Preinflationary and inflationary fast-roll eras and their signatures in the low CMB multipoles

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We study the entire coupled evolution of the inflaton $\phi(t)$ and the scale factor $a(t)$ for general initial conditions $\phi(t_0)$ and $d\phi(t_0)/dt$ at a given initial time $t_0$. The generic early Universe evolution has three stages: decelerated fast roll followed by inflationary fast roll and then inflationary slow roll (an attractor always reached for generic initial conditions). This evolution is valid for all regular inflaton potentials $v(\phi)$. In addition, we find a special (extreme) slow-roll solution starting at $t = -\infty$ in which the fast-roll stages are absent. At some time $t = t_*$, the evolution backwards in time from $t_0$ reaches generically a mathematical singularity where $a(t)$ vanishes and the Hubble parameter becomes singular. We determine the general behavior near the singularity. The classical homogeneous inflaton description turns to be valid for $t - t_* > 10^4 t_{\text{Planck}}$ well before the beginning of inflation, quantum loop effects are negligible there. The singularity is never reached in the validity region of the classical treatment and therefore it is not a real physical phenomenon here. Fast-roll and slow-roll regimes are analyzed in detail including the equation of state evolution, both analytically and numerically. The characteristic time scale of the fast-roll era turns to be $t_1 = (1/m) \sqrt{V(0)}/[3M^2] \sim 10^4 t_{\text{Planck}}$, where $V$ is the double-well inflaton potential, $m$ is the inflaton mass, and $M$ the energy scale of inflation. The whole evolution of the fluctuations along the decelerated and inflationary fast-roll and slow-roll eras is computed. The Bunch-Davies initial conditions are generalized for the present case in which the potential felt by the fluctuations can never be neglected. The fluctuations feel a singular attractive potential near the $t = t_*$ singularity (as in the case of a particle in a central singular potential) with exactly the critical strength ($\sim 1/4$) allowing the fall to the center. Precisely, the fluctuations exhibit logarithmic behavior describing the fall to $t = t_*$. The power spectrum gets dynamically modified by the effect of the fast-roll eras and the choice of Bunch-Davies initial conditions at a finite time through the transfer function $D(k)$ of initial conditions. The power spectrum vanishes at $k = 0, D(k)$ presents a first peak for $k \sim 2/\eta_0$ ($\eta_0$ being the conformal initial time), then oscillates with decreasing amplitude and vanishes asymptotically for $k \to \infty$. The transfer function $D(k)$ affects the low cosmic microwave background multipoles $C_\ell$: the change $\Delta C_\ell/C_\ell$ for $1 \leq \ell \leq 5$ is computed as a function of the starting instant of the fluctuations $t_0$. Cosmic microwave background quadrupole observations indicate large suppressions, which are well reproduced for the range $t_0 - t_* \geq 0.05/m \approx 10^4 t_{\text{Planck}}$.

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I. INTRODUCTION AND SUMMARY OF RESULTS

Since the Universe expands exponentially fast during inflation, gradients are exponentially erased and can be neglected. At the same time, the exponential stretching of spatial lengths classicalizes the physics and allows a classical treatment. One can therefore consider a homogeneous and classical inflaton field which thus determines self-consistently a homogeneous and isotropic Friedmann-Robertson Walker metric sourced by this inflaton.

This treatment is valid for early times well after the Planck time $t = 10^{-44}$ sec, at which the quantum fluctuations are expected to be large and thus a full quantum gravity treatment is required.

In this paper we study the entire coupled evolution of the inflaton field $\phi(t)$ and the scale factor $a(t)$ of the metric for generic initial conditions, fixed by the values of $\phi(t_0)$ and $d\phi(t_0)/dt$ at a given initial time $t_0$.

We show that the generic early Universe evolution has three stages: a decelerated fast-roll stage followed by an inflationary fast-roll stage and then by a slow-roll inflationary regime, which is an attractor always reached for generic initial conditions. This evolution is valid for all regular inflaton potentials. In addition, we find a particular (extreme) slow-roll solution starting from $t = -\infty$ in which the fast-roll stages are absent.

The evolution backwards in time from $t_0$ reaches generically a mathematical singularity at some time $t = t_*$, where
the scale factor $a(t)$ vanishes, and the Hubble parameter becomes singular.

We find the general behavior of the inflaton and the scale factor near the singularity as given by Eqs. (2.14), (2.15), (2.16), and (2.17) and determine the validity of the classical approximation, namely, $(H/M_{Pl})^2 \ll 1$. It must be stressed that such mathematical singularity is attained extrapolating the classical treatment where it is no more valid. The singularity is never reached in the validity region of the classical treatment and therefore such mathematical singularity is not a real physical phenomenon here.

Quantum loops effects turns to be less than 1% for $t - t_s > 10^{-42}$ sec and therefore the classical treatment of the inflaton and the space-time can be trusted well before the beginning of inflation.

The fast-roll (both decelerated and inflationary) and slow-roll regimes are analyzed in detail, with both the exact numerical evolution and an analytic approximation, and the whole equation of state evolution in the three regimes. We consider here the double-well (broken symmetric) fourth order inflaton potential since it gives the best description of the CMB + large scale structure (LSS) data [1,2] within the Ginzburg-Landau effective theory approach we follow.

The characteristic time scale of the fast-roll era turns to be

$$t_1 = (1/m)\sqrt{V(0)/[3M^2]} \sim 10^4 t_{\text{planck}},$$

where $V(0)$ is the double-well inflaton potential at zero inflaton field, $m$ is the inflaton mass and $M$ the energy scale of inflation. The time scale of the inflaton in the extreme slow-roll solution goes as the inverse of $t_1$, namely, $1/[m^2 t_1]$.

We study the whole evolution of the curvature and tensor fluctuations along the three successive regimes: decelerated fast roll followed by inflationary fast roll and then inflationary slow roll, and compute the power spectrum by the end of inflation. The fluctuations feel a singular attractive potential near the $t = t_s$ singularity (as in the case of a particle in a central singular potential) with exactly the critical strength ($-1/4$) for which the fall to the center becomes possible. Precisely, the logarithmic behavior of the fluctuations for $t \rightarrow t_s$ Eq. (4.3) describes the fall to $t = t_s$ for the critical strength of the potential $W_R$ felt by the fluctuations.

We generalize the Bunch-Davies initial conditions (BDic) to the present case in which the potential felt by the fluctuations can never be neglected.

In general, the mode functions for large $k$ behave as free modes since the potential $W_R$ becomes negligible in this limit except at the singularity $t = t_s$. One can then impose Bunch-Davies (BD) conditions for large $k$, which corresponds to assume an initial quantum vacuum Fock state, empty of curvature excitations

$$S_R(k; \eta) \overset{\eta \rightarrow \infty}{=} e^{-ik\eta}$$

and therefore

$$dS_R/d\eta(k; \eta_0) \overset{\eta \rightarrow \infty}{=} -ikS_R(k; \eta_0).$$

Here, $\eta$ stands for the conformal time $d\eta = dt/a(t)$. Equation (1.1) fulfills the Wronskian normalization (that ensures the canonical commutation relations)

$$W[S_R, S_R'] = S_R \frac{dS_R}{d\eta} - \frac{dS_R}{d\eta} S_R' = i.$$  

(1.2)

In asymptotically flat (or conformally flat) regions of the space-time the potential felt by the fluctuations $W_R(\eta)$ vanishes, and the fluctuations exhibit a plane wave behavior for all $k$ (not necessarily large). This is not the case in strong gravity fields or near curvature singularities as in the present cosmological space-time where $W_R(\eta)$ can never be neglected at fixed $k$. However, we can choose BDic at $\eta = \eta_0$ (or equivalently, $t = t_0$) by imposing

$$dS_R/d\eta(k; \eta_0) = -ikS_R(k; \eta_0) \quad \text{for all } k.$$  

(1.3)

That is, we consider the initial value problem for the mode functions giving the values of $S_R(k; \eta)$ and $dS_R/d\eta$ at $\eta = \eta_0$. This condition combined with the Wronskian condition Eq. (1.2) implies that

$$|S_R(k; \eta_0)| = \frac{1}{\sqrt{2k}} \quad \left| \frac{dS_R}{d\eta}(k; \eta_0) \right| = \sqrt{\frac{k}{2}}$$  

(1.4)

which is equivalent to Eq. (1.1) for large $k$.

The power spectrum at the end of slow-roll inflation $P_R(k)$ gets dynamically modified by the effect of the preceding fast-roll eras through the transfer function of initial conditions $D(k)$:

$$P_R(k) = P_R^{BD}(k)[1 + D(k)].$$  

(1.5)

$D(k)$ accounts for the effect of both the initial conditions and the fluctuations evolution during fast roll (before slow roll). $D(k)$ depends on the time $t_0$ at which BDic are imposed.

The power spectrum $P_R^{BD}(k)$ corresponds to start the evolution with pure slow roll from $t_0 \rightarrow -\infty$ and with BDic Eqs. (1.3) and (1.4) imposed there at $t_0 \rightarrow -\infty$, that is, $\eta_0 = -\infty$. $P_R^{BD}(k)$ is given by its customary pure slow-roll expression

$$\log P_R^{BD}(k) = \log A_s(k_0) + (n_s - 1) \log \frac{k}{k_0}$$

$$+ \frac{1}{2} n_{run} \log^2 \frac{k}{k_0} + O\left(\frac{1}{N^2}\right)$$  

(1.6)

where $N$ is the number of inflation e-folds since the pivot cosmic microwave background (CMB) scale $k_0$ exits the horizon. We take here $N = 60$.

Actually, BDic can be imposed at $\eta = \eta_0 = -\infty$ if and only if the inflaton evolution also starts at $\eta = \eta_0 = -\infty$. This only happens for a particular inflaton solution: the
extreme slow-roll solution that we explicitly present and analyze in Sec. III A. In the extreme slow-roll case the fast-roll eras are absent, BDic are imposed at \( t_0 \rightarrow -\infty \) (that is, \( \eta_0 = -\infty \)), then \( D(k) = 0 \) and \( P_R(k) = P_{BDR}^R(k) \). Only in this case the fluctuation power spectrum at the end of inflation is the usual power spectrum \( P_{BDR}^R(k) \) Eq. (4.24).

When BDic are imposed at finite times \( t_0 \), the spectrum is not the usual \( P_{BDR}^R(k) \) but it gets modified by a nonzero transfer function \( D(k) \) Eq. (4.21). The power spectrum \( P_R(k) \) vanishes at \( k = 0 \) and exhibits oscillations which vanish at large \( k \) (see Figs. 6 and 7).

Generically, the power spectrum vanishes at \( k = 0 \), and we thus have

\[
1 + D(k) \xrightarrow{k \to 0} \mathcal{O}(k^{n_r + 1})
\]

as shown in Sec. VA. \( D(k) \) presents a first peak for \( k \sim 2/\eta_0 \) and then oscillates asymptotically with decreasing amplitude such that

\[
D(k) \xrightarrow{k \to 0} \mathcal{O}\left(\frac{1}{k^2}\right).
\]

We solved numerically the fluctuations equation with the BDic Eq. (4.7) covering both the fast-roll and slow-roll regimes, namely, for different initial times \( t_0 \) ranging from the singularity \( \tau = \tau_s \) until the transition time \( \tau_{trans} \) from fast roll to slow roll. That is to say, we solved the fluctuations evolution for BDic imposed at different times in the three eras, and we compare the resulting power spectra among them. We computed the corresponding transfer function, \( D(k) \) for the BDic imposed at the different eras. We depict \( 1 + D(k) \) vs \( k \) for the different values of the time \( t_0 \) where BDic are imposed in Figs. 6.

When the BDic are imposed during the fast-roll stage well before it ends, \( D(k) \) changes much more significantly than along the extreme slow-roll solution. This is due to two main effects: the potential felt by the fluctuations is attractive during fast roll, and \( \eta_0 \), (far from being almost proportional to \( 1/a(\eta) \)), tends to the constant value \( \eta_s \) as \( \tau \to \tau_s \) and \( a(\eta) \to 0 \). The numerical transfer functions \( 1 + D(k) \) obtained from Eqs. (4.12) and (4.21) are plotted in Fig. 6.

We have also computed \( D(k) \) analytically, within the slow-roll approximation, with BDic at finite times \( \eta_0 \), and a simple form is obtained in the scale-invariant case, which is the leading term in the \( 1/N \) expansion:

\[
D(k) = \frac{\cos^2 x}{x^2} - \frac{\sin^2 x}{x^2} + \frac{\sin^2 x}{x^4}, \quad x = k \eta_0.
\]

Different initial times \( t_0 \) lead essentially to a rescaling of \( k \) in \( D(k) \) by a factor \( \eta_0 \) since the conformal time \( \eta \) is almost proportional to \( 1/a(\eta) \) during slow roll [see Fig. 6 and below Eq. (5.7)]. By virtue of the dynamical attractor character of slow roll, the power spectrum when the BDic are imposed at a finite time \( t_0 \) cannot really distinguish between the extreme slow-roll solution or any other solution which is attracted to slow roll well before the time \( t_0 \).

Using the transfer function \( D(k) \) we obtained, we computed the change on the CMB multipoles \( \Delta C_\ell/C_\ell \) for \( \ell = 1, 2 \) and 3 as functions of the starting instant of the fluctuations \( t_0 \). We plot \( \Delta C_\ell/C_\ell \) for \( 1 \leq \ell \leq 5 \) vs \( t_0 - t_s \) in Fig. 9. We see that \( \Delta C_\ell/C_\ell \) is positive for small \( t_0 - t_s \) and decreases with \( t_0 \) becoming then negative. The CMB quadrupole observations indicate a large suppression thus indicating that \( t_0 - t_s \approx 0.05/m = 10 \text{100}_{\text{Planck}} \).

The fact that choosing BDic leads to a primordial power and its respective CMB multipoles which correctly reproduce the observed spectrum justifies the use of BDic.

Besides finding a CMB quadrupole suppression in agreement with observations [1,3–6], we provide here predictions for the dipole and \( \ell \leq 5 \)-multipole suppressions. Forthcoming CMB observations can provide better data to confront our CMB multipole suppression predictions. It will be extremely interesting to measure the primordial dipole and compare with our predicted value.

### II. The Pre-Inflationary and Inflationary Fast-Roll Eras

The current WMAP data are validating the single field slow-roll scenario [7]. Single field slow-roll models provide an appealing, simple and fairly generic description of inflation. This inflationary scenario can be implemented using a scalar field, the inflaton with a Lagrangian density (see, for example, Ref. [1])

\[
\mathcal{L} = a^3(t) \left[ \frac{\dot{\varphi}^2}{2} - \frac{(\nabla \varphi)^2}{2a^2(t)} - V(\varphi) \right],
\]

where \( V(\varphi) \) is the inflaton potential. Since the Universe expands exponentially fast during inflation, gradient terms are exponentially suppressed and can be neglected. At the same time, the exponential stretching of spatial lengths classically the physics and permits a classical treatment. One can therefore consider an homogeneous and classical inflaton field \( \varphi(t) \), which obeys the evolution equation

\[
\ddot{\varphi} + 3H(t)\dot{\varphi} + V'(\varphi) = 0
\]

in the isotropic and homogeneous Friedmann-Robertson-Walker metric which is sourced by the inflaton

\[
ds^2 = dt^2 - a^2(t)dx^2.
\]

\( H(t) = \dot{a}(t)/a(t) \) stands for the Hubble parameter. The energy density and the pressure for a spatially homogeneous inflaton are given by

\[
\rho = \frac{\dot{\varphi}^2}{2} + V(\varphi), \quad p = \frac{\dot{\varphi}^2}{2} - V(\varphi).
\]

Therefore, the scale factor \( a(t) \) obeys the Friedmann equation
\[ H^2(t) = \frac{1}{3M_{pl}^2} \left[ \frac{1}{2} \dot{\varphi}^2 + V(\varphi) \right] . \]  

(2.5)

In order to have a finite number of inflation e-folds, the inflaton potential \( V(\varphi) \) must vanish at its absolute minimum

\[ V'(\varphi_{\text{min}}) = V(\varphi_{\text{min}}) = 0. \]  

(2.6)

Otherwise, inflation continues forever.

We formulate inflation as an effective field theory within the Ginsburg-Landau spirit [1,8,9]. The theory of the second order phase transitions, the Ginsburg-Landau theory of superconductivity, the current-current Fermi theory of weak interactions, the sigma model of pions, nucleons (as skymions), and photons are all successful effective field theories. Our work shows how powerful is the effective theory of inflation to predict observable quantities that can be or will be soon contrasted with experiments.

The effective theory of inflation should be the low energy limit of a microscopic fundamental theory not yet precisely known. The energy scale of inflation \( M \) should be at the grand unified theory (GUT) energy scale in order to reproduce the amplitude of the CMB anisotropies [1]. Therefore, the microscopic theory of inflation is expected to be a GUT in a cosmological space-time. Such a theory of inflation would contain many fields of various spins. However, in order to have a homogeneous and isotropic Universe the expectation value of the energy-momentum tensor of the fields must be homogeneous and isotropic. The inflaton field in the effective theory may be a coarse-grained average of fundamental scalar fields, or a composite (bound state) of fundamental fields of higher spin, just as in superconductivity. The inflaton does not need to be a fundamental field, for example, it may emerge as a condensate of fermion-antifermion pairs (\( \Psi \bar{\Psi} \)) in a GUT in the cosmological background. In order to describe the cosmological evolution is enough to consider the effective dynamics of such condensates. The relation between the effective field theory of inflation and the microscopic fundamental GUT is akin to the relation between the effective Ginzburg-Landau theory of superconductivity and the microscopic BCS theory, or like the relation of the \( O(4) \) sigma model, an effective low energy theory of pions, photons and chiral condensates with QCD [10].

Vector fields have been considered to describe inflation in Ref. [11]. The results for the inflaton should not be very different from the effective inflaton description since the energy-momentum tensor of the vector field is to be taken homogeneous and isotropic. Namely, we are always in the presence of a scalar condensate.

Since the mass of the inflaton is given by \( M^2/M_{pl} \sim 10^{13} \text{ GeV} [1] \), massless fields alone cannot describe inflation which leads to the observed amplitude of the CMB anisotropies.

The classical inflaton potential \( V(\varphi) \) gets modified by quantum loop corrections. We computed relevant quantum loop corrections to inflationary dynamics in Refs. [1,12]. A thorough study of the effect of quantum fluctuations reveals that these loop corrections are suppressed by powers of \( (H/M_{pl})^2 \sim 10^{-9} \), where \( H \) is the Hubble parameter during inflation [1,12]. Therefore, quantum loop corrections are very small, a conclusion that validates the reliability of the classical approximation and of the effective field theory approach to inflationary dynamics. In particular, the (small) one-loop corrections to the potential in an inflationary Universe are very different from the Coleman-Weinberg form [1,12].

We choose the inflaton field initially homogeneous which ensures it is always homogeneous. The fluctuations around are small and give small corrections to the homogeneity of the Universe. The rapid expansion of the Universe, in the inflationary regimes, takes care of the classical fluctuations, quickly flattening an eventually non-homogeneous condensate.

### A. The complete inflaton evolution through the different eras

It is convenient to use the dimensionless variables to analyze the inflaton evolution equations, Eqs. (2.2), (2.3), (2.4), and (2.5), [1]:

\[ \tau = m t, \quad h = \frac{H}{m}, \quad \phi = \frac{\varphi}{M_{pl}}. \]  

(2.7)

The inflaton potential has then the universal form

\[ V(\varphi) = M^4 v \left( \frac{\varphi}{M_{pl}} \right). \]  

(2.8)

where \( M \) is the energy scale of inflation and \( v(\phi) \) is a dimensionless function. Without loss of generality we can set \( v(0) = 0 \) [1]. Moreover, provided \( V'(0) \neq 0 \) we can set without loss of generality \( |V'(0)| = 1/2 \). Namely, we have for small fields,

\[ v(\phi) = v(0) + \frac{1}{2} \phi^2 + O(\phi^3), \]  

(2.9)

where the minus sign in the quadratic term corresponds to new inflation and the plus sign to chaotic inflation.

In these dimensionless variables, the energy density and the pressure for a spatially homogeneous inflaton are given from Eq. (2.4) by

\[ \frac{\rho}{M^4} = \frac{1}{2} \left( \frac{d \phi}{d \tau} \right)^2 + v(\phi), \quad \frac{p}{M^4} = \frac{1}{2} \left( \frac{d \phi}{d \tau} \right)^2 - v(\phi), \]  

(2.10)

and the coupled inflaton evolution Eq. (2.2) and the Friedmann Eq. (2.5) take the form [1],

\[ \frac{d^2 \phi}{d \tau^2} + 3h \frac{d \phi}{d \tau} + v'(\phi) = 0, \]

\[ h^2(\tau) = \frac{1}{3} \left[ \frac{1}{2} \left( \frac{d \phi}{d \tau} \right)^2 + v(\phi) \right]. \]  

(2.11)
These coupled nonlinear differential equations completely define the time evolution of the inflaton field and the scale factor once the initial conditions are given at the initial time $\tau_0$. Namely, the initial conditions are fixed by giving two real numbers, the values of $\phi(\tau_0)$ and $d\phi(\tau_0)/d\tau$.

It follows from Eqs. (2.11) that

$$\frac{d^2 a}{d\tau^2} = \frac{1}{3} \left[ v(\phi) - \frac{d^2 \phi}{d\tau^2} \right] = -\frac{1}{2} \left( \rho + \frac{1}{3} \rho \right).$$

(2.12)

When $d^2 a/d\tau^2 > 0$ the expansion of the Universe accelerates and it is then called inflationary.

The derivative of the Hubble parameter is always negative:

$$\frac{dh}{d\tau} = -\frac{1}{2} \frac{d^2 \phi}{d\tau^2}. \quad \text{(2.13)}$$

Therefore, $h(\tau)$ decreases monotonically with increasing $\tau$. Conversely, if we evolve the solution backwards in time from $\tau_0$, $h(\tau)$ will generically increase without bounds. Namely, at some time $\tau = \tau_*$, $h(\tau)$ can exhibit a singularity where simultaneously $a(\tau_*)$ vanishes.

In fact, the Eqs. (2.11) admit the singular solution for $\tau \to \tau_*$,

$$\phi(\tau) \stackrel{\tau \to \tau_*}{=} \sqrt{\frac{2}{3} \log \frac{\tau - \tau_*}{b}} \to -\infty,$$

$$h(\tau) \equiv \frac{d}{d\tau} \log a(\tau) \stackrel{\tau \to \tau_*}{=} \frac{1}{3(\tau - \tau_*)} \to +\infty,$$

(2.14)

where $b$ is an integration constant. The energy density $\epsilon(\tau)$ and equation of state take the limiting form,

$$\rho(\tau) \stackrel{\tau \to \tau_*}{=} \frac{1}{3(\tau - \tau_*)^2} \to +\infty, \quad \frac{\rho(\tau)}{\rho(\tau)} \to 1. \quad \text{(2.15)}$$

Namely, the limiting equation of state is $p = \epsilon + \rho$.

We have in this regime

$$a(\tau) \stackrel{\tau \to \tau_*}{=} C(\tau - \tau_*)^{1/3} \to 0,$$

(2.16)

where $C$ is some constant. That is, the geometry becomes singular for $\tau \to \tau_*$. The behavior near $\tau_*$ is noninflationary, namely, decelerated, since

$$\frac{d^2 a}{d\tau^2} \stackrel{\tau \to \tau_*}{=} -\frac{2}{9} (\tau - \tau_*)^{-5/3} \to -\infty.$$

(2.17)

For $\tau \to \tau_*$, near the singularity, the potential $v(\phi)$ becomes negligible in Eqs. (2.11). Therefore, Eqs. (2.14), (2.15), (2.16), and (2.17) are valid for all regular potentials $v(\phi)$.

The evolution starts thus by this decelerated fast-roll regime followed by an inflationary fast-roll regime and then by a slow-roll inflationary regime [1]. Recall that the slow-roll regime is an attractor [4], and therefore the inflaton always reaches a slow-roll inflationary regime for generic initial conditions. We display in Fig. 1 the inflaton flow in phase space, namely, $d\phi/d\tau$ vs $\phi$ for different initial conditions. We see that the inflaton always reaches a slow-roll regime for generic initial conditions represented by a quasihorizontal line. Hence, the slow-roll line is an attractor. Ultimately the inflaton reaches asymptotically the absolute minima of $d\phi/d\tau = 0$, $\phi = \phi_{\text{min}} = \sqrt{\Delta N/6} = 19.52\ldots$. The number of e-folds of slow-roll inflation $N_{\text{sr}}$ increases for decreasing initial $\phi > 0$ when $d\phi/d\tau > 0$ initially. The third to seventh trajectories in the upper left part $\phi > 0, d\phi/d\tau > 0$ correspond to $N_{\text{sr}} > 63$.

![Fig. 1 (color online). The complete inflaton flow in phase space. $d\phi/d\tau$ vs $\phi$ for different initial conditions. We see that the inflaton always reaches a slow-roll regime for generic initial conditions represented by a quasihorizontal line. Hence, the slow-roll line is an attractor. Ultimately the inflaton reaches asymptotically the absolute minima of $d\phi/d\tau = 0$, $\phi = \phi_{\text{min}} = \sqrt{\Delta N/6} = 19.52\ldots$. The number of e-folds of slow-roll inflation $N_{\text{sr}}$ increases for decreasing initial $\phi > 0$ when $d\phi/d\tau > 0$ initially. The third to seventh trajectories in the upper left part $\phi > 0, d\phi/d\tau > 0$ correspond to $N_{\text{sr}} > 63$.](063520-5)
Equation (2.13) implies a monotonic decreasing of the expansion rate of the Universe. There are four stages in the Universe evolution described by Eqs. (2.11):

(i) The noninflationary fast-roll stage starting at the singularity \( \tau = \tau_s \) and ending when \( d^2 a/d\tau^2 \) becomes positive [see Eq. (2.12)].

(ii) The inflationary fast-roll stage starts when \( d^2 a/d\tau^2 \) becomes positive and ends at \( \tau = \tau_{\text{trans}} \) when \( \epsilon_v \) becomes smaller than \( 1/N \) [see Eq. (2.19)].

(iii) The inflationary slow-roll stage follows, and it continues as long as \( \epsilon_v < 1/N \) and \( d^2 a/d\tau^2 > 0 \). It ends when \( d^2 a/d\tau^2 \) becomes negative at \( \tau = \tau_{\text{end}} \).

(iv) A matter-dominated stage follows the inflationary era.

The four stages described above correspond to the evolution for generic initial conditions or, equivalently, starting from the singular behavior Eqs. (2.14). In addition, there exists a special (extreme) slow-roll solution starting at \( \tau = -\infty \) where the fast-roll stages are absent. We derive this extreme slow-roll solution in Sec. III A.

As shown in Refs. [1, 2] the double-well (broken symmetric) fourth order potential

\[
V(\varphi) = \frac{1}{4} \lambda \left( \varphi^2 - \frac{m^2}{\lambda} \right)^2 = -\frac{1}{2} m^2 \varphi^2 + \frac{1}{4} \lambda \varphi^4 + \frac{m^4}{4\lambda}
\]

provides a very good fit for the CMB + LSS data, while at the same time being particularly simple, natural, and stable in the Ginsburg-Landau sense. This is a new inflation model with the inflaton rolling from the vicinity of the local maxima of \( V(\varphi) \) at \( \varphi = 0 \) toward the absolute minimum \( \varphi = m/\sqrt{\lambda} \).

The inflaton mass \( m \) and coupling \( \lambda \) are naturally expressed in terms of the two relevant energy scales in this problem: the energy scale of inflation \( M \) and the Planck mass \( M_{\text{pl}} = 2.43534 \times 10^{18} \text{ GeV} \),

\[
m = \frac{M^2}{M_{\text{pl}}}, \quad \lambda = \frac{y^4}{8N} \left( \frac{M}{M_{\text{pl}}} \right)^4.
\]

Here, \( N \sim 60 \) is the number of e-folds since the cosmologically relevant modes exit the horizon until the end of inflation and \( y \sim 1 \) is the quartic coupling.

The MCMC analysis of the CMB + LSS data combined with the theoretical input above yields the value \( y \approx 1.26 \) for the coupling [1, 2]. It turns to be order one consistent with the Ginsburg-Landau formulation of the theory of inflation [1].

This model of new inflation yields as most probable values: \( n_s \approx 0.964, r \approx 0.051 \) [1, 2]. This value for \( r \) is within reach of forthcoming CMB observations. For \( y > 0.431 \) the field exits the horizon in the negative concavity region \( V''(\varphi) < 0 \) intrinsic to new inflation [1]. We find for the best fit [1, 2],

\[
M = 0.543 \times 10^{16} \text{ GeV} \quad \text{for the scale of inflation and} \quad m = 1.21 \times 10^{13} \text{ GeV} \quad \text{for the inflaton mass.}
\]

We consider from now on the quartic broken symmetric potential Eq. (2.20), which becomes using Eq. (2.8)

\[
v(\phi) = \frac{g}{4} \left( \phi^2 - \frac{1}{g} \right)^2 = -\frac{1}{2} \phi^2 + \frac{g}{4} \phi^4 + \frac{1}{4g}
\]

where \( g = \frac{y}{8N} \).

We have two arbitrary real coefficients characterizing the initial conditions. We can choose them as \( b = \tau, \epsilon_v \) [see Eq. (2.14)]. A total number of slow-roll inflation e-folds \( N_{\text{sr}} \approx 63 \) permits to explain the CMB quadrupole suppression [1, 5, 6]. Such requirement fixes the value of \( b \) for a given coupling \( y \).

We integrated numerically Eqs. (2.11) with Eq. (2.14) as initial conditions. We find that \( b = 4.745272 \ldots 10^{-5} \) yields 63 e-folds of inflation during the slow-roll era for \( y = 1.26 \), the best fit to the CMB and LSS data. We find that \( b \) is a monotonically increasing function of the coupling \( y \) for fixed number of slow-roll e-folds. At fixed coupling, \( b \) increases with the number of slow-roll e-folds.

We display in Fig. 2 \( b \) as a function of \( y \) and the number of slow-roll inflation e-folds \( N_{\text{sr}} \).

For this value of \( y \) and 63 e-folds of inflation during the slow roll, fast roll ends by \( \tau = \tau_{\text{trans}} = 0.2487963 \ldots \), in Figs. 3, we depict \( \log a(\tau) \) and \( p(\tau)/\rho(\tau), \log b(\tau), \phi(\tau), \log|\dot{\phi}(\tau)|, \log|N_\epsilon(\tau)| \) vs \( \tau \) until a short time after the end of inflation. We define the time \( \tau_{\text{end}} \) when inflation ends by the condition \( a(\tau_{\text{end}}) = 0 \), which gives \( \tau_{\text{end}} - \tau_s = 18.2547816 \ldots \).

Furthermore, we study in this paper the curvature and tensor fluctuations during the whole inflaton evolution in its three successive regimes: noninflationary fast roll, inflationary fast roll, and inflationary slow roll.

The equation for the scalar curvature fluctuations take in conformal time \( \eta \) and dimensionless variables the form [1]

\[
\left[ \frac{d^2}{d\eta^2} + k^2 - W_\mathcal{R}(\eta) \right] S_\mathcal{R}(k; \eta) = 0,
\]

where \( d\eta = d\tau/a(\tau) \),

\[
W_\mathcal{R}(\eta) = \frac{1}{z} \frac{d^2 z}{d\eta^2} \quad \text{and} \quad z(\eta) = \frac{a(\eta)}{h(\eta)} \frac{d\phi}{d\tau}.
\]

In cosmic time \( \tau \), Eq. (2.24) takes the form

\[
\left[ \frac{d^2}{d\tau^2} + h(\tau) \frac{d}{d\tau} + \frac{k^2}{a^2(\tau)} - V_\mathcal{R}(\tau) \right] S_\mathcal{R}(k; \tau) = 0,
\]

where
FIG. 2 (color online). Left panel: the coefficient $b$ characterizing the initial conditions vs the quartic coupling $y$ for $N_{sr} = 63$ e-folds of slow-roll inflation. Right panel: $b$ vs $N_{sr}$ for $y = 1.26$. The preferred values $y = 1.26$ and $N_{sr}$ are highlighted in both panels.

FIG. 3 (color online). Time evolution during the three eras: noninflationary fast roll, inflationary fast roll and slow roll, and beyond the end of inflation (matter-dominated era). $\log a(\tau)$, $\log h(\tau)$, $\phi(\tau)$, $\log|\phi'(\tau)|$, $\log[N\epsilon_v(\tau)]$ and $p(\tau)/\rho(\tau)$ vs $\tau$. $a(\tau)$ grows monotonically reaching 63 e-folds by the end of inflation. $h(\tau)$ diverges for $\tau - \tau_* = -0.8499574\ldots$ according to Eq. (2.14) and decreases fast during fast roll ($\tau \equiv \tau_{\text{trans}} = 0.2487963\ldots$). Then, $h(\tau)$ decreases slowly during slow roll as discussed in Sec. III B. We depict $h(\tau)$ for short times ($0 < \tau - \tau_* < 0.3$) in Fig. 5. $\phi(\tau)$ diverges for $\tau - \tau_* = -0.8499574\ldots$ according to Eq. (2.14) and decreases fast during fast roll becoming very small during slow roll. After the fast-roll stage where the inflaton field grows according to Eq. (2.14), $\phi(\tau)$ slowly rolls toward its absolute minimum at $\phi_{\text{end}} = \sqrt{8N/y} = 19.52\ldots$. $\log[N\epsilon_v(\tau)]$ vs $\tau - \tau_*$. We have that $\epsilon_v(\tau_e) = 3$ according to Eqs. (2.14) and (2.19). $\epsilon_v(\tau)$ decreases fast during fast roll becoming of the order $1/N$. We define the end of fast roll (and beginning of slow roll) by the condition $N\epsilon_v(\tau) = 1$, which gives $\tau_{\text{trans}} - \tau_* = 0.2487963\ldots$. The equation of state $p(\tau)/\rho(\tau)$ quickly decreases during fast roll from the value $p/\rho = +1$ for $\tau - \tau_* [\text{see Eq. (2.15)}]$ passing through $p/\rho = -1/3$ at the beginning of fast-roll inflation [see Eq. (2.12)], $\tau = \tau_* + 0.0573$, and reaching $p/\rho = -1$ by the beginning of slow roll. $p/\rho$ vanishes again near the end of slow-roll inflation by $\tau_{\text{end}} = \tau_* + 18.698\ldots$.
where $\phi$ stands for $d\phi/d\tau$.

![Diagram](https://via.placeholder.com/150)

**FIG. 4** (color online). The potential $V_R(\tau)$ felt by the fluctuations. Upper plot: $V_R(\tau)$ vs $(\tau - \tau_s)$ in the stage where $V_R(\tau)$ is repulsive ($V_R(\tau) > 0$), which happens for $(\tau - \tau_s) > 0.114$. Notice that $V_R(\tau)$ slowly decreases during the slow-roll stage as $V_R(\tau) \approx 2h^2(\tau) + 1 + O(1/N)$ according to Eq. (2.27) and Fig. 3. Lower plots: Comparison of the exact (numerical) evolution and the analytic approximations Eq. (2.43) during fast roll and slow roll. Left lower plot: $(\tau - \tau_s)^2 V_R(\tau)$ vs $\tau - \tau_s$ in the stage where $V_R(\tau)$ is attractive ($V_R(\tau) < 0$) from the exact (numerical) calculation and from the analytic approximation Eq. (2.43). This happens for $0 \leq (\tau - \tau_s) < 0.114$. Notice that $\lim_{\tau_s \rightarrow \tau_s} (\tau - \tau_s)^2 V_R(\tau) = -1/9$ according to Eq. (4.1). Lower right plot: $V_R(\tau)$ vs $\tau - \tau_s$ when $V_R(\tau) > 0$ from the exact (numerical) calculation and from the analytic approximation Eq. (2.43).
The asymptotic solution of Eqs. (2.30) for $\tau \to \tau_*$ turns to have the dominant form

$$
\phi_1(\tau) \overset{\tau \to \tau_*}{=} (\tau - \tau_*)^2 P_4^\phi \left( \log \frac{\tau - \tau_*}{b} \right), \quad h_1(\tau) \overset{\tau \to \tau_*}{=} (\tau - \tau_*) P_4^b \left( \log \frac{\tau - \tau_*}{b} \right).
$$

(2.31)

where $P_4^\phi(z)$ and $P_4^b(z)$ are fourth degree polynomials in their arguments. The polynomials turn to be of fourth degree because the inflaton potential is of fourth degree. Their explicit expressions follow after calculation

$$
\phi_1(\tau) \overset{\tau \to \tau_*}{=} -\frac{(\tau - \tau_*)^2}{\sqrt{6}} \left[ \frac{g}{18} \left( \log \frac{\tau - \tau_*}{b} \right)^2 + \frac{2}{3} \log^3 \frac{\tau - \tau_*}{b} - \frac{11}{3} \log^2 \frac{\tau - \tau_*}{b} + \frac{49}{9} \log \frac{\tau - \tau_*}{b} - \frac{439}{54} \right]
$$

$$
- \frac{1}{6} \left( \log^2 \frac{\tau - \tau_*}{b} + \frac{1}{3} \log \frac{\tau - \tau_*}{b} - \frac{7}{8} \right) + \frac{1}{8g}.
$$

$$
h_1(\tau) \overset{\tau \to \tau_*}{=} \frac{\tau - \tau_*}{9} \left[ \frac{g}{18} \left( 6 \log \frac{\tau - \tau_*}{b} - 8 \log^3 \frac{\tau - \tau_*}{b} + 8 \log^2 \frac{\tau - \tau_*}{b} - \frac{11}{3} \log \frac{\tau - \tau_*}{b} + \frac{146}{9} \right)
$$

$$
- \log^3 \frac{\tau - \tau_*}{b} + \frac{2}{3} \log \frac{\tau - \tau_*}{b} - \frac{1}{9} + \frac{3}{4g} \right].
$$

(2.32)

As a consequence, the scale factor near the singularity takes the form

$$
a(\tau) \overset{\tau \to \tau_*}{=} C(\tau - \tau_*)^{1/3} \left[ 1 + (\tau - \tau_*)^2 P_4^\phi \left( \log \frac{\tau - \tau_*}{b} \right) \right].
$$

(2.33)

where the coefficients of the fourth order polynomial $P_4^\phi$ can be obtained from Eqs. (2.14) and (2.32).

C. Quantum loop effects and the validity of the classical inflaton picture

When $\tau \to \tau_*$ quantum loop corrections are expected to become very large spoiling the classical description. More precisely, quantum loop corrections are of the order $(H/M_{Pl})^2$ [1]. From Eqs. (2.7) and (2.14) the quantum loop corrections are of the order

$$
\left( \frac{H}{M_{Pl}} \right)^2 \overset{\tau \to \tau_*}{=} \left[ \frac{m}{3(\tau - \tau_*) M_{Pl}} \right]^2 = \left( \frac{1.66 \times 10^{-6}}{\tau - \tau_*} \right)^2
$$

$$
= \frac{1}{9} \left( \tau_{ Planck} \right)^2,
$$

where we used $m = 1.21 \times 10^{13}$ GeV [1].

The characteristic time is here the Planck time

$$
\tau_{ Planck} = m_{ Planck} = \frac{m}{M_{Pl}} = 2.703 \times 10^{-43} \text{ sec} \times m
$$

$$
= 4.97 \times 10^{-6}.
$$

Namely, the quantum loop corrections are less than 1% for times

$$
(\tau - \tau_*) > \frac{10}{3} \tau_{ Planck} = 1.66 \times 10^{-5}.
$$

(2.34)

Therefore, for times $(\tau - \tau_*) > 10^{-5}$ the classical treatment of the inflaton and the space-time presented in Sec. II and II B can be trusted, and we see that the classical description has a wide domain of validity.

The use of a classical and homogeneous inflaton field is justified in the out of equilibrium field theory context as the quantum formation of a condensate during inflation. This condensate turns to obey the classical evolution equations of an homogeneous inflaton [13].

We see from Eq. (2.16) that the inflaton field becomes negative for $\tau \to \tau_*$. But since a condensate field should be always positive, the classical and homogeneous inflaton picture requires

$$
\tau - \tau_* > b.
$$

For the best fit coupling $\gamma = 1.26$ and 63 e-folds of inflation we have

$$
b = 4.745272 \ldots 10^{-5} = 9.55 \tau_{ Planck},
$$

which is consistent with Eq. (2.34). By comparing this value of $b$ with Eq. (2.34) we see that the quantum loop corrections are negligible in the stage where the condensate is already formed.

We can obtain a lower bound on $b$ since $b$ increases with the number of inflation e-folds $N_e$ at fixed inflaton potential and since $N_e$ cannot be smaller than the lower bound provided by flatness and entropy [1].

Although all inflationary solutions obtained evolving backwards in time from the slow-roll stage do reach a zero of the scale factor, such mathematical singularity is attained extrapolating the classical treatment where it is no longer valid. In fact, one never reaches the singularity in the validity region of the classical treatment. In summary, the classical singularity at $\tau = \tau_*$ is not a real physical phenomenon here.

The classical description with the homogeneous inflaton is very good for $\tau - \tau_* > 10 \tau_{ Planck}$ well before the beginning of inflation.

D. The fast-roll regime: analytic approach

As we see from Fig. 3 the inflaton field $\phi(\tau)$ is much smaller than $d\phi/d\tau$ during fast roll. We can therefore approximate the coupled inflaton evolution equation and Friedmann equation Eqs. (2.11) as
The evolution described by Eqs. (2.37), (2.38), (2.39), and (2.40) starts from the mathematical singularity at \( \tau = \tau_0 \) with monotonically decreasing \( d\phi/d\tau \) and \( h(\tau) \) and a monotonically increasing \( \phi(\tau) \) from its initial value \( \phi(\tau_0) = -\infty \).

Slow roll is reached asymptotically for large \( \tau \) since \( d\phi/d\tau \) vanishes for \( \tau \to \tau_0 \to \infty \).

We find for the parameter \( \epsilon_v \) [Eq. (2.19)] and for the equation of state,

\[
\epsilon_v(\tau) = \frac{3}{1 + \sinh^2 u}, \quad \rho(\tau) = \frac{2}{\cosh^2 u} - 1. \tag{2.42}
\]

We see that \( \epsilon_v(\tau) \) monotonically decreases with \( \tau \) and vanishes for \( \tau \to \tau_* \to \infty \). The equation of state \( p/\rho \) smoothly interpolates between \(-1\) at \( \tau = \tau_* \) (extreme noninflationary fast roll) and \(-1\) (slow-roll inflation) for \( \tau \to \tau_* \to \infty \), passing by \( p/\rho = -1/3 \) (the beginning of fast-roll inflation) at \( \tau = \tau_0 = 0.0573 \).

The potential \( V_R(\tau) \) Eq. (2.24) felt by the fluctuations takes here the form

\[
V_R(\tau) = \frac{1}{6g} \left[ 1 - \frac{1}{2\sinh^2 u} - \frac{9}{\cosh^2 u} \right], \quad u = \frac{\tau - \tau_*}{\tau_1}. \tag{2.43}
\]

The limiting values of \( h(\tau), \phi(\tau) \) and \( V_R(\tau) \) for \( \tau \to \infty \) give a reasonable approximation to the numerical results. We have

\[
h(\infty) = \frac{1}{3\tau_1} = \frac{\sqrt{2N}}{3\sqrt{3}}; \quad \phi(\infty) = \frac{2}{3\sqrt{3}} \log \left[ \frac{2\tau_1}{b} \right]; \quad \frac{d\phi}{d\tau}(\infty) = 0, \quad V_R(\infty) = \frac{4N}{3\sqrt{3}}. \tag{2.44}
\]

The characteristic time scale \( \tau_1 \) is generically a small number since according to Eq. (2.38) \( \tau_1 \sim 1/\sqrt{N} \). The value of \( \tau_1 \) for the best fit value for \( y \) is given in Eq. (2.39).

The end of fast roll \( \tau_{\text{trans}} \) can be estimated in this approximation by using Eq. (2.42) for \( \epsilon_v(\tau) \) setting \( \epsilon_v(\tau_{\text{trans}}) = 1/N \). This gives,

\[
\epsilon_v(\tau) \approx 12e^{-(2\tau_{\text{trans}}/\tau_1)} = \frac{1}{N},
\]

\[
\tau_{\text{trans}} \approx \frac{1}{2\tau_1} \ln(12N) = 0.195.
\]

This approximated value for \( \tau_{\text{trans}} \) should be compared with the exact numerical result \( \tau_{\text{trans}} = 0.2487963 \ldots \).

In Figs. 5 we plot \( \ln h(\tau), \ln \phi(\tau), \ln |\phi(\tau)|, \epsilon_v(\tau) \) and \( p(\tau)/\rho(\tau) \) computed numerically and compared using the analytic expressions Eqs. (2.37), (2.38), (2.39), (2.40), (2.41), and (2.42). We compare in Figs. 4 the exact potential \( V_R(\tau) \) with the analytic approximation Eq. (2.43).
We see that the simple analytic formulas Eqs. (2.37), (2.38), (2.39), (2.40), (2.41), (2.42), and (2.43) provide a very good approximation during the fast-roll regime \( \tau \leq t_{\text{trans}} = 0.2487963 \ldots \). In particular, Eq. (2.37) provides an excellent approximation to \( \dot{\phi}(\tau) \) as shown in Fig. 5. In particular, the analytic formulas Eqs. (2.37), (2.38), (2.39), (2.40), (2.41), (2.42), and (2.43) become exact near the singularity at \( \tau = \tau_s \).

**E. The fast-roll regime: numerical solution**

To construct a singular solution we can integrate Eqs. (2.11) backwards in time starting from initial conditions of strong noninflationary fast-roll type, namely,

\[
K = \frac{\dot{\phi}^2}{2\nu(\phi)} \gg 1,
\]

producing a given total number \( N_{\text{sr}} \) of slow-roll inflationary e-folds. For instance, we start from some \( \phi \) and \( \dot{\phi} \) such that \( K = 10^4 \). The time extent backwards from this moment has to be limited so that, integrating back and forth, the required relative accuracy of \( 10^{-12} \) is preserved. We furthermore impose that \( N_{\text{sr}} = 63 \).

We adopt the convention that conformal time \( \eta \) vanishes from below when inflation ends and that \( a(\tau = 0) = 1 \) when there are still \( N = 60 \) e-folds until the end of inflation. This choice of the scale factor normalization seems the most natural. Then, \( \eta \) has a finite nonzero limit \( \eta_s \) as \( \tau \) approaches the time \( \tau_s \) of the singularity, since \( a(\tau) \approx C(\tau - \tau_s)^{1/3} \) as \( \tau \to \tau_s \) according to Eq. (2.16). That is,

\[
\eta = \int_{\tau_{\text{end}}}^{\tau} \frac{d\tau'}{a(\tau')} = \eta_s + \int_{\tau_s}^{\tau} \frac{d\tau'}{a(\tau')}.
\]

The numerics of a fast-roll solution of this type are in Table I where a relative accuracy of \( 10^{-12} \) is preserved.

Using the asymptotic behavior Eq. (2.14) as \( \tau \to \tau_s^+ \) we obtain from Table I

\[
\tau_s = -0.8499574 \ldots,
\]

\[
b = 4.745272 \ldots \times 10^{-5}
\]

and

\[
\eta_s = -15.605614 \ldots.
\]

Slow roll begins at \( \tau_{\text{trans}} = \tau_s + 0.2487963 \ldots = -0.6011611 \ldots \).

The initial value of the ratio

\[
\frac{d\phi}{dt} = \frac{m}{\phi} \frac{\dot{\phi}}{\phi}
\]

has the dimension of mass. The natural mass scale in the problem is here the energy scale of inflation \( M \). Therefore, assuming this ratio of the order \( M \) yields

\[
\frac{\dot{\phi}}{\phi} < \frac{M}{m} \sim 10^3.
\]

Hence, it is natural to start the fast-roll evolution with \( \dot{\phi}/\phi < 10^3 \).


III. THE SLOW-ROLL INFLATIONARY ERA

A. The extreme slow-roll solution

There always exist a special solution of Eqs. (2.11) that starts at $\tau = -\infty$ with vanishing inflaton, vanishing scale factor but nonzero Hubble parameter. More precisely, Eqs. (2.11) can be approximated for small $\phi$ and $\dot{\phi}$ as

$$\frac{d^2 \phi}{d\tau^2} + 3h \frac{d\phi}{d\tau} - \phi = 0, \quad h^2(\tau) = \frac{1}{3} v(0), \quad (3.1)$$

where we used Eqs. (2.9) and (2.11).

Equations (3.1) admit the asymptotic solution for $\tau \to -\infty$

$$\phi(\tau) \to \infty \quad e^{\alpha \tau} \to 0, \quad h(\tau) \to \infty \quad \sqrt{\frac{v(0)}{3}} \to 0, \quad (3.2)$$

where $C_0$ is an integration constant, $v(0) = 2N/y$ for the double-well potential Eq. (2.23) and

$$\alpha \equiv \frac{1}{2} \left[ \sqrt{5v(0)} + 4 - \sqrt{3v(0)} \right] > 0.$$

Notice that $\alpha$ can be expressed in terms of the fast-roll characteristic time scale $\tau_1$ [Eq. (2.38)],

$$\alpha = \frac{1}{2\tau_1} \left[ \sqrt{1 + 4\tau_1^2} - 1 \right] \approx \tau_1$$

since $\tau_1 \approx 0.0592 \ll 1$ [see Eq. (2.39)].

It must be noticed that the characteristic time scale of the inflaton evolution in the extreme slow-roll solution for early times [see Eq. (3.2)]

$$1 = \frac{1}{\alpha} \approx \frac{1}{\tau_1} \gg 1,$$

turns to be the inverse of the characteristic time scale $\tau_1$ of the fast-roll solution and to be very large.

On the contrary, the characteristic time scale of the scale factor evolution in the same regime is very short

$$\sqrt{\frac{3}{v(0)}} = 3\tau_1 \ll 1.$$

The fast-roll stages both noninflationary and inflationary are absent in this solution. The extreme slow-roll solution only possesses the slow-roll inflationary stage followed by the matter-dominated era.

For the value of the coupling $y = 1.2592226\ldots$, we get for the extreme slow-roll solution

$$\alpha = 0.058937108\ldots, \quad \phi_{\text{end}} = 18.5586530\ldots, \quad \phi_{\text{end}} = 0.9415055\ldots \quad (3.3)$$

In Table II we display the values of the relevant magnitudes for this extreme slow-roll solution.

Except for the extreme slow-roll solution, all solutions are of fast-roll type and come from singular values of $\phi$ and $h$ according to Eq. (2.14) as $\tau \to \tau_\ast$ for some finite $\tau_\ast$ characteristic of each particular solution. The slow-roll stage (which starts when $\epsilon \to 1/N$ from above, and ends when again $\epsilon \to 1/N$ from below) of all distinct solutions turns to be almost identical to that of the extreme slow-roll case as one could expect for an attractor.

### Table I

<table>
<thead>
<tr>
<th>$K = 5.3458 \times 10^2$</th>
<th>$K = 10^4$</th>
<th>Inflation start: $\dot{a} = 0^+$</th>
<th>Fast roll $\to$ slow roll</th>
<th>$a = 1$</th>
<th>Inflation end: $\dot{a} = 0^+$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau$</td>
<td>$-0.8499493\ldots$</td>
<td>$-0.8493593\ldots$</td>
<td>$-0.7746494\ldots$</td>
<td>$-0.6011611\ldots$</td>
<td>$0$</td>
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<tr>
<td>$\phi$</td>
<td>$-1.4401237\ldots$</td>
<td>$2.0690604\ldots$</td>
<td>$5.9342489\ldots$</td>
<td>$6.4783577\ldots$</td>
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<tr>
<td>$\dot{\phi}$</td>
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<td>$1365.05241\ldots$</td>
<td>$8.601670\ldots$</td>
<td>$0.9182661\ldots$</td>
<td>$0.3974015\ldots$</td>
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<tr>
<td>$\log a$</td>
<td>$-7.0325621\ldots$</td>
<td>$-5.5999353\ldots$</td>
<td>$-3.9424151\ldots$</td>
<td>$-2.9999999\ldots$</td>
<td>$0$</td>
</tr>
<tr>
<td>$h$</td>
<td>$40984.468\ldots$</td>
<td>$557.30817\ldots$</td>
<td>$6.2560841\ldots$</td>
<td>$5.0295090\ldots$</td>
<td>$4.9653990\ldots$</td>
</tr>
<tr>
<td>$\eta$</td>
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<td>$-15.376218\ldots$</td>
<td>$-15.3549996\ldots$</td>
<td>$-4.0169827\ldots$</td>
<td>$-0.2020609\ldots$</td>
</tr>
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</table>

### Table II

<table>
<thead>
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<th>Inflation start</th>
<th>$a = 1$</th>
<th>Inflation end: $\dot{a} = 0^+$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau$</td>
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<td>$0$</td>
</tr>
<tr>
<td>$\phi$</td>
<td>$10^{-8}$</td>
<td>$6.7484118\ldots$</td>
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<tr>
<td>$\dot{\phi}$</td>
<td>$\alpha 10^{-8} = 5.8937108453\ldots 10^{-10}$</td>
<td>$0.3973834\ldots$</td>
</tr>
<tr>
<td>$\log a$</td>
<td>$-1938.4867948\ldots$</td>
<td>$0$</td>
</tr>
<tr>
<td>$h$</td>
<td>$\sqrt{2N/(3y)} = 5.6361006\ldots$</td>
<td>$4.9653973\ldots$</td>
</tr>
<tr>
<td>$\eta$</td>
<td>$-\infty$ (f.a.p.p)</td>
<td>$-0.2020610\ldots$</td>
</tr>
</tbody>
</table>
B. The inflaton during slow-roll inflation: analytical solution

In the slow-roll regime higher time derivatives can be neglected in the evolution Eqs. (2.11) with the result

$$3h(\tau)\dot{\phi} + u'(\phi) = 0, \quad h^2(\tau) = \frac{u(\phi)}{3}. \quad (3.4)$$

These first order equations can be solved in closed form as

$$N[\phi] = -\int_{\phi_{end}}^{\phi} u(\phi') \frac{d\phi'}{d\tau} \frac{d\phi}{d\phi'}, \quad (3.5)$$

where $N[\phi]$ is the number of e-folds since the field $\phi$ exits the horizon until the end of inflation (where it takes the value $\phi_{end}$).

Equation (3.5) indicates that $N[\phi]$ scales as $\phi^2$ and hence the field $\phi$ is of the order $\sqrt{N} \sim \sqrt{60}$. Therefore, we proposed as universal form for the inflaton potential [1,8]

$$u(\varphi) = N M^4 w(\chi), \quad (3.6)$$

where $\chi$ is the dimensionless, slowly varying field

$$\chi = \frac{\varphi}{\sqrt{N} M_{Pl}} = \frac{\phi}{\sqrt{N}}. \quad (3.7)$$

The equations of motion (2.11) in the field $\chi$ become

$$\mathcal{H}^2(\dot{\tau}) = \frac{1}{3} \left[ \frac{d^2\chi}{d\tau^2} + w'(\chi) \right], \quad \text{with} \quad \mathcal{H} = \frac{h}{\sqrt{N}},$$

$$\frac{1}{N} \frac{d^2\chi}{d\tau^2} + 3H \frac{d\chi}{d\tau} + w'(\chi) = 0, \quad (3.8)$$

and $\dot{\tau}$ stands for the rescaled dimensionless time

$$\dot{\tau} = \frac{\tau}{\sqrt{N}} = \frac{m t}{\sqrt{N}}. \quad (3.9)$$

To leading order in the slow-roll approximation (neglecting $1/N$ corrections), Eqs. (3.8) are solvable in terms of quadratures

$$\dot{\tau} - \dot{\tau}_{trans} = \int_{\chi(\dot{\tau}_{trans})}^{\chi} d\chi' \frac{\sqrt{3w(\chi')}}{w'(\chi')}, \quad (3.10)$$

where $\dot{\tau}_{trans}$ stands for the beginning of slow-roll inflation and we used that

$$\mathcal{H}(\dot{\tau}) = \sqrt{\frac{w(\chi)}{3}} + O\left(\frac{1}{N}\right). \quad (3.10)$$

For the broken symmetric potential Eq. (2.20), from Eqs. (2.10), (3.9), and (3.10), we find

$$\chi(\dot{\tau}) = \chi(\dot{\tau}_{trans}) e^{\sqrt{6/3} (\dot{\tau} - \dot{\tau}_{trans})} + O\left(\frac{1}{N}\right)$$

$$= \sqrt{\frac{8}{y}} e^{-\sqrt{6/3} (\dot{\tau}_{end} - \dot{\tau})} + O\left(\frac{1}{N}\right). \quad (3.11)$$

Inflation ends when the equation of state becomes $p/\rho = -1/3$ [see Eq. (2.12)]. According to Eq. (3.12), this happens when $\dot{\tau}_{end} - \dot{\tau} \sim O(1/\sqrt{N})$. Therefore, expressions Eqs. (3.11) and (3.12) are valid as long as

$$\dot{\tau}_{trans} \leq \dot{\tau} \leq \dot{\tau}_{end} - O\left(\frac{1}{\sqrt{N}}\right) \quad \text{where} \quad O\left(\frac{1}{\sqrt{N}}\right) > 0.$$

That is, Eqs. (3.11) hold while the inflaton is not very near the minimum of the potential $\chi_{end} = \sqrt{8/3}$. By integrating the Hubble parameter $\mathcal{H}(\dot{\tau})$ we obtain for the scale factor $a(\dot{\tau})$

$$\log a(\dot{\tau}_{trans}) = \sqrt{\frac{2}{3y}} N(\dot{\tau} - \dot{\tau}_{trans}) - \frac{N}{8} \chi^2(\dot{\tau}_{trans})$$

$$\times \left[ e^{\sqrt{2y/3}(\dot{\tau} - \dot{\tau}_{trans})} - 1 \right]$$

$$= \sqrt{\frac{2N}{3y}} m(t - t_{trans}) - \frac{1}{8} \left[ \frac{\varphi(t_{trans})}{M_{Pl}} \right]^2$$

$$\times \left[ e^{\sqrt{2y/3} m(t - t_{trans})} - 1 \right]. \quad (3.14)$$

where we used Eqs. (2.7) and (2.11). It must be noticed that $a(\dot{\tau})$ is not exactly a de Sitter scale factor, even in the large $N$ limit at fixed $\dot{\tau}$.

At the end of inflation the number of e-folds is $\ln a \approx 64$, the inflaton is near its minimum

$$\chi = \frac{8}{y} \approx 2.52,$$

$\dot{\chi}$ starts to oscillate around zero and $\mathcal{H}(\dot{\tau})$ begins a rapid decrease (see Fig. 3). At this time the inflaton field is no longer slowly coasting in the $w'(\chi) < 0$ region but rapidly approaching its equilibrium minimum. When inflation ends, the inflaton is at its minimum value up to corrections of order $1/\sqrt{N}$. Therefore, we see from the Friedmann Eqs. (3.8) and (3.11) that

$$\mathcal{H}(\dot{\tau}) = \sqrt{\frac{2}{3y}} \left[ 1 - e^{-\sqrt{2y/3} (\dot{\tau}_{end} - \dot{\tau})} \right] + O\left(\frac{1}{N}\right).$$

$$P \rho(\dot{\tau}) = -1 + \frac{y}{6N} \frac{1}{\sinh^2 \left[ \sqrt{\frac{2}{3y}} (\dot{\tau}_{end} - \dot{\tau}) \right]} + O\left(\frac{1}{N^2}\right). \quad (3.12)$$
Both results are concordant up to the error estimation in comparing them with the numerical solution of Eqs. (2.11). We see in Fig. 3 that the exact formula for the best values relevant time values in the inflaton and fluctuations evolution for the case one has particles falling (or emerging) from a point in space where the potential is singular.

We find from Eqs. (2.26) and (4.1) for the fluctuations near the singularity

\[ S_k (\tau; \eta) = \eta^{-\frac{3}{2}} \sqrt{\eta_0} \exp \left[ \mathcal{A}_R (k) + \mathcal{B}_R (k) \log \frac{\eta}{\eta_0} \right], \]  

where \( \eta_0 \) is the time when the initial conditions will be imposed, \( \mathcal{A}_R \) and \( \mathcal{B}_R \) are complex constants constrained by the Wronskian condition (that ensures the canonical commutation relations) [1]

\[ W (S_R, S^*_R) = \frac{dS_R}{d\eta} - \frac{dS_R}{d\eta} S^*_R = i. \]  

Namely,

\[ 2 \text{Im} [\mathcal{A}_R \mathcal{B}^*_R] = \eta_0. \]  

Precisely, the logarithmic behavior for \( \eta \to 0 \) of the wave function Eq. (4.3) describes the fall to \( \tau = \tau_* = 0 \) for the critical strength of the potential \( W_R (\eta) \). For larger attractive strengths the wave function Eq. (4.3) shows up an oscillatory behavior [3]. Notice, however the physical nature of the process: here we have a time evolution near a classical singularity at a given time while in the potential case one has particles falling (or emerging) from a point in space where the potential is singular.

In general, the mode functions for large \( k \) must behave as free modes (plane waves) since the potential \( W_R (\eta) \) in Eq. (2.24) becomes negligible in this limit except at the singularity \( \tau = \tau_* \). One can then impose Bunch-Davies conditions for large \( k \), which corresponds to assume an initial quantum vacuum Fock state, empty of curvature excitations [1]

\[ S_k (\tau; \eta) \big|_{\eta \to 0} = \exp \left[ -ik \eta \right] \sqrt{2k} \]  

and therefore

\[ \frac{dS_R}{d\eta} (k; \eta_0) \big|_{\eta \to 0} = -ikS_R (k; \eta_0). \]  

Equation (4.6) fulfills the Wronskian normalization Eq. (4.4).
In asymptotically flat (or conformally flat) regions of the space-time the potential felt by the fluctuations vanish and the fluctuations exhibit a plane wave behavior for all $k$ (not necessarily large). This is not the case near strong gravity fields or curvature singularities as in the present cosmological space-time where $W_R(\eta)$ can never be neglected at fixed $k$. However, we can choose BDic at $\eta = \eta_0$ by imposing
\[
\frac{dS_R}{d\eta}(k;\eta_0) = -ikS_R(k;\eta_0) \quad \text{for all } k. \tag{4.7}
\]
That is, we consider the initial value problem for the mode functions giving the values of $S_R(k;\eta)$ and $dS_R/d\eta$ at $\eta = \eta_0$.

Notice that Eq. (4.7) combined with the Wronskian condition Eq. (4.4) implies that
\[
|S_R(k;\eta_0)| = \frac{1}{\sqrt{2k}}, \quad \left| \frac{dS_R}{d\eta}(k;\eta_0) \right| = \sqrt{\frac{k}{2}},
\]
which is equivalent to Eq. (4.6) for large $k$.

Since the mode functions $S_R(k;\eta)$ are defined up to an arbitrary constant phase we can write Eq. (4.3) valid near the metric singularity as
\[
S_R(k;\eta) = i\eta_0\sqrt{2k}\eta_0^{-1} \left( 1 - \left(1 + i\frac{1}{2}k\eta_0 \right) \log \frac{\eta}{\eta_0} \right). \tag{4.8}
\]
In Eq. (4.3) this corresponds to the coefficients
\[
A_R = \frac{1}{\sqrt{2k}}, \quad B_R = \frac{1}{\sqrt{2k}} \left(1 + ik\eta_0 \right).
\]
We have in cosmic time,
\[
S_R(k;\tau) = \frac{1}{\sqrt{2k}} \left( \frac{\tau - \tau_*}{\tau_0} \right)^{1/3} \left[ 1 - \left(1 + i\frac{1}{2}k\tau_0^{2/3} \right) \log \frac{\tau - \tau_*}{\tau_0} \right], \tag{4.9}
\]
where $\tau_0 - \tau_* = (2\eta_0/3)^{3/2}$ for $\tau \rightarrow \tau_*$. Namely, imposing the BDic Eq. (4.7) at small $\eta_0$ where the small $\eta$ behavior Eq. (4.3) applies, yields specific values for the coefficients of the linearly independent solutions $\sqrt{\eta}$ and $\sqrt{\eta} \log \eta$ that we can read from Eqs. (4.8) and (4.9).

For general $\tau_0$ (i.e., $\tau_0$ not near $\tau_*$), the mode functions for $\tau \rightarrow \tau_*$ take the form

\[
\frac{dS_R}{d\tau}(k;\tau) = \frac{1}{\sqrt{2k}} \left( \frac{\tau - \tau_*}{\tau_0} \right)^{1/3} \left[ X(k, \tau) - \left( Y(k, \tau_0) + \frac{i k \tau_0^{2/3}}{X(k, \tau_0)} \log \frac{\tau - \tau_*}{\tau_0} \right) \right]. \tag{4.10}
\]
where we imposed Eq. (4.5), and we have from Eq. (4.9),
\[
X(k, \tau_*) = 1 \quad \text{and} \quad Y(k, \tau_*) = \frac{1}{3}.
\]
Notice that $X(k, \tau_0) > 0$ for $\tau_0 \rightarrow \tau_*$ as we see from Eq. (4.9). Our numerical calculations show that $X(k, \tau_0) > 0$ for all $\tau_0$ and $k$.

### B. The primordial power spectrum, scalar curvature fluctuations and the CMB + LSS data

The power spectrum of curvature perturbations $R$ is given by the expectation value $\langle R^2 \rangle$ in the state with general initial conditions [1]
\[
\langle R^2(\vec{x}, \eta) \rangle = \left( \frac{m}{M_{Pl}} \right)^2 \int_0^\infty \frac{|S_R(k;\eta)|^2 k^2 dk}{\eta^2} \frac{1}{2\pi^2}, \tag{4.11}
\]
where $z(\eta)$ is given by Eq. (2.25). Notice in Eq. (4.11) the factor $(m/M_{Pl})^2$ in the physical power spectrum expressed in terms of the dimensionless quantities used here.

The power spectrum at time $\eta$ is customary defined as the power per unit logarithmic interval in $k$
\[
\langle R^2(\vec{x}, \eta) \rangle = \int_0^\infty \frac{dk}{k} P_R(k, \eta).
\]
Therefore, the scalar power for general initial conditions is given by the fluctuations behavior by the end of inflation [1],
\[
P_R(k) = \left( \frac{m}{M_{Pl}} \right)^2 \frac{k^3}{2\pi^2} \lim_{\eta \rightarrow 0} \frac{|S_R(k;\eta)|^2}{\eta^3} \tag{4.12}
\]
The mode functions $S_R(k;\eta)$ obey the fluctuations Eq. (2.24) where the potential $W_R(\eta)$ [Eq. (2.27)] during slow roll and to leading order in $1/N$ takes the simple form [1],
\[
W_R(\eta) = \frac{2}{\eta^2} \left[ 1 + \frac{3}{2} (3\epsilon - \eta) \right] = \frac{\nu^2 - \frac{1}{2}}{\eta^2},
\]
\[
\nu_R = \frac{3}{2} + 3\epsilon - \eta + O \left( \frac{1}{N^2} \right). \tag{4.13}
\]

### TABLE III. Selected time values for $\gamma = 1.26$ and $N_{eq} = 63$ e-folds of slow-roll inflation. Notice that slow roll starts exactly when fast roll ends. Recall that $\tau = 4.9710^{-6}(t/t_{Plank})$.

<table>
<thead>
<tr>
<th>Event</th>
<th>Time ($\tau$)</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inflation (fast roll)</td>
<td>$\tau = 0$</td>
<td>$\tau_0 = 0.0753 090$</td>
</tr>
<tr>
<td>$V_R(\tau)$ becomes positive</td>
<td>$\tau = 0$</td>
<td>$\tau_0 = 0.0753 090$</td>
</tr>
<tr>
<td>End of fast roll</td>
<td>$\tau = 0.3053$</td>
<td>$\tau_0 = 0.2487 963$</td>
</tr>
<tr>
<td>Maximum of $V_R(\tau)$</td>
<td>$\tau = 0.3503$</td>
<td>$\tau_0 = 0.2487 963$</td>
</tr>
<tr>
<td>End of inflation</td>
<td>$\tau = 18.2547 816$</td>
<td>$\tau_0 = 0.2487 963$</td>
</tr>
</tbody>
</table>
In the slow-roll regime we can consider $\epsilon_\nu$ and $\eta_\nu$ [see Eq. (2.19)] constants in time in Eq. (4.13). During slow roll, the general solution of Eq. (2.24) is then given by

$$S_R(k; \eta) = A_R(k) g_{\nu_0}(k; \eta) + B_R(k) g^*_{\nu_0}(k; \eta),$$

(4.14)

with

$$g_\nu(k; \eta) = \frac{1}{2} i^{\nu + (1/2)} \sqrt{\frac{-\pi \eta}{H^0(\nu)(-k \eta)}},$$

(4.15)

$A_R(k), B_R(k)$ are constants determined by the initial conditions and $H^0(\nu)(z)$ is a Hankel function.

The Wronskian of the solutions $S_R, S'_R$ is given by Eq. (4.4) and

$$W[g_{\nu}, g^*_{\nu}] = i.$$\nonumber

This generically determines that

$$|A_R(k)|^2 - |B_R(k)|^2 = 1.$$\nonumber

(4.16)

For wave vectors deep inside the Hubble radius $|k \eta| \gg 1$ the mode functions $g_\nu(k; \eta)$ have the asymptotic behavior

$$g_\nu(k; \eta) \eta^{-\infty} = \frac{1}{\sqrt{2k}} e^{-i k \eta}, \quad g^*_\nu(k; \eta) \eta^{\infty} = \frac{1}{\sqrt{2k}} e^{i k \eta},$$

(4.17)

while for $\eta \to 0^-$, they behave as

$$g_\nu(k; \eta) \eta^{\eta-\infty} = \frac{\Gamma(\nu)}{\sqrt{2\pi k}} \left( \frac{2}{k \eta} \right)^{\nu-(1/2)}.$$\nonumber

(4.18)

In particular, in the scale-invariant case $\nu = \frac{1}{2}$, which is the leading order in the slow-roll expansion, the mode functions Eqs. (4.15) simplify to

$$g_{1/2}(k; \eta) = \frac{e^{-i k \eta}}{\sqrt{2k}} \left[ 1 - i \frac{\eta}{k \eta} \right].$$\nonumber

(4.19)

As we see from Eq. (2.25), $z(\eta)$ obeys Eq. (2.24) for $k = 0$ and therefore $z(\eta)$ in the slow-roll regime behaves as

$$z(\eta) = \frac{z_0}{(-k_0 \eta)^{\epsilon_{\nu_0}-(1/2)}},$$

(4.20)

where $z_0$ is the value of $z(\eta)$ when the pivot scale $k_0$ exits the horizon, that is at $\eta = -1/k_0$. Combining this result with the small $\eta$ limit Eq. (4.18) we find from Eqs. (4.12) and (4.20),

$$P_R(k) = P^{BD}_R(k)[1 + D(k)].$$

(4.21)

where we introduced the transfer function for the initial conditions of curvature perturbations:

$$D(k) = 2|B_R(k)|^2 - 2 \Re[A_R(k) B^*_R(k)] e^{-3}.$$\nonumber

(4.22)

$D(k)$ is obtained imposing BDic at $\tau = \tau_0$ according to Eq. (4.7).

Notice as shown in Sec. VA that the transfer function $D(k)$ enjoys the properties

$$1 + D(k) = O(k_n, 1), \quad D(k) = O \left( \frac{1}{k^2} \right).$$

(4.23)

$D(k)$ accounts for the effect in the power spectrum both of the initial conditions and of the fluctuations evolution during fast roll (before slow roll). $D(k)$ depends on the time $\tau_0$ at which BDic are imposed.

If one chooses the extreme slow-roll solution presented in Sec. III A and imposes BDic at $\tau_0 = -\infty$ (that is, $\eta_0 = -\infty$) then $D(k) = 0$ and the fluctuation power spectrum at the end of inflation is the usual power spectrum $P_R(k) = P^{BD}_R(k)$.

$P^{BD}_R(k)$ is given by its customary slow-roll expression,

$$\log P^{BD}_R(k) = \log A_\nu(k_0) + (n_s - 1) \log \frac{k}{k_0} + \frac{1}{2} \eta_{run} \log^2 \frac{k}{k_0} + O \left( \frac{1}{N^2} \right).$$

(4.24)

We solved numerically the fluctuations Eq. (2.26) in cosmic time with the BDic Eq. (4.7) covering both the fast-roll and slow-roll regimes. We started at initial times $\tau_0$ ranging from the vicinity of $\tau = \tau_*$ until the transition time $\tau_{trans} = 0.2487963 \ldots$ from fast roll to slow roll. We computed the transfer function $D(k)$ from the mode functions behavior deep during slow-roll inflation from Eqs. (4.12) and (4.21) [1]. In Figs. 6 we depict $1 + D(k)$ vs $k$ for 12 values of the time $\tau_0$ where BDic are imposed.

Notice that when BDic are imposed at finite times $\tau_0$, the spectrum is not the usual $P^{BD}_R(k)$ but it gets modified by a nonzero transfer function $D(k)$ Eq. (4.21). The power spectrum $P_R(k)$ vanishes at $k = 0$ and exhibits oscillations which vanish at large $k$ (see Figs. 6 and 7).

During slow roll different initial times $\tau_0$ lead essentially to a rescaling of $k$ in $D(k)$ by a factor $\eta_0$ since the conformal time $\eta$ is almost proportional to $1/\alpha(\eta)$ during slow roll [see Figs. 7 and 6 and below Eq. (5.7)]. By virtue of the dynamical attractor character of slow roll, the power spectrum when the BDic are imposed at a finite time $\tau_0$ cannot really distinguish between the extreme slow-roll solution (for which slow roll starts from the very beginning $\eta_0 = -\infty$) or any other solution which is attracted to slow roll well before the time $\tau_0$.

C. Accurate numerical computation of the power spectrum and the transfer function $D(k)$ of initial conditions

In order to accurately calculate $n_s$ we proceed as follows: We match the solution $S_R(k; \eta)$ with the slow-roll solution $g_{\nu_0}(k; \eta)$ Eq. (4.15) at the time $\tau_0$ when $N_{sr}$ e-folds of slow roll have still to occur. $\eta$ and $\nu_{\nu_0}$ are computed at this time $\tau_0$. In practice, this corresponds to setting $A_R(k) = 1$, $B_R(k) = 0$ (and therefore $D_R(k) = 0$) in the Bogoliubov transformation Eq. (4.14).

Then, we integrate numerically the fluctuations equations Eq. (2.26). By construction, this produces the stan-
roll at finite times

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FIG. 6 (color online). Numerical transfer function \(1 + D(k)\). Lower left panel: Numerical transfer function \(1 + D(k)\) for BDic at \(\tau = \tau_0 = \tau_s + \Delta \tau\), for different \(\Delta \tau\) values as given in the picture. We see here that the peak of \(1 + D(k)\) grows and moves for larger \(k\) as \(\tau_0\) increases. Here, \(N_{sr} = 63\). Lower right panel: The transfer function \(1 + D(k)\) when the BDic Eq. (4.7) are imposed during slow roll at finite times \(\tau_0\) and \(N_{sr}\) e-folds of slow roll have still to occur. Upper panels: Numerical transfer function \(1 + D(k)\) for BDic at \(\tau = \tau_0 = \tau_s + \Delta \tau\), for different values of \(\Delta \tau\) as given in the picture. We get stronger oscillations in \(1 + D(k)\) for decreasing \(\tau_0\) in the range \(\Delta \tau < 0.04\). Here \(N_{sr} = 63\).

FIG. 7 (color online). Power spectrum with BDic Eq. (4.7) imposed during slow roll when \(N_{sr}\) e-folds of slow-roll inflation have still to occur. We see here the decrease of the power spectrum \(P_R\) as \(k^{n_s - 1}\) multiplied by the oscillations of \(1 + D(k)\). See Eqs. (4.21) and (4.24) and Figs. 6. The nonoscillatory curve corresponds to the usual power with BDic at \(\eta_0 = -\infty\) Eq. (4.24) decreasing as \(k^{n_s - 1}\). The later are imposed the BDic, the smaller is the number of slow-roll e-folds \(N_{sr}\) and the whole \(k\) spectrum shifts to larger \(k\).

dard spectra \(P^{BD}_{R}(k)\) Eq. (4.24) that quickly stabilize as \(N_{sr}\) is increased a few e-folds above \(N = 60\).

It is convenient to introduce the quantity

\[
L_s = \log \left[ \left( \frac{M_{Pl}}{m} \right)^2 A_s(k_0 = m) \right],
\]

(4.25)

with \(k_0 = m\) when \(a(\eta) = 1\), that is \(N = 60\) e-folds before inflation ends. In Table IV we provide \(L_s\) for several values of \(N_{sr}\).

To transform this \(k_0\) in a wave number today we need

- the total redshift from 60 e-folds before inflation ends until today [since we choose \(a(\tau = 0) = 1\) when

| TABLE IV. Exact values of \(L_s = \log[(M_{Pl}/m)^2A_s(k_0 = m)]\) for several values of \(N_{sr}\) from the numerical calculation. The exact values of \(n_s\) vary little with \(N_{sr}\) and are close to the slow-roll approximation value. Also \(n_{run}\) is close to the value in the slow-roll approximation. |
| \(N_{sr}\) | \(L_s\) | \(n_s\) | \(n_{run}\) |
| 61 | 4.6583 | 0.9637 | -0.0007101 |
| 63 | 4.6583 | 0.9641 | -0.0001639 |
| 65 | 4.6584 | 0.9642 | -0.0000165 |
| 67 | 4.6584 | 0.9642 | -0.0000165 |
| 69 | 4.6584 | 0.9642 | -0.0000167 |

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there are still $N = 60$ e-folds until the end of inflation.

- the value of $m$ as determined by the observed value of the amplitude $A_s(k_0)$.

Let $k_{0}^{\text{CMC}}$ be the value of the pivot scale of COSMOMC [that is 50 (Gpc)$^{-1}$ today] 60 e-folds before the end of inflation. Then, we have from Eqs. (4.24) and (4.25),

$$\log A_s(k_0 = m) = L_s + 2 \log \frac{m}{M_{\text{Pl}}} = L_s^{\text{CMC}} + (n_s^{\text{CMC}} - 1) \log \frac{m}{k_{0}^{\text{CMC}}} + \frac{1}{2} n_{\text{run}} \left[ \log \frac{m}{k_{0}^{\text{CMC}}} \right]^2 + O\left( \frac{1}{N^2} \right). \quad (4.26)$$

where $L_s^{\text{CMC}} = \log A_s^{\text{CMC}}(k_0^{\text{CMC}})$ and $n_s^{\text{CMC}}$ are best fit values in a given COSMOMC run. Since the running index $n_{\text{run}}$ is $O(1/N^2)$, we get for $m$,

$$\left( \frac{m}{M_{\text{Pl}}} \right)^2 = \left( \frac{m}{k_{0}^{\text{CMC}}} \right)^{n_s^{\text{CMC}}-1} \exp(L_s^{\text{CMC}} - L_s) \left[ 1 + O\left( \frac{1}{N^2} \right) \right]. \quad (4.27)$$

The wave vectors at $a = 1$ (60 e-folds before inflation ends) and today are related by [1]

$$k^{a-1} = \frac{a^{60}}{a_r^{\text{today}}}, \quad (4.28)$$

where $a_r$ is the scale factor by the end of inflation

$$a_r = 2.5 \times 10^{-29} \sqrt{10^{-4} M_{\text{Pl}}} \times 10^{15} \text{ GeV}, \quad (4.29)$$

and $H_{60}$ is the Hubble parameter 60 e-folds before inflation ends. We thus have for the pivot wave number at $a = 1$

$$k_{0}^{\text{CMC}} \approx 1.46 \ldots \sqrt{\frac{H_{60}}{10^{-4} M_{\text{Pl}}}} \times 10^{15} \text{ GeV} \quad (4.30)$$

and

$$\left( \frac{m}{M_{\text{Pl}}} \right)^{2 - (n_s^{\text{CMC}} - 1)/2} = \left( \frac{16.67 \ldots}{\sqrt{h_{60}}} \right)^{n_s^{\text{CMC}} - 1} \exp(L_s^{\text{CMC}} - L_s),$$

where $h_{60} = H_{60}/M_{\text{Pl}}$.

Notice the small $1/N$ correction $(n_s^{\text{CMC}} - 1)/2$ in the exponent of $m/M_{\text{Pl}}$. Equation (4.26) yields for the best fit COSMOMC run $L_s^{\text{CMC}} = -19.9808 \ldots$ and $n_s = 0.9635 \ldots [1]$:

$$m \approx 4.8114 \ldots 10^{-6} M_{\text{Pl}} = 1.1717 \ldots 10^{13} \text{ GeV}$$

The exact values given above in Table IV

$$A_s = \left( \frac{m}{M_{\text{Pl}}} \right)^2 \exp(L_s), \quad n_s \quad \text{and} \quad n_{\text{run}}$$

are obtained taking into account the fast-roll and slow-roll stages in the numerical calculation. We can compare them to their slow-roll (leading $1/N$) analytic counterparts for the double-well quadratic plus quartic potential, [1]

$$A_s = \frac{N^2}{12 \pi^2} \left( \frac{m}{M_{\text{Pl}}} \right)^2 \left( 1 - \frac{3z}{N} \right)^2, \quad n_s = 1 - \frac{3z}{N} \left( 1 - \frac{1}{N} \right)^2,$$

$$n_{\text{run}} = \frac{2}{N^2} \left( (24z^2 - 35z + 3) \right),$$

where $N = 60$, $z = 0.117446$ and $y = z - 1 - \log z = 1.259222 \ldots$, that is

$$A_s = \frac{N^2}{12 \pi^2} \left( \frac{m}{M_{\text{Pl}}} \right)^2 \exp(4.59536898 \ldots),$$

$$n_s = 0.9635620 \ldots, \quad n_{\text{run}} = -0.000664 \ldots$$

The figure in the exponent is to be compared with the $L_s$ values in Table IV. The agreement with Table IV is quite good, especially for $n_s$.

We now find the exact (numerical) transfer function $D(k)$ for the initial conditions, by simply taking in Eq. (4.21) the ratio of the two power spectra: $P_R(k)$ with BDic at time $\tau_0$ and $P_R^\text{BD}$ (k). In the case of BDic at finite times the result is given in Fig. 5. At the largest value $k/m = 100$ of the wave number interval considered, we have

$$1 + D(100m) = 0.9999994 \ldots, 1.000061 \ldots, 1.000001 \ldots$$

for $N_{\text{bd}} = 61, 63, 65$ and 67, respectively.

This provides a good check of the accuracy of the calculation.

In Figs. 7, 8 we compare the numerically computed $D(k)$ against $D(k\eta_0)$ analytically computed for BDic imposed at time $\eta_0$ during slow roll in Eq. (5.7), Sec. V B. The comparison is performed for BDic imposed when $N_{\text{bd}} = 63$ on the extreme slow-roll solution, which corresponds to $\eta_0 = -4.0202308 \ldots$. We consider two values of $\nu_R; \nu_R = 2 - n_{\text{run}}/2 = 1.5182189 \ldots, n_{\text{run}} = 0.9635620 \ldots$ corresponding to slow roll at leading $1/N$ order, and the exactly scale-invariant case $\nu_R = 3/2$. Notice that in the latter case $D(k\eta_0)$ has the explicit simple analytic form Eq. (5.9).

The maximum of the numerical transfer function 1 + $D(k)$ is located at $k/m = 0.68755 \ldots$ and has the value 1.13218 \ldots. The maximum of 1 + $D(k\eta_0)$, when $\nu_R = 3/2$ is in $k/m = 0.68755 \ldots$ and has the value 1.13009 \ldots. Recall that these values of $k/m$ have the scale fixed by the choice $a = 1$ when $N = 60$ e-folds lack before inflation ends.

Let us now consider the fluctuations on the fast-roll solution of Table II. Since $\eta$ has a finite lower limit, the choice $A_R(k) = 1, B_R(k) = 0$ has little meaning and BDic can be imposed only at a finite time $\tau_0$ later than the
singularity time $\tau_\ast$. If $\tau_0$ is exactly the transition time $\tau_{\text{trans}}$ when $\epsilon_\ast = 1/N$, fast roll ends and slow roll begins, (to proceed for $N_\ast = 63$ e-folds), then $D(k)$ does not differ too much from that computed with the extreme slow-roll solution. This comparison is performed in the lower left panel of Fig. 8. In the right panel $D(k)$ is compared to the $D(k\eta_0)$ for $\nu_R = 3/2$ and $\eta_0 = -4.0169827\ldots$, which is the value of the conformal time at the onset of slow roll (see Table II).

When the BDic are imposed during the fast-roll stage well before it ends, $D(k)$ changes much more significantly than along the extreme slow-roll solution. This is due to two main effects: the potential felt by the fluctuations is attractive during fast roll and $\eta_0$, far from being almost proportional to $1/a(\eta)$, tend to the constant value $\eta_\ast$ as $\tau \to \tau_\ast^+$ and $a(\eta) \to 0$. The numerical transfer functions $1 + D(k)$ obtained from Eqs. (4.12) and (4.21) are plotted in Figs. 6.

The fact that choosing BDic leads to a primordial power and its respective CMB multipoles which correctly reproduce the observed spectrum justifies the use of BDic for the scalar curvature fluctuations.

D. The effect of the fast-roll stage on the low multipoles of the CMB

In the region of the Sachs-Wolfe plateau for $l \lesssim 30$, the matter-radiation transfer function can be set equal to unity and the CMB multipole coefficients $C_l$s are given by [14]

$$C_l = \frac{4\pi}{9} \int_0^\infty \frac{dk}{k} P_X(k)(j_l[k(\eta_0 - \eta_{\text{last}})])^2,$$

where $P_X$ is the power spectrum of the corresponding perturbation, $X = R$ for curvature perturbations and $X = T$ for tensor perturbations, $j_l(x)$ are spherical Bessel functions [15] and $\eta_0 - \eta_{\text{last}}$ is the comoving distance between today and the last scattering surface given by

$$\eta_0 - \eta_{\text{last}} = \frac{1}{H_0} \int_{\tau_{\text{last}}}^1 \frac{da}{\sqrt{\Omega_r + \Omega_M a + \Omega_\Lambda a^4}},$$

where $\Omega_r$, $\Omega_M$, and $\Omega_\Lambda$ stand for the fraction of radiation, matter, and cosmological constant in today’s Universe. We find using $\tau_{\text{last}} = 1100$,
Notice that \( k/H_0 \sim d_H/\lambda_{\text{phys}}(t_0) \) is the ratio between today’s Hubble radius and the physical wavelength. The power spectrum for curvature (\( R \)) is given by Eqs. (4.21), (4.22), (4.23), and (4.24).

Inserting Eq. (4.21) into Eq. (4.31) yields the \( C_\ell \) as the sum of two terms
\[
C_\ell = C_\ell^{BD} + \Delta C_\ell, \quad \frac{\Delta C_\ell}{C_\ell} = \frac{\int_0^\infty D(\kappa x) f_1(x) dx}{\int_0^\infty f_1(x) dx},
\]
where from Eq. (4.33), \( \kappa = H_0/3.296 \ldots \),
\[
f_1(x) = x^{\eta_0-2}[j_1(x)]^2, \tag{4.35}
\]
and \( j_1(x) \) stand for the spherical Bessel functions.

The \( C_\ell^{BD} \)'s correspond to the standard Bunch-Davies power spectrum \( P_{BD}(k) \) Eq. (4.24) and the \( \Delta C_\ell \) exhibit the effect of the transfer function \( D(k) \) on the \( C_\ell \).

Using the transfer function \( D(k) \) obtained above Eq. (4.22), we computed the change on the CMB multipoles \( \Delta C_\ell/C_\ell \) for \( \ell = 1, \ldots, 5 \) as functions of the starting instant of the fluctuations \( \tau_0 \). We plot \( \Delta C_\ell/C_\ell \) for \( 1 \leq \ell \leq 5 \) vs \( \tau_0 - \tau^* \) in Fig. 9. We see that \( \Delta C_\ell/C_\ell \) is positive for small \( \tau_0 - \tau^* \) and decreases with \( \tau_0 \) becoming then negative. The CMB quadrupole observations indicate a large suppression thus indicating that \( \tau_0 - \tau^* \gtrsim 0.05 \approx 10.100\tau_{\text{Planck}} \).

Being \( D(k) < 0 \) for low \( k \) as depicted in Figs. 6, the primordial power at large scales is then suppressed and the low \( C_\ell \) decrease as seen from Eq. (4.34).

\( \Delta C_\ell/C_\ell \) mainly originates from the peak of \( D(k) \) displayed in Figs. 6 whose position moves to smaller \( k \) for decreasing \( \tau_0 \). Therefore, the primordial power suppression is less important for decreasing \( \tau_0 \) and the CMB multipole suppression \( \Delta C_\ell/C_\ell \) less important as depicted in Figs. 9.

For small \( \tau_0 - \tau^* \leq 0.05 \) the peak of \( D(k) \) grows significantly and \( \Delta C_\ell/C_\ell \) become positive, namely, the low CMB multipoles are enhanced.

It should be recalled that the observation of a low CMB quadrupole sparked many different proposals to explain that suppression [16,17].

Besides finding a CMB quadrupole suppression in agreement with observations [1,3–6], we provide here predictions for the dipole and \( \ell \leq 5 \) multipole suppressions. Forthcoming CMB observations can provide better data to confront our CMB multipole suppression predictions.
tions. It will be extremely interesting to measure the primordial dipole and compare with our predicted value.

V. ANALYTIC FORMULAS FOR THE TRANSFER FUNCTION $D(k)$

It is very important to dispose of analytic formulas for the transfer function $D(k)$ in order to better understand the physical origin of its oscillations and properties as well as in the perspective of the MCMC data analysis.

However, the mode Eqs. (2.24) are not solvable in closed form for $k \neq 0$, not even for the approximated inflation solution Eq. (2.37), which leads to the potential $V_R(\tau)$ Eq. (2.43).

The function $D(k)$ must obey the general properties Eq. (4.23).

A. The primordial power spectrum vanishes for $k \to 0$ and becomes the BD power spectrum for $k \to \infty$

The fluctuations Eq. (2.24) can be solved explicitly for $k = 0$

$$s(\eta) = c_1 z(\eta) + c_2 \zeta(\eta) \int_{\eta_0}^{\eta} \frac{d\eta'}{\zeta(\eta')}.$$  \hspace{1cm} (5.1)

where $c_1$ and $c_2$ are arbitrary constants.

The BDic Eq. (4.7) introduce for $k \to 0$ a $1/\sqrt{2k}$ singularity in the mode functions. Thus, the mode functions must have the behavior

$$S_R(k; \eta) \approx \frac{s(\eta)}{\sqrt{2k}} [1 + \mathcal{O}(k)],$$  \hspace{1cm} (5.2)

where $s(\eta)$ is given by Eq. (5.1).

Inserting Eq. (5.2) into the BDic Eq. (4.7) yields for $k \to 0$,

$$s(\eta_0) = 1, \quad \frac{ds(\eta_0)}{d\eta} = 0,$$

which determines the coefficients $c_1$ and $c_2$ in Eq. (5.1).

We finally obtain

$$s(\eta) = \frac{z(\eta)}{z(\eta_0)} - \frac{\zeta(\eta_0) z(\eta)}{z(\eta_0)} \int_{\eta_0}^{\eta} \frac{d\eta'}{\zeta(\eta')}.$$  \hspace{1cm} (5.3)

and using Eq. (4.20) valid for $\eta \to 0^-$ when slow roll applies

$$\lim_{\eta \to 0^-} \frac{s(\eta)}{z(\eta)} = \frac{1}{z(\eta_0)}. $$  \hspace{1cm} (5.4)

The primordial power spectrum for $k \to 0$ follows by inserting Eqs. (5.2) and (5.4) into the general expression Eq. (4.12),

$$P_R(k) = \frac{m}{(M_{Pl})^2} \frac{k^3}{2\pi^2} \lim_{\eta \to 0^-} \left| \frac{S_R(k; \eta)}{z(\eta)} \right|^2.$$  \hspace{1cm} (5.5)

We thus find in general that the power spectrum vanishes as $k^2$ for $k \to 0$, and therefore

$$1 + D(k) = \mathcal{O}(k^{n_0} + 1)$$

as stated in Eq. (4.23). This property is generally true except for the extreme slow-roll inflaton solution (Sec. III A) with BDic imposed at $\eta_0 = -\infty$ in which case $D(k)$ vanishes identically for all $k$.

For growing $k$ the modes exit the horizon later on, during the slow-roll regime where Eq. (4.14) applies. For large $k$ the mode functions $S_R$ as well as $g_{rR}$ behave as plane waves [Eqs. (4.6) and (4.18)], and therefore

$$A_R(k) = 1, \quad B_R(k) = 0. \quad \text{Hence } D(k)^{\to \infty} = 0.$$

B. The transfer function $D(k)$ when BDic are imposed during slow roll

When the BDic Eq. (4.7) are imposed during slow roll at a finite time $\eta_0$ we can use Eq. (4.14) for the mode functions at $\eta = \eta_0$ and we obtain,

$$\frac{e^{-ik\eta_0}}{\sqrt{2k}} = A_R(k)g_{rR}(k; \eta_0) + B_R(k)g^*(rR)(k; \eta_0)$$

$$-i k \frac{e^{-ik\eta_0}}{\sqrt{2k}} = A_R(k)g_{rR}'(k; \eta_0) + B_R(k)g'^*(rR)(k; \eta_0),$$  \hspace{1cm} (5.5)

which determines

$$A_R(k) = \frac{e^{-ik\eta_0}}{i\sqrt{2k}} \left[ g_{rR}(k; \eta_0) + ikg^*(rR)(k; \eta_0) \right],$$

$$B_R(k) = \frac{e^{-ik\eta_0}}{i\sqrt{2k}} \left[ g_{rR}'(k; \eta_0) + ikg'^*(rR)(k; \eta_0) \right].$$  \hspace{1cm} (5.6)

These coefficients satisfy Eq. (4.16) and

$$|A_R(k)|^2 + |B_R(k)|^2 = \frac{1}{k^2} |g_{rR}(k; \eta_0)|^2$$

Notice that the function $g_r(k; \eta)$ Eq. (4.15) and the $k$ factors in Eq. (5.6) combine to produce functions $A_R(k) = A_R(k\eta_0)$ and $B_R(k) = B_R(k\eta_0)$ that only depend on the product $k\eta_0$.

We find from Eqs. (4.22) and (5.6) the corresponding transfer function, which is a function of $k\eta_0$ too,
The functional dependence on $k\eta_0$ confirms the assertion in Sec. IV B that different initial times $\tau_0$ lead to a rescaling in $k$.

In the $k\eta_0 \to \infty$ limit two types of vanishing terms show up in $\tilde{D}(k\eta_0)$: (a) terms that strongly oscillate as $e^{\pm 2ik\eta_0}$ as they tend to zero and (b) nonoscillatory decreasing terms. Under integrals on $k$, the terms of type (a) yield convergent expressions. We derive the nonoscillatory decreasing terms (b) by inserting the asymptotic behavior of the Hankel functions $\text{Eq. (4.15)}$ in Eq. (5.7) with the result

$$\tilde{D}(k\eta_0) = k^{-\infty} \frac{\nu^2}{8(k\eta_0)^4} + \text{terms oscillating as } e^{\pm 2ik\eta_0}. \quad (5.8)$$

However, this approximation will not be valid for large enough $k$ since the modes at small enough wavelength will exit the horizon after the end of slow roll where Eq. (5.8) does not apply anymore. We recall that the occupation number $|B_R(k)|^2$ (and therefore $\tilde{D}(k)$) must decrease faster than $1/k^4$ for $k \to \infty$ in order to ensure finite UV values for the expectation value of the energy-momentum fluctuations [1,18].

The case $\nu_R = 3/2$ is a good approximation which simplifies the expressions above. We obtain in this scale-invariant case:

$$A_R(k) = 1 + \frac{i}{k\eta_0} - \frac{1}{2k^2\eta_0^2}, \quad B_R(k) = -\frac{e^{-2ik\eta_0}}{2k^2\eta_0^2}. \quad (5.9)$$

The transfer function is in this case,

$$\tilde{D}(x) = \cos^2x - \frac{\sin^2x}{x^2} - \frac{\sin^2x}{x^4}, \quad \nu_R = 3/2, \quad x \equiv k\eta_0. \quad (5.9)$$

Equation (5.8) for $\nu = 3/2$ coincides with Eq. (5.9) in the $x \to \infty$ limit, as it must be.

Notice that the simple formula Eq. (5.9) obeys the general properties Eq. (4.23). In particular,

$$\tilde{D}(x) \xrightarrow{x=0} -1 + \frac{4}{9} x^2 + \mathcal{O}(x^4).$$

VI. FIXING THE TOTAL NUMBER OF INFLATION E-FOLDS AND THE BOUND FROM ENTROPY

It is very useful to plot the comoving scales of the cosmological fluctuation wave numbers and the comoving Hubble radius together (see Fig. 10). One sees in this way how and when the cosmological fluctuations cross out and in the Hubble radius. The comoving Hubble radius is defined by $R_H = 1/[a(\tau)H(\tau)]$. We display in Table IV the dependence of $R_H$ on the scale factor $a$ for all the relevant eras of the Universe.

![Graph](image)

**FIG. 10 (color online).** The logarithm of the comoving scales and the logarithm of the comoving Hubble radius $R_H = 1/[aH]$ vs loga.
In summary, assuming that the CMB quadrupole is suppressed because it exited the horizon by the end of fast-roll inflation fixes the total number of inflation e-folds which turns to be

\[ N_{\text{tot}} \approx 63. \]

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