Constrained BSDEs driven by a non quasi-left-continuous random measure and optimal control of PDMPs on bounded domains

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Abstract

We consider an optimal control problem for piecewise deterministic Markov processes (PDMPs) on a bounded state space. The control problem under study is very general: a pair of controls acts continuously on the deterministic flow and on the two transition measures (in the interior and from the boundary of the domain) describing the jump dynamics of the process. For this class of control problems, the value function can be characterized as the unique viscosity solution to the corresponding fully-nonlinear Hamilton-Jacobi-Bellman equation with a non-local type boundary condition.

By means of the recent control randomization method, we are able to provide a probabilistic representation for the value function in terms of a constrained backward stochastic differential equation (BSDE), known as nonlinear Feynman-Kac formula. This result considerably extends the existing literature, where only the case with no jumps from the boundary is considered. The additional boundary jump mechanism is described in terms of a non quasi-left-continuous random measure and induces predictable jumps in the PDMP’s dynamics. The existence and uniqueness results for BSDEs driven by such a random measure are non trivial, even in the unconstrained case, as emphasized in the recent work [2].

Keywords: Backward stochastic differential equations, optimal control problems, piecewise deterministic Markov processes, non quasi-left-continuous random measure, randomization of controls.

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1 Introduction

In this paper we prove that the value function of an infinite-horizon optimal control problem for piecewise deterministic Markov processes on bounded domains can be represented in terms of a suitable backward stochastic differential equation. Piecewise deterministic Markov processes, introduced in [18], evolve by means of random jumps at random times, while the behavior between jumps is described by a deterministic flow. We consider optimal control problems of PDMPs where the control acts continuously on the jump dynamics as well as on the deterministic flow. We deal with PDMPs with bounded state space: whenever the process hits the boundary, it immediately jumps into the interior of the domain. Control problems for this type of processes arise in many contexts, among which operations research, engineering systems and management science, see [18] for a detailed overview. Our aim is to represent the value function by means of an appropriate
BSDE. It is worth mentioning that the probability measures describing the distribution of the controlled PDMP are in general not absolutely continuous with respect to the law of a given, uncontrolled process (roughly speaking, the control problem is non-dominated). This is reflected in the fully nonlinear character of the associated HJB equation, and prevents the use of standard BSDE techniques. For this reason, we shall extend to the present framework the so-called randomization method, recently introduced by [25] in the diffusive context, to represent the solutions of fully nonlinear integro-partial differential equations by means of a new class of BSDEs with nonpositive jumps. The extension of the randomization approach to our PDMPs optimal control problem is particularly delicate due to the presence of the jump mechanism from the boundary. Indeed, since the jumps from the boundary happen at predictable times, the associated BSDE turns out to be driven by a non quasi-left-continuous random measure. For such general backward equations, the existence and uniqueness of a solution is particularly tricky, and counterexamples can be obtained even in simple cases. Only recently, some results have been obtained in this context; in [2], well-posedness is obtained for unconstrained BSDEs in a general non-diffusive framework, under a specific condition involving the Lipschitz constants of the BSDE generator and the size of the predictable jumps. In the present paper we extend the results in [2] to our class of constrained BSDEs, for which the above latter property turns out to be satisfied.

Let us describe our setting in more detail. Let $E$ be an open bounded subset of $\mathbb{R}^d$, with Borel $\sigma$-algebra $\mathcal{E}$. The set $E$ corresponds to the PDMPs state space. Roughly speaking, a controlled PDMP on $(E, \mathcal{E})$ is described by specifying its local characteristics, namely a vector field $h(x, a_0)$, a jump rate $\lambda(x, a_0)$, and two transition probability measures $Q(x, a_0, dy)$ and $R(x, a_1, dy)$ prescribing the positions of the process at the jump times, respectively starting from the interior and from the boundary of the domain. The local characteristics depend on some initial value $x \in E$ and on the parameters $a_0 \in A_0$, $a_1 \in A_1$, where $(A_0, A_0)$ and $(A_1, A_1)$ are two general measurable spaces, denoting respectively the space of control actions in the interior and on the boundary of the domain. The control procedure consists in choosing a pair of strategies: a piecewise open-loop policy controlling the motion in the interior of the domain, i.e. a measurable function only depending on the last jump time $T_n$ and post jump position $E_n$, and a boundary control belonging to the set of feedback policies, that only depends on the position of the process just before the jump time. The above formulation of the control problem is used in many papers as well as books, see for instance [17], [18]. The class of admissible control laws $\mathcal{A}_{ad}$ will be the set of all $A_0 \otimes A_1$-measurable maps $\alpha = (\alpha^0, \alpha^1)$, with $\alpha^1 : \Gamma \to A_1$, and $\alpha^0 : [0, \infty) \times E \to A_0$ such that

$$\alpha^0_t = \alpha^0_0(t, x) \mathbb{1}_{[0,T)}(t) + \sum_{n=1}^{\infty} \alpha^0_n(t - T_n, E_n) \mathbb{1}_{[T_n,T_{n+1})}(t).$$

The controlled process $X$ is defined as

$$X_t = \begin{cases} \phi^\alpha(t, x) & \text{if } t \in [0, T), \\
\phi^\alpha(t - T_n, E_n) & \text{if } t \in [T_n, T_{n+1}), \ n \in \mathbb{N} \setminus \{0\}, \end{cases}$$

where $\phi^\alpha(t, x, \alpha^0_t)$ is the unique solution to the ordinary differential equation on $\mathbb{R}^d$

$$\dot{x}(t) = h(x(t), \alpha^0(t)), \quad x(0) = x.$$  

For every starting point $x \in E$ and for each $\alpha \in \mathcal{A}_{ad}$, by Theorem 3.6 in [21] we can introduce the unique probability measure $\mathbb{P}_{\alpha}$ such that the conditional survival function of the inter-jump times
and the distribution of the post jump positions of \( X \) under \( \mathbb{P}_x^\alpha \) are given by (2.3)-(2.5)-(2.6). Let \( \mathbb{E}_x^\alpha \) denote the expectation under \( \mathbb{P}_x^\alpha \). In the classical infinite-horizon control problem the goal is to minimize over all control laws \( \alpha \) a functional cost of the form

\[
J(x, \alpha) = \mathbb{E}_x^\alpha \left[ \int_0^\infty e^{-\delta s} f(X_s, \alpha_s^0) \, ds + \int_0^\infty e^{-\delta s} c(X_s, \alpha_s^\Gamma(X_s)) \, dp^\ast(s) \right],
\]

where \( f \) is a given real function on \( E \times A_0 \) representing the running cost, \( c \) is a given real function on \( E \times A_\Gamma \) that provides a cost every time the process hits the boundary, \( \delta \in (0, \infty) \) is a discount factor, while the process \( p^\ast_s \) counts the number of times the boundary is hit (see (2.4)). The value function of the control problem is defined in the usual way:

\[
V(x) = \inf_{\alpha \in \mathcal{A}_{ud}} J(x, \alpha), \quad x \in E.
\]

Under suitable assumptions on the cost functions \( f, c \), and on the local characteristics \( h, \lambda, Q, R \), \( V \) is known to be the unique continuous viscosity solution on \([0, \infty) \times E \) of the Hamilton-Jacobi-Bellman (HJB) equation with boundary non-local condition:

\[
\begin{align*}
\delta v(x) &= \inf_{a_0 \in A_0} \left( h(x, a_0) \cdot \nabla v(x) + \lambda(x, a_0) \int_E (v(y) - v(x)) Q(x, a_0, dy) + f(x, a_0) \right), \quad x \in E, \\
v(x) &= \min_{a_\Gamma \in A_\Gamma} \left( \int_E (v(y) - v(x)) R(x, a_\Gamma, dy) + c(x, a_\Gamma) \right), \quad x \in \partial E.
\end{align*}
\]

Our aim is to represent the value function \( V \) by means of an appropriate BSDE. We are interested in the general case where the probability measures \( \{\mathbb{P}_x^\alpha\}_\alpha \) describing the distribution of the controlled process are not absolutely continuous with respect to the law of a given, uncontrolled process. Probabilistic formulae for the value function for non-dominated models have been discovered only in the recent years. In this sense, a key role is played by the randomization method, firstly introduced in [25] to represent the solutions of fully nonlinear integro-partial differential equations related to the classical optimal control for diffusions, and later extended to other types of control problems, see for instance [26], [20], [12], [16], [6], [7]. In the non-diffusive framework, the correct formulation of the randomization method requires some efforts and different techniques from the diffusive case, since the controlled process is described only in terms of its local characteristics and not as a solution to some stochastic differential equation. A first step in the generalization of the randomization method to the non-diffusive framework was done in [4], where a probabilistic representation for the value function associated to an optimal control problem for pure jump Markov processes was provided (notice that in [4] the jump measure of the controlled state process is quasi-left continuous). Afterwards, the randomization techniques have been implemented in [3] to solve PDMPs optimal control problems on unbounded state spaces. In the present paper we are interested to extend those results to the case of optimal control problems for PDMPs on bounded state spaces, where additional forced jumps appear whenever the process hits the boundary. As already mentioned, the jump mechanism from the boundary plays a fundamental role as it leads, among other things, to the study of BSDEs driven by a non quasi-left-continuous random measure.

Let us describe the randomization approach in our framework. The fundamental idea consists in the so-called randomization of the control: roughly speaking, we replace the state trajectory and the associated pair of controls \( (X_s, \alpha_s^0, \alpha_s^\Gamma) \) by an (uncontrolled) PDMP \( (X_s, I_s, J_s) \). The process \( I \) (resp. \( J \)) is chosen to be a pure jump process with values in the space of control actions \( A_0 \) (resp. \( A_\Gamma \)), with an intensity \( \lambda_0(db) \) (resp. \( \lambda_\Gamma(dc) \)), which is arbitrary but finite and with full support. In particular, the PDMP \( (X, I, J) \) is constructed on a new probability space by means of a different...
triplet of local characteristics and takes values on the enlarged space \( E \times A_0 \times A_T \) (or, equivalently, by assigning the compensator \( \tilde{p}(ds \ dy \ db \ dc) \)). For any starting point \((x, a_0, a_T)\) in \( E \times A_0 \times A_T \), we denote by \( \mathbb{P}^{x,a_0,a_T} \) the corresponding law. At this point we introduce an auxiliary optimal control problem where we control the intensity of the processes \( I \) and \( J \): using a Girsanov’s type theorem for point processes, for any pair of predictable, bounded and positive processes \((\nu^0, \nu^T)\), we construct a probability measure \( \mathbb{P}^{x,a_0,a_T} \) under which the compensator of \( I \) (resp. \( J \)) is given by \( \nu^0(db) \lambda_0(db) \ dt \) (resp. \( \nu^T(dc) \lambda_T(dc) \ dt \)).

It is worth mentioning that the applicability of the Girsanov theorem to the present framework, i.e. when the compensator \( \tilde{p} \) is a non quasi-left-continuous random measure, requires the validity of an additional condition involving the intensity control fields \((\nu^0, \nu^T)\) and the predictable jumps of \( \tilde{p} \), see (3.19). The correct formulation of the randomized control problem has to take into account this latter constraint.

The aim of the new control problem (called randomized or dual control problem) is to minimize the functional

\[
J(x, a_0, a_T, \nu^0, \nu^T) = \mathbb{E}^{x,a_0,a_T}_{\nu^0,\nu^T} \left[ \int_{(0,\infty)} e^{-\delta s} f(x_s, I_s) \ ds + \int_{(0,\infty)} e^{-\delta s} c(x_{s-}, J_{s-}) \ dp_s^* \right]
\]  

(1.5)

over all possible choices of \( \nu^0, \nu^T \). Firstly, we give a probabilistic representation of the value function of the randomized control problem, denoted \( V^*(x, a_0, a_T) \), in terms of of a well-posed constrained BSDE. This latter is an equation over infinite horizon of the form (4.3) with the sign constraints (1.3)–(1.5). The random measure \( q = p - \tilde{p} \) driving the BSDE is the compensated measure associated to the jumps of \((X, I, J)\). In particular, the compensator \( \tilde{p} \) has predictable jumps \( \tilde{p}(\{t\} \times dy \ db \ dc) = \mathbbm{1}_{X_{t-} \in \partial E} \). Equation (4.3)–(4.4)–(4.5) is driven by a non quasi-left-continuous random measure; the associated well-posedness results are obtained by means of a penalization approach, by suitably extending the recent existence and uniqueness theorem obtained in [2] for unconstrained BSDEs. Once we achieve the existence and uniqueness of a maximal solution to (4.3)–(4.4)–(4.5), we prove that its component \( Y^{x,a_0,a_T} \) at the initial time represents the randomized value function, i.e. \( Y^{x,a_0,a_T}_0 = V^*(x, a_0, a_T) \). This is the first of our main results, and is the object of Theorem 5.1. Then, we aim at proving that \( Y^{x,a_0,a_T}_0 \) also provides a nonlinear Feynman-Kac representation to the value function (1.3) of our original optimal control problem. To this end, we introduce the deterministic real function on \( E \times A_0 \times A_T \) defined by \( v(x, a_0, a_T) := Y^{x,a_0,a_T}_0 \), and we show that \( v \) is a viscosity solution to (1.4). This second main result of the paper is stated in Theorem 5.1. By the uniqueness of the solution to the HJB equation (1.4) we conclude that

\[
Y^{x,a_0,a_T}_0 = V^*(x, a_0, a_T) = V(x).
\]  

(1.6)

Formula (1.6) gives the desired BSDE representation of the value function for the original control problem. This nonlinear Feynman-Kac formula can be used to design algorithms based on the numerical approximation of the solution to the constrained BSDE (4.3)–(4.4)–(4.5), and therefore to get probabilistic numerical approximations for the value function of the considered optimal control problem. Recently, numerical schemes for constrained BSDEs have been proposed and analyzed in the diffusive framework, see [24], and in the PDMPs context as well, see [1].

The paper is organized as follows. In Section 2 we introduce the optimal control (1.3), and we discuss its solvability. In Section 3 we develop the randomization of the control, by constructing an auxiliary PDMP on \( E \times A_0 \times A_T \) for which we formulate the randomized optimal control problem
In Section 3 we introduce the constrained BSDE (4.3)-(4.4)-(4.5) over infinite horizon, we show that it admits a unique maximal solution $(Y,Z,K)$ in a certain class of processes, and that $Y_0$ coincides with the value function of the randomized optimal control problem. Then, in Section 4 we prove that $Y_0$ also provides a viscosity solution to (4.4). Finally, some technical proofs are collected in the Appendix.

2 Optimal control of PDMPs on bounded domains

In this section we will formulate an optimal control problem for piecewise deterministic Markov processes on bounded domains, and we will discuss its solvability. The PDMP state space $E$ will be an open bounded subset of $\mathbb{R}^d$, and $\mathcal{E}$ the corresponding $\sigma$-algebra. Moreover, we introduce two Borel spaces (i.e. topological spaces homeomorphic to Borel subsets of compact metric spaces) $A_0$, $A_\Gamma$, endowed with their $\sigma$-algebras $A_0$ and $A_\Gamma$, that will be respectively the space of control actions in the interior and on the boundary of the domain. Given a topological space $F$, in the sequel we will denote by $\mathbb{C}_b(F)$ (resp. $\mathbb{C}_b^1(F)$) the set of all bounded continuous functions (resp. all bounded differentiable functions whose derivative is continuous) on $F$.

A controlled PDMP on $(E,\mathcal{E})$ is described by means of a set of local characteristics $(h,\lambda,Q,R)$, with $h,\lambda$ functions on $E \times A_0$, and $Q$, $R$ probability transition measures in $E$ respectively from $E \times A_0$ and from $\partial E \times A_\Gamma$. We will assume the following.

(HhAQR)

(i) $h : E \times A_0 \to \mathbb{R}^d$ and $\lambda : E \times A_0 \to \mathbb{R}_+$ are continuous and bounded functions, Lipschitz continuous on $E$, uniformly in $A_0$.

(ii) $Q$ (resp. $R$) maps $E \times A_0$ (resp. $\partial E \times A_\Gamma$) into the set of probability measures on $(E,\mathcal{E})$, and is a continuous stochastic kernel. Moreover, for all $v \in \mathbb{C}_b(E)$, the maps $(x,a_0) \mapsto \int_E v(y) Q(x,a_0,dy)$ and $(x,a_\Gamma) \mapsto \int_E v(y) R(x,a_\Gamma,dy)$ are Lipschitz continuous in $x$, uniformly in $a_0 \in A_0$ and in $a_\Gamma \in A_\Gamma$, respectively.

We will construct the process $X$ on $E$ in a canonical way. To this end, let $\Omega'$ be the set of sequences $\omega' = (t_n,e_n)_{n \geq 1}$ contained in $((0,\infty) \times E \cup \{(\infty,\Delta)\}$, where $\Delta \notin E$, is adjoined to $E$ as an isolated point, such that $t_n \leq t_{n+1}$, and $t_n < t_{n+1}$ if $t_n < \infty$. We define $\Omega = E \times \Omega'$, where $\omega = (x,\omega') = (x,t_1,e_1,t_2,e_2,...)$. On the sample space $\Omega$ we define the canonical functions $T_n : \Omega \to (0,\infty]$, $E_n : \Omega \to E \cup \{\Delta\}$ as follows: $T_0(\omega) = 0$, $E_0(\omega) = x$, and for $n \geq 1$, $T_n(\omega) = t_n$, $E_n(\omega) = e_n$, and $T_\infty(\omega) = \lim_{n \to \infty} t_n$. We also introduce, for any $B \in \mathcal{E}$, the counting process $N(s,B) = \sum_{n \in \mathbb{N}} \mathbb{1}_{T_n \leq s, E_n \in B}$ and the associated integer-valued random measure on $(0,\infty) \times E$

$$p(ds,dy) = \sum_{n \in \mathbb{N}} \mathbb{1}_{\{T_n,E_n\}}(ds,dy).$$

The class of admissible control law $A_{ad}$ will be the set of all $A_0 \otimes A_\Gamma$-measurable maps $\alpha = (\alpha^0,\alpha^\Gamma)$, where $\alpha^0 : [0,\infty) \times E \to A_0$ is a piecewise open-loop function of the form

$$\alpha^0_t = \alpha^0_0(t,x) \mathbb{1}_{[0,T_1)}(t) + \sum_{n=1}^{\infty} \alpha^0_n(t-T_n,E_n) \mathbb{1}_{[T_n,T_{n+1})}(t), \quad (2.1)$$
and $\alpha^\Gamma : \partial E \to A_\Gamma$ is a feedback policy. We define the controlled process $X : \Omega \times [0, \infty) \to \tilde{E} \cup \{\Delta\}$ setting

$$X_t = \begin{cases} 
\phi^{00}(t, x) & \text{if } t \in [0, T_1), \\
\phi^{00}(t - T_n, E_n) & \text{if } t \in [T_n, T_{n+1}), \; n \in \mathbb{N} \setminus \{0\},
\end{cases} \tag{2.2}$$

where $\phi^U(t, x)$, with $U$ any $A_0$-measurable function, is the unique solution to the ordinary differential equation

$$\dot{y}(t) = h(y(t), U(t)), \quad y(0) = x \in E,$$

on the interval $[0, t_*(x))$, with $t_*(x) = \inf\{t > 0 : y(t) \in \partial E\}$. Finally, we introduce the process

$$p^*_s := \sum_{n=1}^{\infty} 1_{\{s > T_n\}} 1\{X_{T_n} \in \partial E\}, \tag{2.3}$$

that counts the number of times that the process hits the boundary.

Set $F_0 = \mathcal{E} \otimes \{\emptyset, \Omega\}$ and, for all $t \geq 0$, $G_t = \sigma(p((0, s] \times B) : s \in (0, t], B \in \mathcal{E})$. For all $t$, let $F_t$ be the $\sigma$-algebra generated by $F_0$ and $G_t$. In the following all the concepts of measurability for stochastic processes will refer to the right-continuous, natural filtration $\mathbb{F} = (F_t)_{t \geq 0}$. By the symbol $\mathcal{P}$ we will denote the $\sigma$-algebra of $\mathbb{F}$-predictable subsets of $[0, \infty) \times \Omega$.

For every starting point $x \in E$ and for each $\alpha \in A_{ad}$, by Theorem 3.6 in [21], there exists a unique probability measure on $(\Omega, \mathcal{F}_\infty)$, denoted by $\mathbb{P}_x^\alpha$, such that its restriction to $F_0$ is the Dirac measure concentrated at $x$, and the $\mathbb{F}$-compensator under $\mathbb{P}_x^\alpha$ of the measure $p(ds \, dy)$ is

$$p^\alpha(ds \, dy) = \sum_{n=1}^{\infty} 1_{\{T_n, T_{n+1}\}}(s) 1\{\phi^{00}_n(s-T_n, E_n) \in \mathbb{E}\} \cdot \lambda(\phi^{00}(s-T_n, E_n), \alpha_0^{00}(s-T_n, E_n)) Q(\phi^{00}(s-T_n, E_n), \alpha_0^{00}(s-T_n, E_n), dy) \, ds$$

$$+ \sum_{n=1}^{\infty} 1_{\{T_n, T_{n+1}\}}(s) 1\{\phi^{00}_n(s-T_n, E_n) \in \partial \mathbb{E}\}
\cdot R(\phi^{00}(s-T_n, E_n), \alpha^\Gamma(\phi^{00}(s-T_n, E_n)), dy) \, dp^*_s.$$

Arguing as in Proposition 2.2 in [3], one can easily see that under $\mathbb{P}_x^\alpha$, the process $X$ in (2.2) is Markovian with respect to $\mathbb{F}$. In particular, for every $n \geq 1$, the conditional survival function of the inter-jump time $T_{n+1} - T_n$ on $\{T_n < \infty\}$ is

$$\mathbb{P}_x^\alpha(T_{n+1} - T_n > s \mid F_{T_n}) = \exp \left( - \int_{T_n}^{T_n+s} \lambda(\phi^{00}(r-T_n, X_{T_n}), \alpha_0^{00}(r-T_n, X_{T_n})) \, dr \right) \mathbb{1}_{\phi^{00}(s, X_{T_n}) \in \mathbb{E}}, \tag{2.4}$$

and the distribution of the post jump position $X_{T_{n+1}}$ on $\{T_n < \infty\}$ are

$$\begin{aligned}
&\mathbb{P}_x^\alpha(X_{T_{n+1}} \in B \mid F_{T_n}, T_{n+1}, \phi^{00}(T_{n+1} - T_n, X_{T_n}) \in \mathbb{E}) \\
&= Q(\phi^{00}(T_{n+1} - T_n, X_{T_n}), \alpha_0^{00}(T_{n+1} - T_n, X_{T_n}), B), \quad \forall B \in \mathcal{E}, \tag{2.5} \\
&\mathbb{P}_x^\alpha(X_{T_{n+1}} \in B \mid F_{T_n}, T_{n+1}, \phi^{00}(T_{n+1} - T_n, X_{T_n}) \in \Gamma) \\
&= R(\phi^{00}(T_{n+1} - T_n, X_{T_n}), \alpha^\Gamma(\phi^{00}(T_{n+1} - T_n, X_{T_n})), B), \quad \forall B \in \mathcal{E}. \tag{2.6}
\end{aligned}$$
Remark 2.1. Set
\[ t^0_+(x) := \inf\{ t > 0 : \phi^0(t,x) \in \partial E \}. \] (2.7)

Then, for any \( n \in \mathbb{N} \), \( s > 0 \), \( \phi^0(s,X_{T_n}) \in E \) (resp. \( \phi^0(s,X_{T_n}) \in \partial E \)) if and only if \( t^0_+(X_{T_n}) < s \) (resp. that \( t^0_+(X_{T_n}) \geq s \)).

In the classic infinite-horizon control problem one wants to minimize over all control laws \( \alpha \) a functional cost of the form
\[
J(x, \alpha) = \mathbb{E}^x \left[ \int_{(0,\infty)} e^{-\delta s} f(X_s, \alpha^0_s) \, ds + \int_{(0,\infty)} e^{-\delta s} c(X_{s-}, \alpha^\Gamma(X_{s-})) \, dp^*_s \right],
\] (2.8)
where \( f \) is a given real function on \( E \times A_0 \) representing the running cost, \( c \) is a given real function on \( \partial E \times A_\Gamma \) that associates a cost to hitting the active boundary, \( \delta \in (0, \infty) \) is a discounting factor. The value function of the control problem is defined in the usual way:
\[
V(x) = \inf_{\alpha \in \mathcal{A}_{ad}} J(x, \alpha), \quad x \in E.
\] (2.9)

We ask that \( f \) and \( c \) verify the following conditions.

(\text{Hfc}) \quad f : E \times A_0 \to \mathbb{R}_+ \ (\text{resp. } c : \partial E \times A_\Gamma \to \mathbb{R}_+) \text{ is a continuous and bounded function, Lipschitz continuous on } E \ (\text{resp. on } \partial E), \text{ uniformly in } A_0 \ (\text{resp. } A_\Gamma). \text{ In particular,}

\[
\begin{align*}
|f(x,a)| &\leq M_f, \quad \forall x \in E, \ a \in A_0, \\
|c(x,a)| &\leq M_c, \quad \forall x \in \partial E, \ a \in A_\Gamma.
\end{align*}
\]

Moreover, let us set
\[
E_\varepsilon := \left\{ x \in E : \inf_{\alpha^0 \in A_0} t^0_+(x) \geq \varepsilon \right\}.
\]

We will consider the following assumption.

(\text{H0}) \quad \text{There exists } \varepsilon > 0 \text{ such that } R(x,\alpha,E_\varepsilon) = 1 \text{ for all } x \in \partial E \text{ and } \alpha \in \mathcal{A}_\Gamma.

Remark 2.2. Informally, condition (\text{H0}) says that the jumps from the boundary are always to points whose distance from the boundary (as measured by the boundary hitting time \( t^0_+ \)) are uniformly bounded away from zero. Together with the boundedness assumption of the jump rate \( \lambda \) in (\text{Hh\lambda\text{QR}})-(i), it insures that for every starting point \( x \in E \) and admissible control \( \alpha \in \mathcal{A}_{ad} \),
\[
\mathbb{E}^x_\alpha \left[ \sum_{n \geq 1} 1_{\{ t \geq T_n \}} \right] < \infty, \quad \forall t \in \mathbb{R}_+,
\] (2.10)
see the proof of Proposition 24.6 in \[18\]. This implies in particular that
\[
\mathbb{E}^x_\alpha \left[ \sum_{n \geq 1} 1_{\{ t \geq T_n \}} \right] < \infty, \quad \forall t \in \mathbb{R}_+.
\] (2.11)

Let us now define \( W_t := \int_{[0,t]} e^{-\delta s} \, dp^*_s \). From the integration by parts formula for processes of finite variation (see e.g., Proposition 4.5 in \[28\]), together with estimate (2.10), we get
\[
\mathbb{E}^x_\alpha [W_t] = \mathbb{E}^x_\alpha \left[ e^{-\delta t} p_t^* + \int_0^t \delta e^{-\delta s} p_s^* \, ds \right] \leq e^{-\delta t} (\varepsilon^{-1} t + 1) + \int_0^t \delta e^{-\delta s} (\varepsilon^{-1} s + 1) \, ds
\]
Then, setting $C^* := \varepsilon^{-1} + 1$, by the Lebesgue dominated convergence theorem we have

$$\mathbb{E}^x_\alpha \left[ \int_{(0,\infty)} e^{-\delta t} \, dp_t \right] \leq C^*. \tag{2.12}$$

Finally, we assume the following behavior of the state trajectory near the boundary of the state space.

**(HBB)**
For all $x \in \partial E$, if there exists $a_0 \in A_0$ such that $-h(x,a_0) \cdot n(x) \geq 0$, then there exists $a'_0 \in A_0$ such that $-h(x,a'_0) \cdot n(x) > 0$.

**(HBB')**
For all $x \in \partial E$, if $-h(x,a_0) \cdot n(x) \geq 0$ for all $a_0 \in A_0$, then $-h(x,a_0) \cdot n(x) > 0$ for all $a_0 \in A_0$.

Assumptions (HBB)-(HBB') are hypotheses of non-degeneracy defined to remove difficulties arising from trajectories which are tangent to the boundary, see [8] for more details on this subject.

Let us now consider the Hamilton-Jacobi-Bellman (for short, HJB) equation associated to the optimal control problem: this is the following elliptic nonlinear integro-differential equation on $[0, \infty) \times \bar{E}$ with nonlocal boundary conditions

$$H^v(x,v(x),\nabla v(x)) = 0 \quad \text{in } E, \quad \tag{2.13}$$

$$v(x) = F^v(x) \quad \text{on } \partial E, \quad \tag{2.14}$$

where

$$H^v(z,u,p) := \sup_{a_0 \in A_0} \left\{ \delta u - h(z,a_0) \cdot p - f(z,a_0) - \int_E (\psi(y) - \psi(z)) \lambda(z,a_0) Q(z,a_0,dy) \right\},$$

$$F^v(x) := \min_{a_\Gamma \in A_\Gamma} \left\{ c(z,a_\Gamma) + \int_E \psi(y) R(z,a_\Gamma,dy) \right\}.$$

**Definition 2.1.** *(i)* A bounded u.s.c. function $u$ on $\bar{E}$ is a viscosity subsolution of \textit{2.13} \textit{--} \textit{2.14} if and only if, $\forall \phi \in C^1_b(\bar{E})$, if $x_0 \in \bar{E}$ is a global maximum of $u - \phi$ one has

$$H^u(x_0,u(x_0),\nabla \phi(x_0)) \leq 0 \quad \text{if } x_0 \in E,$$

$$\min\{H^u(x_0,u(x_0),\nabla \phi(x_0)),u(x_0) - F^u(x_0)\} \leq 0 \quad \text{if } x_0 \in \partial E.$$

(ii) A bounded l.s.c. function $w$ on $\bar{E}$ is a viscosity supersolution of \textit{2.13} \textit{--} \textit{2.14} if and only if, $\forall \phi \in C^1_b(\bar{E})$, if $x_0 \in \bar{E}$ is a global minimum of $w - \phi$ one has

$$H^w(x_0,w(x_0),\nabla \phi(x_0)) \geq 0 \quad \text{if } x_0 \in E,$$

$$\max\{H^w(x_0,w(x_0),\nabla \phi(x_0)),w(x_0) - F^w(x_0)\} \geq 0 \quad \text{if } x_0 \in \partial E.$$

(iii) A viscosity solution of \textit{2.13} \textit{--} \textit{2.14} is a continuous function which is both subsolution and supersolution of \textit{2.13} \textit{--} \textit{2.14}.

The following theorem collects the results of Theorems 5.8 and 7.5 in [19].

**Theorem 2.1.** Let (HhλQR), (Hfc), (H0), (HBB) and (HBB') hold, and assume that $A_0$, $A_\Gamma$ are compact. Let $V : E \to \mathbb{R}$ be value function of the PDMPs optimal control problem \textit{2.9}.

Then $V$ is a bounded and continuous function, and is the unique viscosity solution of \textit{2.13} \textit{--} \textit{2.14}.
3 The randomized optimal control problem

In the present section we begin to exploit the control randomization method. In the control problem introduced in Section 2 one has to choose two strategies, namely the process $\alpha^0$ acting on the dynamics in the interior of the domain, and the process $\alpha^T$ acting on the boundary jump mechanism. Therefore, differently to the existing literature dealing with the control randomization method (see e.g. [4], [3] for the non-diffusive framework), here the control process to be randomized will be a pair of processes on $A_0 \times A_\Gamma$. More precisely, for any starting point $(x, a_0, a_\Gamma) \in E \times A_0 \times A_\Gamma$, we construct an uncontrolled PDMP $(X, I, J)$ with values in $E \times A_0 \times A_\Gamma$ by specifying its local characteristics, see (3.1)-(3.2)-(3.4)-(3.3) below, (or, equivalently, its compensator $\tilde{p}(ds \ dy \ db \ dc)$.

Then, we formulate an auxiliary (or randomized or dual) optimal control problem where we modify the intensity of the processes $I$ and $J$: for any pair of predictable, bounded and positive random fields $\nu = (\nu^0_t(b), \nu^T_t(b))$ on $((0, \infty) \times A_0, (0, \infty) \times A_\Gamma)$, we look for a probability measure $\mathbb{P}_\nu$ under which the compensator of $I$ and $J$ is multiplied respectively by $\nu^0_t(b)$ and $\nu^T_t(c)$. The probability measure $\mathbb{P}_\nu$ can be constructed by means of a Girsanov type theorem for point processes, see Theorem 4.5 in [21]. Notice that, since $\tilde{p}$ is a non quasi-left-continuous random measure, the above mentioned theorem holds under an additional condition involving the intensity control $\nu$ and the predictable jumps of $\tilde{p}$, see the proof of Proposition 3.1. The correct formulation of our randomized control problem requires some effort to take into account this fact.

3.1 A randomized control system

Let $E$ still be an open bounded subset of $\mathbb{R}^d$ with $\sigma$-algebra $\mathcal{E}$, and $A_0, A_\Gamma$ two Borel spaces with corresponding $\sigma$-algebras $\mathcal{A}_0, \mathcal{A}_\Gamma$. Let moreover $h, \lambda$ be two real functions on $E \times A_0$ and $Q$ (resp. $R$) be a probability transition from $(E \times A_0, \mathcal{E} \otimes \mathcal{A}_0)$ (resp. from $(\partial E \times A_\Gamma, \mathcal{E} \otimes \mathcal{A}_\Gamma)$), satisfying (Hh$\lambda$QR) as before. We denote by $\phi(t, x, a_0)$ the unique solution to the ordinary differential equation

$$\dot{x}(t) = h(x(t), a_0), \quad x(0) = x, \quad a_0 \in A_0.$$ 

Notice that $\phi(t, x, a_0)$ is given by the function $\phi^U(t, x)$, introduced in Section 2 when $U(t) \equiv a_0$.

At this point we introduce two other measures $\lambda_0$ and $\lambda_\Gamma$ on $(A_0, A_0)$ and $(A_\Gamma, A_\Gamma)$, satisfying the following condition:

(H$\lambda_0\lambda_\Gamma$) $\lambda_0$ and $\lambda_\Gamma$ are two finite measures, respectively on $(A_0, A_0)$ and $(A_\Gamma, A_\Gamma)$, with full topological support.

For all $t \geq 0, (a_0, a_\Gamma) \in A_0 \times A_\Gamma$, let us define

$$\tilde{\phi}(t, x, a_0, a_\Gamma) := (\phi(t, x, a_0), a_0, a_\Gamma), \quad x \in E, \quad (3.1)$$

$$\tilde{\lambda}(x, a_0) := \lambda(x, a_0) + \lambda_0(A_0) + \lambda_\Gamma(A_\Gamma), \quad x \in E, \quad (3.2)$$

$$\tilde{R}(x, a_0, a_\Gamma, dy \ db \ dc) := R(x, a_\Gamma, dy) \delta_{a_0}(db) \delta_{a_\Gamma}(dc), \quad x \in \partial E, \quad (3.3)$$

and,

$$\tilde{Q}(x, a_0, a_\Gamma, dy \ db \ dc) := \frac{\lambda(x, a_0) Q(x, a_0, dy) \delta_{a_0}(db) \delta_{a_\Gamma}(dc) + \lambda_0(db) \delta_{a_\Gamma}(dc) \delta_x(dy) + \lambda_\Gamma(dc) \delta_{a_0}(db) \delta_x(dy)}{\lambda(x, a_0)}, \quad x \in E, \quad (3.4)$$
where, for any $F$ topological space, $\delta_a$ denotes the Dirac measure concentrated at some point $a \in F$. Our purpose is to construct a PDMP $(X, I, J)$ with enlarged state space $E \times A_0 \times A_{\Gamma}$ and local characteristics $(\hat{\phi}, \hat{\lambda}, \hat{Q}, \hat{R})$. We construct the process in a canonical way, proceeding as in Section 2. In particular, we define $\Omega'$ as the set of sequences $\omega' = (t_n, e_n, a_n^0, a_n^\Gamma)_{n \geq 1}$ contained in $((0, \infty) \times E \times A_0 \times A_{\Gamma}) \cup \{ (\infty, \Delta, \Delta', \Delta'') \}$, where $\Delta \notin E$, $\Delta' \notin A_0$, $\Delta'' \notin A_{\Gamma}$ are isolated points respectively adjoined to $E$, $A_0$ and $A_{\Gamma}$. In the sample space $\Omega = \Omega' \times E \times A_0 \times A_{\Gamma}$ we define the random variables $T_0(\omega) = 0$, $E_0(\omega) = x$, $A_0^0(\omega) = a_0$, $A_0^\Gamma(\omega) = a_{\Gamma}$, and the sequence of random variables $T_n : \Omega \to (0, \infty]$, $E_n : \Omega \to E \cup \{ \Delta \}$, $A_n^0 : \Omega \to A_0 \cup \{ \Delta' \}$, $A_n^\Gamma : \Omega \to A_{\Gamma} \cup \{ \Delta'' \}$, for $n \geq 1$, by setting $T_n(\omega) = t_n$, $E_n(\omega) = e_n$, $A_n^0(\omega) = a_n^0$, $A_n^\Gamma(\omega) = a_n^\Gamma$, with $T_\infty(\omega) = \lim_{n \to \infty} t_n$. Then, we define the process $(X, I, J)$ on $(E \times A_0 \times A_{\Gamma}) \cup \{ \Delta, \Delta', \Delta'' \}$ as

$$
(X, I, J)_t = \begin{cases} 
(\phi(t - T_n, E_n, A_n^0), A_n^0, A_n^\Gamma) & \text{if } T_n \leq t < T_{n+1}, \text{ for } n \in \mathbb{N}, \\
(\Delta, \Delta', \Delta'') & \text{if } t \geq T_\infty.
\end{cases}
$$

By abuse of notation, in $\Omega$ we define the random measure $p$ on $(0, \infty) \times E \times A$ as

$$
p(ds \, dy \, db \, dc) = \sum_{n \in \mathbb{N}} \mathbb{I}_{[T_n, T_{n+1})}(s) \Lambda(\phi(s - T_n, E_n, A_n^0), A_n^0, A_n^\Gamma, dy \, db \, dc) \, dA_s,
$$

and, for all $t \geq 0$, we introduce the $\sigma$-algebras $\mathcal{G}_t = \sigma(p((0, s] \times G) : s \in (0, t], G \in \mathcal{E} \otimes A_0 \times A_{\Gamma})$, and the $\sigma$-algebra $\mathcal{F}_t$ generated by $\mathcal{F}_0$ and $\mathcal{G}_t$, where $\mathcal{F}_0 = \mathcal{E} \otimes A_0 \otimes A_{\Gamma} \otimes \{ \emptyset, \Omega' \}$. We still denote by $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$ and $\mathcal{P}$ the corresponding filtration and predictable $\sigma$-algebra.

Given any starting point $(x, a_0, a_{\Gamma}) \in E \times A_0 \times A_{\Gamma}$, by Proposition 2.1 in [3], there exists a unique probability measure on $(\Omega, \mathcal{F}_\infty)$, denoted by $\mathbb{P}^{x, a_0, a_{\Gamma}}$, such that its restriction to $\mathcal{F}_0$ is $\delta_{(x, a_0, a_{\Gamma})}$ and the $\mathcal{F}$-compensator of the measure $p(ds \, dy \, db \, dc)$ under $\mathbb{P}^{x, a_0, a_{\Gamma}}$ is the random measure $\tilde{p}(ds \, dy \, db \, dc) = \sum_{n \in \mathbb{N}} \mathbb{I}_{[T_n, T_{n+1})}(s) \Lambda(\phi(s - T_n, E_n, A_n^0), A_n^0, A_n^\Gamma, dy \, db \, dc) \, dA_s$.

where, for all $(x, a_0, a_{\Gamma}) \in E \times A_0 \times A_{\Gamma}$,

$$
\Lambda(x, a_0, a_{\Gamma}, dy \, db \, dc) = \tilde{Q}(x, a_0, a_{\Gamma}, dy \, db \, dc) \, \mathbb{1}_{x \in E} + \tilde{R}(x, a_0, a_{\Gamma}, dy \, db \, dc) \, \mathbb{1}_{x \in A_{\Gamma}},
$$

and $A_s$ is the increasing, predictable process such that, for any $s \geq 0$,

$$
dA_s(\omega) = \tilde{\lambda}(X_{s-}(\omega), I_{s-}(\omega)) \, \mathbb{1}_{X_{s-}(\omega) \in E} \, ds + \mathbb{1}_{X_{s-}(\omega) \in A_{\Gamma}} \, dp^*_s(\omega).
$$

In particular,

$$
\Delta A_t(\omega) = \tilde{p}(\omega, \{ t \} \times E \times A_0 \times A_{\Gamma}) = \mathbb{1}_{X_t(\omega) \in \partial E},
$$

$$
\tilde{p}(\omega, \{ t \} \times dy \, db \, dc) = \tilde{R}(X_t(\omega), I_t-(\omega), J_t-(\omega), dy \, db \, dc) \, \Delta A_t(\omega).
$$

**Remark 3.1.** The $\mathcal{F}$-compensator of the measure $p(ds \, dy \, db \, dc)$ under $\mathbb{P}^{x, a_0, a_{\Gamma}}$ can be decomposed in the following way:

$$
\tilde{p}(\omega, ds \, dy \, db \, dc) = \phi(\omega, t)(dy \, db \, dc) \, dA_s(\omega),
$$

with

$$
\phi(\omega, t)(dy \, db \, dc) := \Lambda(X_t-(\omega), I_t-(\omega), J_t-(\omega), dy \, db \, dc).
$$
The process \((X, I, J)\) is Markovian on \([0, \infty)\) with respect to \(\mathbb{F}\). For every real-valued functions \(\varphi\) defined on \(E \times A_0 \times A_\Gamma\), we define
\[
\mathcal{L}\varphi(x, a_0, a_\Gamma) := h(x, a_0) \cdot \nabla_x \varphi(x, a_0, a_\Gamma) + \int_{E} (\varphi(y, a_0, a_\Gamma) - \varphi(x, a_0, a_\Gamma)) \lambda(x, a_0) Q(x, a_0, dy) \\
+ \int_{A_0} (\varphi(x, b, a_\Gamma) - \varphi(x, a_0, a_\Gamma)) \lambda_0(db) + \int_{A_\Gamma} (\varphi(x, a, c) - \varphi(x, a_0, a_\Gamma)) \lambda_\Gamma(dc), \quad x \in E,
\]
\[
\mathcal{G}\varphi(x, a_0, a_\Gamma) := \int_{E} (\varphi(y, a_0, a_\Gamma) - \varphi(x, a_0, a_\Gamma)) R(x, a_\Gamma, dy), \quad x \in \partial E.
\]
From Theorem 26.14 in [15] it follows that \(\mathcal{L}\) is the extended generator of the process \((X, I, J)\) and \(\mathcal{G}\varphi = 0\) if and only if \(\varphi\) belongs to the domain of \(\mathcal{L}\). In the sequel the operators \(\mathcal{L}\) and \(\mathcal{G}\) will be applied to test functions with suitable regularity.

### 3.2 The randomized optimal control problem

Let us now introduce a randomized optimal control problem associated to the process \((X, I, J)\). For fixed \((x, a_0, a_\Gamma)\), we consider a family of probability measures \(\{\mathbb{P}^{\nu, a_0, a_\Gamma}, \nu \in \mathcal{V}\}\) in the space \((\Omega, \mathcal{F}_\infty)\), whose effect is to change the stochastic intensity of the process \((X, I, J)\).

We formalize our problem. We define \(\mathcal{V} := \mathcal{V}_0 \otimes \mathcal{V}_\Gamma\), where
\[
\mathcal{V}_0 = \{\nu^0 : \Omega \times [0, \infty) \times A_0 \rightarrow (0, \infty) \ \mathcal{P} \otimes \mathcal{A}_0\text{-measurable and bounded}\},
\]
\[
\mathcal{V}_\Gamma = \{\nu^\Gamma : \Omega \times [0, \infty) \times A_\Gamma \rightarrow (0, \infty) \ \mathcal{P} \otimes \mathcal{A}_\Gamma\text{-measurable and bounded}\}.
\]

For every \(\nu = (\nu^0, \nu^\Gamma) \in \mathcal{V}\), let us define, for all \(t \geq 0\), \(x \in E\), \((a_0, a_\Gamma) \in A_0 \times A_\Gamma\),
\[
\hat{\lambda}^\nu(t, x, a_0) := \lambda(x, a_0) + \int_{A_0} \nu^0_t(b) \lambda_0(db) + \int_{A_\Gamma} \nu^\Gamma_t(c) \lambda_\Gamma(dc), \quad (3.12)
\]
\[
\mathcal{Q}^\nu(t, x, a_0, a_\Gamma, dy db dc) := \frac{\lambda(x, a_0) Q(x, a_0, dy) \delta_{a_0}(db) \delta_{a_\Gamma}(dc) + \nu^0_t(b) \lambda_0(db) \delta_{a_\Gamma}(dc) \delta_x(dy) + \nu^\Gamma_t(c) \lambda_\Gamma(dc) \delta_{a_0}(db) \delta_x(dy)}{\lambda^\nu(t, x, a_0)}, \quad (3.13)
\]
and, for all \(t \geq 0\), \(x \in \bar{E}\), \((a_0, a_\Gamma) \in A_0 \times A_\Gamma\),
\[
\Lambda^\nu(t, x, a_0, a_\Gamma, dy db dc) = \hat{\mathcal{Q}}^\nu(t, x, a_0, a_\Gamma, dy db dc) 1_{x \in E} + \tilde{R}(x, a_0, a_\Gamma, dy db dc) 1_{x \in \partial E}. \quad (3.14)
\]

Then, for every \(\nu = (\nu^0, \nu^\Gamma) \in \mathcal{V}\), we consider the predictable random measure
\[
\tilde{\mathcal{P}}^\nu(ds dy db dc) = \sum_{n \in \mathbb{N}} \mathbb{1}_{[\tau_n, \tau_{n+1})}(s) \Lambda^\nu(s, \phi(s - T_n, E_n, A^0_n), A^0_n, A^\Gamma_n, dy db dc) dA^\nu_s, \quad (3.15)
\]
where, for all \((x, a_0, a_\Gamma) \in E \times A_0 \times A_\Gamma\), \(A^\nu_s\) is the increasing, predictable process such that
\[
dA^\nu_s = \hat{\lambda}^\nu(s, X_{\cdot -}, I_{\cdot -}) 1_{X_{\cdot -} \in E} ds + 1_{X_{\cdot -} \in \partial E} dP^*_s, \quad (3.16)
\]
By the Radon-Nikodym theorem one can find three nonnegative functions \(d_1, d_2, d_3\) defined on \(\Omega \times [0, \infty) \times E \times A_0 \times A_\Gamma, \ \mathcal{P} \otimes \mathcal{E} \otimes \mathcal{A}_0 \otimes \mathcal{A}_\Gamma\), such that
\[
d_1(t, y, b, c) \tilde{P}(dt dy db dc) = \delta_{a_\Gamma}(db) \delta_{I_{\cdot -}}(dy) \delta_{X_{\cdot -}}(dc) 1_{X_{\cdot -} \in E} dt,
\]

```
where the latter equality follows from (3.20). On the other hand, by (3.8) we have condition (3.19) follows.

Moreover, (3.9) implies

Indeed, if condition (3.19) holds, then the result would be a direct application of Theorem 4.5 [21].

\[ \alpha \]

with

as compensator under \( P \) probability measure \( \mathbb{P} \).

Remark 3.2. Notice that, by construction, \( d_1(t, y, b, c) \mathbb{1}_{X_t \in \partial E} = d_2(t, y, b, c) \mathbb{1}_{X_t \in \partial E} = 0 \), and \( d_3(t, y, b, c) \mathbb{1}_{X_t \in \partial E} = \mathbb{1}_{X_t \in \partial E} = 0 \).

For any \( \nu \in \mathcal{V} \), consider then the Doléans-Dade exponential local martingale \( L^\nu \) defined by

\[
L^\nu_s = e^{\int_0^s \int_{A_0} (1 - \nu^0_t(b)) \lambda_0(db) \ d\tau} e^{\int_0^s \int_{A_0} (1 - \nu^1_t(c)) \lambda_1(dc) \ d\tau} \ 
\]

\[
\cdot \prod_{n \geq 1 : T_n \leq s} (\nu^0_{T_n}(A_n^0) d_1(T_n, E_n, A_0^0, A_n^0) + \nu^1_{T_n}(A_n^1) d_2(T_n, E_n, A_0^1, A_n^1) + d_3(T_n, E_n, A_0^1, A_n^1)),
\]

for \( s \geq 0 \). We have the following important result.

Proposition 3.1. When \( (L_t^\nu)_{t \geq 0} \) is a true martingale, for every time \( T > 0 \) we can define a probability measure \( \mathbb{P}^{x, a_0, a_T}_{\nu, T} \) equivalent to \( \mathbb{P}^{x, a_0, a_T}_{\nu, T} \) on \( (\Omega, \mathcal{F}_T) \) by

\[
\mathbb{P}^{x, a_0, a_T}_{\nu, T}(d\omega) = L^\nu_T(\omega) \mathbb{P}^{x, a_0, a_T}_{\nu, T}(d\omega).
\]

The restriction of the random measure \( p \) to \( (0, T] \times E \times A_0 \times A_T \) admits \( \tilde{p}^\nu = (\nu d_1 + \nu^1 d_2 + d_3) \tilde{p} \) as compensator under \( \mathbb{P}^{x, a_0, a_T}_{\nu, T} \).

Proof. We shall prove that

\[
\tilde{\nu}_t = 1 \text{ whenever } \alpha_t = 1, \quad (3.19)
\]

with \( \alpha_t := \tilde{p}(\{t\} \times E \times A_0 \times A_T) \), and

\[
\tilde{\nu}_t(y, b, c) := \nu^0_t(b) d_1(t, y, b, c) + \nu^1_t(c) d_2(t, y, b, c) + d_3(t, y, b, c),
\]

\[
\tilde{\nu}_t := \int_{E \times A_0 \times A_T} \tilde{\nu}_t(y, b, c) \tilde{p}(\{t\} \times dy \ db \ dc).
\]

Indeed, if condition (3.19) holds, then the result would be a direct application of Theorem 4.5 [21]. Let us thus show the validity of (3.19). To this end, we start by noticing that, by Remark 3.2,

\[
\tilde{\nu}_s(y, b, c) \mathbb{1}_{X_s \in \partial E} = \mathbb{1}_{X_s \in \partial E}.
\]

(3.20)

Moreover, (3.9) implies

\[
\int_E \tilde{\nu}(t, y, b, c) \tilde{p}(\{t\} \times dy \ db \ dc) = \int_E \tilde{\nu}(t, y, I_{s-}, J_{s-}) R(X_{s-}, J_{s-}, dy) \mathbb{1}_{X_{s-} \in \partial E} = \mathbb{1}_{X_{s-} \in \partial E},
\]

where the latter equality follows from (3.20). On the other hand, by (3.8) we have \( \alpha_t = \mathbb{1}_{X_t \in \partial E} \), and condition (3.19) follows. \( \square \)
In what follows we will denote by \( q = p - \tilde{p} \) the compensated martingale measure associated to \( p \). Following the notation in [22], we introduce the random sets:

\[
D := \{ (\omega, t) : p(\omega, \{ t \} \times E \times A_0 \times A_\Gamma) > 0 \}, \tag{3.21}
\]

\[
J := \{ (\omega, t) : \tilde{p}(\omega, \{ t \} \times E \times A_0 \times A_\Gamma) > 0 \}, \tag{3.22}
\]

\[
K := \{ (\omega, t) : \tilde{p}(\omega, \{ t \} \times E \times A_0 \times A_\Gamma) = 1 \}. \tag{3.23}
\]

Notice that, by (3.8), for any \( t \geq 0 \),

\[
J = K = \{ (\omega, t) : \Delta A_t(\omega) = 1 \} = \{ (\omega, t) : X_{t-}(\omega) \in \partial E \}. \tag{3.24}
\]

We also denote by \( G^2_{x,a_0,a_1} (q; 0, T) \), \( T > 0 \), the set of \( \mathcal{P}_T \otimes \mathcal{B}(E) \otimes A_0 \otimes A_\Gamma \)-measurable maps \( Z : \Omega \times [0, T] \times E \times A_0 \times A_\Gamma \to \mathbb{R} \) with \( ||Z||^2_{G^2_{x,a_0,a_1}, (q; 0, T)} < \infty \), where

\[
||Z||^2_{G^2_{x,a_0,a_1}, (q; 0, T)} := \mathbb{E}^{x,a_0,a_1} \left[ \int_{\{0,T\}} \int_{E \times A_0 \times A_\Gamma} |Z_t(y,b,c) - \hat{Z}_t|^2 \tilde{p}(dt\,dy\,db\,dc) + \sum_{s \in (0,T]} |\hat{Z}_s|^2 (1 - \Delta A_t) \right], \tag{3.25}
\]

with

\[
\hat{Z}_t := \int_{E \times A_0 \times A_\Gamma} Z_t(y,b,c) \tilde{p}\{\{t\} \times dy\,db\,dc), \quad 0 \leq t \leq T. \tag{3.26}
\]

We set \( G^2_{x,a_0,a_1, loc} (q; 0, T) := \bigcap_{T > 0} G^2_{x,a_0,a_1} (q; 0, T) \).

**Lemma 3.2.** Let \( H \in G^2_{x,a_0,a_1} (q; 0, T) \). Then

\[
(i) \quad ||H||^2_{G^2_{x,a_0,a_1}, (q; 0, T)} = \mathbb{E}^{x,a_0,a_1} \left[ \sum_{s \in (0,T]} \left| \int_{E \times A_0 \times A_\Gamma} H_s(y,b,c) q\{\{s\} \times dy\,db\,dc \right|^2 \right]
\]

\[
(ii) \quad ||H||^2_{G^2_{x,a_0,a_1}, (q; 0, T)} = \mathbb{E}^{x,a_0,a_1} \left[ \int_{\{0,T\}} \int_{E \times A_0 \times A_\Gamma} |H_t(y,b,c) - \hat{H}_t 1_{\{t\}}|^2 \tilde{p}(dt\,dy\,db\,dc) \right].
\]

**Proof.** Concerning item (i), we have

\[
\left| \int_{E \times A_0 \times A_\Gamma} H_s(y,b,c) q\{\{s\} \times dy\,db\,dc \right|^2 = \int_{E \times A_0 \times A_\Gamma} |H_s(y,b,c) - \hat{H}_s|^2 p\{\{s\} \times dy\,db\,dc \right) - \int_{E \times A_0 \times A_\Gamma} \hat{H}_s^2 p\{\{s\} \times dy\,db\,dc \right) + |\hat{H}_s|^2 + 2 \int_{E \times A_0 \times A_\Gamma} \hat{H}_s H_s(y,b,c) p\{\{s\} \times dy\,db\,dc \right) - 2 \hat{H}_s \int_{E \times A_0 \times A_\Gamma} H_s(y,b,c) p\{\{s\} \times dy\,db\,dc \right),
\]

and the conclusion follows from (3.8) and (3.26). Regarding item (ii), (3.25) can be rewritten

\[
\mathbb{E}^{x,a_0,a_1} \left[ \int_{\{0,T\}} \int_{E \times A_0 \times A_\Gamma} |H_t(y,b,c) - \hat{H}_t 1_{\{t\}}|^2 \tilde{p}(dt\,dy\,db\,dc) + \sum_{0 < t \leq T} |\hat{H}_t|^2 (1 - \Delta A_t) 1_{\{t\}} 1_{\{t\}} \right],
\]

see e.g. Remark 2.6 in [5], and we conclude by (3.24). \( \square \)

For any stopping time \( \tau \), we denote by \( \{ \tau \} \) the random set \( \{ (\omega, \tau(\omega)) \} \subset \Omega \times [0, \infty] \).
Proposition 3.3. Let $D$ be the random set in (3.21). Then $D = K \cup (\cup_n \{T^n\})$ up to an evanescent set, where $(T^n)_n$ are totally inaccessible times such that $\{T^n\} \cap \{T^m\} = \emptyset$, $n \neq m$.

Proof. Set $\tilde{p}^c := p\tilde{I}_J$ and $\tilde{p}^c := \tilde{p}\tilde{I}_J$. The measure $\tilde{p}^c$ is the compensator of $p^c$, see paragraph b) in [22]. We have

$$\tilde{p}^c(ds \, dy \, db \, dc) = \tilde{Q}(X_s, I_{s-}, J_{s-}, dy \, db \, dc) 1_{X_s \in E} \, ds.$$

We remark that $D \cap J^c = \{ (\omega, t) : p^c(\omega, \{t\} \times \mathbb{R}) > 0 \}$. On the other hand, being $\tilde{p}^c$ absolutely continuous with respect to the Lebesgue measure, we have $\{ (\omega, t) : p^c(\omega, \{t\} \times \mathbb{R}) > 0 \} = \cup_n \{T^n\}$, for some $(T^n)_n$ totally inaccessible times, see, e.g., Assumption (A) in [15]. Therefore, since $J = K$ by (3.24), we have $D = K \cup (D \cap J^c)$, and the conclusion follows.

Let us set $q^\nu := p - \tilde{p}^\nu$, and denote by $\mathbb{E}_x^\nu_{\cdot}$ the expectation operator under $\mathbb{P}_x^\nu_{\cdot}$.

Lemma 3.4. Let assumptions (HhλQR) and (Hλ0λΓ) hold. Then, for every $(x, a_0, a_T) \in E \times A_0 \times A_T$ and $\nu \in V$, under the probability $\mathbb{P}_x^{a_0, a_T}$, the process $(L^\nu_t)_{t \geq 0}$ is a martingale.

Moreover, for every time $T > 0$, $L^\nu_T$ is square integrable, and, for every $H \in G_{x, a_0, a_T}^2(q; 0, T)$, the process

$$M^\nu_t := \int_{[0, \infty]} \int_{E \times A_0 \times A_T} H_s(y, b, c) \, q^\nu(ds \, dy \, db \, dc), \quad t \in [0, T], \quad (3.27)$$

is a square integrable $\mathbb{P}_x^{a_0, a_T}$-martingale on $[0, T]$.

Proof. The first part of the result is a consequence of Theorem 5.2 in [21]. The square integrability property of $L^\nu_T$ can be proved arguing as in the proof of Lemma 3.2 in [4]. Finally, Proposition 3.71-(a) in [22] implies that the stochastic integral

$$\int_{(t, T]} \int_{E \times A_0 \times A_T} H_s(y, b, c) \, q(ds \, dy \, db \, dc)$$

is well-defined, and, by Proposition 3.66 in [22], the process

$$M_t := \int_{[0, \infty]} \int_{E \times A_0 \times A_T} H_s(y, b, c) \, q(ds \, dy \, db \, dc), \quad t \in [0, T],$$

is a square integrable $\mathbb{P}_x^{a_0, a_T}$-martingale. Using the Burkholder-Davis-Gundy and Cauchy-Schwarz inequalities, together with the square integrability of $L^\nu_T$, we see that (3.27) is a square integrable $\mathbb{P}_x^{a_0, a_T}$-martingale.

In order to get a suitable probability measure on $(\Omega, F_\infty)$, we can extend the previous construction to the infinite horizon. The following result is based on the Kolmogorov extension theorem for product spaces, see e.g. Theorem 1.1.10 in [30], and can be proved proceeding as in Proposition 3.2 in [3].

Proposition 3.5. Let assumptions (HhλQR) and (Hλ0λΓ) hold. Then, for every $(x, a_0, a_T) \in E \times A_0 \times A_T$ and $\nu \in V$, there exists a unique probability $\mathbb{P}_x^{a_0, a_T}$ on $(\Omega, F_\infty)$, under which the random measure $\tilde{p}^\nu$ in (3.15) is the compensator of the measure $p$ in (3.14) on $(0, \infty) \times E \times A_0 \times A_T$. Moreover, for any time $T > 0$, the restriction of $\mathbb{P}_x^{a_0, a_T}$ on $(\Omega, F_T)$ is given by the probability measure $\mathbb{P}_x^{a_0, a_T}_{\nu, T}$ in (3.18).
Finally, for every \((x, a_0, a_T) \in E \times A_0 \times A_T\) and \(\nu \in \mathcal{V}\), we introduce the functional cost
\[
J(x, a_0, a_T, \nu) := \mathbb{E}_\nu^{x, a_0, a_T} \left[ \int_{(0, \infty)} e^{-\delta s} f(X_s, I_s) \, ds + \int_{(0, \infty)} e^{-\delta s} c(X_{s-}, J_{s-}) \, dp_s \right],
\]
and the randomized (or dual) value function
\[
V^*(x, a_0, a_T) := \inf_{\nu \in \mathcal{V}} J(x, a_0, a_T, \nu).
\]

4 Constrained BSDEs and the representation of \(V^*\)

In the present section we introduce a BSDE with two sign constraints on its martingale part, that will provide a probabilistic representation formula for the randomized value function introduced in (3.29). We study the well-posedness of this constrained BSDE in an appropriate sense: proceeding by a means of a penalization argument, we prove that the admits a unique maximal solution. Notice that, differently from the previous literature, the penalized BSDE is of non-standard type since it is driven by a non quasi-left-continuous random measure. The proof of existence and uniqueness of a solution to such general equation is a difficult task, and counterexamples can be obtained even in simple cases, see [13]. Only recently, some results have been obtained in this context, see [14], [15] and [2]. In [2], well-posedness is obtained in a general non-diffusive context under a specific condition involving the Lipschitz constants of the generator and the size of the predictable jumps. The above mentioned property turns out to be satisfied in our PDMPs framework, and the existence and uniqueness theorem in [2] provides a fundamental tool in our penalization approach.

Throughout this section the random measures \(p, \bar{p}\) and \(q\), as well as the randomized control setting \(\Omega, \mathbb{F}, (X, I, J), \mathbb{F}^{x, a_0, a_T}, \) are the same as in Section 3.1. We recall that \(\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}\) is the augmentation of the natural filtration generated by \(p\), and that \(\mathcal{P}_T, T > 0\), denotes the \(\sigma\)-field of \(\mathbb{F}\)-predictable subsets of \([0, T] \times \Omega\). For any \(\beta \geq 0\) and any predictable increasing process \(A\), we denote by \(\mathcal{E}^\beta\) the Doléans-Dade exponential of the process \(\beta A\), which is given by
\[
\mathcal{E}^\beta_t = e^{\beta A_t} \prod_{0 < s \leq t} (1 + \beta \Delta A_s) e^{-\beta \Delta A_s}.
\]

In particular, \(d\mathcal{E}^\beta_s = \mathcal{E}^\beta_s \, dA_s\), \(\mathcal{E}^\beta_s \geq 1\).

**Remark 4.1.** Let \(\mathcal{E}_t^\beta\) be the process introduced in (4.1). For any càdlàg process \(C\), the Itô’s formula applied to \(\mathcal{E}_s^\beta |C|\) reads
\[
d(\mathcal{E}_s^\beta |C|) = \mathcal{E}_s^\beta \, d|C| + (C_s - C_{s-})^2 \, d\mathcal{E}_s^\beta
\]
\[
= 2 \mathcal{E}_s^\beta C_{s-} \, dC_s + \mathcal{E}_s^\beta (\Delta C_s)^2 + \beta \mathcal{E}_s^\beta |C_{s-}|^2 \, dA_s
\]
\[
= 2 \mathcal{E}_s^\beta C_s \, dC_s + \mathcal{E}_s^\beta (\Delta C_s)^2 + \beta \mathcal{E}_s^\beta (1 + \beta \Delta A_s)^{-1} |C_{s-}|^2 \, dA_s,
\]

where the latter equality follows from the fact that \(\mathcal{E}_s^\beta = \mathcal{E}_s^\beta (1 + \beta \Delta A_s)^{-1}\).

We will need to ask the following additional assumption on the process \(p^\beta\).

**(H0')** For any \((x, a_0, a_T) \in E \times A_0 \times A_T\), \(t \in \mathbb{R}_+, \beta > 0\), there exists some constant \(C_\beta(t)\), only depending on \(t\) and \(\beta\), such that
\[
\mathbb{E}^{x, a_0, a_T} \left[ (1 + p^\beta_t) (1 + \beta p^\beta_t) \right] \leq C_\beta(t).
\]

For any \((x, a_0, a_T) \in E \times A_0 \times A_T\) and \(\beta \geq 0\), we introduce the following notation.
• $L^2_{x,a_0,a_1}(\mathcal{F}_\tau)$, the set of $\mathcal{F}_\tau$-measurable random variables $\xi$ such that $E^{x,a_0,a_1}[|\xi|^2] < \infty$; here $\tau \geq 0$ is an $\mathbb{F}$-stopping time.

• $S^\infty$ the set of real-valued càdlàg adapted processes $Y = (Y_t)_{t \geq 0}$ which are uniformly bounded.

• $L^{2,\beta}_{x,a_0,a_1}(p^*; 0, T), T > 0$, the set of real-valued progressive processes $Y = (Y_t)_{0 \leq t \leq T}$ such that

$$ ||Y||^2_{L^{2,\beta}_{x,a_0,a_1}(p^*; 0, T)} := E^{x,a_0,a_1} \left[ \int_{(0, T]} \mathcal{E}_t^\beta |Y_t|^2 \, dA_t \right] < \infty. $$

• $G^{2,\beta}_{x,a_0,a_1}(q; 0, T), T > 0$, the set of $\mathcal{P}_T \otimes \mathcal{E} \otimes A_0 \otimes A_\Gamma$-measurable maps $Z : \Omega \times [0, T] \times E \times A_0 \times A_\Gamma \to \mathbb{R}$ such that

$$ ||Z||^2_{G^{2,\beta}_{x,a_0,a_1}(q; 0, T)} := E^{x,a_0,a_1} \left[ \int_{(0, T]} \mathcal{E}_t^\beta \int_{E \times A_0 \times A_\Gamma} |Z_t(y, b, c) - \tilde{Z}_t I_K(t)|^2 \, (dt \, dy \, db \, dc) \right] $$

is finite, where $\tilde{Z}_t$ is the process introduced in (3.26). We also define $G^{2,\beta}_{x,a_0,a_1,loc}(q) := \cap_{T > 0} G^{2,\beta}_{x,a_0,a_1}(q; 0, T)$.

• $L^2(\lambda_0)$ (resp. $L^2(\lambda_\Gamma)$), the set of $A_0$-measurable maps $\psi : A_0 \to \mathbb{R}$ (resp. $A_\Gamma$-measurable maps $\psi : A_\Gamma \to \mathbb{R}$) such that

$$ |\psi|^2_{L^2(\lambda_0)} := \int_{A_0} |\psi(b)|^2 \lambda_0(db) < \infty \quad \text{(resp. } |\psi|^2_{L^2(\lambda_\Gamma)} := \int_{A_\Gamma} |\psi(c)|^2 \lambda_\Gamma(dc) < \infty \text{)}.$$

• $L^2(\phi_{\omega,t}) = L^2(E \times A_0 \times A_\Gamma, \mathcal{E} \otimes A_0 \otimes A_\Gamma, \phi_{\omega,t}(dy \, db \, dc))$, for any $(\omega, t) \in \Omega \times \mathbb{R}_+$, the set of $\mathcal{E} \otimes A_0 \otimes A_\Gamma$-measurable maps $\zeta : E \times A_0 \times A_\Gamma \to \mathbb{R}$ such that

$$ |\zeta|^2_{L^2(\phi_{\omega,t})} := \int_{E \times A_0 \times A_\Gamma} |\zeta(y, b, c)|^2 \phi_{\omega,t}(dy \, db \, dc) < \infty,$$

where $\phi_{\omega,t}(dy \, db \, dc)$ is the random measure introduced in (3.11).

• $K^2_{x,a_0,a_1}(0, T), T > 0$, the set of nondecreasing càdlàg predictable processes $K = (K_t)_{0 \leq t \leq T}$ such that $K_0 = 0$ and $E^{x,a_0,a_1}[|K_T|^2] < \infty$. We also define $K^2_{x,a_0,a_1,loc} := \cap_{T > 0} K^2_{x,a_0,a_1}(0, T)$.

We aim at studying the following family of BSDEs with partially nonnegative jumps over an infinite horizon, parametrized by $(x, a_0, a_1)$: $\mathbb{P}^{x,a_0,a_1}$-a.s.,

\begin{align}
Y^{x,a_0,a_1}_s &= Y^{x,a_0,a_1}_T - \delta \int_{(s, T]} Y^{x,a_0,a_1}_{r^-} \, dr + \int_{(s, T]} f(X_r, I_r) \, dr + \int_{(s, T]} c(X_{r^-}, J_{r^-}) \, dp^s_r \\
&\quad - \int_{(s, T]} \int_{A_0} Z^{x,a_0,a_1}_r (X_r, b, J_r) \lambda_0(db) \, dr - \int_{(s, T]} \int_{A_\Gamma} Z^{x,a_0,a_1}_r (X_r, I_r, c) \lambda_\Gamma(dc) \, dr \\
&\quad - (K^{x,a_0,a_1}_T - K^{x,a_0,a_1}_s) \\
&\quad - \int_{(s, T]} \int_{E \times A_0 \times A_\Gamma} Z^{x,a_0,a_1}_r (y, b, c) \, q(dy \, db \, dc), \quad 0 \leq s \leq T < \infty, \quad (4.3)
\end{align}

with the constraints

\begin{align}
Z^{x,a_0,a_1}_s (X_{s^-}, b, J_{s^-}) \geq 0, \quad d\mathbb{P}^{x,a_0,a_1}_s \lambda_0(db) \text{ -a.e. on } [0, \infty) \times \Omega \times A_0, \quad (4.4)
\end{align}
where $\delta$ is the positive parameter introduced in Section 2.

We look for a maximal solution $(Y_s^{x, a_0, \alpha r}, Z_s^{x, a_0, \alpha r}, K_s^{x, a_0, \alpha r}) \in S^\infty \times \mathcal{G}_x^{2, a_0, \alpha r, \text{loc}}(q) \times K_x^{2, a_0, \alpha r, \text{loc}}$ to (4.3)-(4.4)-(4.5), in the sense that for any other solution $(\bar{Y}, \bar{Z}, \bar{K}) \in S^\infty \times \mathcal{G}_x^{2, a_0, \alpha r, \text{loc}}(q) \times K_x^{2, a_0, \alpha r, \text{loc}}$, we have $Y_t^{x, a_0, \alpha r} \geq \bar{Y}_t$, $\mathbb{P}^{x, a_0, \alpha r}$-a.s., for all $t \geq 0$. The existence of the maximal solution to (4.3)-(4.4)-(4.5) will be proved in the sequel by a penalization approach.

### 4.1 The penalized BSDE on finite horizon

We start by considering, for fixed $T > 0$, the family of penalized BSDEs on $[0, T]$ with zero final cost at time $T$, associated to (4.3)-(4.4)-(4.5), parametrized by the integer $n \geq 1$: $\mathbb{P}^{x, a_0, \alpha r}$-a.s.

\[
Y_s^{T, n, x, a_0, \alpha r} = -\delta \int_{(s, T]} Y_r^{T, n, x, a_0, \alpha r} \, dr + \int_{(s, T]} f(X_r, I_r) \, dr + \int_{(s, T]} c(X_{r^-}, J_r) \, dp_r^s \\
- n \int_{(s, T]} \int_{A_0} [Z_r^{T, n, x, a_0, \alpha r}(X_r, b, J_r)]^- \lambda_0(db) \, dr - \int_{(s, T]} \int_{A_0} Z_r^{T, n, x, a_0, \alpha r}(X_r, b, J_r) \lambda_0(db) \, dr \\
- n \int_{(s, T]} \int_{A_0} [Z_r^{T, n, x, a_0, \alpha r}(X_r, I_r, c)]^- \lambda_T(dc) \, dr - \int_{(s, T]} \int_{A_T} Z_r^{T, n, x, a_0, \alpha r}(X_r, I_r, c) \lambda_T(dc) \, dr \\
- \int_{(s, T]} \int_{E \times A_0 \times A_T} Z_r^{T, n, x, a_0, \alpha r}(y, b, c) q(dr \, dy \, db \, dc), \quad 0 \leq s \leq T, \tag{4.6}
\]

where $[z]^-$ = max$(-z, 0)$ is the negative part of $z$. We set

\[
\mathbb{H}_2^{2, \beta, a_0, \alpha r}(q; 0, T) := \mathbb{L}_x^{2, \beta, a_0, \alpha r}(p^*; 0, T) \times \mathcal{G}_x^{2, \beta, a_0, \alpha r}(q; 0, T),
\]

and, for every $(Y, Z) \in \mathbb{H}_2^{2, \beta, a_0, \alpha r}(q; 0, T)$, denote

\[
\|(Y, Z)\|_{\mathbb{H}_2^{2, \beta, a_0, \alpha r}(q; 0, T)} := \|Y\|_{\mathbb{L}_x^{2, \beta, a_0, \alpha r}(p^*; 0, T)} + \|Z\|_{\mathcal{G}_x^{2, \beta, a_0, \alpha r}(q; 0, T)}.
\]

Notice that the space $\mathbb{H}_2^{2, \beta, a_0, \alpha r}(q; 0, T)$, endowed with the topology induced by $\| \cdot \|_{\mathbb{H}_2^{2, \beta, a_0, \alpha r}(q; 0, T)}$, is an Hilbert space, provided we identify pairs of processes $(Y, Z), (Y', Z')$ satisfying $\|(Y - Y', Z - Z')\|_{\mathbb{H}_2^{2, \beta, a_0, \alpha r}(q; 0, T)} = 0$.

**Definition 4.1.** A solution to equation (4.6) with data $(\delta, \beta, f, c)$ is a pair $(Y, Z) \in \mathbb{H}_2^{2, \beta, a_0, \alpha r}(q; 0, T)$ satisfying equation (4.6). We say that equation (4.6) admits a unique solution if, given two solutions $(Y, Z), (Y', Z') \in \mathbb{H}_2^{2, \beta, a_0, \alpha r}(q; 0, T)$, we have $(Y, Z) = (Y', Z')$ in $\mathbb{H}_2^{2, \beta, a_0, \alpha r}(q; 0, T)$.

**Remark 4.2.** Given a solution $(Y, Z)$ to equation (4.6) with data $(\delta, \beta, f, c)$, since $\mathcal{E}_t^\beta \geq 1$, we have

\[
\|Z\|_{\mathcal{G}_x^{2, \beta, a_0, \alpha r}(q; 0, T)} \leq \|Z\|_{\mathbb{H}_2^{2, \beta, a_0, \alpha r}(q; 0, T)} < \infty. \tag{4.7}
\]

Therefore, the process $(Z_t^1)_{t \geq 0}$ belongs to the space $\mathcal{G}_x^{2, a_0, \alpha r}(q; 0, T)$. In particular, the stochastic integral $\int_{[0, T]} \int_{E \times A_0 \times A_T} Z_s(y, b, c) q(ds \, dy \, db \, dc)$ in (4.6) is well-defined, and the process $M_t := \int_{[0, t]} \int_{E \times A_0 \times A_T} Z_s(y, b, c) q(ds \, dy \, db \, dc)$, $t \in [0, T]$, is a square integrable martingale (see Proposition 3.66 in [22]), with

\[
\mathbb{E}^{x, a_0, \alpha r}[M_T^2] = \mathbb{E}^{x, a_0, \alpha r}[(M, M)_t] = \|Z\|_{\mathcal{G}_x^{2, a_0, \alpha r}(q; 0, T)}^2. \tag{4.8}
\]
Remark 4.3. Uniqueness in $G_{x,a_0,a_t}^2(q; 0, T)$ means that $||Z - Z'||_{G_{x,a_0,a_t}^2(q; 0, T)} = 0$. In particular, recalling Lemma 3.3(ii), this implies that there is a predictable process $(l_t)_{t \geq 0}$ such that

$$Z_t(y, b, c) - Z'_t(y, b, c) = l_t \mathbb{1}_K(t), \quad d\mathbb{P} \tilde{p}(dt dy db dc) - a.e.,$$

see also Remark 3.1 in [5].

At this point, we introduce the functions $f^n : E \times A_0 \times L^2(\lambda_0) \times L^2(\lambda_\Gamma) \rightarrow \mathbb{R}, \tilde{f}^n : \mathbb{R}^+ \times E \times A_0 \times A_\Gamma \times L^2(\phi, t) \rightarrow \mathbb{R}$, respectively defined as

$$f^n(x, a_0, \psi, \phi) := f(x, a_0) - \int_{A_0} \left\{ n[\psi(b)] - \psi(b) \right\} \lambda_0(db) - \int_{A_\Gamma} \left\{ n[\phi(c)] - \phi(c) \right\} \lambda_\Gamma(dc),$$

$$\tilde{f}^n(t, x, a_0, a_\Gamma, \zeta) := e^{-\delta t} f^n(x, a_0, \zeta(x, \cdot, a_\Gamma), \zeta(x, a_0, \cdot)) \mathbb{1}_{x \in E} + e^{-\delta t} c(x, a_\Gamma) \mathbb{1}_{x \in \partial E}. \quad (4.10)$$

Remark 4.4. The penalized BSDE (4.6) can be rewritten as follows: $\mathbb{P}^{x,a_0,a_t} - a.e.,$

$$\tilde{Y}_s^{T,n,x,a_0,a_t} = \int_{\{s, T\}} \tilde{f}^n(r, s, X_{r-}, I_{r-}, J_{r-}, Z_r^{T,n,x,a_0,a_t}) dA_r$$

$$- \int_{\{s, T\}} e^{-\delta(r-s)} \int_{E \times A_0 \times A_\Gamma} Z_r^{T,n,x,a_0,a_t}(y, b, c) q(dr dy db dc), \quad s \in [0, T],$$

where $\tilde{f}^n$ is the deterministic function defined in (4.10).

Remark 4.5. Under assumptions (Hh$\lambda$QR), (H$\lambda$,$\lambda_\Gamma$), (H0), (H0') and (Hfc), for any $n \in \mathbb{N}$, the generator $\tilde{f}^n$ of the BSDE (4.11) satisfies the following properties.

(i) For any $Z : \Omega \times [0, T] \times E \times A_0 \times A_\Gamma \rightarrow \mathbb{R}$ predictable, $\tilde{f}^n(t, X_{t-}(\omega), I_{t-}(\omega), J_{t-}(\omega), Z_{\omega,t}(\cdot, \cdot, \cdot))$ is predictable.

(ii) For some nonnegative constant $L_n$, depending on $n$, we have

$$\left| \tilde{f}^n(t, X_{t-}(\omega), I_{t-}(\omega), J_{t-}(\omega), \zeta') - \tilde{f}^n(t, X_{t-}(\omega), I_{t-}(\omega), J_{t-}(\omega), \zeta) \right| \leq$$

$$L_n \left( \int_{E \times A_0 \times A_\Gamma} \left| \tilde{\zeta}(y, b, c) - \Delta A_t(\omega) \right| \int_{E \times A_0 \times A_\Gamma} \tilde{\zeta}(y, b, c) \phi_{\omega,t}(dy db dc) \right)^2 \phi_{\omega,t}(dy db dc) \right)^{1/2}, \quad (4.13)$$

for all $(\omega, t) \in \Omega \times [0, T], \zeta, \zeta' \in L^2(\phi_{\omega,t})$, where we have set $\tilde{\zeta} = \zeta - \zeta'$;

(iii) $\mathbb{E} \left[ 1 + \sum_{0 < t \leq T} |\Delta A_t|^2 \right] \int_0^T \mathcal{E}_t^\beta \tilde{f}^n(t, X_{t-}, I_{t-}, J_{t-}, 0) \right| dA_t \right] < \infty.
Proof. See Section A.3.

At this point, let us set
\[ \beta_0^n := \frac{2(L_n + \varepsilon)^2}{1 - \varepsilon}, \quad \varepsilon \in (0, 1), \]
where \( L_n \) is the Lipschitz constant in (4.12). With the help of Proposition 4.1 we can apply Theorem 4.1 in [2] in order to obtain existence and uniqueness for the penalized BSDE (4.6).

**Theorem 4.2.** Let Hypotheses (HhdQ0R), (H0), (H0'), (H\(L_0\lambda\Gamma\)) and (Hfc) hold. Then, for every \((x, a_0, a\Gamma, n, T) \in E \times A_0 \times A\Gamma \times \mathbb{N} \times (0, \infty)\), there exists a unique solution \((Y_{T,n,x,a_0,a\Gamma}, Z_{T,n,x,a_0,a\Gamma}) \in L^{2,\beta}_{x,a_0,a\Gamma}(p^*; 0, T) \times G^{2,\beta}_{x,a_0,a\Gamma}(q; 0, T)\) to equation (4.6) for \( \beta \geq \beta_0^n \).

Moreover, the following uniform estimate holds:
\[ Y_{T,n,x,a_0,a\Gamma} \leq \frac{M_f}{\delta} + C^* M_c, \quad \forall s \in [0, T], \quad (4.14) \]
where \( C^* \) is the constant defined in (2.12).

Proof. The existence and uniqueness result follows from Theorem 4.1 in [2], with \( \Delta A_t(\omega) = \mathbb{1}_{X_{t-}(\omega) \in T} \). Indeed, from Proposition 4.1 we see that the generator \( \hat{f}^n \) of the BSDE in the reformulated form (4.11) satisfies the assumptions of Theorem 4.1 in [2], namely properties (i)-(ii)-(iii) at page 3 in [2]. Notice that in our framework the Lipschitz constant of \( \hat{f}^n \) with respect to \( Y \), that we will denote \( L_n \), is identically zero. In particular, the technical hypothesis of Theorem 4.1 in [2], i.e. the existence of \( \varepsilon \in (0, 1) \) such that
\[ 2 L_y^2 |\Delta A_t|^2 \leq 1 - \varepsilon, \quad \mathbb{P}\text{-a.s.,} \forall t \in [0, T], \]
here is automatically verified. Then, we can apply Theorem 4.1 in [2] and conclude that, for every \((x, a_0, a\Gamma, n, T) \in E \times A_0 \times A\Gamma \times \mathbb{N} \times (0, \infty)\), there exists a unique solution \((Y_{T,n}, Z_{T,n}) = (Y_{T,n,x,a_0,a\Gamma}, Z_{T,n,x,a_0,a\Gamma}) \in L^{2,\beta}_{x,a_0,a\Gamma}(q; 0, T)\) to equation (4.6) for \( \beta \geq \beta_0^n \).

It remains to prove uniform estimate (4.14). To this end, for any \( \nu \in \mathcal{Y}_n \), let us introduce the compensated martingale measure \( q^\nu(ds dy db dc) = q(ds dy db dc) - \int (\nu^0(b) - 1) d_1(s, y, b, c) + (\nu^0(c) - 1) d_1(s, y, b, c) \) \( \bar{p}(ds dy db dc) \) under \( \mathbb{P}^{x,a_0,a\Gamma}_\nu \). Taking the expectation in (4.11) under \( \mathbb{P}^{x,a_0,a\Gamma}_\nu \), conditional to \( \mathcal{F}_s \), and since \( Z_{T,n}^\nu \) is \( G^{2,\beta}_{x,a_0,a\Gamma}(q; 0, T) \) from Lemma 3.4 we get that, \( \mathbb{P}^{x,a_0,a\Gamma}_\nu \)-a.s.,
\[ Y_{T,n}^\nu = -\mathbb{E}_\nu^{x,a_0,a\Gamma}\left[ \int_{(s, T]} e^{-\delta(r-s)} \left\{ n[Z_{T,n}^\nu(X_r, b, J_r)] + \nu^0_r(b) Z_{T,n}^\nu(X_r, b, J_r) \right\} \lambda_0(db) \right| \mathcal{F}_s \]
\[ \quad - \mathbb{E}_\nu^{x,a_0,a\Gamma}\left[ \int_{(s, T]} e^{-\delta(r-s)} \left\{ n[Z_{T,n}^\nu(X_r, I_r, c)] + \nu^0_r(c) Z_{T,n}^\nu(X_r, I_r, c) \right\} \lambda_1(dc) \right| \mathcal{F}_s \]
\[ \quad + \mathbb{E}_\nu^{x,a_0,a\Gamma}\left[ \int_{(s, T]} e^{-\delta(r-s)} f(X_r, I_r) \right| \mathcal{F}_s \right] \quad s \in [0, T]. \quad (4.15) \]

Estimate (4.14) directly follows from the elementary numerical inequality \( n[z]^- + \nu z \geq 0 \) for all \( z \in \mathbb{R}, \nu \in (0, n] \), and the boundedness of \( f \) and \( c \).
4.2 The penalized BSDE on infinite horizon

At this point we consider the family of penalized BSDEs on $[0, \infty)$ associated to (4.3)-(4.4)-(4.5):

$$Y_{n,x,a}^{n,x,a,\Gamma} = Y_{n,x,a}^{n,x,a,\Gamma} - \delta \int_{(s,T]} Y_{n,x,a}^{n,x,a,\Gamma} \, dr + \int_{(s,T]} f(X_t, I_t) \, dr + \int_{(s,T]} c(X_{r-}, J_{r-}) \, dp_r$$

$$- n \int_{(s,T]} \int_{A_0} [Z_{n,x,a}^{n,x,a,\Gamma}(X_t, b, J_r)]^{-} \lambda_0(db) \, dr - \int_{(s,T]} \int_{A_0} Z_{n,x,a}^{n,x,a,\Gamma}(X_t, b, J_r) \lambda_0(db) \, dr$$

$$- \int_{(s,T]} \int_{A_\Gamma} Z_{n,x,a}^{n,x,a,\Gamma}(y, b, c) \, q(dr \, db \, dc), \quad 0 \leq s \leq T < \infty.$$  (4.16)

**Remark 4.6.** The penalized BSDE (4.16) can be rewritten as follows: for any positive $T < \infty$, $\mathbb{P}^{n,x,a,\Gamma}$-a.s.,

$$Y_{n,x,a}^{n,x,a,\Gamma} = Y_{n,x,a}^{n,x,a,\Gamma} e^{-\delta(T-s)} + \int_{(s,T]} \tilde{f}^n(r-s, X_{r-}, I_{r-}, J_{r-}, Z_{r}^{n,x,a,\Gamma}) \, dA_r$$

$$- \int_{(s,T]} e^{-\delta(r-s)} \int_{E \times A_0 \times A_\Gamma} Z_{n,x,a}^{n,x,a,\Gamma}(y, b, c) \, q(dr \, db \, dc), \quad s \in [0, T].$$  (4.17)

where $\tilde{f}^n$ is the deterministic function defined in (4.10).

By means of Theorem 4.2 we can prove that there exists a unique solution to equation (4.16) in a suitable sense; moreover, we will give an explicit representation to (4.16) in terms of a family of randomized control problems. To this end, for every integer $n \geq 1$, let us denote by $\mathcal{Y}^n$ the subset of elements $\nu \in \mathcal{V}$ valued in $(0, n]$. We set

$$K_{t}^{n,x,a,\Gamma} := n \int_{0}^{t} \left( \int_{A_0} [Z_{s}^{n,x,a,\Gamma}(X_{s}, b, J_s)]^{-} \lambda_0(db) + \int_{A_\Gamma} [Z_{s}^{n,x,a,\Gamma}(X_{s}, I_s, c)]^{-} \lambda_\Gamma(dc) \right) \, ds.$$  (4.18)

We start by giving a technical result.

**Lemma 4.3.** Assume that Hypotheses (HhλQR), (H0), (H0'), (Hλ0λΓ) and (Hfc) hold. Fix $T > 0$, $(n, x, a, \Gamma) \in \mathbb{N} \times E \times A_0 \times A_\Gamma$, and let $(Y_{n,x,a,\Gamma}, Z_{n,x,a,\Gamma})$ be the unique solution to (4.16). Then, for any $t \in (0, T]$, the following identity holds:

$$|Y_{t}^{n,x,a,\Gamma}|^2 + \sum_{s \in \{t, T\}} \left| \int_{E \times A_0 \times A_\Gamma} Z_{s}^{n,x,a,\Gamma}(y, b, c) \, q(\{s\} \times dy \, db \, dc) \right|^2$$

$$= |Y_{T}^{n,x,a,\Gamma}|^2 - \sum_{s \in \{t, T\}} |c(X_{s-}, J_{s-})|^2 \mathbb{1}_{X_{s-} \in \partial E}$$

$$+ 2 \sum_{s \in \{t, T\}} \left( \int_{E \times A_0 \times A_\Gamma} Z_{s}^{n,x,a,\Gamma}(y, b, c) \, q(\{s\} \times dy \, db \, dc) \right) c(X_{s-}, J_{s-}) \mathbb{1}_{X_{s-} \in \partial E}$$

$$+ 2 \int_{(t, T]} Y_{s}^{n,x,a,\Gamma} f(X_s, I_s) \, ds + 2 \int_{(0, T]} Y_{s}^{n,x,a,\Gamma} c(X_{s-}, J_{s-}) \, dp_s^*$$

$$- 2 \delta \int_{(t, T]} |Y_{s}^{n,x,a,\Gamma}|^2 \, ds - 2 \int_{(t, T]} Y_{s}^{n,x,a,\Gamma} \, dK_{s}^{n,x,a,\Gamma}.$$
\[-2 \int_{[t,T]} Y_{s}^{n,x,a_0,\alpha, \Gamma} \int_{E \times A_0 \times A_{\Gamma}} Z_{s}^{n,x,a_0,\alpha, \Gamma} (y, b, c) \, q(ds \, dy \, db \, dc), \quad s \in (t, T]. \tag{4.19} \]

**Proof.** See Section A.2. \[\square\]

Using Lemma 4.3, one can prove the following well-posedness result for the BSDE (4.16).

**Proposition 4.4.** Let Hypotheses (Hh\(\lambda QR\), (H0), (H0\'), (H\(\lambda_0 \alpha \Gamma\)) and (Hfc) hold. Then, for every \((x,a_0,\alpha, n) \in E \times A_0 \times A_{\Gamma} \times \mathbb{N}\), for any \(\beta \geq \beta_0\), there exists a unique solution \((Y_{n,x,a_0,\alpha, \Gamma}, Z_{n,x,a_0,\alpha, \Gamma}) \in S^{\infty} \times G_{n,n,x,a_0,\alpha, \Gamma}^{2,\beta}(q)\) to (4.16). Moreover, \((Y_{n,x,a_0,\alpha, \Gamma}, Z_{n,x,a_0,\alpha, \Gamma})\) admits the following explicit representation: \(\mathbb{P}^{x,a_0,\alpha, \Gamma}\)-a.s., for all \(s \geq 0\),

\[Y_{s}^{n,x,a_0,\alpha, \Gamma} = \operatorname{ess inf}_{\nu \in Y^n} \bigg[ \int_{(s,\infty)} e^{-\delta(r-s)} f(X_r, I_r) \, dr + \int_{(s,\infty)} e^{-\delta(r-s)} c(X_r-, J_r-) \, dp_r^{\ast} \bigg| \mathcal{F}_{s}\bigg]. \tag{4.20} \]

**Proof.** See Section A.3. \[\square\]

**Remark 4.7.** Let \((Y_{n,x,a_0,\alpha, \Gamma}, Z_{n,x,a_0,\alpha, \Gamma})\) be the unique solution to (4.16) for some \((x,a_0,\alpha, n) \in E \times A_0 \times A_{\Gamma} \times \mathbb{N}, \beta \geq \beta_0\). By Remark 4.2, \(Z_{n,x,a_0,\alpha, \Gamma} \in G_{n,n,x,a_0,\alpha, \Gamma}^{2,\beta}(q)\).

The following a priori uniform estimate on the sequence \((Z_{n,x,a_0,\alpha, \Gamma}, K_{n,x,a_0,\alpha, \Gamma})_{n \geq 0}\) will be fundamental in order to provide the existence and uniqueness of the maximal solution to the constrained BSDE (4.3)-(4.4)-(4.5).

**Lemma 4.5.** Assume that hypotheses (Hh\(\lambda QR\), (H0\), (H0\'), (H\(\lambda_0 \alpha \Gamma\)) and (Hfc) hold. For every \((x,a_0,\alpha, n) \in E \times A_0 \times A_{\Gamma} \times \mathbb{N}\) and \(\beta \geq \beta_0\), let \((Y_{n,x,a_0,\alpha, \Gamma}, Z_{n,x,a_0,\alpha, \Gamma}) \in S^{\infty} \times G_{n,n,x,a_0,\alpha, \Gamma}^{2,\beta}(q)\) be the unique solution to (4.16). Then, for every \(T > 0\), there exists a constant \(C\) depending only on \(M_f, M_c, \delta, T, C^\ast\), such that

\[\|Z_{n,x,a_0,\alpha, \Gamma}\|_{G_{n,n,x,a_0,\alpha, \Gamma}^{2,\beta}(q;0,T)}^2 + \|K_{n,x,a_0,\alpha, \Gamma}\|_{K_{n,n,x,a_0,\alpha, \Gamma}^{2,\beta}(0,T)}^2 \leq C. \tag{4.21} \]

**Proof.** See Section A.4. \[\square\]

### 4.3 BSDE representation of the randomized value function

We are ready to state the main result of the section, i.e. the well-posedness of the constrained BSDE (4.3)-(4.4)-(4.5), and a probabilistic representation of the randomized value function \(V^\ast\) introduced in Section 3.2 by means of its maximal solution.

**Theorem 4.6.** Under Hypotheses (Hh\(\lambda QR\)), (H0), (H0\'), (H\(\lambda_0 \alpha \Gamma\)) and (Hfc), for every \((x,a_0,\alpha, \Gamma) \in E \times A_0 \times A_{\Gamma}\), there exists a unique maximal solution \((Y_{x,a_0,\alpha, \Gamma}, Z_{x,a_0,\alpha, \Gamma}, K_{x,a_0,\alpha, \Gamma}) \in S^{\infty} \times G_{x,a_0,\alpha, \Gamma}^{2,\beta}(q)\times K_{x,a_0,\alpha, \Gamma}^{2,\beta}(0,T)\) to the BSDE with partially nonnegative jumps (4.3)-(4.4)-(4.5). In particular,

(i) \(Y_{x,a_0,\alpha, \Gamma}\) is the nondecreasing limit of \((Y_{n,x,a_0,\alpha, \Gamma})_{n}\);  
(ii) \(Z_{x,a_0,\alpha, \Gamma}\) is the weak limit of \((Z_{n,x,a_0,\alpha, \Gamma})_{n}\) in \(G_{x,a_0,\alpha, \Gamma}^{2,\beta}(q)\);  
(iii) \(K_{s,x,a_0,\alpha, \Gamma}\) is the weak limit of \((K_{s,x,a_0,\alpha, \Gamma})_{s}\) in \(L_{x,a_0,\alpha, \Gamma}(\mathcal{F}_s)\), for any \(s \geq 0\).
Moreover, \( Y_{s}^{x,a_{0},a_{\Gamma}} \) has the explicit representation:

\[
Y_{s}^{x,a_{0},a_{\Gamma}} = \text{ess inf}_{\nu \in \mathcal{V}} \mathbb{E}_{\nu}^{x,a_{0},a_{\Gamma}} \left[ \int_{(s,\infty)} e^{-\delta(r-s)} f(X_{r}, I_{r}) \, dr + \int_{(s,\infty)} e^{-\delta(r-s)} c(X_{r-}, J_{r-}) \, dp_{r}^{*} \big| F_{s} \right], \quad \forall \ s \geq 0.
\]

(4.22)

In particular, setting \( s = 0 \), we have the following representation formula for the value function of the randomized control problem:

\[
V^{*}(x,a_{0},a_{\Gamma}) = Y_{0}^{x,a_{0},a_{\Gamma}}, \quad (x,a_{0},a_{\Gamma}) \in E \times A_{0} \times A_{\Gamma}.
\]

(4.23)

Proof. The key ingredients in the proof of the existence result are the well-posedness of the penalized BSDE (4.6), given in Theorem 4.2 and the uniform estimate (4.21), proved in Lemma 4.5. The representation formula (4.22) and the maximality property follow instead from the penalized BSDE (4.6), given in Theorem 4.2, and the uniform estimate (4.21), proved in Lemma 4.5.

Regarding the uniqueness result, let \((Y,Z,K)\) and \((Y',Z',K')\) be two maximal solutions of (4.3)-(4.4)-(4.5). The component \(Y\) is unique by definition. Let us now consider the difference between the two backward equations. We get: \( \mathbb{P}^{x,a_{0},a_{\Gamma}} \)-a.s.

\[
\int_{[0,t]} \int_{E \times A_{0} \times A_{\Gamma}} (Z_{s}(y,b,c) - Z'_{s}(y,b,c)) \, q(ds \, dy \, db \, dc) \quad (4.24)
\]

\[
= (K_{t} - K'_{t}) - \int_{[0,t]} \int_{A_{0}} (Z_{s}(X_{s},b,J_{s}) - Z'_{s}(X_{s},b,J_{s})) \, \lambda_{0}(db) \, ds
\]

\[
- \int_{[0,t]} \int_{A_{\Gamma}} (Z_{s}(X_{s}, I_{s}, c) - Z'_{s}(X_{s}, I_{s}, c)) \, \lambda_{\Gamma}(dc) \, ds, \quad 0 \leq t \leq T < \infty. \quad (4.25)
\]

The right-hand side of (4.24) is a predictable process, therefore it has no totally inaccessible jumps (see, e.g., Proposition 2.24, Chapter I, in [23]); on the other hand, by Proposition 3.3, together with (3.21)-(3.23), it follows that the left-hand side of (4.24) is a jump process with only totally inaccessible jumps. This implies that \( Z = Z' \) in the sense of Remark 4.3 and as a consequence the component \(K\) is unique as well.

\[
\square
\]

5 A BSDE representation for the value function

Our principal goal is to prove that the value function \( V \) of our optimal control problem (2.9) can be represented by means of the maximal solution to the BSDE with nonnegative jumps (1.3)-(1.4)-(1.5). Theorem 4.6 insures that, under Hypotheses (HhλQR), (H0), (H0'), (Hλ0λΓ) and (Hfc), there exists a unique maximal solution \( \left( Y_{x,a_{0},a_{\Gamma}}, Z_{x,a_{0},a_{\Gamma}}, K_{x,a_{0},a_{\Gamma}} \right) \) on \((\Omega,\mathcal{F},\mathbb{P}^{x,a_{0},a_{\Gamma}})\) to (1.3)-(1.4)-(1.5). Let \( v : E \times A_{0} \times A_{\Gamma} \to \mathbb{R} \) be the deterministic function defined by

\[
v(x,a_{0},a_{\Gamma}) := Y_{0}^{x,a_{0},a_{\Gamma}}, \quad (x,a_{0},a_{\Gamma}) \in E \times A_{0} \times A_{\Gamma}.
\]

(5.1)

Our main result is as follows.

Theorem 5.1. Let Hypotheses (HhλQR), (H0), (H0'), (Hλ0λΓ) and (Hfc) hold. Then
(i) The function $v$ in (5.1) does not depend on its last arguments:

$$v(x,a_0,a_{\Gamma}) = v(x,a_0',a_{\Gamma}'), \quad x \in E, \ (a_0,a_0') \in A_0, \ (a_{\Gamma},a_{\Gamma}') \in A_{\Gamma}, \quad (5.2)$$

(ii) By abuse of notation, let us define the function $v$ on $E$ by

$$v(\cdot):=v(\cdot,a_0,a_{\Gamma}), \quad \text{for any} \ (a_0,a_{\Gamma}) \in A_0 \times A_{\Gamma}. \quad (5.3)$$

Then $v$ is a viscosity solution to (2.13)-(2.14).

In particular, under the hypotheses of Theorem 2.1, $v$ provides the unique viscosity solution to (2.13)-(2.14), is continuous in $E$ and coincides with the value function $V$ of the PDMPs optimal control problem. Therefore $V$ admits the probabilistic representation (5.1). Moreover, by Theorem 4.6 the randomized value function $V^*$ coincides with the value function $V$ of the original control problem. These considerations are formalized in the following corollary.

**Corollary 5.2.** Assume that Hypotheses (HhλQR), (H0), (H0'), (Hλ0,λ_Γ), (Hfc), (HBB), (HBB*) hold, and that $A_0,A_{\Gamma}$ are compact. Then the value function $V$ of the optimal control problem defined in (2.9) admits the Feynman-Kac representation formula:

$$V(x) = Y^x,a_0,a_{\Gamma}_0, \quad (x,a_0,a_{\Gamma}) \in E \times A_0 \times A_{\Gamma}. \quad (5.4)$$

We devote the rest of the paper to prove Theorem 5.1.

### 5.1 The identification property of $(Y^x,a_0,a_{\Gamma})_{s \geq 0}$

In order to prove Theorem 5.1, a fundamental preliminary result consists in showing that the maximal solution to the constrained BSDE (4.3)-(4.4)-(4.5) satisfies the following identification property: for every $(x,a_0,a_{\Gamma}) \in E \times A_0 \times A_{\Gamma}$, $p^{x,a_0,a_{\Gamma}}$-a.s., $s \geq 0$,

$$Y^x,a_0,a_{\Gamma}_s = v(X_s,I_s,J_s), \quad (5.5)$$

with $v$ is the deterministic function introduced in (5.1).

Notice that, by Theorem 4.6 $Y^x,a_0,a_{\Gamma}$ is the pointwise limit of the solution $Y^{n,x,a_0,a_{\Gamma}}$ to equation (4.16). Therefore, in order to prove property (5.5) for $Y^x,a_0,a_{\Gamma}$ it would be enough to prove the analogous result for the solution $Y^{n,x,a_0,a_{\Gamma}}$ to the penalized BSDE (4.16). To this end, for every $n \in \mathbb{N}$, let us define the deterministic function $v^n$ on $E \times A_0 \times A_{\Gamma}$ as

$$v^n(x,a_0,a_{\Gamma}) := Y^{n,x,a_0,a_{\Gamma}}_0, \quad (x,a_0,a_{\Gamma}) \in E \times A_0 \times A_{\Gamma}. \quad (5.6)$$

By (5.6) and (4.14), one easily see that

$$|v^n(x,a_0,a_{\Gamma})| \leq \frac{M_f}{\delta} + C^* M_c, \quad \forall (x,a_0,a_{\Gamma}) \in E \times A_0 \times A_{\Gamma}. \quad (5.6)$$

We have the following result.
Lemma 5.3. Let Hypotheses (HhλQR), (H0), (H0’), (Hλ0λT), and (Hfc) hold. For any \((x,a_0,a_T,n) \in E \times A_0 \times A_T \times \mathbb{N}\), let \((Y_{n,T},Z_{n,T})\) be the solution to the penalized BSDE (4.16). Then the function \(v^n\) in (5.6) is such that, \(\mathbb{P}^x,a_0,a_T\)-a.s., for every \(s \geq 0\),

\[
Y^n_{s,T} = v^n(x_s, I_s, J_s). \tag{5.7}
\]

Proof. Let us fix \((x,a_0,a_T,n) \in E \times A \times \mathbb{N}\), and let \((Y^n,Z^n) = (Y_{n,T},Z_{n,T})\) be the solution to the penalized BSDE (4.16). By Proposition 4.1, there exists a sequence \((Y^{n,T},Z^{n,T})\) in \(S^\infty \times G^2_{x,a_0,a_T,loc}\) such that, when \(T\) goes to infinity, \((Y^{n,T},Z^{n,T})\) converges \(\mathbb{P}^x,a_0,a_T\)-a.s. to \((Y^n,Z^n)\) and \((Z^{n,T})\) converges to \((Z^n)\) in \(G^2_{x,a_0,a_T,loc}\). Let us now fix \(T,S > 0, S < T\), and consider the BSDE solved by \((Y^{n,T},Z^{n,T})\) on \([0,S]\). Then, we know from the fixed point argument giving the well-posedness of the penalized BSDE (4.16) (see the proof of Theorem 4.1 in [2]) that there exists a sequence \((Y^{n,T,k},Z^{n,T,k})\) in \(S^\infty \times G^2_{x,a_0,a_T,loc}\) converging to \((Y^{n,T},Z^{n,T})\) in \(S^\infty \times G^2_{x,a_0,a_T,loc}\). Let us define \(v^{n,T}(x,a_0,a_T) := Y^{n,T}_0\), \(v^{n,T,k}(x,a_0,a_T) := Y^{n,T,k}_0\). For \(k = 0\), we have, \(\mathbb{P}^x,a_0,a_T\)-a.s.,

\[
Y^{n,T,1}_t = \mathbb{E}^{x,a_0,a_T}\left[\int_{(t,S]} f(X_r, I_r) \, dr + \int_{(t,S]} c(X_r, J_r) \, dp_r^x \bigg| \mathcal{F}_t\right], \quad t \in [0,S].
\]

Then, from the Markov property of \((X,I,J)\) we get, \(\mathbb{P}^x,a_0,a_T\)-a.s., \(Y^{n,T,1}_t = v^{n,T,1}(X_t,I_t,J_t)\), and in particular

\[
\Delta Y^{n,T,1}_t = -c(X_{t-}, J_{t-}) \Delta p_t^x + \int_{E \times A_0 \times A_T} Z^{n,T,1}_t(y,b,c) q(\{t\} \times dy \, dc)
\]

\[
= -c(X_{t-}, J_{t-}) \Delta p_t^x + Z^{n,T,1}_t(X_t,I_t,J_t) - Z^{n,T,1}_t(X_{t-}, I_{t-}, J_{t-}) \Delta p_t^x, \quad 0 \leq t \leq S.
\]

This gives

\[
Z^{n,T,1}_t(y,b,c) - Z^{n,T,1}_t(X_{t-} \in \partial E) = v^{n,T,1}_t(y,b,c) - v^{n,T,1}_t(X_{t-}, I_{t-}, J_{t-}) - c(X_{t-}, J_{t-}) \mathbb{1}_{X_{t-} \in \partial E}.
\]

We now consider the inductive step: \(1 \leq k \in \mathbb{N}\), and assume that \(\mathbb{P}^x,a_0,a_T\)-a.s.,

\[
Y^{n,T,k}_t = v^{n,T,k}(X_t,I_t,J_t) \tag{5.9}
\]

\[
Z^{n,T,k}_t(y,b,c) - Z^{n,T,k}_t(X_{t-} \in \partial E) = v^{n,T,k}_t(y,b,c) - v^{n,T,k}_t(X_{t-}, I_{t-}, J_{t-}) - c(X_{t-}, J_{t-}) \mathbb{1}_{X_{t-} \in \partial E}. \tag{5.10}
\]

Then, plugging (5.9) in (5.8) and computing the conditional expectation as before, by the Markov property of \((X,I)\) we achieve that, \(\mathbb{P}^x,a_0,a_T\)-a.s., \(Y^{n,T,k+1}_t = v^{n,T,k+1}(X_t,I_t,J_t)\). Then,
applying the Itô formula to \(|Y_t^{n,T,k} - Y_t^{n,T}|^2\) and taking the supremum of \(t\) between 0 and \(S\), one can show that
\[
\mathbb{E}^{x,a_0,a_T} \left[ \sup_{0 \leq t \leq S} |Y_t^{n,T,k} - Y_t^{n,T}|^2 \right] \to 0 \text{ as } k \to \infty.
\]
Therefore, \(v^{n,T,k}(x,a_0,a_T) \to v^{n,T}(x,a_0,a_T)\) as \(k\) goes to infinity, for all \((x,a_0,a_T) \in E \times A_0 \times A_T\), from which it follows that, \(\mathbb{P}^{x,a_0,a_T}\)-a.s.,
\[
Y_t^{n,T,x,a_0,a_T} = v^{n,T}(X_t, I_t, J_t).
\]
Finally, we have that \((Y^{n,T,x,a})_t\) converges \(\mathbb{P}^{x,a_0,a_T}\)-a.s. to \((Y^{n,x,a_0,a_T})\) uniformly on compact sets of \(\mathbb{R}\). Thus, \(v^{n,T}(x,a_0,a_T) \to v^n(x,a_0,a_T)\) as \(T\) goes to infinity, for all \((x,a_0,a_T) \in E \times A_0 \times A_T\), and this gives identification \((5.7)\).

### 5.2 The non-dependence of the function \(v\) on the variable \(a\).

Let us now prove item (i) of Theorem \(5.1\). Notice that, by \((4.23)\) and \((5.1)\), \(v\) coincides with the value function \(V^*\) of the randomized control problem introduced in Section \(3.2\). Therefore, to prove \((5.2)\) we have to show that \(V^*(x,a_0,a_T)\) does not depend on \((a_0,a_T)\). This is guaranteed by the next proposition. Indeed, identity \((5.11)\) would imply that
\[
V^*(x,a_0,a_T') \leq J(x,a_0,a_T,v^0,v^\Gamma), \quad x \in E, \quad (a_0,a_0') \in A_0, (a_T,a_T') \in A_T,
\]
and by the arbitrariness of \((v^0,v^\Gamma)\) one can conclude that
\[
V^*(x,a_0,a_T') \leq V^*(x,a_0,a_T) \quad x \in E, \quad (a_0,a_0') \in A_0, (a_T,a_T') \in A_T.
\]

**Proposition 5.4.** Let Hypotheses \((Hh\lambda QR)\), \((H0)\), \((H\lambda_0\lambda_T)\), and \((Hfc)\) hold. Fix \(x \in E\), \((a_0,a_0') \in A_0, (a_T,a_T') \in A_T\), and \((v^0,v^\Gamma) \in \mathcal{V}\). Then, there exist a pair of sequences \((v^{0,\varepsilon},v^{\Gamma,\varepsilon})\) such that
\[
\lim_{\varepsilon \to 0^+} J(x,a_0,a_T',v^{0,\varepsilon},v^{\Gamma,\varepsilon}) = J(x,a_0,a_T,v^0,v^\Gamma).
\]

**Proof.** One can proceed as in the proof of the corresponding result in the context of PDMPs with no jumps from the boundary, see Proposition 5.6 in \(3\). Since the presence of predictable jumps does not induce here any additional technical difficulty, we refer the reader to Section 6.1 in \(3\). \(\square\)

Taking into account \((5.3)\), identity \((5.5)\) gives: \(\mathbb{P}^{x,a_0,a_T}\)-a.s.,
\[
v(X_s) = Y_s^{x,a_0,a_T}, \quad \forall s \geq 0.
\]

In particular, we have the following result.

**Lemma 5.5.** Let Hypotheses \((Hh\lambda QR)\), \((H0)\), \((H\lambda_0\lambda_T)\), and \((Hfc)\) hold. Then the function \(v\) in \((5.3)\) is bounded and continuous.

**Proof.** By \((4.23)\), \((5.12)\) and recalling the definition of \(V^*\) in \((3.29)\), we have
\[
v(x) = V^*(x,a_0,a_T) = \inf_{\nu \in \mathcal{V}} \mathbb{E}^{x,a_0,a_T}_{\nu} \left[ \int_{0}^{\infty} e^{-\delta s} f(X_s, I_s) \, ds + \int_{0}^{\infty} e^{-\delta s} c(X_{s-}, J_{s-}) \, dp_s \right].
\]
The boundedness of \( v \) then directly follows from the boundedness of \( f \) and \( c \). In particular,
\[
|v(x)| \leq \frac{M}{\delta} + C x M_c, \quad \forall x \in E.
\]

Let us now prove the continuity property of \( v \). We proceed as in [19], Section 5. Let \( B(E) \) be the set of all bounded functions on \( E \). Fix \((a_0, a_T) \in A_0 \times A_T\), and define the deterministic operator \( G : B(E) \to B(E) \) as \( G \varphi(x) := \inf_{\nu \in \mathcal{V}} G_{\nu} \varphi(x) \), where
\[
G_{\nu} \varphi(x) := \mathbb{E}^{x,a_0,a_T}_{\nu} \left[ \int_{[0,T_1]} e^{-\delta s} f(X_s, I_s) \, ds + \int_{[0,T_1]} e^{-\delta s} c(X_{s-}, J_{s-}) \, dp^s_s + e^{-\delta T_1} \varphi(X_{T_1}) \right],
\]
with \( T_1 \) the first jump time of the PDMP \((X, I, J)\) under \( \mathbb{P}^{x,a_0,a_T}_{\nu} \). Set \( t^*_n(x) := \inf\{t \geq 0 : \{X_t \notin E, (X_0, I_0, J_0) = (x, a_0, a_T)\}, \mathbb{P}^{x,a_0,a_T}_{\nu}\text{-a.s.}\} \), and consider the sequence of Borel-measurable functions \((v_n)_{n \in \mathbb{Z}_+}\) defined by
\[
v_{n+1}(x) = G v_n(x) := \inf_{\nu \in \mathcal{V}} \left\{ \int_0^{t^*_n(x)} \chi\nu(s) f^{\nu}_0(X_s, I_s) \, ds + \chi\nu(t^*_n(x)) F^\nu(X_{t^*_n}, J_{t^*_n}) \right\},
\]
where \( \chi\nu(s) := e^{-\delta s} e^{-\int_0^s \tilde{\lambda}(t,X_t,I_t) \, dt} \) and, for any \( \psi \in B(E) \),
\[
f^{\nu}_0(X_s, I_s) = f(X_s, I_s) + \int_E \psi(y) \lambda(X_s, I_s) Q(X_s, I_s, dy)
\]
\[
F^\nu(X_{s-}, J_{s-}) = c(X_{s-}, J_{s-}) + \int_E \psi(y) R(X_{s-}, J_{s-}, dy).
\]

If we prove that \( G \) is a two-stage contraction mapping, then by the strong Markov property of the PDMP \((X, I, J)\) it would follow that \( v \) is the unique fixed point of \( G \), and therefore \( v(x) = \lim_{n \to \infty} v_n(x) \), see Corollary 5.6 in [19]. Then, the continuity property of \( v \) in \( E \) would follow from the existence of two monotone sequences of continuous functions converging to \( v \), one from above and one from below, see Lemmas 5.9 and 5.10 in [19].

It remains to prove that \( G^2 \) is a contraction in \( E \). To this end, it would be enough to show that, for any \( \psi_1, \psi_2 \in B(E) \), \( |G^2_{\nu} \psi_1 - G^2_{\nu} \psi_2| \leq \rho |\psi_1 - \psi_2| \) for some constant \( \rho < 1 \), independent on \( \nu \), where \( ||\psi|| = \max_{x \in E} \psi(x) \), \( \psi \in B(E) \). Denoting by \( T_2 \) the second jump time of \((X, I, J)\), we have
\[
G^2_{\nu} \psi(x) := \mathbb{E}^{x,a_0,a_T}_{\nu} \left[ \int_{[0,T_2]} e^{-\delta s} f(X_s, I_s) \, ds + \int_{[0,T_2]} e^{-\delta s} c(X_{s-}, J_{s-}) \, dp^s_s + e^{-\delta T_2} \psi(X_{T_2}) \right],
\]
so that \( |G^2_{\nu} \psi_1 - G^2_{\nu} \psi_2| \leq \mathbb{E}^{x,a_0,a_T}_{\nu} [ e^{-\delta T_2} ||\psi_1 - \psi_2|| ] \). The fact that \( \mathbb{E}^{x,a_0,a_T}_{\nu} [ e^{-\delta T_2} ] \leq \rho < 1 \) is a consequence of Hypothesis (H0), see the proof of Proposition 46.17 in [18] for more details.

### 5.3 Viscosity properties of the function \( v \)

It remains to prove item (ii) of Theorem 5.1, i.e. that the function \( v \) in (5.3) provides a viscosity solution to (2.13)-(2.14). To this end, we first recall an useful result.

**Proposition 5.6.** A function \( u \in \mathcal{C}_b(\bar{E}) \) (resp. \( w \in \mathcal{C}_b(\bar{E}) \)) is a sub- (resp. super-) solution to (2.13)-(2.14) if and only if, for any \( \phi \in \mathcal{C}_b(\bar{E}) \), for any \( x_0 \) global maximum (resp. global minimum) point of \( u - \phi \) (resp. \( w - \phi \)),
\[
H^\phi(x_0, \phi(x_0), \nabla \phi(x_0)) \leq 0 \quad \text{if} \ x_0 \in E,
\]
At this point, applying Itô’s formula to $e^x$ with (5.13), it follows that

$$
\begin{align*}
\min\{H^\phi(x_0, \phi(x_0), \nabla\phi(x_0)), \phi(x_0) - F^\phi(x_0)\} \leq 0 & \quad \text{if } x_0 \in \partial E, \\
\text{resp.} \quad H^\phi(x_0, \phi(x_0), \nabla\phi(x_0)) \geq 0 & \quad \text{if } x_0 \in E, \\
\max\{H^\phi(x_0, \phi(x_0), \nabla\phi(x_0)), \phi(x_0) - F^\phi(x_0)\} \geq 0 & \quad \text{if } x_0 \in \partial E.
\end{align*}
$$

Proof. See the proof of Proposition II.1 in [29].

We separate the proof of viscosity subsolution and supersolution properties, which are different.

In particular the supersolution property is more delicate and should take into account the maximality property of $Y^{x,a_0,a_I}$. Notice that, by Lemma 5.5, it is enough to verify the viscosity sub- and super-solution properties for $v$ in the sense of Proposition 5.6.

**Proof of the viscosity subsolution property to (2.13)-(2.14).**

**Proposition 5.7.** Let assumptions (HhλQR), (H0), (H0′), (Hλ0λΓ), and (Hfc) hold. Then, the function $v$ in (5.3) is a viscosity subsolution to (2.13)-(2.14).

Proof. Let $\bar{x} \in \bar{E}$, and let $\varphi \in C^1(\bar{E})$ be a test function such that

$$
0 = (v - \varphi)(\bar{x}) = \max(v - \varphi)(y). \tag{5.13}
$$

**Case 1:** $\bar{x} \in E$. Fix $(a_0, a_I) \in A_0 \times A_I$, set $\eta = \frac{1}{4}d(\bar{x}, \delta E)$, and

$$
\tau := \inf\{t \geq 0 : |\phi(t, \bar{x}, a_0) - \bar{x}| \geq \eta\}. \tag{5.14}
$$

Let $h > 0$. Let $Y^{x,a_0,a_I}$ be the unique maximal solution to (4.3)-(4.4)-(4.5) under $P^{x,a_0,a_I}$. We apply the Itô formula to $e^{-\delta t} Y^{x,a_0,a_I}_t$ between 0 and $\theta := \tau \wedge h \wedge T_1$, where $T_1$ denotes the first jump time of $(X, I, J)$. From the constraints (4.4)-(4.5) and the fact that $K$ is a nondecreasing process, it follows that $P^{x,a_0,a_I}$-a.s.,

$$
\begin{align*}
Y^{x,a_0,a_I}_0 & \leq e^{-\delta \theta_m} Y^\phi_\theta + \int_{(0,\theta]} e^{-\delta r} f(X_r, I_r) \, dr + \int_{(0,\theta]} e^{-\delta r} c(X_{r-}, J_{r-}) \, dp^*_r \\
& \quad - \int_{(0,\theta]} e^{-\delta r} \int_{E \times A_0 \times A_I} Z^x c_{\bar{a}a} q(dr \, dy \, dc).
\end{align*}
$$

Applying the expectation with respect to $P^{x,a_0,a_I}$, from the identification property (5.12), together with (5.13), it follows that

$$
\varphi(\bar{x}) \leq \mathbb{E}^{x,a_0,a_I}\left[ e^{-\delta \theta} \varphi(X_\theta) + \int_{(0,\theta]} e^{-\delta r} f(X_r, I_r) \, dr + \int_{(0,\theta]} e^{-\delta r} c(X_{r-}, J_{r-}) \, dp^*_r \right].
$$

At this point, applying Itô’s formula to $e^{-\delta r} \varphi(X_r)$ between 0 and $\theta$, we get

$$
\begin{align*}
\frac{1}{h} \mathbb{E}^{x,a_0,a_I}\left[ \int_{(0,\theta]} e^{-\delta r} [\delta \varphi(X_r) - L^{I_r} \varphi(X_r) - f(X_r, I_r)] \, dr \right] \\
\leq \frac{1}{h} \mathbb{E}^{x,a_0,a_I}\left[ \int_{(0,\theta]} e^{-\delta r} [R^{I_r} \varphi(X_{r-}) + c(X_{r-}, J_{r-})] \chi_{X_{r-} \in \partial E} \, dp^*_r \right], \tag{5.15}
\end{align*}
$$

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where
\[ \mathcal{L}^I_r \varphi(X_r) := h(X_r, I_r) \cdot \nabla \varphi(X_r) + \int_E (\varphi(y) - \varphi(X_r)) \lambda(X_r, I_r) Q(X_r, I_r, dy), \]
\[ \mathcal{R}^I_r \varphi(X_{r-}) := \int_E (\varphi(y) - \varphi(X_{r-})) R(X_{r-}, J_{r-}, dy). \]

Now we notice that, for every \( r \in [0, \theta] \), \((X_{r-}, I_{r-}, J_{r-}) = (\phi(r, \bar{x}, a_0), a_0, a_\Gamma)\), \(\mathbb{P}^{\bar{x}, a_0, a_\Gamma}\)-a.s., with \(\phi(r, \bar{x}, a_0) \in E\). In particular the right-hand side of \(5.20\) is zero. Taking into account the continuity on \(E\) of the maps \(r \mapsto \phi(r, \bar{x}, a_0), z \mapsto \delta \varphi(z) - \mathcal{L}^{a_0} \varphi(z) - f(z, a_0)\), we see that for any \(\varepsilon > 0\),
\[ -\varepsilon + \frac{\delta \varphi(\bar{x}) - \mathcal{L}^{a_0} \varphi(\bar{x}) - f(\bar{x}, a_0)}{h} \frac{1 - e^{-\theta \delta}}{\delta} \leq 0. \]  

Set \(g(\theta) := \frac{1-e^{-s \delta}}{\delta}, \theta \in R_+\). Recalling that the the distribution density of \(T_1\) under \(\mathbb{P}^{\bar{x}, a_0, a_\Gamma}\) is given by
\[ f_{T_1}(s) = \lambda(\phi(s, \bar{x}, a_0), a_0) \exp \left( -\int_0^s \lambda(\phi(r, \bar{x}, a_0), a_0) dr \right) \mathbb{1}_{\phi(s, \bar{x}, a_0) \in E}, \]
and taking \(m > M\), we have
\[ \frac{\mathbb{E}^{\bar{x}, a_0, a_\Gamma}[g(\theta)]}{h} = \frac{1}{h} \int_0^h g(s) f_{T_1}(s) ds + \frac{g(h)}{h} \mathbb{P}^{\bar{x}, a_0, a_\Gamma}[T_1 > h] \]
\[ = \int_0^h \frac{1 - e^{-s \delta}}{\delta h} \lambda(\phi(s, \bar{x}, a_0), a_0) e^{-\int_0^s \lambda(\phi(r, \bar{x}, a_0), a_0) dr} \mathbb{1}_{\phi(s, \bar{x}, a_0) \in E} ds \]
\[ + \frac{1 - e^{-h \delta}}{\delta h} e^{-\int_0^h \lambda(\phi(r, \bar{x}, a_0), a_0) dr} \mathbb{1}_{\phi(h, \bar{x}, a_0) \in E}. \]  

where, \(\lambda\) is the function introduced in \(3.2\). By the boundedness of \(\lambda, \lambda_0\) and \(\lambda_\Gamma\), it is easy to see that the two terms in the right-hand side of \(5.19\) converge respectively to zero and one when \(h\) goes to zero. Thus, passing into the limit in \(5.21\) as \(h\) goes to zero we obtain
\[ \delta \varphi(\bar{x}) - h(\bar{x}, a_0) \cdot \nabla \varphi(\bar{x}) - \int_E (\varphi(y) - \varphi(\bar{x})) \lambda(\bar{x}, a_0) Q(\bar{x}, a_0, dy) - f(\bar{x}, a_0) \leq 0. \]

From the arbitrariness of \(a_0 \in A_0\), we conclude that \(H^\varphi(\bar{x}, \varphi(\bar{x}), \nabla \varphi(\bar{x})) \leq 0\).

Case 2: \(\bar{x} \in \partial E\). If \(\varphi(\bar{x}) - F(\bar{x}) \leq 0\) we have finished. Otherwise, suppose that \(\varphi(\bar{x}) - F(\bar{x}) > 0\). We argue similarly to the Case 1. Let \((x_m)_m\) in \(E\) such that \(x_m \xrightarrow{m \to \infty} \bar{x}\). Fix \((a_0, a_\Gamma) \in A_0 \times A_\Gamma\). Let \(\eta\) be a positive constant, and \(\tau_m := \inf\{t \geq 0 : |\phi(t, x_m, a_0) - x_m| \geq \eta\}\). Let moreover \((h_m)_m\) be a strictly positive sequence such that
\[ h_m \xrightarrow{m \to \infty} 0, h_m < \tau_m, m \geq M, \text{ for some } M \in \mathbb{N}. \]

Let \(Y_{x_m, a_0, a_\Gamma}\) be the unique maximal solution to \(1.3\) and \(4.4\) under \(\mathbb{P}^{x_m, a_0, a_\Gamma}\). We apply the Itô formula to \(e^{-\theta_m} Y_{x_m, a_0, a_\Gamma}\) between 0 and \(\theta_m := \tau_m \wedge h_m \wedge T_1\), where \(T_1\) denotes the first jump time of \((X, I, J)\). Proceeding as in Case 1, we get
\[ \frac{1}{h_m} \mathbb{E}^{x_m, a_0, a_\Gamma}[e^{-\delta \tau_m} [\delta \varphi(X_{\tau_m}) - \mathcal{L}^I \varphi(X_{\tau_m}) - f(X_{\tau_m}, I_{\tau_m})] dr] \]
\[ \leq \frac{1}{h_m} \mathbb{E}^{x_m, a_0, a_\Gamma}[e^{-\theta_m} [\mathcal{R}^{J_{\theta_m} - \varphi}(X_{\theta_m}) + c(X_{\theta_m}, J_{\theta_m})] \mathbb{1}_{X_{\theta_m} \in \partial E}], \]  

(5.20)
where $\mathcal{L}^r$ and $\mathcal{R}^r$ are the operators defined respectively in (5.16) and (5.17). Now we notice that, for every $r \in [0, \theta]$, $(X_{r-}, I_{r-}, J_{r-}) = (\phi(r, x_m, a_0), a_0, a_r)$, $\mathbb{P}^{x_m, a_0, a_r}$-a.s., with $|\phi(r, x_m, a_0) - x_m| < \eta$. Notice that $\Omega$ is the disjoint union of $\Omega_1, \Omega_2, \Omega_3$, with $\Omega_1 = \{\theta_m < T_1, X_{T_1-} \in E\}$, $\Omega_2 = \{\theta_m = T_1, X_{T_1-} \in E\}$, $\Omega_3 = \{\theta_m = T_1, X_{T_1-} \in \partial E\}$. On $\Omega_3$, (5.20) gives
\[
0 \leq \mathbb{E}^{x_m, a_0, a_r} \left[ e^{-\delta T_1} \left| \mathcal{R}^{J_{T_1-}} \varphi(X_{T_1-}) + c(X_{T_1-}, J_{T_1-}) \right| \right] \leq 2||\varphi||_{\infty} + M_c,
\]
that always holds true. On the other hand, on $\Omega_1 \cup \Omega_2$, the right-hand side of (5.20) is zero. In this case, proceeding as in Case 1, we get for any $\varepsilon > 0$,
\[
-\varepsilon + \delta \varphi(x_m) - \mathcal{L}^{a_0} \varphi(x_m) - f(x_m, a_0) \mathbb{E}^{x_m, a_0, a_r} \left[ \frac{1 - e^{-\delta \theta_m}}{\delta} \right] \leq 0.
\]
Passing into the limit in (5.21) as $m$ goes to infinity, arguing as in (5.19), we obtain
\[
\delta \varphi(\bar{x}) - h(\bar{x}, a_0) \cdot \nabla \varphi(\bar{x}) - \int_{E} (\varphi(y) - \varphi(\bar{x})) \lambda(\bar{x}, a_0) Q(\bar{x}, a_0, dy) - f(\bar{x}, a_0) \leq 0,
\]
and we conclude that $H^\varphi(\bar{x}, \varphi(\bar{x}), \nabla \varphi(\bar{x})) \leq 0$ from the arbitrariness of $a_0 \in A_0$.

\section*{Proof of the viscosity supersolution property to (2.13)-(2.14).}

\begin{proposition}
Let assumptions (HhλQR), (H0), (H0'), (Hλ0λr), and (Hfc) hold. Then, the function $v$ in (5.3) is a viscosity supersolution to (2.13)-(2.14).
\end{proposition}

\begin{proof}
Let $\bar{x} \in \bar{E}$, and let $\varphi \in C^1(\bar{E})$ be a test function such that
\[
0 = (v - \varphi)(\bar{x}) = \min_{\bar{x} \in \bar{E}}(v - \varphi)(x).
\]

\textbf{Case 1:} $\bar{x} \in E$. Notice that we can assume w.l.o.g. that $\bar{x}$ is a strict minimum of $v - \varphi$. As a matter of fact, one can subtract to $\varphi$ a positive cut-off function which behaves as $|x - \bar{x}|^2$ when $|x - \tilde{x}|^2$ is small, and that regularly converges to 1 as $|x - \tilde{x}|^2$ increases to 1. Then, for every $\eta > 0$, we can define
\[
0 < \beta(\eta) := \inf_{x \in B^c(\bar{x}, \eta) \cap E} (v - \varphi)(x),
\]
where $B(\bar{x}, \eta) := \{y \in E : |\bar{x} - y| < \eta\}$.

We will show the result by contradiction. Assume thus that $H^\varphi(\bar{x}, \varphi(\bar{x}), \nabla \varphi(\bar{x})) < 0$. Then by the continuity of $H$, there exists $\eta > 0$, $\beta(\eta) > 0$ and $\varepsilon \in (0, \beta(\eta))$ such that
\[
H^\varphi(y, \varphi(y), \nabla \varphi(y)) \leq -\varepsilon, \quad \text{for all } y \in B(\bar{x}, \eta).
\]

Let us fix $T > 0$ and define $\theta := \tau \wedge T$, where $\tau = \inf\{t \geq 0 : X_t \notin B(\bar{x}, \eta)\}$. Moreover, let us fix $(a_0, a_r) \in A_0 \times A_r$, and consider the solution $Y^{n, \bar{x}, a_0, a_r}$ to the penalized (4.19), under the probability $\mathbb{P}^{\bar{x}, a_0, a_r}$. Notice that
\[
\mathbb{P}^{\bar{x}, a_0, a_r}(\tau = 0) = \mathbb{P}^{\bar{x}, a_0, a_r}(X_0 \notin B(\bar{x}, \eta)) = 0.
\]

We apply Itô’s formula to $e^{-\delta t} Y^{n, \bar{x}, a_0, a_r}_t$ between 0 and $\theta$. Then, proceeding as in the proof of formula (4.20) in Proposition 4.4, we get the following inequality:
\[
Y^{n, \bar{x}, a_0, a_r}_0 \geq \inf_{\nu \in \mathcal{V}^n} \mathbb{E}^{\bar{x}, a_0, a_r}[e^{-\delta \theta} Y^{n, \bar{x}, a_0, a_r}_\theta + \int_{(0, \theta]} e^{-\delta r} f(X_r, I_r) dr + \int_{(0, \theta]} e^{-\delta r} c(X_r, J_r) d\nu_r].
\]
Recall that $Y^{n,\bar{x},a_0,a_T}$ converges decreasingly to the maximal solution $Y^{x_m,a_0,a_T}$ to the constrained BSDE (4.3)-(4.4)-(4.5). By the identification property (5.12), together with (5.22) and (5.23), from inequality (5.24) we get that there exists a strictly positive, predictable and bounded function $\nu \in \mathcal{V}$ such that

$$
\phi(x) \geq \mathbb{E}_\nu^{\bar{x},a_0,a_T} \left[ e^{-\delta \theta} \Phi(X_\theta) + \beta e^{-\delta \theta} \mathbb{1}_{[\tau \leq T]} \right] + \mathbb{E}_\nu^{\bar{x},a_0,a_T} \left[ \int_{(0,\theta]} e^{-\delta r} f(X_r, I_r) dr + \int_{(0,\theta]} e^{-\delta r} c(X_r, J_r, I_r) d\nu^* \right] - \frac{\varepsilon}{2 \delta}.
$$

At this point, applying Itô’s formula to $e^{-\delta r} \phi(X_r)$ between 0 and $\theta$, we get

$$
\mathbb{E}_\nu^{\bar{x},a_0,a_T} \left[ \int_{(0,\theta]} e^{-\delta r} [\delta \phi(X_r) - \mathcal{L}^I_r \phi(X_r) - f(X_r, I_r)] dr \right] - \mathbb{E}_\nu^{\bar{x},a_0,a_T} \left[ \int_{(0,\theta]} e^{-\delta r} [\mathcal{R}^{J-}_r \phi(X_r) - c(X_r, J_r, I_r)] \mathbb{1}_{X_r \in \partial E} d\nu^* \right] \\
= -\frac{\varepsilon}{2 \delta} + \mathbb{E}_\nu^{\bar{x},a_0,a_T} \left[ \int_{(0,\theta]} e^{-\delta r} \mathbb{1}_{\tau \leq T} \right] - \mathbb{E}_\nu^{\bar{x},a_0,a_T} \left[ \int_{(0,\theta]} e^{-\delta r} \mathbb{1}_{\tau > T} \right] \\
\geq 0,
$$

(5.25)

where $\mathcal{L}^I_r$ and $\mathcal{R}^{J-}_r$ are defined respectively in (5.16) and (5.17). Notice that, for $r \in [0, \theta]$, $X_r \in B(\bar{x}, \eta) \subset E$. In particular, $[\mathcal{R}^{J-}_r \phi(X_r) - c(X_r, J_r, I_r)] \mathbb{1}_{X_r \in \partial E} = 0$. Moreover,

$$
\delta \phi(X_r) - \mathcal{L}^I_r \phi(X_r) - f(X_r, I_r) \leq \delta \phi(X_r) - \inf_{b \in A_0} \{ \mathcal{L}^b \phi(X_r) + f(X_r, b) \} \\
= H^\phi(X_r, \phi(X_r), \nabla \phi(X_r)) \leq -\varepsilon,
$$

and therefore, from (5.25) we obtain

$$
0 \leq \frac{\varepsilon}{2 \delta} + \mathbb{E}_\nu^{\bar{x},a_0,a_T} \left[ -\varepsilon \int_{(0,\theta]} e^{-\delta r} dr - \beta e^{-\delta \theta} \mathbb{1}_{\tau \leq T} \right] \\
= -\frac{\varepsilon}{2 \delta} + \mathbb{E}_\nu^{\bar{x},a_0,a_T} \left[ \left( \frac{\varepsilon}{\delta} - \beta \right) e^{-\delta \theta} \mathbb{1}_{\tau \leq T} + \frac{\varepsilon}{\delta} e^{-\delta \theta} \mathbb{1}_{\tau > T} \right] \\
\leq -\frac{\varepsilon}{2 \delta} + \frac{\varepsilon}{\delta} \mathbb{E}_\nu^{\bar{x},a_0,a_T} \left[ e^{-\delta T} \mathbb{1}_{\tau > T} \right] \\
\leq -\frac{\varepsilon}{2 \delta} + \frac{\varepsilon}{\delta} e^{-\delta T}.
$$

Letting $T$ go to infinity we achieve the contradiction: $0 \leq -\frac{\varepsilon}{2 \delta}$. 

**Case 2:** $\bar{x} \in \partial E$. As in the previous case, we can assume w.l.o.g. that $\bar{x}$ is a strict minimum of $v - \phi$. Then, for every $\eta > 0$, we can define

$$
0 < \beta(\eta) := \inf_{x \in B^c(\bar{x}, \eta) \cap E} (v - \phi)(x),
$$

(5.26)

where $\bar{B}(\bar{x}, \eta) := \{ y \in E : |\bar{x} - y| < \eta \}$.

If $\phi(\bar{x}) - F^\phi(\bar{x}) \geq 0$ we have finished. Otherwise, assume that $\phi(\bar{x}) - F^\phi(\bar{x}) < 0$. We will show the result by contradiction. Assume thus that $H^\phi(\bar{x}, \phi(\bar{x}), \nabla \phi(\bar{x})) < 0$. Then by the continuity of $H$ and $F$, there exists $\eta > 0$, $\beta(\eta) > 0$ and $\varepsilon \in (0, \beta(\eta) \delta]$ such that

$$
H^\phi(y, \phi(y), \nabla \phi(y)) \leq -\varepsilon, \quad \phi(y) - F^\phi(y) \leq -\varepsilon, \quad \text{for all } y \in \bar{B}(\bar{x}, \eta).
$$
Let us fix $T > 0$ and define $\theta := \tau \wedge T$, where $\tau = \inf\{t \geq 0 : X_t \notin \bar{B}(\bar{x}, \eta)\}$. Arguing as in Case 1, we get

$$
0 \leq \frac{\varepsilon}{2\delta} + \mathbb{E}_{\nu}^{x,\alpha,\gamma} \left[ -\varepsilon \int_{(0,\theta]} e^{-\delta r} \, dr - \varepsilon \int_{(0,\theta]} e^{-\delta r} \, dp_r - \beta e^{-\delta \theta} \mathbb{1}_{\{\tau \leq T\}} \right]
$$

for some $\nu \in \mathcal{V}$. Noticing that, for $r \in [0, \theta]$,

$$
\delta \varphi(X_r) - \mathcal{L}^{I_r} \varphi(X_r) - f(X,r,I_r) \leq \delta \varphi(X_r) - \inf_{b \in A_0} \{ \mathcal{L}^b \varphi(X_r) + f(X_r,b) \} \leq -\varepsilon,
$$

from (5.27) we obtain

$$
0 \leq \frac{\varepsilon}{2\delta} + \mathbb{E}_{\nu}^{x,\alpha,\gamma} \left[ -\varepsilon \int_{(0,\theta]} e^{-\delta r} \, dr - \varepsilon \int_{(0,\theta]} e^{-\delta r} \, dp_r - \beta e^{-\delta \theta} \mathbb{1}_{\{\tau \leq T\}} \right]
$$

Letting $T$ go to infinity we get the contradiction: $0 \leq -\frac{\varepsilon}{2\delta}$.

**Appendix**

**A Technical proofs of Sections 4.1 and 4.2**

**A.1 Proof of Proposition 4.1**

Let us fix $n \in \mathbb{N}$. Property (i) directly follows from definition (4.10). Concerning property (ii), let us fix $(\omega,t) \in \Omega \times [0,T]$, $\zeta, \zeta' \in \mathbb{L}^2(\phi_{\omega,t})$. We have

$$
|\tilde{f}^n(t,X_{t-}(\omega),I_{t-}(\omega),J_{t-}(\omega),\zeta') - \tilde{f}^n(t,X_{t-}(\omega),I_{t-}(\omega),J_{t-}(\omega),\zeta)|
\leq |f^n(X_{t-}(\omega),I_{t-}(\omega),\zeta'(X_{t-}(\omega)),J_{t-}(\omega),\zeta'(X_{t-}(\omega),I_{t-}(\omega),\cdot)) - f^n(X_{t-}(\omega),I_{t-}(\omega),\zeta'(X_{t-}(\omega),\cdot),J_{t-}(\omega),\cdot))| \mathbb{1}_{X_{t-}(\omega) \in E}
\leq L_n \left( \int_{A_0} |	ilde{\zeta}(X_{t-}(\omega),b,J_{t-}(\omega))|^2 \mathbb{1}_{X_{t-}(\omega) \in E} \lambda_0(\lambda_0) + \int_{A_1} |	ilde{\zeta}(X_{t-}(\omega),I_{t-}(\omega),c)|^2 \mathbb{1}_{X_{t-}(\omega) \in E} \lambda_1(d\lambda_1) \right)
= L_n \left( \int_{E \times A_0 \times A_1} |	ilde{\zeta}(y,b,c)|^2 \mathbb{1}_{X_{t-}(\omega) \in E} \lambda_0(\lambda_0) \delta_{X_{t-}(\omega)}(E) + \lambda_1(d\lambda_0) \delta_{X_{t-}(\omega)}(E) \delta_{X_{t-}(\omega)}(d\lambda_0) \right)
\leq 2L_n \left( \int_{E \times A_0 \times A_1} |	ilde{\zeta}(y,b,c)|^2 \mathbb{1}_{X_{t-}(\omega) \in E} \int_{E \times A_0 \times A_1} \tilde{\zeta}(y',b',c') \phi_{\omega,t}(dy',db',dc') \right).
so that \((A.1)\) reads

\[
\leq 2 L_n \left( \int_{E \times A_0 \times A_T} |\zeta(y, b, c) - \Delta A_t(\omega) \int_{E \times A_0 \times A_T} \zeta(y', b', c') \phi_{\omega,t}(dy', db', dc')^2 \phi_{\omega,t}(dy \, db \, dc) \right),
\]

where \(L_n\) is the constant defined in Remark 4.3. Therefore property (ii) holds with \(L^n_x := 2 L_n\).

Finally, let us consider property (iii). Recalling the definitions of \(E^\beta\) and of \(A\) given respectively in (4.10) and in (5.7), and using the boundedness of \(f\) and \(c\), we get

\[
E^{x,0,a_T} \left[ (1 + \sum_{0<t\leq T} |\Delta A_t|^2) \int_{[0,T]} E^\beta_t \left| \tilde{f}^n(t, X_{t-}, I_{t-}, J_{t-}, 0) \right|^2 dA_t \right]
\]

\[
= E^{x,0,a_T} \left[ (1 + \sum_{0<t\leq T} |\Delta A_t|^2) \int_{[0,T]} E^\beta_t \left( e^{-\delta t} f(X_{t-}, I_{t-}) 1_{X_{t-} \in E} + e^{-\delta t} c(X_{t-}, J_{t-}) 1_{X_{t-} \in \partial E} \right)^2 dA_t \right]
\]

\[
\leq (M^n_2 \lor M^n_2) E^{x,0,a_T} \left[ (1 + p^n_T) \int_{[0,T]} E^\beta_t dA_t \right]
\]

\[
\leq (M^n_2 \lor M^n_2) E^{x,0,a_T} \left[ (1 + p^n_T) \left( \int_{[0,T]} e^{\beta A_t^c} dA_t^c + \int_{[0,T]} e^{\beta p^n_T} \prod_{s \in [0,t]} (1 + \beta) e^{-\delta p^n_s} dp^n_s \right)^2 \right]
\]

\[
\leq (M^n_2 \lor M^n_2) E^{x,0,a_T} \left[ (1 + p^n_T) \left( ||\lambda||_\infty T e^{\beta ||\lambda||_\infty T} + (1 + \beta)p^n_T \right)^2 \right]
\]

\[
= c_1(T)(1 + E^{x,0,a_T} [p^n_T]) + c_2 E^{x,0,a_T} \left[ (1 + p^n_T) (1 + \beta)p^n_T \right],
\]

where we have set \(c_1(T) = (M^n_2 \lor M^n_2)||\lambda||_\infty T e^{\beta ||\lambda||_\infty T}, \ c_2 = (M^n_2 \lor M^n_2)\). The conclusion follows from (2.10) and hypothesis (H0').

**A.2 Proof of Lemma 4.3**

Fix \(T > 0\), \((n, x, a_0, a_T) \in N \times E \times A_0 \times A_T\), and let \((Y^n, Z^n) = (Y^n_{r,x,a_0,a_T}, Z^n_{r,x,a_0,a_T})\) be the solution to (4.10), and \(K^n = K^n_{r,x,a_0,a_T}\) the process in (4.18). We start by noticing that \((Y^n, Z^n)\) satisfies (4.17). Then, applying Itô’s formula (4.2) to the process \(|Y^n_r|^2\), between \(t\) and \(T\), we get: \(E^{x,a_T}\)-a.s.,

\[
|Y^n_T|^2 = |Y^n_T|^2 - 2 \int_{(t,T]} Y^n_s \, dY^n_s - \sum_{s \in (t,T]} |\Delta Y^n_s|^2. \quad (A.1)
\]

On the other hand,

\[
|\Delta Y^n_s|^2 = \left| \int_{E \times A_0 \times A_T} Z^n_s(y, b, c) q(\{s\} \times dy \, db \, dc) \right|^2 + |c(X_{s-}, J_{s-})|^2 1_{X_{s-} \in \partial E}
\]

\[
- 2 \left( \int_{E \times A_0 \times A_T} Z^n_s(y, b, c) q(\{s\} \times dy \, db \, dc) \right) c(X_{s-}, J_{s-}) 1_{X_{s-} \in \partial E},
\]

so that (A.1) reads

\[
|Y^n_T|^2 + \sum_{s \in (t,T]} \left| \int_{E \times A_0 \times A_T} Z^n_s(y, b, c) q(\{s\} \times dy \, db \, dc) \right|^2
\]

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A.3 Proof of Proposition 4.4

Uniqueness. Fix $n \in \mathbb{N}$, $\beta \geq \beta^n$, $(x, a_0, a_r) \in E \times A_0 \times A_r$, and consider two solutions $(Y^1, Z^1) = (Y^{1,n,x,a_0,a_r}, Z^{1,n,x,a_0,a_r})$, $(Y^2, Z^2) = (Y^{2,n,x,a_0,a_r}, Z^{2,n,x,a_0,a_r}) \in S^\infty \times \mathcal{G}^{2,\delta}_{x,a_0,a_r}(q)$ of (4.16). Set $Y = Y^2 - Y^1, Z = Z^2 - Z^1$. Let $0 \leq s \leq T < \infty$. Recalling formula (4.19) and the definition of $K^{n,x,a_0,a_r}$ in (4.18), we get: $[\mathbb{P}^x,a_0,a_r]$, a.s.,

$$e^{-2s} |\tilde{Y}_s|^2 + \sum_{r \in \{s\}} e^{-2s} r \left| \int_{E \times a_0 \times A_r} \tilde{Z}_r(y, b, c) q(\{r\} \times dy dB dc) \right|^2$$

$$= e^{-2s} T |\tilde{Y}_T|^2 - 2 \int_{\{s\}} e^{-2s} \tilde{Y}_s \left( \frac{1}{2} \right) \int_{E \times a_0 \times A_r} \tilde{Z}_r(y, b, c) q(dr dy dB dc)$$

$$- 2n \int_{\{s\}} \int_{A_0} e^{-2s} \tilde{Y}_s \left( \left[ Z^1_r(X_s, b, J_r)^- - Z^1_r(X_s, b, J_r)^- \right] \lambda_0(db) dr \right)$$

$$- 2 \int_{\{s\}} \int_{A_r} e^{-2s} \tilde{Y}_s \left( \left[ Z^2_r(X_s, b, J_r)^- - Z^2_r(X_s, b, J_r)^- \right] \lambda_0(dc) dr \right)$$

$$- 2 \int_{\{s\}} \int_{A_r} e^{-2s} \tilde{Y}_s \left( \left[ Z^1_r(X_s, I_s, J_r)^- - Z^1_r(X_s, I_s, J_r)^- \right] \lambda_0(dc) dr \right).$$

(A.3)

Proceeding as in the proof of Proposition 4.4 in [2], one can exhibit a pair $\nu = (\nu^0, \nu^\Gamma, \nu^I, \nu^H) \in V^n$, depending on $\epsilon \in (0, 1)$, such that $[\mathbb{P}^x,a_0,a_r]$-a.s.,

$$e^{-2s} |\tilde{Y}_s|^2 \leq e^{-2s} T |\tilde{Y}_T|^2$$

$$- 2 \int_{\{s\}} \int_{E \times A_0 \times A_r} e^{-2s} \tilde{Y}_s \left( \frac{1}{2} \right) \int_{E \times A_0 \times A_r} \tilde{Z}_r(X_s, b) q^\nu(ds dy dB dc) + 2 \frac{\epsilon}{\delta_0} (\lambda_0(A_0) + \lambda_0(A_r)),$$ (A.4)

where $q^\nu(ds dy dB) := p(ds dy dB) - (\nu^0 \epsilon (b) d_1(s, y, b) + \nu^\Gamma \epsilon (c) d_2(s, y, b, c) + d_3(s, y, b)) \tilde{p}(ds dy dB dc)$ is the compensated martingale measure of $p$ under the probability measure $\mathbb{P}^x,a_0,a_r$ on $(\Omega, F_\infty)$, whose restriction to $(\Omega, F_T)$ is given by $\mathbb{P}^x,a_0,a_r = L^{\nu}_T \mathbb{P}^x,a_0,a_r$, see Proposition 3.5. By Lemma 3.4 together with Remark 3.7, taking the conditional expectation on $F_s$, and using the arbitrariness of $\epsilon$, we achieve

$$e^{-2s} |\tilde{Y}_s|^2 \leq e^{-2s} T \mathbb{E}^{x,a_r}[|\tilde{Y}_T|^2| F_s].$$ (A.5)

In particular, $(e^{-2s} |\tilde{Y}_s|^2)_{t \geq 0}$ is a uniformly integrable submartingale, therefore $e^{-2s} |\tilde{Y}_s|^2 \to \xi_\infty \in L^1(\Omega, F, \mathbb{P}^{x,a_0,a_r})$, as $s \to \infty$. Using again the boundedness of $\tilde{Y}$, we obtain that $\xi_\infty = 0$, which implies $\tilde{Z} = 0$. Finally, plugging $\tilde{Y} = 0$ into (A.3) we conclude that $\tilde{Z} = 0$.

Existence. Fix $n \in \mathbb{N}$, $(x, a_0, a_r) \in E \times A_0 \times A_r$, $\beta \geq \beta^n$. For $T > 0$, let $(Y^T, Z^T) = (Y^{T,n,x,a_0,a_r}, Z^{T,n,x,a_0,a_r})$ denote the unique solution to the penalized BSDE (4.16) in $L^{2,\beta}_{x,a_0,a_r}(p^*, 0, T) \times \mathbb{R}^d$. Let $Y$ be the unique solution to the penalized BSDE (4.16) in $L^{2,\beta}_{x,a_0,a_r}(p^*, 0, T) \times \mathbb{R}^d$.
we see that, for any $0 \leq Y, b, c$

**Step 1. Convergence of $(Y^T)_T$.** Let $T, T' > 0$, with $T < T'$, and $s \in [0, T]$. We have

$$|Y^T_{s} - Y^T_{s}|^2 \leq e^{-2\delta(T-s)} \mathbb{E}_{[s]}^{x,0,a,T} \left[|Y^T_{T'} - Y^T_{T}|^2 \right]_{T,T' \to \infty} \to 0,$$  

(A.6)

where the convergence result follows from (4.14). Let us now consider the sequence of real-valued càdlàg adapted processes $(Y^T)_T$. It follows from (A.6) that, for any $t \geq 0$, the sequence $(Y^T_t(\omega))_T$ is Cauchy for almost every $\omega$, so that it converges $\mathbb{P}^{\omega, a}$-a.s. to some $\mathcal{F}_T$-measurable random variable $Y_t$, which is bounded from the right-hand side of (4.14). Moreover, using again (A.6) and (4.14), we see that, for any $0 \leq S < T \land T'$, with $T, T' > 0$, we have

$$\sup_{0 \leq t \leq S} |Y^T_{t} - Y^T_{t}| \leq e^{-\delta(T\land T'-S)} \left( \frac{M_T}{\delta} + C^* M_c \right)_{T,T' \to \infty} 0.$$  

(A.7)

Since each $Y^T$ is a càdlàg process, it follows that $Y$ is càdlàg, as well. Finally, from estimate (A.7) we see that $Y$ is uniformly bounded and therefore belongs to $S^{\infty}$.

**Step 2. Convergence of $(Z^T)_T$.** Let $S, T, T' > 0$, with $S < T < T'$. Then, applying Itô’s formula to $e^{-2\delta s}|Y^T_{t} - Y^T_{t}|^2$ between 0 and $S$, we find identity (A.3) with $Y = Y^T - Y^{T'}$, $Z = Z^T - Z^{T'}$, $Z^1 = Z^T$, $Z^2 = Z^{T'}$. Taking the expectation, and recalling the elementary inequality $bc \leq b^2 + c^2/4$, for any $b, c \in \mathbb{R}$, we get

$$\frac{1}{2} \mathbb{E}_{[0,S]}^{x,0,a,T} \left[ \sum_{r \in (0,S]} e^{-2\delta r} \int_{E \times A_0 \times A_T} (Z^T_r(y, b, c) - Z^T_r(y, b, c)) q(\{r\} \times dy \, db \, dc) \right]^2 \leq \frac{1}{2} \mathbb{E}_{[0,S]}^{x,0,a,T} \left[ Y^T_S - Y^T_S \right]^2$$

$$+ 4(n^2 + 1) (\lambda_0(A_0) + \lambda_T(A_T)) \mathbb{E}_{[0,S]}^{x,0,a,T} \left[ \int_0^S e^{-2\delta r} |Y^T_r - Y^T_r|^2 \, dr \right]_{T,T' \to \infty} S^{\infty},$$

where the convergence result to zero follows from estimate (A.7). Then, for any $S > 0$, we see that $(Z^T_{[0,S]})_{T>S}$ is a Cauchy sequence in the Hilbert space $\mathcal{G}_x^{2,a,T}(q; \mathbf{0}, \mathbf{S})$. Therefore, we deduce that there exists $\tilde{Z}^S \in \mathcal{G}_x^{2,a,a_T}(q; \mathbf{0}, \mathbf{S})$ such that $(Z^T_{[0,S]})_{T>S}$ converges to $\tilde{Z}^S$ in $\mathcal{G}_x^{2,a,a_T}(q; \mathbf{0}, \mathbf{S})$. Notice that $\tilde{Z}^S_{[0,S]} = \tilde{Z}^S_S$, for any $0 \leq S \leq S' < \infty$. Hence, we define $Z_s = Z^S_s$ for all $s \in [0, S]$ and for any $S > 0$. For any $S > 0$, $(Z^T_{[0,S]})_{T>S}$ converges to $Z_{[0,S]}$ in $\mathcal{G}_x^{2,a,a_T}(q; \mathbf{0}, \mathbf{S})$, i.e. recalling Lemma 4.2(i),

$$\mathbb{E}_{[0,S]}^{x,0,a,T} \left[ \sum_{r \in (0,S]} \int_{E \times A_0 \times A_T} (Z^T_r(y, b, c) - Z_r(y, b, c)) q(\{r\} \times dy \, db \, dc) \right]^2_{T \to \infty} 0$$

(A.8)

Now, fix $S \in [0, T]$ and consider the BSDE satisfied by $(Y^T, Z^T)$ on $[0, S]$: $\mathbb{P}^{x,a}$-a.s.,

$$Y^T_t = Y^T_S - \delta \int_{[t,S]} Y^T_r \, dr + \int_{[t,S]} f(X_r, I_r) \, dr + \int_{[t,S]} c(X_r, J_r) \, dp_r^*$$

$$- \int_{[t,S]} \int_{E \times A_0 \times A_T} Z^T_r(y, b, c) q(\{r\} \times dy \, db \, dc)$$

$$- \int_{[t,S]} \int_{A_0} \{n[Z^T_r(X_r, b, J_r)] + Z^T_r(X_r, b, J_r)\} \lambda_0(\{db\}) \, dr$$
By construction, we have convergence theorem that, for all
Since
From the elementary numerical inequality:
From (A.8) and (A.7), we can pass to the limit in the above BSDE by letting
unique solution to (4.16). For any
P
under
is in
From (4.20) and (4.17) we get:
We take the expectation in (4.17) under \( P^x, \alpha, \Gamma \), conditional to \( F_s \). Recalling that
is in \( G^2_{x, \alpha, \Gamma, \text{loc}}(q; 0, T) \), from Lemma 3.4 we get: \( P^x, \alpha, \Gamma \)-a.s., for all \( \nu \in \mathcal{V}^n \),

\[
Y_{s}^{n, x, \alpha, \Gamma} = \int_{(T-s]} e^{-\delta (T-s)} Y^{n, x, \alpha, \Gamma}_T + \int_{(s, T]} e^{-\delta (r-s)} f(X_r, I_r) \, dr + \int_{(s, T]} e^{-\delta (r-s)} c(X_{r-}, J_r) \, dp^*_r \bigg| F_s \\
- \mathbb{E}_P^{x, \alpha, \Gamma} \left[ \int_{[s, T]} e^{-\delta (s-r)} \{ n[Z_r^{n, x, \alpha, \Gamma}(X_r, b, J_r)] - \nu^{T}(b) Z^{n, x, \alpha, \Gamma}_r(X_r, b, J_r) \} \lambda_0(\{db\}) \, dr \bigg| F_s \right] \\
- \mathbb{E}_P^{x, \alpha, \Gamma} \left[ \int_{[s, T]} e^{-\delta (s-r)} \{ n[Z_r^{n, x, \alpha, \Gamma}(X_r, I_r, c)] - \nu^{T}(c) Z^{n, x, \alpha, \Gamma}_r(X_r, I_r, c) \} \lambda_1(\{dc\}) \, dr \bigg| F_s \right].
\]

From the elementary numerical inequality: \( n[z] - \nu z \geq 0 \) for all \( z \in \mathbb{R}, \nu \in (0, n] \), we deduce by (A.9) that, for all \( \nu \in \mathcal{V}^n \),

\[
Y_{s}^{n} \leq \mathbb{E}_P^{x, \alpha, \Gamma} \left[ e^{-\delta (T-s)} Y^{n}_T + \int_{(s, T]} e^{-\delta (r-s)} f(X_r, I_r) \, dr + \int_{(s, T]} e^{-\delta (r-s)} c(X_{r-}, J_r) \, dp^*_r \bigg| F_s \right].
\]

Since \( Y^n \) is in \( S^\infty \), sending \( T \to \infty \), we obtain from the conditional version of Lebesgue dominated convergence theorem that, for all \( \nu \in \mathcal{V}^n \),

\[
Y_{s}^{n} \leq \mathbb{E}_P^{x, \alpha, \Gamma} \left[ \int_{(s, \infty)} e^{-\delta (r-s)} f(X_r, I_r) \, dr + \int_{(s, \infty)} e^{-\delta (r-s)} c(X_{r-}, J_r) \, dp^*_r \bigg| F_s \right].
\]

so that

\[
Y_{s}^{n} \leq \text{ess inf}_{\nu \in \mathcal{V}^n} \mathbb{E}_P^{x, \alpha, \Gamma} \left[ \int_{(s, \infty)} e^{-\delta (r-s)} f(X_r, I_r) \, dr + \int_{(s, \infty)} e^{-\delta (r-s)} c(X_{r-}, J_r) \, dp^*_r \bigg| F_s \right]. \tag{A.10}
\]

On the other hand, for \( \varepsilon \in (0, 1) \), let us consider the process \( \nu^{\varepsilon} \) := \((\nu^{0, \varepsilon}, \nu^{T, \varepsilon}) \in \mathcal{V}^n \) defined by:

\[
\nu^{0, \varepsilon}(b) = n \mathbb{1}_{\{Z^n_0(X_{s-b}, I_{s-c}) \leq 0\}} + \varepsilon \mathbb{1}_{\{0 < Z^n_0(X_{s-b}, I_{s-c}) < 1\}} + \varepsilon Z^n_0(X_{s-b}, I_{s-c})^{-1} \mathbb{1}_{\{Z^n_0(X_{s-b}, I_{s-c}) \geq 1\}},
\]

\[
\nu^{T, \varepsilon}(c) = n \mathbb{1}_{\{Z^n_T(X_{s-b}, I_{s-c}) \leq 0\}} + \varepsilon \mathbb{1}_{\{0 < Z^n_T(X_{s-b}, I_{s-c}) < 1\}} + \varepsilon Z^n_0(X_{s-b}, I_{s-c})^{-1} \mathbb{1}_{\{Z^n_T(X_{s-b}, I_{s-c}) \geq 1\}}.
\]

By construction, we have

\[
n[Z^n_0(X_{s-b}, I_{s-c})] + \nu^{0, \varepsilon}(b) Z^n_0(X_{s-b}, I_{s-c}) \leq \varepsilon, \quad s \geq 0, b \in A_0,
\]

\[
n[Z^n_T(X_{s-b}, I_{s-c})] + \nu^{T, \varepsilon}(c) Z^n_T(X_{s-b}, I_{s-c}) \leq \varepsilon, \quad s \geq 0, c \in A_T.
\]
and thus for this choice of $\nu = \nu^x$ in (A.9):

$$Y_{s}^{n,x,a_0,a_{\Gamma}} \geq -\varepsilon \frac{1 - e^{-\delta(T-s)}}{\delta} (\lambda_{\Gamma}(A_{\Gamma}) + \lambda_{0}(A_{0})) + \mathbb{E}_{\nu^x}^{x,a_0,a_{\Gamma}} \left[ e^{-\delta(T-s)} Y_{T}^{n,x,a_0,a_{\Gamma}} + \int_{(s,T]} e^{-\delta(r-s)} f(X_{r}, I_{r}) \, dr + \int_{(s,\infty)} e^{-\delta(r-s)} c(X_{r-}, J_{r-}) \, dp_{r}^{*} \right].$$

Letting $T \to \infty$, since $f, c$ are bounded and $Y_{s}^{n,x,a} \in S^{\infty}$, it follows from the conditional version of Lebesgue dominated convergence theorem that

$$Y_{s}^{n,x,a} \geq \text{ess inf}_{\nu \in V^{n}} \mathbb{E}_{\nu}^{x,a_0,a_{\Gamma}} \left[ \int_{(s,\infty)} e^{-\delta(r-s)} f(X_{r}, I_{r}) \, dr + \int_{(s,\infty)} e^{-\delta(r-s)} c(X_{r-}, J_{r-}) \, dp_{r}^{*} \right] - \frac{\varepsilon}{\delta} (\lambda_{0}(A_{0}) + \lambda_{\Gamma}(A_{\Gamma})).$$

(A.11)

Taking into account the arbitrariness of $\varepsilon$, the required representation of $Y_{s}^{n,x,a}$ follows from (A.11) and (A.10).

A.4 Proof of Lemma 4.5

Fix $T > 0$. In what follows we shall denote by $C > 0$ a generic positive constant depending on $M_{f}, M_{c}, C^{*}, \delta$ and $T$, which may vary from line to line. Let $(Y_{n}, Z_{n}) = (Y_{n,x,a_0,a_{\Gamma}}, Z_{n,x,a_0,a_{\Gamma}})$ be the solution to (4.16), and let $K_{n} = K_{n,x,a_0,a_{\Gamma}}$ be the process in (4.18). By Lemma 4.3, applying the expectation with respect to $\mathbb{P}^{x,a_0,a_{\Gamma}}$ in identity (4.19) with $t = 0$, and taking into account Lemma 3.2 we obtain

$$\mathbb{E}^{x,a_0,a_{\Gamma}} \left[ \int_{(0,T]} \int_{E \times A_{0} \times A_{\Gamma}} |Z_{s}^{n}(y, b, c) - \hat{Z}_{s}^{n} K_{s}(s)|^{2} \tilde{p}(ds \, dy \, db) \right]$$

$$\leq \mathbb{E}^{x,a_0,a_{\Gamma}} \left[ |Y_{T}^{n}|^{2} \right] - 2 \mathbb{E}^{x,a_0,a_{\Gamma}} \left[ \int_{(0,T]} Y_{s}^{n} \, dK_{s}^{n} \right]$$

$$+ 2 \mathbb{E}^{x,a_0,a_{\Gamma}} \left[ \sum_{s \in (0,T]} \left( \int_{E \times A_{0} \times A_{\Gamma}} Z_{s}^{n}(y, b, c) q(s) \times \, dy \, db \, dc \right) e(X_{s-}, J_{s-}) \mathbb{1}_{X_{s-} \in \partial E} \right]$$

$$+ 2 \mathbb{E}^{x,a_0,a_{\Gamma}} \left[ \int_{(0,T]} Y_{s}^{n} f(X_{s}, I_{s}) \, ds \right] + 2 \mathbb{E}^{x,a_0,a_{\Gamma}} \left[ \int_{(0,T]} Y_{s}^{n} c(X_{s-}, J_{s-}) \, dp_{s}^{*} \right].$$

Using the elementary inequality $2a b \leq \gamma a^{2} + \frac{1}{\gamma} b^{2}$, with $\gamma \in \mathbb{R}_{+} \setminus \{0\}$, $\gamma < 1$, we get

$$(1 - \gamma) \mathbb{E}^{x,a_0,a_{\Gamma}} \left[ \int_{(0,T]} \int_{E \times A_{0} \times A_{\Gamma}} |Z_{s}^{n}(y, b, c) - \hat{Z}_{s}^{n} K_{s}(s)|^{2} \tilde{p}(ds \, dy \, db) \right]$$

$$\leq \mathbb{E}^{x,a_0,a_{\Gamma}} \left[ |Y_{T}^{n}|^{2} \right] - 2 \mathbb{E}^{x,a_0,a_{\Gamma}} \left[ \int_{(0,T]} Y_{s}^{n} \, dK_{s}^{n} \right] + \frac{1}{\gamma} \mathbb{E}^{x,a_0,a_{\Gamma}} \left[ \sum_{s \in (0,T]} |c(X_{s-}, J_{s-})|^{2} \mathbb{1}_{X_{s-} \in \partial E} \right]$$

$$+ 2 \mathbb{E}^{x,a_0,a_{\Gamma}} \left[ \int_{(0,T]} Y_{s}^{n} f(X_{s}, I_{s}) \, ds \right] + 2 \mathbb{E}^{x,a_0,a_{\Gamma}} \left[ \int_{(0,T]} Y_{s}^{n} c(X_{s-}, J_{s-}) \, dp_{s}^{*} \right].$$
Set now \( C_Y := \frac{M_f}{\gamma} + C^* M_c \). Recalling the uniform estimate \((4.14)\) on \( Y^n \), we obtain

\[
(1 - \gamma) \mathbb{E}^{x,a_0,a_T} \left[ \int_{(0,T]} \int_{E \times A_0 \times A_f} \left| Z^n_s(y, b, c) - \hat{Z}^n_s 1_K(s) \right|^2 \tilde{p}(ds dy db dc) \right] 
\leq \frac{1}{\gamma} M^2 C^*(T) + C Y^n + 2C_Y (M_f T + M_e C^*(T)) + 2C_Y \mathbb{E}^{x,a_0,a_T} [K^n_T], \tag{A.12}
\]

where \( C^*(t) \) is the deterministic function defined in \((2.10)\). On the other hand, from \((4.16)\), we get:

\[
K^n_T = Y^n_T - Y^n_0 - \delta \int_{(0,T]} Y^n_{s,x,a} \, ds + \int_{(0,T]} f(x_s, I_s) \, ds + \int_{(0,T]} c(X_{s-}, J_{s-}) \, dp^*_s 
- \int_{(0,T]} \int_{A_0} Z^n_s(X_s, b, J_s) \lambda_0(\delta b) \, ds - \int_{(0,T]} \int_{A_T} Z^n_s(X_s, I_s, c) \lambda_0(\delta c) \, ds 
- \int_{(0,T]} \int_{E \times A_0 \times A_T} Z^n_s(y, b, c) q(ds dy db dc). \tag{A.13}
\]

Using again the inequality \( 2ab \leq \frac{1}{\eta} a^2 + \eta b^2 \), for any \( \eta = \alpha, k > 0 \), and taking the expected value in \((A.13)\), we get

\[
2 \mathbb{E}^{x,a_0,a_T} [K^n_T] \leq 4C_Y + 2\delta C_Y T + 2M_f T + 2M_e C^*(T)
+ \frac{T}{\alpha} \lambda_0(A_0) + \alpha \mathbb{E}^{x,a_0,a_T} \left[ \int_{(0,T]} \int_{A_0} |Z^n_s(X_s, b, J_s)|^2 \lambda_0(\delta b) \, ds \right] 
+ \frac{T}{k} \lambda_1(A_0) + k \mathbb{E}^{x,a_0,a_T} \left[ \int_{(0,T]} \int_{A_T} |Z^n_s(X_s, I_s, c)|^2 \lambda_1(\delta c) \, ds \right]. \tag{A.14}
\]

Plugging \((A.14)\) into \((A.12)\), we have

\[
(1 - \gamma) \mathbb{E}^{x,a_0,a_T} \left[ \int_{(0,T]} \int_{E \times A_0 \times A_T} \left| Z^n_s(y, b, c) - \hat{Z}^n_s 1_K(s) \right|^2 \tilde{p}(ds dy db) \right] \leq C +
+ (\alpha \vee k) C_Y (1 + 2T) \left( \int_{(0,T]} \int_{A_0} |Z^n_s(X_s, b, J_s)|^2 \lambda_0(\delta b) + \int_{A_T} |Z^n_{s,x,a}(X_s, I_s, c)|^2 \lambda_1(\delta c) \, ds \right),
\]

and choosing \( \alpha = k = \frac{1 - \gamma}{2C_Y (1 + 2T)} \), we get

\[
(1 - \gamma) \mathbb{E}^{x,a_0,a_T} \left[ \int_{(0,T]} \int_{E \times A_0 \times A_T} \left| Z^n_s(y, b, c) - \hat{Z}^n_s 1_K(s) \right|^2 \tilde{p}(ds dy db) \right] \leq C,
\]

that gives the required uniform estimate for \((Z^n)\), and also \((K^n)\) by \((A.13)\).

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**References**


