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Dipartimento di / Department of Matematica e Applicazioni
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REDUCED FUSION SYSTEMS ON SYLOW 3-SUBGROUPS OF THE MCLAUGHLIN GROUP

Cognome / Surname: BACCANELLI Nome / Name: ELISA
Matricola / Registration number: 787751

Tutore / Tutor: Prof.ssa FRANCESCA DALLA VOLTA
Supervisor: Prof.ssa CLARA FRANCHI
Coordinatore / Coordinator: Prof. ROBERTO PAOLETTI

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Introduction

Nowadays, the theory of fusion systems finds applications in different areas of mathematics: e.g., in finite group theory, to study p-local theory of finite groups (in particular, it is expected that an approach via fusion systems will allow to semplify the proof of the classification of finite simple groups); in representation theory, to study modular representations of finite groups; in topology, to study p-completed classifying spaces of finite groups.

The theory of fusion systems is rather "young", i.e., it was founded quite recently and it has been developing very quickly; yet, its roots spread deep in a further past. Indeed, the basic ideas standing behind fusion systems stretch back to the works on fusion in finite groups by, e.g., W. Burnside and F.G. Frobenius, and later by J.L. Alperin (see, e.g., [2]). In those works the word "fusion" denotes the behavior of the p-subgroups (and their elements) of a finite group G with respect to G-conjugation. Later, R. Solomon came across with the first example of what is now called "exotic fusion system" (cf. [29]): i.e., the fusion of p-elements is incompatible with being induced by conjugation in a finite group lying above.

The first formalization of the theory of fusion systems dates back to the work of L. Puig during the '90s, which was published only some years later (see, e.g., [27]). Actually, he used the name "Frobenius category" instead of "fusion system". In the meanwhile, before the publication, other mathematicians started working independently on the same ideas, so that now there is not yet a commonly accepted notation. Of course, the definitions and results worked out in such work are all equivalent, and now people are working to provide a homogeneous and harmonic framework to the theory of fusion systems: see, e.g., the book of M. Aschbacher, R. Kessar, and B. Oliver (see [6]), and the book of D. Craven (see [12]).

The idea of fusion systems is to axiomatize and generalize the notions and fundamental properties of conjugation in finite groups: given a prime p and a finite p-group S, a fusion system $\mathcal{F}$ on S is a category whose objects are all subgroups of S, and whose morphisms are injective group homomorphisms satisfying certain conditions (see Definition 1.6). In particular, all conjugations via elements of S are morphisms in $\mathcal{F}$. The phylosophy behind fusion systems is to "force" morphisms to behave like conjugations, in order to generalize the fusion of S and its subgroups in a finite group G containing S. In this way, while dealing with fusion in finite groups one may "forget" about the group G standing above, and focus only on p-groups. Still, with this approach one recovers a big load of information on the p-local behavior of G.

In fact, in order to study effectively fusion of groups via fusion systems, the "plain" definition of fusion system is not enough, as it is somehow too permissive: for example, the category whose objects are all subgroups of a p-group S with morphisms any monomorphism between these subgroups is a (very uninteresting) fusion system. Therefore, one needs to add some further technical conditions, leading to the definition of saturated fusion system (see Definition 1.17). In fact, such definition was taylored according to the following example: the fusion system $\mathcal{F}_S(G)$ (where G is a finite group with Sylow p-subgroup S), whose morphisms are precisely the conjugations with elements of G. The fusion system $\mathcal{F}_S(G)$ describes how the subgroups of S are related by G-conjugation; in fact, since all Sylow p-subgroups of G are conjugate, $\mathcal{F}_S(G)$ determines how all p-subgroups of G are related by G-conjugation: in finite group theory this is called p-fusion pattern of G.

Let S be a p-group. As we have seen above, every finite group G which contains S comes endowed with the fusion system $\mathcal{F}_S(G)$. Conversely, I. Leary and R. Stancu proved that for every fusion system $\mathcal{F}$ on S there exists a (possibly infinite) group G containing S such that $\mathcal{F} = \mathcal{F}_S(G)$ (see [19]). Yet, R. Solomon discovered the existence of fusion systems which are not "induced" by a finite group G (see [29]). This discovery dates back to the '70s, before fusion systems were formalized. Later, such fusion systems were called exotic fusion systems, as they appear rather rarely.

The exotic fusion systems discovered by Solomon are saturated fusion systems on $\mathrm{Spin}_7(q)$, with $q = 2^k$ for some odd k. This result is very interesting also because it is the only known example of exotic fusion systems on 2-groups. On the other hand, many examples of exotic fusion systems on p-groups, with p odd, are now known. E.g., while classifying all saturated fusion systems on Sylow p-subgroups of $SL_3(p)$, A. Ruiz and A. Viruel found three exotic fusion systems when $p = 7$ (see [28]). As for groups of order a power of three, A. Diaz, A. Ruiz and A. Viruel proved that there are several infinite families of exotic fusion systems over 3-groups of rank 2 (see [14]). In [22], B. Oliver found many families of exotic fusion systems on a non abelian 3-group with a unique abelian subgroup of index 3. In [8], M. Clelland and C. Parker discovered two other infinite families of exotic fusion systems on two class of groups consisting of certain amalgams, which include also 3-groups.

It is worth stressing that if all fusion systems were induced by finite groups, the theory of fusion systems would provide only a more effective language to describe fusion in finite groups, and nothing more. But since there exist also exotic fusion systems, the study of fusion systems may lead to the discovery of new algebraic structures.

The current research on fusion systems concernes mainly the classification of saturated fusion systems on "small" p-groups. For example, the classification is completed in the case of metacyclic p-groups, extraspecial p-groups of exponent p and order p^3, and 2-groups of rank 2 (see [30, 28, 13]). The classification in the case of p-groups (p odd) with rank 3 or 4 is still a work in progress by D. Benson and C. Parker.

The goal of this Thesis is to contribute to the classification of saturated fusion systems on "small" p-groups, by classifying all reduced fusion systems on a Sylow 3-subgroup S of the sporadic McLaughlin group Mc, where a saturated fusion system is said to be reduced if it has no non trivial normal subgroups (see Definitions 1.29 and 1.30 for more details). The McLaughlin group Mc has order $2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$ and was discovered by J. McLaughlin in 1969 (see [21]); the group S may be considered "small", as it has order 3^6 and rank 4.

In Chapter 1 we recall some definitions and basic facts on fusion systems, which are required in the Thesis. In this Chapter we make reference to [12].

In Chapter 2 we study the structure of S and its subgroups, listing some properties which will be useful in the sequel. In particular, we show that S splits as semidirect product: in fact, $S = A \rtimes B$, where $A \simeq C_3^4$ and $B \simeq C_3^2$ (see (2.1.1)), but also $S = E \rtimes T$, where E is the extraspecial group of order 3^5 and exponent 3, and $T \simeq C_3$ (see (2.2.1)).

In Chapter 3 we study a particular class of subgroups of S, i.e., the $\mathcal{F}$-essential subgroups, where $\mathcal{F}$ is a saturated fusion system on S (see Definition 1.26 for the definition of $\mathcal{F}$-essential subgroup). The $\mathcal{F}$-essential subgroups are very important in the study of saturated fusion systems, as by the "fusion systems version" of the Alperin's Fusion Theorem (see [12, Theorem 1.23]) every isomorphism in a saturated fusion system $\mathcal{F}$ on a finite p-group S is the composition of restrictions of automorphisms of $\mathcal{F}$-essential subgroups of S and of S itself: thus, $\mathcal{F}$ is "generated" by these automorphisms. In particular, when $\mathcal{F}$ is a reduced fusion system on S, we prove that S has exactly two $\mathcal{F}$-essential subgroups, A and E (see Proposition 3.12).

In Chapter 4 we study $\mathrm{Out}_{\mathcal{F}}(A)$, $\mathrm{Out}_{\mathcal{F}}(E)$ and $\mathrm{Out}_{\mathcal{F}}(S)$, the groups of the outer automorphisms in $\mathcal{F}$ of A, E and S. Since the inner automorphisms are always morphisms in a fusion system, we may conveniently study the groups of outer automorphisms in $\mathcal{F}$ instead of the whole groups of automorphisms in $\mathcal{F}$. In particular, we determine all the 3-tuples $(\mathrm{Out}_{\mathcal{F}}(S), \mathrm{Out}_{\mathcal{F}}(A), \mathrm{Out}_{\mathcal{F}}(E))$, where $\mathcal{F}$ is a reduced fusion system on S, with the following procedure. First, we show that $\mathrm{Out}_{\mathcal{F}}(A)$ is a subgroup of $GL_4(3)$, such that $O_3(\mathrm{Out}_{\mathcal{F}}(A)) = 1$, the Sylow 3-subgroups of $\mathrm{Out}_{\mathcal{F}}(A)$ have order 3^2, and the centralizer in A of a Sylow 3-subgroup of $\mathrm{Out}_{\mathcal{F}}(A)$ has order 3 (cf. Lemma 4.8). We use this information to determine all possible groups $\mathrm{Out}_{\mathcal{F}}(A)$, which are listed in Theorem 4.11. Pairwise, we show that $\mathrm{Out}_{\mathcal{F}}(E)$ is a subgroup of the general symplectic linear group $\mathrm{GSp}_4(3)$ not contained in $\mathrm{Sp}_4(3)$, such that $O_3(\mathrm{Out}_{\mathcal{F}}(E)) = 1$, and the Sylow 3-subgroups of $\mathrm{Out}_{\mathcal{F}}(E)$ have order 3 (cf. Lemma 4.15). Using this information we determine all possible groups $\mathrm{Out}_{\mathcal{F}}(E)$ (cf. Lemmas 4.18–4.22). Moreover, we prove that $\mathrm{Out}_{\mathcal{F}}(S)$ is a 2-group which is isomorphic to a Sylow 2-subgroup of $\mathrm{Aut}_{\mathcal{F}}(S)$ (cf. Lemma 4.3), and the Sylow 2-subgroups of the normalizer in $\mathrm{Out}_{\mathcal{F}}(A)$, respectively in $\mathrm{Out}_{\mathcal{F}}(E)$, of its Sylow 3-subgroup are isomorphic to $\mathrm{Out}_{\mathcal{F}}(S)$ (cf. Lemmas 4.6–4.7). The information obtained so far allows us to determine uniquely the aforementioned triplets (cf. Theorem 4.23). We study only reduced fusion systems on S as they "correspond" in some sense to simple groups. Since such 3-tuples determine the structure of $\mathcal{F}$ by the Alperin's Fusion Theorem, this completes the classification of the reduced fusion systems on S.

In Chapter 5, we prove that if $\mathcal{F}$ is "induced" by some finite group, then $\mathcal{F} = \mathcal{F}_S(G)$ for some finite almost simple group G, such that the generalized Fitting subgroup $F^*(G)$ is a simple group and the Sylow 3-subgroups of G are isomorphic to S and contained in $F^*(G)$ (cf. Proposition 5.3). Thus, we search for all finite simple groups K which contain S as Sylow 3-subgroup, and we show that K is isomorphic to one of the following (cf. Theorem 5.13):

- the McLaughlin group Mc;
- the Conway group Co_2;
- the classical group $PSU_4(3)$;
- the linear group $PSL_6(q)$, where $3 \mid q-1$ and $9 \nmid q-1$;
- the classical group $PSU_6(q)$, where $3 \mid q+1$ and $9 \nmid q+1$.

Then we determine the triplets $(\mathrm{Out}_{\mathcal{F}}(S), \mathrm{Out}_{\mathcal{F}}(A), \mathrm{Out}_{\mathcal{F}}(E))$ in the case when $\mathcal{F} = \mathcal{F}_S(G)$ and G is a finite almost simple group with $F^*(G)$ one of the simple groups listed in Theorem 5.13 and Sylow 3-subgroups contained in $F^*(G)$. In fact, four of the 3-tuples listed in Theorem 4.23 are "induced" by no such group G: we show that the associated fusion systems are exotic (see Theorem 5.15).

Chapter 1

Fusion systems - Definitions & basic facts

First of all, we outline the notation we will use while dealing with finite groups, and we provide some definitions. We make group homomorphisms act on the right: namely, if $\phi\colon G \to H$ is a homomorphism of groups, we denote the image of an element $x \in G$ via ϕ by $x\phi$. In particular, for any $x \in G$, the map $c_x\colon G \to G$ denotes the right conjugation by x, i.e.,

$$gc_x = g^x = x^{-1}gx, \qquad \text{for any } g \in G.$$

Thus, the commutator between two elements $x, y \in G$ is

$$[x, y] = x^{-1}y^{-1}xy = x^{-1}x^y.$$

For any subgroup H of G, we denote the image of H via c_x by H^x. Given two homomorphisms $\phi_1 : G \to H$ and $\phi_2 : H \to K$, we denote their composition by $\phi_1\phi_2 : G \to K$, where ϕ_1 acts first, i.e., $x(\phi_1\phi_2) = (x\phi_1)\phi_2$, for any $x \in G$.

We denote the generalized Fitting subgroup of a group G by $F^*(G)$, and we denote the Frattini subgroup of a group G by $\Phi(G)$. It is well known that if G is a finite p-group, then

$$\Phi(G) = G^p \cdot G',$$

where

$$G^p = \langle x^p \mid x \in G\rangle \quad \text{and} \quad G' = \langle [x, y] | x, y \in G\rangle$$

(i.e., G' is the derived subgroup of G).

If p is a prime number and G is a finite group, we denote with $O_p(G)$ the largest normal p-subgroup of G, with $O_{p'}(G)$ the largest normal subgroup of G whose order is not divisible by p, and with $O^{p'}(G)$ the smallest normal subgroup N of G such that G/N is a p'-group (i.e., p does not divide $|G/N|$).

For a group G, the notation $G = N.H$ means that there exists a (not necessarily split) short exact sequence

$$1 \to N \to G \to H \to 1.$$

Definition 1.1. Let p be a prime. The *p-rank* of a finite group G is the maximal dimension as $\mathbb{F}_p$-vector space of an elementary abelian p-subgroup of G. We denote the p-rank of G by $m_p(G)$.

Definition 1.2. Let G be a finite group, and let p be a prime which divides the order of G. Set

$$J(G) = \langle P \leq G \mid P \text{ is elementary abelian of } p\text{−rank } m_p(G)\rangle.$$

The group $J(G)$ is called the *Thompson subgroup* of G for the prime p.

By [4, 32.1], $J(G)$ is characteristic in G for every prime p dividing the order of G.

Definition 1.3. Let p be a prime. A p-group P is said to be *extraspecial* if $Z(P) = P' = \Phi(P) \simeq C_p$.

Definition 1.4. Let P and Q be groups. Suppose we identify a central subgroup C of P (i.e., $C \leq Z(P)$) with a central subgroup D of Q via the isomorphism $\varphi : C \to D$. The (external) *central product* of P and Q (with respect to φ), denoted by $P * Q$, is the quotient of the direct product $P \times Q$ by the subgroup $\{(g, (g\varphi)^{-1}) |\, g \in C\}$.

In particular, $P * Q$ has normal subgroups $\tilde{P}$ and $\tilde{Q}$ (isomorphic to P and Q respectively), such that $P * Q = \tilde{P}\tilde{Q}$, $\tilde{P}$ and $\tilde{Q}$ centralize each other, and $\tilde{P} \cap \tilde{Q} \simeq C \simeq D$. With an abuse of notation, we identify $\tilde{P}$ with P, $\tilde{Q}$ with Q, and $\tilde{P} \cap \tilde{Q}$ with C and D.

Definition 1.5. A group G is said to be *almost simple* if there exists a simple non-abelian group P such that $P \leq G \leq \mathrm{Aut}(P)$.

1.1 Fusion Systems

From now on, p will denote a prime number, and S a finite p-group.

Definition 1.6. A *fusion system* $\mathcal{F}$ on S is a category, whose objects are all subgroups of S, and whose morphism sets $\mathrm{Hom}_{\mathcal{F}}(Q, R)$ are set of injective group homomorphisms $Q \to R$ (with composition of morphisms given by the usual composition of group homomorphisms), satisfying the following properties:

- $\mathrm{Hom}_{\mathcal{F}}(S, S)$ contains all conjugation automorphisms c_x, $x \in S$;

- for any $\phi \in \mathrm{Hom}_{\mathcal{F}}(Q, R)$, the isomorphism $Q \to Q\phi$ belongs to $\mathrm{Hom}_{\mathcal{F}}(Q, Q\phi)$;

- $\mathcal{F}$ is closed with respect to inversion, i.e., if $\phi \in \mathrm{Hom}_{\mathcal{F}}(Q, R)$ is an isomorphism, then $\phi^{-1} \in \mathrm{Hom}_{\mathcal{F}}(R, Q)$.

If $\phi : Q \to R$ lies in $\mathcal{F}$, we will say that ϕ is a $\mathcal{F}$-morphism.

Example 1.7. The category whose objects are all subgroups of S, and whose morphisms are all injective group homomorphisms between subgroups of S, is the largest fusion system on S.

Example 1.8. Suppose S is a finite p-subgroup of a group G. Consider the category $\mathcal{F} = \mathcal{F}_S(G)$ whose objects are all subgroups of S, with morphism sets

$$\mathrm{Hom}_{\mathcal{F}}(Q, R) = \{(c_x)_{|Q} \mid x \in G, Q^x \leq R\}.$$

Then $\mathcal{F}$ is a fusion system on S. If we take $G = S$, we obtain the smallest fusion system on S, usually denoted by $\mathcal{F}_S(S)$.

Here we list some basic properties of fusion systems. In this section, we refer to [12, Chapter 4] (note that in the statements of [12] P denotes what here is denoted by S, and viceversa).

Proposition 1.9. *Let $\mathcal{F}$ be a fusion system on S, and let $\phi : Q \to R$ be a morphism in $\mathcal{F}$. Let P be any subgroup of Q, and let T be any subgroup of S containing $P\phi$. Then there is a $\mathcal{F}$-morphism $\psi : P \to T$ such that ϕ and ψ agree on P. Thus, given a $\mathcal{F}$-morphism, one may restrict the domain arbitrarily, and extend or constrict the codomain to any overgroup of the image of the restriction.*

Proof. Let P be a subgroup of Q, and let T be a subgroup of S containing $P\phi$. Consider the inclusion maps $\imath_1 : P \to Q$, and $\imath_2 : P\phi \to T$. The maps $\imath_1$ and $\imath_2$ are morphisms in $\mathcal{F}_S(S)$. Since $\mathcal{F}_S(S)$ is the smallest fusion system on S, then $\imath_1$ and $\imath_2$ lie in $\mathcal{F}$. Thus, the composition map $\imath_1\phi\imath_2$ also lies in $\mathcal{F}$, and we may take $\psi = \imath_1\phi\imath_2$. This yields the claim. □

For any $P \leq S$, let $\mathrm{Aut}_{\mathcal{F}}(P)$ be the set of all $\mathcal{F}$-automorphisms of P. Moreover, set

$$\mathrm{Aut}_S(P) = \{(c_x)_{|P} \mid x \in N_S(P)\}.$$

Definition 1.10. For a subgroup P of S, let

$$c_P : N_S(P) \to \mathrm{Aut}_S(P)$$

be the group homomorphism sending $x \in N_S(P)$ to $(c_x)_{|P} \in \mathrm{Aut}_S(P)$.

Thus, one has that

$$\mathrm{Aut}_S(P) \simeq N_S(P)/C_S(P).$$

In particular, $\mathrm{Aut}_S(P)$ is a p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(P)$. We write $\mathrm{Out}_{\mathcal{F}}(P)$ for the group of outer automorphisms of P that lie in $\mathcal{F}$. By the definition of fusion system, it follows that

$$\mathrm{Out}_{\mathcal{F}}(P) = \mathrm{Aut}_{\mathcal{F}}(P)/\mathrm{Inn}(P).$$

Set

$$\mathrm{Out}_S(P) = \mathrm{Aut}_S(P)/\mathrm{Inn}(P).$$

If $\phi : Q \to R$ is an $\mathcal{F}$-isomorphism, then we say that Q and R are $\mathcal{F}$-isomorphic or $\mathcal{F}$-conjugate.

Lemma 1.11. *The relation of $\mathcal{F}$-conjugacy is an equivalence relation on the set of all subgroups of S.*

Lemma 1.12. *Let $\mathcal{F}$ be a fusion system on S. Every morphism in $\mathcal{F}$ decomposes as the composition of a $\mathcal{F}$-isomorphism with an inclusion map.*

Thus, as a consequence of Lemma 1.12 we may reduce to study only isomorphisms between subgroups of S while studying fusion systems on S.

Proposition 1.13. *Let $\mathcal{F}$ be a fusion system on S. If Q and R are subgroups of S, $\mathcal{F}$-conjugate under an isomorphism ϕ, then $\mathrm{Aut}_{\mathcal{F}}(Q)$ and $\mathrm{Aut}_{\mathcal{F}}(R)$ are isomorphic under the map $\hat{\phi}$ sending any $\psi \in \mathrm{Aut}_{\mathcal{F}}(P)$ to $\phi^{-1}\psi\phi \in \mathrm{Aut}_{\mathcal{F}}(Q)$.*

By Proposition 1.9, one may extend arbitrarily the codomain of any $\mathcal{F}$-morphism. On the other hand, the opposite is not true, i.e., one can not extend arbitrarily the domain of a $\mathcal{F}$-morphism. For example, if P is a Sylow 2-subgroup of S_4, conjugation by the element $(2,3,4)$ acts by cycling the elements $(1,2)(3,4)$, $(1,3)(2,4)$, and $(1,4)(2,3)$, and hence has order 3; but it does not lift to an automorphism of P, because $\mathrm{Aut}(P)$ has order 8.

Suppose that the $\mathcal{F}$-isomorphism $\phi : Q \to R$ has an extension $\bar{\phi} : \bar{Q} \to \bar{R}$, with domain $\bar{Q} > Q$. Thus, $N_{\bar{Q}}(Q) > Q$, and ϕ extends its domain to a subgroup of $N_S(Q)$. Hence ϕ extends to a (proper) overgroup of Q in S if, and only if, it extends to a (proper) overgroup of Q in $N_S(Q)$. Since extensions inside $N_S(Q)$ are easier to handle, hereafter we will deal with this kind of extensions.

Proposition 1.14. *Let $\mathcal{F}$ be a fusion system on S, and let $\phi : Q \to R$ be a $\mathcal{F}$-isomorphism. Suppose that ϕ extends to a $\mathcal{F}$-morphism $\bar{\phi} : P \to S$, with $P \leq N_S(Q)$. Then the image of $\bar{\phi}$ is contained in $N_S(R)$, and*

$$Pc_Q \leq \mathrm{Aut}_S(Q) \cap \mathrm{Aut}_S(R)^{\phi^{-1}}.$$

Proof. Let $x \in P$. For any $g \in Q$, one has that $g^x \in Q$. Thus,

$$(g\bar{\phi})^{x\bar{\phi}} = (g^x)\bar{\phi} \in R.$$

Hence $x\bar{\phi} \in N_S(R)$.

Let $\hat{\phi} : \mathrm{Aut}_{\mathcal{F}}(Q) \to \mathrm{Aut}_{\mathcal{F}}(R)$ be the map sending $\psi \in \mathrm{Aut}_{\mathcal{F}}(Q)$ to $\phi^{-1}\psi\phi \in \mathrm{Aut}_{\mathcal{F}}(R)$. We claim that $\bar{\phi}c_R = c_Q\hat{\phi}$ on P, i.e., $x(\bar{\phi}c_R) = x(c_Q\hat{\phi})$ for every $x \in P$. For every $t \in R$, one has that

$$t(x(\bar{\phi}c_R)) = t((x\bar{\phi})c_R) = t(c_{x\bar{\phi}})_{|R} = t^{x\bar{\phi}}.$$

Since ϕ is an isomorphism, there exists $z \in Q$ such that $z\phi = t$. One has that

$$\begin{aligned} t(x(c_Q\hat{\phi})) &= t((c_x)_{|Q}\hat{\phi}) = t(\phi^{-1}(c_x)_{|Q}\phi) \\ &= z((c_x)_{|Q}\phi) = (z^x)\phi = (z^x)\bar{\phi} \\ &= t^{x\bar{\phi}}, \end{aligned}$$

and this yields the claim.

Since $x\bar{\phi} \in N_S(R)$, in particular $(x\bar{\phi})c_R \in \mathrm{Aut}_S(R)$. Then

$$(Pc_Q)^{\phi} = P(c_Q\hat{\phi}) = (P\bar{\phi})c_R \leq \mathrm{Aut}_S(R).$$

Moreover, $Pc_Q \leq \mathrm{Aut}_S(Q)$ and the proof is completed. □

Definition 1.15. Let Q,R, and ϕ be as in the statement of Proposition 1.14. We denote with N_ϕ the preimage of $\mathrm{Aut}_S(Q) \cap \mathrm{Aut}_S(R)^{\phi^{-1}}$ under the map c_Q. Equivalently,

$$N_\phi = \{x \in N_S(Q) \mid \exists\, y \in N_S(R) : (g^x)\phi = (g\phi)^y, \forall\, g \in Q\}.$$

By Proposition 1.14, one can not extend (inside $N_S(Q)$) the isomorphism ϕ to a subgroup of S containing properly N_ϕ.

Definition 1.16. Let $\mathcal{F}$ be a fusion system on S. A subgroup Q of S is said to be *receptive* if every $\mathcal{F}$-isomorphism with image Q is extensible to N_ϕ.

1.2 Saturated Fusion Systems

The idea of fusion systems is to provide a model for fusion in finite groups, in particular for conjugation of p-subgroups of a Sylow p-subgroup of a finite group. In order to develope a rich theory, the "plain" definition of fusion system is not enough. One needs some further technical restrictions in the definition of fusion systems, leading to the definition of a saturated fusion system. The main example of saturated fusion system is the fusion system

$\mathcal{F}_S(G)$, where S is a Sylow p-subgroup of a finite group G. Classical results about fusion in a Sylow p-subgroup S of a finite group G can be interpreted as results about the fusion system $\mathcal{F}_S(G)$: indeed, in this category every morphism between subgroups of S comes from conjugations with elements of the group G. Thus, the category $\mathcal{F}_S(G)$ describes how subgroups of S are related by conjugations with elements of G; in particular, since all Sylow p-subgroups of G are conjugate in G, $\mathcal{F}_S(G)$ determines how all p-subgroups of G are related by conjugations with elements of G.

Definition 1.17. Let P be a subgroup of S, and let $P^{\mathcal{F}}$ be the $\mathcal{F}$-conjucacy class of P, i.e., the set of subgroups of S which are $\mathcal{F}$-isomorphic to P.

- P is said to be *fully centralized* in $\mathcal{F}$ if $|C_S(P)| \geq |C_S(Q)|$ for every $Q \in P^{\mathcal{F}}$;
- P is said to be *fully normalized* in $\mathcal{F}$ if $|N_S(P)| \geq |N_S(Q)|$ for every $Q \in P^{\mathcal{F}}$;
- P is said to be *fully automized* in $\mathcal{F}$ if $\mathrm{Aut}_S(P)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(P)$;
- P is said to be *strongly $\mathcal{F}$-closed* if, for any subgroup R of P and for any $\mathcal{F}$-morphism $\phi : R \to S$, $R\phi$ is contained in P.

Definition 1.18. A fusion system $\mathcal{F}$ on S is said to be *saturated* if every $\mathcal{F}$-conjugacy class of subgroups of S contains a subgroup that is both receptive and fully automized.

Proposition 1.19. *Let G be a finite group, and let S be a Sylow p-subgroup of G. The fusion system $\mathcal{F}_S(G)$ is saturated.*

Proof. Set $\mathcal{F} = \mathcal{F}_S(G)$. Let R be a subgroup of S, let P be a Sylow p-subgroup of G containing a Sylow p-subgroup of $N_G(R)$, and let $g \in G$ such that $P^g = S$. Set $Q = R^g$: then $N_S(Q)$ is a Sylow p-subgroup of $N_G(Q)$, as $N_P(R)$ is a Sylow p-subgroup of $N_G(R)$. Hence Q is receptive (see [12, Proposition 4.10]).

Moreover,

$$N_S(Q)C_G(Q)/C_G(Q) \simeq N_S(Q)/(C_G(Q) \cap N_S(Q)) = N_S(Q)/C_S(Q)$$

is a Sylow p-subgroup of $\mathrm{Aut}_G(Q)$. Thus, Q is fully automized and the claim follows. □

The following comes from [12, Theorem 4.21]

Theorem 1.20. *Let $\mathcal{F}$ be a saturated fusion system on S. Then:*

1. *a subgroup of S is fully centralized if and only if it is receptive;*
2. *a subgroup of S is fully normalized if and only if it is both receptive and fully automized.*

1.3 Alperin's Fusion Theorem

The aim of this section is to state the Alperin's Fusion Theorem, which sets that in a saturated fusion system $\mathcal{F}$ on S any isomorphism may be decomposed as product of restrictions of automorphisms of certain peculiar subgroups of S.

Definition 1.21. Let $\mathcal{F}$ be a fusion system on S, and let P be a subgroup of S. P is said to be *$\mathcal{F}$-centric* if $C_S(Q) = Z(Q)$, for every $Q \in P^{\mathcal{F}}$.

Definition 1.22. Let G be a finite group and let p be a prime which divides the order of G. A proper subgroup H of G is said to be *strongly p-embedded* in G if H contains a Sylow p-subgroup of G, and $p \nmid |H \cap H^g|$ for any $g \in G \backslash H$.

Lemma 1.23. *Let G be a finite group, let p a prime which divides the order of G, let $P \in \mathrm{Syl}_p(G)$ and set $X = \langle N_G(Q) \mid Q \leq P \text{ and } Q \neq 1\rangle$. Then G has a strongly p-embedded subgroup if, and only if, $X \neq G$.*

Proof. Suppose that G has a strongly p-embedded subgroup H. Without loss of generality, we may assume that $P \leq H$. Hence $N_G(Q) \leq H$, for every $1 \neq Q \leq P$: indeed, if there exists $1 \neq Q \leq P$ such that $N_G(Q) \nleq H$, and $g \in N_G(Q) \setminus H$, then $Q \leq (H \cap H^g)$, a contradiction, since H is strongly p-embedded. It follows that $X \leq H < G$.

Now suppose that $X < G$. We claim that X is a strongly p-embedded subgroup of G. Let $g \in G$ such that $p \,|\, |X \cap X^g|$, and let Q be a Sylow p-subgroup of $X \cap X^g$ and R be a Sylow p-subgroup of X^g containing Q. If $Q < R$, then $N_R(Q) > Q$, a contradiction, since $N_R(Q)$ is a p-subgroup of $X \cap X^g$. Thus, $Q = R$. There exist $x \in X$, $y \in X^g$, such that $P^x = Q = P^{gy}$. Hence $gyx^{-1} \in N_G(P) \leq X$. Let $k \in X$ such that $y = k^g$, and let $h \in X$ such that $gyx^{-1} = h$. Direct computations show that $g = k^{-1}hx \in X$: therefore X is a strongly p-embedded subgroup of G, and the proof is completed. □

Corollary 1.24. *Let G be a finite group, and let p be a prime such that $p \mid |G|$ and $p^2 \nmid |G|$. Let P be a Sylow p-subgroup of G. Then G has a strongly p-embedded subgroup if, and only if, P is not normal in G. In particular, if P is not normal in G, $N_G(P)$ is a strongly p-embedded subgroup of G.*

Remark 1.25. Let G be a finite group, and let p be a prime which divides the order of G. If G contains strongly p-embedded subgroups, Lemma 1.23 implies that $O_p(G) = 1$.

Definition 1.26. A subgroup P of S is said to be *$\mathcal{F}$-essential* if P is $\mathcal{F}$-centric and $\mathrm{Out}_{\mathcal{F}}(P)$ contains a strongly p-embedded subgroup.

Remark 1.27. One has the following:

i. the property of being $\mathcal{F}$-essential is invariant by isomorphism;

ii. yet, the property of being $\mathcal{F}$-essential and fully normalized is invariant only by conjugation.

Now we state a version of Alperin's Fusion Theorem due to L. Puig (cf. [12, Theorem 4.51])

Theorem 1.28 (Alperin's Fusion Theorem)**.** *Let $\mathcal{F}$ be a saturated fusion system on S, let $\mathcal{S}$ denote the set of all fully normalized, $\mathcal{F}$-essential subgroups of S, and let Q and R be two subgroups of S, with $\phi\colon Q \to R$ a $\mathcal{F}$-isomorphism. Then there exist*

1. *a sequence of $\mathcal{F}$-isomorphic subgroups $Q = Q_0, Q_1, \ldots, Q_{n+1} = R$;*
2. *a sequence $S_1, S_2, \ldots, S_n$ of elements of $\mathcal{S}$, with $Q_{i-1}, Q_i \leq S_i$;*
3. *a sequence of $\mathcal{F}$-automorphisms ϕ_i of S_i such that $Q_{i-1}\phi_i = Q_i$;*
4. *a $\mathcal{F}$-automorphism ψ of S (mapping Q_n to Q_{n+1});*

such that

$$(\phi_1\phi_2\cdots\phi_n\psi)_{|Q} = \phi. \tag{1.3.1}$$

Proof. First, we claim that if $\theta \in \mathrm{Aut}_{\mathcal{F}}(S)$ and $\phi \in \mathrm{Aut}_{\mathcal{F}}(E)$ for some fully normalized, $\mathcal{F}$-essential subgroup E of S, then there exists $\rho' \in \mathrm{Aut}_{\mathcal{F}}(\tilde{E})$, with $\tilde{E}$ an other fully normalized, $\mathcal{F}$-essential subgroup of S, such that

$$\theta\rho = \rho'\theta$$

on $E\theta^{-1}$. Indeed, one has that

$$u(\theta\rho) = u(\theta\rho\theta^{-1}\theta)$$

for any $u \in E\theta^{-1}$. Since $\theta \in \mathrm{Aut}_{\mathcal{F}}(S)$, it follows that

$$N_S(E)\theta^{-1} = N_S(E\theta^{-1}),$$

and since E is fully normalized, also $E\theta^{-1}$ is fully normalized. By Remark 1.27, $E\theta^{-1}$ is $\mathcal{F}$-essential. Thus, the claim holds, with $\tilde{E} = E\theta^{-1}$ and $\rho' = \theta\rho\theta^{-1}$.

This proves that if α and β are $\mathcal{F}$-isomorphisms which decompose as in (1.3.1), then also $\alpha\beta$ (when defined), α^{-1} and β^{-1} decompose as in (1.3.1).

We proceed by induction on $|S : Q|$. If $S = Q$, then $\phi \in \mathrm{Aut}_{\mathcal{F}}(S)$, $n = 0$ and the statement holds. Thus, we may assume that $Q \lneq S$.

Suppose that R is fully normalized; then by Theorem 1.20 R is fully automized. There exists $\alpha \in \mathrm{Aut}_{\mathcal{F}}(R)$ such that $\mathrm{Aut}_S(Q)^{\phi\alpha} \leq \mathrm{Aut}_S(R)$. Set $\bar{\phi} = \phi\alpha$: then $N_{\bar{\phi}} = N_S(Q)$. Since $\phi^{-1}\bar{\phi} \in \mathrm{Aut}_{\mathcal{F}}(R)$, there exists $\chi \in \mathrm{Aut}_{\mathcal{F}}(R)$ such that $\phi\chi = \bar{\phi}$. Thus, the $\mathcal{F}$-isomorphism $\phi\chi$ extends to

$\overline{\phi\chi} : N_S(Q) \to S$. Since $Q \lneq N_S(Q)$, we may apply the inductive hypothesis to $\overline{\phi\chi}$: this means that $\overline{\phi\chi}$, and hence also $\phi\chi$, decomposes as in (1.3.1).

It remains to show that χ decomposes as in (1.3.1), since then also $\phi = (\phi\chi)\chi^{-1}$ decomposes as in (1.3.1). If R is not centric, then $RC_S(R) \gneq R$ and by the inductive hypothesis $\bar{\chi}$ (which extends χ to $RC_S(R)$) decomposes as in (1.3.1); it follows that also χ decomposes in such a way.

Otherwise, by [12, Proposition 4.48], there exist two sequences of subgroups

$$\mathrm{Aut}_S(R) = A_1, A_2, ..., A_n = \mathrm{Aut}_S(R)^\chi$$

and $B_1, ..., B_{n-1}$ such that:

1. $B_i \leq A_i, A_{i+1}$, for every $i < n$;

2. $\mathrm{Aut}_R(R) < B_i$ for every i.

We may replace the groups A_i with Sylow p-subgroups of $\mathrm{Aut}_{\mathcal{F}}(R)$ containing each A_i: then for every i there exists an element $\chi_i \in \mathrm{Aut}_{\mathcal{F}}(R)$ such that $A_1^{\chi_i} = A_i$ (where χ_1 is the identity and $\chi_{n-1} = \chi$). Setting $\theta_i = \chi_{i+1}\chi_i^{-1}$, it follows that $\chi_{i+1} = \theta_i\theta_{i-1}...\theta_1$ for every $1 \leq i < n-1$.

For every i, we claim that $R \lneq N_{\theta_i}$, so that every θ_i extends to a proper overgroup of R and hence decomposes as in (1.3.1): since the composition of the θ_i is χ, then this claim would imply that also χ decomposes as in (1.3.1). Since

$$\begin{aligned}(N_{\theta_i}/Z(R))^{\chi_{i+1}} &= (\mathrm{Aut}_S(R) \cap \mathrm{Aut}_S(R)^{\theta_i^{-1}})^{\chi_{i+1}} \\ &= A_{i+1} \cap A_i \geq B_i > \mathrm{Aut}_R(R),\end{aligned}$$

one has that $N_{\theta_i}/Z(R)$ properly contains $\mathrm{Aut}_R(R) = N_S(R)/Z(R)$. Thus, $R \lneq N_{\theta_i}$, and the claim holds.

Finally, we remove the assumption that R is fully normalized. Let P be a fully normalized subgroup of S, and let $\nu : R \to P$ be a $\mathcal{F}$-isomorphism. Then ν and $\phi\nu$ have a decomposition of the required form. It follows that ϕ decomposes has in (1.3.1), and this completes the proof. □

As a consequence of Alperin's Fusion Theorem, in order to study saturated fusion systems on a group S, one may reduce to study $\mathcal{F}$-essential and fully normalized subgroups of S, and their $\mathcal{F}$-automorphism groups.

1.4 Reduced Fusion Systems

Among all saturated fusion systems, there are some with peculiar characteristics, which will turn out to be useful in our study.

Definition 1.29. Let $\mathcal{F}$ be a fusion system on S, and let H be a subgroup of S. We say that H is *normal* in $\mathcal{F}$ (and we write $H \trianglelefteq \mathcal{F}$) if $H \trianglelefteq S$, and every morphism $\phi \in \mathrm{Hom}_{\mathcal{F}}(R, P)$ extends to a morphism $\beta \in \mathrm{Hom}_{\mathcal{F}}(HR, HP)$, such that $\beta_{|H} \in \mathrm{Aut}_{\mathcal{F}}(H)$.

Definition 1.30. Let $\mathcal{F}$ be a saturated fusion system on S. The largest subgroup of S that is normal in $\mathcal{F}$ is denoted by $O_p(\mathcal{F})$. We say that $\mathcal{F}$ is *reduced* if $O_p(\mathcal{F}) = 1$.

The following comes from [12, Exercise 9.3].

Proposition 1.31. *Let $\mathcal{F}$ be a saturated fusion system on S. Suppose S has a unique $\mathcal{F}$-essential subgroup H. Then $H \trianglelefteq \mathcal{F}$.*

Proof. First of all, $H \trianglelefteq S$ by Remark 1.27. In order to prove the statement, we need to show that for every homomorphism $\phi \in \mathrm{Hom}_{\mathcal{F}}(R, P)$ there exists a homomorphism $\beta \in \mathrm{Hom}_{\mathcal{F}}(HR, HP)$, such that $\beta_{|R} = \phi$ and $\beta_{|H} \in \mathrm{Aut}_{\mathcal{F}}(H)$.

We claim that every $\mathcal{F}$-homomorphism $\alpha\colon H \to Q$, for $Q \leq S$, normalizes H. Indeed, $H\alpha \simeq H$, so that $H\alpha$ is $\mathcal{F}$-essential by Remark 1.27; thus $\alpha(H) = H$ by hypothesis.

By Alperin's Fusion Theorem (cf. Theorem 1.28), there exist

- a sequence of $\mathcal{F}$-isomorphic subgroups $R = Q_0, Q_1, \ldots, Q_{n+1} = P$ contained in H;
- a sequence of automorphisms $\phi_i \in \mathrm{Aut}_{\mathcal{F}}(H)$ such that $Q_{i-1}\phi_i = Q_i$;
- a $\mathcal{F}$-automorphism ψ of S mapping Q_n to $R\phi$;

such that $(\phi_1\phi_2 \cdots \phi_n\psi\iota)_{|R} = \phi$, where $\iota\colon R\phi \hookrightarrow P$. Set $\beta = \phi_1\phi_2 \cdots \phi_n\psi\iota$. Then $\beta \in \mathrm{Hom}_{\mathcal{F}}(HR, HP) = \mathrm{Aut}_{\mathcal{F}}(H)$, and $\beta_{|R} = \phi$. □

1.5 Surjectivity Property

Now we introduce the surjectivity property, which provides a criterion to extend certain $\mathcal{F}$-automorphisms between subgroups of S.

If R is a subgroup of S with $Q \leq R \leq N_S(Q)$, we denote by $\mathrm{Aut}_{\mathcal{F}}(Q \leq R)$ the set of all $\mathcal{F}$-automorphisms of R that restrict to automorphisms of Q.

Remark 1.32. Let Q be a subgroup of S, let R such that $QC_S(Q) \leq R \leq N_S(Q)$, and let $\alpha \in \mathrm{Aut}_{\mathcal{F}}(Q \leq R)$. Obviously, $\alpha_{|Q} \in \mathrm{Aut}_{\mathcal{F}}(Q)$ by definition. Let $r \in R$: the conjugation map c_r restricted to Q lies in $\mathrm{Aut}_R(Q)$. Since the group $\mathrm{Inn}(R)$ is normal in $\mathrm{Aut}_{\mathcal{F}}(R)$, one has that $\alpha^{-1}c_r\alpha \in \mathrm{Inn}(R)$. Hence $(\alpha^{-1}c_r\alpha)_{|Q} \in \mathrm{Aut}_R(Q)$, and the restriction map

$$\mathrm{res} : \mathrm{Aut}_{\mathcal{F}}(Q \leq R) \to N_{\mathrm{Aut}_{\mathcal{F}}(Q)}(\mathrm{Aut}_R(Q))$$

is well defined.

Definition 1.33. Let $\mathcal{F}$ be a fusion system on S, and let Q be a subgroup of S. We say that Q has the *surjectivity property* for $\mathcal{F}$ if, for every subgroup R of S such that $QC_S(Q) \leq R \leq N_S(Q)$, the restriction map res is surjective.

Thus, if Q has the surjectivity property and $QC_S(Q) \leq R \leq N_S(Q)$, any $\mathcal{F}$-automorphism ϕ of Q that normalizes $\mathrm{Aut}_R(Q)$ extends to some $\mathcal{F}$-automorphisms ψ of R.

Remark 1.34. If Q is receptive, then by definition Q has the surjectivity property.

Lemma 1.35. *Let $\mathcal{F}$ be a saturated fusion system on S, and let Q be a fully $\mathcal{F}$-normalized subgroup of S. Then Q has the surjectivity property.*

Proof. It follows by Remark 1.34 and Theorem 1.20. $\square$

Chapter 2

Sylow 3-subgroups of the McLaughlin group Mc

Let S be a Sylow 3-subgroup of the McLaughlin sporadic group Mc. The main goal of the Thesis is to study the reduced fusion systems on S. Therefore, the first step will be the study of the group structure of S. In particular, we produce two presentations of S (which we will use in different situations), and we study the structure of some relevant subgroups of S.

Our references on the structure of the group Mc are [1, 9, 17].

2.1 The group S

The McLaughlin group Mc has order $2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$. Thus, S has order 3^6. Moreover, a maximal subgroup of Mc is isomorphic to the group

$$H = (C_3 \times C_3 \times C_3 \times C_3) \rtimes M_{10},$$

where M_{10} is the Mathieu group of degree 10. Since the order of M_{10} is $2^4 \cdot 3^2 \cdot 5$, the group H contains a Sylow 3-subgroup of Mc.

The Mathieu group M_{10} contains the alternating group A_6. Since the order of A_6 is $2^3 \cdot 3^2 \cdot 5$, the group

$$P = (C_3 \times C_3 \times C_3 \times C_3) \rtimes A_6$$

contains a Sylow 3-subgroup of Mc. Thus, we may assume, without loss of generality, that S is contained in P. The alternating group A_6 has a unique representation of dimension 4 over the field with 3 elements. Hence, such representation determines the structure of P.

The alternating group A_6 has the following presentation:

$$A_6 = \left\langle \sigma_1, \sigma_2 \mid \sigma_1^2 = \sigma_2^4 = (\sigma_1\sigma_2)^5 = (\sigma_1\sigma_2^2)^5 = 1 \right\rangle,$$

where one may choose $\sigma_1 = (2,3)(4,5)$, and $\sigma_2 = (1,2,3,4)(5,6)$. Moreover, a Sylow 3-subgroup of A_6 is $\langle (2,6,3),(1,5,4)\rangle \simeq C_3 \times C_3$, where

$$\begin{aligned}(2,6,3) &= \sigma_1\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2^3\sigma_1\sigma_2\sigma_1\sigma_2^3\sigma_1\sigma_2^3 =: \tau_1 \quad \text{and}\\ (1,5,4) &= (2,6,3)^{\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2^3} =: \tau_2.\end{aligned}$$

Then S is isomorphic to $(C_3 \times C_3 \times C_3 \times C_3) \rtimes \langle \tau_1, \tau_2\rangle$.

Let $\varphi : A_6 \to GL_4(3)$ be the aforementioned representation of A_6. Then

$$\sigma_1\varphi = \begin{pmatrix} 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{pmatrix}, \quad \text{and} \quad \sigma_2\varphi = \begin{pmatrix} -1 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & -1 & 0 & 0\\ 0 & 1 & 1 & -1\end{pmatrix}.$$

Moreover, explicit computations show that

$$\tau_1\varphi = \begin{pmatrix} 1 & 1 & -1 & -1\\ 1 & 0 & 0 & 0\\ 0 & -1 & -1 & 1\\ -1 & 1 & 0 & 1\end{pmatrix}, \quad \text{and} \quad \tau_2\varphi = \begin{pmatrix} 1 & -1 & 0 & -1\\ 0 & 0 & 0 & -1\\ -1 & 0 & 0 & 0\\ 1 & -1 & 1 & 0\end{pmatrix}.$$

Identify the group $V = C_3 \times C_3 \times C_3 \times C_3$ with the $\mathbb{F}_3$-vector space of dimension 4, and let $\{w_1, w_2, w_3, w_4\}$ be the canonical basis, where we consider the w_i as row vectors. With a harmless abuse of notation, we identify τ_1 with $\tau_1\varphi$, and τ_2 with $\tau_2\varphi$. Thus, the action of the group $\langle \tau_1, \tau_2\rangle$ over V is given by multiplication row vector$\times$matrix. In particular, one has that

$$\begin{aligned} w_1^{\tau_1} &= w_1w_2w_3^{-1}w_4^{-1} & \qquad w_1^{\tau_2} &= w_1w_2^{-1}w_4^{-1}\\ w_2^{\tau_1} &= w_1 & w_2^{\tau_2} &= w_4^{-1}\\ w_3^{\tau_1} &= w_2^{-1}w_3^{-1}w_4 & w_3^{\tau_2} &= w_1^{-1}\\ w_4^{\tau_1} &= w_1^{-1}w_2w_4 & w_4^{\tau_2} &= w_1w_2^{-1}w_3 \end{aligned}$$

(where we keep the multiplicative notation). Then S has the following presentation: $S = \langle w_1, w_2, w_3, w_4, \tau_1, \tau_2 \mid \mathcal{R}\rangle$, where $\mathcal{R}$ is the set of defining relations

$$\mathcal{R} = \left\{ \begin{array}{l} w_1^3 = w_2^3 = w_3^3 = w_4^3 = \tau_1^3 = \tau_2^3 = 1,\\ [w_1, w_2] = [w_1, w_3] = [w_1, w_4] = 1\\ [w_2, w_3] = [w_2, w_4] = [w_3, w_4] = [\tau_1, \tau_2] = 1,\\ [w_1, \tau_1] = w_2w_3^{-1}w_4^{-1},\ [w_2, \tau_1] = w_1w_2^{-1}\\ [w_3, \tau_1] = w_2^{-1}w_3w_4,\ [w_4, \tau_1] = w_1^{-1}w_2,\\ [w_1, \tau_2] = w_2^{-1}w_4^{-1},\ [w_2, \tau_2] = w_2^{-1}w_4^{-1},\\ [w_3, \tau_2] = w_1^{-1}w_3^{-1},\ [w_4, \tau_2] = w_1w_2^{-1}w_3w_4^{-1} \end{array} \right\}. \tag{2.1.1}$$

From now on, let $A = \langle w_1, w_2, w_3, w_4\rangle$ and $B = \langle \tau_1, \tau_2\rangle$; hence

$$S = A \rtimes B.$$

2.2 A second presentation of S

We use the previous presentation (2.1.1) of S to show that S is the central product of two copies of the extraspecial group P of order 27 and exponent 3 (whose centers we will identify), extended by an automorphism of order 3 normalizing the two copies of P and acting on them "in the same way". Thus, we shall produce a new presentation of S.

Notice that there is (up to isomorphism) a unique extraspecial group of order 27 and exponent 3. It has the following presentation:

$$P = \langle x, y, z \mid x^3 = y^3 = z^3 = 1, [x, y] = z, [x, z] = [y, z] = 1\rangle.$$

Proposition 2.1. *S is isomorphic to the group $T = \langle x, y, z, a, b, t \mid \mathcal{P}\rangle$, where $\mathcal{P}$ is the set of defining relations*

$$\mathcal{P} = \left\{ \begin{array}{l} x^3 = y^3 = z^3 = a^3 = b^3 = t^3 = 1, \\ [x, y] = [a, b] = z \\ [x, z] = [y, z] = [a, z] = [b, z] = [t, z] = 1 \\ [x, a] = [x, b] = [y, a] = [y, b] = 1 \\ [x, t] = [a, t] = 1,\ [y, t] = xz,\ [b, t] = az \end{array} \right\}. \tag{2.2.1}$$

In particular, S is the central product of two copies of the extraspecial group of order 27 and exponent 3, extended by an automorphism of order 3.

Proof. Clearly, T is isomorphic to the central product of two extraspecial groups

$$P_1 = \langle x, y, z \mid x^3 = y^3 = z^3 = 1, [x, y] = z, [x, z] = [y, z] = 1\rangle$$

and

$$P_2 = \langle a, b, c \mid a^3 = b^3 = c^3 = 1, [a, b] = c, [a, c] = [b, c] = 1\rangle,$$

(with respect to the isomorphism $\varphi : Z(P_1) \to Z(P_2)$ defined by $[x, y]\varphi = [a, b]$), extended with an automorphism t of order 3 normalizing P_1 and P_2, and acting on them "in the same way".

Explicit computations show that $\langle x, z, a, t\rangle$ is elementary abelian of order 3^4, and that it is normal in T. Thus,

$$T = \langle x, z, a, t\rangle \rtimes \langle y, b\rangle.$$

In the notation of (2.1.1), let

$$v = w_1 w_3^{-1} w_4^{-1},\ w = w_1^{-1} w_3^{-1},\ \text{and } u = w_2.$$

One verifies that

$$S = \langle v, w, u, v^{\tau_1} v^{-1}\rangle \rtimes \langle \tau_1, \tau_2\rangle.$$

Direct computations show that $v \in C_A(\tau_2)$, $w \in C_A(\tau_1)$, $v^{\tau_1}v^{-1} = w^{\tau_2}w^{-1} \in Z(S)$, $u^{\tau_1}u^{-1} = (v^{-1})^{\tau_1}$, and $u^{\tau_2}u^{-1} = (w^{-1})^{\tau_2}$. Then the map $\alpha : T \to S$, defined on the generators of T by:

$$\begin{aligned}
x\alpha &= v,\\
y\alpha &= \tau_1,\\
z\alpha &= v^{\tau_1}v^{-1},\\
a\alpha &= w,\\
b\alpha &= \tau_2,\\
t\alpha &= u,
\end{aligned}$$

and extended by linearity to the whole T is an isomorphism, and $S \simeq T$. $\square$

Hence we have produced an other presentation (2.2.1) of S.

From now on, set

$$E = \langle x, y, a, b\rangle.$$

Clearly, E is the extraspecial group of order 3^5 and exponent 3, and

$$S = E \rtimes \langle t\rangle.$$

Here we state the formula for the commutator of two elements of S, which will turn to be useful in the sequel.

Lemma 2.2. *Let $\varphi : \mathbb{Z} \to \mathbb{Z}$ be the map defined by $\varphi(m) = \lfloor 3n/2 \rfloor$, where n is the rest of the division of m by 3, and $\lfloor \alpha \rfloor$ denotes the low integer part of α, and let $h, k \in S$. Then, in the notation of* (2.2.1)*, one may write in a unique way*

$$h = x^{n_1}y^{n_2}a^{n_3}b^{n_4}t^{n_5}z^{n_6},$$

and

$$k = x^{m_1}y^{m_2}a^{m_3}b^{m_4}t^{m_5}z^{m_6},$$

for some $n_i, m_i \in \{0, 1, 2\}$. Then

$$\begin{aligned}
[h, k] &= [x^{n_1}y^{n_2}a^{n_3}b^{n_4}t^{n_5}z^{n_6}, x^{m_1}y^{m_2}a^{m_3}b^{m_4}t^{m_5}z^{m_6}] =\\
&= x^{n_2m_5-n_5m_2} \cdot a^{n_4m_5-n_5m_4} \cdot z^{l},
\end{aligned} \tag{2.2.2}$$

where

$$\begin{aligned}
l = {} & m_5\varphi(n_4) + m_5\varphi(n_2) - n_5\varphi(m_4) - n_5\varphi(m_2)+\\
& + n_3m_4 - n_4m_3 + n_1m_2 - n_2m_1.
\end{aligned}$$

Proof. Direct computations. $\square$

In the following result we list some further properties of the group S.

Lemma 2.3. *Let S be as above. Then, in the notation of* (2.2.1)*, the following hold:*

i $A = \langle x, a, t, z\rangle$*,* $S' = A \cap E = \langle x, a, z\rangle$*, and* $Z(S) = \langle z\rangle$*;*

ii. $Z(S) = S^3$*;*

iii. S *has exponent 9;*

iv. $S' = [S, A]$*;*

v. $S' = \Phi(S) = Z_2(S)$*;*

vi. $m_3(S) = 4$.

Proof. From Presentation (2.2.1), one see that

$$A = \langle x, a, t, z\rangle,$$

and

$$S' = \langle x, a, z\rangle.$$

Let $h = x^{n_1}y^{n_2}a^{n_3}b^{n_4}t^{n_5}z^{n_6} \in S$, with $n_i \in \{0, 1, 2\}$. Direct computations and Formula (2.2.2) implies that $h \in Z(S)$ if, and only if, $n_1 = n_2 = n_3 = n_4 = n_5 = 0$. Thus, $Z(S) = \langle z\rangle$, and statement *i* holds.

Now, let $k = x^{m_1}y^{m_2}a^{m_3}b^{m_4}t^{m_5}z^{m_6} \in S$, with $m_i \in \{0, 1, 2\}$. By direct computations and Formula (2.2.2),

$$h \cdot k = x^{-n_1-m_1-n_5m_2}y^{n_2+m_2}a^{-n_3-m_3-n_5m_4}b^{n_4+m_4}t^{n_5+m_5}z^{\alpha},$$

where

$$\alpha = n_4n_5m_4 + n_5m_4^2 + n_2n_5m_2 + n_5m_2^2 - n_2m_1 - n_4m_3 + n_6 + m_6 \\ - n_5\varphi(m_2) - n_5\varphi(m_4).$$

In particular,

$$h^3 = z^{2n_5(\varphi(n_2)+\varphi(n_4)+\varphi(2n_2)+\varphi(2n_4))} \in Z(S),$$

and statements *ii* and *iii* hold.

One has that

$$S' = [S, S] = [BA, BA] = [B, BA]^A \cdot [A, BA] = [A, B];$$

moreover, $[A, S] = [A, AB] = [A, B]$. Hence, $S' = [A, S]$. This shows statement *iv*.

Since $S^3 = Z(S)$ by statement *ii*, it follows that $\Phi(S) = S'$. Direct computations with Formula (2.2.2) imply that $Z_2(S) = S'$, and this yields statement *v*.

Finally, since $A \leq S$ and $m_3(\mathrm{Mc}) = 4$ (see [17, Table 5.6.1]), claim *vi* follows and the proof is completed. □

Lemma 2.4. *For every $x \in S \smallsetminus A$, the order of $C_A(x)$ is 3^2. Moreover, $C_A(B) = Z(S)$.*

Proof. Let $w \in A$, and let $\tau \in B$. Thus, $w \in C_A(\tau)$ if, and only if, $w^\tau = w$. Explicit computations show that the elements of B are:

$$\tau_1 = \begin{pmatrix} 1 & 1 & 2 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 1 \\ 2 & 1 & 0 & 1 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 2 \\ 2 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 2 & 2 & 1 \end{pmatrix},$$
$$\tau_4 = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 2 & 1 & 2 & 0 \\ 2 & 1 & 0 & 1 \\ 0 & 2 & 0 & 0 \end{pmatrix}, \quad \tau_5 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 2 & 2 \\ 1 & 2 & 1 & 0 \\ 2 & 0 & 1 & 2 \end{pmatrix}, \quad \tau_6 = \begin{pmatrix} 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 0 \\ 2 & 0 & 1 & 2 \\ 2 & 0 & 0 & 0 \end{pmatrix},$$
$$\tau_7 = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 1 & 2 & 0 & 2 \\ 2 & 2 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \tau_8 = \begin{pmatrix} 2 & 1 & 2 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 2 & 2 & 1 & 1 \end{pmatrix}, \quad \tau_9 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We may write $w \in A$ as $w = (a_1, a_2, a_3, a_4)$, where $a_i \in \mathbb{F}_3$. Explicit computations show that $w \in C_A(\tau_1) \Leftrightarrow a_1 = a_3$ and $a_2 = a_4$. Thus, one has that

$$C_A(\tau_1) = \{(a_1, a_2, a_1, a_2) \mid a_1, a_2 \in \mathbb{F}_3\},$$

and $|C_A(\tau_1)| = 9$. Similar computations show that $|C_A(\tau_i)| = 9$, for every $i \in \{1, \dots, 8\}$.

Now, let $x \in S \smallsetminus A$, and write $x = v\tau$, with $v \in A$ and $\tau \in B$. Since A is abelian, for every $w \in A$ one has that

$$[x, w] = [v\tau, w] = [v, w]^\tau \cdot [\tau, w] = [\tau, w].$$

Hence $C_A(x) = C_A(\tau)$, and $|C_A(x)| = 3^2$ by the previous part of the proof.

Let $w \in C_A(B)$, and let $x \in S$. Again, we write $x = v\tau$, with $v \in A$ and $\tau \in B$. One has that

$$wx = wv\tau = vw\tau = v\tau w = xw.$$

Then $C_A(B) \leq Z(S)$. Since $|Z(S)| = 3$ by Lemma 2.3, one obtains $C_A(B) = Z(S)$. □

Lemma 2.5. *A is the unique abelian subgroup of S of order at least 3^4; in particular, A is the Thompson subgroup of S (for the prime 3), and A is characteristic in S. Moreover, $A = C_S(A)$.*

Proof. Let P be an abelian subgroup of S such that $|P| \geq 3^4$, and suppose that $P \neq A$. Let $k \in P \smallsetminus A$. Since P is abelian,

$$P \cap A = C_P(k) \cap A = C_A(k).$$

Hence $|P \cap A| = 3^2$ by Lemma 2.4, and $PA = S$. Then $P \cap A \leq Z(S)$, as A and P are abelian, a contradiction, since $|Z(S)| = 3$ and $|P \cap A| = 3^2$. Then A is the unique abelian subgroup of S of order at least 3^4. Furthermore, since S has 3-rank equal to 4 and $A \simeq C_3 \times C_3 \times C_3 \times C_3$, $A = J(S)$ and A is characteristic in S.

Since A is abelian, one has that $C_S(A) \geq A$. Suppose that $C_S(A) \gneq A$. Thus, there exists $x \in C_S(A) \setminus A$. Then $\langle A, x \rangle$ is an abelian subgroup of S with order greater then 3^4, and this is a contradiction. $\square$

Lemma 2.6. *Let P be an abelian subgroup of S of order 3^3, not contained in A. The following hold:*

i. if $|P \cap A| = 3^2$, then $P \cap A = C_A(h)$ for some $h \notin A$, and $P = \langle C_A(h), h \rangle$;

ii. if $|P \cap A| = 3$, then $P \cap A = Z(S)$;

iii. P is self-centralizing, i.e., $C_S(P) = P$.

Proof. Set $P_0 = P \cap A$. The order of P_0 is either 3^2 or 3. We proceed according to $|P_0|$.

1. Suppose $|P_0| = 3^2$.

$$\begin{tikzcd}
 & S \arrow[d, dash, "3"] & \\
 & PA \arrow[dl, dash] \arrow[dr, dash, "3"] & \\
P \arrow[dr, dash, "3"'] & & A \arrow[dl, dash] \\
 & P_0 \arrow[d, dash, "3^2"] & \\
 & 1 &
\end{tikzcd}$$

Take $h \in P \setminus A$. In particular, $h \notin P_0$. Thus, $P = \langle P_0, h \rangle$. Moreover, $P_0 \leq C_A(h)$, as $P_0 \leq A$ is abelian. Then by Lemma 2.4, $P_0 = C_A(h)$, and claim i follows. Moreover, $C_S(P) \geq P$, since P is abelian. Suppose that $C_S(P) \gneq P$. Since $C_S(P) \cap A = C_A(h) = P_0$, one has that $|C_S(P)| = 3^4$ and $S = C_S(P)A$. Hence $P_0 = C_A(h) \leq Z(S)$, a contradiction, since $|P_0| = 3^2$ and $|Z(S)| = 3$.

$$
\begin{tikzcd}[column sep=small]
 & S \arrow[dl, dash, "3^2"'] \arrow[dr, dash, "3^2"] & \\
C_S(P) \arrow[dr, dash] & & A \arrow[dl, dash] \\
 & P_0 \arrow[d, dash, "3^2"] & \\
 & 1 &
\end{tikzcd}
$$

Therefore $C_S(P) = P$ and statement *iii* holds.

2. Suppose $|P_0| = 3$.

$$
\begin{tikzcd}[column sep=small]
 & S = PA \arrow[dl, dash] \arrow[dr, dash, "3^2"] & \\
P \arrow[dr, dash, "3^2"'] & & A \arrow[dl, dash] \\
 & P_0 \arrow[d, dash, "3"] & \\
 & 1 &
\end{tikzcd}
$$

Since $PA = S$ and A and P are abelian, $P_0 \leq Z(S)$. Since $|Z(S)| = |P_0| = 3$, one obtains that $P_0 = Z(S)$, and claim *ii* follows. Moreover, since A is abelian, one has that $C_S(P) \cap A \leq Z(S)$. Hence $C_S(P) \cap A = Z(S)$ and $|C_S(P)| \leq 3^3$. As $P \leq C_S(P)$, claim *iii* follows, and the proof is completed. $\square$

Lemma 2.7. *Let P be an abelian subgroup of S of order 3^3, not contained in A. Then P is not normal in S.*

Proof. Set $P_0 = P \cap A$. We proceed with a case-by-case analysis.

1. Assume $|P_0| = 3$. Then $P = \langle P_0, h, k \rangle$, for some $h, k \notin A$, and $P_0 = Z(S)$ (see Lemma 2.6). Hence

$$g \in N_A(P) \Leftrightarrow [g,h] \in A \cap P = Z(S) \quad \text{and} \quad [g,k] \in A \cap P = Z(S).$$

Moreover, one has that

$$[g,h] \in Z(S) \Leftrightarrow [gZ(S), hZ(S)] = 1 \Leftrightarrow gZ(S) \in C_{A/Z(S)}(hZ(S)).$$

By a similar argument,

$$[g,k] \in Z(S) \Leftrightarrow gZ(S) \in C_{A/Z(S)}(kZ(S)).$$

Since $S = PA$, one has that $S/Z(S) = \langle A/Z(S), hZ(S), kZ(S)\rangle$. Then

$$\begin{aligned} g \in N_A(P) \quad \Leftrightarrow \quad & gZ(S) \in C_{A/Z(S)}(hZ(S)) \cap C_{A/Z(S)}(kZ(S)) \leq \\ & \leq Z(S/Z(S)) = Z_2(S)/Z(S). \end{aligned}$$

Thus, since $Z_2(S) \leq N_A(P)$, it follows that $N_A(P) = Z_2(S) \lneq A$, and the statement holds.

2. Assume $|P_0| = 3^2$. Then $P = \langle P_0, h\rangle$, for some $h \notin A$, and $P_0 = C_A(h)$ (see Lemma 2.6). Hence $PA = \langle A, h\rangle$, and $|PA| = 3^5$. Since $P_0 = C_A(h)$, one has that $P_0 \leq Z(PA)$. Let $k \in Z(PA)$: in particular, $k \in C_S(P) \cap C_S(A)$. By Lemma 2.6, $C_S(P) = P$. Then Lemma 2.4 implies that $k \in P \cap A = C_A(h)$. Hence $C_A(h) = Z(PA)$. Moreover, $P/P_0 = \langle hP_0\rangle$ and $PA/P_0 = \langle A/P_0, hP_0\rangle$. Then

$$\begin{aligned} g \in N_A(P) \quad & \Leftrightarrow \quad [g,h] \in A \cap P = P_0 \\ & \Leftrightarrow \quad [gP_0, hP_0] = 1 \\ & \Leftrightarrow \quad gP_0 \in C_{A/P_0}(hP_0) \leq Z(PA/P_0) = \frac{Z_2(PA)}{P_0}. \end{aligned}$$

Thus, $N_A(P) \leq Z_2(PA)$. Since $Z_2(S) \leq N_A(P)$, one has that $Z_2(S) \leq Z_2(PA)$. Hence, $3^3 \leq |Z_2(PA)| \leq 3^5$.

Let $w \in A$; in the notation of (2.2.1), we may write

$$w = x^{n_1} y^0 a^{n_3} b^0 t^{n_5} z^{n_6} = x^{n_1} a^{n_3} t^{n_5} z^{n_6},$$

and

$$h = x^{m_1} y^{m_2} a^{m_3} b^{m_4} t^{m_5} z^{m_6},$$

for some $n_i, m_i \in \{0,1,2\}$. Since $h \notin A$, $(m_2, m_4) \neq (0,0)$. Then Formula (2.2.2) implies that

$$[w,h] = x^{-n_5 m_2} \cdot a^{-n_5 m_4} \cdot z^{-n_5[3m_4/2] - n_5[3m_2/2] + n_3 m_4 + n_1 m_2},$$

and

$$[[w,h],h] = z^{-n_5 m_4^2 - n_5 m_2^2}.$$

It follows that $[w,h] \in C_A(h)$ if, and only if, $-n_5(m_4^2 + m_2^2) = 3n$, for some $n \in \mathbb{Z}$. If $(m_2, m_4, n_5) = (0,1,1)$, then $-n_5(m_4^2 + m_2^2) \neq 3n$, for all $n \in \mathbb{Z}$, and $[w,h] \notin C_A(h)$.

Suppose $|Z_2(PA)| = 3^5$ or $|Z_2(PA)| = 3^4$. Thus, PA/P_0 is abelian, and therefore $[PA, PA] \leq P_0 = C_A(h)$, in contradiction with the previous part of the proof. Then $|Z_2(PA)| = 3^3$, $Z_2(PA) = Z_2(S)$, $N_A(P) = Z_2(S) \lneq A$, and the statement holds. □

Lemma 2.8. *$E \cup A$ covers the set of all elements of S of order 3.*

Proof. Let $h \in S$; we may write (in a unique way) $h = x^{n_1}y^{n_2}a^{n_3}b^{n_4}t^{n_5}z^{n_6}$, for some $n_i \in \{0, 1, 2\}$. One has that $h^3 = z^{2n_5\beta}$, where

$$\beta = \varphi(n_2) + \varphi(n_4) + \varphi(2n_2) + \varphi(2n_4)$$

(see the proof of Lemma 2.3). Thus, $h^3 = 1$ if, and only if, $3 \mid 2n_5$ or $3 \mid \beta$. In the first case, $3 \mid n_5$, then $n_5 = 0$ and $h = x^{n_1}y^{n_2}a^{n_3}b^{n_4}z^{n_6} \in E$; in the second case, $(n_2, n_4) = (0, 0)$, then $h = x^{n_1}a^{n_3}t^{n_5}z^{n_6} \in A$, and the proof is completed. $\square$

Lemma 2.9. *Let P be an elementary abelian subgroup of S of order 3^3. Then $P \leq E$ or $P \leq A$.*

Proof. One may write (in a unique way) every element h_1 of E and h_2 of A as

$$h_1 = x^{m_1}y^{m_2}a^{m_3}b^{m_4}z^{m_5} \qquad \text{and} \qquad h_2 = x^{n_1}a^{n_2}t^{n_3}z^{n_4}, \tag{2.2.3}$$

for some $m_i, n_i \in \{0, 1, 2\}$.

Suppose that $P \nleq E$ and $P \nleq A$. By Lemma 2.6, $Z(S) \leq P$. We claim that $|P \cap E| = 3^2$: indeed, $|P \cap E| > 3$, as otherwise the order of the product EP would be too big; on the other hand $|P \cap E| < 3^3$, as $P \nleq E$. In particular, one may pick an element $s \in E$ such that $P \cap E = \langle s, z \rangle$.

Now pick an element r such that $P = \langle r, s, z \rangle$. Thus, one has that $r \notin E$. By Lemma 2.8, one knows that $E \cup A$ covers the set of all elements of S of order 3. Hence, necessarily $r \in A$. On the other hand, this implies that $s \notin A$. Therefore, by (2.2.3) one may write

$$r = x^{n_1}a^{n_2}t^{n_3}z^{n_4} \qquad \text{and} \qquad s = x^{m_1}y^{m_2}a^{m_3}b^{m_4}z^{m_5}, \tag{2.2.4}$$

for some $n_i, m_i \in \{0, 1, 2\}$. By Formula (2.2.2), from (2.2.4) one obtains

$$[r, s] = x^{-n_3m_2} \cdot a^{-n_3m_4} \cdot z^{-n_3\varphi(m_4)-m_2\varphi(n_3)+n_2m_4+n_1m_2}.$$

Hence,

$$[r, s] = 1 \Leftrightarrow \begin{cases} 3 \,|\, n_3m_2 \\ 3 \,|\, n_3m_4 \\ 3 \,|\, l \end{cases}$$

where $l = -n_3\varphi(m_4) - m_2\varphi(n_3) + n_2m_4 + n_1m_2$.

Since $r \in A \setminus E$, $n_3 \neq 0$. Similarly, $(m_2, m_4) \neq (0, 0)$, as $s \in E \setminus A$. If $3 \,|\, n_3m_2$, then either $(n_3, m_2) = (1, 0)$ or $(n_3, m_2) = (2, 0)$: in both cases, we get that $[r, s] \neq 1$, since $3 \,|\, n_3m_4$ if, and only if, $m_4 = 0$, in contradiction with the condition $(m_2, m_4) \neq (0, 0)$. This is a contradiction, since P is abelian. This yields the claim. $\square$

Lemma 2.10. *The derived subgroup S' is the unique elementary abelian normal subgroup of S of order 3^3.*

Proof. Let P be an elementary abelian normal subgroup of S of order 3^3. In particular, Lemma 2.7 implies that $P \leq A$. If $P \leq E$, then $P = E \cap A = \langle x, a, z\rangle = S'$. Suppose $P \nleq E$. Therefore there exists $h \in P$, such that we may write (in a unique way)

$$h = x^{n_1} a^{n_2} t^{n_3} z^{n_4},$$

for some $n_i \in \{0, 1, 2\}$, with $n_3 \neq 0$. Since $P \trianglelefteq S$, $[h, k] \in P$ for every $k \in S$, and direct computations show that x, a, z must be contained in P. Then $S' = \langle x, a, z\rangle \lneq P$, and $|P| > 3^3$, in contradiction with the hypothesis. It follows that $P = S'$, and the proof is completed. □

2.3 Maximal subgroups of S

In order to provide a complete description of the subgroups of S, we study the maximal subgroups of S. Every maximal subgroup H of S contains $\Phi(S)$, as $S/H \simeq C_3$ is elementary abelian. Hence the set of the subgroups of S of index 3 is in bijection with the set of the subgroups of $S/\Phi(S)$ of index 3.

First we determine the subgroups of $S/\Phi(S)$ of index 3. Notice that $S/\Phi(S)$ is a 3-dimensional vector space over the field with 3 elements. Recall that, following the notation of (2.2.1), $\Phi(S) = \langle x, a, z\rangle$. Hence

$$S/\Phi(S) = \langle y\Phi(S), b\Phi(S), t\Phi(S)\rangle.$$

Set $S/\Phi(S) = \bar{S}$, $y\Phi(S) = \bar{y}$, $b\Phi(S) = \bar{b}$, and $t\Phi(S) = \bar{t}$. Then

$$\bar{S} = \left\{x_1\bar{y} + x_2\bar{b} + x_3\bar{t}\right\}, \quad \text{where } x_i \in \mathbb{F}_3. \tag{2.3.1}$$

Let $\bar{H}$ be a subgroup of $\bar{S}$ of index 3. Then $\bar{H}$ is a 2-dimensional subspace of $\bar{S}$, and $\bar{H}$ is the kernel of a linear application $\varphi : \bar{S} \to \mathbb{F}_3$ (of rank 1). In particular, φ can be represented by a matrix $(\alpha, \beta, \gamma) \in \mathrm{Mat}_{1,3}(\mathbb{F}_3)$. Hence, in the notation of (2.3.1),

$$\bar{H} = \ker(\varphi) = \left\{x_1\bar{y} + x_2\bar{b} + x_3\bar{t} \in \bar{S} | \alpha x_1 + \beta x_2 + \gamma x_3 = 0\right\}.$$

The number of 2-dimensional subspaces $\bar{H}$ of $\bar{S}$ is the number of the equations $\alpha x_1 + \beta x_2 + \gamma x_3 = 0$ (modulo $\mathbb{F}_3^*$), in which $(\alpha, \beta, \gamma) \neq (0, 0, 0)$ (indeed in this case φ has rank 0). Hence there are $\dfrac{(3 \cdot 3 \cdot 3) - 1}{2} = 13$ subspaces of $\bar{S}$ of dimension 2. These subspaces $\bar{H}_i$ are the following:

1. if $x_1 = 0 \Rightarrow \bar{H}_1 = \{x_2\bar{b} + x_3\bar{t}\} = \langle \bar{b}, \bar{t}\rangle$;

2. if $x_2 = 0 \Rightarrow \bar{H_2} = \{x_1\bar{y} + x_3\bar{t}\} = \langle \bar{y}, \bar{t} \rangle$;

3. if $x_3 = 0 \Rightarrow \bar{H_3} = \{x_1\bar{y} + x_2\bar{b}\} = \langle \bar{y}, \bar{b} \rangle$;

4. if $x_1 + x_2 = 0 \Rightarrow x_2 = -x_1 \Rightarrow \bar{H_4} = \{x_1\bar{y} - x_1\bar{b} + x_3\bar{t}\} = \langle \bar{y}\bar{b^2}, \bar{t} \rangle$;

5. if $x_1 + x_3 = 0 \Rightarrow x_3 = -x_1 \Rightarrow \bar{H_5} = \{x_1\bar{y} + x_2\bar{b} - x_1\bar{t}\} = \langle \bar{y}\bar{t^2}, \bar{b} \rangle$;

6. if $x_2 + x_3 = 0 \Rightarrow x_3 = -x_2 \Rightarrow \bar{H_6} = \{x_1\bar{y} + x_2\bar{b} - x_2\bar{t}\} = \langle \bar{y}, \bar{b}\bar{t^2} \rangle$;

7. if $x_1+x_2+x_3 = 0 \Rightarrow x_3 = 2x_1+2x_2 \Rightarrow \bar{H_7} = \{x_1\bar{y}+x_2\bar{b}+2(x_1+x_2)\bar{t}\} = \{x_1\bar{y} + x_2\bar{b} + 2x_1\bar{t} + 2x_2\bar{t}\} = \langle \bar{y}\bar{t^2}, \bar{b}\bar{t^2} \rangle$;

8. if $x_1 + 2x_2 = 0 \Rightarrow x_1 = x_2 \Rightarrow \bar{H_8} = \{x_1\bar{y} + x_1\bar{b} + x_3\bar{t}\} = \langle \bar{y}\bar{b}, \bar{t} \rangle$;

9. if $x_1 + 2x_3 = 0 \Rightarrow x_1 = x_3 \Rightarrow \bar{H_9} = \{x_1\bar{y} + x_2\bar{b} + x_1\bar{t}\} = \langle \bar{y}\bar{t}, \bar{b} \rangle$;

10. if $x_2 + 2x_3 = 0 \Rightarrow x_2 = x_3 \Rightarrow \bar{H_{10}} = \{x_1\bar{y} + x_2\bar{b} + x_2\bar{t}\} = \langle \bar{y}, \bar{b}\bar{t} \rangle$;

11. if $x_1+x_2+2x_3 = 0 \Rightarrow x_3 = x_1+x_2 \Rightarrow \bar{H_{11}} = \{x_1\bar{y}+x_2\bar{b}+x_1\bar{t}+x_2\bar{t}\} = \langle \bar{y}\bar{t}, \bar{b}\bar{t} \rangle$;

12. if $x_1+2x_2+x_3 = 0 \Rightarrow x_2 = x_1+x_3 \Rightarrow \bar{H_{12}} = \{x_1\bar{y}+x_1\bar{b}+x_3\bar{b}+x_3\bar{t}\} = \langle \bar{y}\bar{b}, \bar{b}\bar{t} \rangle$;

13. if $2x_1+x_2+x_3 = 0 \Rightarrow x_1 = x_2+x_3 \Rightarrow \bar{H_{13}} = \{x_2\bar{y}+x_3\bar{y}+x_2\bar{b}+x_3\bar{t}\} = \langle \bar{y}\bar{b}, \bar{y}\bar{t} \rangle$.

Now we can determine the 13 subgroups H_i of S of order 3^5, preimages of the $\bar{H}_i$ in the projection of S on $S/\Phi(S)$:

1. $H_1 = \langle b, t, a, x, z \rangle$;

2. $H_2 = \langle y, t, a, x, z \rangle$;

3. $H_3 = \langle y, b, a, x, z \rangle$;

4. $H_4 = \langle yb^2, t, a, x, z \rangle$;

5. $H_5 = \langle yt^2, b, a, x, z \rangle$;

6. $H_6 = \langle y, bt^2, a, x, z \rangle$;

7. $H_7 = \langle yt^2, bt^2, a, x, z \rangle$;

8. $H_8 = \langle yb, t, a, x, z \rangle$;

9. $H_9 = \langle yt, b, a, x, z \rangle$;

10. $H_{10} = \langle y, bt, a, x, z \rangle$;

11. $H_{11} = \langle yt, bt, a, x, z \rangle$;

12. $H_{12} = \langle yb, bt, a, x, z \rangle$;

13. $H_{13} = \langle yb, yt, a, x, z \rangle$.

Notice that $E = H_3 = \langle y, b, a, x, z \rangle$. Direct calculations show that E is the unique maximal subgroup of S of exponent 3: thus, E is characteristic in S.

Chapter 3

$\mathcal{F}$-essential subgroups of S

By Alperin's fusion theorem, one knows that the first step to study a (saturated) fusion system on a p-group is to sort out the $\mathcal{F}$-essential subgroups of such group. Therefore, we proceed with the study of the $\mathcal{F}$-essential subgroups of S, where $\mathcal{F}$ is a saturated fusion system on S.

First of all, we recall a result on p-groups which will be used in the sequel.

Lemma 3.1. *Let p be a prime number, P be a finite p-group and G be a subgroup of* $\mathrm{Aut}(P)$. *Let*

$$P_0 \leq P_1 \leq ... \leq P_m = P$$

be a sequence of normal subgroups of P, all G-invariant, such that P_0 is contained in $\Phi(P)$. Let H be the subgroup of those $g \in G$ that act as the identity on P_i/P_{i-1}, for any $1 \leq i \leq m$. Then H is a normal p-subgroup of G.

Proof. See [16, Theorems 5.1.4 and 5.3.2]. □

Lemma 3.2. *The group A is a $\mathcal{F}$-centric subgroup of S for every fusion system $\mathcal{F}$ on S.*

Proof. It follows by Lemma 2.5. □

Remark 3.3. If a subgroup P of S is properly contained in A, one has that $C_S(P) \geq A$. Then P is not $\mathcal{F}$-centric for any fusion system $\mathcal{F}$ on S.

From now on, for a subgroup P of S, we set $P_0 = P \cap A$.

Lemma 3.4. *Let $\mathcal{F}$ be a fusion system on S, and let P be a subgroup of S, $\mathcal{F}$-essential and not abelian. Suppose that $Z(P) \leq A$. Then $Z_2(S) \leq P_0$.*

Proof. Consider the normal series

$$1 < Z(P) < P. \tag{3.0.1}$$

Since P is $\mathcal{F}$-essential, $Z(S) \leq Z(P)$. Thus, for every $h \in Z_2(S)$, one has that

$$[h, S] \leq Z(S) \leq Z(P) \leq A.$$

We claim that h centralizes the series (3.0.1). Since $Z_2(S) \leq A$ by Lemma 2.3 and $Z(P) \leq A$ by hypothesis, one has that h acts as the identity on $Z(P)$. Since $[h, P] \leq [h, S] \leq Z(P)$, one has that h acts as the identity on $P/Z(P)$, and the claim follows.

By Lemma 3.1 and Remark 1.25, one has that the conjugation map $c_h \in O_p(\mathrm{Aut}_{\mathcal{F}}(P)) = \mathrm{Inn}(P)$. Hence $Z_2(S) \leq P$, and since $Z_2(S) \leq A$ the statement holds. □

Now we proceed with the study of the $\mathcal{F}$-essential subgroups of S according to the order of such subgroups.

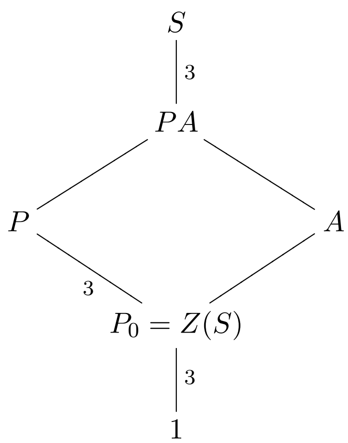
3.1 $\mathcal{F}$-essential subgroups of S of order 3^2

Proposition 3.5. *Let P be a subgroup of S of order 3^2. Then P is not $\mathcal{F}$-essential for any fusion system $\mathcal{F}$ on S.*

Proof. If $P \leq A$, by Remark 3.3 one has that P is not $\mathcal{F}$-essential for any fusion system $\mathcal{F}$ on S. Suppose that $P \nleq A$. Therefore, $|P_0|$ is either 1 or 3.

1. Assume $|P_0| = 1$, and suppose that P is $\mathcal{F}$-essential for some fusion system $\mathcal{F}$ on S. Hence $Z(S) \leq Z(P)$. Since $Z(S) \leq A$, one has that $Z(S) \leq P_0$, a contradiction, as $|P_0| = 1$ and $|Z(S)| = 3$. Then P is not $\mathcal{F}$- essential for any fusion system $\mathcal{F}$ on S.

2. Assume $|P_0| = 3$, and suppose that P is $\mathcal{F}$-essential for some fusion system $\mathcal{F}$ on S. Thus $Z(S) \leq Z(P)$, and then $Z(S) \leq P_0$. Since $|Z(S)| = |P_0| = 3$, it follows that $Z(S) = P_0$.



By Remark 1.25, $O_3(\mathrm{Aut}_{\mathcal{F}}(P)) = \mathrm{Inn}(P)$.

- If $P \simeq C_9$, one has that $|P^3| = 3$. Moreover, $P^3 \leq S^3$ and $|P^3| = |S^3| = 3$. Hence $P^3 = S^3 = Z(S)$ (see Lemma 2.3), and $Z(S)$ is characteristic in P. Let $g \in Z_2(S) \backslash P_0$. Then:
  - c_g is the identity on $P_0 = Z(S)$;
  - c_g is the identity on $P/Z(S)$, as $[g, P] \leq [g, S] \leq Z(S)$.

  By Lemma 3.1, $c_g \in O_p(\mathrm{Aut}_{\mathcal{F}}(P)) = \mathrm{Inn}(P)$, i.e., $g \in P \cap A = P_0$, a contradiction. Therefore P is not $\mathcal{F}$- essential for any fusion system $\mathcal{F}$ on S.
- If $P \simeq C_3 \times C_3$, then $\mathrm{Aut}(P) \simeq \mathrm{GL}_2(3)$. Moreover, $[Z_2(S), P] \leq Z(S) \leq P$, and then $Z_2(S)$ acts by conjugation on P. Since $Z_2(S)/C_{Z_2(S)}(P) \lesssim GL_2(3)$ and $|\mathrm{GL}_2(3)| = 3 \cdot 2^4$, we get that $|C_{Z_2(S)}(P)| \geq 9$, whence a contradiction, since P is $\mathcal{F}$-centric and then $C_{Z_2(S)}(P) \leq P \cap A = P_0$. $\square$

3.2 $\mathcal{F}$-essential subgroups of S of order 3^3

Lemma 3.6. *Let P be a non abelian subgroup of S of order 3^3. Then P is not $\mathcal{F}$-essential for any fusion system $\mathcal{F}$ on S.*

Proof. Obviously $P \nleq A$, as P is not abelian. Suppose that P is $\mathcal{F}$-essential for some fusion system $\mathcal{F}$ on S. Hence $Z(S) \leq Z(P)$. Then $Z(S) \leq P_0$, as $Z(S) \leq A$. Since P is not abelian, one has that $|Z(P)| = 3 = |Z(S)|$, and $Z(P) = Z(S) \leq A$. By Lemma 3.4, $Z_2(S) \leq P_0$, a contradiction, since $|P_0| \leq 3^2$ and $|Z_2(S)| = 3^3$. Then the statement holds. $\square$

The following Lemma is needed for the study of $\mathcal{F}$-essential subgroups of order 3^3 and 3^4.

Lemma 3.7. *Let H be a subgroup of $GL_3(3) \times C_2$, and suppose that H has Sylow 3-subgroups of order 3^2. Then $O_3(H) \neq 1$.*

Proof. Clearly, it suffices to show that the statement holds when H lies in $GL_3(3)$.

It is well known that $GL_3(3) = SL_3(3) \times C_2$ (cf., e.g., [9, p. 13]). Therefore the maximal subgroups of $GL_3(3)$ different to $SL_3(3)$ are the direct product of a maximal subgroup of $SL_3(3)$ with $Z(SL_3(3))$. The maximal subgroups of $SL_3(3)$ are listed in [9, p. 13]. Then the maximal subgroups of $GL_3(3)$ are (isomorphic to):

- $SL_3(3)$;
- $(C_3^2 \rtimes 2S_4) \times C_2$
- $C_{13} \rtimes C_6$;
- $S_4 \times C_2$.

Therefore $H \leq SL_3(3)$ or $H \leq (C_3^2 \rtimes 2S_4) \times C_2$.

Suppose that $H \leq (C_3^2 \rtimes 2S_4) \times C_2$, and set $N = C_3^2$. If $N \leq H$, then $N \trianglelefteq H$ and $O_3(H) \neq 1$. If $N \nleq H$, let P be a Sylow 3-subgroup of H. Then $|P \cap N| = 3$ (as otherwise NP would be a subgroup of $(C_3^2 \rtimes 2S_4) \times C_2$ of order 3^4, and this is not possible). For any $x \in P \cap N$ and for any $h \in H$, one has that $x^h \in N \cap H$. Moreover, $N \cap H = N \cap P$, as otherwise $N \leq H$. Then $(P \cap N) \trianglelefteq H$, and $O_3(H) \neq 1$.

If $H \leq SL_3(3)$, the claim follows with a similar argument as above. □

Lemma 3.8. *Let $\mathcal{F}$ be a saturated fusion system on S, and let P be a $\mathcal{F}$-essential subgroup of S, of order 3^3. Then the following hold:*

i. *P is elementary abelian;*

ii. *$|N_S(P)/P| = 3$;*

iii. *$|P_0| = 3^2$.*

Proof. Suppose that $P^3 \neq 1$. Since $|S^3| = 3$ and $S^3 = Z(S)$, one has that $P^3 = S^3 = Z(S)$, and $Z(S)$ is characteristic in P. Let $g \in Z_2(S) \backslash P_0$. Then c_g is the identity on $Z(S)$ and on $P/Z(S)$. By Lemma 3.1, this implies that $c_g \in O_p(\mathrm{Aut}_{\mathcal{F}}(P))$, a contradiction, since $O_p(\mathrm{Aut}_{\mathcal{F}}(P)) = \mathrm{Inn}(P)$ by Remark 1.25. Therefore P is elementary abelian, and the proof of point i is completed.

Thus, $\mathrm{Aut}_{\mathcal{F}}(P) \lesssim GL_3(3)$. Since P is a proper subgroup of S, one has that $N_S(P) \gneq P$, and $3 \leq |N_S(P)/P| \leq 3^3$.

Suppose that P is fully normalized. By Theorem 1.20, P is fully automized. Therefore $\mathrm{Aut}_S(P) \simeq N_S(P)/P$ is a Sylow 3-subgroup of $\mathrm{Aut}_{\mathcal{F}}(P)$.

If $|N_S(P)/P| = 3^3$, then $N_S(P) = S$ and $P \trianglelefteq S$, a contradiction to Lemma 2.7. If $|N_S(P)/P| = 3^2$, then $\mathrm{Aut}_{\mathcal{F}}(P)$ is isomorphic to a subgroup of $GL_3(3)$ with Sylow 3-subgroups of order 3^2. Then $O_3(\mathrm{Aut}_{\mathcal{F}}(P)) \neq 1$ by Lemma 3.7. Since P is abelian, $\mathrm{Aut}_{\mathcal{F}}(P) = \mathrm{Out}_{\mathcal{F}}(P)$. Thus, $\mathrm{Out}_{\mathcal{F}}(P)$ has not strongly 3-embedded subgroups by Remark 1.25, and P is not $\mathcal{F}$-essential, a contradiction to the hypothesis.

Thus, $|N_S(P)/P| = 3$ for every fully normalized $\mathcal{F}$-essential subgroup P of S of order 3^3, and $|N_S(P)|$ is as smallest as possible. Then $|N_S(P)/P| = 3$ for every $\mathcal{F}$-essential subgroup P of S of order 3^3 and this completes the proof of statement ii.

Now suppose that $|P_0| = 3$. Hence $|P \cap Z_2(S)| = 3$, as $P_0 = Z(S)$ (see Lemma 2.6) and $Z(S) \leq Z_2(S)$. Since $S = PA$, by the proof of Lemma 2.7 one has that $N_S(P) = PN_A(P) = PZ_2(S)$. Then $|N_S(P)| = 3^5$ and $|N_S(P)/P| = 3^2$, a contradiction to statement ii. Then $|P_0| = 3^2$, and this completes the proof. □

Proposition 3.9. *Let P be a subgroup of S of order 3^3. Then P is not $\mathcal{F}$-essential for any saturated fusion system $\mathcal{F}$ on S.*

Proof. Suppose that P is $\mathcal{F}$-essential for some saturated fusion system $\mathcal{F}$ on S. By Lemma 3.6 and Lemma 3.8, P is elementary abelian. Hence, Lemma 2.9 implies that $P \leq A$ or $P \leq E$. The former case leads to a contradiction, as $C_S(P) > P$ (see Remark 3.3). In the latter case, since $Z(S) \leq P$ (as P is $\mathcal{F}$-essential) and $E/Z(S)$ is abelian, it follows that $P \trianglelefteq E$, and hence $|N_S(P)/P| \geq 3^2$, in contradiction with Lemma 3.8. □

3.3 $\mathcal{F}$-essential subgroups of S of order 3^4

Proposition 3.10. *Let P be a non abelian subgroup of S, of order 3^4. Then P is not $\mathcal{F}$-essential for any saturated fusion system $\mathcal{F}$ on S.*

Proof. Suppose that P is $\mathcal{F}$-essential for some saturated fusion system $\mathcal{F}$ on S. Since $|P_0|$ is either 3^2 or 3^3, we proceed according to the order of P_0.

1. Assume $|P_0| = 3^2$.

$$\begin{tikzcd}
& S \arrow[dl, dash] \arrow[dr, dash, "3^2"] & \\
P \arrow[dr, dash, "3^2"'] & & A \arrow[dl, dash, "3^2"] \\
& P_0 \arrow[d, dash, "3"] & \\
& Z(S) \arrow[d, dash, "3"] & \\
& 1 &
\end{tikzcd}$$

As P is $\mathcal{F}$-essential, $Z(S) \leq Z(P)$. Since P is not abelian, the order of $Z(P)$ is either 3 or 3^2. If $|Z(P)| = 3$, then $Z(S) = Z(P)$ and $Z(P) \leq A$. Hence by Lemma 3.4, $Z_2(S) \leq P_0$, a contradiction, as $|Z_2(S)| = 3^3$ and $|P_0| = 3^2$. Then $|Z(P)| = 3^2$. If $Z(P) = P_0$, then $Z(P) \leq A$. Again, by Lemma 3.4, $Z_2(S) \leq P_0$, a contradiction. Hence $Z(P) \neq P_0$. One has that $P/Z(P) \simeq C_3 \times C_3$, as P is not abelian. Hence $\Phi(P) \leq Z(P)$. If $\Phi(P) = Z(P)$, then $\Phi(P) \nleq P_0$, a contradiction, as $P/P_0 \simeq S/A \simeq B$ is elementary abelian. Hence $\Phi(P) \lneq Z(P)$ and $|\Phi(P)| = 3$.

$$
\begin{tikzcd}
& P \arrow[dl, dash, "3^2"'] \arrow[dr, dash, "3^2"] & \\
Z(P) \arrow[dr, dash, "3"'] & & P_0 \arrow[dl, dash, "3"] \\
& \Phi(P) = Z(S) \arrow[d, dash, "3"] & \\
& 1 &
\end{tikzcd}
$$

One has that $Z(S) < Z(P)$ and $Z(S) < P_0$, hence $Z(S) \leq Z(P) \cap P_0$, and in particular $Z(S) = Z(P) \cap P_0$. On the other hand, $\Phi(P) < Z(P)$ and $\Phi(P) < P_0$, hence $\Phi(P) \leq Z(P) \cap P_0$, and in particular $\Phi(P) = Z(P) \cap P_0$. It follows that $Z(S) = \Phi(P)$ and $Z(S)$ is characteristic in P.

Now, let $h \in Z_2(S) \backslash P_0$: c_h is the identity on $Z(S)$ and on $P/Z(S)$, hence by Lemma 3.1 $c_h \in O_p(\mathrm{Aut}_{\mathcal{F}}(P)) = \mathrm{Inn}(P)$, a contradiction.

Then P is not $\mathcal{F}$-essential for any saturated fusion system $\mathcal{F}$ on S, and the statement holds.

2. Assume $|P_0| = 3^3$.

$$
\begin{tikzcd}
& S \arrow[d, dash, "3"] & \\
& PA \arrow[dl, dash] \arrow[dr, dash, "3"] & \\
P \arrow[dr, dash, "3"'] & & A \arrow[dl, dash] \\
& P_0 = Z_2(S) \arrow[d, dash, "3^2"] & \\
& Z(S) \arrow[d, dash, "3"] & \\
& 1 &
\end{tikzcd}
$$

By hypothesis, P is not abelian. Moreover, P_0 is abelian and maximal in P. Therefore $P_0 = C_P(P_0)$. Thus, $Z(P) \leq P_0 \leq A$, and by lemma 3.4, $Z_2(S) \leq P_0$. In particular, $P_0 = Z_2(S)$. It follows that $P = \langle Z_2(S), h \rangle$, for some $h \notin A$. Since $Z_2(S) \trianglelefteq P$, one may write (in a unique way) every $k \in P$ as $k = h^n \cdot r$, for some $n \in \mathbb{Z}$ and $r \in Z_2(S)$. For every $h^n \cdot r, h^m \cdot s \in P$ (with $n, m \in \mathbb{Z}$ and $r, s \in Z_2(S)$), one has that

$$\begin{aligned} [h^n \cdot r, h^m \cdot s] &= [h^n, s] \cdot [h^n, h^m] \cdot [r, s] \cdot [r, h^m] = \\ &= [h^n, s] \cdot [r, h^m] \in Z(S), \end{aligned}$$

as $r, s \in Z_2(S)$. Hence $P' = [Z_2(S) \cdot \langle h \rangle, Z_2(S) \cdot \langle h \rangle] \leq Z(S)$. Thus, $P' = Z(S)$, as P is not abelian and $|Z(S)| = 3$. Moreover, $[P, S] \leq S' = Z_2(S) \leq P$. Hence $P \trianglelefteq S$.

Consider the map

$$\varphi : \mathrm{Aut}_{\mathcal{F}}(P) \to \mathrm{Aut}(P/P') \times \mathrm{Aut}(P') ,$$

sending $\alpha \in \mathrm{Aut}_{\mathcal{F}}(P)$ to $(\bar{\alpha}, \alpha_{|P'})$, where $\bar{\alpha}$ is the map induced by α on the quotient P/P'.

One has that

$$\mathrm{Aut}(P/P') \simeq GL_3(3) \quad \text{and} \quad \mathrm{Aut}(P') \simeq C_2,$$

as $P/P' \simeq C_3 \times C_3 \times C_3$ (recall that $P' = Z(S) = S^3$) and $P' \simeq C_3$. By Lemma 3.1, $\ker(\varphi)$ is a normal 3-subgroup of $\mathrm{Aut}_{\mathcal{F}}(P)$. Since P is $\mathcal{F}$-essential, by Remark 1.25 one has that $O_3(\mathrm{Aut}_{\mathcal{F}}(P)) = \mathrm{Inn}(P)$. We claim that $\mathrm{Inn}(P) = \ker(\varphi)$. Obviously, $\mathrm{Inn}(P)$ centralizes $P' = Z(S)$. Moreover, for every $g \in P$, one has that $[g, P] \leq P'$, as P/P' is abelian. Thus, $\mathrm{Inn}(P)$ centralizes P/P', and this yields the claim. Then $\mathrm{Aut}_{\mathcal{F}}(P)/\ker(\varphi) = \mathrm{Out}_{\mathcal{F}}(P)$ is isomorphic to a subgroup of $GL_3(3) \times C_2$. $\mathcal{F}$ is saturated and P is fully normalized (as $P \trianglelefteq S$), hence by Theorem 1.20 P is fully automized . Therefore $\mathrm{Aut}_S(P) \simeq N_S(P)/C_S(P) = S/Z(P)$ is a Sylow 3-subgroup of $\mathrm{Aut}_{\mathcal{F}}(P)$. Since P is not abelian, $|Z(P)|$ is either 3 or 3^2. Anyhow, $\mathrm{Out}_S(P) = \mathrm{Aut}_S(P)/\mathrm{Inn}(P)$ is a Sylow 3-subgroup of $\mathrm{Out}_{\mathcal{F}}(P)$, and $|\mathrm{Out}_S(P)| = 3^2$. Hence $\mathrm{Out}_{\mathcal{F}}(P)$ is isomorphic to a subgroup of $GL_3(3) \times C_2$, with Sylow 3-subgroups of order 3^2: Lemma 3.7 implies that $O_3(\mathrm{Out}_{\mathcal{F}}(P)) \neq 1$. Thus, by Remark 1.25 one has that $\mathrm{Out}_{\mathcal{F}}(P)$ does not contain strongly 3-embedded subgroups. Then P is not $\mathcal{F}$-essential for any saturated fusion system $\mathcal{F}$ on S, and the statement holds. $\square$

3.4 $\mathcal{F}$-essential subgroups of order 3^5

Recall that, in the notation of 2.2.1, $E = \langle x, y, a, b \rangle$.

Proposition 3.11. *Let P be a maximal subgroup of S, with $P \neq E$. Then P is not $\mathcal{F}$-essential for any saturated fusion system $\mathcal{F}$ on S.*

Proof. Suppose that P is $\mathcal{F}$-essential for some saturated fusion system $\mathcal{F}$ on S. Since a group can not be the union of two proper subgroups considered as sets, it follows that P is not contained in $E \cup A$; thus, P has exponent 9 by Lemma 2.8, and $P^3 = S^3 = Z(S)$ is characteristic in P. Then $Z_2(S)$ centralizes the series

$$1 < Z(S) < P,$$

and since P is $\mathcal{F}$-essential, Lemma 3.1 implies that $Z_2(S) \leq P$.

Suppose that $P' = S' = Z_2(S)$: then S centralizes the series

$$1 < Z(S) < P' < P,$$

and again since P is $\mathcal{F}$-essential, Lemma 3.1 implies that $S \leq P$, and this is not possible. Hence $P' < S'$.

Since $|E \cap P| = 3^4$ and $Z_2(S) \leq (E \cap P)$, one has that $E \cap P = \langle x, a, z, s\rangle$; without loss of generality, we may suppose that $s = y^{m_2} b^{m_4}$, for some $m_2, m_4 \in \{0, 1, 2\}$, with $(m_2, m_4) \neq (0, 0)$.

The maximality of $E \cap P$ in P implies that $P = \langle x, a, z, s, r\rangle$, where $r \in P \smallsetminus E$; we may suppose that $r = y^{n_2} b^{n_4} t^{n_5}$, for some $n_2, n_4, n_5 \in \{0, 1, 2\}$, with $n_5 \neq 0$.

Formula (2.2.2) implies that

$$[r, s] = x^{-n_5 m_2} \cdot a^{-n_5 m_4} \cdot z^{-n_5 [3m_4/2] - n_5 [3m_2/2]} \notin Z(S),$$

and at least one among the commutators $[s, x] = z^{-m_2}$ and $[s, a] = z^{-m_4}$ is not trivial. Therefore $|P'| \geq 9$, and then $|P'| = 9$ for the previous part of the proof.

Suppose that $|Z(P)| = 3$; hence $Z(P) = Z(S)$, as P is $\mathcal{F}$-essential. Moreover $Z_2(S) \leq Z_2(P)$; if $Z_2(S) \lneq Z_2(P)$, then $Z_2(P)$ is maximal in P and $P/Z(P)$ is abelian: thus, $P' \leq Z(P)$, a contradiction, as $|Z(P)| = 3$ and $|P'| = 9$. Therefore $Z_2(S) = Z_2(P)$. Then S centralizes the characteristic series

$$1 < Z(S) < Z_2(S) < P,$$

and since P is $\mathcal{F}$-essential, Lemma 3.1 implies that $S \leq P$, and this is not possible.

Thus, $|Z(P)| \geq 9$. Commutator calculus show that

$$Z(P) = \{x^{\alpha_1} a^{\alpha_3} z^{\alpha_6} \,|\, \alpha_3 m_4 + \alpha_1 m_2 = 0 \text{ and } \alpha_3 n_1 + \alpha_1 n_2 = 0\}.$$

Hence $|Z(P)| = 3^{1+d}$, where d is the dimension of the space of the solutions in $\mathbb{Z}/3\mathbb{Z}$ of

$$\begin{cases} \alpha_3 m_4 + \alpha_1 m_2 = 0 \\ \alpha_3 n_1 + \alpha_1 n_2 = 0 \end{cases}.$$

Since $|Z(P)| \geq 9$, one has that $d \geq 1$. Then

$$\begin{pmatrix} m_4 & m_2 \\ n_1 & n_2 \end{pmatrix}$$

has rank equal to one, i.e., (m_4, m_2) and (n_1, n_2) are linearly dependent. Therefore $r = s^{\beta} t^{n_5}$, for some $\beta \in \{0, 1, 2\}$, and $t \in P$. It follows that $A \leq P$, and $P = A\langle s\rangle$.

Now we claim that $Z_2(S)/Z(S) = Z(P/Z(S))$. Obviously, $Z_2(S)/Z(S)$ is contained in $Z(P/Z(S))$. Let $h = x^{\beta_1} \cdot y^{\beta_2} \cdot a^{\beta_3} \cdot b^{\beta_4} \cdot t^{\beta_5} \cdot z^{\beta_6} \in P$ such that $hZ(S) \in Z(P/Z(S))$. One has that

$$[h, t] \in Z(S) \Leftrightarrow \beta_2 = \beta_4 = 0, \text{ and } \beta_5 = 0.$$

Thus, $h \in Z_2(S)$, and the claim follows.

Hence $Z_2(S)$ is characteristic in P, and this yields a contradiction, since S centralizes the series

$$1 < Z(S) < Z_2(S) < P.$$

Then P is not $\mathcal{F}$-essential for any saturated fusion system $\mathcal{F}$ on S. $\square$

3.5 Existence of $\mathcal{F}$-essential subgroups

Proposition 3.12. *Let $\mathcal{F}$ be a reduced fusion system on S. Then S has exactly two $\mathcal{F}$-essential subgroups, E and A.*

Proof. Let P be a $\mathcal{F}$-essential subgroup of S. By Proposition 3.5, Proposition 3.9, Proposition 3.10, and Proposition 3.11, one has that either $P = A$ or $P = E$. On the other hand, by Proposition 1.31 both A and E are $\mathcal{F}$-essential subgroups of S, and the statement holds. $\square$

Chapter 4

Automorphism groups in $\mathcal{F}$

By Alperin's Fusion Theorem, the structure of a saturated fusion system $\mathcal{F}$ on S is determined by the structure of the $\mathcal{F}$-automorphism groups of the $\mathcal{F}$-essential subgroups of S, and of S itself. If $\mathcal{F}$ is a reduced fusion system on S, A and E are the unique $\mathcal{F}$-essential subgroups of S (see Proposition 3.12). Thus, the goal of this Chapter is the study of the $\mathcal{F}$-automorphism groups of A, E, and S. In particular, since the inner automorphisms are always morphisms in a fusion system, we study the groups of outer automorphisms in $\mathcal{F}$ instead of the whole groups of automorphisms in $\mathcal{F}$. At the end of the Chapter we state the main Theorem, in which we determine all acceptable 3-tuples $(\mathrm{Out}_{\mathcal{F}}(S), \mathrm{Out}_{\mathcal{F}}(A), \mathrm{Out}_{\mathcal{F}}(E))$, with $\mathcal{F}$ a reduced fusion system on S, and this completes the classification of the reduced fusion systems on S.

First, we recall some facts on coprime actions which will be used in the sequel. The following comes from [4, Theorem 18.1].

Theorem 4.1 (Schur-Zassenhaus Theorem)**.** *Let G be a finite group, let H be a normal subgroup of G, and assume that:*

- $(|H|, |G/H|) = 1$*;*
- *either H or G/H is solvable.*

Then:

1. *G splits over H, i.e., there exists $M \leq G$, such that $M \cap H = 1$ and $G = HM$;*
2. *G is transitive on the complements to H in G.*

Lemma 4.2 (Coprime Action)**.** *Let G be a finite group, let T be a p-subgroup of G, and let $N \trianglelefteq G$ such that $p \nmid |N|$. Then*

$$N_{G/N}(TN/N) = N_G(T)N/N.$$

Proof. Obviously, $N_G(T)N/N \leq N_{G/N}(TN/N)$.

Let $g \in G$ such that $gN \in N_{G/N}(TN/N)$: hence g acts by conjugation on TN. Since $T^g \leq (TN)^g = TN$, by Schur-Zassenhaus Theorem there exists $h \in TN$ such that $T^g = T^h$. Therefore $T^{gh^{-1}} = T$, and $gh^{-1} \in N_G(T)$. We may write $h = tn$, for some $t \in T$, $n \in N$. Then

$$g \in N_G(T)h = N_G(T)tn \leq N_G(T)N,$$

and $gN \in N_G(T)N/N$: this completes the proof. □

From now on, let $\mathcal{F}$ be a reduced fusion system on S. We start the investigation of the $\mathcal{F}$-automorphism groups with the study of the $\mathcal{F}$-automorphism groups of S.

Lemma 4.3. *The group* $\mathrm{Out}_{\mathcal{F}}(S)$ *is a 2-group isomorphic to a Sylow 2-subgroup of* $\mathrm{Aut}_{\mathcal{F}}(S)$*.*

Proof. By definition of saturated fusion system, $\mathrm{Aut}_S(S) = \mathrm{Inn}(S)$ is a Sylow 3-subgroup of $\mathrm{Aut}_{\mathcal{F}}(S)$. Then $\mathrm{Out}_{\mathcal{F}}(S) = \mathrm{Aut}_{\mathcal{F}}(S)/\mathrm{Inn}(S)$ is a $3'$-group. Hence by Schur-Zassenhaus Theorem, one has that

$$\mathrm{Aut}_{\mathcal{F}}(S) = \mathrm{Inn}(S) \rtimes K,$$

with K a subgroup of $\mathrm{Aut}_{\mathcal{F}}(S)$ isomorphic to $\mathrm{Aut}_{\mathcal{F}}(S)/\mathrm{Inn}(S) = \mathrm{Out}_{\mathcal{F}}(S)$.

Consider the homomorphism

$$\psi : K \to \mathrm{Aut}(S/E) \times \mathrm{Aut}(E/S') \times \mathrm{Aut}(S'/Z(S)) \times \mathrm{Aut}(Z(S)),$$

sending $\alpha \in K$ to $(\alpha_1, \alpha_2, \alpha_3, \alpha_{|Z(S)})$, where $\alpha_1, \alpha_2, \alpha_3$ are the maps induced by α respectively on the quotients S/E, E/S', and $S'/Z(S)$. Hence ψ is injective, since K is a $3'$-group and by Lemma 3.1 the kernel of ψ is a 3-group. One has that

$$\mathrm{Aut}(S/E) \simeq \mathrm{Aut}(Z(S)) \simeq C_2 \text{ and } \mathrm{Aut}(E/S') \simeq \mathrm{Aut}(S'/Z(S)) \simeq GL_2(3),$$

as $S/E \simeq Z(S) \simeq C_3$, and $E/S' \simeq S'/Z(S) \simeq C_3 \times C_3$. Thus, $\mathrm{Out}_{\mathcal{F}}(S)$ is isomorphic to a subgroup of $C_2 \times GL_2(3) \times GL_2(3) \times C_2$. It follows that $\mathrm{Out}_{\mathcal{F}}(S)$ is a 2-group. In particular, $\mathrm{Out}_{\mathcal{F}}(S)$ is isomorphic to a Sylow 2-subgroup of $\mathrm{Aut}_{\mathcal{F}}(S)$. □

Lemma 4.4. *Let α be a $\mathcal{F}$-automorphism of S of order a power of 2, such that its restriction to A or to E is the identity respectively on A or on E. Then α is the identity on S.*

Proof. One has that $C_S(A) \leq A$ and $C_S(E) \leq E$, since A and E are $\mathcal{F}$-essential. Hence the claim follows by Thompson $A \times B$ Lemma (see, e.g., [4, Lemma 24.2]). □

Now we study the $\mathcal{F}$-automorphism groups of E and A, and their relationships with $\mathrm{Out}_{\mathcal{F}}(S)$.

Lemma 4.5. *Let T be a Sylow 2-subgroup of $N_{\mathrm{Aut}_{\mathcal{F}}(E)}(\mathrm{Aut}_S(E))$, and let Q be a Sylow 2-subgroup of $N_{\mathrm{Out}_{\mathcal{F}}(E)}(\mathrm{Out}_S(E))$. Then $T \simeq Q$.*

Proof. Set $G = \mathrm{Aut}_{\mathcal{F}}(E)$, $H = \mathrm{Aut}_S(E)$, and $K = \mathrm{Inn}(E)$. Then $K \leq H \leq G$, $K \trianglelefteq G$, and $T \cap K = 1$, as K is a 3-group. This implies that $T \simeq TK/K$, and TK/K is a Sylow 2-subgroup of $N_G(H)/K$. By Lemma 4.2, $N_G(H)/K = N_{G/K}(H/K)$, and the claim follows. $\square$

Lemma 4.6. *The group $\mathrm{Out}_{\mathcal{F}}(S)$ is isomorphic to a Sylow 2-subgroup of the normalizer $N_{\mathrm{Aut}_{\mathcal{F}}(A)}(\mathrm{Aut}_S(A))$.*

Proof. Set $N = N_{\mathrm{Aut}_{\mathcal{F}}(A)}(\mathrm{Aut}_S(A))$. Let H be a Sylow 2-subgroup of $\mathrm{Aut}_{\mathcal{F}}(S)$, and T be a Sylow 2-subgroup of N. Hence by Lemma 4.3, $H \simeq \mathrm{Out}_{\mathcal{F}}(S)$. Let

$$\mathrm{res} : \mathrm{Aut}_{\mathcal{F}}(S) \to \mathrm{Aut}_{\mathcal{F}}(A)$$

be the restriction map sending any $\phi \in \mathrm{Aut}_{\mathcal{F}}(S)$ to $\phi_{|A} \in \mathrm{Aut}_{\mathcal{F}}(A)$. By Lemma 4.4, the map $\mathrm{res}_{|H} : H \to \mathrm{Aut}_{\mathcal{F}}(A)$ is injective. Thus, H is isomorphic to a 2-subgroup of $\mathrm{Aut}_{\mathcal{F}}(A)$; moreover, by Remark 1.32 one has that $H \lesssim T$.

Since A is fully $\mathcal{F}$-normalized, Lemma 1.35 implies that A has the surjectivity property. Hence the restriction map

$$\mathrm{res} : \mathrm{Aut}_{\mathcal{F}}(S) \twoheadrightarrow N$$

is surjective, i.e., every $\mathcal{F}$-automorphism of A which lies in N extends to a $\mathcal{F}$-automorphism of S. Let M be the full preimage of T under the restriction map res, i.e., the set of all $\mathcal{F}$-automorphisms of S whose restrictions to A lie in T. Then the map

$$\mathrm{res}_{|M} : M \twoheadrightarrow T$$

is surjective, and $M/\ker(\mathrm{res}_{|M}) \simeq T$. Obviously, $|M/\ker(\mathrm{res}_{|M})| = |T|$ divides $|\mathrm{Aut}_{\mathcal{F}}(S)|$. Hence $|T| \leq |H|$, since T is a 2-group and H is a Sylow 2-subgroup of $\mathrm{Aut}_{\mathcal{F}}(S)$. Moreover, one has that $|H| \leq |T|$, since $H \lesssim T$. Hence $|H| = |T|$ and $\mathrm{Out}_{\mathcal{F}}(S) \simeq H \simeq T$. $\square$

Lemma 4.7. *The group $\mathrm{Out}_{\mathcal{F}}(S)$ is isomorphic to a Sylow 2-subgroup of the normalizer $N_{\mathrm{Out}_{\mathcal{F}}(E)}(\mathrm{Out}_S(E))$.*

Proof. Let H be a Sylow 2-subgroup of $\mathrm{Aut}_{\mathcal{F}}(S)$, and T be a Sylow 2-subgroup of $N_{\mathrm{Aut}_{\mathcal{F}}(E)}(\mathrm{Aut}_S(E))$. By Lemma 4.5, T is isomorphic to a Sylow 2-subgroup K of $N_{\mathrm{Out}_{\mathcal{F}}(E)}(\mathrm{Out}_S(E))$. We proceed as in the proof of Lemma 4.6. Hence $\mathrm{Out}_{\mathcal{F}}(S) \simeq T \simeq K$, and the statement holds. $\square$

We collect the information about $\mathrm{Aut}_{\mathcal{F}}(A)$ in the following Lemma.

Lemma 4.8. *Let $\mathcal{F}$ be a reduced fusion system on S. Then $\mathrm{Aut}_{\mathcal{F}}(A)$ is isomorphic to a subgroup of $GL_4(3)$, $O_3(\mathrm{Aut}_{\mathcal{F}}(A)) = 1$, $\mathrm{Aut}_S(A)$ is a Sylow 3-subgroup of $\mathrm{Aut}_{\mathcal{F}}(A)$, $|\mathrm{Aut}_S(A)| = 3^2$, and, in the notation of (2.2.1), $\mathrm{Aut}_S(A) \simeq \langle y, b\rangle$. In particular, $|C_A(\mathrm{Aut}_S(A))| = 3$.*

Proof. Since $A \simeq \mathbb{F}_3^4$, it follows that $\mathrm{Aut}_{\mathcal{F}}(A) \lesssim GL_4(3)$, and since A is $\mathcal{F}$-essential and abelian, $O_3(\mathrm{Aut}_{\mathcal{F}}(A)) = 1$ by Remark 1.25. The definition of saturated fusion system implies that $\mathrm{Aut}_S(A)$ is a Sylow 3-subgroup of $\mathrm{Aut}_{\mathcal{F}}(A)$, since A is characteristic in S. One has that $\mathrm{Aut}_S(A) \simeq S/A \simeq B$, and in the notation of (2.2.1), $S/A = \langle yA, bA\rangle \simeq \langle y, b\rangle$: hence $|\mathrm{Aut}_S(A)| = 3^2$, $\mathrm{Aut}_S(A) \simeq \langle y, b\rangle$, and $|C_A(\mathrm{Aut}_S(A))| = 3$ by Lemma 2.4. □

Using the information collected in Lemma 4.8, we determine all possible $\mathrm{Aut}_{\mathcal{F}}(A)$, with $\mathcal{F}$ a reduced fusion system on S.

The following comes from [20, pp. 375-389] and [7, Table 8.8].

Proposition 4.9. *Let M be a maximal subgroup of $GL_4(3)$. Then one of the following holds:*

1. $M \simeq C_3^3 \rtimes (GL_3(3) \times C_2)$;
2. $M \simeq C_3^4 \rtimes (GL_2(3) \times GL_2(3))$;
3. $M \simeq GL_2(9) \rtimes C_2$;
4. $M \simeq (GL_2(3) \times GL_2(3)) \rtimes C_2$;
5. $M \simeq (SO_4^-(3).C_2).C_2 \simeq (A_6 \rtimes C_4).C_2 \simeq (C_2 \times M_{10}) \rtimes C_2$;
6. $M \simeq (SO_4^+(3).C_2).C_2$;
7. $M \simeq SL_4(3)$.

The following comes from [7, Table 8.12].

Proposition 4.10. *Let M be a maximal subgroup of $Sp_4(3)$. Then one of the following holds:*

1. $M \simeq 3^{1+2} \rtimes (C_2 \times SL_2(3))$, *where 3^{1+2} is an extraspecial group of order 3^3;*
2. $M \simeq C_3^3 \rtimes GL_2(3)$;
3. $M \simeq (SL_2(3) \times SL_2(3)) \rtimes C_2$;
4. $M \simeq SL_2(9) \rtimes C_2 \simeq 2A_6 \rtimes C_2$;
5. $M \simeq 2_-^{1+4}.A_5$, *where 2_-^{1+4} is an extraspecial group of order 2^5 and exponent 4.*

Theorem 4.11. *Let $\mathcal{F}$ be a reduced fusion system on S, and set $H = \mathrm{Aut}_{\mathcal{F}}(A)$. Let $P \in \mathrm{Syl}_3(H)$, and let $T \in \mathrm{Syl}_2(N_H(P))$. Then one of the following holds:*

1. *$H \simeq (C_2 \times M_{10}) \rtimes C_2 \simeq (C_2 \times (A_6.C_2)) \rtimes C_2$, and $T \simeq C_2 \times QD_{16}$;*
2. *$H \simeq A_6 \rtimes C_4$, and $T \simeq C_8 \times C_2$;*
3. *$H \simeq C_2 \times S_6$, and $T \simeq D_8 \times C_2$;*
4. *$H \simeq C_2 \times M_{10} \simeq C_2 \times (A_6.C_2)$, and $T \simeq Q_8 \times C_2$;*
5. *$H \simeq C_2 \times A_6$, and $T \simeq C_4 \times C_2$;*
6. *$H \simeq S_6$, and $T \simeq D_8$;*
7. *$H \simeq M_{10} \simeq A_6.C_2$, and $T \simeq Q_8$;*
8. *$H \simeq A_6$, and $T \simeq C_4$.*

Proof. Let M be a maximal subgroup of $GL_4(3)$ that contains H. Using Proposition 4.9, we proceed with a case-by-case analysis.

If $M \simeq C_3^3 \rtimes (GL_3(3) \times C_2)$, set $U = C_3^3$. Then $H \cap U = 1$, as otherwise $H \cap U$ would be a non trivial normal 3-subgroup of H, in contradiction with Lemma 4.8. Thus, H is isomorphic to a subgroup of $GL_3(3) \times C_2$, but Lemma 3.7 implies that $O_3(H) \neq 1$, in contradiction with Lemma 4.8.

If $M \simeq C_3^4 \rtimes (GL_2(3) \times GL_2(3))$, set $U = C_3^4$. Then $H \cap U = 1$ as before. Thus, H is isomorphic to a subgroup of $GL_2(3) \times GL_2(3)$; in particular, P is conjugate to

$$\left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & y & 1 \end{pmatrix} \mid x, y \in \mathbb{F}_3 \right\}.$$

It follows that $C_A(P)$ has dimension 2, in contradiction with Lemma 4.8.

If $M \simeq (GL_2(3) \times GL_2(3)) \rtimes C_2$, then $P \leq GL_2(3) \times GL_2(3)$ and $C_A(P)$ has dimension 2 as before.

Similarly, if $M \simeq GL_2(9) \rtimes C_2$, then $P \leq GL_2(9)$ and $C_A(P)$ has dimension 2.

If $M \simeq (C_2 \times M_{10}) \rtimes C_2 \simeq (A_6 \rtimes C_4).C_2)$, then M contains a normal subgroup N such that $N \simeq A_6$ and $|M : N| = 8$. Thus, $P \leq (H \cap N)$; since $H \cap N \trianglelefteq H$, it follows that $P \ntrianglelefteq (H \cap N)$, as otherwise $P \trianglelefteq H$. The only subgroup of A_6 with not normal Sylow 3-subgroups of order 3^2 is A_6 itself (see, e.g., [9, p. 4]): therefore $H \cap N = N$, and $N \leq H$. Moreover, $M/N \simeq D_8$: this implies that M contains exactly 8 conjugacy classes of subgroups which contain N. Computations with the computer algebra system GAP show explicitly that the only cases for H (and consequently for T) are the ones listed in the statement.

```
M:=Filtered(MaximalSubgroupClassReps(GeneralLinearGroup(4,3)),
x->Size(x)=2880);
[ <group of 4x4 matrices of size 2880 over GF(3)> ]
l:=LatticeSubgroups(M[1]);
<subgroup lattice of <matrix group of size 2880
with 2 generators>, 167 classes, 8063 subgroups>
sl:=List([1..167],x->ConjugacyClassesSubgroups(l)[x][1]);
f:=Filtered(sl,x->Size(SylowSubgroup(x,3))=9 and
IsNormal(x,SylowSubgroup(x,3))=false);
List(f,x->[StructureDescription(x),StructureDescription
(SylowSubgroup(Normalizer(x,SylowSubgroup(x,3)),2))]);
[ [ "A6", "C4" ], [ "C2 x A6", "C4 x C2" ], [ "S6", "D8" ],
  [ "A6 . C2", "Q8" ], [ "A6 : C4", "C8 x C2" ],
  [ "C2 x S6", "C2 x D8" ], [ "C2 x (A6 . C2)", "C2 x Q8" ],
  [ "(C2 x (A6 . C2)) : C2", "C2 x QD16" ] ]
IdGroup(f[4]);
[ 720, 765 ]
IdGroup(MathieuGroup(10));
[ 720, 765 ]
```

If $M \simeq (SO_4^+(3).C_2).C_2$, then $P \leq SO_4^+(3)$ and $H \cap SO_4^+(3)$ is a subgroup of $SO_4^+(3)$ which order is divisible by 3^2. Using Lemma 1.23, GAP computations show explicitly that $SO_4^+(3)$ and its subgroups with orders divisible by 3^2 do not have strongly 3-embedded subgroups; this implies that H has not strongly 3-embedded subgroups, in contradiction with the definition of $\mathcal{F}$-essential subgroup.

If $M \simeq SL_4(3)$, then H is contained in one of the maximal subgroups K of $SL_4(3)$, which are listed in [7, p. 381]. Assuming $K \simeq C_3^3 \rtimes GL_3(3)$, $K \simeq (C_3^4 \rtimes SL_2(3)^2) \rtimes C_2$, $K \simeq (SL_2(9).C_4).C_2$ or $K \simeq SO_4^+(3).C_2$ leads to a contradiction as before. Suppose $K \simeq \mathrm{Sp}_4(3).C_2$: thus, $H \cap \mathrm{Sp}_4(3)$ is contained in one of the maximal subgroups R of $\mathrm{Sp}_4(3)$, which are listed in Proposition 4.10. Because of the order, $R \simeq 3^{1+2} \rtimes (C_2 \times SL_2(3))$, $R \simeq C_3^3 \rtimes GL_2(3)$, $R \simeq (SL_2(3) \times SL_2(3)) \rtimes C_2$ or $R \simeq SL_2(9) \rtimes C_2$. The first two cases lead to a contradiction since H should be isomorphic respectively to a subgroup of $C_2 \times SL_2(3)$ or $GL_2(3)$ (as otherwise $O_3(H) \neq 1$), and this is not possible because of the order, while the latter two cases lead to a contradiction since $C_A(P)$ should have dimension 2. Finally, suppose $R \simeq SO_4^-(3).C_2 \simeq A_6 \rtimes C_4$; then $R \leq (A_6 \rtimes C_4).C_2 \simeq (C_2 \times M_{10}) \rtimes C_2$, and this completes the proof. □

By Lemma 4.6, Theorem 4.11 yields the following.

Proposition 4.12. *Let $\mathcal{F}$ be a reduced fusion system on S. Then one of the following holds:*

1. $\mathrm{Out}_{\mathcal{F}}(S) \simeq C_2 \times QD_{16}$;

2. $\mathrm{Out}_{\mathcal{F}}(S) \simeq C_2 \times C_8$*;*

3. $\mathrm{Out}_{\mathcal{F}}(S) \simeq C_2 \times D_8$*;*

4. $\mathrm{Out}_{\mathcal{F}}(S) \simeq C_2 \times Q_8$*;*

5. $\mathrm{Out}_{\mathcal{F}}(S) \simeq C_2 \times C_4$*;*

6. $\mathrm{Out}_{\mathcal{F}}(S) \simeq D_8$*;*

7. $\mathrm{Out}_{\mathcal{F}}(S) \simeq Q_8$*;*

8. $\mathrm{Out}_{\mathcal{F}}(S) \simeq C_4$*.*

Recall that the *general symplectic group* $\mathrm{GSp}_4(3)$ is the group of all linear transformations of the vector space of dimension 4 over the field with 3 elements that leave invariant modulo scalars a non degenerate antisymmetric form. Note that the vector space $E/Z(E)$ equipped with the map

$$\begin{gathered} f : E/Z(E) \times E/Z(E) \to \mathbb{F}_3 \\ ([x_1Z(E), x_2Z(E)]) \mapsto [x_1, x_2] \end{gathered}$$

is a symplectic space. Let $\{xZ(E), aZ(E), bZ(E), yZ(E)\}$ be an ordered basis for $E/Z(E)$; thus,

$$J = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{4.0.1}$$

is the matrix of the symplectic form f with respect to the aforementioned basis (we choose such a basis since J is the "matrix of the symplectic space" $\mathrm{Sp}_4(3)$ used by GAP).

Lemma 4.13. *The group* $\mathrm{Out}(E)$ *is a subgroup of* $GL_4(3)$*, isomorphic to* $\mathrm{GSp}_4(3)$*.*

Proof. Let

$$\psi : \mathrm{Aut}(E) \to \mathrm{Aut}(E/Z(E))$$

be the map induced by the canonical projection $E \to E/Z(E)$. Since $Z(E) = \Phi(E)$, one has that $E/Z(E) \simeq \mathbb{F}_3^4$, and then $\mathrm{Aut}(E/Z(E)) \simeq GL_4(3)$. By [4, Exercise 5, p. 116], $\ker(\psi) = \mathrm{Inn}(E)$. Thus $\mathrm{Out}(E)$ is isomorphic to a subgroup of $GL_4(3)$.

Now, let

$$\varphi\colon \mathrm{Aut}(E) \to \mathrm{Aut}(Z(E))$$

be the map sending $\alpha \in \mathrm{Aut}(E)$ to its restriction to $Z(E)$. Since $Z(E) \simeq C_3$, it follows that $\mathrm{Aut}(Z(E)) \simeq C_2$. It is easy to see that the map φ is surjective, thus $\mathrm{Aut}(E)/\ker(\varphi) \simeq C_2$. Obviously, $\mathrm{Inn}(E) \leq \ker(\varphi)$. By [4, Exercise 5, p. 116], $\ker(\varphi)/\mathrm{Inn}(E) \simeq \mathrm{Sp}_4(3)$. Hence $\mathrm{Out}(E)$ contains a normal subgroup of index 2 which is isomorphic to $\mathrm{Sp}_4(3)$.

The map $\alpha\colon E \to E$ such that $x\alpha = x^{-1}$, $y\alpha = y$, $a\alpha = a^{-1}$, and $b\alpha = b$ is an automorphism of E of order 2, sending z to z^{-1}. Hence $\alpha \notin \ker(\varphi)$, and $\mathrm{Aut}(E) = \ker(\varphi)\langle\alpha\rangle$; in particular, $\alpha \notin \mathrm{Inn}(E)$.

Let $\{xZ(E), aZ(E), bZ(E), yZ(E)\}$ be an ordered basis of the symplectic space $E/Z(E)$. The matrix

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

represents the image of α under ψ, with respect to the aforementioned basis of $E/Z(E)$. Then

$$\mathrm{Out}(E) \simeq \mathrm{Sp}_4(3) \rtimes \langle \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \rangle = \mathrm{GSp}_4(3).$$

and the proof is completed. □

Lemma 4.14. *The group* $\mathrm{Out}_{\mathcal{F}}(E)$ *should not be contained in* $\mathrm{Sp}_4(3)$.

Proof. By [4, Exercise 5, p. 116], one has that

$$C_{\mathrm{Aut}(E)}(Z(E))/\mathrm{Inn}(E) \simeq \mathrm{Sp}_4(3).$$

Let $\alpha \in \mathrm{Aut}_{\mathcal{F}}(E)$. Thus,

$$\alpha\mathrm{Inn}(E) \in \mathrm{Sp}_4(3) \Leftrightarrow \alpha \in C_{\mathrm{Aut}(E)}(Z(E)).$$

Suppose that $\mathrm{Out}_{\mathcal{F}}(E) \leq \mathrm{Sp}_4(3)$; hence every $\alpha \in \mathrm{Aut}_{\mathcal{F}}(E)$ acts as the identity map on $Z(E)$.

Now, let T be a Sylow 2-subgroup of $N_{\mathrm{Aut}_{\mathcal{F}}(A)}(\mathrm{Aut}_S(A))$. GAP computations show explicitly that there exists $\beta \in T$ such that, in the notation of (2.2.1), $z\beta = z^{-1}$. Since A has the surjectivity property, we may extend β to $\hat{\beta} \in \mathrm{Aut}_{\mathcal{F}}(S)$. Since E is characteristic in S, we may restrict $\hat{\beta}$ to E, and again $z\hat{\beta}_{|E} = z\beta = z^{-1}$, in contradiction with the hypothesis that $\mathrm{Out}_{\mathcal{F}}(E) \leq \mathrm{Sp}_4(3)$. This completes the proof. □

We collect the information about $\mathrm{Out}_{\mathcal{F}}(E)$ in the following Lemma.

Lemma 4.15. *Let* $\mathcal{F}$ *be a reduced fusion system on* S. *Then* $\mathrm{Out}_{\mathcal{F}}(E)$ *is isomorphic to a subgroup of* $\mathrm{GSp}_4(3)$, $\mathrm{Out}_S(E) \nleq \mathrm{Sp}_4(3)$, $\mathrm{Out}_S(E)$ *is a Sylow 3-subgroup of* $\mathrm{Out}_{\mathcal{F}}(E)$, $|\mathrm{Out}_S(E)| = 3$, *and, in the notation of* (2.2.1), $\mathrm{Out}_S(E) \simeq \langle t\rangle$. *Moreover,* $O_3(\mathrm{Out}_{\mathcal{F}}(E)) = 1$.

Proof. By Lemma 4.13 and Lemma 4.14, $\mathrm{Out}_{\mathcal{F}}(E) \lesssim \mathrm{GSp}_4(3)$ and $\mathrm{Out}_{\mathcal{F}}(E)$ is not contained in $\mathrm{Sp}_4(3)$. Since E is characteristic in S, the definition of saturated fusion system implies that $\mathrm{Aut}_S(E)$ is a Sylow 3-subgroup of $\mathrm{Aut}_F(E)$: thus, $\mathrm{Out}_S(E) = \mathrm{Aut}_S(E)/\mathrm{Inn}(E)$ is a Sylow 3-subgroup of $\mathrm{Out}_{\mathcal{F}}(E)$. Since $\mathrm{Aut}_S(E) \simeq S/Z(E)$, one has that $\mathrm{Out}_S(E) \simeq S/E$: thus, $|\mathrm{Out}_S(E)| = 3$ and, in the notation of (2.2.1), $\mathrm{Out}_S(E) \simeq \langle t\rangle$. Moreover, $O_3(\mathrm{Out}_{\mathcal{F}}(E)) = 1$ by Remark 1.25, and this completes the proof. □

Now, using the information collected in Lemma 4.13, we determine all possible outer automorphism groups $\mathrm{Out}_{\mathcal{F}}(E)$.

Remark 4.16. Set $H = \mathrm{Out}_{\mathcal{F}}(E)$, $G = \mathrm{Sp}_4(3)$, and let $P \in \mathrm{Syl}_3(H)$. Then $P \in \mathrm{Syl}_3(H \cap G)$, and $O_3(H \cap G) = 1$, as otherwise P would be normal in H.

Lemma 4.17. *Let $L = UK$, where $U \simeq 2_-^{1+4}$ is normal in L, $K \simeq 2S_5$, and $C_L(U) = U \cap K = Z(U) = Z(K) = Z(L)$. Then L is uniquely determined, L is a maximal subgroup of $\mathrm{GSp}_4(3)$, and if T is a Sylow 3-subgroup of K the following hold:*

i. L *has 2-rank* 3 *and* UK' *has 2-rank* 2;

ii. $U/Z(U)$ *is an irreducible* $\mathbb{Z}_2[K]$*-module;*

iii. $N_U(T) = C_U(T) \cong D_8$ *and* $[U, T] \cong Q_8$;

iv. $U = [U, T] * C_U(T)$;

v. $C_L([U, T]) \cong SD_{16}$;

vi. $C_L(C_U(T)) \cong GL_2(3)$;

vii. $N_L([U, T]) = UN_K(T)$.

Proof. First of all note that L is uniquely determined, since the condition

$$C_L(U) = U \cap K = Z(U) = Z(K) = Z(L)$$

implies that $K/Z(K) \simeq S_5$ acts faithfully on $U/Z(U)$, and this action respects the structure of orthogonal space of $U/Z(U)$; thus, the representation of $K/Z(K)$ on $U/Z(U)$ affords the character ϕ_4 of S_5 in the notation of [1]: this yields claim *ii*, as the representation which affords the character ϕ_4 is irreducible. In particular, by Proposition 4.10, L is isomorphic to a maximal subgroup of $\mathrm{GSp}_4(3)$ of order 3840. By [17, Theorem 4.10.5], $m_2(\mathrm{Sp}_4(3)) = 2$: thus, since

$$L' = UK' = UA_4 \leq \mathrm{Sp}_4(3),$$

we get claim *i*. Using the following procedure with GAP we get claims $iii - vi$.

```
Q:=[[-1,0,0,0],
    [0,-1,0,0],
    [0,0,1,0],
    [0,0,0,1]]*Z(3)^0;
G:=Subgroup(GeneralLinearGroup(4,3),
Union([Q], SmallGeneratingSet(SymplecticGroup(4,3))));
mG:=Filtered(MaximalSubgroupClassReps(G),x->Size(x)=3840);
[ <matrix group of size 3840 with 2 generators> ]
L:=mG[1];
ns:=Filtered(NormalSubgroups(L),x->Size(x)=32);
[ <matrix group of size 32 with 6 generators> ]
U:=ns[1];
T:=SylowSubgroup(L,3);
cUT:=Centralizer(U,T);
StructureDescription(cUT);
"D8"
nUT:=Normalizer(U,T);
StructureDescription(nUT);
"D8"
cUT=nUT;
true
commUT:=CommutatorSubgroup(U,T);
StructureDescription(commUT);
"Q8"
Centralizer(U,cUT)=commUT;
true
cLcommUT:=Centralizer(L,commUT);
StructureDescription(cLcommUT);
"QD16"
cLUT:=Centralizer(L,cUT);
StructureDescription(cLUT);
"GL(2,3)"
```

Finally, since U is normal in L, $N_L(T) \leq N_L([U,T])$. Thus, $UN_K(T)$ is contained in $N_L([U,T])$. Moreover, $N_L([U,T])/U$ is isomorphic to a subgroup of S_5 and contains $N_K(T)U/U \simeq N_K(T)/Z(K)$, which is the normalizer of the Sylow 3-subgroup $TZ(K)/Z(K)$; hence $N_K(T)/Z(K) \simeq D_{12}$, $N_K(T)/Z(K)$ is isomorphic to a maximal subgroup of S_5, and $N_L([U,T])/U = N_K(T)U/U$. This yields claim *vii*, and the proof is completed. □

Lemma 4.18. *Set $H = \mathrm{Out}_{\mathcal{F}}(E)$, let T be a Sylow 3-subgroup of H, and let L be as in the statement of Lemma 4.17. Assume $H \leq L$, and let X be a Sylow 2-subgroup of $N_H(T)$. Then, one of the following holds:*

i. H is isomorphic to a subgroup of K;

ii. $H \simeq UH_1$, *where* H_1 *is a subgroup of* K, *and* $N_H(T) = C_U(T)N_{H_1}(T)$;

iii. $Z(L) \not\leq H$, $Z(H) \leq C_U(T)$, *and either* $H \simeq 2A_4$ *and* $X \simeq C_2$, *or* $H \simeq 2S_4$ *and* $X \simeq C_2 \times C_2$;

iv. $H \cap U$ *is a maximal subgroup of* $C_U(T)$, $C_H(H \cap U)$ *has index at most* 2 *in* H, *and* $H/(H \cap U)$ *is isomorphic to* A_4 *or* S_4;

v. $H = [U, T]N_H(T)$, *and* $N_H(T)/Z(U)$ *is isomorphic to* C_3, S_3, C_6, *or* D_{12};

vi. $H \cap U = [U, T]R$, *where* R *is an elementary abelian maximal subgroup of* $C_U(T)$, *and* $H/H \cap U$ *is isomorphic to* A_4 *or* S_4;

vii. $H \cap U = [U, T]R$, *where* R *is a cyclic maximal subgroup of* $C_U(T)$, $H = [U, T]N_H(T)$, *and* $N_H(T)/R$ *is isomorphic to* C_3, S_3, C_6, *or* D_{12};

viii. $H = L$, *and* $X \simeq C_2 \times QD_{16}$.

Proof. Suppose $U \leq H$. Then by Dedekind's modular low,

$$H = H \cap UK = U(H \cap K),$$

and $H \cap K$ is a subgroup of K whose order is divisible by 3. Set $H_1 = H \cap K$. By Dedekind's modular low and Lemma 4.17, we get

$$N_H(T) = N_U(T)N_{H_1}(T) = C_U(T)N_{H_1}(T).$$

Thus, claim *ii* holds.

Now, suppose $U \not\leq H$. Then, $H \cap U$ is a proper subgroup of U, $H \cap U$ is normal in H, and in particular $H \cap U$ is normalized by T. Set

$$C = C_H((H \cap U)/(H \cap U)').$$

One has that C is normal in H, and C contains $H \cap U$. If $Z(U) \not\leq H \cap U$, then $H \cap U$ has order 2, as U has 2-rank equal to 2. Hence $H \cap U \leq Z(H)$. Moreover, $U/Z(U)$ is not irreducible as $\mathbb{Z}_2[H]$-module. Since $H/(H \cap U)$ is isomorphic to a subgroup of S_5 with order divisible by 3, and by Lemma 4.2 $H/(H \cap U)$ has non-normal Sylow 3-subgroups, it follows that H is isomorphic to one of the following: $2A_4, 2S_4, 2A_5, 2S_5$. The last two cases lead to a contradiction, as $U/Z(U)$ would be irreducible as $\mathbb{Z}_2[H]$-module (cf. [1]), and this yields claim *iii*.

Now, assume that $Z(U) \leq H$. If $Z(U) = H \cap U$, then $H/Z(U)$ is isomorphic to a subgroup of L/U. In particular, $H \lesssim 2S_5$, and we get claim *i*. Thus, assume that $H \cap U > Z(U)$. If $|H \cap U| = 4$, then $H \cap U \leq C_U(T)$. By Lemma 4.2, $T(H \cap U)/(H \cap U)$ is not normal in $H/(H \cap U)$, and hence $H/(H \cap U)$ is isomorphic to one of the following groups: S_5, A_5, S_4, A_4. The first two cases lead to a contradiction, since $U/Z(U)$ would be irreducible as $\mathbb{Z}_2[H]$-module. Hence, $H/(H \cap U)$ is isomorphic either to S_4 or to A_4, and $C_H(H \cap U)$ has index at most 2 in H. This yields claim *iv*.

Let V be a subgroup of U, such that $Z(U) \leq V$ and V is normalized by T. Thus, $V/Z(U)$ is a $\mathbb{Z}_2[T]$-module. If T acts trivially on $V/Z(U)$, then $V \leq C_U(T)$. Otherwise, by Maschke's Theorem, $V/Z(U)$ is the direct sum of irreducible $\mathbb{Z}_2[T]$-modules, and T acts non-trivially at least on one of such modules. Hence, without loss of generality, we may suppose that $V/Z(U)$ is irreducible. Thus, $V = [V,T] \leq [U,T]$. Since also the 2-dimension space $[U,T]/Z(U)$ is an irreducible $\mathbb{Z}_2[T]$-module, it follows that $V = [U,T]$. In particular, this implies that the only subgroups of U containing $Z(U)$ and normalized by T are U, $[U,T]$, $C_U(T)$ and its subgroups.

Assume $|H \cap U| = 8$. If $H \cap U = C_U(T)$, then $H/(H \cap U)$ is isomorphic either to A_4 or to S_4, and $C_H(H \cap U)$ has index at most 2 in H, as in the previous point. Thus, $|C_H(H \cap U)| \geq 96$, and this is in contradiction with Lemma 4.17, stating that $C_L(C_U(T)) \simeq GL(2,3)$. Therefore, one obtains $H \cap U = [U,T]$ by the above paragraph. In particular,

$$H \leq N_L([U,T]) = UN_K(T).$$

Thus, $H/(H \cap U)$ is isomorphic to a subgroup of D_{12} (see the proof of Lemma 4.17), and Lemma 4.2 implies that $H = [U,T]N_H(T)$. Since $N_H(T) \cap [U,T] = Z(U)$, one has that $N_H(T)/Z(U)$ is isomorphic to a subgroup of D_{12} with order divisible by 3. Hence, one of the following holds: $N_H(T) = Z(U)T$, $H = [U,T]T \cong Q_8 \rtimes C_3$ and $X = Z(U) \cong C_2$; or $N_H(T)/Z(U) \cong C_3 \times C_2$ and $|X| = 4$; or $N_H(T)/Z(U) \cong S_3$ and $|X| = 4$; or $|N_H(T)| = 12$ and $|X| = 8$. This yields claim *v*.

Finally, assume $|H \cap U| = 16$. Since $Z(U) \leq H \cap U$ and $H \cap U$ is normalized by T, it follows that

$$H \cap U = [U,T]R,$$

where R is a maximal subgroup of $C_U(T)$. Hence $R = Z(H \cap U)$ is a normal subgroup of H. If $T(H \cap U)/H \cap U$ is not normal in $H/H \cap U$, then $H/H \cap U$ is isomorphic either to A_4 or to S_4. Otherwise, if $T(H \cap U)/H \cap U$ is normal in $H/H \cap U$, then by Lemma 4.2 one has that

$$H = (H \cap U)N_H(T) = N_H(T)[U,T].$$

Moreover, since $N_H(T) \cap ([U,T]R) = R$, it follows that $N_H(T)/R \simeq H/H \cap U$ is isomorphic to a subgroup of S_5 with normal Sylow 3-subgroup. Suppose that R is cyclic. Then R is normalized by $N_K(T)$, as R is characteristic in $C_U(T) = N_U(T)$. With the same argument used at the end of the proof of Lemma 4.17 we get

$$N_L([U,T]R) = UN_K(T).$$

Since $H \le N_L([U,T]R)$, one obtains

$$H/(H\cap U) \simeq HU/U \le UN_K(T)/U \simeq N_K(T)/(N_K(T)\cap U),$$

so that $T(H\cap U)/(H\cap U)$ is normal in $H/(H\cap U)$, and this completes the proof of claim *vii*. Consequently, in case *vi* one has that R is elementary abelian.

Finally, if $H = L$, direct GAP computations show that $X \simeq C_2 \times QD_{16}$, and this completes the proof. □

Lemma 4.19. *Set $H = \mathrm{Out}_{\mathcal{F}}(E)$, and $G = \mathrm{Sp}_4(3)$. Suppose that $H \cap G$ is contained in $3^{1+2} \rtimes (C_2 \times SL_2(3))$. Then one of the following holds:*

1. *$H\cap G \simeq SL_2(3)$ and $H \simeq SL_2(3).C_2$, or;*
2. *$H \cap G \simeq SL_2(3)\times C_2$ and $H \simeq (SL_2(3)\times C_2).C_2$.*

Proof. Set $N = 3^{1+2}$; then $(H\cap G)\cap N = 1$, as otherwise $O_3(H\cap G) \ne 1$, and $H \cap G \lesssim C_2 \times SL_2(3)$. It follows that either $H \cap G \simeq SL_2(3)$ and $H \simeq SL_2(3).C_2$, or $H\cap G \simeq SL_2(3)\times C_2$ and $H \simeq (SL_2(3)\times C_2).C_2$. □

Lemma 4.20. *Set $H = \mathrm{Out}_{\mathcal{F}}(E)$, and $G = \mathrm{Sp}_4(3)$. Suppose that $H \cap G$ is contained in $C_3^3 \rtimes GL_2(3)$. Then one of the followings hold:*

1. *$H\cap G \simeq GL_2(3)$ and $H \simeq GL_2(3).C_2$, or;*
2. *$H \cap G \simeq SL_2(3)$ and $H \simeq SL_2(3).C_2$.*

Proof. Set $N = C_3^3$, and let $P \in \mathrm{Syl}_3(H)$. Then $(H \cap G)\cap N = 1$, as otherwise $O_3(H\cap G) \ne 1$. Thus, $H \cap G \lesssim GL_2(3)$. It follows that either $H \cap G \simeq GL_2(3)$, and $H \simeq GL_2(3).C_2$, or $H \cap G \simeq SL_2(3)$, and $H \simeq SL_2(3).C_2$. □

Lemma 4.21. *Set $H = \mathrm{Out}_{\mathcal{F}}(E)$, and $G = \mathrm{Sp}_4(3)$. Suppose that $H \cap G$ is contained in $2A_6 \rtimes C_2$. Then one of the following holds:*

1. *$H\cap G \simeq 2A_5$ and $H \simeq 2A_5.C_2$;*
2. *$H\cap G \simeq 2S_5$ and $H \simeq 2S_5.C_2$;*
3. *$H\cap G \simeq 2A_4$ and $H \simeq 2A_4.C_2$;*
4. *$H\cap G \simeq 2S_4$ and $H \simeq 2S_4.C_2$;*
5. *$H\cap G \simeq 2A_4 \times C_2$ and $H \simeq (2A_4\times C_2).C_2$;*
6. *$H\cap G \simeq 2A_4 * C_4$ and $H \simeq (2A_4 * C_4).C_2$;*
7. *$H\cap G \simeq (2A_4\times C_2)\rtimes C_2$ and $H \simeq ((2A_4 \times C_2)\rtimes C_2).C_2$;*

8. $H \cap G \simeq (2A_4 * C_4) \rtimes C_2$ *and* $H \simeq ((2A_4 * C_4) \rtimes C_2).C_2$.

Proof. The group $(H \cap G)Z(G)/Z(G)$ is contained in the maximal subgroup S_6 of $\mathrm{PSp}_4(3)$ (see [9, p. 26] and Proposition 4.10). Thus, by Lemma 4.2 and Remark 4.16, $(H \cap G)Z(G)/Z(G)$ is isomorphic to A_5, S_5, A_4, S_4, $A_4 \times C_2$, or $S_4 \times C_2$ (see, e.g., [9, p. 4]).

Since $\mathrm{Sp}_4(3)$ has 2-rank equal to 2 (see [17, Theorem 4.10.2]), it follows that:

- if $(H \cap G)Z(G)/Z(G) \simeq A_5$, then $H \cap G \simeq 2A_5$ and $H \simeq 2A_5.C_2$;
- if $(H \cap G)Z(G)/Z(G) \simeq S_5$, then $H \cap G \simeq 2S_5$ and $H \simeq 2S_5.C_2$;
- if $(H \cap G)Z(G)/Z(G) \simeq A_4$, then $H \cap G \simeq 2A_4$ and $H \simeq 2A_4.C_2$;
- if $(H \cap G)Z(G)/Z(G) \simeq S_4$, then $H \cap G \simeq 2S_4$ and $H \simeq 2S_4.C_2$;
- if $(H \cap G)Z(G)/Z(G) \simeq A_4 \times C_2$, then either $H \cap G \simeq 2A_4 \times C_2$ and $H \simeq (2A_4 \times C_2).C_2$, or $H \cap G \simeq 2A_4 * C_4$ and $H \simeq (2A_4 * C_4).C_2$, as there are two conjugacy classes of maximal subgroups of S_6 isomorphic to $S_4 \times C_2$, and their preimages in $\mathrm{Sp}_4(3)$ are not isomorphic (see [9, p. 4]);
- if $(H \cap G)Z(G)/Z(G) \simeq S_4 \times C_2$, then either $H \cap G \simeq (2A_4 \times C_2) \rtimes C_2$ and $H \simeq ((2A_4 \times C_2) \rtimes C_2).C_2$, or $H \cap G \simeq (2A_4 * C_4) \rtimes C_2$ and $H \simeq ((2A_4 * C_4) \rtimes C_2).C_2$.

This completes the proof. $\square$

Set
$$\eta = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$
and
$$\chi = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$
In particular, $\eta \in \mathrm{Sp}_4(3)$ and $\chi \notin \mathrm{Sp}_4(3)$ (where the matrix J of the symplectic form is as in (4.0.1)).

By Proposition 4.10, the group $(SL_2(3) \times SL_2(3)) \rtimes \langle \eta \rangle$ is a maximal subgroup of $\mathrm{Sp}_4(3)$.

Set $R = \langle \eta, \chi \rangle \simeq C_2 \times C_2$, and $M = (SL_2(3) \times SL_2(3)) \rtimes R$. Then M is a maximal subgroup of $\mathrm{GSp}_4(3)$.

Lemma 4.22. *Set $H = \mathrm{Out}_{\mathcal{F}}(E)$, and $G = \mathrm{Sp}_4(3)$. Let η, χ, R, and M be as above. Suppose $H \leq M$, and set $K = S_1 \times S_2 = SL_2(3) \times SL_2(3)$. Then one of the following holds:*

1. *$|H \cap S_i| = 2$ for both $i = 1, 2$, and $H = (H \cap K)W$, with W a subgroup of R of order at least 2;*

2. *$|H \cap S_i| = 8$ for both $i = 1, 2$, and either $H \simeq (Q_8 \times Q_8) \rtimes S_3$ or $H \simeq (Q_8 \times Q_8) \rtimes (S_3 \times C_2)$;*

3. *$|H \cap S_1| = 2$ and $|H \cap S_2| = 8$ (or viceversa), and $H \simeq (C_2 \times Q_8) \rtimes S_3$.*

Proof. Let T be a Sylow 3-subgroup of H; up to conjugation, we may suppose that $T = \langle t\mathrm{Inn}(E)\rangle$. With respect to the aforementioned basis $\{xZ(E), aZ(E), bZ(E), yZ(E)\}$ of $E/Z(E)$,

$$T = \langle \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \rangle.$$

Therefore, explicit computations show that T is normalized by R. Moreover, $T \leq H \cap K$, and by Frattini's Argument (see, e.g., [4, 6.2]),

$$H = (H \cap K)N_H(T).$$

Let $Y \in \mathrm{Syl}_2(N_H(T))$: thus, $N_H(T) = T \rtimes Y$; in particular,

$$H = (H \cap K)Y.$$

One has that $N_K(T) = Z(K)P$, where P is the Sylow 3-subgroup of K which contains T. In particular, $Z(K)$ is the unique Sylow 2-subgroup of $N_K(T)$, and $Z(K)R \simeq D_8 \times C_2$ is the Sylow 2-subgroup of $N_M(T)$.

Since $H \cap K \trianglelefteq H$ and T is not normal in H, it follows that T is not normal in $H \cap K$. Thus, $|H \cap K| > 6$.

Suppose $|H \cap K| = 12$. The alternating group A_4 is the unique group of order 12 with no normal Sylow 3-subgroups. Then $H \cap K \simeq A_4$, and the (unique) Sylow 2-subgroup Q of $H \cap K$ is isomorphic to $C_2 \times C_2$. Moreover, $(H \cap K) \cap Z(K) = 1$, as $Z(A_4) = 1$. Since $Z(K) \simeq C_2 \times C_2$, it follows that $QZ(K) \simeq C_2^4$, a contradiction, as $m_2(\mathrm{GSp}_4(3)) = 3$ (cf. [17, Theorem 4.10.5]). Thus, necessarily $|H \cap K| > 12$.

If $H \cap S_i = 1$ for some $i \in \{1, 2\}$, then $H \cap K \simeq SL_2(3)$, as $|H \cap K| \geqslant 24$. In particular,

$$N_{H \cap K}(T) = TZ(H \cap K) \simeq C_3 \times C_2.$$

Since a Sylow 2-subgroup of $N_H(T)$ is contained in $Z(H \cap K)R \simeq C_2^3$, also the Sylow 2-subgroups of $N_H(T)$ are elementary abelian, in contradiction with Lemma 4.7 and Proposition 4.12.

Thus, necessarily $H \cap S_i \neq 1$ for both $i = 1, 2$. Suppose there exists $k \in H \cap S_1$, such that $|\langle k \rangle| = 3$. Hence

$$k = \begin{pmatrix} x_1 & x_2 & 0 & 0 \\ x_3 & x_4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

for some $x_i \in \mathbb{F}_3$. This yields a contradiction, as T and $\langle k \rangle$ are Sylow 3-subgroups of H, but they are not conjugate in M. Therefore, $|H \cap S_i| = 2^\alpha$, for some α. Since $H \cap S_i \trianglelefteq H \cap K$, in particular $H \cap S_i$ is normalized by $T \simeq C_3$. The only subgroups of S_i with order a power of 2 which are normalized by T are $Z(S_i) \simeq C_2$ and the Sylow 2-subgroup of S_i (isomorphic to Q_8), as $S_i \simeq SL_2(3)$. This implies that $Z(K) \leq H$.

If $|H \cap S_i| = 2$ for both $i = 1, 2$, then $|(H \cap K)/Z(K)| \leq 12$, as otherwise $|H \cap S_1| > 2$ or $|H \cap S_2| > 2$. Since T is not normal in $H \cap K$, one has that $|(H \cap K)/Z(K)| = 12$, and $(H \cap K)/Z(K) \simeq A_4$. Let

$$\pi : H \cap K \twoheadrightarrow (H \cap K)/Z(K)$$

be the canonical projection, and let Q be the full preimage of the Sylow 2-subgroup of $(H \cap K)/Z(K)$ under the map π: thus, $|Q| = 16$. Recall that the Sylow 2-subgroup P of K is isomorphic to $Q_8 \times Q_8$, and for every $k \in P \smallsetminus Z(K)$ one has that k has order 4. This implies that $Q/Z(K) \simeq C_2 \times C_2$, and

$$Q = Z(K)\langle g, h \rangle,$$

for some $g, h \notin Z(K)$. Note that $Q/Z(K)$ has 3 proper subgroups, $\langle g \rangle Z(K)$, $\langle h \rangle Z(K)$, and $\langle gh \rangle Z(K)$, and T acts on these subgroups as a 3-cycle (as $(H \cap K)/Z(K) \simeq A_4$). Suppose that $g^t \in \langle h \rangle Z(K)$: thus, $g^t = h^\alpha a$, for some $\alpha \in \{1, 3\}$ and $a \in Z(K)$, and

$$g^2 = (g^2)^t = (g^t)^2 = (h^\alpha a)^2 = h^{2\alpha} = h^2.$$

If $gh = hg$, then $Z(K) \leq \langle g, h \rangle \simeq C_4 \times C_2$, and $|Q| = 8$, a contradiction. If $g^t \in \langle gh \rangle Z(K)$, similar computations hold. Therefore $\langle g, h \rangle \simeq Q_8$, and $Q \simeq Q_8 \times C_2$. Moreover, $H \cap K \simeq SL_2(3) \times C_2$. Since $H = (H \cap K)N_H(T)$, $H \nleq \mathrm{Sp}_4(3)$ (cf. Lemma 4.13), and $Z(K) \leq H$, one obtains that $\chi \in H$ or $\eta\chi \in H$, and

$$H = (H \cap K)W,$$

with W a subgroup of R of order at least 2.

If $|H \cap S_i| = 8$ for both $i = 1, 2$, then

$$H \cap K = ((H \cap S_1) \times (H \cap S_2))T \simeq (Q_8 \times Q_8) \rtimes C_3,$$

and either $H \simeq (Q_8 \times Q_8) \rtimes S_3$ or $H \simeq (Q_8 \times Q_8) \rtimes (S_3 \times C_2)$.

Finally, if $|H \cap S_1| = 2$ and $|H \cap S_2| = 8$, then

$$H \cap K = ((H \cap S_1) \times (H \cap S_2))T \simeq (C_2 \times Q_8) \rtimes C_3.$$

Thus, $C_2 \times Q_8 \trianglelefteq H$, and H contains χ or $\eta\chi$. The element $\eta\chi$ acts on $Z(K) = Z(S_1) \times Z(S_2)$ and interchanges $Z(S_1)$ with $Z(S_2)$: this is not possible, as $Z(S_2)$ is generated by the square of an element of order 4 of $H \cap K$, and $Z(S_1)$ is not. $\square$

Now we are ready to state the main Theorem. In order to determine completely and uniquely the structure of $\mathrm{Out}_{\mathcal{F}}(E)$ we need GAP, which assignes the isomorphism class of every finite "small" group a label $[n, m]$, where n is the order of the group, and m is the "position" of the group in the GAP list of all groups of order n.

Theorem 4.23. *Let $\mathcal{F}$ be a reduced fusion system on a Sylow 3-subgroup S of the McLaughlin group* Mc. *Then the triplet* $(\mathrm{Out}_{\mathcal{F}}(S), \mathrm{Out}_{\mathcal{F}}(A), \mathrm{Out}_{\mathcal{F}}(E))$ *associated to $\mathcal{F}$ is one of the following:*

i. $(C_4, A_6, SL_2(3) \rtimes C_2 = [48, 33])$;

ii. $(C_4, A_6, C_2.S_4 = [48, 28])$;

iii. $(C_4 \times C_2, C_2 \times A_6, (SL_2(3) \rtimes C_2) \rtimes C_2 = [96, 192])$;

iv. $(C_4 \times C_2, C_2 \times A_6, SL_2(5) \rtimes C_2 = [240, 93])$;

v. $(D_8, S_6, GL_2(3) \rtimes C_2 = [96, 193])$;

vi. $(D_8, S_6, (C_2 \times SL_2(3)) \rtimes C_2 = [96, 190])$;

vii. $(D_8, S_6, ((Q_8 \times Q_8) \rtimes C_3) \rtimes C_2 = [384, 18130])$;

viii. $(Q_8, M_{10}, (C_2.S_4) \rtimes C_2 = [96, 191])$;

ix. $(Q_8, M_{10}, (SL_2(3) \rtimes C_2) \rtimes C_2 = [96, 201])$;

x. $(Q_8, M_{10}, C_2.S_5 = [240, 89])$;

xi. $(C_2 \times D_8 C_2 \times S_6, (C_2 \times GL_2(3)) \rtimes C_2 = [192, 1485])$;

xii. $(C_2 \times D_8, C_2 \times S_6, (((Q_8 \times Q_8) \rtimes C_3) \rtimes C_2) \rtimes C_2 = [768, 1086054])$;

xiii. $(C_2 \times Q_8, C_2 \times M_{10}, ((C_2.S_4) \rtimes C_2) \rtimes C_2 = [192, 1483])$;

xiv. $(C_2 \times Q_8, C_2 \times M_{10}, (C_2.S_5) \rtimes C_2 = [480, 947])$;

xv. $(C_2 \times C_8, A_6 \rtimes C_4, (SL_2(3) \rtimes C_4) \rtimes C_2 = [192, 963])$;

xvi. $(C_2 \times QD_{16}, (C_2 \times M_{10}) \rtimes C_2, ((((C_8 \times C_2) \rtimes C_2) \rtimes C_3) \rtimes C_2) \rtimes C_2 = [384, 18045])$;

xvii. $(C_2 \times QD_{16}, (C_2 \times M_{10}) \rtimes C_2, C_2 \times SD_{16}, 2_-^{1+4} * 2S_5)$.

Proof. Let T be a Sylow 3-subgroup of $\mathrm{Out}_{\mathcal{F}}(E)$, and let X be a Sylow 2-subgroup of $N_{\mathrm{Out}_{\mathcal{F}}(E)}(T)$. Up to conjugation, $T = \langle t\mathrm{Inn}(E)\rangle$. Recall that, by Lemma 4.14, $\mathrm{Out}_{\mathcal{F}}(E) \not\leq \mathrm{Sp}_4(3)$.

Suppose that, following the notation of Lemma 4.17, $\mathrm{Out}_{\mathcal{F}}(E) \leq L = UK$. If $\mathrm{Out}_{\mathcal{F}}(E) \simeq L \simeq 2_-^{1+4} * 2S_5$, then $X \simeq C_2 \times QD_{16}$ (cf. Lemma 4.18, statement *viii*).

If $\mathrm{Out}_{\mathcal{F}}(E)$ is isomorphic to a subgroup of K (cf. Lemma 4.18, statement *i*), GAP computations show that $\mathrm{Out}_{\mathcal{F}}(E) \simeq C_2.S_4 = SL_2(3).C_2 = [48, 28]$, and $X \simeq C_4$.

```
t:=[[1,0,0,0],
    [0,1,0,0],
    [0,1,1,0],
    [1,0,0,1]]*Z(3)^0;
T:=Subgroup(GeneralLinearGroup(4,3),[t]);
mG:=Filtered(MaximalSubgroups(G),x->Size(x)=3840 and
    IsSubgroup(x,T)=true);
L:=mG[1];
ns:=Filtered(NormalSubgroups(L),x->Size(x)=32);
[ <matrix group of size 32 with 6 generators> ]
U:=ns[1];
m:=Filtered(MaximalSubgroups(L),x->Size(x)=240 and
   IsSubgroup(x,T)=true);
K:=m[1];
maxK:=Filtered(MaximalSubgroups(K),x->IsSubgroup(x,T)=true));
smaxK:=List([1..4],x->Filtered(Union
(ConjugacyClassesSubgroups
(LatticeSubgroups(maxK[x]))),y->IsSubgroup(y,T)=true));
sK:=Union(smaxK);
sgK:=Filtered(sK,x->IsSubgroup(DerivedSubgroup(K),x)=false
and IsNormal(x,T)=false);
List(sgK,x->[IdGroup(x),StructureDescription(x),
StructureDescription(SylowSubgroup(Normalizer(x,T),2))]);
[ [ [ 48, 28 ], "C2 . S4 = SL(2,3) . C2", "C4" ],
  [ [ 48, 28 ], "C2 . S4 = SL(2,3) . C2", "C4" ] ]
```

If $\mathrm{Out}_{\mathcal{F}}(E) \simeq UH_1$, with $H_1 \leq K$ (cf. Lemma 4.18, statement *ii*), GAP computations show that one of the following hold:

1a. $\mathrm{Out}_{\mathcal{F}}(E) \simeq ((Q_8 \times Q_8) \rtimes C_3) \rtimes C_2) \rtimes C_2 = [768, 1086054]$, and $X \simeq C_2 \times D_8$;

2a. $\mathrm{Out}_{\mathcal{F}}(E) \simeq ((((C_8 \times C_2) \rtimes C_2) \rtimes C_3) \rtimes C_2) \rtimes C_2 = [384, 18045]$, and $X \simeq C_2 \rtimes QD_{16}$;

3a. $\mathrm{Out}_{\mathcal{F}}(E) \simeq (C_2 \times GL_2(3)) \rtimes C_2 = [192, 1485]$, and $X \simeq C_2 \times D_8$;

4a. $\mathrm{Out}_{\mathcal{F}}(E) \simeq (((C_8 \times C_2) \rtimes C_2) \rtimes C_3) \rtimes C_2 = [192, 1018]$, and $X \simeq QD_{16}$.

By Lemma 4.7 and Proposition 4.12, the last of these four cases leads to a contradiction, and hence it is not acceptable.

```
ssK:=Filtered(sK,x->IsSubgroup(DerivedSubgroup(K),x)=false);
s2:=List(ssK, x->Subgroup(L,Union
(SmallGeneratingSet(U),SmallGeneratingSet(x))));
s2f:=List(s2, x->[IdGroup(x),StructureDescription(x),
StructureDescription(SylowSubgroup(Normalizer(x,T),2))]);
[ [ [ 768, 1086054 ], "(((Q8xQ8):C3):C2):C2", "C2xD8" ],
  [ [ 768, 1086054 ], "(((Q8xQ8):C3):C2):C2", "C2xD8" ],
  [ [ 384, 18045 ], "((((C8xC2):C2):C3):C2):C2", "C2xQD16" ],
  [ [ 192, 1485 ], "(C2xGL(2,3)):C2", "C2xD8" ],
  [ [ 192, 1018 ], "(((C8xC2):C2):C3):C2", "QD16" ] ]
```

If $\mathrm{Out}_{\mathcal{F}}(E) \simeq 2A_4$ or $\mathrm{Out}_{\mathcal{F}}(E) \simeq 2S_4$ (cf. Lemma 4.18, statement *iii*), then respectively $X \simeq C_2$ or $X \simeq C_2 \times C_2$, in contradiction with Lemma 4.7 and Proposition 4.12.

If $\mathrm{Out}_{\mathcal{F}}(E) \cap U$ is a maximal subgroup of $C_U(T)$, Lemma 4.18, statement *iv*, implies that $\mathrm{Out}_{\mathcal{F}}(E)/(\mathrm{Out}_{\mathcal{F}}(E) \cap U)$ is isomorphic to A_4 or to S_4. GAP computations show that one of the following hold:

1b. $\mathrm{Out}_{\mathcal{F}}(E) \simeq (C_2 \times SL_2(3)) \rtimes C_2 = [96, 190]$, and $X \simeq D_8$;

2b. $\mathrm{Out}_{\mathcal{F}}(E) \simeq C_2 \times GL_2(3)$, and $X \simeq C_2 \times C_2 \times C_2$;

3b. $\mathrm{Out}_{\mathcal{F}}(E) \simeq C_2 \times SL_2(3)$, and $X \simeq C_2 \times C_2$.

By Lemma 4.7 and Proposition 4.12, the last two cases lead to a contradiction, and hence they are not acceptable.

```
maxL:=Filtered(MaximalSubgroups(L),x->IsSubgroup(x,T)=true);
smaxL:=List([1..8],x->Filtered(Union(ConjugacyClassesSubgroups
(LatticeSubgroups(maxL[x]))),y->IsSubgroup(y,T)=true));
sL:=Union(smaxL);
sgL:=Filtered(sL,x->IsSubgroup(DerivedSubgroup(K),x)=false and
IsNormal(x,T)=false);
F1:=Filtered(sgL,x->Size(Intersection(x,U))=4 and Size(x)=96);
List(F1,x->StructureDescription(x/Intersection(x,U)));
[ "S4", "S4", "S4", "S4", "S4", "S4", "S4", "S4" ]
List(F1,x->[IdGroup(x),StructureDescription(x),
StructureDescription
(SylowSubgroup(Normalizer(x,SylowSubgroup(x,3)),2))]);
[ [ [ 96, 190 ], "(C2 x SL(2,3)) : C2", "D8" ],
  [ [ 96, 189 ], "C2 x GL(2,3)", "C2 x C2 x C2" ],
```

```
  [ [ 96, 189 ], "C2 x GL(2,3)", "C2 x C2 x C2" ],
  [ [ 96, 190 ], "(C2 x SL(2,3)) : C2", "D8" ],
  [ [ 96, 190 ], "(C2 x SL(2,3)) : C2", "D8" ],
  [ [ 96, 190 ], "(C2 x SL(2,3)) : C2", "D8" ],
  [ [ 96, 189 ], "C2 x GL(2,3)", "C2 x C2 x C2" ],
  [ [ 96, 189 ], "C2 x GL(2,3)", "C2 x C2 x C2" ] ]
F2:=Filtered(sgL,x->Size(Intersection(x,U))=4 and Size(x)=48);
List(F2,x->StructureDescription(x/Intersection(x,U)));
[ "A4", "A4", "A4", "A4", "A4", "A4", "A4", "A4" ]
List(F2,x->[IdGroup(x),StructureDescription(x),
StructureDescription
(SylowSubgroup(Normalizer(x,SylowSubgroup(x,3)),2))]);
[ [ [ 48, 32 ], "C2 x SL(2,3)", "C2 x C2" ],
  [ [ 48, 32 ], "C2 x SL(2,3)", "C2 x C2" ],
  [ [ 48, 32 ], "C2 x SL(2,3)", "C2 x C2" ],
  [ [ 48, 32 ], "C2 x SL(2,3)", "C2 x C2" ],
  [ [ 48, 32 ], "C2 x SL(2,3)", "C2 x C2" ],
  [ [ 48, 32 ], "C2 x SL(2,3)", "C2 x C2" ],
  [ [ 48, 32 ], "C2 x SL(2,3)", "C2 x C2" ],
  [ [ 48, 32 ], "C2 x SL(2,3)", "C2 x C2" ] ]
```

If $|\mathrm{Out}_{\mathcal{F}}(E) \cap U| = 8$, and $\mathrm{Out}_{\mathcal{F}}(E) \simeq [U,T]N_{\mathrm{Out}_{\mathcal{F}}(E)}(T)$ (cf. Lemma 4.18, statement v), then GAP computations show that one of the following holds:

1c. $\mathrm{Out}_{\mathcal{F}}(E) \simeq (C_2.S_4) \rtimes C_2 = [96, 192]$, and $X \simeq C_4 \times C_2$;

2c. $\mathrm{Out}_{\mathcal{F}}(E) \simeq (C_2.S_4) \rtimes C_2 = [96, 191]$, and $X \simeq Q_8$;

3c. $\mathrm{Out}_{\mathcal{F}}(E) \simeq C_2 \times GL_2(3)$, and $X \simeq C_2 \times C_2 \times C_2$;

4c. $\mathrm{Out}_{\mathcal{F}}(E) \simeq GL_2(3)$, and $X \simeq C_2 \times C_2$;

5c. $\mathrm{Out}_{\mathcal{F}}(E) \simeq C_2.S_4 = [48, 28]$, and $X \simeq C_4$;

6c. $\mathrm{Out}_{\mathcal{F}}(E) \simeq SL_2(3) \rtimes C_2 = [48, 33]$, and $X \simeq C_4$.

The third and the fourth cases lead to a contradiction, by Lemma 4.7 and Proposition 4.12.

```
nT:=Normalizer(L,T);
lnT:=LatticeSubgroups(nT);
<subgroup lattice of <matrix group of
 size 96 with 5 generators>, 68 classes, 186 subgroups>
snT:=List([1..68],x->ConjugacyClassesSubgroups(lnT)[x][1]);
sg1nT:=Filtered(snT,x->IsSubgroup(x,T)=true and
Size(Intersection(x,U))=2);
sg2nT:=Filtered(sg1nT,x->IsSubgroup
```

```
(DerivedSubgroup(L),x)=false);
s1L:=AsSet(List(sg2nT,
x->Subgroup(L,Union(SmallGeneratingSet(x),
   SmallGeneratingSet(commUT)))));
List([1..7],x->[IdGroup(s1L[x]),StructureDescription(s1L[x]),
StructureDescription(SylowSubgroup(Normalizer(s1L[x],T),2))]);
[ [ [ 96, 192 ], "(C2 . S4 = SL(2,3) . C2) : C2", "C4 x C2" ],
  [ [ 96, 191 ], "(C2 . S4 = SL(2,3) . C2) : C2", "Q8" ],
  [ [ 96, 189 ], "C2 x GL(2,3)", "C2 x C2 x C2" ],
  [ [ 48, 29 ], "GL(2,3)", "C2 x C2" ],
  [ [ 48, 28 ], "C2 . S4 = SL(2,3) . C2", "C4" ],
  [ [ 48, 33 ], "SL(2,3) : C2", "Q8" ],
  [ [ 48, 29 ], "GL(2,3)", "C2 x C2" ] ]
```

If $\mathrm{Out}_{\mathcal{F}}(E) \cap U = [U,T]R$, with R an elementary abelian maximal subgroup of $C_U(T)$, then $\mathrm{Out}_{\mathcal{F}}(E)/(\mathrm{Out}_{\mathcal{F}}(E) \cap U)$ is isomorphic to A_4 or to S_4 (cf. Lemma 4.18, statement *vi*). GAP computations show that either $\mathrm{Out}_{\mathcal{F}}(E) \simeq ((Q_8 \times Q_8) \rtimes C_3) \rtimes C_2$ and $X \simeq C_2 \times C_2 \times C_2$ (in contradiction with Lemma 4.7 and Proposition 4.12), or $\mathrm{Out}_{\mathcal{F}}(E) \simeq ((Q_8 \times Q_8) \rtimes C_3) \rtimes C_2 = [384, 18130]$ and $X \simeq D_8$.

```
f:=Filtered(sL,x->IsSubgroup(DerivedSubgroup(L),x)=false and
IsNormal(x,T)=false);
mcUT:=MaximalSubgroupClassReps(cUT);
List(mcUT,x->Exponent(x));
[ 2, 2, 4 ]
R1:=mcUT[1];
f1:=Filtered(f,x->Size(x)=384 and Size(Intersection(x,U))=16
and Intersection(x,U)=Subgroup(L,
Union(SmallGeneratingSet(commUT),SmallGeneratingSet(R1))));
List(f1,x->[IdGroup(x),StructureDescription(x),
StructureDescription(SylowSubgroup(Normalizer(x,T),2))]);
[ [ [ 384, 18130 ], "((Q8 x Q8) : C3) : C2", "D8" ],
  [ [ 384, 18131 ], "((Q8 x Q8) : C3) : C2", "C2 x C2 x C2" ] ]
f2:=Filtered(f,x->Size(x)=192 and Size(Intersection(x,U))=16
and Intersection(x,U)=Subgroup(L,
Union(SmallGeneratingSet(commUT),SmallGeneratingSet(R1))));
[ ]
R2:=mcUT[2];
f3:=Filtered(f,x->Size(x)=384 and Size(Intersection(x,U))=16
and Intersection(x,U)=Subgroup(L,
Union(SmallGeneratingSet(commUT),SmallGeneratingSet(R2))));
List(f3,x->[IdGroup(x),StructureDescription(x),
StructureDescription(SylowSubgroup(Normalizer(x,T),2))]);
[ [ [ 384, 18131 ], "((Q8 x Q8) : C3) : C2", "C2 x C2 x C2" ],
```

```
  [ [ 384, 18130 ], "((Q8 x Q8) : C3) : C2", "D8" ] ]
f4:=Filtered(f,x->Size(x)=192 and Size(Intersection(x,U))=16
and Intersection(x,U)=Subgroup(L,
Union(SmallGeneratingSet(commUT),SmallGeneratingSet(R2))));
[  ]
```

If $\mathrm{Out}_{\mathcal{F}}(E) = [U,T]RN_{\mathrm{Out}_{\mathcal{F}}(E)}(T)$, with R a cyclic maximal subgroup of $C_U(T)$ (cf. Lemma 4.18, statement vii), then GAP computations show that one of the following holds:

1d. $\mathrm{Out}_{\mathcal{F}}(E) \simeq ((C_2.S_4) \rtimes C_2) \rtimes C_2 = [192, 965]$, and $X \simeq QD_{16}$;

2d. $\mathrm{Out}_{\mathcal{F}}(E) \simeq (SL_2(3) \rtimes C_4) \rtimes C_2 = [192, 988]$, and $X \simeq QD_{16}$;

3d. $\mathrm{Out}_{\mathcal{F}}(E) \simeq ((C_8 \times C_2) \rtimes C_2) \rtimes C_3 = [96, 74]$, and $X \simeq C_8$;

4d. $\mathrm{Out}_{\mathcal{F}}(E) \simeq C_2 \times GL_2(3)$, and $X \simeq C_2 \times C_2 \times C_2$;

5d. $\mathrm{Out}_{\mathcal{F}}(E) \simeq ((C_2.S_4) \rtimes C_2) \rtimes C_2 = [192, 1483]$, and $X \simeq C_2 \times Q_8$;

6d. $\mathrm{Out}_{\mathcal{F}}(E) \simeq (((C_8 \times C_2) \rtimes C_2) \rtimes C_3) \rtimes C_2 = [192, 963]$, and $X \simeq C_8 \times C_2$;

7d. $\mathrm{Out}_{\mathcal{F}}(E) \simeq (C_2 \times SL_2(3)) \rtimes C_2 = [96, 190]$, and $X \simeq D_8$;

8d. $\mathrm{Out}_{\mathcal{F}}(E) \simeq (C_2.S_4) \rtimes C_2 = [96, 192]$, and $X \simeq C_4 \times C_2$;

9d. $\mathrm{Out}_{\mathcal{F}}(E) \simeq (SL_2(3) \rtimes C_2) \rtimes C_2 = [96, 201]$, and $X \simeq Q_8$;

10d. $\mathrm{Out}_{\mathcal{F}}(E) \simeq GL_2(3) \rtimes C_2 = [96, 193]$, and $X \simeq D_8$.

Cases 1d-4d lead to a contradiction, by Lemma 4.7 and Proposition 4.12.

```
sg3nT:=Filtered(snT,x->IsSubgroup(x,T)=true and
Size(Intersection(x,U))=4);
sg4nT:=Filtered(sg3nT,x->IsSubgroup
(DerivedSubgroup(L),x)=false);
s7L:=AsSet(List(sg4nT,x->Subgroup(L,Union(SmallGeneratingSet(x),
SmallGeneratingSet(commUT)))));
List([1..10],x->[IdGroup(s7L[x]),StructureDescription(s7L[x]),
StructureDescription(SylowSubgroup(Normalizer(s7L[x],T),2))]);
[ [ [ 192, 965 ], "((C2.S4=SL(2,3).C2):C2):C2", "QD16" ],
  [ [ 192, 963 ], "(SL(2,3):C4):C2", "C8xC2" ],
  [ [ 96, 74 ], "((C8xC2):C2):C3", "C8" ],
  [ [ 192, 1483 ], "((C2.S4=SL(2,3).C2):C2):C2", "C2xQ8" ],
  [ [ 96, 190 ], "(C2 x SL(2,3)) : C2", "D8" ],
  [ [ 96, 192 ], "(C2 . S4 = SL(2,3) . C2) : C2", "C4 x C2" ],
  [ [ 96, 189 ], "C2 x GL(2,3)", "C2 x C2 x C2" ],
  [ [ 192, 988 ], "(((C8 x C2) : C2) : C3) : C2", "QD16" ],
  [ [ 96, 201 ], "(SL(2,3) : C2) : C2", "Q8" ],
  [ [ 96, 193 ], "GL(2,3) : C2", "D8" ] ]
```

Suppose that $\mathrm{Out}_{\mathcal{F}}(E) \cap \mathrm{Sp}_4(3)$ is contained in the maximal subgroups $3^{1+2} \rtimes (C_2 \times SL_2(3))$ or $C_3^3 \rtimes GL_2(3)$ of $\mathrm{Sp}_4(3)$. Thus, $\mathrm{Out}_{\mathcal{F}}(E)$ is contained in a maximal subgroup of $\mathrm{GSp}_4(3)$ of order 2592. One has that $\mathrm{Out}_{\mathcal{F}}(E) \simeq SL_2(3).C_2$, or $\mathrm{Out}_{\mathcal{F}}(E) \simeq (SL_2(3) \times C_2).C_2$, or $\mathrm{Out}_{\mathcal{F}}(E) \simeq GL_2(3).C_2$ (see Lemmas 4.19 and 4.20). GAP computations show that $\mathrm{Out}_{\mathcal{F}}(E) \simeq GL_2(3)$ and $X \simeq C_2 \times C_2$, in contradiction with Lemma 4.7 and Proposition 4.12.

```
M:=Filtered(MaximalSubgroups(G),x->Size(x)=2592 and
IsSubgroup(x,T)=true);
[ <matrix group of size 2592 with 2 generators>,
  <matrix group of size 2592 with 2 generators>,
  <matrix group of size 2592 with 2 generators>,
  <matrix group of size 2592 with 2 generators>,
  <matrix group of size 2592 with 2 generators> ]
IsConjugate(G,M[1],M[2]);
false
IsConjugate(G,M[1],M[3]);
false
IsConjugate(G,M[1],M[4]);
false
IsConjugate(G,M[1],M[5]);
false
IsConjugate(G,M[2],M[3]);
true
IsConjugate(G,M[2],M[4]);
true
IsConjugate(G,M[2],M[5]);
true
maxM1:=Filtered(MaximalSubgroups(M[1]),
x->IsSubgroup(x,T)=true);
smaxM1:=List([1..10],x->Filtered(Union
(ConjugacyClassesSubgroups(LatticeSubgroups(maxM1[x]))),
y->IsSubgroup(y,T)=true));
sM1:=Union(smaxM1);
f:=Filtered(sM1,x->IsSubgroup(SymplecticGroup(4,3),x)=false and
IsNormal(x,T)=false);
f1:=Filtered(f,x->StructureDescription
  (Intersection(x,SymplecticGroup(4,3)))="SL(2,3)");
[  ]
f2:=Filtered(f,x->StructureDescription
(Intersection(x,SymplecticGroup(4,3)))="SL(2,3) x C2");
[  ]
f3:=Filtered(f,x->StructureDescription
(Intersection(x,SymplecticGroup(4,3)))="GL(2,3)");
```

```
[ ]
maxM2:=Filtered(MaximalSubgroups(M[2]),
x->IsSubgroup(x,T)=true);
smaxM2:=List([1..7],x->Filtered(Union
(ConjugacyClassesSubgroups(LatticeSubgroups(maxM2[x]))),
y->IsSubgroup(y,T)=true));
sM2:=Union(smaxM2);
F:=Filtered(sM2,x->IsSubgroup(SymplecticGroup(4,3),x)=false
and IsNormal(x,T)=false);
F1:=Filtered(F,x->StructureDescription
(Intersection(SymplecticGroup(4,3),x))="SL(2,3)");
List(F1,x->[StructureDescription(x),
StructureDescription(SylowSubgroup(Normalizer
(x,SylowSubgroup(x,3)),2))]);
[ [ "GL(2,3)", "C2 x C2" ], [ "GL(2,3)", "C2 x C2" ],
  [ "GL(2,3)", "C2 x C2" ], [ "GL(2,3)", "C2 x C2" ],
  [ "GL(2,3)", "C2 x C2" ], [ "GL(2,3)", "C2 x C2" ],
  [ "GL(2,3)", "C2 x C2" ], [ "GL(2,3)", "C2 x C2" ],
  [ "GL(2,3)", "C2 x C2" ], [ "GL(2,3)", "C2 x C2" ],
  [ "GL(2,3)", "C2 x C2" ], [ "GL(2,3)", "C2 x C2" ],
  [ "GL(2,3)", "C2 x C2" ], [ "GL(2,3)", "C2 x C2" ],
  [ "GL(2,3)", "C2 x C2" ], [ "GL(2,3)", "C2 x C2" ],
  [ "GL(2,3)", "C2 x C2" ], [ "GL(2,3)", "C2 x C2" ] ]
F2:=Filtered(F,x->StructureDescription(Intersection
(SymplecticGroup(4,3),x))="SL(2,3) x C2");
[ ]
F3:=Filtered(F,x->StructureDescription(Intersection
(SymplecticGroup(4,3),x))="GL(2,3)");
[ ]
```

Suppose that $\mathrm{Out}_{\mathcal{F}}(E) \cap \mathrm{Sp}_4(3)$ is contained in the maximal subgroup $2A_6 \rtimes C_2$ of $\mathrm{Sp}_4(3)$ (cf. Lemma 4.21). Then $\mathrm{Out}_{\mathcal{F}}(E)$ is contained in a maximal subgroup of $\mathrm{GSp}_4(3)$ of order 2880, and GAP computations show that one of the following holds:

1e. $\mathrm{Out}_{\mathcal{F}}(E) \simeq SL_2(3) \rtimes C_2 = [48, 33]$, and $X \simeq C_4$;

2e. $\mathrm{Out}_{\mathcal{F}}(E) \simeq C_2.S_42 = [48, 28]$, and $X \simeq C_4$;

3e. $\mathrm{Out}_{\mathcal{F}}(E) \simeq (C_2.S_4) \rtimes C_2 = [96, 192]$, and $X \simeq C_4 \times C_2$;

4e. $\mathrm{Out}_{\mathcal{F}}(E) \simeq SL_2(5) \rtimes C_2 = [240, 93]$, and $X \simeq C_4 \times C_2$;

5e. $\mathrm{Out}_{\mathcal{F}}(E) \simeq (C_2.S_4) \rtimes C_2 = [96, 191]$, and $X \simeq Q_8$;

6e. $\mathrm{Out}_{\mathcal{F}}(E) \simeq (SL_2(3) \rtimes C_2) \rtimes C_2 = [96, 201]$, and $X \simeq Q_8$;

7e. $\mathrm{Out}_{\mathcal{F}}(E) \simeq C_2.S_5 = SL_2(5).C_2 = [240, 89]$, and $X \simeq Q_8$;

8e. $\mathrm{Out}_{\mathcal{F}}(E) \simeq (C_2.S_5) \rtimes C_2 = [480, 947]$, and $X \simeq C_2 \times Q_8$.

```
Filtered(MaximalSubgroupClassReps(G),x->Size(x)=2880);
[ <matrix group of size 2880 with 2 generators> ]
M:=Filtered(MaximalSubgroups(G),x->Size(x)=2880 and
IsSubgroup(x,T)=true);
maxM:=Filtered(MaximalSubgroups(M[1]),
x->IsSubgroup(x,T)=true);
smaxM:=List([1..10],x->Filtered(Union
(ConjugacyClassesSubgroups(LatticeSubgroups(maxM[x]))),
y->IsSubgroup(y,T)=true));
sM:=Union(smaxM);
F:=Filtered(sM,x->IsSubgroup(SymplecticGroup(4,3),x)=false
and IsNormal(x,T)=false);
F1:=Filtered(F,x->Size(x)=240);
List(F1,x->[IdGroup(x),StructureDescription(x),
StructureDescription(SylowSubgroup(Normalizer(x,T),2))]);
[ [ [ 240, 93 ], "SL(2,5) : C2", "C4 x C2" ],
  [ [ 240, 93 ], "SL(2,5) : C2", "C4 x C2" ],
  [ [ 240, 89 ], "C2 . S5 = SL(2,5) . C2", "Q8" ],
  [ [ 240, 89 ], "C2 . S5 = SL(2,5) . C2", "Q8" ],
  [ [ 240, 89 ], "C2 . S5 = SL(2,5) . C2", "Q8" ],
  [ [ 240, 93 ], "SL(2,5) : C2", "C4 x C2" ] ]
F2:=Filtered(F,x->Size(x)=480);
List(F2,x->[IdGroup(x),StructureDescription(x),
StructureDescription(SylowSubgroup(Normalizer(x,T),2))]);
  [ [ [ 480, 947 ], "(C2 . S5 = SL(2,5) . C2) : C2", "C2 x Q8" ],
    [ [ 480, 947 ], "(C2 . S5 = SL(2,5) . C2) : C2", "C2 x Q8" ],
    [ [ 480, 947 ], "(C2 . S5 = SL(2,5) . C2) : C2", "C2 x Q8" ] ]
F3:=Filtered(F,x->Size(x)=48);
List(F3,x->[IdGroup(x),StructureDescription(x),
StructureDescription(SylowSubgroup(Normalizer(x,T),2))]);
[ [ [ 48, 29 ], "GL(2,3)", "C2 x C2" ],
  [ [ 48, 28 ], "C2 . S4 = SL(2,3) . C2", "C4" ],
  [ [ 48, 33 ], "SL(2,3) : C2", "C4" ],
  [ [ 48, 33 ], "SL(2,3) : C2", "C4" ],
  [ [ 48, 33 ], "SL(2,3) : C2", "C4" ],
  [ [ 48, 33 ], "SL(2,3) : C2", "C4" ],
  [ [ 48, 28 ], "C2 . S4 = SL(2,3) . C2", "C4" ],
  [ [ 48, 28 ], "C2 . S4 = SL(2,3) . C2", "C4" ],
  [ [ 48, 29 ], "GL(2,3)", "C2 x C2" ],
  [ [ 48, 33 ], "SL(2,3) : C2", "C4" ],
  [ [ 48, 33 ], "SL(2,3) : C2", "C4" ],
```

```
  [ [ 48, 29 ], "GL(2,3)", "C2 x C2" ] ]
F4:=Filtered(F,x->Size(x)=96);
List(F4,x->[IdGroup(x),StructureDescription(x),
StructureDescription(SylowSubgroup(Normalizer(x,T),2))]);
[ [ [ 96, 192 ], "(C2 . S4 = SL(2,3) . C2) : C2", "C4 x C2" ],
  [ [ 96, 192 ], "(C2 . S4 = SL(2,3) . C2) : C2", "C4 x C2" ],
  [ [ 96, 192 ], "(C2 . S4 = SL(2,3) . C2) : C2", "C4 x C2" ],
  [ [ 96, 201 ], "(SL(2,3) : C2) : C2", "Q8" ],
  [ [ 96, 191 ], "(C2 . S4 = SL(2,3) . C2) : C2", "Q8" ],
  [ [ 96, 191 ], "(C2 . S4 = SL(2,3) . C2) : C2", "Q8" ],
  [ [ 96, 192 ], "(C2 . S4 = SL(2,3) . C2) : C2", "C4 x C2" ],
  [ [ 96, 191 ], "(C2 . S4 = SL(2,3) . C2) : C2", "Q8" ],
  [ [ 96, 191 ], "(C2 . S4 = SL(2,3) . C2) : C2", "Q8" ],
  [ [ 96, 201 ], "(SL(2,3) : C2) : C2", "Q8" ],
  [ [ 96, 192 ], "GL(2,3) : C2", "C4 x C2" ],
  [ [ 96, 192 ], "GL(2,3) : C2", "C4 x C2" ],
  [ [ 96, 192 ], "GL(2,3) : C2", "C4 x C2" ],
  [ [ 96, 191 ], "(C2 . S4 = SL(2,3) . C2) : C2", "Q8" ],
  [ [ 96, 191 ], "(C2 . S4 = SL(2,3) . C2) : C2", "Q8" ],
  [ [ 96, 201 ], "(SL(2,3) : C2) : C2", "Q8" ],
  [ [ 96, 192 ], "GL(2,3) : C2", "C4 x C2" ],
  [ [ 96, 192 ], "(C2 . S4 = SL(2,3) . C2) : C2", "C4 x C2" ] ]
```

Finally, suppose that, in the notation of Lemma 4.22, $\mathrm{Out}_{\mathcal{F}}(E) \leq K \rtimes R$. Thus, $\mathrm{Out}_{\mathcal{F}}(E) \cap K$ is isomorphic to $SL_2(3) \times C_2$, $(Q_8 \times Q_8) \rtimes C_3$, or $(C_2 \times Q_8) \rtimes C_3$ (see the proof of Lemma 4.22). GAP computations show that one of the following holds:

1f. $\mathrm{Out}_{\mathcal{F}}(E) \simeq (C_2 \times GL_2(3)) \rtimes C_2 = [192, 1485]$, and $X \simeq C_2 \times D_8$;

2f. $\mathrm{Out}_{\mathcal{F}}(E) \simeq (C_2 \times SL_2(3)) \rtimes C_2 = [96, 190]$, and $X \simeq D_8$;

3f. $\mathrm{Out}_{\mathcal{F}}(E) \simeq (((Q_8 \times Q_8) \rtimes C_3) \rtimes C_2) \rtimes C_2 = [768, 1086054]$, and $X \simeq C_2 \times D_8$;

4f. $\mathrm{Out}_{\mathcal{F}}(E) \simeq ((Q_8 \times Q_8) \rtimes C_3) \rtimes C_2 = [384, 18130]$, and $X \simeq D_8$;

5f. $\mathrm{Out}_{\mathcal{F}}(E) \simeq C_2 \times GL_2(3)$, and $X \simeq C_2 \times C_2 \times C_2$;

6f. $\mathrm{Out}_{\mathcal{F}}(E) \simeq ((Q_8 \times Q_8)) \rtimes C_3) \rtimes C_2 = [384, 18131]$, and $X \simeq C_2 \times C_2 \times C_2 \times C_2$.

The last two cases are in contradiction with Lemma 4.7 and Proposition 4.12, and hence not acceptable.

```
Filtered(MaximalSubgroupClassReps(G),x->Size(x)=2304);
[ <matrix group of size 2304 with 2 generators> ]
M:=Filtered(MaximalSubgroups(G),x->Size(x)=2304 and
IsSubgroup(x,T)=true);
Filtered(MaximalSubgroupClassReps(G),x->Size(x)=2304);
[ <matrix group of size 2304 with 2 generators> ]
maxM:=Filtered(MaximalSubgroups(M[1]),x->IsSubgroup(x,T)=true);
smaxM:=List([1..7],x->Filtered(Union(ConjugacyClassesSubgroups
(LatticeSubgroups(maxM[x]))),y->IsSubgroup(y,T)=true));
sM:=Union(smaxM);
F:=Filtered(sM,x->IsSubgroup(SymplecticGroup(4,3),x)=false and
IsNormal(x,T)=false);
F1:=Filtered(F,x->Size(Intersection(x,K))=48);
List(F1,x->[IdGroup(x),StructureDescription(x),
StructureDescription(SylowSubgroup(Normalizer(x,T),2))]);
[ [ [ 192, 1485 ], "(C2 x GL(2,3)) : C2", "C2 x D8" ],
  [ [ 96, 189 ], "C2 x GL(2,3)", "C2 x C2 x C2" ],
  [ [ 96, 190 ], "(C2 x SL(2,3)) : C2", "D8" ],
  [ [ 96, 189 ], "C2 x GL(2,3)", "C2 x C2 x C2" ],
  [ [ 96, 189 ], "C2 x GL(2,3)", "C2 x C2 x C2" ],
  [ [ 96, 189 ], "C2 x GL(2,3)", "C2 x C2 x C2" ],
  [ [ 192, 1485 ], "(C2 x GL(2,3)) : C2", "C2 x D8" ],
  [ [ 96, 189 ], "C2 x GL(2,3)", "C2 x C2 x C2" ],
  [ [ 96, 190 ], "(C2 x SL(2,3)) : C2", "D8" ],
  [ [ 192, 1485 ], "(C2 x GL(2,3)) : C2", "C2 x D8" ],
  [ [ 96, 189 ], "C2 x GL(2,3)", "C2 x C2 x C2" ],
  [ [ 96, 190 ], "(C2 x SL(2,3)) : C2", "D8" ],
  [ [ 96, 189 ], "C2 x GL(2,3)", "C2 x C2 x C2" ],
  [ [ 96, 189 ], "C2 x GL(2,3)", "C2 x C2 x C2" ],
  [ [ 96, 189 ], "C2 x GL(2,3)", "C2 x C2 x C2" ] ]
F2:=Filtered(F,x->Size(Intersection(x,K))=192);
List(F2,x->[IdGroup(x),StructureDescription(x),
StructureDescription(SylowSubgroup(Normalizer(x,T),2))]);
[ [ [ 768, 1086054 ], "(((Q8xQ8):C3):C2):C2", "C2xD8" ],
  [ [ 384, 18131 ], "((Q8xQ8):C3):C2", "C2xC2xC2" ],
  [ [ 768, 1086054 ], "(((Q8xQ8):C3):C2):C2", "C2xD8" ],
  [ [ 384, 18131 ], "((Q8xQ8):C3):C2", "C2xC2xC2" ],
  [ [ 768, 1086054 ], "(((Q8xQ8):C3):C2):C2", "C2xD8" ],
  [ [ 384, 18131 ], "((Q8xQ8):C3):C2", "C2xC2xC2" ],
  [ [ 384, 18130 ], "((Q8xQ8):C3):C2", "D8" ] ]
```

Thus, we have determined all acceptable $\mathrm{Out}_{\mathcal{F}}(E)$, and combining the previous part of the proof with Lemma 4.6, Lemma 4.7, and Theorem 4.11 it follows that the acceptable 3-tuples $(\mathrm{Out}_{\mathcal{F}}(S), \mathrm{Out}_{\mathcal{F}}(A), \mathrm{Out}_{\mathcal{F}}(E))$ are the ones listed in the statement, and the proof is completed. $\square$

Chapter 5

Sources of the reduced fusion systems on S

The aim of this Chapter is to find which reduced fusion systems on S are "induced" by finite groups containing S as Sylow 3-subgroup, and which one are exotic. In fact, we show that a reduced fusion system on S which is not exotic is the fusion system of an almost simple group containing S as Sylow 3-subgroup.

Remark 5.1. Let $H \simeq A_6$, let P be a Sylow 3-subgroup of H, and let Q be a subgroup of P, such that $|Q| = 3$. Then Q is not normal in $N_H(P)$.

Proof. We may assume that $P = \langle (1, 2, 3), (4, 5, 6)\rangle$. Then

$$N_H(P) = \langle (1, 2, 3), (4, 5, 6), (1, 4, 2, 5)(3, 6)\rangle,$$

and conjugation with the element $(1, 4, 2, 5)(3, 6)$ interchanges the subgroup $\langle (1, 2, 3)\rangle$ with the subgroup $\langle (4, 5, 6)\rangle$, and the subgroup $\langle (1, 2, 3)(4, 5, 6)\rangle$ with the subgroup $\langle (1, 3, 2)(4, 5, 6)\rangle$. $\square$

Lemma 5.2. *Let $\mathcal{F}$ be a reduced fusion system on S. Then S is the unique non trivial strongly $\mathcal{F}$-closed subgroup of S.*

Proof. Suppose there exists $1 \neq P \lneq S$, such that P is strongly $\mathcal{F}$-closed: in particular, by definition $P \trianglelefteq S$. Set $Y = P \cap A$. Since P is normal in S, it follows that $Z(S) \cap P \neq 1$. Thus, $Z(S) \leq P$, as $|Z(S)| = 3$. Since $Z(S) \leq A$, it follows that $Y \neq 1$. Moreover, Y is $\mathrm{Aut}_{\mathcal{F}}(A)$-invariant, since P is strongly $\mathcal{F}$-closed. By Theorem 4.11, $\mathrm{Aut}_{\mathcal{F}}(A) \gtrsim A_6$: thus, $\mathrm{Aut}_{\mathcal{F}}(A)$ acts irreducibly on A, as the action of A_6 on A is irreducible (see, e.g., [1]). Therefore $Y = A$, and $P \geqslant A$.

Suppose that $P = A$. Since A is abelian, one has that $P \trianglelefteq \mathcal{F}$ (cf. [12, Corollary 5.38]), a contradiction, as $\mathcal{F}$ is reduced. Hence $P \gneq A$. Set $N = N_{\mathrm{Aut}_{\mathcal{F}}(A)}(\mathrm{Aut}_S(A))$. Every automorphism in N extends to a automorphism of S, as A has the surjectivity property by Lemma 1.35. Thus, we may consider the action of N on S/A. Since the Sylow 3-subgroups of $\mathrm{Aut}_{\mathcal{F}}(A)$ have order 3^2 (see Lemma 4.8), and $\mathrm{Aut}_{\mathcal{F}}(A) \gtrsim A_6$ (see Theorem 4.11), Remark 5.1 implies that there are no proper subgroups of S containing A which are normalized by N. It follows that N acts irreducibly on S/A. Hence P/A is not $\mathrm{Aut}_{\mathcal{F}}(A)$-invariant, a contradiction, since P is strongly $\mathcal{F}$-closed. Thus, $P = S$ and the claim holds. □

Proposition 5.3. *Let $\mathcal{F}$ be a reduced fusion system on S, and suppose $\mathcal{F}$ is the fusion system of a finite group. Then $\mathcal{F}$ is the fusion system of an almost simple group G, such that $G/F^*(G)$ is a $3'$-group.*

Proof. Suppose that G is a finite group of minimal order such that $\mathcal{F} = \mathcal{F}_S(G)$. It is easy to show that $\mathcal{F}_S(G) = \mathcal{F}_{\bar{S}}(\bar{G})$, where $\bar{G} = G/K$, K is a normal $3'$-subgroup of G, and $\bar{S} = SK/K$; in particular, one may consider $\bar{G} = G/O_{3'}(G)$. Then the minimal choice of G implies that $O_{3'}(G) = 1$.

Let K be a minimal normal subgroup of G; hence 3 divides $|K|$, and $S \cap K \neq 1$ (as otherwise the product SK would be a subgroup of G with Sylow 3-subgroups too big). Since $S \cap K$ is strongly $\mathcal{F}$-closed, Lemma 5.2 implies that $S \cap K = S$ and $S \leq K$. It follows that $K = O^{3'}(G)$ and K is the unique minimal normal subgroup of G. Thus, K is a direct product of isomorphic simple groups, and such groups are not abelian (as $S \leq K$ is not abelian); in particular, $S \neq K$, as otherwise K would be the direct product of six copies of C_3.

If K has more than one factor, then S factorizes as a direct product of non-trivial subgroups. Suppose $S = P_1 \times P_2$; since $Z(P_1) \leq Z(S)$, $Z(P_2) \leq Z(S)$, and $|Z(S)| = 3$, it follows that $Z(P_1) = Z(P_2) = Z(S)$ and $P_1 \cap P_2 \neq 1$, a contradiction. Hence K is simple and the claim follows by the uniqueness of K. □

By the proof of the previous Proposition, if $\mathcal{F} = \mathcal{F}_S(G)$ for a finite almost simple group G containing S as Sylow 3-subgroup, then $F^*(G) = K$ is a simple group with Sylow 3-subgroups isomorphic to S. Thus, it is enough to determine the finite simple groups with Sylow 3-subgroups isomorphic to S.

5.1 Sporadic Groups

Proposition 5.4. *The only sporadic groups whose Sylow 3-subgroups are isomorphic to S are the McLaughlin group* Mc*, and the Conway group* Co_2.

Proof. By [4, Table 16.3], the sporadic groups whose Sylow 3-subgroups have order 3^6 are the McLaughlin group Mc, the Conway group Co_2, and the Harada-Norton group F_5. The Sylow 3-subgroups of Mc are isomorphic to S by construction. One has that Mc is contained in Co_2 (see [9, p. 154]), hence also the Sylow 3-subgroups of Co_2 are isomorphic to S. GAP computations show that the elements of order 3 of a Sylow 3-subgroup P of the group F_5 are 512, while those of the group S are 296 (see Lemma 2.8): then the Sylow 3-subgroups of F_5 are not isomorphic to S.

```
F5:=SmallSimpleGroup(273030912000000);
P:=SylowSubgroup(F5,3);
f:=Filtered(P,x->Order(x)=3);
Length(f);
512
```

This yields the claim. □

5.2 Alternating Groups

Lemma 5.5. *The alternating groups whose Sylow 3-subgroups have order 3^6 are A_{15}, A_{16}, and A_{17}.*

Proof. Since $|A_n| = n!/2$, the claim follows by direct calculations. □

Proposition 5.6. *There are no alternating groups whose Sylow 3-subgroups are isomorphic to S.*

Proof. A Sylow 3-subgroup P of the alternating group A_{15} is the group

$$\langle (1,2,3),(4,5,6),(7,8,9),(10,11,12),(13,14,15)\rangle \rtimes \langle (1,7,4)(2,8,5)(3,9,6)\rangle.$$

Hence $P \simeq C_3^5 \rtimes C_3$, and $m_3(P) = 5$. Since $m_3(S) = 4$, P and S are not isomorphic.

With a similar argument, it follows that the Sylow 3-subgroups of A_{16} and A_{17} have respectively 3-rank equal to 6 and 7, and then they are not isomorphic to S. This yields the claim. □

5.3 Groups of Lie Type

Throughout this Section, we refer to [17, §2.2]. For every group of Lie type K, there exists a unique universal version K_u, such that there is an epimorphism $K_u \to K$ (cf. [17, Theorem 2.2.6]). In particular, if K is simple, one has $K \simeq K_u/Z(K_u)$.

For a simple group of Lie type $K(q)$ over a field of order q, one has that

$$|Z(K_u(q))| \cdot |K(q)| = q^N \prod_{i=1}^{m} (q^{d_i} - \epsilon_i), \tag{5.3.1}$$

where $\epsilon_i \in \{\pm 1\}$, and N, m, d_i, and ϵ_i depend on the type of the group $K(q)$ (see [17, Table 2.2]). Further, one may decompose every $(q^{d_i} - \epsilon_i)$ as product of cyclotomic polynomials evaluated in q. Thus,

$$|Z(K_u(q))| \cdot |K(q)| = q^N \prod_{k \in \mathbf{H}} \Phi_k(q), \tag{5.3.2}$$

where N and the index set $\mathbf{H}$ depend on the type of the group $K(q)$.

Lemma 5.7. *The simple groups of Lie type over a field of characteristic 3 whose Sylow 3-subgroups have order* 3^6 *are* $PSL_2(3^6)$, $PSL_3(3^2)$, $PSL_4(3)$, $G_2(3)$, $PSU_3(3^2)$, *and* $PSU_4(3)$.

Proof. It follows by [4, Table 16.2] and Equation (5.3.2). □

Proposition 5.8. *The only simple group of Lie type over a field of characteristic 3 whose Sylow 3-subgroups are isomorphic to S is the group* $PSU_4(3)$.

Proof. By Lemma 5.7, one knows which simple groups of Lie type over a field of characteristic 3 have Sylow 3-subgroups of order 3^6. We proceed with a case-by-case analysis.

1. It is well known that $|Z(SL_2(3^6)| = (2, 3^6 - 1) = 2$, hence a Sylow 3-subgroup of $PSL_2(3^6)$ is isomorphic to a Sylow 3-subgroup of $SL_2(3^6)$, under the projection map

$$SL_2(3^6) \to SL_2(3^6)/Z(SL_2(3^6)).$$

   A Sylow 3-subgroup of $SL_2(3^6)$ is the group of the upper unitriangular matrices

$$U = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mid a \in \mathbb{F}_{3^6} \right\} \simeq \mathbb{F}_{3^6}.$$

   Since U is abelian, one has that even the Sylow 3-subgroups of $PSL_2(3^6)$ are abelian, and hence not isomorphic to S.

2. Since $|Z(SL_3(3^2)| = (3, 3^2 - 1) = 1$, it follows that $PSL_3(3^2) = SL_3(3^2)$. A Sylow 3-subgroup of $SL_3(3^2)$ is the group of the upper unitriangular matrices

$$U = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{F}_{3^2} \right\}.$$

   Since $|Z(U)| = 9$ and $|Z(S)| = 3$, U and S are not isomorphic.

3. It is well known that $|Z(SL_4(3)| = (4, 2) = 2$, hence a Sylow 3-subgroup of $PSL_4(3)$ is isomorphic to a Sylow 3-subgroup of $SL_4(3)$, under the projection map

$$SL_4(3) \to SL_4(3)/Z(SL_4(3)).$$

Let T be a Sylow 3-subgroup of $SL_4(3)$. By [34, Lemma 2.1.3], [34, Lemma 2.1.4] and [34, Proposition 2.1.8], $T = \langle x, y, z, a, b, t \mid \mathcal{P} \rangle$, where $\mathcal{P}$ is the set of defining relations

$$\mathcal{P} = \left\{ \begin{array}{l} x^3 = y^3 = z^3 = a^3 = b^3 = t^3 = 1, \\ [x, y] = [a, b] = z \\ [x, z] = [y, z] = [a, z] = [b, z] = [t, z] = 1 \\ [x, a] = [x, b] = [y, a] = [y, b] = 1 \\ [x, t] = [a, t] = 1,\ [y, t] = xz,\ [b, t] = a^{-1}z^{-1} \end{array} \right\}.$$

Thus, the Sylow 3-subgroups of $SL_4(3)$ are not isomorphic to S by (2.2.1).

4. One has that $M = (3_+^{1+2} \times C_3 \times C_3) \rtimes 2S_4$ is isomorphic to a maximal subgroup of $G_2(3)$ (see [9, p. 61]). Hence at least a Sylow 3-subgroup of $G_2(3)$ is contained in M. Then the Sylow 3-subgroups of $G_2(3)$ are isomorphic to $(3_+^{1+2} \times C_3 \times C_3) \rtimes C_3$, and they have more than one elementary abelian subgroup of order 3^4. By Lemma 2.5, the Sylow 3-subgroups of $G_2(3)$ are not isomorphic to S.

5. It is well known that $|Z(SU_3(3^2))| = (3, 3^2-1) = 1$. Hence $PSU_3(3^2) = SU_3(3^2)$. Since $SU_3(3^2) \leq SL_3(3^2)$ and the Sylow 3-subgroups of $SU_3(3^2)$ and $SL_3(3^2)$ have order 3^6, it follows that a Sylow 3-subgroup of $SU_3(3^2)$ is isomorphic to the ones of $SL_3(3^2)$, and hence by point 2 not isomorphic to S.

6. Since $PSU_4(3)$ is contained in Mc (see [9, p. 100]), obviously the Sylow 3-subgroups of $PSU_4(3)$ are isomorphic to S.

This completes the proof. □

Now we proceed with the analysis in the case of simple groups of Lie type over a field of characteristic not 3.

Definition 5.9. Let $K_u(q)$ be the universal version of a simple group $K(q)$ over a field of order q, and let p be a prime such that p divides $|K_u(q)|$ and p does not divide q. We denote with m_0 the multiplicative period of $[q]_p$, i.e., the multiplicative period of q modulo p.

Proposition 5.10. *Let $K(q)$ be a simple group of Lie type over a field of order q, and let $K_u(q)$ be the universal version of $K(q)$. Let p be a prime such that p divides the order of $K(q)$ and p does not divide q. Then, in the notation of Definition 5.9, $m_p(K_u(q)) = m_0$. Moreover, $m_p(K(q))$ is either m_0 or $m_0 - 1$, and in the latter case p divides $|Z(K_u)|$.*

Proof. See [17, Theorem 4.10.3]. □

Formula (5.3.2) and Proposition 5.10 imply the following.

Lemma 5.11. *Let $K(q)$ be a simple group of Lie type over a field of characteristic not 3 such that $m_3(K(q)) = 4$, and assume that the Sylow 3-subgroups of $K(q)$ have order 3^6. Then one of the following holds:*

1. $K(q) = E_6(q)$*, where* $3 \mid q+1$ *and* $9 \nmid q+1$*;*
2. $K(q) = F_4(q)$*, where either* $3 \mid q-1$ *and* $9 \nmid q-1$ *or* $3 \mid q+1$ *and* $9 \nmid q+1$*;*
3. $K(q) = {}^2E_6(q)$*, where* $3 \mid q-1$ *and* $9 \nmid q-1$*;*
4. $K(q) = PSL_6(q)$*, where* $3 \mid q-1$ *and* $9 \nmid q-1$*;*
5. $K(q) = PSU_6(q)$*, where* $3 \mid q+1$ *and* $9 \nmid q+1$*.*

Proof. By applying Proposition 5.10 and Formula (5.3.2) to our case, one obtains the statement. □

Proposition 5.12. *Let $K(q)$ be a group of Lie type over a field of characteristic not 3, whose Sylow 3-subgroups are isomorphic to S. Then one of the following holds:*

1. $K(q) = PSL_6(q)$*, where* $3 \mid q-1$ *and* $9 \nmid q-1$*, or;*
2. $K(q) = PSU_6(q)$*, where* $3 \mid q+1$ *and* $9 \nmid q+1$*.*

Proof. By Lemma 5.11, we know which groups of Lie type over a field of characteristic not 3 have 3-rank equal to 4, and Sylow 3-subgroups of order 3^6. We proceed with a case-by-case analysis.

The Sylow 3-subgroups of the group $F_4(2)$ are isomorphic to the Sylow 3-subgroups of $PSL_4(3)$ (see [9, p. 170]). Hence by Proposition 5.8, the Sylow 3-subgroups of $F_4(2)$ are not isomorphic to S. Let $q = 2^n$, such that either $3 \mid q-1$ and $9 \nmid q-1$, or $3 \mid q+1$ and $9 \nmid q+1$; since $F_4(2) \leq F_4(q)$, at least a Sylow 3-subgroup of $F_4(q)$ is contained in $F_4(2)$. It follows that the Sylow 3-subgroups of $F_4(q)$ are isomorphic to the ones of $F_4(2)$, and hence not isomorphic to S. Now let p be a prime, with $p \neq 2, 3$. The group $(C_3 \times C_3 \times C_3) \rtimes SL_3(3)$ is (up to isomorphism) a subgroup of $F_4(p)$ (see [32, p. 160]). Let $q = p^n$, such that either $3 \mid q-1$ and $9 \nmid q-1$, or $3 \mid q+1$ and $9 \nmid q+1$. Since $F_4(p) \leq F_4(q)$, the Sylow 3-subgroups of $F_4(q)$ are isomorphic to $Q = (C_3 \times C_3 \times C_3) \rtimes P$, where P is a Sylow 3-subgroup of $SL_3(3)$. We claim that Q is not isomorphic to S. Suppose that S is isomorphic to Q: then one may write S as the semidirect product of a normal subgroup R isomorphic to $C_3 \times C_3 \times C_3$, and a subgroup T isomorphic to P. By Lemma 2.10, one has that S' is the unique normal elementary abelian subgroup of S of order 3^3: hence $R = S'$. If $S = S' \rtimes T$, then $T \simeq S/S' = S/\Phi(S)$ is elementary abelian. Since $|Z(P)| = 3$, one has that T and P are not isomorphic. Then Q is not isomorphic to S. This completes the proof in the case of $F_4(q)$.

By [32, p. 168] and [32, p. 173], $F_4(q)$ is contained in $E_6(q)$ and in ${}^2E_6(q)$. Thus, by the previous part of the proof the Sylow 3-subgroups of $E_6(q)$ and ${}^2E_6(q)$ are not isomorphic to S.

Since the projective special unitary group $PSU_6(q)$ is defined over the field with q^2 elements, it follows that $PSU_6(q) \leq PSL_6(q^2)$. Let q be a power of a prime such that 3 divides $q+1$ and 9 does not divide $q+1$: obviously 3 divides q^2-1 and 9 does not divide q^2-1. By [17, Table 2.2] and Proposition 5.10, the Sylow 3-subgroups of $PSL_6(q^2)$ have order 3^6 and 3-rank equal to 4, and it suffices to study the Sylow 3-subgroups of $PSL_6(q)$, where q is a power of a prime such that 3 divides $q-1$ and 9 does not divide $q-1$. Let $\zeta \in \mathbb{F}_q$, such that $\zeta^3 = 1$, and let D be the group of the diagonal matrices of $GL_6(q)$: then a Sylow 3-subgroup of the group $D \cap SL_6(q)$ is isomorphic to C_3^5, and it is generated, e.g., by the matrices

$$v_1 = \begin{pmatrix} \zeta & 0 & 0 & 0 & 0 & 0 \\ 0 & \zeta^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \zeta & 0 & 0 & 0 \\ 0 & 0 & 0 & \zeta^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$v_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \zeta & 0 \\ 0 & 0 & 0 & 0 & 0 & \zeta^{-1} \end{pmatrix}, \quad v_4 = \begin{pmatrix} \zeta & 0 & 0 & 0 & 0 & 0 \\ 0 & \zeta & 0 & 0 & 0 & 0 \\ 0 & 0 & \zeta & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$v_5 = \begin{pmatrix} \zeta & 0 & 0 & 0 & 0 & 0 \\ 0 & \zeta & 0 & 0 & 0 & 0 \\ 0 & 0 & \zeta & 0 & 0 & 0 \\ 0 & 0 & 0 & \zeta & 0 & 0 \\ 0 & 0 & 0 & 0 & \zeta & 0 \\ 0 & 0 & 0 & 0 & 0 & \zeta \end{pmatrix}.$$

Identify S_6 with the group of the permutation matrices of $GL_6(q)$. Since a Sylow 3-subgroup of $GL_6(q)$ is contained in $N_{GL_6(q)}(D) = D \rtimes S_6$ (cf. [17, Theorem 4.10.2]), it follows that a Sylow 3-subgroup of $SL_6(q)$ is the group $\langle v_1, v_2, v_3, v_4, v_5, v_6, v_7 \rangle$, where v_6 and v_7 are respectively the matrices "associated" to the 3-cycles $(1,2,3)$ and $(4,5,6)$, i.e.,

$$v_6 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad v_7 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Set $Z = Z(SL_6(q))$. Thus, a Sylow 3-subgroup of $PSL_6(q)$ is the group $T = \langle v_1Z, v_2Z, v_3Z, v_4Z, v_6Z, v_7Z\rangle$. Since $\langle v_1Z, v_2Z, v_3Z, v_4Z\rangle \simeq (C_3)^4$, $\langle v_6Z, v_7Z\rangle$ is isomorphic to a Sylow 3-subgroup of A_6, and A_6 has a unique representation of dimension four over the field with 3 elements (see [1]), T is isomorphic to S, and this completes the proof. $\square$

We summarize the results obtained so far in this Chapter in the following.

Theorem 5.13. *Let G be a finite simple group with Sylow 3-subgroups isomorphic to S. Then G is one of the following:*

1. *the McLaughlin group* Mc*;*
2. *the Conway group* Co_2*;*
3. $PSU_4(3)$*,*
4. $PSL_6(q)$*, where* $3 \mid q-1$ *and* $9 \nmid q-1$*;*
5. $PSU_6(q)$*, where* $3 \mid q+1$ *and* $9 \nmid q+1$.

5.4 The correspondence between triplets and groups

In this Section we determine the triplets $(\mathrm{Out}_{\mathcal{F}}(S), \mathrm{Out}_{\mathcal{F}}(A), \mathrm{Out}_{\mathcal{F}}(E))$ in the case when $\mathcal{F} = \mathcal{F}_S(G)$ and G is a finite almost simple group with Sylow 3-subgroups isomorphic to S, such that $F^*(G)$ is simple and the Sylow 3-subgroups of G are contained in $F^*(G)$. Moreover, we prove that four of the triplets listed in Theorem 4.23 are "induced" by exotic fusion systems.

We proceed as follows: for such a finite almost simple group, we compute explicitly with GAP the triplet $(\mathrm{Out}_{\mathcal{F}}(S), \mathrm{Out}_{\mathcal{F}}(A), \mathrm{Out}_{\mathcal{F}}(E))$, where $\mathcal{F} = \mathcal{F}_S(G)$. In this case by definition one has

$$\begin{aligned} \mathrm{Aut}_{\mathcal{F}}(S) &\simeq N_G(S)/C_G(S), \\ \mathrm{Aut}_{\mathcal{F}}(A) &\simeq N_G(A)/C_G(A), \\ \mathrm{Aut}_{\mathcal{F}}(E) &\simeq N_G(E)/C_G(E). \end{aligned}$$

Since for every such G one has $C_G(S) = Z(S)$, and also $C_G(A) = Z(A)$ and $C_G(E) = Z(E)$, it follows that

$$\begin{aligned} \mathrm{Out}_{\mathcal{F}}(S) &\simeq N_G(S)/S, \\ \mathrm{Out}_{\mathcal{F}}(A) &\simeq N_G(A)/A, \\ \mathrm{Out}_{\mathcal{F}}(E) &\simeq N_G(E)/E. \end{aligned}$$

The computation performed with GAP leads to the following table, where we use the ATLAS notation to indicate the different extensions of the same simple group, and in the third column we indicate the position of the triplet in the list of Theorem 4.23.

G	$(\mathrm{Out}_{\mathcal{F}}(S), \mathrm{Out}_{\mathcal{F}}(A), \mathrm{Out}_{\mathcal{F}}(E))$	$n°$
Mc	$(Q_8, M_{10}, [240, 89])$	x
Mc.2	$(C_2 \times Q_8, C_2 \times M_{10}, [480, 947])$	xiv
Co_2	$(C_2 \times QD_{16}, (C_2 \times M_{10}) \rtimes C_2, 2_-^{1+4} * 2S_5)$	$xvii$
$PSU_4(3)$	$(C_4, A_6, [48, 28])$	ii
$PSU_4(3).2_1$	$(C_2 \times C_4, C_2 \times A_6, [96, 192])$	iii
$PSU_4(3).4$	$(C_2 \times C_8, A_6 \rtimes C_4, [192, 963])$	xv
$PSU_4(3).2_2$	$(D_8, S_6, [96, 190])$	vi
$PSU_4(3).(2^2)_{122}$	$(C_2 \times D_8, C_2 \times S_6, [192, 1485])$	xi
$PSU_4(3).2_3$	$(Q_8, M_{10}, [96, 191])$	$viii$
$PSU_4(3).(2^2)_{133}$	$(C_2 \times Q_8, C_2 \times M_{10}, [192, 1483])$	$xiii$
$PSU_4(3).D_8$	$(C_2 \times QD_{16}, (C_2 \times M_{10}) \rtimes C_2, [384, 18045])$	xvi
$PSU_6(2)$	$(D_8, S_6, [384, 18130])$	vii
$PSU_6(2).2$	$(C_2 \times D_8, C_2 \times S_6, [768, 1086054])$	xii

Remark 5.14. The configurations obtained with $PSU_6(2)$ and its extension $PSU_6(2).2$ can be obtained also with $PSL_6(4)$ and its extension $PSL_6(4)\langle\phi\rangle$, where ϕ is a field automorphism of order 2. Moreover, there is evidence that the same holds for all groups $PSU_6(q)$ (where $3 \mid q+1$ and $9 \nmid q+1$) and $PSL_6(q)$ (where $3 \mid q-1$ and $9 \nmid q-1$), and their (acceptable) extensions.

Theorem 5.15. *Let $\mathcal{F}$ be a reduced fusion system on S, such that the 3-tuple $(\mathrm{Out}_{\mathcal{F}}(S), \mathrm{Out}_{\mathcal{F}}(A), \mathrm{Out}_{\mathcal{F}}(E))$ is one of the following:*

$$\begin{aligned} &(C_4, A_6, SL_2(3) \rtimes C_2 = [48, 33]), \\ &(C_2 \times C_4, C_2 \times A_6, SL_2(5) \rtimes C_2 = [240, 93]), \\ &(Q_8, M_{10}, (SL_2(3) \rtimes C_2) \rtimes C_2 = [96, 201]), \\ &(D_8, S_6, GL_2(3) \rtimes C_2 = [96, 193]). \end{aligned}$$

Then $\mathcal{F}$ is exotic.

Proof. Suppose that $\mathcal{F}$ is not exotic. Then by Proposition 5.3 and Theorem 5.13, there exists a finite almost simple group G with $F^*(G)$ isomorphic to one of the simple groups listed in Theorem 5.13 and Sylow 3-subgroups contained in $F^*(G)$, such that $\mathcal{F} = \mathcal{F}_S(G)$. By the previous Table, it follows that $F^*(G)$ is not isomorphic to Mc, Co_2, or $PSU_4(3)$. Hence $F^*(G)$ is isomorphic either to $PSL_6(q)$, for some q such that $3 \mid q-1$ and $9 \nmid q-1$, or to $PSU_6(q)$, for some q such that $3 \mid q+1$ and $9 \nmid q+1$.

If $F^*(G) \simeq PSL_6(q)$, for some q such that $3 \mid q-1$ and $9 \nmid q-1$, direct calculations show that, in the notation of the proof of Proposition 5.12, the map $\alpha : S \to T$, defined on the generators of S by:

$$\begin{aligned} x\alpha &= v_1 Z, \\ y\alpha &= v_6 Z, \\ a\alpha &= v_7^{-1} v_3^{-1} v_7 Z, \\ b\alpha &= v_7 Z, \\ z\alpha &= v_4^{-1} Z, \\ t\alpha &= v_2 Z, \end{aligned}$$

and extended by linearity to the whole S is an isomorphism; in particular,

$$E = \langle v_1 Z, v_6 Z, v_7^{-1} v_3^{-1} v_7 Z, v_7 Z, v_4^{-1} Z \rangle,$$

and

$$Z(E) = \langle v_4^{-1} Z \rangle.$$

Since $N_{F^*(G)}(E) \leq N_{F^*(G)}(Z(E))$ and every $g \in N_{F^*(G)}(Z(E))$ respects the eigenspaces of $\langle v_4^{-1} Z \rangle$, it follows that

$$\begin{aligned} N_{F^*(G)}(E) \leq & \left\{ \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} Z \,|\, X, Y \in GL_3(q),\ \det(A) = \det(B)^{-1} \right\} \\ & \rtimes \left\langle \begin{pmatrix} 0 & I_3 \\ I_3 & 0 \end{pmatrix} Z \right\rangle. \end{aligned}$$

By [7, Table 8.3], one has that both

$$N_{F^*(G)}(\langle v_1 Z, v_6 Z \rangle) \cap C_{F^*(G)}(\langle v_7^{-1} v_3^{-1} v_7 Z, v_7 Z \rangle),$$

and

$$N_{F^*(G)}(\langle v_7^{-1} v_3^{-1} v_7 Z, v_7 Z \rangle) \cap C_{F^*(G)}(\langle v_1 Z, v_6 Z \rangle)$$

contain a subgroup isomorphic to Q_8. This implies that $N_{F^*(G)}(E)$ contains a subgroup isomorphic to $Q_8 \times Q_8$. Since E is a 3-group, also $N_{F^*(G)}(E)/E$ contains a subgroup isomorphic to $Q_8 \times Q_8$.

If $F^*(G) \simeq PSU_6(q)$, for some q such that $3 \mid q+1$ and $9 \nmid q+1$, with a similar argument one obtains that $N_{F^*(G)}(E)/E$ contains a subgroup isomorphic to $Q_8 \times Q_8$ (cf. [7, Table 8.5]).

Thus, we get a contradiction, as the Sylow 2-subgroups of $\mathrm{Out}_{\mathcal{F}}(E)$ have orders lower than 2^6, and the proof is completed. □

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