t)) \int_{\Delta_e} \big[\psi(p) - \psi(\rho)\big] R(\rho, \alpha^n(t); \mathrm{d}p) \Big\} \, \mathrm{d}t \geqslant h_n \end{aligned} \tag{3.3.14}$$

where $h_n \to 0$ as $n \to +\infty$. Let us define the following quantities

$$F_n \coloneqq n \int_0^{1/n} F(\rho, \alpha^n(t))\, \mathrm{d}t$$

$$K_n \coloneqq n \int_0^{1/n} \Big\{ \rho \mathbf{f}(\alpha^n(t)) + r(\rho, \alpha^n(t)) \int_{\Delta_e} \big[\psi(p) - \psi(\rho)\big] R(\rho, \alpha^n(t); \mathrm{d}p) \Big\} \, \mathrm{d}t$$

and the set $C(\rho) \coloneqq \{(F(\rho, u), \rho\mathbf{f}(u) + r(\rho, u) \int_{\Delta_e} \big[\psi(p) - \psi(\rho)\big] R(\rho, u; \mathrm{d}p)), u \in U\}$. Notice that $(F_n, K_n) \in \overline{\mathrm{co}}\, C(\rho)$ for all $n \in \mathbb{N}$ and $\overline{\mathrm{co}}\, C(\rho)$ is compact since $C(\rho)$ is bounded. Hence there is a subsequence, still denoted by (F_n, K_n) that converges to some $(F, K) \in \overline{\mathrm{co}}\, C(\rho)$. Therefore, taking the limit as n goes to infinity in (3.3.14) we get

$$\beta\psi(\rho) - F \cdot \mathrm{D}\psi(\rho) - K \geqslant 0$$

so that

$$\beta\psi(\rho) + \sup_{(F,K) \in \overline{\mathrm{co}}\, C(\rho)} \{-F \cdot \mathrm{D}\psi(\rho) - K\} \geqslant 0.$$

Finally, noticing that

$$\sup_{(F,K) \in \overline{\mathrm{co}}\, C(\rho)} \{-F \cdot \mathrm{D}\psi(\rho) - K\} = H(\rho, \mathrm{D}\psi(\rho), \psi)$$

we get the desired supersolution property for v. □

3.4 Existence of an ordinary optimal control

We want now to prove that under some additional assumptions there exists an optimal ordinary control $\mathbf{u}^\star \in \mathcal{U}_{ad}$ such that the minimum in (3.1.5) is achieved. Thanks to Theorem 3.2.1 this optimal control exists if and only if there exists an optimal policy $\mathbf{a}^\star = (a_0, a_1, \dots) \in \mathcal{A}_{ad}$ such that for all $n \in \mathbb{N}_0$ the functions a_n take values in the set A of ordinary controls. Since we already established the existence of a stationary optimal policy made of relaxed controls, we want to find an analogous policy made of ordinary controls.

First, we need to find $\alpha^\star \in A$ such that for each fixed $\nu \in \Delta_e$ the functional

$$\mathcal{J}(\nu, \alpha) = \int_0^\infty e^{-\beta t} L(\phi_\nu^\alpha(t), \chi_\nu^\alpha(t), \alpha(t), v)\, \mathrm{d}t, \quad \alpha \in A \tag{3.4.1}$$

reaches its infimum (the function v appearing as the last argument of the function L is the value function of the separated problem characterized in the previous section). If this is the case, then an optimal stationary policy $\mathbf{a}^\star \in \mathcal{A}_{ad}$ is granted by standard results in discrete-time control theory, as stated in Theorem 3.4.2.

Theorem 3.4.1. *Let Assumptions 3.1.1 and 3.1.2 hold and suppose that for each $\rho \in \Delta_e$ and $s \in [0,1]$ the set*

$$C(\rho, s) = \{(f, g, l) \in \Delta_e \times [0,1] \times \mathbb{R} \text{ s.t. } \\ f = F(\rho, u), g = -r(\rho, u)s, l \geqslant L(\rho, s, u, v), u \in U\}$$

is convex.

Then for each fixed $\nu \in \Delta_e$ there exists $\alpha^\star \in A$ such that the infimum of the functional $\mathcal{J}$ appearing in (3.4.1) is achieved.

Proof. Fix $\nu \in \Delta_e$ and let us write (3.4.1) in a lighter way, suppressing the explicit mention of the value function v and the dependence on the control α and ν of the functions ϕ_ν^α and χ_ν^α. We will then write

$$\mathcal{J}(\alpha) = \int_0^\infty e^{-\beta t} L(\phi(t), \chi(t), \alpha(t))\, \mathrm{d}t, \quad \alpha \in A.$$

Let $\alpha_n \in A$, $n \in \mathbb{N}$ be a minimizing sequence for $\mathcal{J}$ (i. e. $\mathcal{J}(\alpha_n) \to \inf_\alpha \mathcal{J}(\alpha)$ as $n \to +\infty$) and let $(\phi_n, \chi_n)_{n\in\mathbb{N}}$ be the corresponding trajectories of the flow and the survival distribution of the first jump time of the PDP. For each $n \in \mathbb{N}$, $\phi_n \in \mathrm{C}([0,+\infty); \Delta_e)$ and $\chi_n \in \mathrm{C}([0,+\infty); [0,1])$. It can be easily checked that both sequences $(\phi_n)_{n\in\mathbb{N}}$ and $(\chi_n)_{n\in\mathbb{N}}$ are uniformly bounded and equicontinuous on each compact subset of $[0,+\infty)$, hence by Ascoli-Arzelà theorem we get that there exist $\phi \in \mathrm{C}([0,+\infty); \Delta_e)$ and $\chi \in \mathrm{C}([0,+\infty); [0,1])$ such that, up to a subsequence, $\phi_n \to \phi$ and $\chi_n \to \chi$ uniformly on each compact subset of $[0,+\infty)$.

Let us now define for all $t \geqslant 0$

- $F_n(t) = F(\phi_n(t), \alpha_n(t))$,
- $G_n(t) = -r(\phi_n(t), \alpha_n(t))\chi_n(t)$,
- $L_n(t) = L(\phi_n(t), \chi_n(t), \alpha_n(t))$.

Denoting by L^1_β the weighted L^1 space (with weight given by the discount factor β), it can be easily shown that, for each $n \in \mathbb{N}$, $F_n \in \mathrm{L}^1_\beta([0,+\infty); \Delta_e)$, $G_n \in \mathrm{L}^1_\beta([0,+\infty); \mathbb{R})$, $L_n \in \mathrm{L}^1_\beta([0,+\infty); \mathbb{R})$ and that the sequences $(F_n)_{n\in\mathbb{N}}$, $(G_n)_{n\in\mathbb{N}}$ and $(L_n)_{n\in\mathbb{N}}$ are uniformly bounded and uniformly integrable. Hence there exist $\hat{F} \in \mathrm{L}^1_\beta([0,+\infty); \Delta_e)$, $\hat{G} \in \mathrm{L}^1_\beta([0,+\infty); \mathbb{R})$ and $\hat{L} \in \mathrm{L}^1_\beta([0,+\infty); \mathbb{R})$ such that, up to a subsequence, $F_n \rightharpoonup \hat{F}$, $G_n \rightharpoonup \hat{G}$ and $L_n \rightharpoonup \hat{L}$ weakly in L^1_β.

By Mazur's Theorem (see e. g. [16, Corollary 3.8, p. 61], or [66, Theorem 2, p. 120]), there exist sequences, still denoted by $(F_n)_{n\in\mathbb{N}}$, $(G_n)_{n\in\mathbb{N}}$ and $(L_n)_{n\in\mathbb{N}}$, that are convex combinations of the elements of the original ones, such that $F_n \to \hat{F}$, $G_n \to \hat{G}$ and $L_n \to \hat{L}$ strongly in L^1_β and also, again up to a subsequence, a.e. in $[0,+\infty)$. Thanks to the hypotheses we have that the functions F, L and $-r(\rho, u)s$ are continuous on the compact set $\Delta_e \times [0,1] \times U$ and it can be proved that the sets $C(\rho, s)$ are closed for each $\rho \in \Delta_e$ and $s \in [0,1]$ (see e. g. [23, 8.5.vi, p. 296]). Therefore, for almost all $t \geqslant 0$ the triple $(\hat{F}(t), \hat{G}(t), \hat{L}(t))$ belongs to the set $C(\phi(t), \chi(t))$ and we can apply standard measurable selection theorems (see e. g. [23, 8.2.ii, p. 277], or [48, Corollary 2.26, p. 102]) to obtain a measurable function $\alpha^\star$ such that

- $\hat{F}(t) = F(\phi(t), \alpha^\star(t))$,

- $\hat{G}(t) = -r(\phi(t), \alpha^\star(t))\chi(t)$,
- $\hat{L}(t) = L(\phi(t), \chi(t), \alpha^\star(t)) + z(t)$,

where z is a non-negative function defined on $[0, +\infty)$.

Now it remains to prove that $\alpha^\star$ is optimal for the functional $\mathcal{J}$. Let $(\gamma_{k,n})$, where $n \in \mathbb{N}$ and $k \geqslant n$, be the system of non-negative numbers of Mazur's Theorem, such that for each $n \in \mathbb{N}$

$$\sum_{k=n}^{K_n} \gamma_{k,n} = 1, \quad L(\phi(t), \chi(t), \alpha^\star(t)) = \lim_{n\to+\infty} \sum_{k=n}^{K_n} \gamma_{k,n} L(\phi_k(t), \chi_k(t), \alpha_k(t)). \tag{3.4.2}$$

First of all, let us notice that z has to be zero a.e. in $[0, +\infty)$. If this were not the case, we would reach a contradiction (arguing as in the following lines) with the fact that $(\alpha_n)_{n\in\mathbb{N}}$ is a minimizing sequence for $\mathcal{J}$. Since the function L is bounded by some constant $K > 0$ and obviously the function $Ke^{-\beta t} \in \mathrm{L}^1([0, +\infty))$, we can apply Fatou's Lemma to obtain

$$\begin{aligned}
\mathcal{J}(\alpha^\star) &= \int_0^\infty e^{-\beta t} L(\phi(t), \chi(t), \alpha^\star(t)) \\
&\leqslant \liminf_{n\to+\infty} \sum_{k=n}^{K_n} \gamma_{kn} \int_0^\infty e^{-\beta t} L(\phi_k(t), \chi_k(t), \alpha_k(t)) \\
&= \liminf_{n\to+\infty} \sum_{k=n}^{K_n} \gamma_{kn} \mathcal{J}(\alpha_k) = \inf_\alpha \mathcal{J}(\alpha).
\end{aligned} \tag{3.4.3}$$

The claim follows since clearly $\inf_\alpha \mathcal{J}(\alpha) \leqslant \mathcal{J}(\alpha^\star)$. □

Remark 3.4.1. Convexity of the sets $C(\rho, s)$ is guaranteed, for instance, when

- $U \subset \mathbb{R}$ is a closed interval.
- Matrix coefficients $\lambda_{ij}(u)$ are linear in u, for all $i, j \in I$, $i \neq j$.
- The functions $u \mapsto f(i, u)$ are convex for each $i \in I$.

We are now ready to state the main result of this Section. To be fully precise its proof would require to formulate the entire control problem in a broader setting. This should be done to allow for more general control policies, namely universally measurable ones. However, this formulation does not pose any particular problem (the interested reader may consult [13]) and it is irrelevant to the results of this thesis. Therefore, we will omit all unnecessary technical details.

Theorem 3.4.2. *For each initial law $\mu \in \Delta$ there exists an optimal ordinary stationary policy $\mathbf{a}^\star \in \mathcal{A}_{ad}$ (with corresponding optimal ordinary control $\mathbf{u}^\star \in \mathcal{U}_{ad}$), i. e. an admissible policy with values in the set of ordinary controls A such that*

$$V(\mu) = J(\mu, \mathbf{u}^\star) = \sum_{y\in O} \mu\big(h^{-1}(y)\big) \bar{J}(H_y[\mu], \mathbf{a}^\star) = \sum_{y\in O} \mu\big(h^{-1}(y)\big) v(H_y[\mu]).$$

Proof. Let $\mu \in \Delta$ be fixed. Thanks to Theorem 3.4.1, to the fact that the function $\mathcal{J}$ appearing in (3.4.1) is measurable and to the fact that Δ_e and A are Borel spaces, standard selection theorems (see e. g. [13, Prop. 7.50]) ensure that there exists a universally measurable selector $a^u \colon \Delta_e \to A$ such that for all $\nu \in \Delta_e$

$$v(\nu) = \mathcal{J}\big(\nu, a^u(\nu)\big) = \inf_{\alpha \in A} \mathcal{J}(\nu, \alpha).$$

Let Q be the probability measure on Δ_e defined in (3.2.11) and let us define the optimal strategy $\mathbf{a}^u = (a^u, a^u, \dots)$. Thanks to [68, Th. 3.1] we can conclude that there exists a stationary policy $\mathbf{a}^\star \in \mathcal{A}_{ad}$ such that

$$\bar{J}(\cdot, \mathbf{a}^u) = \bar{J}(\cdot, \mathbf{a}^\star) \quad Q-\text{a.s.}$$

Since Q is concentrated at points $\{H_y[\mu]\}_{y \in O}$ we get that for all $y \in O$

$$v(H_y[\mu]) = \bar{J}(H_y[\mu], \mathbf{a}^u) = \bar{J}(H_y[\mu], \mathbf{a}^\star)$$

and the claim follows immediately. □

3.5 A numerical example

In this Section we provide an explicit and rather simple example of optimal control problem with noise-free partial observation of a Markov chain. We are able to provide the value function and to write down the ordinary optimal control, whose existence was established in Section 3.4.

Let us fix $I := \{1, 2, 3\}$ and $O := \{a, b\}$ as the state spaces of the unobserved Markov chain X and of the observed process Y respectively. We recall that to solve the optimal control problem we first put ourselves in a canonical framework for the Markov chain X. Therefore, all the following processes are defined as in Section 3.1. In this example, we assume that the observed process satisfies $Y_t = h(X_t),\ t \geqslant 0$, with function $h \colon I \to O$ given by

$$h(i) = \begin{cases} a, & \text{if } i = 1, 2 \\ b, & \text{if } i = 3 \end{cases}.$$

Hence, we have a perfect observation whenever $Y_t = b$ for some $t \geqslant 0$, i. e. if this is the case we know that $X_t = 3$ almost surely. On the contrary, we face uncertainty about the true state of the Markov chain X at some time $t \geqslant 0$ if we have $Y_t = a$.

Next, we define the space of control actions U, the controlled rate transition matrix $\Lambda(u)$ associated to the Markov chain X and the cost function $f \colon I \times U \to \mathbb{R}$ of our optimal control problem. We choose them as specified in Remark 3.4.1, in order to be sure about the existence of an optimal control.

$$\Lambda(u) := \begin{bmatrix} -2u & u & u \\ 0 & -u & u \\ u & 0 & -u \end{bmatrix} \qquad \mathbf{f}(u) := \begin{bmatrix} u^2+1 \\ u^2+1 \\ u^2 \end{bmatrix} \qquad u \in U := [0, 1].$$

We recall that, since the set I is finite, we identify the function f with the vector-valued function $\mathbf{f}$.

From the data of our problem given above, we can specify the state space Δ_e (called the effective simplex) of the filtering process and explicitly compute its local characteristics, defined in (3.1.12).

The effective simplex Δ_e is given by $\Delta_e = \Delta_a \cup \Delta_b \subset \mathbb{R}^3$, where $\Delta_a := \mathcal{P}(\{1,2\}) = \{(p, 1-p, 0), p \in [0,1]\}$ and $\Delta_b := \mathcal{P}(\{3\}) = \{(0,0,1)\}$. We will identify probability measures $\nu \in \Delta_a$ with vectors of the form $(p, 1-p, 0) \in \mathbb{R}^3$ for some $p \in [0,1]$. The local characteristics of the filtering process, that is a PDP, are given, for all $u \in [0,1]$, by

$$F(\nu, u) = \begin{cases} \begin{bmatrix} -up & up & 0 \end{bmatrix}, & \text{if } (p, 1-p, 0) = \nu \in \Delta_a \\ \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}, & \text{if } (0,0,1) = \nu \in \Delta_b \end{cases}$$
$$r(\nu, u) = u, \quad \nu \in \Delta_e$$
$$R(\nu, u, \mathrm{d}\rho) = \begin{cases} \delta_{[0\,0\,1]}(\mathrm{d}\rho), & \text{if } (p, 1-p, 0) = \nu \in \Delta_a \\ \delta_{[1\,0\,0]}(\mathrm{d}\rho), & \text{if } (0,0,1) = \nu \in \Delta_b \end{cases}.$$

Notice that, in this case, the ODE (3.1.9). hence the filtering equation, admits an explicit solution, given, for all relaxed controls $m \in \mathcal{M}$, by

$$\phi_\nu^m(t) = \begin{cases} \begin{bmatrix} p(t) & 1-p(t) & 0 \end{bmatrix}, & \text{if } (p, 1-p, 0) = \nu \in \Delta_a \\ \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, & \text{if } (0,0,1) = \nu \in \Delta_b \end{cases}, \quad t \geqslant 0$$

with $p(t) = p \, \exp\Big\{-\int_0^t \int_U u \, m(s;\, \mathrm{d}u) \, \mathrm{d}s\Big\}$, $t \geqslant 0$.

Having written the local characteristics of the filtering process, we can provide the explicit form of the operator $\mathcal{G}$, defined in (3.2.31), and of the HJB equation (3.3.8) satisfied by the value function v of the separated optimal control problem, defined in (3.2.10).

First of all, let us notice that we can identify any function $w \colon \Delta_e \to \mathbb{R}$ with the pair $(\hat{w}, w_b)$, where $\hat{w} \colon [0,1] \to \mathbb{R}$ and $w_b \in \mathbb{R}$. In fact,

$$w(\nu) = \begin{cases} w(p, 1-p, 0) =: \hat{w}(p), & \text{if } (p, 1-p, 0) = \nu \in \Delta_a \\ w(0,0,1) =: w_b, & \text{if } (0,0,1) = \nu \in \Delta_b \end{cases}.$$

Thanks to this identification, for any function $w \in \mathrm{B}_b(\Delta_e)$, we can write the operator $\mathcal{G}$ as

$$\mathcal{G}w(\nu) = \begin{cases} \inf_{\alpha \in A} \int_0^{+\infty} e^{-\int_0^t [\beta + \alpha(s)] \, \mathrm{d}s} \big[\alpha(t)^2 + \alpha(t) w_b + 1\big] \, \mathrm{d}t, & \text{if } \nu \in \Delta_a \\ \inf_{\alpha \in A} \int_0^{+\infty} e^{-\int_0^t [\beta + \alpha(s)] \, \mathrm{d}s} \big[\alpha(t)^2 + \alpha(t) \hat{w}(1)\big] \, \mathrm{d}t, & \text{if } \nu \in \Delta_b \end{cases} \tag{3.5.1}$$

where A is the set of ordinary controls, i. e. $A := \{\alpha \colon [0, +\infty) \to U, \text{ measurable}\}$. The HJB equation is given by

$$\begin{cases} \sup_{u \in [0,1]} \big\{-u^2 + 2pu\hat{v}'(p) - u[v_b - \hat{v}(p)]\big\} + \beta \hat{v}(p) = 0, & \text{if } \nu \in \Delta_a \\ \sup_{u \in [0,1]} \big\{-u^2 - u[\hat{v}(1) - v_b]\big\} + \beta v_b = 0, & \text{if } \nu \in \Delta_b \end{cases}. \tag{3.5.2}$$

We now search for an explicit solution of the fixed point problem and the HJB equation. Notice that, since the cost function f is non-negative, we have that $v \geqslant 0$. Moreover, in (3.5.1) there is no dependence on the parameter $p \in [0,1]$ defining all probability measures $\nu \in \Delta_a$. This suggests that the value function v of the separated problem, satisfying $v = \mathcal{G}v$, should be a constant function.

For $\nu \in \Delta_b$ the fixed point relationship $v = \mathcal{G}v$ takes the form

$$v_b = \inf_{\alpha \in A} \int_0^{+\infty} e^{-\int_0^t [\beta + \alpha(s)]\,\mathrm{d}s} \big[\alpha(t)^2 + \alpha(t)\hat{v}(1)\big]\,\mathrm{d}t. \tag{3.5.3}$$

It is clear that, since $v_b \geqslant 0$, if we choose the constant control $\alpha(t) = 0$, $t \geqslant 0$, we achieve the minimum in (3.5.3), with value $v_b = 0$. Substituting this into (3.5.2) we get

$$\begin{cases} \sup\limits_{u \in [0,1]} \big\{-u^2 + 2up\hat{v}'(p) + u\hat{v}(p)\big\} + \beta\hat{v}(p) = 0, & \text{if } \nu \in \Delta_a \\ \sup\limits_{u \in [0,1]} \big\{-u^2 - u\hat{v}(1)\big\} = 0, & \text{if } \nu \in \Delta_b \end{cases}. \tag{3.5.4}$$

After some computations, we obtain that the second equality in (3.5.4) is satisfied for all possible values of $\hat{v}(1)$, while the ODE appearing as first equality admits the constant solution $\hat{v}(p) = 2\big(\sqrt{\beta^2+1} - \beta\big)$, $p \in [0,1]$. Hence, we have that the value function of the separated problem is given by

$$v(\nu) = \begin{cases} 2\big(\sqrt{\beta^2+1} - \beta\big), & \text{if } \nu \in \Delta_a \\ 0, & \text{if } \nu \in \Delta_b \end{cases}. \tag{3.5.5}$$

The value function V of the optimal control problem with noise-free partial observation for the Markov chain X, defined in (3.1.5), is given for all $\mu \in \mathcal{P}(I)$ by

$$V(\mu) = V(q_1, q_2, 1 - q_1 - q_2) = \begin{cases} 2\big(\sqrt{\beta^2+1} - \beta\big)(q_1 + q_2), & \text{if } q_1 + q_2 \neq 0 \\ 0, & \text{if } q_1 + q_2 = 0 \end{cases} \tag{3.5.6}$$

where $\mu = (q_1, q_2, 1 - q_1 - q_2)$ for some $q_1, q_2 \in [0,1]$, $q_1 + q_2 \leqslant 1$.

At this point, we can also provide an explicit optimal policy $\mathbf{a}^\star \in \mathcal{A}_{ad}$, with corresponding optimal control $\mathbf{u}^\star \in \mathcal{U}_{ad}$. It is clear that if $Y_0 = b$ the optimal policy $\mathbf{a}^\star$ is composed by the single constant optimal control $\alpha_b^\star(t) = 0$, $t \geqslant 0$, i.e. $\mathbf{a}^\star = \alpha_b^\star$. In fact, given the structure of the controlled rate transition matrix, applying this policy means that the Markov chain X remains in the state 3 forever and we sustain no cost, since $f(3,0) = 0$. If, instead, $Y_0 = a$ the optimal policy is given by $\mathbf{a}^\star = (\alpha_a^\star, \alpha_b^\star)$, where $\alpha_a^\star \in A$ solves

$$2\big(\sqrt{\beta^2+1} - \beta\big) = \int_0^{+\infty} e^{-\int_0^t [\beta + \alpha_a^\star(s)]\,\mathrm{d}s} \big[\alpha_a^{\star 2} + 1\big]\,\mathrm{d}t. \tag{3.5.7}$$

Notice that this characterization comes from the fixed point equation $v = \mathcal{G}v$. The optimal policy $\alpha_a^\star$, whose existence is granted by Theorem 3.4.1, guarantees that we achieve the minimum cost up to the first jump time of the Markov chain X into the state 3. This jump time is finite almost surely, as can be easily checked. Therefore, when X jumps into the state 3, i.e. the observed process takes value b, we can choose the optimal control $\alpha_b^\star$ and stop the motion of the Markov chain in that state at zero cost.

Searching for constant solutions to (3.5.7) we are able to obtain that the optimal control $\alpha_a^\star$ is given by $\alpha_a^\star(t) \coloneqq \sqrt{\beta^2+1} - \beta$, $t \geqslant 0$.

CHAPTER 4

Optimal control: the infinite dimensional case

In this Chapter we return to the setting of Chapter 2, i. e. the case where the unobserved process $X = (X_t)_{t \geqslant 0}$ is a continuous-time pure jump Markov process and we study an optimal control problem on infinite time horizon with partial observation in the same setting of Chapter 3.

We recall that the aim of our control problem is to optimize the dynamics of the unobserved process X through the actions described by a *control process* $\mathbf{u} = (u_t)_{t \geqslant 0}$, with values in the set of Borel probability measures $\mathcal{P}(U)$ on a measurable space $(U, \mathcal{U})$, the *space of control actions*. A priori, we are selecting control actions specified by *relaxed controls*. We anticipate that, while we are able to guarantee the existence of a (relaxed) optimal control, we can not provide the existence of an ordinary optimal control. The control process is chosen in a specific class of *admissible controls*. At any time the chosen control action shall be based on the information provided by the observed process $Y = (Y_t)_{t \geqslant 0}$, that will be of noise-free type as in the previous Chapters. The choice of the control process is done following a performance criterion that, also in this Chapter, is the minimization of a discounted cost functional.

Throughout this Chapter we will assume that I and O are complete and separable metric spaces, equipped with their respective Borel σ-algebras $\mathcal{I} := \mathcal{B}(I)$ and $\mathcal{O} := \mathcal{B}(O)$. The set U is a compact metric space equipped with its Borel σ-algebra $\mathcal{U} := \mathcal{B}(U)$. This assumption entails that $\mathcal{P}(U)$ is a compact metric space, too. As in the previous Chapters, we are given a function $h \colon I \to O$ that gives the values of the observed process Y as a deterministic transformation of the values assumed by the unobserved process X. We consider this function to be surjective, without loss of generality. We remind that, in general, h can be constant or one-to-one, but we will exclude these cases in what follows.

Our analysis will follow essentially the same steps of Chapter 3. In Section 4.1 we formulate our optimal control problem with partial observation in a canonical setting for the pure jump Markov process X. Thanks to the filtering process we are able to rewrite this control problem in an equivalent one with complete observation, where the new *state variable* is the filtering process itself, in place of the unobserved process X. Also in this case, we will need to reformulate our control problem, introducing a separated discrete-time control problem for the filtering process.

The separated control problem will be formulated in Section 4.2 in a canonical setting for the PDP given by the filtering process. We will prove that the original and the separated control problem are linked in such a way, that we are able to write down a formula providing an equality involving the value function V of the original control problem and the value function v of the separated control problem.

The original value function V is indirectly characterized by studying the value function v of the separated control problem. This is done in Section 4.3, where we obtain that v is the unique fixed point of a certain contraction mapping. Contrary to what we proved in Section 3.3, we are not able to characterize v as a viscosity solution to some *Hamilton-Jacobi-Bellman* equation, since this equation would be defined in an infinite dimensional space (consider that the effective simplex Δ_e introduced in Remark 2.2.2 is a subset of the *a priori* infinite dimensional metric space $\mathcal{M}(I)$). The study of HJB equations in infinite dimensional spaces is a recent subject in the literature, but still limited to some particular cases. For an up-to-date reference to this kind of equations, studied in connection with stochastic optimal control problems, see [35]. In the context of control problems with partial observation, see also [7, 40, 49].

Finally, in Section 4.4 we make some comments on the rôle of the function h in the control problem.

4.1 The jump Markov process optimal control problem

We will shortly introduce the setting under which we will formulate the optimal control problem for the unobserved pure jump Markov process. The construction is almost identical to that provided in Section 3.1 for the Markov chain optimal control problem. We will return to the notation adopted in Chapter 2.

The aim of this Section is to provide a canonical framework for a continuous time pure jump Markov process described by an initial law and a *controlled rate transition measure* on I. By this we mean that we have a transition measure λ from $(I \times U, \mathcal{I} \otimes \mathcal{U})$ into $(I, \mathcal{I})$ such that

$$\lambda(x, u, \{x\}) = 0, \quad x \in I, \, u \in U.$$

To simplify the notation it is convenient to define the controlled jump rate function $\lambda \colon I \times U \to [0, +\infty)$ as

$$\lambda(x, u) \coloneqq \lambda(x, u, I), \quad x \in I, \, u \in U.$$

It will always be clear from the context if λ refers to the rate transition measure or the jump rate function.

We introduce the following Assumption.

Assumption 4.1.1.

1. For each $x \in I$, $A \in \mathcal{I}$ the map $u \mapsto \lambda(x, u, A)$ is continuous on U (hence bounded and uniformly continuous on U).

2. $\displaystyle\sup_{(x,u) \in I \times U} \lambda(x, u) < +\infty.$

We are now ready to build the probability space on which the processes X, Y, $\mathbf{u}$ are defined. As recalled at the beginning of this Chapter, the construction is almost identical to that provided in Section 3.1. Let us define Ω as the set

$$\Omega = \{\omega = (i_0, t_1, i_1, t_2, i_2, \dots)\colon \\ i_0 \in I, i_n \in I, t_n \in (0, +\infty], t_n < +\infty \Rightarrow t_n < t_{n+1}, n \in \mathbb{N}\}.$$

For each $n \in \mathbb{N}$ we introduce the following random variables

$$T_0(\omega) = 0; \quad T_n(\omega) = t_n; \quad T_\infty(\omega) = \lim_{n\to\infty} T_n(\omega); \quad \xi_0(\omega) = i_0; \quad \xi_n(\omega) = i_n$$

and we define the random measure on $\big((0, +\infty) \times I,\, \mathcal{B}\big((0, +\infty)\big) \otimes \mathcal{I}\big)$

$$n(\omega, \mathrm{d}t\,\mathrm{d}z) = \sum_{n\in\mathbb{N}} \delta_{\left(T_n(\omega),\, \xi_n(\omega)\right)}(\mathrm{d}t\,\mathrm{d}z)\mathbb{1}_{\{T_n<+\infty\}}(\omega), \quad \omega \in \Omega$$

with associated natural filtration $\mathcal{N}_t = \sigma\big(n\big((0, t] \times A\big),\, 0 \leqslant s \leqslant t,\, A \in \mathcal{I}\big)$. Finally, let us specify the σ-algebras

$$\mathcal{X}_0^\circ = \sigma(\xi_0); \qquad \mathcal{X}_t^\circ = \sigma(\mathcal{X}_0 \cup \mathcal{N}_t); \qquad \mathcal{X}^\circ = \sigma\Big(\bigcup_{t\geqslant 0} \mathcal{X}_t\Big).$$

The unobserved process X is defined as

$$X_t(\omega) = \begin{cases} \xi_n(\omega), & t \in \big[T_n(\omega), T_{n+1}(\omega)\big),\, n \in \mathbb{N}_0,\, T_n(\omega) < +\infty \\ i_\infty, & t \in \big[T_\infty(\omega), +\infty),\, T_\infty(\omega) < +\infty \end{cases}$$

where $i_\infty \in I$ is an arbitrary state, that is irrelevant to specify. Next, we define the observed process Y and its natural filtration $(\mathcal{Y}_t^\circ)_{t\geqslant 0}$ as

$$Y_t(\omega) = h(X_t(\omega)),\, t \geqslant 0,\, \omega \in \Omega; \qquad \mathcal{Y}_t^\circ = \sigma\big(Y_s,\, 0 \leqslant s \leqslant t\big),\, t \geqslant 0.$$

As we already pointed out in Section 2.2, we can equivalently describe this process via a MPP $(\eta_n, \tau_n)_{n\in\mathbb{N}}$ together with the initial condition $\eta_0 = h(\xi_0) = Y_0$. Each σ-algebra $\mathcal{Y}_t^\circ$, $t \geqslant 0$ is the smallest σ-algebra generated by the union of $\sigma(\eta_0)$ and the σ-algebra at time t of the natural filtration of the MPP $(\eta_n, \tau_n)_{n\in\mathbb{N}}$.

The control processes $\mathbf{u}$ that we want to consider are based on the information coming from the observed process Y. We will pick them in the following class of *admissible controls*

$$\mathcal{U}_{ad} = \Big\{\mathbf{u}\colon \Omega \times [0, +\infty) \to \mathcal{P}(U),\, (\mathcal{Y}_t^\circ)_{t\geqslant 0} - \text{predictable}\Big\}. \tag{4.1.1}$$

Concerning the choice of $\mathcal{P}(I)$ as target space for control processes, i. e. the choice of *relaxed controls*, see Remark 3.1.1.

In Section 3.1 we noticed that predictable processes with respect to the natural filtration of a point process admit a precise description (see Theorem 1.1.2 and Remark 1.1.1). Thus, control processes in the class $\mathcal{U}_{ad}$ can be characterized by a sequence of Borel-measurable functions $(u_n)_{n\in\bar{\mathbb{N}}_0}$, with $u_n\colon [0, +\infty) \times O \times \big((0, +\infty) \times O\big)^n \to \mathcal{P}(U)$ for each $n \in \bar{\mathbb{N}}_0$, so that we can write

$$u_t(\omega) = u_0(t, Y_0(\omega))\mathbb{1}(0 \leqslant t \leqslant \tau_1(\omega)) + \\ \sum_{n=1}^{\infty} u_n(t, Y_0(\omega), \tau_1(\omega), Y_{\tau_1}(\omega), \dots, \tau_n(\omega), Y_{\tau_n}(\omega))\mathbb{1}(\tau_n(\omega) < t \leqslant \tau_{n+1}(\omega)) + \\ u_\infty(t, Y_0(\omega), \tau_1(\omega), Y_{\tau_1}(\omega), \dots)\mathbb{1}(t > \tau_\infty(\omega)), \tag{4.1.2}$$

where $\tau_\infty(\omega) = \lim_{n\to\infty} \tau_n(\omega)$, $\omega \in \Omega$. Also in this Chapter, the reader must always remember this kind of decomposition, since it will be of fundamental importance in the analysis of the optimal control problem. Moreover, we will frequently switch between the notation $(u_t)_{t\geqslant 0}$ and $(u_n)_{n\in\bar{\mathbb{N}}_0}$ and, to simplify matters, we will often use the more compact writing $u_n(\cdot)$ instead of $u_n(\cdot, Y_0(\omega), \dots, \tau_n(\omega), Y_{\tau_n}(\omega))$, $n \in \mathbb{N}_0$.

The dynamics of the unobserved process will be specified by the initial distribution μ, a probability measure on I, and by the following random measure depending on the chosen control process $\mathbf{u} \in \mathcal{U}_{ad}$.

$$\nu^{\mathbf{u}}(\omega;\, \mathrm{d}t\, \mathrm{d}z) = \mathbb{1}_{t<T_\infty(\omega)} \int_U \lambda(X_{t-}(\omega), \mathrm{u}, \mathrm{d}z)\, u_t(\omega\,; \mathrm{du})\, \mathrm{d}t \tag{4.1.3}$$

for any $\omega \in \Omega$ and $\mathbf{u} \in \mathcal{U}_{ad}$. For sake of simplicity, we will drop ω in what follows.

Now set P_0 as the probability measure on $\left(\Omega, \mathcal{X}_0^\circ\right)$ such that $X_0 = \xi_0$ has law μ. It is easy to see that the previously described setting is equivalent to that provided in Assumption 1.1.3. In fact, one can show that the random measure $\nu^{\mathbf{u}}$ is $(\mathcal{X}_t^\circ)_{t\geqslant 0}$–predictable and satisfies (1.1.3), i. e.

1. $\nu^{\mathbf{u}}\big(\{t\} \times I\big) \leqslant 1$,
2. $\nu^{\mathbf{u}}\big([T_\infty, +\infty) \times I\big) = 0$.

Therefore, by Theorem 1.1.9, there exists a unique probability measure $\mathrm{P}_\mu^{\mathbf{u}}$ on $\left(\Omega, \mathcal{X}^\circ\right)$, such that $\mathrm{P}_\mu^{\mathbf{u}}|_{\mathcal{X}_0^\circ} = \mathrm{P}_0$ and $\nu^{\mathbf{u}}$ is the $\left(\mathrm{P}_\mu^{\mathbf{u}}, \mathcal{X}_t^\circ\right)$–predictable projection of n. Once specified the control $\mathbf{u} \in \mathcal{U}_{ad}$ and consequently the probability measure $\mathrm{P}_\mu^{\mathbf{u}}$, it follows from the second part of Assumption 4.1.1 that the point process n is $\mathrm{P}_\mu^{\mathbf{u}}$-a.s. non-explosive, i. e. that $T_\infty = +\infty$, $\mathrm{P}_\mu^{\mathbf{u}}$-a.s. . For this reason we will drop the term $\mathbb{1}_{t<T_\infty}$ appearing in (3.1.3) and, since also $\tau_\infty = +\infty$ $\mathrm{P}_\mu^{\mathbf{u}}$-a.s., we will avoid specifying the function u_∞ in (3.1.2).

Finally, we define for each probability measure μ on I and $\mathbf{u} \in \mathcal{U}_{ad}$ the completions of the natural filtrations of the processes X and Y as follows.

- $\mathcal{X}^{\mu,\mathbf{u}}$ is the $\mathrm{P}_\mu^{\mathbf{u}}$-completion of $\mathcal{X}^\circ$ ($\mathrm{P}_\mu^{\mathbf{u}}$ is extended to $\mathcal{X}^{\mu,\mathbf{u}}$ in the natural way).
- $\mathcal{Z}^{\mu,\mathbf{u}}$ is the family of elements of $\mathcal{X}^{\mu,\mathbf{u}}$ with zero $\mathrm{P}_\mu^{\mathbf{u}}$ – probability.
- $\mathcal{Y}_t^{\mu,\mathbf{u}} := \sigma(\mathcal{Y}_t^\circ, \mathcal{Z}^{\mu,\mathbf{u}})$, for $t \geqslant 0$.

$(\mathcal{Y}_t^{\mu,\mathbf{u}})_{t\geqslant 0}$ is called the *natural completed filtration* of Y.

As we anticipated at the beginning of this Chapter, we choose control actions in order to minimize, for all possible choices of the initial distribution μ of the process X, the following *cost functional*

$$J(\mu, \mathbf{u}) = \mathrm{E}_\mu^{\mathbf{u}}\left[\int_0^\infty e^{-\beta t} \int_U f(X_t, \mathrm{u})\, u_t(\mathrm{du})\, \mathrm{d}t\right] \tag{4.1.4}$$

where f is called *cost function* and $\beta > 0$ is a fixed constant called *discount factor*. In other words, we want to characterize the *value function*

$$V(\mu) = \inf_{\mathbf{u}\in\mathcal{U}_{ad}} J(\mu, \mathbf{u}). \tag{4.1.5}$$

We make the following assumption on the cost function f, ensuring that the functional J is well defined (and also bounded).

Assumption 4.1.2. The function $f\colon I \times U \to \mathbb{R}$ is bounded and uniformly continuous. In particular, for some constant $C_f > 0$ it holds that

$$\sup_{(x,u) \in I \times U} |f(x,u)| \leqslant C_f, \tag{4.1.6}$$

We can transform the problem formulated above into a complete observation problem by means of the *filtering process*. Similarly to what we did in Section 3.1 we provide without proof some results concerning this process. They can be obtained as slight generalizations of the corresponding statements that the reader can found in Chapter 2.

The filtering process is defined as the $\mathcal{P}(I)$–valued process given by

$$\mathrm{P}_\mu^{\mathbf{u}}(X_t \in A \mid \mathcal{Y}_t^{\mu,\mathbf{u}}), \quad t \geqslant 0,\ A \in \mathcal{I}.$$

As we already know, the true image set of this process is the so called *effective simplex* Δ_e, defined as

$$\Delta_e \coloneqq \bigcup_{y \in O} \Delta_y, \quad \Delta_y \coloneqq \{\nu \in \mathcal{P}(I) \colon \nu\big(h^{-1}(y)^c\big) = 0\},\ y \in O \tag{4.1.7}$$

It is a proper subset of $\mathcal{P}(I)$ unless the function h is constant. Moreover, the sets Δ_y are clearly pairwise disjoint. We recall that the effective simplex can be regarded as a topological space (Δ_e, τ_e) under the relative topology τ_e induced on Δ_e by the total variation norm, with which we endow the space $\mathcal{M}(I)$.

It is worth noticing that the filtering process is a $(\mathcal{Y}_t^{\mu,\mathbf{u}})_{t \geqslant 0}$ – adapted process and since $(\mathcal{Y}_t^{\mu,\mathbf{u}})_{t \geqslant 0}$ is right continuous we can choose a $(\mathcal{Y}_t^{\mu,\mathbf{u}})_{t \geqslant 0}$ – progressive version. We will assume this whenever needed.

We want to state an explicit equation satisfied by the filtering process. To do so, let us define for each $y \in O$ the map $F_y \colon \mathcal{M}(I) \times U \to \mathcal{M}(I)$ as

$$F_y(\nu, u) \coloneqq \mathcal{B}_y^u \nu - \nu\, \mathcal{B}_y^u \nu(I), \quad \nu \in \mathcal{M}(I),\ u \in U \tag{4.1.8}$$

where the controlled operator $\mathcal{B}_y^u$ is defined for all $y \in O$ and $u \in U$ by

$$\mathcal{B}_y^u \nu(\mathrm{d}z) \coloneqq \mathbb{1}_{h^{-1}(y)}(z) \int_I \lambda(x, u, \mathrm{d}z)\, \nu(\mathrm{d}x) - \lambda(z, u)\nu(\mathrm{d}z), \quad \nu \in \mathcal{M}(I). \tag{4.1.9}$$

It is clear that, for each fixed $y \in O$, the map $u \mapsto F_y(\nu, u)$ is measurable for all $\nu \in \mathcal{M}(I)$. Moreover, we can provide the following generalization of Proposition 2.2.2, concerning the Lipschitz continuity of F_y.

Proposition 4.1.1. *Under Assumption 4.1.1, for each fixed $y \in O$ the map F_y is locally Lipschitz continuous in ν uniformly in u, i. e. there exists a constant $L_F > 0$ such that*

$$\sup_{u \in U} \|F(\nu, u) - F(\rho, u)\|_{TV} \leqslant L_F \|\nu - \rho\|_{TV}, \quad \textit{for all } \nu, \rho \in \mathcal{M}(I). \tag{4.1.10}$$

Thanks to this, we can generalize Theorem 2.2.3 to get the following result.

Theorem 4.1.2. *For all $y \in O$, $\rho \in \Delta_y$ and all measurable $m \colon [0, +\infty) \to \mathcal{P}(U)$, the ODE*

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} z(t) = \displaystyle\int_U F_y(z(t), u)\, m(t\,; \mathrm{d}u), & t \geqslant 0 \\ z(0) = \rho \end{cases} \tag{4.1.11}$$

admits a unique global solution $z\colon [0,+\infty) \to \mathcal{M}(I)$. *Moreover* $z(t) \in \Delta_y$ *for all* $t \geqslant 0$ *and, if* z_1 *and* z_2 *are solutions to (4.1.11) with* $z_1(0) = \rho_1 \in \Delta_y$, $z_2(0) = \rho_2 \in \Delta_y$, *we have that the estimate*

$$\|z_1(t) - z_2(t)\|_{TV} \leqslant \|\rho_1 - \rho_2\|_{TV} e^{L_F t}, \quad t \geqslant 0 \tag{4.1.12}$$

holds for all measurable $m\colon [0,+\infty) \to \mathcal{P}(U)$.

Remark 4.1.1. Similarly to what we did following Remark 2.2.2, in the remainder of this Chapter we will denote the solution z by $\phi^m_{y,\rho}(\cdot)$, to stress the dependence on $\rho \in \Delta_y$ and the measurable function m. By standard results on ODE, $(t,\rho) \mapsto \phi^m_{y,\rho}(t)$ is continuous for each $y \in O$ and it enjoys the flow property, i. e. $\phi^m_{y,\phi^m_y(s,\rho)}(t) = \phi^m_{y,\rho}(t+s)$, for $t, s \geqslant 0$. The function $y \mapsto \phi^m_{y,\cdot}(\cdot)$ is called the *controlled flow* associated with the vector field F_y on Δ_y and the control function m. To simplify the notation, it is convenient to define a *global controlled flow* ϕ^m on Δ_e setting $\phi^m_\rho(t) = \phi^m_{y,\rho}(t)$, if $\rho \in \Delta_y$. In this way, for all fixed control functions m and $t \geqslant 0$, $\rho \mapsto \phi^m_\rho(t)$ is a function $\Delta_e \to \Delta_e$ leaving each set Δ_y invariant. Finally, we can associate to the global flow a *global controlled vector field* $F\colon \Delta_e \times U \to \Delta_e$ defined as

$$F(\nu,u) := F_y(\nu,u) = \mathcal{B}^u_y \nu - \nu\, \mathcal{B}^u_y \nu(I), \quad \nu \in \Delta_y,\, u \in U. \tag{4.1.13}$$

We are now ready to state the filtering equation, which can be deduced from Theorem 2.1.6.

Theorem 4.1.3 (Filtering equation). *For all* $\omega \in \Omega$ *define* $\tau_0(\omega) \equiv 0$ *and for fixed* $\mathbf{u} \in \mathcal{U}_{ad}$ *the stochastic process* $\pi^{\mu,\mathbf{u}} = (\pi^{\mu,\mathbf{u}}_t)_{t\geqslant 0}$ *as the unique solution of the following system of ODEs*

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\pi^{\mu,\mathbf{u}}_t(\omega) = \displaystyle\int_U F(\pi^{\mu,\mathbf{u}}_t(\omega), \mathfrak{u})\, u_t(\omega\,; \mathrm{d}\mathfrak{u}), \quad t \in [\tau_n(\omega), \tau_{n+1}(\omega)),\, n \in \mathbb{N}_0 \\ \pi^{\mu,\mathbf{u}}_0(\omega) = H_{Y_0(\omega)}[\mu] \\ \pi^{\mu,\mathbf{u}}_{\tau_n}(\omega) = H_{Y_{\tau_n(\omega)}(\omega)}\big[\Lambda\big(\pi^{\mu,\mathbf{u}}_{\tau_n^-(\omega)}(\omega), u_{\tau_n^-(\omega)}(\omega)\big)\big],\, n \in \mathbb{N}. \end{cases} \tag{4.1.14}$$

where F *is the vector field defined in (4.1.8),* H *is the operator given in Definition 2.1.1,* $\Lambda\colon \Delta_e \times \mathcal{P}(U) \to \mathcal{M}_+(I)$ *is defined as*

$$\Lambda(\nu,u) := \mathbb{1}_{h^{-1}(y)^c}(z) \int_I \int_U \lambda(x,\mathfrak{u},\mathrm{d}z)\, u(\mathrm{d}\mathfrak{u})\, \nu(\mathrm{d}x), \quad \nu \in \Delta_y,\, u \in \mathcal{P}(U) \tag{4.1.15}$$

and the quantity $\pi^{\mu,\mathbf{u}}_{\tau_n^-(\omega)}(\omega)$ *is defined as*

$$\pi^{\mu,\mathbf{u}}_{\tau_n^-(\omega)}(\omega) := \lim_{t\to\tau_n(\omega)^-} \pi^{\mu,\mathbf{u}}_t(\omega), \quad \textit{on } \{\omega \in \Omega\colon \tau_n(\omega) < +\infty\}.$$

Then, $\pi^{\mu,\mathbf{u}}$ *is* $(\mathcal{Y}^\circ_t)_{t\geqslant 0}$*–adapted and is a modification of the filtering process, i. e.*

$$\pi^{\mu,\mathbf{u}}_t(A) = \mathrm{P}^{\mathbf{u}}_\mu(X_t \in A \mid \mathcal{Y}^{\mu,\mathbf{u}}_t), \quad \mathrm{P}^{\mathbf{u}}_\mu\textit{–a.s.},\, t \geqslant 0,\, A \in \mathcal{I}.$$

Remark 4.1.2. Thanks to the structure of admissible controls shown in (4.1.2) we can write (4.1.14) as

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\pi^{\mu,\mathbf{u}}_t = \displaystyle\int_U F(\pi^{\mu,\mathbf{u}}_t, \mathfrak{u})\, u_n(t, Y_0, \dots, \tau_n, Y_{\tau_n}\,; \mathrm{d}\mathfrak{u}), \quad t \in [\tau_n, \tau_{n+1}),\, n \in \mathbb{N}_0 \\ \pi^{\mu,\mathbf{u}}_0 = H_{Y_0}[\mu] \\ \pi^{\mu,\mathbf{u}}_{\tau_n} = H_{Y_{\tau_n}}\big[\Lambda\big(\pi^{\mu,\mathbf{u}}_{\tau_n^-}, u_{n-1}(\tau_n^-, Y_0, \dots, \tau_{n-1}, Y_{\tau_{n-1}})\big)\big],\, n \in \mathbb{N} \end{cases}$$

We can characterize the filtering process as a *Piecewise Deterministic Process* (PDP). Its characteristic triple (F, r, R) is given by the controlled vector field F defined in (4.1.13), a controlled jump rate function $r\colon \Delta_e \times U \to [0, +\infty)$ and a controlled stochastic kernel R, i. e. a probability transition kernel from $(\Delta_e \times U, \mathcal{B}(\Delta_e) \otimes \mathcal{U})$ to $(\Delta_e, \mathcal{B}(\Delta_e))$. We define the functions in this triple as

$$\begin{aligned} F(\nu, u) &:= F_y(\nu, u) = \mathcal{B}_y^u \nu - \nu\, \mathcal{B}_y^u \nu(I), \quad \nu \in \Delta_y,\, u \in U \\ r(\nu, u) &:= -\mathcal{B}_y^u \nu(I) = \int_I \lambda\big(x, u, h^{-1}(y)^c\big)\, \nu(\mathrm{d}x), \quad \nu \in \Delta_y,\, u \in U \\ R(\nu, u, D) &:= \int_O \mathbb{1}_D\big(H_\upsilon[\Lambda(\nu, u)]\big)\, \rho(\nu, \mathrm{d}\upsilon), \quad \nu \in \Delta_y,\, D \in \mathcal{B}(\Delta_e),\, u \in U \end{aligned} \tag{4.1.16}$$

where ρ is a transition probability from $\big(\Delta_e \times U, \mathcal{B}(\Delta_e) \otimes \mathcal{U}\big)$ into $(O, \mathcal{O})$ defined for all $\nu \in \Delta_y,\, u \in U$ and all $B \in \mathcal{O}$ as

$$\rho(\nu, u, B) := \begin{cases} \dfrac{1}{r(\nu, u)} \displaystyle\int_I \lambda\big(x, u, h^{-1}(B \setminus \{y\})\big)\, \nu(\mathrm{d}x), & \text{if } r(\nu, u) > 0 \\ q_y(B), & \text{if } r(\nu, u) = 0 \end{cases} \tag{4.1.17}$$

where $(q_y)_{y \in O}$ is a family of probability measures, each concentrated on the level set $h^{-1}(y),\, y \in O$, whose exact values are irrelevant.

Since for any given $\nu \in \Delta_y$ and $u \in U$ the probability $\rho(\nu, u, \cdot)$ is concentrated on the set $O \setminus \{y\}$, the probability $R(\nu, u, \cdot)$ is concentrated on $\Delta_e \setminus \Delta_y$.

It is important to notice that under Assumption 4.1.1 r is Lipschitz continuous uniformly in u, i. e.

$$\sup_{u \in U} |r(\rho, u) - r(\vartheta, u)| \leqslant L_r \|\rho - \vartheta\|_{TV}, \quad \text{for all } \rho, \vartheta \in \Delta_y,\, y \in O, \tag{4.1.18}$$

for some constant $L_r > 0$. We also have that for some $C_r > 0$

$$\sup_{(\rho, u) \in \Delta_e \times U} |r(\rho, u)| \leqslant C_r. \tag{4.1.19}$$

We can now state the characterization of the filtering process as a PDP. This is a generalization of Theorem 2.2.7. However, notice that we lose the Markov property with respect to the natural filtration of the observed process, a loss due to the structure of admissible controls.

Theorem 4.1.4. *For every $\nu \in \Delta_e$ and all $\mathbf{u} \in \mathcal{U}_{ad}$ the filtering process $\pi^{\nu,\mathbf{u}} = (\pi_t^{\nu,\mathbf{u}})_{t \geqslant 0}$ defined on the probability space $(\Omega, \mathcal{X}^\circ, \mathrm{P}_\nu^{\mathbf{u}})$ and taking values in Δ_e is a controlled* Piecewise Deterministic Process *with respect to the triple (F, r, R) defined in (4.1.16) and with starting point ν.*

More specifically, we have that for all $n \in \mathbb{N}_0$ and $\mathrm{P}_\nu^{\mathbf{u}}$–a.s.

$$\pi_t^{\nu,\mathbf{u}} = \phi^{u_n}_{\pi_{\tau_n}^{\nu,\mathbf{u}}}(t - \tau_n), \quad \text{on } \{\tau_n < +\infty\},\, t \in [\tau_n, \tau_{n+1}) \tag{4.1.20}$$

$$\begin{aligned} &\mathrm{P}_\nu^{\mathbf{u}}(\tau_{n+1} - \tau_n > t,\, \tau_n < +\infty \mid \mathcal{Y}_{\tau_n}^{\nu,\mathbf{u}}) = \\ &\mathbb{1}_{\tau_n < +\infty} \exp\bigg\{-\int_0^t \int_U r\big(\phi^{u_n(\cdot + \tau_n)}_{\pi_{\tau_n}^{\nu,\mathbf{u}}}(s), \mathfrak{u}\big)\, u_n(s + \tau_n\, ; \mathrm{d}\mathfrak{u})\, \mathrm{d}s\bigg\}, \quad t \geqslant 0 \end{aligned} \tag{4.1.21}$$

$$\mathrm{P}^{\mathbf{u}}_{\nu}(\pi^{\nu,\mathbf{u}}_{\tau_{n+1}} \in D,\, \tau_{n+1} < +\infty \mid \mathcal{Y}^{\nu,\mathbf{u}}_{\tau^-_{n+1}}) =$$
$$\mathbb{1}_{\tau_{n+1}<+\infty} \int_U R\big(\phi^{u_n(\cdot\,+\tau_n)}_{\pi^{\nu,\mathbf{u}}_{\tau_n}}(\tau^-_{n+1} - \tau_n), \mathrm{u}; D\big)\, u_n(\tau^-_{n+1}\,; \mathrm{du}), \quad D \in \mathcal{B}(\Delta_e) \tag{4.1.22}$$

where, for each $n \in \mathbb{N}_0$, $\phi^{u_n}_{\pi^{\nu,\mathbf{u}}_{\tau_n}}$ *is the flow starting from* $\pi^{\nu,\mathbf{u}}_{\tau_n}$ *and determined by the controlled vector field* F *under the action of the control function* $u_n(\cdot, Y_0, \dots, \tau_n, Y_{\tau_n})$.

Remark 4.1.3. We point out once more (cfr. Remark 3.1.3) that equations (4.1.20)–(4.1.22) are changed with respect to the standard formulation with piecewise open-loop controls. This is necessary because of the chosen type of control processes, i. e. the class $\mathcal{U}_{ad}$. Another difference with respect to the usual definition of a PDP is the absence in our model of a boundary behavior of the PDP, i. e. the specification of a transition kernel giving the post-jump position of the process in case it touches the boundary.

Clearly enough, also in the infinite dimensional setting we do not have that the transition measure R is a Feller kernel. Analogously to what we did in Proposition 3.1.4, we can state a weaker form of this property.

Proposition 4.1.5. *Let Assumption 4.1.1 hold. Then for every bounded and continuous function* $w\colon \Delta_e \to \mathbb{R}$ *and* $u \in U$ *the function* $\rho \mapsto r(\rho, u) \int_{\Delta_e} w(p) R(\rho, u; \mathrm{d}p)$ *is bounded and continuous on* Δ_e.

Proposition 4.1.5 establishes continuity of the map therein considered for each fixed control parameter $u \in U$, but this is not enough for our purposes. In the following Sections we will invoke whenever needed the following condition.

Assumption 4.1.3. For every bounded and continuous function $w\colon \Delta_e \to \mathbb{R}$ the map $\rho \mapsto r(\rho, u) \int_{\Delta_e} w(p) R(\rho, u; \mathrm{d}p)$ is continuous on Δ_e uniformly in $u \in U$.

Remark 4.1.4. It is important to notice that Assumption 4.1.3 is satisfied in some situations, e. g. in the Markov chain setting studied in Chapter 3, as Proposition 3.1.4 shows. It can be proved that it holds also in the case where the rate transition measure λ is absolutely continuous with respect to some fixed measure on $(I, \mathcal{I})$ and the resulting transition density is uniformly bounded from above and bounded away from zero.

To conclude this Section, we make some concluding remarks on the cost functional J defined in (4.1.4). Let us fix some more notation, first. Let us consider a measurable space $(E, \mathcal{E})$ and for each $\nu \in \mathcal{P}(I)$ and all $\varphi\colon I \times E \to \mathbb{R}$ bounded and measurable, let us denote by $\nu(\varphi;\, \cdot)\colon E \to \mathbb{R}$ the following function

$$\nu(\varphi;\, e) \coloneqq \int_I \varphi(x, e)\, \nu(\mathrm{d}x), \quad e \in E.$$

Since the control processes $\mathbf{u}$ are $(\mathcal{Y}^\circ_t)_{t \geqslant 0}$ – predictable and we know that the filtering process $\pi^{\mu,\mathbf{u}}$ provides us with the conditional law of X_t given $\mathcal{Y}^{\mu,\mathbf{u}}_t$, for all $t \geqslant 0$, an easy application of the Fubini-Tonelli Theorem and of the freezing lemma shows that

$$J(\mu, \mathbf{u}) = \mathrm{E}^{\mathbf{u}}_{\mu}\left[\int_0^\infty e^{-\beta t} \int_U \pi^{\mu,\mathbf{u}}_t(f;\, \mathrm{u})\, u_t(\mathrm{du})\, \mathrm{d}t\right]. \tag{4.1.23}$$

In this way, our control problem depends on completely observable quantities, since the new state process $\pi^{\mu,\mathbf{u}}$ and the control process $\mathbf{u}$ depend on the observed process

Y. The optimal control problem for the pure jump Markov process X becomes an optimal control problem for the PDP $\pi^{\mu,\mathbf{u}}$. As we did at the end of Section 3.1, we can exploit the structure of admissible controls $\mathbf{u} = (u_n)_{n\in\mathbb{N}_0} \in \mathcal{U}_{ad}$ to rewrite J as a discrete-time cost functional.

$$J(\mu, \mathbf{u}) = \mathrm{E}_\mu^{\mathbf{u}}\Big[\sum_{n=0}^{+\infty} \int_{\tau_n}^{\tau_{n+1}} e^{-\beta t} \int_U \pi_t^{\mu,\mathbf{u}}(f;\, \mathfrak{u})\, u_n(t\,; \mathrm{d}\mathfrak{u})\, \mathrm{d}t\Big]$$
$$= \mathrm{E}_\mu^{\mathbf{u}}\Big[\sum_{n=0}^{+\infty} e^{-\beta\tau_n} \int_0^{+\infty} e^{-\beta t} \chi_{\pi_{\tau_n}^{\mu,\mathbf{u}}}^{u_n(\cdot\,+\,\tau_n)}(t) \int_U \phi_{\pi_{\tau_n}^{\mu,\mathbf{u}}}^{u_n(\cdot\,+\,\tau_n)}(f;\, \mathfrak{u}, t)\, u_n(t + \tau_n\,; \mathrm{d}\mathfrak{u})\, \mathrm{d}t\Big]$$
$$= \mathrm{E}_\mu^{\mathbf{u}}\Big[\sum_{n=0}^{+\infty} e^{-\beta\tau_n} g\big(\pi_{\tau_n}^{\mu,\mathbf{u}}, u_n(\cdot + \tau_n, Y_0, \tau_1, Y_{\tau_1}, \dots, \tau_n, Y_{\tau_n})\big)\Big] \tag{4.1.24}$$

where the function g (that will be defined precisely in Section 4.2) represents the double integral appearing in the second line and $\chi_{\pi_{\tau_n}^{\mu,\mathbf{u}}}^{u_n(\cdot\,+\,\tau_n)}$ is the survival distribution appearing in (4.1.21).

The reformulated problem does not fit in the framework of a classical discrete-time optimal control problem (see e. g. [13]) for the same reasons already expressed at the end of Section 3.1. The problem should be based only on the discrete-time process given by the pairs of jump times and jump locations of the filtering process $\pi^{\mu,\mathbf{u}}$ (notice that in (4.1.24) also the process Y appears) which, in turn, should not depend on the initial law of the process X and on the control trajectory $\mathbf{u}$. Moreover, the class of admissible controls $\mathcal{U}_{ad}$ is not adequate for a discrete-time problem, since its policies should be functions depending at each time step exclusively on the past trajectory of a discrete-time process (in this case, the one based on the filtering process, as explained above). It is immediate to see that this is not the case for (4.1.24), since each of the functions u_n depends on a continuous-time variable and on the positions of the process Y.

The solution to these issues is represented by a separated discrete-time control problem based on the filtering process and related to the present one.

4.2 The separated optimal control problem

In this Section we will reformulate the original optimal control problem into a discrete-time one based on the filtering process. Also in the infinite dimensional setting of this Chapter, such a reformulation will fall in the framework of [13]. Therefore, we will be able to use the same results used in Section 3.2 to study the value function V defined in (4.1.5). We will prove the equivalence between the original control problem and the separated one. In particular, we will show that the value function V can be indirectly characterized by its counterpart in the separated problem, that will be studied in the next Section.

Analogously to what we did in Section 3.2, we choose as *action space* the space of *relaxed controls*

$$\mathcal{M} = \{m \colon [0, +\infty) \to \mathcal{P}(U),\ \text{measurable}\}. \tag{4.2.1}$$

We recall that $\mathcal{M}$ is compact under the *Young topology* (see e. g. [32]). The set of *ordinary controls*

$$A = \{\alpha \colon [0, +\infty) \to U,\ \text{measurable}\} \tag{4.2.2}$$

can be identified as a subset of $\mathcal{M}$ via the function $t \mapsto \delta_{\alpha(t)}$, $\alpha \in A$, where δ_u denotes the Dirac probability measure concentrated at the point $u \in U$. We recall that, thanks to [67, Lemma 1], A is a *Borel space* when endowed with the coarsest σ-algebra such that the maps

$$\alpha \mapsto \int_0^{+\infty} e^{-t}\psi(t, \alpha(t))\,\mathrm{d}t$$

are measurable for all $\psi\colon [0, +\infty) \times U \to \mathbb{R}$, bounded and measurable. The class of *admissible policies* $\mathcal{A}_{ad}$ for the discrete-time optimal control problem is given by

$$\mathcal{A}_{ad} = \{\mathbf{a} = (a_n)_{n \in \bar{\mathbb{N}}_0}, a_n\colon \Delta_e \times \big((0, +\infty) \times \Delta_e\big)^n \to \mathcal{M} \text{ measurable } \forall n \in \bar{\mathbb{N}}_0\}. \tag{4.2.3}$$

We introduce now the separated optimal control problem. In this problem the state to be controlled is represented by the filtering process, therefore we put ourselves in a canonical framework for this process. The construction is exactly the same provided in Section 3.2 and we repeat it here for the reader's convenience.

- $\bar{\Omega} = \{\bar{\omega}\colon [0, +\infty) \to \Delta_e,\ \text{cádlág}\}$ denotes the canonical space for Δ_e – valued PDPs. We define $\bar{\pi}_t(\bar{\omega}) = \bar{\omega}(t)$, for $\bar{\omega} \in \bar{\Omega}$, $t \geqslant 0$, and

$$\begin{aligned}
\bar{\tau}_0(\bar{\omega}) &= 0,\\
\bar{\tau}_n(\bar{\omega}) &= \inf\{t > \bar{\tau}_{n-1}(\bar{\omega}) \text{ s.t. } \bar{\pi}_t(\bar{\omega}) \neq \bar{\pi}_{t-}(\bar{\omega})\}, \quad n \in \mathbb{N},\\
\bar{\tau}_\infty(\bar{\omega}) &= \lim_{n\to\infty} \bar{\tau}_n(\bar{\omega}).
\end{aligned}$$

- The family of σ-algebras $(\bar{\mathcal{F}}_t^\circ)_{t \geqslant 0}$ given by

$$\bar{\mathcal{F}}_t^\circ = \sigma(\bar{\pi}_s, 0 \leqslant s \leqslant t), \quad \bar{\mathcal{F}}^\circ = \sigma(\bar{\pi}_s, s \geqslant 0),$$

  is the natural filtration of the process $\bar{\pi} = (\bar{\pi}_t)_{t \geqslant 0}$.

- For every $\nu \in \Delta_e$ and all $\mathbf{a} \in \mathcal{A}_{ad}$ we denote by $\bar{\mathrm{P}}_\nu^{\mathbf{a}}$ the probability measure on $(\bar{\Omega}, \bar{\mathcal{F}}^\circ)$ such that the process $\bar{\pi}$ is a PDP, starting from the point ν and with characteristic triple (F, r, R). With this, we mean that for all $n \in \mathbb{N}_0$ and $\bar{\mathrm{P}}_\nu^{\mathbf{a}}$–a.s.

$$\bar{\pi}_t = \phi^{a_n}_{\bar{\pi}_{\bar{\tau}_n}}(t - \bar{\tau}_n), \quad \text{on } \{\bar{\tau}_n < +\infty\},\ t \in [\bar{\tau}_n, \bar{\tau}_{n+1}). \tag{4.2.4}$$

$$\begin{aligned}
&\bar{\mathrm{P}}_\nu^{\mathbf{a}}(\bar{\tau}_{n+1} - \bar{\tau}_n > t,\ \bar{\tau}_n < +\infty \mid \bar{\mathcal{F}}^\circ_{\bar{\tau}_n}) =\\
&\qquad \mathbb{1}_{\bar{\tau}_n < +\infty} \exp\left\{-\int_0^t \int_U r(\phi^{a_n}_{\bar{\pi}_{\bar{\tau}_n}}(t), \mathrm{u})\, a_n(s\,; \mathrm{du})\, \mathrm{d}s\right\}, \quad t \geqslant 0.
\end{aligned} \tag{4.2.5}$$

$$\begin{aligned}
&\bar{\mathrm{P}}_\nu^{\mathbf{a}}(\bar{\pi}_{\bar{\tau}_{n+1}} \in D,\ \bar{\tau}_{n+1} < +\infty \mid \bar{\mathcal{F}}^\circ_{\bar{\tau}^-_{n+1}}) =\\
&\mathbb{1}_{\bar{\tau}_{n+1} < +\infty} \int_U R(\phi^{a_n}_{\bar{\pi}_{\bar{\tau}_n}}(\bar{\tau}^-_{n+1} - \bar{\tau}_n), \mathrm{u}; D)\, a_n(\bar{\tau}^-_{n+1} - \bar{\tau}_n\,; \mathrm{du}), \quad D \in \mathcal{B}(\Delta_e).
\end{aligned} \tag{4.2.6}$$

  where we simplified the notation by indicating $a_n = a_n(\bar{\pi}_0, \dots, \bar{\tau}_n, \bar{\pi}_{\bar{\tau}_n})$ and, for each $n \in \mathbb{N}_0$, we denoted by $\phi^{a_n}_{\bar{\pi}_{\bar{\tau}_n}}$ the flow starting from $\bar{\pi}_{\bar{\tau}_n}$ and determined by the controlled vector field F under the action of the relaxed control $a_n(\bar{\pi}_0, \dots, \bar{\tau}_n, \bar{\pi}_{\bar{\tau}_n})$. We recall that the probability measure $\bar{\mathrm{P}}_\nu^{\mathbf{a}}$ always exists by the canonical construction of a PDP (see Section 1.2).

- For every $Q \in \mathcal{P}(\Delta_e)$ and every $\mathbf{a} \in \mathcal{A}_{ad}$ we define a probability $\bar{\mathrm{P}}^{\mathbf{a}}_Q$ on $(\bar{\Omega}, \bar{\mathcal{F}}^\circ)$ by $\bar{\mathrm{P}}^{\mathbf{a}}_Q(C) = \int_{\Delta_e} \bar{\mathrm{P}}^{\mathbf{a}}_\nu(C)\, Q(\mathrm{d}\nu)$ for $C \in \bar{\mathcal{F}}^\circ$. This means that Q is the initial distribution of $\bar{\pi}$ under $\bar{\mathrm{P}}^{\mathbf{a}}_Q$.

- Let $\bar{\mathcal{F}}^{Q,\mathbf{a}}$ be the $\bar{\mathrm{P}}^{\mathbf{a}}_Q$-completion of $\bar{\mathcal{F}}^\circ$. We still denote by $\bar{\mathrm{P}}^{\mathbf{a}}_Q$ the measure naturally extended to this new σ-algebra. Let $\bar{\mathcal{Z}}^{Q,\mathbf{a}}$ be the family of sets in $\bar{\mathcal{F}}^{Q,\mathbf{a}}$ with zero $\bar{\mathrm{P}}^{\mathbf{a}}_Q$-probability and define

$$\bar{\mathcal{F}}^{Q,\mathbf{a}}_t = \sigma(\bar{\mathcal{F}}^\circ_t \cup \bar{\mathcal{Z}}^{Q,\mathbf{a}}), \quad \bar{\mathcal{F}}_t = \bigcap_{\substack{Q \in \mathcal{P}(\Delta_e) \\ \mathbf{a} \in \mathcal{A}_{ad}}} \bar{\mathcal{F}}^{Q,\mathbf{a}}_t, \quad t \geqslant 0.$$

$(\bar{\mathcal{F}}_t)_{t\geqslant 0}$ is called the *natural completed filtration* of $\bar{\pi}$. By a slight generalization of Theorem 1.2.2 it is right-continuous.

The PDP $(\bar{\Omega}, \bar{\mathcal{F}}, (\bar{\mathcal{F}}_t)_{t\geqslant 0}, (\bar{\pi}_t)_{t\geqslant 0}, (\bar{\mathrm{P}}^{\mathbf{a}}_\nu)^{\mathbf{a}\in\mathcal{A}_{ad}}_{\nu\in\Delta_e})$ constructed as above admits the controlled characteristic triple (F, r, R) defined in (4.1.16). To simplify the notation, let us introduce the function χ^m_ρ, depending on $\rho \in \Delta_e$ and $m \in \mathcal{M}$, given by

$$\chi^m_\rho(t) = \exp\biggl\{-\int_0^t \int_U r(\phi^m_\rho(s), \mathrm{u})\, m(s\,; \mathrm{du})\, \mathrm{d}s\biggr\}, \quad t \geqslant 0. \tag{4.2.7}$$

In this way, we can write (3.2.4) as

$$\bar{\mathrm{P}}^{\mathbf{a}}_\nu(\bar{\tau}_{n+1} - \bar{\tau}_n > t \mid \bar{\mathcal{F}}^\circ_{\bar{\tau}_n}) = \chi^{a_n}_\nu(t), \quad t \geqslant 0, \text{ on } \{\bar{\tau}_n < +\infty\}.$$

Notice that χ^m_ρ solves the ODE

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} z(t) = -z(t) \displaystyle\int_U r(\phi^m_\rho(t), \mathrm{u})\, m(t\,; \mathrm{du}), & t \geqslant 0 \\ z(0) = 1 \end{cases} \tag{4.2.8}$$

The observed process $\bar{Y}$ can be defined on $\bar{\Omega}$ as follows. Let us introduce the measurable function $\mathrm{proj}_{\mathrm{Y}} \colon \Delta_e \to O$ given by

$$\mathrm{proj}_{\mathrm{Y}}(p) = y, \quad \text{if } p \in \Delta_y, \text{ for some } y \in O$$

and set

$$\bar{Y}_t(\bar{\omega}) = \begin{cases} \mathrm{proj}_{\mathrm{Y}}(\bar{\pi}_0(\bar{\omega})), & t \in \bigl[0, \bar{\tau}_1(\bar{\omega})\bigr) \\ \mathrm{proj}_{\mathrm{Y}}(\bar{\pi}_{\bar{\tau}_n(\bar{\omega})}(\bar{\omega})), & t \in \bigl[\bar{\tau}_n(\bar{\omega}), \bar{\tau}_{n+1}(\bar{\omega})\bigr),\ n \in \mathbb{N},\ \bar{\tau}_n(\bar{\omega}) < +\infty \\ o_\infty, & t \in \bigl[\bar{\tau}_\infty(\bar{\omega}), +\infty),\ \bar{\tau}_\infty(\bar{\omega}) < +\infty \end{cases},$$

where $o_\infty \in O$ is an arbitrary state, that is irrelevant to specify, since under Assumption 4.1.1 for each fixed $\nu \in \Delta_e$ and $\mathbf{a} \in \mathcal{A}_{ad}$ we have that $\bar{\tau}_\infty = +\infty$, $\bar{\mathrm{P}}^{\mathbf{a}}_\nu$–a.s.. In other words, the observed process is $\bar{\mathrm{P}}^{\mathbf{a}}_\nu$–a.s. non explosive.

Next, we define the cost functional associated to the separated optimal control problem. Let us recall that, in the end, we want this problem to be equivalent to the original optimal control problem for the unobserved process X. Consequently we define the new cost functional $\bar{J}$ in analogy to the form of the original cost functional J shown in (4.1.23). To this purpose, let us introduce the discrete-time one-stage cost function $g\colon \Delta_e \times \mathcal{M} \to \mathbb{R}$ as

$$g(\nu, m) = \int_0^{+\infty} e^{-\beta t} \chi_\nu^m(t) \int_U \phi_\nu^m(f;\, \mathrm{u}, t)\, m(t\,; \mathrm{d}\mathrm{u})\, \mathrm{d}t. \tag{4.2.9}$$

We define the cost functional $\bar{J}$ as

$$\bar{J}(\nu, \mathbf{a}) = \bar{\mathrm{E}}_\nu^{\mathbf{a}}\Big[\sum_{n=0}^{+\infty} e^{-\beta \bar{\tau}_n} g\big(\bar{\pi}_{\bar{\tau}_n}, a_n(\bar{\pi}_{\bar{\tau}_0}, \dots, \bar{\tau}_n, \bar{\pi}_{\bar{\tau}_n})\big)\Big], \quad \nu \in \Delta_e,\ \mathbf{a} \in \mathcal{A}_{ad}. \tag{4.2.10}$$

Finally, we define the value function of the separated problem as

$$v(\nu) = \inf_{\mathbf{a} \in \mathcal{A}_{ad}} \bar{J}(\nu, \mathbf{a}). \tag{4.2.11}$$

As we anticipated earlier, we need to establish a connection between the cost functionals J and $\bar{J}$, respectively given in (4.1.23) and (4.2.10). In a similar manner to Theorem 3.2.1, this link will be given by constructing corresponding admissible controls in $\mathcal{U}_{ad}$ and admissible policies in $\mathcal{A}_{ad}$.

Theorem 4.2.1. *Fix $\mu \in \mathcal{P}(I)$ and let $Q \in \mathcal{P}(\Delta_e)$ the Borel probability measure on Δ_e concentrated on the set $\bigcup_{y \in O}\{H_y[\mu]\}$, defined as*

$$Q \coloneqq \mu \circ h^{-1} \circ \mathcal{H}^{-1}, \quad \mathcal{H}(y) \coloneqq H_y[\mu]. \tag{4.2.12}$$

For all $\mu \in \mathcal{P}(I)$ and all $\mathbf{u} \in \mathcal{U}_{ad}$ there exists an admissible policy $\mathbf{a} \in \mathcal{A}_{ad}$ such that the laws of $\pi^{\mu,\mathbf{u}}$ under $\mathrm{P}_\mu^{\mathbf{u}}$ and of $\bar{\pi}$ under $\bar{\mathrm{P}}_Q^{\mathbf{a}}$ are the same. Moreover, for such an admissible policy

$$J(\mu, \mathbf{u}) = \int_{\Delta_e} \bar{J}(\nu, \mathbf{a})\, Q(\mathrm{d}\nu) = \int_O \bar{J}(H_y[\mu], \mathbf{a})\, \mu \circ h^{-1}(\mathrm{d}y). \tag{4.2.13}$$

Viceversa, for all $\mu \in \mathcal{P}(I)$ and all $\mathbf{a} = (a_n)_{n \in \mathbb{N}} \in \mathcal{A}_{ad}$ there exists an admissible control $\mathbf{u} \in \mathcal{U}_{ad}$ such that the same conclusions hold.

Proof. Let us start from the first part of the theorem. Given an admissible control $\mathbf{u} \in \mathcal{U}_{ad}$ we are able to construct a corresponding admissible policy in the same way as did in the proof of Theorem 3.2.1. Let us define the functions $a_n\colon \Delta_e \times \big((0, +\infty) \times \Delta_e\big)^n \to \mathcal{M}$ as

$$a_n(\nu_0, \dots, s_n, \nu_n)(t\,; \mathrm{du}) = u_n\big(t +\ s_n, \mathrm{proj}_{\mathrm{Y}}(\nu_0), \dots, s_n, \mathrm{proj}_{\mathrm{Y}}(\nu_n)\,; \mathrm{du}\big)$$

for all possible sequences $(\nu_i)_{i=0}^n \subset \Delta_e$ and $(s_i)_{i=1}^n \subset (0, +\infty)$.

Thanks to the fact that $\mathrm{proj}_{\mathrm{Y}}$ is Borel-measurable and that $\mathcal{M}$ is a Borel space, we can apply [67, Lemma 3(i)] and it follows that each function a_n is measurable. Therefore we have that $\mathbf{a} = (a_n)_{n \in \mathbb{N}_0} \in \mathcal{A}_{ad}$.

The laws of $\pi^{\mu,\mathbf{u}}$ under $\mathrm{P}_\mu^{\mathbf{u}}$ and $\bar{\pi}$ under $\bar{\mathrm{P}}_Q^{\mathbf{a}}$ are determined respectively by the finite-dimensional distributions of the stochastic processes $\{\pi_0^{\mu,\mathbf{u}}, \tau_1, \pi_{\tau_1}^{\mu,\mathbf{u}}, \dots\}$ and $\{\bar{\pi}_0, \bar{\tau}_1, \bar{\pi}_{\bar{\tau}_1}, \dots\}$ and by the flows associated to the controlled vector fields $F^{\mathbf{u}}$ and $F^{\mathbf{a}}$. These laws, in turn, can be expressed via the initial distributions of $\pi_0^{\mu,\mathbf{u}}$ and $\bar{\pi}_0$

and the conditional distributions of the sojourn times and post-jump locations, i. e. for $t \geqslant 0$, $D \in \mathcal{B}(\Delta_e)$ and $n \in \mathbb{N}$ the quantities

$$\mathrm{P}^{\mathbf{u}}_{\mu}(\tau_n - \tau_{n-1} > t,\, \tau_{n-1} < +\infty \mid \pi^{\mu,\mathbf{u}}_0, \dots, \tau_{n-1}, \pi^{\mu,\mathbf{u}}_{\tau_{n-1}}); \tag{4.2.14}$$

$$\bar{\mathrm{P}}^{\mathbf{a}}_{Q}(\bar{\tau}_n - \bar{\tau}_{n-1} > t,\, \bar{\tau}_{n-1} < +\infty \mid \bar{\pi}_0, \dots, \bar{\tau}_{n-1}, \bar{\pi}_{\bar{\tau}_{n-1}}); \tag{4.2.15}$$

$$\mathrm{P}^{\mathbf{u}}_{\mu}(\pi^{\mu,\mathbf{u}}_{\tau_n} \in D,\, \tau_n < +\infty \mid \pi^{\mu,\mathbf{u}}_0, \dots, \pi^{\mu,\mathbf{u}}_{\tau_{n-1}}, \tau_n); \tag{4.2.16}$$

$$\bar{\mathrm{P}}^{\mathbf{a}}_{Q}(\bar{\pi}_{\bar{\tau}_n} \in D,\, \bar{\tau}_n < +\infty \mid \bar{\pi}_0, \dots, \bar{\pi}_{\bar{\tau}_{n-1}}, \bar{\tau}_n). \tag{4.2.17}$$

It suffices to prove that under the two different probability measures $\mathrm{P}^{\mathbf{u}}_{\mu}$ and $\bar{\mathrm{P}}^{\mathbf{a}}_{Q}$ the initial laws of $\pi^{\mu,\mathbf{u}}_0$ and $\bar{\pi}_0$ are equal, since the proof of equivalence between (4.2.14)–(4.2.15) and (4.2.16)–(4.2.17) is identical to that provided in the proof of Theorem 3.2.1. This is immediate, as for fixed $D \in \mathcal{B}(\Delta_e)$ we have that

$$\begin{aligned}
&\mathrm{P}^{\mathbf{u}}_{\mu}(\pi^{\mu,\mathbf{u}}_0 \in D) = \mathrm{P}^{\mathbf{u}}_{\mu}(H_{Y_0}[\mu] \in D) = \mathrm{P}^{\mathbf{u}}_{\mu}\big(Y_0 \in \mathcal{H}^{-1}(D)\big) = \\
&\mathrm{P}^{\mathbf{u}}_{\mu}\big(X_0 \in h^{-1}(\mathcal{H}^{-1}(D))\big) = \mu\big(h^{-1}(\mathcal{H}^{-1}(D))\big) = Q(D)
\end{aligned}$$

while $\bar{\mathrm{P}}^{\mathbf{a}}_{Q}(\bar{\pi}_0 \in D) = Q(D)$, by definition of $\bar{\mathrm{P}}^{\mathbf{a}}_{Q}$.

We are left to prove (4.2.13). Fix $\mu \in \mathcal{P}(I)$ and $\mathbf{u} \in \mathcal{U}_{ad}$ with corresponding $\mathbf{a} \in \mathcal{A}_{ad}$ defined as above. Let us define the function $\Phi\colon \bar{\Omega} \to \mathbb{R}$ as

$$\begin{aligned}
\Phi(\bar{\omega}) &= \sum_{n=0}^{+\infty} e^{-\beta\bar{\tau}_n(\bar{\omega})} g\big(\bar{\pi}_{\bar{\tau}_n(\bar{\omega})}(\bar{\omega}), a_n(\dots, \bar{\tau}_n(\bar{\omega}), \bar{\pi}_{\tau_n(\bar{\omega})}(\bar{\omega})\big) \\
&= \sum_{n=0}^{+\infty} e^{-\beta\bar{\tau}_n(\bar{\omega})} g\big(\bar{\pi}_{\bar{\tau}_n(\bar{\omega})}(\bar{\omega}), u_n(\cdot + \bar{\tau}_n(\bar{\omega}), \dots, \bar{\tau}_n(\bar{\omega}), \mathrm{proj}_{\mathrm{Y}}(\bar{\pi}_{\tau_n(\bar{\omega})}(\bar{\omega}))\big).
\end{aligned}$$

Thanks to Assumptions 4.1.1 and 4.1.2 this function is bounded. Since for each $n \in \mathbb{N}_0$ the functions a_n (equivalently u_n) are measurable it is also $\bar{\mathcal{F}}$-measurable.

Now, take $\bar{\omega} = \pi^{\mu,\mathbf{u}}(\omega)$, $\omega \in \Omega$. It is clear that for all $t \geqslant 0$ we have $\bar{\pi}_t(\bar{\omega}) = \bar{\omega}(t) = \pi^{\mu,\mathbf{u}}_t(\omega)$ and also, by definition of the jump times $(\bar{\tau}_n)_{n\in\mathbb{N}_0}$, that $\bar{\tau}_n(\bar{\omega}) = \tau_n(\omega)$, $\mathrm{P}^{\mathbf{u}}_{\mu}$-a.s.. Then, we get that $\mathrm{P}^{\mathbf{u}}_{\mu}$-a.s.

$$\begin{aligned}
\Phi(\pi^{\mu,\mathbf{u}}(\omega)) &= \sum_{n=0}^{+\infty} e^{-\beta\tau_n(\omega)} g\big(\pi^{\mu,\mathbf{u}}_{\tau_n(\omega)}(\omega), u_n(\cdot + \tau_n(\omega)\big) \\
&= \sum_{n=0}^{+\infty} e^{-\beta\tau_n(\omega)} g\big(\pi^{\mu,\mathbf{u}}_{\tau_n(\omega)}(\omega), u_n(\cdot + \tau_n(\omega), \dots, \tau_n(\omega), Y_{\tau_n(\omega)}(\omega))\big),
\end{aligned}$$

hence, comparing this result with (4.1.24) and applying the Fubini-Tonelli Theorem we obtain

$$\begin{aligned}
J(\mu, \mathbf{u}) &= \int_{\Omega} \Phi(\pi^{\mu,\mathbf{u}}(\omega))\mathrm{P}^{\mathbf{u}}_{\mu}(\mathrm{d}\omega) = \int_{\bar{\Omega}} \Phi(\bar{\omega})\bar{\mathrm{P}}^{\mathbf{a}}_{Q}(\mathrm{d}\bar{\omega}) = \int_{\bar{\Omega}} \Phi(\bar{\omega})\int_{\Delta_e} \bar{\mathrm{P}}^{\mathbf{a}}_{\nu}(\mathrm{d}\bar{\omega})\, Q(\mathrm{d}\nu) \\
&= \int_{\Delta_e} \left\{\int_{\bar{\Omega}} \Phi(\bar{\omega})\bar{\mathrm{P}}^{\mathbf{a}}_{\nu}(\mathrm{d}\bar{\omega})\right\} Q(\mathrm{d}\nu) = \int_{\Delta_e} \bar{J}(\nu, \mathbf{a})\, Q(\mathrm{d}\nu)
\end{aligned}$$

by definition of the functional $\bar{J}$.

To prove the second part of the theorem, fix $\mu \in \mathcal{P}(I)$ and $\mathbf{a} = (a_n)_{n\in\mathbb{N}} \in \mathcal{A}_{ad}$. Also in this case, the construction of a corresponding admissible control is analogous to that given in the proof of Theorem 3.2.1. We define, for each possible sequence $b_0, b_1, \dots \in O$ and $s_1, \dots \in (0, +\infty)$ the following quantities by recursion for all $n \in \mathbb{N}$

$$\begin{aligned} & p_0 = p_0(b_0) = H_{b_0}[\mu] \\ & p_n = p_n(b_0, s_1, \dots, s_n, b_n) = \\ & \begin{cases} H_{b_n}\left[\phi^{a_{n-1}}_{p_{n-1}}(s_n^- - s_{n-1}) \int_U \Lambda(\mathfrak{u})\, a_{n-1}(s_n^- - s_{n-1}\,; \mathrm{d}\mathfrak{u})\right], & \text{if } s_1 < \dots < s_n \\ \rho, & \text{otherwise} \end{cases} \end{aligned}$$

Here $s_0 = 0$ and $\rho \in \Delta_e$ is an arbitrarily chosen value.

For all $n \in \mathbb{N}_0$ we define the functions $u_n \colon [0, +\infty) \times O \times \big((0, +\infty) \times O\big)^n \to \mathcal{P}(U)$ as

$$u_n(t, b_0, \dots, s_n, b_n\,; \mathrm{d}\mathfrak{u}) = \begin{cases} a_n(p_0, \dots, s_n, p_n)(t - s_n\,; \mathrm{d}\mathfrak{u}), & \text{if } t \geqslant s_n, \\ \mathfrak{u}, & \text{if } t < s_n, \end{cases}$$

where $\mathfrak{u} \in U$ is some fixed value that is irrelevant to specify. Thanks to the fact that each of the functions $(b_0, \dots, s_n, b_n) \mapsto p_n$ is Borel-measurable and that $\mathcal{M}$ is a Borel space, we can use [67, Lemma 3(ii)] to conclude that all the functions u_n are Borel-measurable and therefore $\mathbf{u} = (u_n)_{n \in \mathbb{N}_0} \in \mathcal{U}_{ad}$.

Now the proof follows the same steps of the first part. Equality between the laws of $\pi^{\mu,\mathbf{u}}$ under $\mathrm{P}^{\mathbf{u}}_\mu$ and $\bar{\pi}$ under $\mathrm{P}^{\mathbf{a}}_Q$ is established by proving equivalence between the initial distributions of the two processes (that have not changed from the first part of the proof) and of the conditional distributions

$$\begin{aligned} & \mathrm{P}^{\mathbf{u}}_\mu(\tau_n - \tau_{n-1} > t,\, \tau_{n-1} < +\infty \mid Y_0, \dots, \tau_{n-1}, Y_{\tau_{n-1}}); \\ & \bar{\mathrm{P}}^{\mathbf{a}}_Q(\bar{\tau}_n - \bar{\tau}_{n-1} > t,\, \bar{\tau}_{n-1} < +\infty \mid \bar{Y}_0, \dots, \bar{\tau}_{n-1}, \bar{Y}_{\bar{\tau}_{n-1}}); \\ & \mathrm{P}^{\mathbf{u}}_\mu(\pi^{\mu,\mathbf{u}}_{\tau_n} \in D,\, \tau_n < +\infty \mid Y_0, \dots, Y_{\tau_{n-1}}, \tau_n); \\ & \bar{\mathrm{P}}^{\mathbf{a}}_Q(\bar{\pi}_{\bar{\tau}_n} \in D,\, \bar{\tau}_n < +\infty \mid \bar{Y}_0, \dots, \bar{Y}_{\bar{\tau}_{n-1}}, \bar{\tau}_n), \end{aligned}$$

where $t \geqslant 0$, $D \in \mathcal{B}(\Delta_e)$ and $n \in \mathbb{N}$.

Finally, to prove (4.2.13) it suffices to define $\Phi \colon \bar{\Omega} \to \mathbb{R}$ as

$$\Phi(\bar{\omega}) = \sum_{n=0}^{+\infty} e^{-\beta \bar{\tau}_n(\bar{\omega})} g\big(\bar{\pi}_{\bar{\tau}_n(\bar{\omega})}(\bar{\omega}), u_n(\cdot + \bar{\tau}_n, \bar{Y}_0(\bar{\omega}), \dots, \bar{\tau}_n(\bar{\omega}), \bar{Y}_{\tau_n(\bar{\omega})}(\bar{\omega})\big).$$

and notice that $p_n(\bar{Y}_0, \dots, \bar{\tau}_n, \bar{Y}_{\bar{\tau}_n}) = \bar{\pi}_{\bar{\tau}_n}$, so that we can write

$$\Phi(\bar{\omega}) = \sum_{n=0}^{+\infty} e^{-\beta \bar{\tau}_n(\bar{\omega})} g\big(\bar{\pi}_{\bar{\tau}_n(\bar{\omega})}(\bar{\omega}), a_n(\bar{\pi}_0(\bar{\omega}), \dots, \bar{\tau}_n(\bar{\omega}), \bar{\pi}_{\tau_n(\bar{\omega})}(\bar{\omega})\big).$$

The desired equality follows from the same reasoning as in the first part of the proof. □

Remark 4.2.1. It is clear that both the classes $\mathcal{U}_{ad}$ and $\mathcal{A}_{ad}$ are strictly larger than the corresponding classes of *piecewise open-loop controls* that are standard in PDP optimal control problems. See Remark 3.2.1 for more details on this.

We can focus now on the analysis of the auxiliary PDP optimal control problem. What we are aiming at is to prove that its value function v is the unique fixed point of the operator $\mathcal{T}\colon \mathrm{B}_b(\Delta_e) \to \mathrm{B}_b(\Delta_e)$, defined for all $\nu \in \Delta_e$ as

$$\begin{aligned}\mathcal{T}w(\nu) &:= \inf_{m\in\mathcal{M}} \int_0^\infty \int_U e^{-\beta t} L(\phi_\nu^m(t), \chi_\nu^m(t), \mathfrak{u}, w)\, m(t; \mathrm{d}\mathfrak{u})\, \mathrm{d}t \\ &:= \inf_{m\in\mathcal{M}} \int_0^\infty \int_U e^{-\beta t} \chi_\nu^m(t) \Big[\phi_\nu^m(f;\, \mathfrak{u}, t) + \\ &\qquad r(\phi_\nu^m(t), \mathfrak{u}) \int_{\Delta_e} w(p) R(\phi_\nu^m(t), \mathfrak{u}; \mathrm{d}p) \Big]\, m(t; \mathrm{d}\mathfrak{u})\, \mathrm{d}t \end{aligned} \tag{4.2.18}$$

The operator $\mathcal{T}$ is a contraction under Assumptions 4.1.1 and 4.1.2, as it can be shown analogously to Lemma 3.2.2. Therefore, we just need to show that v is a fixed point of $\mathcal{T}$. We will invoke results from [13]. In fact, our problem is an instance of a *lower semicontinuous model*. The next results follow the same reasoning of Section 3.2 and are stated without proof, being almost identical.

Proposition 4.2.2. *Under Assumptions 4.1.1, 4.1.2 and 4.1.3 there exists an optimal policy $\mathbf{a}^\star \in \mathcal{A}_{ad}$, i. e. a policy such that*

$$v(\nu) = \bar{J}(\nu, \mathbf{a}^\star), \quad \textit{for all } \nu \in \Delta_e.$$

Moreover, this policy is stationary, the value function v is lower semicontinuous and it is the unique fixed point of the operator $\mathcal{T}$.

We get rid of relaxed controls, thanks to the following theorem.

Theorem 4.2.3. *Let us define the operator $\mathcal{G}\colon \mathrm{B}_b(\Delta_e) \to \mathrm{B}_b(\Delta_e)$ as*

$$\mathcal{G}w(\nu) = \inf_{\alpha\in A} \int_0^\infty e^{-\beta t} L(\phi_\nu^\alpha(t), \chi_\nu^\alpha(t), \alpha(t), w)\, \mathrm{d}t, \quad \nu \in \Delta_e \tag{4.2.19}$$

where the infimum is taken among all possible ordinary controls in the set A, defined in (4.2.2).

Under Assumptions 4.1.1, 4.1.2 and 4.1.3 the operator $\mathcal{G}$ is a contraction and v is its unique fixed point.

The final statement of this Section gives an equality between the two value functions V and v. This proves the equivalence of the original control problem for the unobserved process X and the auxiliary control problem for the filtering process $\bar{\pi}$.

Theorem 4.2.4. *Under Assumptions 4.1.1, 4.1.2 and 4.1.3, for all $\mu \in \mathcal{P}(I)$ we have that*

$$V(\mu) = \int_O v(H_y[\mu])\, \mu \circ h^{-1}(\mathrm{d}y). \tag{4.2.20}$$

In the next Section we will characterize the value function v (and, indirectly, the original value function V).

4.3 Characterization of the value function

In this Section we will prove that the value function v of the separated problem is continuous, characterizing it as the unique fixed point of the operator $\mathcal{G}$ in the space of bounded and continuous functions on Δ_e.

Let us denote by $\mathrm{C}_b(\Delta_e)$ the space of bounded continuous functions on Δ_e equipped with the usual sup norm. To prove continuity of v we need to show that $\mathcal{G}$ maps the space $\mathrm{C}_b(\Delta_e)$ into itself and that v is its unique fixed point in that space (recall that we already established that $\mathcal{G}$ is a contraction). The following version of the *Dynamic Programming Principle* is an important piece to get continuity of v. We omit its proof since it is identical to that of Proposition 3.3.1.

Proposition 4.3.1 (Dynamic Programming Principle)**.** *For all functions $w \in \mathrm{B}_b(\Delta_e)$ and all $T > 0$ the function $\mathcal{G}w$ satisfies the following identity*

$$\mathcal{G}w(\nu) = \inf_{\alpha \in A} \left\{ \int_0^T e^{-\beta t} L(\phi_\nu^\alpha(t), \chi_\nu^\alpha(t), \alpha(t), w) \, \mathrm{d}t + e^{-\beta T} \chi_\nu^\alpha(T) \mathcal{G}w(\phi_\nu^\alpha(T)) \right\}. \tag{4.3.1}$$

The following Lemma is a key tool to be used in the proof of the next Proposition.

Lemma 4.3.2. *Let $T > 0$ and $w \in \mathrm{C}_b(\Delta_e)$ be fixed and define*

$$\mathcal{J}_{T,w}(\nu, \alpha) := \int_0^T e^{-\beta t} L(\phi_\nu^\alpha(t), \chi_\nu^\alpha(t), \alpha(t), w) \, \mathrm{d}t, \quad \nu \in \Delta_e, \, \alpha \in A. \tag{4.3.2}$$

Then, under Assumptions 4.1.1, 4.1.2 and 4.1.3, the map $\nu \mapsto \mathcal{J}_{T,w}(\nu, \alpha)$ is continuous on Δ_e, uniformly with respect to $\alpha \in A$.

Proof. Let $\varepsilon > 0$, $\alpha \in A$ and $\nu, \rho \in \Delta_e$ be fixed. First of all, we need to estimate the quantity

$$\left| L(\phi_\nu^\alpha(t), \chi_\nu^\alpha(t), \alpha(t), w) - L(\phi_\rho^\alpha(t), \chi_\rho^\alpha(t), \alpha(t), w) \right|.$$

We have that for all $t \geqslant 0$

$$\begin{aligned}
&\left| L(\phi_\nu^\alpha(t), \chi_\nu^\alpha(t), \alpha(t), w) - L(\phi_\rho^\alpha(t), \chi_\rho^\alpha(t), \alpha(t), w) \right| \\
&\leqslant \left| \chi_\nu^\alpha(t) - \chi_\rho^\alpha(t) \right| \left| \phi_\nu^\alpha(f;\, \alpha(t), t) + r(\phi_\nu^\alpha(t), \alpha(t)) \int_{\Delta_e} w(p) R(\phi_\nu^\alpha(t), \alpha(t); \mathrm{d}p) \right| \\
&\quad + \chi_\rho^\alpha(t) \left| \phi_\nu^\alpha(f;\, \alpha(t), t) - \phi_\rho^\alpha(f;\, \alpha(t), t) \right| \\
&\quad + \chi_\rho^\alpha(t) \left| r(\phi_\nu^\alpha(t), \alpha(t)) \int_{\Delta_e} w(p) R(\phi_\nu^\alpha(t), \alpha(t); \mathrm{d}p) \right. \\
&\qquad\qquad \left. - r(\phi_\rho^\alpha(t), \alpha(t)) \int_{\Delta_e} w(p) R(\phi_\rho^\alpha(t), \alpha(t); \mathrm{d}p) \right|.
\end{aligned}$$

Thanks to Assumptions 4.1.1, 4.1.2 the first summand satisfies

$$\left| \phi_\nu^\alpha(f;\, \alpha(t), t) + r(\phi_\nu^\alpha(t), \alpha(t)) \int_{\Delta_e} w(p) R(\phi_\nu^\alpha(t), \alpha(t); \mathrm{d}p) \right| \leqslant K,$$

where $K > 0$ is a constant depending on C_f and C_r defined in (4.1.6) and (4.1.19) and on $\sup_{\vartheta \in \Delta_e} |w(\vartheta)|$. Furthermore, a repeated application of (4.1.12) and of Gronwall's Lemma shows that for all $t \geqslant 0$

$$\left| \chi_\nu^\alpha(t) - \chi_\rho^\alpha(t) \right| \leqslant \frac{L_r}{L_F} (e^{L_F T} - 1) e^{C_r T} \|\nu - \rho\|_{TV}$$

where L_F and L_r are the constants defined in (4.1.10) and (4.1.18) respectively.

As for the second summand, observe that $\chi_\nu^\alpha(t) \leqslant 1$. Thanks to Assumption 4.1.2, we can apply (4.1.12) to obtain

$$\left|\phi_\nu^\alpha(f;\,\alpha(t),t) - \phi_\rho^\alpha(f;\,\alpha(t),t)\right| \leqslant C_f e^{L_F t}\|\nu-\rho\|_{TV}.$$

Finally, by Assumption 4.1.3 we can find δ such that the third summand can be estimated as

$$\begin{aligned}\Bigg| r(\phi_\nu^\alpha(t),\alpha(t)) \int_{\Delta_e} w(p) R(\phi_\nu^\alpha(t),\alpha(t);\mathrm{d}p)\\ - r(\phi_\nu^\alpha(t),\alpha(t)) \int_{\Delta_e} w(p) R(\phi_\nu^\alpha(t),\alpha(t);\mathrm{d}p)\Bigg| < \frac{\varepsilon}{2K_\beta}\end{aligned}$$

as soon as $\|\phi_\nu^\alpha(t) - \phi_\rho^\alpha(t)\|_{TV} < \eta$ for some $\eta > 0$, i. e. – applying again (4.1.12) – as soon as $\|\nu-\rho\|_{TV} < \frac{\eta}{e^{L_F T}}$. Here $K_\beta := \frac{e^{\beta T}-1}{\beta}$ is a constant introduced for convenience, as will be clear later. Notice that η depends on ε, β and T, but not on α.

Collecting all the computations made so far we get

$$\begin{aligned}&\left|L(\phi_\nu^\alpha(t),\chi_\nu^\alpha(t),\alpha(t),w) - L(\phi_\rho^\alpha(t),\chi_\rho^\alpha(t),\alpha(t),w)\right| \leqslant\\ &\qquad \frac{L_r}{L_F} e^{C_r T}(e^{L_F T}-1)(K + C_f e^{L_F T})\|\nu-\rho\|_{TV} + \frac{\varepsilon}{2K_\beta}.\end{aligned}$$

We are now in a position to prove our claim. It suffices to notice that

$$\begin{aligned}&\left|\mathcal{J}_{T,w}(\nu,\alpha) - \mathcal{J}_{T,w}(\rho,\alpha)\right|\\ &\leqslant \int_0^T e^{-\beta t}\left|L(\phi_\nu^\alpha(t),\chi_\nu^\alpha(t),\alpha(t),w) - L(\phi_\rho^\alpha(t),\chi_\rho^\alpha(t),\alpha(t),w)\right|\mathrm{d}t\\ &\leqslant \frac{e^{-\beta T}-1}{\beta}\left[\frac{L_r}{L_F}e^{C_r T}(e^{L_F T}-1)(K + C_f e^{L_F T})\|\nu-\rho\|_{TV} + \frac{\varepsilon}{2K_\beta}\right]\\ &= C\|\nu-\rho\|_{TV} + \frac{\varepsilon}{2}\end{aligned} \tag{4.3.3}$$

where we defined $C := K_\beta\big[\frac{L_r}{L_F}e^{C_r T}(e^{L_F T}-1)(K + C_f e^{L_F T})\big]$. Notice, again, that C is a constant depending on the functions F, r, f, w and on the constants β, T but not on $\alpha \in A$. We conclude observing that, thanks to (4.3.3), we have $\left|\mathcal{J}_{T,w}(\nu,\alpha) - \mathcal{J}_{T,w}(\rho,\alpha)\right| < \varepsilon$ as soon as we take $\|\nu-\rho\|_{TV} < \min\{\frac{\eta}{e^{L_F T}}, \frac{\varepsilon}{2C}\} =: \delta$ (recall that η depends on ε, β, T). ☐

The following Proposition establishes a fundamental fact: the operator $\mathcal{G}$ maps bounded and continuous functions into bounded and continuous functions. Thanks to this Proposition we are able to prove that the value function v is the unique fixed point of $\mathcal{G}$ in the space $\mathrm{C}_b(\Delta_e)$.

Proposition 4.3.3. *Under Assumptions 4.1.1, 4.1.2 and 4.1.3, for each function* $w \in \mathrm{C}_b(\Delta_e)$ *we have that* $\mathcal{G}w \in \mathrm{C}_b(\Delta_e)$.

Proof. Let us choose $\nu_0, \rho_0 \in \Delta_e$, such that for some $\delta > 0$ small enough $\|\nu_0 - \rho_0\|_{TV} < \delta$. Let $\varepsilon > 0$, $T > 0$ be arbitrarily fixed and choose $\alpha_0^\varepsilon \in A$ such that

$$\mathcal{G}w(\rho_0) + \frac{\varepsilon}{4K_\beta} \geqslant \int_0^T e^{-\beta t} L(\phi_{\rho_0}^{\alpha_0^\varepsilon}(t), \chi_{\rho_0}^{\alpha_0^\varepsilon}(t), \alpha_0^\varepsilon(t), w)\,\mathrm{d}t + e^{-\beta T}\chi_{\rho_0}^{\alpha_0^\varepsilon}(T)\mathcal{G}w(\phi_{\rho_0}^{\alpha_0^\varepsilon}(T)) \tag{4.3.4}$$

according to the Dynamic Programming Principle, where $K_\beta > 0$ is a constant that we will fix later. From (4.3.4) we easily get

$$\begin{aligned}\mathcal{G}w(\nu_0) - \mathcal{G}w(\rho_0) &\leqslant \mathcal{J}_{T,w}(\nu_0, \alpha_0^\varepsilon) - \mathcal{J}_{T,w}(\rho_0, \alpha_0^\varepsilon) + \frac{\varepsilon}{4K_\beta} \\ &\quad + e^{-\beta T}\big[\chi_{\nu_0}^{\alpha_0^\varepsilon}(T)\mathcal{G}w(\phi_{\nu_0}^{\alpha_0^\varepsilon}(T)) - \chi_{\rho_0}^{\alpha_0^\varepsilon}(T)\mathcal{G}w(\phi_{\rho_0}^{\alpha_0^\varepsilon}(T))\big] \\ &\leqslant \big|\mathcal{J}_{T,w}(\nu_0, \alpha_0^\varepsilon) - \mathcal{J}_{T,w}(\rho_0, \alpha_0^\varepsilon)\big| \\ &\quad + e^{-\beta T}\big|\chi_{\nu_0}^{\alpha_0^\varepsilon}(T) - \chi_{\rho_0}^{\alpha_0^\varepsilon}(T)\big| \sup_{\vartheta\in\Delta_e}\big|\mathcal{G}w(\vartheta)\big| \\ &\quad + e^{-\beta T}\big[\mathcal{G}w(\phi_{\nu_0}^{\alpha_0^\varepsilon}(T)) - \mathcal{G}w(\phi_{\rho_0}^{\alpha_0^\varepsilon}(T))\big] + \frac{\varepsilon}{4K_\beta}\end{aligned}$$

where $\mathcal{J}_{T,w}$ was defined in (4.3.2) and $\sup_{\vartheta\in\Delta_e}\big|\mathcal{G}w(\vartheta)\big| < +\infty$ since w is bounded and $\mathcal{G}$ maps bounded functions into bounded functions.

We need to provide an estimate for the terms appearing in the last lines of the previous equation. We know from Lemma 4.3.2 that for a suitably chosen δ we have

$$\big|\mathcal{J}_{T,w}(\nu_0, \alpha_0^\varepsilon) - \mathcal{J}_{T,w}(\rho_0, \alpha_0^\varepsilon)\big| < \frac{\varepsilon}{4K_\beta}.$$

Recall that this fact is independent of α_0^ε. From the proof of Lemma 4.3.2 we know that

$$\big|\chi_{\nu_0}^{\alpha_0^\varepsilon}(T) - \chi_{\rho_0}^{\alpha_0^\varepsilon}(T)\big| \leqslant K\delta$$

for a specific constant K depending on the functions F, r, f and T but not on α_0^ε.

Taking into account what we have obtained so far, we get

$$\begin{aligned}\mathcal{G}w(\nu_0) - \mathcal{G}w(\rho_0) &< K\delta e^{-\beta T}\sup_{\vartheta\in\Delta_e}\big|\mathcal{G}w(\vartheta)\big| \\ &\qquad + e^{-\beta T}\big[\mathcal{G}w(\phi_{\nu_0}^{\alpha_0^\varepsilon}(T)) - \mathcal{G}w(\phi_{\rho_0}^{\alpha_0^\varepsilon}(T))\big] + \frac{\varepsilon}{2K_\beta}.\end{aligned}$$

Now, let $\nu_1 \coloneqq \phi_{\nu_0}^{\alpha_0^\varepsilon}(T)$, $\rho_1 \coloneqq \phi_{\rho_0}^{\alpha_0^\varepsilon}(T)$. Notice that $\|\nu_1 - \rho_1\|_{TV} \leqslant e^{L_F T}\delta$, thanks to (4.1.12). Choose $\alpha_1^\varepsilon \in A$ such that

$$\mathcal{G}w(\rho_1) + \frac{\varepsilon}{4K_\beta} \geqslant \int_0^T e^{-\beta t} L(\phi_{\rho_1}^{\alpha_1^\varepsilon}(t), \chi_{\rho_1}^{\alpha_1^\varepsilon}(t), \alpha_1^\varepsilon(t), w)\,\mathrm{d}t + e^{-\beta T}\chi_{\rho_1}^{\alpha_1^\varepsilon}(T)\mathcal{G}w(\phi_{\rho_1}^{\alpha_1^\varepsilon}(T)) \tag{4.3.5}$$

according to the Dynamic Programming Principle. We can apply the same estimates as above to $\mathcal{G}w(\nu_1) - \mathcal{G}w(\rho_1)$ and obtain

$$\begin{aligned}\mathcal{G}w(\nu_0) - \mathcal{G}w(\rho_0) &< \left(\frac{\varepsilon}{2K_\beta} + K\delta e^{-\beta T}\sup_{\vartheta\in\Delta_e}\big|\mathcal{G}w(\vartheta)\big|\right)(1 + e^{-\beta T}) \\ &\qquad + e^{-2\beta T}\big[\mathcal{G}w(\phi_{\nu_1}^{\alpha_1^\varepsilon}(T)) - \mathcal{G}w(\phi_{\rho_1}^{\alpha_1^\varepsilon}(T))\big].\end{aligned}$$

Proceeding in this way, for all $n \in \mathbb{N}$ we can pick a sequence of control functions

$\alpha_0^\varepsilon, \dots, \alpha_n^\varepsilon \in A$ such that

$$\begin{aligned}\mathcal{G}w(\nu_0) - \mathcal{G}w(\rho_0) &< \left(\frac{\varepsilon}{2K_\beta} + K\delta e^{-\beta T} \sup_{\vartheta\in\Delta_e} \left|\mathcal{G}w(\vartheta)\right|\right) \sum_{k=0}^{n} e^{-k\beta T} \\ &\quad + e^{-(n+1)\beta T}\left[\mathcal{G}w(\phi_{\nu_n}^{\alpha_n^\varepsilon}(T)) - \mathcal{G}w(\phi_{\rho_n}^{\alpha_n \varepsilon}(T))\right] \\ &\leqslant \left(\frac{\varepsilon}{2K_\beta} + K\delta e^{-\beta T} \sup_{\vartheta\in\Delta_e} \left|\mathcal{G}w(\vartheta)\right|\right) \frac{1-e^{-(n+1)\beta T}}{1-e^{-\beta T}} \\ &\quad + 2e^{-(n+1)\beta T} \sup_{\vartheta\in\Delta_e} \left|\mathcal{G}w(\vartheta)\right|\end{aligned}$$

where $\nu_k = \phi_{\nu_{k-1}}^{\alpha_{k-1}^\varepsilon}(T)$, $\rho_k = \phi_{\rho_{k-1}}^{\alpha_{k-1}^\varepsilon}(T)$ with $k = 1, \dots, n$. It is clear that we obtain the same estimate reversing the roles of ν_0 and ρ_0, hence for all $n \in \mathbb{N}$ we get

$$\begin{aligned}&\left|\mathcal{G}w(\nu_0) - \mathcal{G}w(\rho_0)\right| \\ &< \left(\frac{\varepsilon}{2K_\beta} + K\delta e^{-\beta T} \sup_{\vartheta\in\Delta_e} \left|\mathcal{G}w(\vartheta)\right|\right) \frac{1-e^{-(n+1)\beta T}}{1-e^{-\beta T}} + 2e^{-(n+1)\beta T} \sup_{\vartheta\in\Delta_e} \left|\mathcal{G}w(\vartheta)\right|.\end{aligned} \tag{4.3.6}$$

Now, let us choose $N \in \mathbb{N}$ such that $2e^{-(N+1)\beta T} \sup_{\vartheta\in\Delta_e}\left|\mathcal{G}w(\vartheta)\right| < \frac{\varepsilon}{4}$, fix $K_\beta = \frac{1-e^{-(N+1)\beta T}}{1-e^{-\beta T}}$ and take δ such that $K\delta e^{-\beta T} \sup_{\vartheta\in\Delta_e}\left|\mathcal{G}w(\vartheta)\right| K_\beta < \frac{\varepsilon}{4}$. From (4.3.6) we get that $\left|\mathcal{G}w(\nu_0) - \mathcal{G}w(\rho_0)\right| < \varepsilon$ and, being ε arbitrary, we conclude. □

Theorem 4.3.4. *Under Assumptions 4.1.1, 4.1.2 and 4.1.3 the value function v of the separated problem is the unique fixed point of the operator $\mathcal{G}$ in the space of bounded and continuous functions on Δ_e.*

Proof. As in the proof of the corresponding Theorem 3.3.4, we just need to put together the following facts: v is the unique fixed point of $\mathcal{G}$ in the space $\mathrm{B}_b(\Delta_e)$; the operator $\mathcal{G}\colon \mathrm{C}_b(\Delta_e) \to \mathrm{C}_b(\Delta_e)$ is a contraction mapping; Proposition 4.3.3. □

4.4 Some comments on the observed process

In this brief Section we highlight what happens to the results proved in this Chapter and, consequently, in Chapter 3 if we allow the function h, providing the observed process, to be one-to-one or constant. We recall that these cases were excluded from our analysis since the control problem arising from them is not of true partial observation nature. However, the assumption that h is neither one-to-one nor constant is not used in any of the proofs contained in Chapters 3 and 4 and, in fact, the reader may check that they still hold even in these cases, leading to the results shown below.

The case where h is one-to-one is associated to a control problem with complete observation. As explained in Section 2.4, both the observed process Y and the filtering process can be identified with the pure-jump Markov process X. Therefore, our control problem falls in the framework of optimal control problems for continuous-time pure-jump processes, that have been treated, for instance, in [54, 68].

The case where h is constant is associated to a control problem with no information, in particular a deterministic one. The class of admissible controls $\mathcal{U}_{ad}$ given in (4.1.1) can be identified with the set of relaxed controls $\mathcal{M}$, introduced in (4.2.1). In fact, our admissible controls have to be predictable with respect to the natural filtration $(\mathcal{Y}_t^\circ)_{t\geqslant 0}$ of the process Y, but since the σ–algebras $\mathcal{Y}_t^\circ$ coincide with the trivial one for all $t \geqslant 0$, we have that an admissible control is just a relaxed control.

As said in Section 2.4, the filtering process $\pi^{\mu,\mathbf{u}}$ coincides with the law of the unobserved process X under the probability measure $\mathrm{P}_\mu^{\mathbf{u}}$, for all $\mu \in \mathcal{P}(I)$ and $\mathbf{u} \in \mathcal{U}_{ad}$. Moreover, it satisfies the following evolution equation

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\pi_t^{\mu,\mathbf{u}} = \mathcal{L}_{\mathbf{u}}^\star \pi_t^{\mu,\mathbf{u}}, \quad t \geqslant 0 \\ \pi_0^{\mu,\mathbf{u}} = \mu \end{cases} \tag{4.4.1}$$

where $(\mathcal{L}_u^\star)_{u\in U}$ is the family of controlled adjoint operators associated to the family of controlled infinitesimal generators $(\mathcal{L}_u)_{u\in U}$ of the process X. In particular, the adjoint operator $\mathcal{L}_u^\star$ is defined by

$$\mathcal{L}_u^\star \nu(\mathrm{d}z) = \int_I \lambda(x,u,\mathrm{d}z)\,\nu(\mathrm{d}x) - \lambda(z,u)\nu(\mathrm{d}z), \quad \nu \in \mathcal{P}(I),\, u \in U$$

where λ is the controlled rate transition measure of the process X.

Thanks to (4.4.1), for all $\mu \in \mathcal{P}(I)$ and $\mathbf{u} \in \mathcal{U}_{ad}$ we can write the cost functional, introduced in (4.1.4), as

$$\begin{aligned} J(\mu,\mathbf{u}) &= \mathrm{E}_\mu^{\mathbf{u}}\left[\int_0^\infty e^{-\beta t}\int_U f(X_t,\mathrm{u})\,u_t(\mathrm{du})\,\mathrm{d}t\right] \\ &= \int_0^\infty e^{-\beta t}\int_U\int_I f(x,\mathrm{u})\,\pi_t^{\mu,\mathbf{u}}(\mathrm{d}x)\,u_t(\mathrm{du})\,\mathrm{d}t \\ &= \int_0^\infty e^{-\beta t}\int_U \tilde{f}(\pi_t^{\mu,\mathbf{u}},\mathrm{u})\,u_t(\mathrm{du})\,\mathrm{d}t \end{aligned}$$

where $\tilde{f}(\nu,u) := \int_I f(x,u)\,\nu(\mathrm{d}x)$, $\nu \in \mathcal{P}(I)$, $u \in U$. Hence, our optimal control problem can be reformulated as a deterministic optimal control problem.

If we assume that the process X takes values on $\mathbb{R}^n$ and that its law admits density with respect to the Lebesgue measure, then (4.4.1) can be rewritten as a PIDE for the density process, called the *controlled Fokker-Planck equation*. Sometimes, also evolution equations as (4.4.1) in the space $\mathcal{P}(I)$ are called Fokker-Planck equations. Optimal control problems for Fokker-Planck equations have been studied in the literature, usually for probability density functions of diffusion processes. However, the focus of those researches is mainly targeted to ensure the existence of an optimal control or to find optimality conditions (see e. g. [2, 36]). Optimal control of Fokker-Planck equations has also been studied in connection to mean-field games, for instance in [12].

As a final remark, we notice that in the case of constant h, no characterization of the value function is possible through a fixed point argument.

Conclusions and future developments

Optimal control problems with partial observation are still the subject of intense investigation in the literature. As pointed out in the Introduction, various techniques are today available to solve these problems and the search for new methods is still ongoing. In addition, a wide range of applications offers new impulses to analyze this kind of problems and poses interesting questions. For instance, one may think of problems in finance concerning portfolio optimization or risk minimization where latent unobservable variables may influence the price processes of the traded assets. In economics, optimal control problems with delay, due e. g. for a change in taxation or the effect of a new law, may be of partial observation nature due to unobserved factors that influence macroeconomic variables. In engineering, dynamical systems that are not fully observable or are affected by unknown stochastic parameters are common.

This thesis aims to deal with this kind of optimal control problems in a specific situation, that of an unobserved process of pure-jump type and a noise-free observation. The purpose is to fill, albeit in part, a void in the analysis of this kind of models and to promote further investigations on noise-free problems. It is not so infrequent to deal with models where the observation is truly of noise-free type or the noise affecting it may be considered negligible with respect to the randomness of the whole system under investigation. In this case, the opportunity to treat such a model as a noise-free one may result in a simplification of the analysis. We point out the possible advantages of our noise-free model by recapitulating the main results of this thesis.

In Chapter 2 we provided an explicit filtering equation and characterized the filtering process as a Piecewise Deterministic Markov Process. Such an explicit result is not always obtainable in filtering problems. Moreover, the fairly simple structure of the filtering equation gives the opportunity to numerically approximate its solution or, in some cases, to have an explicit closed formula for it. Characterizing the filtering process as a PDMP has the important consequence that many results on this class of processes are readily usable, e. g. the structure of its extended generator, or formulas to compute distributions or expectations of functionals of a PDMP.

In Chapter 3 we studied an infinite-horizon optimal control problem with fixed discount factor for a continuous-time homogeneous Markov chain. We provided an explicit structure of admissible controls, something that only on very specific situations can be obtained, and we investigated the discrete-time structure of the control problem. We also showed that the value function is the unique fixed point of a suitable contraction mapping. Again, this is extremely useful from a computational point of view, since the optimization algorithm can be loosely stated as follows: one minimizes the cost functional of the associated deterministic optimal control problem (the one solved between two consecutive jump times) for every possible initial state of the filtering process and then one optimizes the cost functional of the discrete-time stochastic problem, choosing controls based at each time step on past and present jump times and positions of the filtering process. Moreover, the characterization of the value function as the unique constrained viscosity solution of an integro-differential Hamilton Jacobi Bellman equation puts these results into the general framework for optimal control problem that we can find in the literature. We also showed that under suitable assumptions a piecewise open-loop optimal control exists, hence we are able to implement in real-world applications our control strategies.

In Chapter 4 we studied an optimal control problem in the same setting as in Chapter 3 for a continuous-time pure jump Markov process. We showed that also in this more complicated situation the same conclusions of Chapter 3 hold, apart from the characterization of the value function as unique solution of a HJB equation. Even though the problem analyzed in Chapter 4 is an infinite-dimensional one, we are able to express the optimal control problem as a discrete-time one, with all the connected advantages of this approach already pointed out.

The implications of our results from a computational point of view are not to be underestimated, especially if we think about applications. In finance or engineering, for instance, it is essential to write down algorithms enabling to compute the value function or even provide optimal controls.

It is clear that analyses of optimal control problems with partial noise-free observation can be (and, in our opinion, should be) broadened to a wider range of models. As examples we can think of

- A signal process given by a PDMP, a diffusion process or a Lévy process.
- Optimal control problems with a different kind of noise-free observation, e. g. the running maximum of a real-valued signal process.
- Optimal switching problems where, for instance, the observation process solves an ODE governed by coefficients that randomly commute according to an unobserved Markov chain. In this situation, the controller wants to govern the dynamic of the Markov chain so that the system switches between different regimes in an optimal way with respect to some criterion.
- Jump Markov linear systems, that are physical systems described by a stochastic linear dynamic model whose behavior is governed by an underlying jump Markov process.

At this point it should be plain that optimal control problems with partial noise-free observation represent a subject worth to be studied. Applications are possible in a wide range of fields, such as economics, finance and engineering, and various models can be described. In turn, these models can be more general than the present one, that can represent a starting point, say a reference, for future studies. From a mathematical point of view, challenges presented by noise-free models are undoubtedly intriguing.

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