SCUOLA DI DOTTORATO
UNIVERSITÀ DEGLI STUDI DI MILANO-BICOCCA

Department of

Mathematics and its Applications

PhD program in Pure and Applied Mathematics Cycle XXIX

THESIS TITLE

PERIODIC SOLUTIONS OF ORDINARY DIFFERENTIAL SYSTEMS WITH CURVATURE-LIKE PRINCIPAL PART

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ACADEMIC YEAR 2015/2016

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Introduction

Given a function

$$G : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}\,,$$

such that $G(t,s)$ is T-periodic in t, the study of T-periodic solutions $u : \mathbb{R} \to \mathbb{R}^n$ for the system

$$-u''(t) = \nabla_s G(t, u(t))$$

constitutes a widely investigated subject and a typical area of application of variational methods, as u is a T-periodic solution if and only if its restriction to $[0,T]$ is a critical point of the functional

$$\left\{ u \mapsto \frac{1}{2} \int_0^T |u'|^2 \, dt - \int_0^T G(t,u) \, dt \right\}$$

defined on u's in $W^{1,2}(0,T;\mathbb{R}^n)$ such that $u(0) = u(T)$ (see e.g. [30, 35]).

Among several possible assumptions on G, special attention has been devoted, starting from [34], to the case in which $G(t,s) \approx |s|^\beta$ as $|s| \to \infty$, with $\beta > 2$.

Many refinements and generalizations have been produced since that paper. First of all, if $\nabla_s G$ is allowed to be discontinuous in t (typically, $\nabla_s G(\cdot, s) \in L^1_{loc}(\mathbb{R};\mathbb{R}^n)$), it is equivalent to consider G just defined on $]0,T[\times\mathbb{R}^n$. Then, among very recent contributions, let us mention [28], where it is proved that there exists a nonconstant T-periodic solution u of

$$-\left(|u'|^{p-2}u'\right)' = \nabla_s G(t,u)\,,$$

with $1 < p < \infty$, provided that:

(a) *the function $G(\cdot, s)$ is measurable for every $s \in \mathbb{R}^n$, $G(t,0) = 0$ and $G(t,\cdot)$ is of class C^1 for a.e. $t \in]0,T[$ and, for every $r > 0$, there exists $\alpha_r \in L^1(0,T)$ satisfying*

$$|\nabla_s G(t,s)| \le \alpha_r(t) \qquad \textit{for a.e. } t \in]0,T[\textit{ and every } s \in \mathbb{R}^n \textit{ with } |s| \le r;$$

(b) *$G(t,s) \ge 0$ for a.e. $t \in]0,T[$ and every $s \in \mathbb{R}^n$;*

(c) *there exist* $\beta > p$, $\alpha \in L^1(0,T)$ *and a measurable subset* E *of* $]0,T[$ *with positive measure such that*

$$\beta\, G(t,s) - \nabla_s G(t,s)\cdot s \le (1+|s|^p)\alpha(t) \qquad \text{for a.e. } t\in]0,T[\text{ and every } s\in\mathbb{R}^n\,,$$

$$\limsup_{|s|\to\infty} \frac{\beta\, G(t,s) - \nabla_s G(t,s)\cdot s}{|s|^p} \le 0 \qquad \text{for a.e. } t\in]0,T[\,,$$

$$\liminf_{|s|\to\infty} \frac{G(t,s)}{|s|^p} > 0 \qquad \text{for a.e. } t\in E\,;$$

(d) *we have*

$$\lim_{s\to 0} \frac{G(t,s)}{|s|^p} = 0 \qquad \text{uniformly for a.e. } t\in]0,T[\,.$$

Let us observe that, since the functional

$$\left\{u \mapsto \frac{1}{p}\int_0^T |u'|^p\,dt\right\}$$

is convex, we have that $u \in W^{1,p}(0,T;\mathbb{R}^n)$ with $u(0)=u(T)$ is a critical point of

$$\left\{u \mapsto \frac{1}{p}\int_0^T |u'|^p\,dt - \int_0^T G(t,u)\,dt\right\}$$

if and only if

$$\frac{1}{p}\int_0^T |v'|^p\,dt + \int_0^T \nabla_s G(t,u)\cdot(u-v)\,dt \ge \frac{1}{p}\int_0^T |u'|^p\,dt \qquad \text{for any } v\in W^{1,p}(0,T;\mathbb{R}^n) \text{ with } v(0)=v(T)\,.$$

Coming back to the general study of periodic solutions, not necessarily in the case started by [34], a certain attention has been recently devoted to the limit case as $p\to 1$ (see e.g. [33]). In this case the term

$$\int_0^T |u'|\,dt$$

defined for u's in $W^{1,1}(0,T;\mathbb{R}^n)$ with $u(0)=u(T)$ has to be substituted by the term

$$|u'|(]0,T[) + |u(0_+) - u(T_-)|$$

defined for u's in $BV(0,T;\mathbb{R}^n)$ (see also the next Proposition 1.2.2). Therefore, one looks for $u\in BV(0,T;\mathbb{R}^n)$ such that

$$|v'|(]0,T[) + |v(0_+) - v(T_-)| + \int_0^T \nabla_s G(t,u)\cdot(u-v)\,dt \ge |u'|(]0,T[) + |u(0_+) - u(T_-)| \qquad \text{for any } v\in BV(0,T;\mathbb{R}^n)\,.$$

Now one could ask whether it is possible to find a nonconstant solution under the same assumptions (a)–(d) with p replaced by 1.

The adaptation is not standard and the point is assumption (a). More precisely, the most conservative approach, with respect to the case $p > 1$, would be to define $f : BV(0,T;\mathbb{R}^n) \to \mathbb{R}$ as

$$f(u) = |u'|(]0,T[) + |u(0_+) - u(T_-)| - \int_0^T G(t,u)\,dt\,.$$

Then f turns out to be locally Lipschitz and nonsmooth critical point theory allows to treat this level of regularity. However, the Palais-Smale condition fails, because the BV norm is too strong with respect to the lack of uniform convexity of the principal part of the functional.

This difficulty was already recognized and overcome in [17], if u is defined on an open subset of $\mathbb{R}^N$ with $N \geq 2$ (so we have a PDE instead of an ODE and the assumptions on G have to be naturally adapted). The device is to extend the functional to $L^{1^*} = L^{\frac{N}{N-1}}$ with value $+\infty$ outside BV. In this way the functional becomes a C^1 perturbation of a convex and lower semicontinuous functional, a class still covered by nonsmooth critical point theory, and now the Palais-Smale condition can be proved (see e.g. [17, Theorem 6.2]). This kind of device has been also applied in [6, 15, 27, 31].

The same idea, when $N = 1$, would suggest to extend f to L^∞ with value $+\infty$ outside BV, but now L^∞ is not so well behaved as $L^{\frac{N}{N-1}}$ with $N \geq 2$ and the Palais-Smale condition still fails. To recover a more comfortable Lebesgue space, one could extend the functional to L^q with $1 < q < \infty$, again with value $+\infty$ outside BV, but now assumption (a) is not enough to guarantee that the functional is lower semicontinuous on L^q, as we will see in Remark 2.2.10. The lack of lower semicontinuity is a serious difficulty in view of direct methods. By the way, also [33], which treats a different problem, requires a stronger version of (a), with $\alpha_r \in L^q$ with $q > 1$.

The purpose of this thesis is to propose a different functional approach that allows to prove the required result (see Theorem 2.2.1 and Remark 2.2.6) and also other existence and multiplicity results under different behaviors of G.

The starting point is a device introduced in [12] and largely exploited. Given a discontinuous function $f : X \to]-\infty, +\infty]$, it is often convenient to consider the epigraph of f

$$\mathrm{epi}\,(f) = \{(u,\lambda) \in X \times \mathbb{R} : \ f(u) \leq \lambda\}$$

and then the continuous function $\mathscr{G}_f : \mathrm{epi}\,(f) \to \mathbb{R}$ defined as $\mathscr{G}_f(u,\lambda) = \lambda$. In this way the study of a general function can be reduced, to a certain extent, to that of a continuous function.

In our case this device does not improve the situation, because the lack of lower semicontinuity of f becomes a lack of completeness of epi$\,(f)$. However, a variation of this idea will solve the problem.

Since our functional f has two parts, say $f = f_0 + f_1$ with

$$f_0(u) = |u'|(]0,T[) + |u(0_+) - u(T_-)|\,, \qquad f_1(u) = -\int_0^T G(t,u)\,dt\,,$$

we can consider the epigraph just of f_0

$$\mathrm{epi}\,(f_0) = \{(u,\lambda) \in BV(0,T;\mathbb{R}^n) \times \mathbb{R} :\ f_0(u) \le \lambda\}$$

which is complete also under very weak norms, say that of $L^1(0,T;\mathbb{R}^n) \times \mathbb{R}$. Then define $\mathscr{F} : \mathrm{epi}\,(f_0) \to \mathbb{R}$ by $\mathscr{F}(u,\lambda) = \lambda + f_1(u)$, which turns out to be continuous with respect to the topology of $L^1(0,T;\mathbb{R}^n) \times \mathbb{R}$ as the convergence in this topology, restricted to epi$\,(f_0)$, implies a BV bound.

In the end, the solutions u of the periodic problem will be obtained as "critical points" of $\mathscr{F}$ of the form $(u, f_0(u))$.

In Chapter 1 we review some known results and describe the general functional setting. In Chapter 2 we treat the problem addressed in the Introduction, while in Chapter 3 we adapt to our setting a result of [29] concerning the case $p = 2$ (see Theorems 3.1.4 and 3.2.1). Finally, in Chapter 4 we treat the case in which $G(t,s) \approx |s|$ as $|s| \to \infty$, under a suitable nonresonance condition (see Theorem 4.2.2).

Chapter 1

Auxiliary results and general setting

In this first chapter we review some general facts, which will be useful in the following, and we formulate the general setting of the problem.

1 Some auxiliary results

1.1 Functions with bounded variation

We refer the reader to [2, 23].

Definition 1.1.1. *Let Ω be an open subset of $\mathbb{R}$. We denote by $BV(\Omega;\mathbb{R}^n)$ the set of u's in $L^1(\Omega;\mathbb{R}^n)$ such that*

$$\sup\left\{\int_\Omega u\cdot v'\,dt:\ v\in C_c^1(\Omega;\mathbb{R}^n)\,,\ |v(t)|\le 1\ \forall t\in\Omega\right\}<+\infty\,.$$

If $u\in BV(\Omega;\mathbb{R}^n)$, it turns out that the distributional derivative u' is a vector Radon measure with bounded total variation. The Lebesgue's decomposition and Radon-Nikodym theorem then allow to write

$$du' = u'_a\,d\mathscr{L}^1 + \frac{u'_s}{|u'_s|}\,d|u'_s|\,.$$

In particular, we have

$$|u'|(\Omega) = \int_\Omega |u'_a|\,dt + |u'_s|(\Omega)\,.$$

In the case $\Omega =]a,b[$, we will write $L^1(a,b;\mathbb{R}^n)$, $BV(a,b;\mathbb{R}^n)$ instead of $L^1(]a,b[;\mathbb{R}^n)$, $BV(]a,b[;\mathbb{R}^n)$. Moreover, we will denote by $\|u\|_p$ the usual norm in L^p.

Proposition 1.1.2. *For every $T > 0$ and $u \in BV(0,T;\mathbb{R}^n)$, we have*

$$\operatorname*{ess\,sup}_{]0,T[} \left| u - \frac{1}{T}\int_0^T u\,dt \right| \le |u'|(]0,T[)\,,$$

$$\operatorname*{ess\,sup}_{]0,T[} |u| \le \frac{1}{T}\int_0^T |u|\,dt + |u'|(]0,T[)\,,$$

$$\operatorname*{ess\,inf}_{]0,T[} |u| \ge \frac{1}{T}\int_0^T |u|\,dt - |u'|(]0,T[)\,.$$

Proof. For every $u \in BV(0,T;\mathbb{R}^n)$, $\alpha \in \mathbb{R}^n$ and a.e. $t, s \in]0,T[$, we have

$$\alpha \cdot (u(s) - u(t)) \le |\alpha|\,|u'|(]0,T[)\,.$$

Integrating in dt, we get

$$\alpha \cdot \left(Tu(s) - \int_0^T u(t)\,dt \right) \le T\,|\alpha|\,|u'|(]0,T[) \qquad \text{for a.e. } s \in]0,T[\,,$$

whence

$$\left| u(s) - \frac{1}{T}\int_0^T |u|\,dt \right| \le |u'|(]0,T[) \qquad \text{for a.e. } s \in]0,T[$$

by the arbitrariness of α. Then the assertions easily follow. □

1.2 Lower semicontinuity

We refer the reader to [2, 23].

Definition 1.1.3. *Let X be a set and let $f\colon X \to [-\infty,+\infty]$ be a function. We define the* epigraph *of f as*

$$\operatorname{epi}(f) := \{(u,\lambda) \in X \times \mathbb{R} : \; f(u) \le \lambda\}\,.$$

We also consider the function $\mathscr{G}_f\colon \operatorname{epi}(f) \to \mathbb{R}$ defined as $\mathscr{G}_f(u,\lambda) = \lambda$.

Definition 1.1.4. *Let X be a topological space. A function $f\colon X \to [-\infty,+\infty]$ is said to be* lower semicontinuous *if $\operatorname{epi}(f)$ is closed in $X \times \mathbb{R}$.*

Proposition 1.1.5. *Let X be a topological space and let $f\colon X \to [-\infty,+\infty]$ be a function. Then the following facts are equivalent:*

(*a*) *f is lower semicontinuous;*

(*b*) *for every $c \in \mathbb{R}$, the set $\{u \in X : \; f(u) > c\}$ is open in X;*

(*c*) *for every $c \in \mathbb{R}$, the set $\{u \in X : \; f(u) \le c\}$ is closed in X.*

1.3 Nonsmooth analysis

We refer the reader to [4, 8, 10, 16, 19, 25, 26].
Let X be a metric space endowed with the distance d. We will denote by $\mathrm{B}_\delta(u)$ the open ball of center u and radius δ. Moreover, $X \times \mathbb{R}$ will be endowed with the distance

$$d(\,(u,\lambda)\,,\,(v,\mu)\,) = \left(d(u,v)^2 + (\lambda-\mu)^2\right)^{1/2}$$

and $\mathrm{epi}\,(f)$ with the induced distance.

The next notion has been independently introduced in [10, 16] and in [26], while a variant has been proposed in [25].

Definition 1.1.6. *Let $f\colon X \to \mathbb{R}$ be a continuous function and let $u \in X$. We denote by $|df|(u)$ the supremum of the σ's in $[0,+\infty[$ such that there exist $\delta > 0$ and a continuous map*

$$\mathscr{H}\colon\ \mathrm{B}_\delta(u) \times [0,\delta] \to X$$

satisfying

$$d(\mathscr{H}(v,t),t) \le t\,, \qquad f(\mathscr{H}(v,t)) \le f(v) - \sigma t\,,$$

for every $v \in \mathrm{B}_\delta(u)$ and $t \in [0,\delta]$.

The extended real number $|df|(u)$ is called the weak slope of f at u.

Proposition 1.1.7. *Let X be an open subset of a normed space and let $f\colon X \to \mathbb{R}$ be of class C^1. Then we have $|df|\,(u) = \|f'(u)\|$ for any $u \in X$.*

Proposition 1.1.8. *Let $f\colon X \to \mathbb{R}$ be a continuous function, $u \in X$ and $\lambda \in \mathbb{R}$.*

Then we have

$$|d\mathscr{G}_f|(u,f(u)) = \begin{cases} \dfrac{|df|(u)}{\sqrt{1+\left(|df|(u)\right)^2}} & \text{if } |df|(u) < +\infty\,, \\ 1 & \text{if } |df|(u) = +\infty\,, \end{cases}$$

$$|d\mathscr{G}_f|(u,\lambda) = 1 \qquad \text{if } f(u) < \lambda\,.$$

This proposition allows to define, in a consistent way, the weak slope of a general function. Since $\mathscr{G}_f$ is Lipschitz continuous of constant 1, it is easily seen that $|d\mathscr{G}_f|\,(u,\lambda) \le 1$ for any $(u,\lambda) \in \mathrm{epi}\,(f)$.

Definition 1.1.9. *Let $f\colon X \to [-\infty, +\infty]$ be a function and let $u \in X$ with $f(u) \in \mathbb{R}$. We set*

$$|df|(u) := \begin{cases} \dfrac{|d\mathscr{G}_f|(u, f(u))}{\sqrt{1 - \left(|d\mathscr{G}_f|(u, f(u))\right)^2}} & \text{if } |d\mathscr{G}_f|(u, f(u)) < 1\,, \\ +\infty & \text{if } |d\mathscr{G}_f|(u, f(u)) = 1\,. \end{cases}$$

Proposition 1.1.10. *Let $f\colon X \to [-\infty, +\infty]$ be a function, let*

$$D = \{u \in X : \ f(u) < +\infty\}$$

and denote by $\overline{f}$ the restriction of f to D.

Then, for every $u \in D$ with $f(u) > -\infty$, we have

$$\left|d\overline{f}\right|(u) = |df|\,(u)\,.$$

Proof. We have $\operatorname{epi}\left(\overline{f}\right) = \operatorname{epi}(f)$ and $\mathscr{G}_{\overline{f}} = \mathscr{G}_f$. □

Proposition 1.1.11. *Let $f\colon X \to [-\infty, +\infty]$ be a function and $\beta : X \to \mathbb{R}$ a Lipschitz continuous function of constant L. Let*

$$Y = \{u \in X : \ f(u) \leq \beta(u)\}$$

and denote by $\overline{f}$ the restriction of f to Y.

Then, for every $u \in Y$ with $f(u) > -\infty$ and $|df|\,(u) > L$, we have

$$\left|d\overline{f}\right|(u) \geq |df|\,(u)\,.$$

Proof. See [13, Proposition 3.2]. □

Proposition 1.1.12. *Let $f\colon X \to [-\infty, +\infty]$ be a function and let $g : X \to \mathbb{R}$ be a Lipschitz continuous function of constant L.*

Then, for every $u \in X$ with $f(u) \in \mathbb{R}$, we have

$$|df|\,(u) - L \leq |d(f+g)|\,(u) \leq |df|\,(u) + L\,.$$

Proof. See [19, Proposition 1.6]. □

Definition 1.1.13. *Let $f\colon X \to [-\infty, +\infty]$ be a function. We say that $u \in X$ is a* (lower) critical point *of f if $f(u) \in \mathbb{R}$ and $|df|(u) = 0$. We say that $c \in \mathbb{R}$ is a* (lower) critical value *of f if there exists $u \in X$ such that $f(u) = c$ and $|df|(u) = 0$.*

Definition 1.1.14. *Let $f\colon X \to [-\infty, +\infty]$ be a function and let $c \in \mathbb{R}$. We say that f satisfies the* Palais-Smale condition at level c *($(PS)_c$, for short), if every sequence (u_k) in X, with $f(u_k) \to c$ and $|df|(u_k) \to 0$, admits a convergent subsequence in X.*

Definition 1.1.15. *Let $f\colon X \to [-\infty, +\infty]$ be a function, let $\bar{u} \in X$ and $c \in \mathbb{R}$. We say that f satisfies the* Cerami-Palais-Smale condition at level c *($(CPS)_c$, for short), if every sequence (u_k) in X, with $f(u_k) \to c$ and $\big(1 + d(u_k, \bar{u})\big)|df|(u_k) \to 0$, admits a convergent subsequence in X.*

Since

$$\big(1 + d(u_k, \hat{u})\big)|df|(u_k) \le \big(1 + d(\bar{u}, \hat{u})\big)\big(1 + d(u_k, \bar{u})\big)|df|(u_k)\,,$$

it is easily seen that $(CPS)_c$ is independent of the choice of the point $\bar{u}$. It is also clear that $(PS)_c$ implies $(CPS)_c$.

Being a generalization of $\|f'(u)\|$, the weak slope $|df|\,(u)$ cannot have a rich calculus. For this reason, an auxiliary concept is sometimes useful.

From now on in this subsection, we assume that X is a normed space over $\mathbb{R}$ and $f\colon X \to [-\infty, +\infty]$ a function.

The next notion has been introduced in [4].

Definition 1.1.16. *For every $u \in X$ with $f(u) \in \mathbb{R}$, $v \in X$ and $\varepsilon > 0$, let $f^0_\varepsilon(u; v)$ be the infimum of r's in $\mathbb{R}$ such that there exist $\delta > 0$ and a continuous map*

$$\mathcal{V}\colon \big(B_\delta(u, f(u)) \cap \operatorname{epi}(f)\big)\times]0, \delta] \to B_\varepsilon(v)$$

satisfying

$$f(z + t\mathcal{V}((z, \mu), t)) \le \mu + rt$$

whenever $(z, \mu) \in B_\delta(u, f(u)) \cap \operatorname{epi}(f)$ and $t \in]0, \delta]$.

Then let

$$f^0(u; v) = \sup_{\varepsilon > 0} f^0_\varepsilon(u; v)\,.$$

Let us recall that the function $f^0(u; \cdot) : X \to [-\infty, +\infty]$ is convex, lower semicontinuous and positively homogeneous of degree 1. Moreover $f^0(u; 0) \in \{0, -\infty\}$.

Definition 1.1.17. *For every $u \in X$ with $f(u) \in \mathbb{R}$, we set*

$$\partial f(u) = \big\{\alpha \in X' :\ \langle \alpha, v\rangle \le f^0(u; v) \quad \forall v \in X\big\}\,.$$

This kind of subdifferential is suitably related to the weak slope, because of the next result.

Theorem 1.1.18. *For every $u \in X$ with $f(u) \in \mathbb{R}$, the following facts hold:*

(a) $|df|(u) < +\infty \iff \partial f(u) \neq \emptyset$;

(b) $|df|(u) < +\infty \implies |df|(u) \geq \min\{\, \|\alpha\| : \alpha \in \partial f(u)\,\}$.

Proposition 1.1.19. *Assume there exists $D \subseteq X$ such that $f\big|_D$ is real valued and continuous, while $f = +\infty$ on $X \setminus D$.*

Then for every $u \in D$, $v \in X$ and $\varepsilon > 0$ we have that $f^0_\varepsilon(u; v)$ is the infimum of the r's in $\mathbb{R}$ such that there exist $\delta > 0$ and a continuous map

$$\mathcal{V} \colon \big(B_\delta(u) \cap D\big) \times]0, \delta] \to B_\varepsilon(v)$$

satisfying

$$f(z + t\mathcal{V}(z,t)) \leq f(z) + rt$$

whenever $z \in B_\delta(u) \cap D$ and $t \in]0, \delta]$.

Remark 1.1.20. *If f is convex, then ∂f agrees with the subdifferential of convex analysis. If f is locally Lipschitz, then f^0 and ∂f agree with Clarke's notions [8]. In particular, $f^0(u, \cdot)$ also is Lipschitz continuous and for every $u, v \in X$ we have*

$$\begin{aligned} f^0(u; v) &= \limsup_{(z,t)\to(u,0_+)} \frac{f(z+tv) - f(z)}{t} \\ &= \limsup_{(z,w,t)\to(u,v,0_+)} \frac{f(z+tw) - f(z)}{t}\,, \end{aligned}$$

$$(-f)^0(u; v) = f^0(u; -v)\,,$$

$$\big\{(u, v) \mapsto f^0(u; v)\big\} \text{ is upper semicontinuous}.$$

2 The general setting

Let us introduce the general setting that will be considered from now on.

2.1 The principal part

Throughout the thesis, we assume that $\Psi : \mathbb{R}^n \to \mathbb{R}$ satisfies:

(Ψ) *the function* Ψ *is convex, with* $\Psi(0) = 0$*, and there exists* $\nu > 0$ *satisfying*

$$\nu|\xi| - \frac{1}{\nu} \leq \Psi(\xi) \leq \frac{1}{\nu}\big(1 + |\xi|\big) \qquad \text{for every } \xi \in \mathbb{R}^n\,.$$

Let us also introduce the recession function $\Psi^\infty : \mathbb{R}^n \to \mathbb{R}$ defined as:

$$\Psi^\infty(\xi) := \lim_{\tau \to +\infty} \frac{\Psi(\tau\xi)}{\tau}\,.$$

It is well known (see e.g. [2, 11, 23]) that Ψ^∞ is convex and positively homogeneous of degree 1.

Proposition 1.2.1. *The following facts hold:*

(*a*) *the function* Ψ *is Lipschitz continuous of constant* $1/\nu$*; in particular, it follows that*

$$\Psi(\xi) \leq \frac{1}{\nu}|\xi| \qquad \text{for every } \xi \in \mathbb{R}^n\,;$$

(*b*) *for every* $\varepsilon > 0$ *there exists* $M_\varepsilon > 0$ *such that*

$$|\Psi(2\xi) - 2\Psi(\xi)| \leq \varepsilon\Psi(\xi) + M_\varepsilon \qquad \text{for every } \xi \in \mathbb{R}^n\,;$$

(*c*) *the function* Ψ^∞ *itself is Lipschitz continuous with the same constant* $1/\nu$ *and we have*

$$\nu|\xi| \leq \Psi^\infty(\xi) \leq \frac{1}{\nu}\,|\xi| \qquad \text{for every } \xi \in \mathbb{R}^n\,;$$

(*d*) *for every* $\varepsilon > 0$ *there exists* $M_\varepsilon > 0$ *such that*

$$(1 - \varepsilon)\Psi^\infty(\xi) - M_\varepsilon \leq \Psi(\xi) \leq \Psi^\infty(\xi) \qquad \text{for every } \xi \in \mathbb{R}^n\,.$$

Proof. (*a*)&(*c*) For every $\xi_0, \xi_1 \in \mathbb{R}^n$ and $\tau \geq 1$, we have

$$\xi_1 = \xi_0 + \frac{1}{\tau}\,\{[\xi_0 + \tau(\xi_1 - \xi_0)] - \xi_0\}\,,$$

whence, by the convexity of Ψ,

$$\Psi(\xi_1) \leq \Psi(\xi_0) + \frac{1}{\tau}\,\{\Psi(\xi_0 + \tau(\xi_1 - \xi_0)) - \Psi(\xi_0)\}\,,$$

which is equivalent to

$$\Psi(\xi_0) + \tau[\Psi(\xi_1) - \Psi(\xi_0)] \leq \Psi(\xi_0 + \tau(\xi_1 - \xi_0)) .$$

By the upper estimate in assumption (Ψ), it follows

$$\Psi(\xi_0) + \tau[\Psi(\xi_1) - \Psi(\xi_0)] \leq \frac{1}{\nu}(1 + |\xi_0 + \tau(\xi_1 - \xi_0)|) ,$$

whence

$$\frac{1}{\tau}\Psi(\xi_0) + \Psi(\xi_1) - \Psi(\xi_0) \leq \frac{1}{\nu}\left(\frac{1}{\tau} + \left|\frac{1}{\tau}\xi_0 + \xi_1 - \xi_0\right|\right) .$$

Going to the limit as $\tau \to +\infty$, we get

$$\Psi(\xi_1) - \Psi(\xi_0) \leq \frac{1}{\nu}|\xi_1 - \xi_0| ,$$

whence the Lipschitz continuity of Ψ, as we can exchange ξ_0 with ξ_1.

Consequently, we also have

$$\left|\frac{\Psi(\tau\xi_1)}{\tau} - \frac{\Psi(\tau\xi_0)}{\tau}\right| \leq \frac{1}{\tau}\frac{1}{\nu}\tau|\xi_1 - \xi_0| = \frac{1}{\nu}|\xi_1 - \xi_0| .$$

Going to the limit as $\tau \to +\infty$, we obtain

$$|\Psi^\infty(\xi_1) - \Psi^\infty(\xi_0)| \leq \frac{1}{\nu}|\xi_1 - \xi_0| ,$$

that means that Ψ^∞ also is Lipschitz continuous of constant $1/\nu$.

Since

$$\nu|\xi| - \frac{1}{\tau\nu} \leq \frac{\Psi(\tau\xi)}{\tau} \leq \frac{1}{\nu}\left(\frac{1}{\tau} + |\xi|\right) ,$$

going to the limit as $\tau \to +\infty$ the double estimate on Ψ^∞ also follows.

(b) Because of assumption (Ψ), it is equivalent to prove that

$$\lim_{|\xi|\to\infty} \frac{\Psi(2\xi) - 2\Psi(\xi)}{|\xi|} = 0 .$$

Let (τ_k) be a sequence with $\tau_k \to +\infty$ and (v_k) a sequence in $\mathbb{R}^n$ with $|v_k| = 1$. Up to a subsequence, we may assume that $v_k \to v$.

Since Ψ is Lipschitz continuous of constant $1/\nu$, we get

$$\begin{aligned}
\frac{\Psi(2\tau_k v_k) - 2\Psi(\tau_k v_k)}{\tau_k} &= \frac{\Psi(2\tau_k v) - 2\Psi(\tau_k v)}{\tau_k} + \frac{\Psi(2\tau_k v_k) - \Psi(2\tau_k v)}{\tau_k} \\
&\qquad - 2\,\frac{\Psi(\tau_k v_k) - \Psi(\tau_k v)}{\tau_k} \\
&\leq \frac{\Psi(2\tau_k v) - 2\Psi(\tau_k v)}{\tau_k} + \frac{1}{\tau_k}\left(\frac{2}{\nu}\tau_k|v_k - v| + \frac{2}{\nu}\tau_k|v_k - v|\right) \\
&= 2\left[\frac{\Psi(2\tau_k v)}{2\tau_k} - \frac{\Psi(\tau_k v)}{\tau_k}\right] + \frac{4}{\nu}|v_k - v| ,
\end{aligned}$$

whence

$$\limsup_k \frac{\Psi(2\tau_k v_k) - 2\Psi(\tau_k v_k)}{\tau_k} \leq 2\Psi^\infty(v) - 2\Psi^\infty(v) = 0\,.$$

The lower limit can be treated in a similar way.

(d) If $\tau \geq 1$ and $\xi \in \mathbb{R}^n$, we have

$$\xi = \Big(1 - \frac{1}{\tau}\Big)0 + \frac{1}{\tau}(\tau\xi)\,,$$

whence

$$\Psi(\xi) \leq \frac{1}{\tau}\Psi(\tau\xi)\,.$$

Going to the limit as $\tau \to +\infty$, we get $\Psi(\xi) \leq \Psi^\infty(\xi)$. On the other hand, assume for a contradiction that there exist $\varepsilon > 0$ and (ξ_k) such that

$$(1-\varepsilon)\Psi^\infty(\xi_k) - k > \Psi(\xi_k)\,.$$

First of all, it follows $|\xi_k| \to \infty$. If $\xi_k = \tau_k v_k$ with $\tau_k \to +\infty$, $|v_k| = 1$ and $v_k \to v$, we have

$$(1-\varepsilon)\Psi^\infty(v_k) > \frac{\Psi(\tau_k v_k)}{\tau_k} \geq \frac{\Psi(\tau_k v)}{\tau_k} - \frac{1}{\nu}\,|v_k - v|\,,$$

whence

$$(1-\varepsilon)\Psi^\infty(v) \geq \Psi^\infty(v)$$

and a contradiction follows. □

Now let $T > 0$ and let

$$\check{f}_0\,,\ \ f_0 : L^1(0,T;\mathbb{R}^n) \to]-\infty,+\infty]$$

be the functionals defined as

$$\check{f}_0(u) = \begin{cases} \displaystyle\int_0^T \Psi(u')\,dt & \text{if } u \in W^{1,1}_T(0,T;\mathbb{R}^n)\,, \\ +\infty & \text{if } u \in L^1(0,T;\mathbb{R}^n) \setminus W^{1,1}_T(0,T;\mathbb{R}^n)\,, \end{cases}$$

$$f_0(u) = \begin{cases} \displaystyle\int_0^T \Psi(u'_a)\,dt + \int_{]0,T[} \Psi^\infty\left(\frac{u'_s}{|u'_s|}\right) d|u'_s| & \\ \qquad\qquad +\Psi^\infty\big(u(0_+) - u(T_-)\big) & \text{if } u \in BV(0,T;\mathbb{R}^n)\,, \\ +\infty & \text{if } u \in L^1(0,T;\mathbb{R}^n) \setminus BV(0,T;\mathbb{R}^n)\,, \end{cases}$$

where

$$W^{1,1}_T(0,T;\mathbb{R}^n) = \big\{u \in W^{1,1}(0,T;\mathbb{R}^n) :\ u(0) = u(T)\big\}\ .$$

Proposition 1.2.2. *The functional f_0 is the lower semicontinuous envelope of the functional $\check{f}_0$.*

Proof. It is well known that

$$\begin{cases} \displaystyle\int_0^T \Psi(u'_a)\,dt + \int_{]0,T[} \Psi^\infty\left(\frac{u'_s}{|u'_s|}\right) d|u'_s| & \\ \qquad\qquad +\Psi^\infty\big(u(0_+)\big) + \Psi^\infty\big(-u(T_-)\big) & \text{if } u \in BV(0,T;\mathbb{R}^n)\,, \\ +\infty & \text{if } u \in L^1(0,T;\mathbb{R}^n) \setminus BV(0,T;\mathbb{R}^n)\,, \end{cases}$$

is the lower semicontinuous envelope of

$$\begin{cases} \displaystyle\int_0^T \Psi(u')\,dt & \text{if } u \in W_0^{1,1}(0,T;\mathbb{R}^n)\,, \\ +\infty & \text{if } u \in L^1(0,T;\mathbb{R}^n) \setminus W_0^{1,1}(0,T;\mathbb{R}^n)\,, \end{cases}$$

(see e.g. [3, Corollary 11.3.1]).

It is easily seen that $f_0 \le \check{f}_0$. To prove the lower semicontinuity of f_0, consider a sequence (u_k) converging to u in $L^1(0,T;\mathbb{R}^n)$. Without loss of generality, we may assume that $\sup\limits_k f_0(u_k) < +\infty$. Then from assumption ($\Psi$) we infer that (u_k) is bounded in $BV(0,T;\mathbb{R}^n)$, so that $u \in BV(0,T;\mathbb{R}^n)$ and $(u_k(T_-))$ is convergent, up to a subsequence, to some y in $\mathbb{R}^n$.

If we set

$$\begin{aligned} v_k(t) &= u_k(t) - u_k(T_-)\,, \\ v(t) &= u(t) - y\,, \end{aligned}$$

we have

$$\begin{aligned} \Psi^\infty\left(u(0_+) - u(T_-)\right) &= 2\Psi^\infty\left(\frac{1}{2}\left(u(0_+) - y\right) + \frac{1}{2}\left(y - u(T_-)\right)\right) \\ &\le \Psi^\infty\left(u(0_+) - y\right) + \Psi^\infty\left(y - u(T_-)\right) \\ &= \Psi^\infty\left(v(0_+)\right) + \Psi^\infty\left(-v(T_-)\right)\,, \end{aligned}$$

whence

$$\begin{aligned} f_0(u) &\le \int_0^T \Psi(v'_a)\,dt + \int_{]0,T[} \Psi^\infty\left(\frac{v'_s}{|v'_s|}\right) d|v'_s| + \Psi^\infty\big(v(0_+)\big) + \Psi^\infty\big(-v(T_-)\big) \\ &\le \liminf_k \left[\int_0^T \Psi((v_k)'_a)\,dt + \int_{]0,T[} \Psi^\infty\left(\frac{(v_k)'_s}{|(v_k)'_s|}\right) d|(v_k)'_s| + \Psi^\infty\big(v_k(0_+)\big)\right] \\ &= \liminf_k\, f_0(u_k)\,. \end{aligned}$$

Let now $u \in BV(0,T;\mathbb{R}^n)$ and set

$$v(t) = u(t) - u(T_-)\,.$$

There exists a sequence (v_k) in $W^{1,1}_0(0,T;\mathbb{R}^n)$ such that

$$\begin{aligned}\lim_k \int_0^T \Psi(v_k')\,dt &= \int_0^T \Psi(v_a')\,dt + \int_{]0,T[} \Psi^\infty\left(\frac{v_s'}{|v_s'|}\right)\,d|v_s'| + \Psi^\infty\big(v(0_+)\big) \\ &= f_0(u)\,.\end{aligned}$$

If we set

$$u_k(t) = v_k(t) + u(T_-)\,,$$

we have $u_k \in W^{1,1}_T(0,T;\mathbb{R}^n)$ and

$$\lim_k \check{f}_0(u_k) = \lim_k \int_0^T \Psi(v_k')\,dt = f_0(u)\,.$$

□

For every $u \in BV(0,T;\mathbb{R}^n)$, we also have

$$f_0(u) = \int_0^T \Psi(u_a')\,dt + \int_{[0,T[} \Psi^\infty\left(\frac{u_s'}{|u_s'|}\right)\,d|u_s'|\,,$$

after extending u to $]-T,T[$ by $u(t+T) = u(t)$ for a.e. $t \in]-T,0[$. With this extension, it turns out that $u(T_-) = u(0_-)$.

We also denote by f_0^∞ the corresponding functional induced by Ψ^∞ instead of Ψ (then $(\Psi^\infty)^\infty = \Psi^\infty$), namely

$$f_0^\infty(u) = \begin{cases} \displaystyle\int_{]0,T[} \Psi^\infty\left(\frac{u'}{|u'|}\right)\,d|u'| & \\ \qquad\qquad +\Psi^\infty\big(u(0_+) - u(T_-)\big) & \text{if } u \in BV(0,T;\mathbb{R}^n)\,, \\ +\infty & \text{if } u \in L^1(0,T;\mathbb{R}^n) \setminus BV(0,T;\mathbb{R}^n)\,, \end{cases}$$

and we denote by $\hat{f}_0$ the functional induced by $\widehat{\Psi}(\xi) = |\xi|$, namely

$$\hat{f}_0(u) = \begin{cases} \displaystyle\int_{]0,T[} |u'| + \big|u(0_+) - u(T_-)\big| & \text{if } u \in BV(0,T;\mathbb{R}^n)\,, \\ +\infty & \text{if } u \in L^1(0,T;\mathbb{R}^n) \setminus BV(0,T;\mathbb{R}^n)\,. \end{cases}$$

Proposition 1.2.3. *The following facts hold:*

(a) *the functional f_0 is convex and lower semicontinuous, namely* epi (f_0) *is a closed and convex subset of $L^1(0,T;\mathbb{R}^n) \times \mathbb{R}$;*

(*b*) *we have*

$$\nu \hat{f}_0(u) - \frac{T}{\nu} \le f_0(u) \le \frac{1}{\nu}\,\hat{f}_0(u) \qquad \forall u \in L^1(0,T;\mathbb{R}^n)\,;$$

(*c*) *for every* $\varepsilon > 0$ *there exists* $M_\varepsilon > 0$ *such that*

$$|f_0(2u) - 2f_0(u)| \le \varepsilon f_0(u) + M_\varepsilon \qquad \forall u \in BV(0,T;\mathbb{R}^n)\,,$$
$$(1-\varepsilon) f_0^\infty(u) - TM_\varepsilon \le f_0(u) \le f_0^\infty(u) \qquad \forall u \in BV(0,T;\mathbb{R}^n)\,;$$

(*d*) *for every* $(u,\lambda) \in \mathrm{epi}\,(f_0)$ *and every* $R > 0$, *we have*

$$\sup\{\, \hat{f}_0(v) : \ (v,\eta) \in \mathrm{epi}\,(f_0)\ ,\ \|v-u\|_1^2 + |\eta - \lambda|^2 \le R^2 \,\} < +\infty\,,$$
$$\sup\{\, \|v\|_\infty : \ (v,\eta) \in \mathrm{epi}\,(f_0)\ ,\ \|v-u\|_1^2 + |\eta - \lambda|^2 \le R^2 \,\} < +\infty\,.$$

Proof. We already know that f_0 is convex and lower semicontinuous.
By assumption (Ψ) and Proposition 1.2.1, for every $u \in BV(0,T;\mathbb{R}^n)$ we have

$$\begin{aligned} f_0(u) &\ge \nu \int_0^T |u_a{}'|\,dt - \frac{T}{\nu} + \nu \int_{]0,T[} d|u_s{}'| + \nu|u(0_+) - u(T_-)|\,, \\ &= \nu \hat{f}_0(u) - \frac{T}{\nu}\,. \end{aligned}$$

The upper estimate and assertion (*c*) can be proved in a similar way.

Finally, if $f_0(v) \le \eta \le \lambda + R$, from assertion (*b*) we infer a bound for $\hat{f}_0(v)$. Since $\|v\|_1 \le \|u\|_1 + R$, we deduce from Proposition 1.1.2 a bound for v in $L^\infty(0,T;\mathbb{R}^n)$. □

2.2 The lower order term

We also consider $G :]0,T[\times \mathbb{R}^n \to \mathbb{R}$ satisfying:

(G_b) *the function* $G(\cdot, s)$ *is measurable for every* $s \in \mathbb{R}^n$, $G(t,0) = 0$ *for a.e.* $t \in]0,T[$ *and, for every* $r > 0$, *there exists* $\alpha_r \in L^1(0,T)$ *satisfying*

$$|G(t,s) - G(t,\sigma)| \le \alpha_r(t)|s - \sigma|$$

for a.e. $t \in]0,T[$ *and every* $s, \sigma \in \mathbb{R}^n$ *with* $|s| \le r$ *and* $|\sigma| \le r$.

From (G_b) it follows that $G(t,\cdot)$ is locally Lipschitz, for a.e. $t \in]0,T[$. Then, according to Remark 1.1.20, for a.e. $t \in]0,T[$ and every $s, \sigma \in \mathbb{R}^n$ we have

$$\begin{aligned} G^0(t,s;\sigma) &= \limsup_{\substack{\hat{s}\to s\\ \tau\to 0_+}} \frac{G(t,\hat{s}+\tau\sigma) - G(t,\hat{s})}{\tau} \\ &= \limsup_{\substack{\hat{s}\to s\\ \hat{\sigma}\to\sigma\\ \tau\to 0_+}} \frac{G(t,\hat{s}+\tau\hat{\sigma}) - G(t,\hat{s})}{\tau}\,, \end{aligned} \tag{1.2.4}$$

$$(-G)^0(t,s;\sigma) = G^0(t,s;-\sigma)\,. \tag{1.2.5}$$

If $G(t,\cdot)$ is of class C^1, then

$$G^0(t,s;\sigma) = \nabla_s G(t,s)\cdot\sigma\,.$$

Taking into account that $BV(0,T;\mathbb{R}^n) \subseteq L^\infty(0,T;\mathbb{R}^n)$, we define the functional $f_1\colon BV(0,T;\mathbb{R}^n) \to \mathbb{R}$ by

$$f_1(u) := -\int_0^T G(t,u)\,dt\,.$$

2.3 The problem and the functional setting

According to the Introduction, we are interested in the solutions $u \in BV(0,T;\mathbb{R}^n)$ of the hemivariational inequality

$$(HI) \qquad f_0(v) + \int_0^T G^0(t,u;u-v)\,dt \geq f_0(u) \qquad \text{for every } v \in BV(0,T;\mathbb{R}^n)\,.$$

Then we introduce the functional $\mathscr{F}\colon L^1(0,T;\mathbb{R}^n)\times\mathbb{R} \to]-\infty,+\infty]$ defined as

$$\mathscr{F}(u,\lambda) := \begin{cases} \lambda + f_1(u) & \text{if } (u,\lambda)\in \operatorname{epi}(f_0)\,,\\ +\infty & \text{otherwise}\,. \end{cases}$$

The space $L^1(0,T;\mathbb{R}^n)\times\mathbb{R}$ will be endowed with the norm

$$\|(u,\lambda)\| = \left(\|u\|_1^2 + \lambda^2\right)^{\frac{1}{2}}\,,$$

while the dual space of $L^1(0,T;\mathbb{R}^n)\times\mathbb{R}$ will be identified with $L^\infty(0,T;\mathbb{R}^n)\times\mathbb{R}$, so that

$$\langle (w,\mu),(u,\lambda)\rangle = \int_0^T w\cdot u\,dt + \mu\lambda\,,$$
$$\|(w,\mu)\| = \left(\|w\|_\infty^2 + \mu^2\right)^{\frac{1}{2}}\,.$$

In the following, we will also consider the functional $f : BV(0,T;\mathbb{R}^n) \to \mathbb{R}$ defined as

$$f(u) := f_0(u) + f_1(u)\,.$$

This subsection is devoted to general properties of the functional $\mathscr{F}$, that are implied just by assumptions (Ψ) and (G_b).

Theorem 1.2.6. *The functional $\mathscr{F}$ is lower semicontinuous and bounded from below on bounded subsets. Moreover its restriction to* $\operatorname{epi}(f_0)$ *is continuous.*

Proof. If (v_k, η_k) is a sequence convergent to (u, λ) in $\mathrm{epi}\,(f_0)$, from Proposition 1.2.3 we infer that (v_k) is bounded in $L^\infty(0,T;\mathbb{R}^n)$. From assumption (G_b) it follows that $f_1(v_k) \to f_1(u)$. Therefore the restriction of $\mathscr{F}$ to $\mathrm{epi}\,(f_0)$ is continuous.

Again by Proposition 1.2.3 we know that the set $\mathrm{epi}\,(f_0)$ is closed in $L^1(0,T;\mathbb{R}^n)\times\mathbb{R}$, so that $\mathscr{F}$ is lower semicontinuous, and that $f_0(u)$ and $\|u\|_\infty$ are bounded on bounded subsets of $\mathrm{epi}\,(f_0)$. □

Theorem 1.2.7. *Let (u_k, λ_k) be a sequence in $\mathrm{epi}\,(f_0)$ such that (u_k, λ_k) is bounded in $L^1(0,T;\mathbb{R}^n)\times\mathbb{R}$.*

Then (u_k, λ_k) admits a convergent subsequence in $L^1(0,T;\mathbb{R}^n)\times\mathbb{R}$.

Proof. Since $f_0(u_k) \le \lambda_k$, from Proposition 1.2.3 we infer that $\hat{f}_0(u_k)$ is bounded, hence that (u_k) is bounded in $BV(0,T;\mathbb{R}^n)$. Then the assertion follows from the compact embedding of $BV(0,T;\mathbb{R}^n)$ in $L^1(0,T;\mathbb{R}^n)$. □

Theorem 1.2.8. *For every $(u,\lambda) \in \mathrm{epi}\,(f_0)$, the following facts hold:*

(a) *for every $(v,\eta) \in \mathrm{epi}\,(f_0)$, we have*

$$\mathscr{F}^0((u,\lambda);(v,\eta)-(u,\lambda)) \le \eta - \lambda + \int_0^T G^0(t,u;u-v)\,dt\,;$$

(b) *if $(w,\mu) \in \partial\mathscr{F}(u,\lambda)$, then we have $\mu \le 1$ and*

$$(1-\mu)f_0(v) + \int_0^T G^0(t,u;u-v)\,dt \ge (1-\mu)\lambda + \int_0^T w\cdot(v-u)\,dt \quad \forall v \in BV(0,T;\mathbb{R}^n)\,;$$

(c) *if $(w,\mu) \in \partial\mathscr{F}(u,\lambda)$ with $f_0(u) < \lambda$, then we have $\mu = 1$.*

Proof. We aim to apply Proposition 1.1.19 with $D = \mathrm{epi}\,(f_0)$. Let $(v,\eta) \in \mathrm{epi}\,(f_0)$ and let $\varepsilon > 0$. Let also $\sigma > 0$. We claim that there exists $\delta > 0$ such that

$$\frac{f_1(z+\tau(v-z)) - f_1(z)}{\tau} = -\int_0^T \frac{G(t,z+\tau(v-z)) - G(t,z)}{\tau}\,dt < \int_0^T (-G)^0(t,u;v-u)\,dt + \sigma\,,$$

whenever $(z,\mu) \in \mathrm{epi}\,(f_0)$ with $\|z-u\|_1^2 + |\mu-\lambda|^2 < \delta^2$ and $0 < \tau \le \delta$. Actually, assume for a contradiction that $(z_k,\mu_k) \in \mathrm{epi}\,(f_0)$ and $\tau_k > 0$ satisfy

$$\|z_k-u\|_1^2 + |\mu_k-\lambda|^2 \to 0\,,\;\; \tau_k \to 0$$

and

$$-\int_0^T \frac{G(t,z_k+\tau_k(v-z_k))-G(t,z_k)}{\tau_k}\,dt \geq \int_0^T (-G)^0(t,u;v-u)\,dt+\sigma\,.$$

Without loss of generality, we may assume that (z_k) is convergent to u a.e. in $]0,T[$. Moreover, according to Proposition 1.2.3 we have that (z_k) is bounded in $L^\infty(0,T;\mathbb{R}^n)$. From assumption (G_b) we infer that

$$\frac{G(t,z_k+\tau_k(v-z_k))-G(t,z_k)}{\tau_k} \geq -\|v-z_k\|_\infty\,\alpha_r(t)$$

for a suitable $\alpha_r \in L^1(0,T)$. Then, from Fatou's Lemma and (1.2.4) we deduce that

$$\limsup_k \left[-\int_0^T \frac{G(t,z_k+\tau_k(v-z_k))-G(t,z_k)}{\tau_k}\,dt\right] \leq \int_0^T (-G)^0(t,u;v-u)\,dt$$

and a contradiction follows. Therefore the claim is proved.

Now, let us define the continuous map

$$\mathscr{V}:\mathrm{epi}\,(f_0)\times]0,+\infty[\to L^1(0,T;\mathbb{R}^n)\times\mathbb{R}$$

by

$$\mathscr{V}\big((z,\mu),\tau\big)=\big(v-z,\eta-\mu\big)\,.$$

By reducing δ, we may assume that

$$\|\mathscr{V}\big((z,\mu),\tau\big)-(v-u,\eta-\lambda)\|<\varepsilon\,,\;\;|\mu-\lambda|<\sigma$$

whenever $\|z-u\|_1^2+|\mu-\lambda|^2<\delta^2$. Moreover, if $0<\tau\leq\delta$ we have

$$\begin{aligned}\mathscr{F}\big((z,\mu)+\tau\mathscr{V}((z,\mu),\tau)\big) &= \mu+\tau(\eta-\mu)+f_1\big(z+\tau(v-z)\big)\\ &= \mu+f_1(z)+\tau(\eta-\mu)+f_1\big(z+\tau(v-z)\big)-f_1(z)\\ &= \mathscr{F}(z,\mu)+\tau\left((\eta-\mu)+\frac{f_1\big(z+\tau(v-z)\big)-f_1(z)}{\tau}\right)\\ &\leq \mathscr{F}(z,\mu)+\tau\left((\eta-\lambda)+\int_0^T(-G)^0(t,u;u-v)\,dt+2\sigma\right)\,.\end{aligned}$$

It follows

$$\begin{aligned}\mathscr{F}^0_\varepsilon((u,\lambda);(v,\eta)-(u,\lambda)) &\leq \eta-\lambda+\int_0^T(-G)^0(t,u;v-u)\,dt+2\sigma\\ &= \eta-\lambda+\int_0^T G^0(t,u;u-v)\,dt+2\sigma\,,\end{aligned}$$

whence

$$\mathscr{F}_\varepsilon^0((u,\lambda);(v,\eta)-(u,\lambda)) \leq \eta - \lambda + \int_0^T G^0(t,u;u-v)\,dt$$

by the arbitrariness of σ. Then assertion (a) follows.

If $(w,\mu) \in \partial\mathscr{F}(u,\lambda)$, for every $(v,\eta) \in \mathrm{epi}\,(f_0)$ we have

$$\int_0^T w\cdot(v-u)\,dt + \mu(\eta-\lambda) \leq \mathscr{F}^0((u,\lambda);(v,\eta)-(u,\lambda)) \leq \eta - \lambda + \int_0^T G^0(t,u;u-v)\,dt\,,$$

whence

$$(1-\mu)\eta + \int_0^T G^0(t,u;u-v)\,dt \geq (1-\mu)\lambda + \int_0^T w\cdot(v-u)\,dt\,.$$

If we choose $(v,\eta) = (u,\eta)$ with $\eta \to +\infty$, we infer that $\mu \leq 1$. On the other hand, we can also choose $(v,\eta) = (v, f_0(v))$ and assertion (b) follows.

Finally, if $f_0(u) < \lambda$ we can choose $v = u$ in assertion (b), obtaining

$$(1-\mu)f_0(u) \geq (1-\mu)\lambda\,,$$

namely $(1-\mu)(\lambda - f_0(u)) \leq 0$, which implies $\mu \geq 1$. □

The next result provides the crucial information, that allows to solve (HI) by the study of the functional $\mathscr{F}$. However, by the results of [32], one can guess that $|d\mathscr{F}|\,(u,f_0(u)) = 0$ carries much more information than just (HI).

Corollary 1.2.9. *The following facts hold:*

(a) *if $(u,\lambda) \in \mathrm{epi}\,(f_0)$ with $f_0(u) < \lambda$, then we have $|d\mathscr{F}|\,(u,\lambda) \geq 1$;*

(b) *if $u \in BV(0,T;\mathbb{R}^n)$ and $|d\mathscr{F}|\,(u,f_0(u)) < +\infty$, then there exist $w \in L^\infty(0,T;\mathbb{R}^n)$ and $\mu \leq 1$ such that*

$$\|w\|_\infty^2 + \mu^2 \leq (|d\mathscr{F}|\,(u,f_0(u)))^2$$

and

$$(1-\mu)f_0(v) + \int_0^T G^0(t,u;u-v)\,dt \geq (1-\mu)f_0(u) + \int_0^T w\cdot(v-u)\,dt \quad \forall v \in BV(0,T;\mathbb{R}^n)\,.$$

Proof. The assertions are consequences of the previous theorem and Theorem 1.1.18. □

Chapter 2

Superlinear lower order terms

This chapter is devoted to the case in which $G(t,s) \approx |s|^\beta$ as $|s| \to \infty$, with $\beta > 1$. In other words, $G(t,\cdot)$ is "superlinear" at infinity, while Ψ is "linear" at infinity.

1 The generalized linking theorem

We following concept has been introduced in [22].

Definition 2.1.1. *Let D, S, A, B be four subsets of X, with $S \subseteq D$, $B \subseteq A$ and $S \cap A = B \cap D = \emptyset$. We say that (D,S)* links *(A,B) if, for every deformation $\eta : D \times [0,1] \to X \setminus B$ with $\eta(u,t) = u$ on $S \times [0,1]$, it holds $\eta(D \times \{1\}) \cap A \neq \emptyset$.*

Now let us mention an interesting extension of the celebrated mountain pass theorem [1].

Theorem 2.1.2. *Let X be a complete metric space and let $f : X \to \mathbb{R}$ be a continuous function. Let D, S, A, B be four subsets of X such that (D,S) links (A,B) and such that*

$$\sup_S f < \inf_A f \,, \qquad \sup_D f < \inf_B f$$

(we agree that $\sup \emptyset = -\infty$, $\inf \emptyset = +\infty$).

If f satisfies $(CPS)_c$ for any c with

$$\inf_A f \leq c \leq \sup_D f \,,$$

then f admits a critical value c with

$$\inf_A f \leq c \leq \sup_D f \,.$$

Proof. According to [9], the $(CPS)_c$ condition is just the $(PS)_c$ condition with respect to an auxiliary distance which keeps the completeness of X and does not change the critical points of f and the topology of X. Then the assertion follows from [22, Theorems 3.1 and 3.9]. □

Corollary 2.1.3. *Let X be a Banach space, let C be a closed subset of X and let $f : X \to]-\infty, +\infty]$ be a function such that f is real valued and continuous on C, while $f = +\infty$ on $X \setminus C$. Assume that*

$$X = X_- \oplus X_+$$

with X_- finite dimensional and X_+ closed in X. Suppose also that there exist $0 < r_+ < r_-$ and $\check{u} \in X \setminus X_-$ such that

$$\sup_{D_- \cup H} f < \inf_{S_+} f \,, \qquad \sup_Q f < +\infty \,,$$

where

$$\begin{aligned} S_+ &= \{u \in X_+ : \ \|u\| = r_+\} \,, \\ Q &= \{u + t\check{u} : \ u \in X_- \,, \ t \geq 0 \,, \ \|u + t\check{u}\| \leq r_-\} \,, \\ H &= \{u + t\check{u} : \ u \in X_- \,, \ t \geq 0 \,, \ \|u + t\check{u}\| = r_-\} \,, \\ D_- &= \{u \in X_- : \ \|u\| \leq r_-\} \,. \end{aligned}$$

If f satisfies $(CPS)_c$ for any c with

$$\inf_{S_+} f \leq c \leq \sup_Q f \,,$$

then f admits a critical value c with

$$\inf_{S_+} f \leq c \leq \sup_Q f \,.$$

Proof. It is well known (see e.g. [22]) that $(Q, D_- \cup H)$ links $(S_+, \emptyset)$. On the other hand, $Q \subseteq C$ and then $(Q, D_- \cup H)$ links $(S_+ \cap C, \emptyset)$ in the metric space C. Moreover C is complete and, by Proposition 1.1.10, there is no change in critical points and $(CPS)_c$, if we restrict f to C. Then the assertion follows from Theorem 2.1.2. □

2 Existence of a periodic solution

Throughout this section, we still assume that

$$\Psi : \mathbb{R}^n \to \mathbb{R}\,, \qquad G :]0,T[\times\mathbb{R}^n \to \mathbb{R}$$

satisfy conditions (Ψ) and (G_b). We also require that:

(P) *we have* $G(t,s) \geq 0$ *for a.e.* $t \in]0,T[$ *and every* $s \in \mathbb{R}^n$*;*

(P_∞) *there exist* $\beta > 1$*,* $\alpha \in L^1(0,T)$ *and a measurable subset* E *of* $]0,T[$ *with positive measure such that*

$$\beta\, G(t,s) + G^0(t,s;-s) \leq (1+|s|)\,\alpha(t) \qquad \text{for a.e. } t \in]0,T[\text{ and every } s \in \mathbb{R}^n\,,$$

$$\limsup_{|s|\to\infty} \frac{\beta\, G(t,s) + G^0(t,s;-s)}{|s|} \leq 0 \qquad \text{for a.e. } t \in]0,T[\,,$$

$$\limsup_{|s|\to\infty} \frac{\beta\, G(t,s) + G^0(t,s;-s)}{|s|} < 0 \qquad \text{for a.e. } t \in E\,;$$

(P_0) *there exists* $p \geq 1$ *such that:*

- (i) $\liminf\limits_{\xi\to 0} \frac{\Psi(\xi)}{|\xi|^p} > 0$*;*
- (ii) $\lim\limits_{s\to 0} \frac{G(t,s)}{|s|^p} = 0$ *for a.e.* $t \in]0,T[$*;*
- (iii) *there exist* $r > 0$ *and* $\hat{\alpha}_r \in L^1(0,T)$ *such that* $G(t,s) \leq |s|^p \hat{\alpha}_r(t)$ *for a.e.* $t \in]0,T[$ *and every* $s \in \mathbb{R}^n$ *with* $|s| \leq r$*.*

About (i) of assumption (P_0), the typical cases are:

$$\Psi(\xi) = |\xi| \qquad \text{with } p = 1\,,$$
$$\Psi(\xi) = \sqrt{1+|\xi|^2} - 1 \qquad \text{with } p = 2\,.$$

In the case $p = 1$, assumption (iii) of (P_0) is implied by (G_b).
If $G(t,\cdot)$ is of class C^1 for a.e. $t \in]0,T[$, assumption (P_∞) is equivalent to:

$$\beta\, G(t,s) - \nabla_s G(t,s)\cdot s \leq (1+|s|)\,\alpha(t) \qquad \text{for a.e. } t \in]0,T[\text{ and every } s \in \mathbb{R}^n\,,$$

$$\limsup_{|s|\to\infty} \frac{\beta\, G(t,s) - \nabla_s G(t,s)\cdot s}{|s|} \leq 0 \qquad \text{for a.e. } t \in]0,T[\,,$$

$$\limsup_{|s|\to\infty} \frac{\beta\, G(t,s) - \nabla_s G(t,s)\cdot s}{|s|} < 0 \qquad \text{for a.e. } t \in E\,.$$

Let us state our main result.

Theorem 2.2.1. *Under assumptions* (Ψ), (G_b), (P), (P_∞) *and* (P_0), *there exists a non-constant* $u \in BV(0,T;\mathbb{R}^n)$ *satisfying* (HI).

For the proof we need some lemmas.

Lemma 2.2.2. *Under assumptions* (Ψ), (G_b) *and* (P_∞), *the functional* $\mathscr{F}$ *satisfies* $(PS)_c$ *for any* $c \in \mathbb{R}$.

Proof. Let (u_k, λ_k) be a sequence in $\mathrm{epi}\,(f_0)$ with

$$\mathscr{F}(u_k, \lambda_k) = \lambda_k + f_1(u_k) \to c\,, \qquad |d\mathscr{F}|(u_k, \lambda_k) \to 0\,.$$

From Corollary 1.2.9 we infer that $\lambda_k = f_0(u_k)$ eventually as $k \to \infty$, so that $f(u_k) \to c$, and that there exist $w_k \in L^\infty(0,T;\mathbb{R}^n)$ and $\mu_k \leq 1$ such that

$$\|w_k\|_\infty^2 + {\mu_k}^2 \leq \Big(|d\mathscr{F}|\big(u_k, f_0(u_k)\big)\Big)^2\,,$$

$$\begin{aligned}(1-\mu_k)f_0(v) &+ \int_0^T G^0(t,u_k;u_k - v)\,dt \\ &\geq (1-\mu_k)f_0(u_k) + \int_0^T w_k\cdot(v-u_k)\,dt \qquad \forall v \in BV(0,T;\mathbb{R}^n)\,.\end{aligned}$$

The choice $v = 2u_k$ yields

$$(1-\mu_k)f_0(2u_k) + \int_0^T G^0(t,u_k;-u_k)\,dt \geq (1-\mu_k)f_0(u_k) + \int_0^T w_k\cdot u_k\,dt\,,$$

whence, taking into account (P_∞),

$$\begin{aligned}\int_0^T \alpha(1+|u_k|)\,dt &\geq \int_0^T \left[\beta\, G(t,u_k) + G^0(t,u_k;-u_k)\right]\,dt \\ &\geq -\beta\, f(u_k) + \beta\, f_0(u_k) \\ &\qquad - (1-\mu_k)f_0(2u_k) + (1-\mu_k)f_0(u_k) + \int_0^T w_k\cdot u_k\,dt \\ &= [\beta - 1 + \mu_k]f_0(u_k) + \int_0^T w_k\cdot u_k\,dt \\ &\qquad - (1-\mu_k)[f_0(2u_k) - 2f_0(u_k)] - \beta\, f(u_k)\,.\end{aligned}$$

On the other hand, by Proposition 1.2.3 we have

$$|f_0(2u_k) - 2f_0(u_k)| \leq \varepsilon f_0(u_k) + M_\varepsilon\,,$$

whence

$$\int_0^T \alpha(1+|u_k|)\,dt \geq \int_0^T \left[\beta\, G(t,u_k) + G^0(t,u_k;-u_k)\right]\,dt \\ \geq [\beta - 1 + \mu_k - (1-\mu_k)\varepsilon] f_0(u_k) + \int_0^T w_k \cdot u_k\,dt \\ - (1-\mu_k)M_\varepsilon - \beta\, f(u_k)\,. \tag{2.2.3}$$

We claim that (u_k) is bounded in $L^\infty(0,T;\mathbb{R}^n)$. Assume, for a contradiction, that $u_k = \tau_k v_k$ with $\tau_k \to +\infty$ and $\|v_k\|_\infty = 1$. From (2.2.3) it follows

$$[\beta - 1 + \mu_k - (1-\mu_k)\varepsilon]\,\frac{f_0(u_k)}{\tau_k} \\ \leq \int_0^T \frac{\beta\, G(t,u_k) + G^0(t,u_k;-u_k)}{\tau_k}\,dt - \int_0^T w_k \cdot v_k\,dt + \frac{(1-\mu_k)M_\varepsilon + \beta\, f(u_k)}{\tau_k}\,.$$

Since $\|w_k\|_\infty \to 0$, $\mu_k \to 0$ and $f(u_k) \to c$, it follows

$$\limsup_k\,[\beta - 1 + \mu_k - (1-\mu_k)\varepsilon]\,\frac{f_0(u_k)}{\tau_k} \\ \leq \limsup_k \int_0^T \frac{\beta\, G(t,u_k) + G^0(t,u_k;-u_k)}{\tau_k}\,dt\,. \tag{2.2.4}$$

On the other hand, from (2.2.3) we also infer

$$\tau_k^{-1}\,[\beta - 1 + \mu_k - (1-\mu_k)\varepsilon]\,f_0(\tau_k v_k) \\ \leq \int_0^T \alpha\left(\tau_k^{-1} + |v_k|\right)\,dt - \int_0^T w_k \cdot v_k\,dt + \tau_k^{-1}(1-\mu_k)M_\varepsilon + \tau_k^{-1}\beta\, f(u_k)\,.$$

Since $\beta > 1$, $\|w_k\|_\infty \to 0$, $\mu_k \to 0$, $f(u_k) \to c$ and ε is arbitrary, it follows that

$$\limsup_k \frac{f_0(\tau_k v_k)}{\tau_k} < +\infty$$

hence, by Proposition 1.2.3,

$$\limsup_k \hat{f}_0(v_k) < +\infty\,.$$

Therefore (v_k) is bounded in $BV(0,T;\mathbb{R}^n)$ hence convergent, up to a subsequence, to some $v \in BV(0,T;\mathbb{R}^n)$ a.e. in $]0,T[$.

From (P_∞) we infer that

$$\frac{\beta\, G(t,\tau_k v_k) + G^0(t,\tau_k v_k;-\tau_k v_k)}{\tau_k} \leq \alpha\left(\tau_k^{-1} + |v_k|\right) \qquad \text{a.e. in }]0,T[\,,$$

$$\limsup_k \frac{\beta\, G(t,\tau_k v_k) + G^0(t,\tau_k v_k;-\tau_k v_k)}{\tau_k} \leq 0 \qquad \text{a.e. in }]0,T[\,.$$

From Fatou's lemma we deduce that

$$\limsup_k \int_0^T \frac{\beta\, G(t,u_k) + G^0(t,u_k;-u_k)}{\tau_k}\,dt \leq 0\,,$$

hence, by (2.2.4),

$$\limsup_k \frac{f_0(\tau_k v_k)}{\tau_k} \leq 0\,.$$

By Proposition 1.2.3 it follows

$$\lim_k \hat{f}_0(v_k) = 0\,,$$

so that $\|v_k - v\|_\infty \to 0$ and v is constant a.e. In particular, $v \neq 0$ a.e. in $]0,T[$.

Again from (P_∞) and Fatou's lemma now we infer that

$$\limsup_k \int_{]0,T[\setminus E} \frac{\beta\, G(t,\tau_k v_k) + G^0(t,\tau_k v_k;-\tau_k v_k)}{\tau_k}\,dt \leq 0\,,$$
$$\limsup_k \int_E \frac{\beta\, G(t,\tau_k v_k) + G^0(t,\tau_k v_k;-\tau_k v_k)}{\tau_k}\,dt < 0\,,$$

whence

$$\limsup_k \frac{f_0(\tau_k v_k)}{\tau_k} < 0\,,$$

which implies

$$\limsup_k \hat{f}_0(v_k) < 0$$

and a contradiction follows. Therefore (u_k) is bounded in $L^\infty(0,T;\mathbb{R}^n)$.

From assumption (G_b) it follows that $(f_1(u_k))$ is bounded. Since $\lambda_k + f_1(u_k) \to c$, we infer that (λ_k) also is bounded. By Theorem 1.2.7 we conclude that (u_k,λ_k) admits a convergent subsequence in $L^1(0,T;\mathbb{R}^n)\times\mathbb{R}$. □

Lemma 2.2.5. *Under assumptions (G_b), (P) and (P_∞), we have*

$$\liminf_{|s|\to\infty} \frac{G(t,s)}{|s|^\beta} > 0 \qquad \text{for a.e. } t \in E\,.$$

In particular,

$$\lim_{|s|\to\infty} \frac{G(t,s)}{|s|} = +\infty \qquad \text{for a.e. } t \in E\,.$$

Proof. Let $t \in E$ be such that $G(t,\cdot)$ is locally Lipschitz, $G(t,s) \geq 0$ for any $s \in \mathbb{R}^n$ and

$$\limsup_{|s|\to\infty} \frac{\beta\, G(t,s) + G^0(t,s;-s)}{|s|} < 0\,.$$

Then let $S > 0$ be such that

$$\beta\, G(t,s) + G^0(t,s;-s) < 0 \qquad \text{whenever } |s| \geq S\,.$$

It follows $G^0(t,s;-s) < 0$ whenever $|s| \geq S$, hence $G(t,s) > 0$ whenever $|s| \geq S$, as s cannot be a minimum point of $G(t,\cdot)$. If we set $\gamma(\tau) = \tau^{-\beta}\, G(t,\tau s)$, it follows

$$\begin{aligned}\gamma^0(\tau;-1) &\leq \tau^{-\beta} G^0(t,\tau s;-s) + \beta\tau^{-\beta-1}\, G(t,\tau s) \\ &= \tau^{-\beta-1}\, \left(G^0(t,\tau s;-\tau s) + \beta\, G(t,\tau s)\right) < 0\end{aligned}$$

whenever $\tau|s| \geq S$. From Lebourg's theorem [8] we infer that

$$\gamma(1) \geq \gamma(\tau_0) \qquad \text{whenever } 1 \geq \tau_0 \text{ and } \tau_0|s| \geq S\,,$$

whence

$$G(t,s) \geq S^{-\beta}|s|^\beta\, G\left(t, \frac{S}{|s|}\, s\right) \qquad \text{whenever } |s| \geq S$$

and the assertion follows. □

Remark 2.2.6. *Under assumptions (G_b) and (P), condition (P_∞) can be reformulated in several equivalent ways. For instance, condition (P_∞) holds if and only if*

$(\widetilde{P}_\infty)$ *there exist $\beta > 1$, $\alpha \in L^1(0,T)$ and a measurable subset E of $]0,T[$ with positive measure such that*

$$\beta\, G(t,s) + G^0(t,s;-s) \leq (1+|s|)\,\alpha(t) \qquad \textit{for a.e. } t \in]0,T[\textit{ and every } s \in \mathbb{R}^n\,,$$

$$\limsup_{|s|\to\infty} \frac{\beta\, G(t,s) + G^0(t,s;-s)}{|s|} \leq 0 \qquad \textit{for a.e. } t \in]0,T[\,,$$

$$\lim_{|s|\to\infty} \frac{G(t,s)}{|s|} = +\infty \qquad \textit{for a.e. } t \in E\,.$$

Proof. Just the previous proof shows that (P_∞) implies $(\widetilde{P}_\infty)$. Conversely, assume $(\widetilde{P}_\infty)$. If $1 < \hat\beta < \beta$, because of (P) we also have

$$\hat\beta\, G(t,s) + G^0(t,s;-s) \leq (1+|s|)\,\alpha(t) \qquad \text{for a.e. } t \in]0,T[\text{ and every } s \in \mathbb{R}^n\,,$$

$$\limsup_{|s|\to\infty} \frac{\hat\beta\, G(t,s) + G^0(t,s;-s)}{|s|} \leq 0 \qquad \text{for a.e. } t \in]0,T[\,.$$

We can also write

$$\limsup_{|s|\to\infty} \left(\frac{(\beta-\hat\beta)\, G(t,s)}{|s|} + \frac{\hat\beta\, G(t,s) + G^0(t,s;-s)}{|s|}\right) \leq 0\,,$$

whence

$$\lim_{|s|\to\infty} \frac{\hat{\beta}\, G(t,s) + G^0(t,s;-s)}{|s|} = -\infty \qquad \text{for a.e. } t \in E$$

and (P_∞) follows (with β replaced by $\hat{\beta}$). $\square$

Now let

$$\begin{aligned}
\widehat{X}_- &= \left\{ u \in L^1(0,T;\mathbb{R}^n) :\ u \text{ is constant a.e.} \right\} , \\
\widehat{X}_+ &= \left\{ u \in L^1(0,T;\mathbb{R}^n) :\ \int_0^T u(t)\, dt = 0 \right\} , \\
X_- &= \widehat{X}_- \times \{\, 0 \,\} , \\
X_+ &= \widehat{X}_+ \times \mathbb{R} ,
\end{aligned}$$

so that

$$L^1(0,T;\mathbb{R}^n) \times \mathbb{R} = X_- \oplus X_+$$

with X_- finite dimensional and X_+ closed in $L^1(0,T;\mathbb{R}^n) \times \mathbb{R}$.

Let also $\hat{u} \in \widehat{X}_+ \setminus \{0\}$ be defined as

$$\hat{u}(t) = \left(\sin\left(\frac{2\pi}{T}\, t \right), 0, \cdots, 0 \right)$$

and let $\check{u} \in X_+ \setminus \{0\}$ be defined as

$$\check{u} = \left(\hat{u}, \frac{1}{\nu}\, \hat{f}_0(\hat{u}) \right) .$$

Lemma 2.2.7. *Under assumptions (Ψ), (G_b), (P) and (P_∞), we have*

$$\begin{aligned}
&\mathscr{F}(u,\lambda) \le 0 \qquad \textit{for every } (u,\lambda) \in X_- , \\
&\sup_B \mathscr{F} < +\infty \qquad \textit{for every bounded subset } B \textit{ of } X_- + [0,+\infty[\check{u} , \\
&\lim_{\substack{\|(u,\lambda)\|\to\infty \\ (u,\lambda)\in X_- + [0,+\infty[\check{u}}} \mathscr{F}(u,\lambda) = -\infty .
\end{aligned}$$

Proof. From assumption (P) it follows that

$$\mathscr{F}(u,\lambda) \le 0 \qquad \text{for every } (u,\lambda) \in X_- .$$

Moreover, by Proposition 1.2.3 we have

$$f_0(c + \tau\hat{u}) \le \frac{1}{\nu}\, \hat{f}_0(c + \tau\hat{u}) = \frac{\tau}{\nu}\, \hat{f}_0(\hat{u}) ,$$

hence

$$(c,0)+\tau\check{u}=\left(c+\tau\hat{u},\frac{\tau}{\nu}\,\hat{f}_0(\hat{u})\right)\in \operatorname{epi}(f_0)\qquad\text{for every } c\in\widehat{X}_-\text{ and }\tau\geq 0\,.$$

Since $\widehat{X}_-\oplus\mathbb{R}\hat{u}$ is a finite dimensional subspace of $BV(0,T;\mathbb{R}^n)$, if $\|c+\tau\hat{u}\|_1$ is bounded, then $(c+\tau\hat{u})$ is bounded in $BV(0,T;\mathbb{R}^n)$, which implies that $f_1(c+\tau\hat{u})$ is bounded. Therefore $\mathscr{F}$ is bounded on every bounded subset of $X_-+[0,+\infty[\check{u}$.

Now assume, for a contradiction, that

$$\inf_k \mathscr{F}\left(c_k+\tau_k\hat{u},\frac{\tau_k}{\nu}\,\hat{f}_0(\hat{u})\right)>-\infty$$

with $c_k\in\mathbb{R}^n$, $\tau_k\geq 0$ and $\|c_k+\tau_k\hat{u}\|_1^2+\left(\frac{\tau_k}{\nu}\,\hat{f}_0(\hat{u})\right)^2\to\infty$.

Again, if $\|c_k+\tau_k\hat{u}\|_1$ is bounded, then $(c_k+\tau_k\hat{u})$ is bounded in $BV(0,T;\mathbb{R}^n)$, which implies that $\frac{\tau_k}{\nu}\,\hat{f}_0(\hat{u})=\frac{1}{\nu}\,\hat{f}_0(\tau_k\hat{u})$ is bounded in $\mathbb{R}$ and a contradiction follows. Therefore $\|c_k+\tau_k\hat{u}\|_1\to\infty$.

Moreover

$$\inf_k\left(\frac{1}{\nu}\,\hat{f}_0(c_k+\tau_k\hat{u})-\int_0^T G\left(t,c_k+\tau_k\hat{u}\right)\,dt\right)=\inf_k\left(\frac{\tau_k}{\nu}\,\hat{f}_0(\hat{u})+f_1\left(c_k+\tau_k\hat{u}\right)\right)>-\infty\,.$$

Let us write $c_k+\tau_k\hat{u}=\varrho_k v_k$ with $\varrho_k\to+\infty$ and $\|v_k\|_1=1$. We have

$$\liminf_k\left(\frac{1}{\nu}\,\hat{f}_0(v_k)-\int_0^T\frac{G\left(t,\varrho_k v_k\right)}{\varrho_k}\,dt\right)\geq 0$$

hence, being $\widehat{X}_-\oplus\mathbb{R}\hat{u}$ a finite dimensional subspace of $BV(0,T;\mathbb{R}^n)$,

$$\limsup_k\int_0^T\frac{G\left(t,\varrho_k v_k\right)}{\varrho_k}\,dt<+\infty\,.$$

On the other hand, up to a subsequence, (v_k) is convergent a.e. in $]0,T[$ to some $v\in\widehat{X}_-\oplus\mathbb{R}\hat{u}$ with $\|v\|_1=1$. In particular, $v\neq 0$ a.e. in $]0,T[$.

From assumption (P), Lemma 2.2.5 and Fatou's lemma we infer that

$$\lim_k\int_0^T\frac{G\left(t,\varrho_k v_k\right)}{\varrho_k}\,dt=+\infty$$

and a contradiction follows. □

Lemma 2.2.8. *Under assumptions* (Ψ) *and* (i) *of* (P_0), *there exists* $\hat{\nu}>0$ *such that*

$$\Psi(\xi)\geq\hat{\nu}\,\varphi_p(|\xi|)\qquad\textit{for any }\xi\in\mathbb{R}^n\,,$$

where

$$\varphi_p(\tau) = (1 + |\tau|^p)^{\frac{1}{p}} - 1 \,.$$

Then we have

$$f_0(u) \geq \hat{\nu} T \varphi_p \left(\frac{1}{T} \hat{f}_0(u) \right) \qquad \text{for any } u \in BV(0,T;\mathbb{R}^n)\,.$$

Proof. It is clear that there exists $\hat{\nu} > 0$ such that

$$\Psi(\xi) \geq \hat{\nu}\, \varphi_p(|\xi|) \qquad \text{for any } \xi \in \mathbb{R}^n \,.$$

It follows $\Psi^\infty(\xi) \geq \hat{\nu}\,|\xi|$. Then, for every $u \in BV(0,T;\mathbb{R}^n)$, we have

$$f_0(u) \geq \hat{\nu} \left(\int_0^T \varphi_p(|u_a'|)\, dt + |u_s'|(]0,T[) + \big|u(0_+) - u(T_-)\big| \right) .$$

On the other hand, if

$$\lambda \in \partial \varphi_p \left(\frac{1}{T} \hat{f}_0(u) \right) ,$$

we have

$$\varphi_p(|u_a'|) \geq \varphi_p \left(\frac{1}{T} \hat{f}_0(u) \right) + \lambda \left(|u_a'| - \frac{1}{T} \hat{f}_0(u) \right) \qquad \text{a.e. in }]0,T[\,,$$

whence

$$\begin{aligned} \frac{1}{T} \int_0^T \varphi_p(|u_a'|)\, dt &\geq \varphi_p \left(\frac{1}{T} \hat{f}_0(u) \right) + \lambda \left(\frac{1}{T} \int_0^T |u_a'|\, dt - \frac{1}{T} \hat{f}_0(u) \right) \\ &= \varphi_p \left(\frac{1}{T} \hat{f}_0(u) \right) - \frac{\lambda}{T} \left(|u_s'|(]0,T[) + \big|u(0_+) - u(T_-)\big| \right) . \end{aligned}$$

We infer that

$$\begin{aligned} T\, \varphi_p \left(\frac{1}{T} \hat{f}_0(u) \right) &\leq \int_0^T \varphi_p(|u_a'|)\, dt + \lambda \left(|u_s'|(]0,T[) + \big|u(0_+) - u(T_-)\big| \right) \\ &\leq \int_0^T \varphi_p(|u_a'|)\, dt + |u_s'|(]0,T[) + \big|u(0_+) - u(T_-)\big| \,, \end{aligned}$$

as $\lambda \leq 1$, and the assertion follows. □

Lemma 2.2.9. *Under assumptions* (Ψ), (G_b) *and* (P_0), *we have*

$$\liminf_{\substack{\|(u,\lambda)\| \to 0 \\ (u,\lambda) \in X_+}} \frac{\mathscr{F}(u,\lambda)}{\|(u,\lambda)\|^p} > 0 \,.$$

Proof. It is enough to show that

$$\liminf_{\substack{\|(u,\lambda)\|\to 0\\ (u,\lambda)\in X_+\\ f_0(u)\le\lambda}} \frac{\lambda + f_1(u)}{(\|u\|_\infty^2+\lambda^2)^{p/2}} > 0\,.$$

According to Lemma 2.2.8, we may assume that $\lambda \ge 0$ and, of course, $\lambda \le 1$. It follows

$$\nu \hat{f}_0(u) - \frac{T}{\nu} \le f_0(u) \le \lambda \le 1\,,$$

so that $\hat{f}_0(u)$ is bounded.

If $\|u\|_\infty \le \lambda$, we have

$$\frac{\lambda}{(\|u\|_\infty^2+\lambda^2)^{p/2}} \ge \frac{\lambda^p}{(2\lambda^2)^{p/2}} = 2^{-p/2}\,.$$

If $\|u\|_\infty \ge \lambda$, we have

$$\frac{\lambda}{(\|u\|_\infty^2+\lambda^2)^{p/2}} \ge \frac{f_0(u)}{(2\|u\|_\infty^2)^{p/2}}\,.$$

On the other hand, from Proposition 1.1.2 and Lemma 2.2.8 we infer that

$$\frac{f_0(u)}{(2\|u\|_\infty^2)^{p/2}} \ge \frac{\hat{\nu}T}{2^{p/2}T^p}\,\frac{\varphi_p\left(\frac{1}{T}\hat{f}_0(u)\right)}{\left(\frac{1}{T}\hat{f}_0(u)\right)^p}\,.$$

Therefore

$$\liminf_{\substack{\|(u,\lambda)\|\to 0\\ (u,\lambda)\in X_+\\ f_0(u)\le\lambda}} \frac{\lambda}{(\|u\|_\infty^2+\lambda^2)^{p/2}} > 0\,.$$

On the other hand, we have

$$\frac{|f_1(u)|}{(\|u\|_\infty^2+\lambda^2)^{p/2}} \le \frac{|f_1(u)|}{\|u\|_\infty^p}$$

and from Lemma 2.2.8 we infer that $\|(u,\lambda)\| \to 0$ implies $\|u\|_\infty \to 0$.

From (P_0) it easily follows that

$$\lim_{\|u\|_\infty\to 0} \frac{\displaystyle\int_0^T G(t,u)\,dt}{\|u\|_\infty^p} = 0\,,$$

whence

$$\lim_{\substack{\|(u,\lambda)\|\to 0\\ (u,\lambda)\in X_+\\ f_0(u)\le\lambda}} \frac{f_1(u)}{(\|u\|_\infty^2+\lambda^2)^{p/2}} = 0\,.$$

Then the assertion follows. □

Proof of Theorem 2.2.1.
We aim to apply Corollary 2.1.3 to

$$\mathscr{F}\colon L^1(0,T;\mathbb{R}^n)\times\mathbb{R}\to\mathbb{R}\,.$$

By Proposition 1.2.3 the set $\mathrm{epi}\,(f_0)$ is closed in $L^1(0,T;\mathbb{R}^n)\times\mathbb{R}$ and by Theorem 1.2.6 the restriction of $\mathscr{F}$ to $\mathrm{epi}\,(f_0)$ is continuous. Moreover, $\mathscr{F}$ satisfies $(PS)_c$ for any $c\in\mathbb{R}$ by Lemma 2.2.2.

By Lemma 2.2.9, there exists $r_+>0$ such that

$$\inf_{S_+}\mathscr{F}>0\,.$$

On the other hand, by Lemma 2.2.7,

$$\sup_{D_-\cup H}\mathscr{F}\le 0\,,$$

provided that r_- is large enough, and $\mathscr{F}$ is bounded from above on Q for any $r_->0$.

From Corollary 2.1.3 we infer that there exists a critical point (u,λ) of $\mathscr{F}$ with $\mathscr{F}(u,\lambda)>0$. From Corollary 1.2.9 we infer that u is a solution of (HI), while $\mathscr{F}(u,\lambda)>0$ and (P) imply that u is not constant. □

Remark 2.2.10. *Let Ψ be a function satisfying (Ψ), let $\beta\ge 1$, let $G\colon\,]0,1[\times\mathbb{R}\to\mathbb{R}$ be the function defined by*

$$G(t,s)=t^{-\alpha(s)}\,|s|^{\beta}\,,$$

where

$$\alpha(s)=\frac{2}{\pi}\arctan(|s|^{\beta})\,,$$

and let f_0, f_1 be defined as before.

Then the following facts hold:

(a) *the function G satisfies (G_b) and (P);*

(b) *for every $q<\infty$, the functional*

$$f\colon L^q(0,1;\mathbb{R})\to]-\infty,+\infty]$$

defined as

$$f(u)=\begin{cases} f_0(u)+f_1(u) & \text{if } u\in BV(0,1;\mathbb{R})\,,\\ +\infty & \text{otherwise}\,,\end{cases}$$

is not lower semicontinuous;

(c) *if $\beta > p \geq 1$, then G also satisfies (P_∞) and (P_0).*

Proof. We have $0 \leq \alpha(s) < 1$, hence

$$\max_{|s|\leq S} \alpha(s) < 1 \qquad \text{for any } S > 0\,.$$

Moreover $G(t,\cdot)$ is of class C^1 on $\mathbb{R}\setminus\{0\}$, so that

$$G^0(t,s;\sigma) = D_sG(t,s)\sigma = t^{-\alpha(s)}\left(\beta|s|^{\beta-2}s - \frac{2}{\pi}\,\frac{\beta|s|^{2\beta-2}s}{1+|s|^{2\beta}}\,\log t\right)\sigma \qquad \forall s \neq 0\,.$$

Then it is easily seen that G satisfies (G_b) and (P).

Now define $u_k \in BV(0,1;\mathbb{R})$ by

$$u_k(t) = \begin{cases} \log k & \text{if } 0 < t \leq \dfrac{1}{k}\,,\\ 0 & \text{if } \dfrac{1}{k} < t < 1\,. \end{cases}$$

It is easily seen that $u_k \to 0$ in any $L^q(0,1;\mathbb{R})$ with $q < \infty$. On the other hand, we have

$$f_0(u_k) \leq \frac{1}{\nu}\,\hat{f}_0(u_k) = \frac{2}{\nu}\,\log k\,,$$

while

$$\int_0^1 G(t,u_k)\,dt = \frac{k^{\alpha(\log k)-1}}{1-\alpha(\log k)}\,(\log k)^\beta\,.$$

Since

$$\limsup_{s\to+\infty}\,(\alpha(s)-1)s < +\infty\,,$$

we have

$$\limsup_k\, k^{\alpha(\log k)-1} = \lim_k \exp\left[(\alpha(\log k)-1)\,\log k\right] < +\infty\,,$$

hence

$$\lim_k (f_0(u_k) + f_1(u_k)) = -\infty\,.$$

Therefore f is not lower semicontinuous.

If $\beta > p \geq 1$, it is easily seen that G also satisfies (P_0). Moreover, we have

$$G^0(t,s;-s) = -D_sG(t,s)s = -t^{-\alpha(s)}\left(\beta|s|^\beta - \frac{2}{\pi}\,\frac{\beta|s|^{2\beta}}{1+|s|^{2\beta}}\,\log t\right) \qquad \forall s \neq 0\,,$$

hence

$$\beta G(t,s) + G^0(t,s;-s) = \frac{2\beta}{\pi}\,t^{-\alpha(s)}\,\frac{|s|^{2\beta}}{1+|s|^{2\beta}}\,\log t < 0 \qquad \forall s \neq 0\,,$$

as $\log t < 0$. By Remark 2.2.6, assumption (P_∞) also holds with $E =]0,1[$. ∎

Chapter 3

Many solutions near the origin

This chapter is devoted to a case whose model is

$$\Psi(\xi) = \sqrt{1 + |\xi|^2} - 1\,, \qquad G(t,s) = |s|\,.$$

1 A D.C. Clark type result

Among multiplicity results of critical points in the coercive case, the classical theorem by D.C. Clark (see eg. [35]) has been the object of several developments (see [24, 29]), also in the direction of a local analysis in a neighborhhod of 0, so that the behavior of the functional at infinity has no relevance.

We aim to propose, by a completely different proof, the extension of the abstract result of [29] to continuous functionals and then apply it to the existence of solutions of (HI).

Let X be a metric space endowed with the distance d and let $\Phi\colon X \to X$ be an isometry such that $\Phi^2 = \mathrm{Id}$.

Definition 3.1.1. *A function $f\colon X \to [-\infty, +\infty]$ is said to be* Φ-invariant*, if*

$$f(\Phi(u)) = f(u) \qquad \textit{for any } u \in X\,.$$

If $S \subseteq \mathbb{R}^m$ is symmetric with respect to the origin, a map $\psi\colon S \to X$ is said to be Φ-equivariant*, if*

$$\psi(-x) = \Phi(\psi(x)) \qquad \textit{for any } x \in S\,.$$

Finally, we set

$$\mathrm{Fix}(X) = \{\, u \in X : \ \Phi(u) = u \,\}\,.$$

The next result, contained in [19, Theorem 2.5], is the natural extension of D.C. Clark's theorem to continuous functionals.

Theorem 3.1.2. *Let $f\colon X \to \mathbb{R}$ be a continuous and Φ-invariant function and let $m \geq 1$. Assume that:*

(a) *f is bounded from below;*

(b) *there exists a continuous Φ-equivariant map $\psi\colon S^{m-1} \to X$ such that*

$$\sup\{\, f(\psi(x)) : \ x \in S^{m-1}\,\} < \inf\{\, f(v) : \ v \in \mathrm{Fix}(X)\,\}\ ,$$

(where S^{m-1} is the $(m-1)$-dimensional sphere and we agree that $\inf\emptyset = +\infty$);

(c) *X is complete and, for every $c \in \mathbb{R}$ with $c < \inf\{\, f(v) : \ v \in \mathrm{Fix}(X)\,\}$, the function f satisfies $(CPS)_c$.*

Then there exist at least m distinct pairs $\{u_1, \Phi(u_1)\}, \ldots, \{u_m, \Phi(u_m)\}$ of critical points of f with

$$f(u_j) < \inf\{\, f(v) : \ v \in \mathrm{Fix}(X)\,\} \qquad \forall j = 1, \ldots, m\,.$$

If assumption (b) is satisfied for any $m \geq 1$, then a further information can be provided, in the line of [24, Proposition 2.2].

Theorem 3.1.3. *Let $f\colon X \to \mathbb{R}$ be a continuous and Φ-invariant function. Assume that:*

(a) *f is bounded from below;*

(b) *for every $m \geq 1$ there exists a continuous and Φ-equivariant map $\psi_m\colon S^{m-1} \to X$ such that*

$$\sup\{\, f(\psi_m(x)) : \ x \in S^{m-1}\,\} < \inf\{\, f(v) : \ v \in \mathrm{Fix}(X)\,\}\ ;$$

(c) *X is complete and, for every $c \in \mathbb{R}$ with $c < \inf\{\, f(v) : \ v \in \mathrm{Fix}(X)\,\}$, the function f satisfies $(CPS)_c$.*

Then there exists a sequence (u_k) of critical points of f such that

$$f(u_k) < \inf\{\, f(v) : \ v \in \mathrm{Fix}(X)\,\}\ , \qquad f(u_k) \to \inf\{\, f(v) : \ v \in \mathrm{Fix}(X)\,\}\ .$$

Proof. It is enough to combine the general technique of [19, Theorem 2.5] with the argument of [24, Proposition 2.2]. □

Our purpose is to prove a variant, related to [29, Theorem 1.1], where the Palais-Smale condition is assumed also at the level

$$\inf\{\, f(v):\ v \in \mathrm{Fix}(X)\,\}\ .$$

Theorem 3.1.4. *Let $f\colon X \to \mathbb{R}$ be a continuous and Φ-invariant function. Assume that:*

- (a) *f is bounded from below and $\mathrm{Fix}(X) \neq \emptyset$;*
- (b) *for every $m \geq 1$ there exists a continuous and Φ-equivariant map $\psi_m\colon S^{m-1} \to X$ such that*

$$\sup\{\, f(\psi_m(x)):\ x \in S^{m-1}\,\} < \inf\{\, f(v):\ v \in \mathrm{Fix}(X)\,\}\ ;$$

- (c) *X is complete and, for every $c \in \mathbb{R}$ with $c \leq \inf\{\, f(v):\ v \in \mathrm{Fix}(X)\,\}$, the function f satisfies $(PS)_c$.*

Then, at least one of the following facts holds:

- (i) *there exists a sequence (u_k) of critical points of f such that*

$$f(u_k) < \inf\{\, f(v):\ v \in \mathrm{Fix}(X)\,\}\ , \qquad d(u_k, \mathrm{Fix}(X)) \to 0\,;$$

- (ii) *there exists $\overline{r} > 0$ such that, for every $r \in]0, \overline{r}]$, there exists a critical point u of f with*

$$f(u) = \inf\{\, f(v):\ v \in \mathrm{Fix}(X)\,\}\ , \qquad d(u, \mathrm{Fix}(X)) = r\,.$$

Proof. Let us set

$$b = \inf\{\, f(v):\ v \in \mathrm{Fix}(X)\,\}$$

and argue by contradiction. Since (i) is false, there exists $\overline{r} > 0$ such that

$$\Big[d(u, \mathrm{Fix}(X)) \leq \overline{r} \text{ and } f(u) < b\Big] \Rightarrow |df|(u) > 0\,.$$

Since (ii) also is false, there exists $0 < r \leq \overline{r}$ such that

$$\Big[d(u, \mathrm{Fix}(X)) = r \text{ and } f(u) = b\Big] \Rightarrow |df|(u) > 0\,.$$

In particular, we have

$$\Big[d(u, \mathrm{Fix}(X)) \leq r \text{ and } f(u) < b\Big] \Rightarrow |df|(u) > 0\,,$$

$$\Big[d(u, \mathrm{Fix}(X)) = r \text{ and } f(u) \leq b\Big] \Rightarrow |df|(u) > 0\,.$$

Because of $(PS)_c$ and the boundedness from below of f, there exists $\sigma > 0$ such that

$$\Big[r \leq d(u, \mathrm{Fix}(X)) \leq r + \sigma \text{ and } f(u) \leq b + \sigma^2 \Big] \Rightarrow |df|(u) \geq \sigma\,.$$

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a function of class C^1 such that

$$\begin{aligned} \varphi(\tau) &= 0 && \text{whenever } \tau \leq r\,, \\ \varphi(\tau) &= -\frac{\sigma^2}{4} && \text{whenever } \tau \geq r + \sigma\,, \\ -\frac{\sigma}{2} &\leq \varphi'(\tau) \leq 0 && \text{whenever } \tau \in \mathbb{R}\,, \end{aligned}$$

and denote by $\overline{f}$ the restriction of

$$\{u \mapsto f(u) + \varphi(d(u, \mathrm{Fix}(X)))\}$$

to

$$Y := \left\{ v \in X : \; f(v) \leq b + \frac{\sigma^2}{8} \right\}.$$

We aim to apply Theorem 3.1.3 to $\overline{f} : Y \to \mathbb{R}$. Taking into account Propositions 1.1.11 and 1.1.12, we infer that

$$\begin{aligned} \left|d\overline{f}\right|(u) &= |df|\,(u) && \text{if } u \in Y \text{ with } d(u, \mathrm{Fix}(X)) \notin [r, r+\sigma]\,, \\ \left|d\overline{f}\right|(u) &\geq \frac{\sigma}{2} && \text{if } u \in Y \text{ with } d(u, \mathrm{Fix}(X)) \in [r, r+\sigma]\,, \\ \overline{f}(u) &\leq b - \frac{\sigma^2}{8} && \text{if } u \in Y \text{ with } d(u, \mathrm{Fix}(X)) \geq r + \sigma\,. \end{aligned}$$

Then it is easy to check that all the assumptions of Theorem 3.1.3 are satisfied with

$$\inf \{ \overline{f}(v) : \; v \in \mathrm{Fix}(Y) \} = b\,.$$

Let (u_k) be a sequence of critical points of $\overline{f}$, hence of f, with $\overline{f}(u_k) < b$ and $\overline{f}(u_k) \to b$. Then we have $d(u_k, \mathrm{Fix}(X)) \leq r$ and $f(u_k) < b$, eventually as $k \to \infty$. A contradiction follows. □

Corollary 3.1.5. *Under the same assumptions of Theorem 3.1.4, there exists a convergent sequence (u_k) of critical points of f such that*

$$f(u_k) \leq \inf \{ f(v) : \; v \in \mathrm{Fix}(X) \}\,, \quad d(u_k, \mathrm{Fix}(X)) > 0\,, \quad d(u_k, \mathrm{Fix}(X)) \to 0\,.$$

Proof. We have only to observe that, if (u_k) is a sequence of critical points of f such that

$$f(u_k) \leq \inf \{ f(v) : \; v \in \mathrm{Fix}(X) \}\,,$$

then by the Palais-Smale condition and the boundedness from below of f, the sequence (u_k) admits a convergent subsequence. □

2 Existence of infinitely many periodic solutions

Throughout this section, we still assume that

$$\Psi : \mathbb{R}^n \to \mathbb{R}$$

satisfies condition (Ψ). Since we are interested in a result in a neighborhood of the origin, here we suppose that

$$G :]0,T[\times \{s \in \mathbb{R}^n : \ |s| < r\} \to \mathbb{R}\,,$$

for some $T > 0$, $r > 0$. We also assume that:

(G_0) *the function* $G(\cdot, s)$ *is measurable for every* $s \in \mathbb{R}^n$ *with* $|s| < r$, $G(t, 0) = 0$ *for a.e.* $t \in]0,T[$ *and there exists* $\check{\alpha} \in L^1(0,T)$ *satisfying*

$$|G(t, s) - G(t, \sigma)| \leq \check{\alpha}(t)|s - \sigma|$$

for a.e. $t \in]0,T[$ *and every* $s, \sigma \in \mathbb{R}^n$ *with* $|s| < r$ *and* $|\sigma| < r$*;*

(B_0) *the following conditions hold:*

(i) *we have* $\lim\limits_{\xi\to 0} \frac{\Psi(\xi)}{|\xi|} = 0$*;*

(ii) *there exists a measurable subset* E *of* $]0,T[$ *with positive measure such that*

$$\liminf_{s\to 0} \frac{G(t,s)}{|s|} \geq 0 \qquad \text{for a.e. } t \in]0,T[\,,$$
$$\liminf_{s\to 0} \frac{G(t,s)}{|s|} > 0 \qquad \text{for a.e. } t \in E\,;$$

(B_e) *we have*

$$\Psi(-\xi) = \Psi(\xi) \qquad \text{for every } \xi \in \mathbb{R}^n;\ ,$$
$$G(t, -s) = G(t, s) \qquad \text{for a.e. } t \in]0,T[\text{ and every } s \in \mathbb{R}^n \text{ with } |s| < r\,.$$

About the function Ψ, a typical example is

$$\Psi(\xi) = \sqrt{1 + |\xi|^2} - 1\,.$$

Let us state the main result.

Theorem 3.2.1. *Under assumptions* (Ψ), (G_0), (B_0) *and* (B_e), *there exists a sequence* (u_k) *in* $BV(0,T;\mathbb{R}^n) \setminus \{0\}$ *of solutions of* (HI) *with* $\|u_k\|_\infty \to 0$.

We will prove the result first in a particular case, then in the general case.

2.1 Proof in a particular case

Throughout this subsection, we also assume that $G(t,s)$ is defined for any $s \in \mathbb{R}^n$, with $G(t,-s) = G(t,s)$ for a.e. $t \in]0,T[$ and every $s \in \mathbb{R}^n$, satisfies (G_b) and:

(G_∞) *we have*

$$\limsup_{|s|\to\infty} \frac{G(t,s)}{|s|} < 0 \qquad \text{for a.e. } t \in]0,T[$$

and there exists $\tilde{\alpha} \in L^1(0,T)$ *such that*

$$G(t,s) \leq (1+|s|)\,\tilde{\alpha}(t) \qquad \text{for a.e. } t \in]0,T[\text{ and every } s \in \mathbb{R}^n\,.$$

Then we consider f_0, f_1 and

$$\mathscr{F} : L^1(0,T;\mathbb{R}^n) \times \mathbb{R} \to]-\infty,+\infty]$$

as before and denote by $\widehat{\mathscr{F}}$ its restriction to $\mathrm{epi}\,(f_0)$.

Since Ψ is even, we can define an isometry

$$\Phi : \mathrm{epi}\,(f_0) \to \mathrm{epi}\,(f_0)$$

by

$$\Phi(u,\lambda) = (-u,\lambda)\,.$$

It is easily seen that $\Phi^2 = \mathrm{Id}$ and that

$$\widehat{\mathscr{F}}(\Phi(u,\lambda)) = \widehat{\mathscr{F}}(u,\lambda)\,,$$

as $G(t,\cdot)$ also is even. Moreover,

$$\mathrm{Fix}(\mathrm{epi}\,(f_0)) = \{0\} \times [0,+\infty[$$

and

$$\min\left\{\widehat{\mathscr{F}}(u,\lambda) : \ (u,\lambda) \in \mathrm{Fix}(\mathrm{epi}\,(f_0))\right\} = 0\,.$$

Lemma 3.2.2. *Under assumptions (Ψ), (G_b) and (G_∞), for every $c \in \mathbb{R}$ the set*

$$\left\{(u,\lambda) \in \mathrm{epi}\,(f_0) : \ \widehat{\mathscr{F}}(u,\lambda) \leq c\right\}$$

is compact in $L^1(0,T;\mathbb{R}^n) \times \mathbb{R}$. In particular, $\widehat{\mathscr{F}}$ is bounded from below.

Proof. By Theorem 1.2.7, it is enough to show that

$$\left\{(u,\lambda) \in \mathrm{epi}\,(f_0) : \ \widehat{\mathscr{F}}(u,\lambda) \leq c\right\}$$

is bounded in $L^1(0,T;\mathbb{R}^n) \times \mathbb{R}$.

Assume, for a contradiction, that (u_k,λ_k) is a sequence in $\mathrm{epi}\,(f_0)$ with

$$\|u_k\|_1^2 + \lambda_k^2 \to +\infty\,, \qquad \lambda_k - \int_0^T G(t,u_k)\,dt \leq c\,.$$

Let us write $(u_k,\lambda_k) = \tau_k(v_k,\mu_k)$ with

$$\tau_k \to +\infty\,, \qquad \|v_k\|_1^2 + \mu_k^2 = 1\,,$$

so that, by Proposition 1.2.3,

$$\nu\,\hat{f}_0(v_k) - \frac{T}{\tau_k \nu} \leq \frac{f_0(u_k)}{\tau_k} \leq \mu_k\,.$$

Therefore, up to a subsequence, (μ_k) is convergent to $\mu \geq 0$ and (v_k) is bounded in $L^\infty(0,T;\mathbb{R}^n)$ and convergent a.e. in $]0,T[$ to $v \in BV(0,T;\mathbb{R}^n)$ with $\|v\|_1^2 + \mu^2 = 1$.

Moreover, we have

$$\mu_k - \int_0^T \frac{G(t,\tau_k v_k)\,dt}{\tau_k}\,dt \leq \frac{c}{\tau_k}\,,$$

whence

$$\liminf_k \int_0^T \frac{G(t,\tau_k v_k)\,dt}{\tau_k}\,dt \geq \mu\,.$$

On the other hand, from assumption (G_∞) we infer that

$$\frac{G(t,\tau_k v_k)}{\tau_k} \leq \left(\frac{1}{\tau_k} + |v_k|\right)\tilde{\alpha}(t) \qquad \text{for a.e. } t \in]0,T[\,,$$

$$\limsup_k \frac{G(t,\tau_k v_k)}{\tau_k} \leq 0 \qquad \text{for a.e. } t \in]0,T[\,,$$

$$\limsup_k \frac{G(t,\tau_k v_k)}{\tau_k} < 0 \qquad \text{for a.e. } t \in]0,T[\text{ with } v(t) \neq 0\,.$$

By Fatou's lemma we first deduce that

$$\limsup_k \int_0^T \frac{G(t,\tau_k v_k)\,dt}{\tau_k}\,dt \leq 0\,,$$

whence $\mu = 0$, which in turn implies $v(t) \neq 0$ on a set of positive measure, hence

$$0 = \mu \leq \limsup_k \int_0^T \frac{G(t,\tau_k v_k)\,dt}{\tau_k}\,dt < 0$$

and a contradiction follows. Therefore, the set

$$\left\{(u,\lambda) \in \mathrm{epi}\,(f_0):\ \widehat{\mathscr{F}}(u,\lambda) \leq c\right\}$$

is bounded, hence compact, in $L^1(0,T;\mathbb{R}^n) \times \mathbb{R}$.

From Theorem 1.2.6 we infer that $\widehat{\mathscr{F}}$ is bounded from below. □

Lemma 3.2.3. *Under assumptions* (Ψ), (G_b) *and* (G_∞), *for every* $c \in \mathbb{R}$ *the functional* $\widehat{\mathscr{F}}$ *satisfies* $(PS)_c$.

Proof. It easily follows from Lemma 3.2.2. □

Lemma 3.2.4. *Under assumptions* (Ψ), (G_0), (B_0) *and* (B_e), *for every* $m \geq 1$ *there exists a continuous and* Φ*-equivariant map* $\psi_m\colon S^{m-1} \to \mathrm{epi}\,(f_0)$ *such that*

$$\sup\{\,\widehat{\mathscr{F}}(\psi_m(x)):\ x \in S^{m-1}\,\} < 0\,.$$

Proof. Let V be the linear subspace of $W_0^{1,1}(0,T;\mathbb{R}^n)$ spanned by

$$\left(\sin\left(\frac{\pi}{T}t\right),0,\ldots,0\right),\left(\sin\left(2\frac{\pi}{T}t\right),0,\ldots,0\right),\ldots,\left(\sin\left(m\frac{\pi}{T}t\right),0,\ldots,0\right)\,,$$

let $\|\ \|$ be any norm on V and let

$$S = \{v \in V:\ \|v\| = 1\}\,.$$

Since

$$f_0(v) = \int_0^T \Psi(v')\,dt \qquad \text{for any } v \in V\,,$$

the map $\psi : S \to \mathrm{epi}\,(f_0)$ defined as

$$\psi(v) = (\tau v, f_0(\tau v))$$

is clearly continuous and Φ-equivariant, for any $\tau > 0$. It is enough to show that

$$\widehat{\mathscr{F}}(\psi(v)) < 0 \qquad \text{for any } v \in S\,,$$

provided that τ is small enough.

Assume, for a contradiction, that $\tau_k > 0$ and $v_k \in V$ satisfy

$$\tau_k \to 0\,, \qquad \|v_k\| = 1\,, \qquad \int_0^T \Psi(\tau_k v_k')\,dt - \int_0^T G(t,\tau_k v_k)\,dt \geq 0\,.$$

Then, up to a subsequence, (v_k) is strongly convergent in $W^{1,1}_0(0,T;\mathbb{R}^n)$ to some $v \in S$. From Proposition 1.2.1 and (B_0) it is easy to deduce that

$$\lim_k \frac{\displaystyle\int_0^T \Psi(\tau_k v_k')\,dt}{\tau_k} = 0\,.$$

On the other hand, $v \neq 0$ a.e. in $]0,T[$, so that (G_0) and (B_0) imply that

$$\frac{G(t,\tau_k v_k)}{\tau_k} \geq -\breve{\alpha}\,|v_k| \qquad \text{for a.e. } t \in]0,T[\,,$$
$$\liminf_k \frac{G(t,\tau_k v_k)}{\tau_k} \geq 0 \qquad \text{for a.e. } t \in]0,T[\,,$$
$$\liminf_k \frac{G(t,\tau_k v_k)}{\tau_k} > 0 \qquad \text{for a.e. } t \in E\,.$$

Therefore,

$$\liminf_k \frac{\displaystyle\int_0^T G(t,\tau_k v_k)\,dt}{\tau_k} > 0$$

and a contradiction follows. □

Proof of Theorem 3.2.1.
We aim to apply Corollary 3.1.5 to

$$\widehat{\mathscr{F}} : \mathrm{epi}\,(f_0) \to \mathbb{R}\,.$$

By Lemmas 3.2.2, 3.2.3 and 3.2.4, all the assumptions are satisfied. Therefore, there exists a sequence (u_k,λ_k) of critical points of $\widehat{\mathscr{F}}$ with

$$\lambda_k - \int_0^T G(t,u_k)\,dt \leq 0\,, \qquad \|u_k\|_1^2 + (\lambda_k^-)^2 > 0\,, \qquad \|u_k\|_1^2 + (\lambda_k^-)^2 \to 0\,.$$

Then $\|u_k\|_1 \to 0$ and from Corollary 1.2.9 we infer that $\lambda_k = f_0(u_k)$ and u_k is a solution of (HI) with $u_k \neq 0$. By Lemma 3.2.2, the sequence $(f_0(u_k))$ is bounded. Then (u_k) is bounded in $L^\infty(0,T;\mathbb{R}^n)$ by Proposition 1.2.3, whence

$$\limsup_k f_0(u_k) \leq \lim_k \int_0^T G(t,u_k)\,dt = 0\,.$$

Since Ψ is even, we have $\Psi(\xi) \geq 0$ for any $\xi \in \mathbb{R}^n$. Therefore

$$\lim_k f_0(u_k) = 0\,.$$

Moreover, by Propositions 1.1.2 and 1.2.3, for every $]a,b[\subseteq]0,T[$ we have

$$\begin{aligned}\operatorname*{ess\,sup}_{]a,b[}|u_k| &\leq \frac{1}{b-a}\int_a^b |u_k|\,dt + |u_k'|(]a,b[)\\ &\leq \frac{1}{b-a}\int_a^b |u_k|\,dt + \frac{1}{\nu}\left[\int_a^b \Psi(u_a')\,dt + \int_{]a,b[}\Psi^\infty\left(\frac{u_s'}{|u_s'|}\right)d|u_s'| + \frac{b-a}{\nu}\right]\\ &\leq \frac{1}{b-a}\int_a^b |u_k|\,dt + \frac{1}{\nu}\,f_0(u_k) + \frac{b-a}{\nu^2}\,,\end{aligned}$$

whence

$$\limsup_k\left(\operatorname*{ess\,sup}_{]a,b[}|u_k|\right) \leq \frac{b-a}{\nu^2}\,.$$

By the arbitrariness of $]a,b[$ we infer that

$$\lim_k \|u_k\|_\infty = 0\,.$$

□

2.2 Proof in the general case

Let $\vartheta\colon\mathbb{R}\to[0,1]$ be a function of class C^1 such that

$$\begin{aligned}\vartheta(\tau) &= 1 \qquad \text{if } \tau\leq\frac{r^2}{4}\,,\\ \vartheta(\tau) &= 0 \qquad \text{if } \tau\geq\frac{9r^2}{16}\,.\end{aligned}$$

Then define $\widehat{G}:]0,T[\times\mathbb{R}^n\to\mathbb{R}$ as

$$\widehat{G}(t,s)=\begin{cases}G(t,\vartheta(|s|^2)s)-(1-\vartheta(|s|^2))\sqrt{1+|s|^2} & \text{if } |s|<r\\ -(1-\vartheta(|s|^2))\sqrt{1+|s|^2} & \text{if } |s|\geq r\,.\end{cases}$$

It is easily seen that $\widehat{G}$ satisfies all the assumptions required in the particular case. Therefore we can apply Theorem 3.2.1 with G replaced by $\widehat{G}$. Since $\|u_k\|_\infty\to 0$, we have that, eventually as $k\to\infty$, u_k is also a solution of (HI) with the original G instead of $\widehat{G}$.

Chapter 4

Asymptotically linear lower order terms

This chapter is devoted to the case in which $G(t,s) \approx |s|$ as $|s| \to \infty$, so that a nonresonance condition will be imposed.

1 The cohomology of suitable pairs

In the following, H^* will denote Alexander-Spanier cohomology with coefficients in $\mathbb{Z}_2$.

Theorem 4.1.1. *Let X be a normed space over $\mathbb{R}$ and let S be a compact and symmetric subset of X with $0 \notin S$.*

Then there exists $m \geq 0$ such that $H^m(X,S) \neq \{0\}$.

Proof. Let Index denote the $\mathbb{Z}_2$-cohomological index of [20, 21]. Since S is compact with $0 \notin S$, we have $\text{Index}(S) < \infty$. From [14, Theorem 2.7] it follows that (X,S) links $(X \setminus S, \emptyset)$ in the sense of [14, Definition 2.3], which is just the assertion. □

Theorem 4.1.2. *Let X be a metric space, $f : X \to \mathbb{R}$ a continuous function and let $a \in \mathbb{R}$ and $b \in]-\infty, +\infty]$ with $a \leq b$. Assume that f has no critical points u with $a < f(u) \leq b$, that $(CPS)_c$ holds and that $\{u \in X : \ f(u) \leq c\}$ is complete whenever $c \in [a, b[$.*

Then $H^*(\{u \in X : \ f(u) \leq b\}, \{u \in X : \ f(u) \leq a\})$ is trivial.

Proof. As in the proof of Theorem 2.1.2, by [9] $(CPS)_c$ becomes $(PS)_c$ with respect to an auxiliary distance. Then the assertion is contained in [18, Theorem 2.7]. □

Theorem 4.1.3. *Let X be a metric space, $f : X \to \mathbb{R}$ a continuous function, $\beta : X \longrightarrow \mathbb{R}$ be a Lipschitz function of constant L, let $a \in \mathbb{R}$ and let $\varepsilon > 0$ be such that* (*a*) *the set*

$$\Sigma := \{u \in X : \; a \leq f(u) \leq \beta(u)\}$$

is complete;

(*b*) *we have*

$$\inf \{|df|\,(u) : \; u \in \Sigma\,, \; f(u) \leq a + \varepsilon\} > 0\,;$$

(*c*) *we have*

$$\inf \left\{|df|\,(u) : \; u \in \Sigma\,, \; f(u) \geq \beta(u) - \frac{2}{5}\,\varepsilon\right\} > L\,.$$

Define

$$\begin{aligned}
A &= \{u \in \Sigma : \; f(u) \leq a + \varepsilon\}\,, \\
\Sigma' &= \left\{u \in X : \; a + \frac{1}{5}\,\varepsilon \leq f(u) \leq \beta(u) - \frac{1}{5}\,\varepsilon\right\}\,, \\
A' &= \left\{u \in \Sigma' : \; f(u) \leq a + \frac{4}{5}\,\varepsilon\right\}\,, \\
\Sigma'' &= \left\{u \in X : \; a + \frac{2}{5}\,\varepsilon \leq f(u) \leq \beta(u) - \frac{2}{5}\,\varepsilon\right\}\,, \\
A'' &= \left\{u \in \Sigma'' : \; f(u) \leq a + \frac{3}{5}\,\varepsilon\right\}\,.
\end{aligned}$$

Then the inclusions $(\Sigma'', A'') \subseteq (\Sigma', A')$ *and* $(\Sigma', A') \subseteq (\Sigma, A)$ *induce isomorphisms in cohomology.*

Proof. See [7, Lemma 3.1], where the assertion is proved for homology, but no change is required to treat cohomology. □

2 Existence of a periodic solution

Throughout this section, we still assume that

$$\Psi : \mathbb{R}^n \to \mathbb{R}$$

satisfies condition (Ψ). We also suppose that

$$G :]0, T[\times \mathbb{R}^n \to \mathbb{R}$$

satisfies:

(G) *the function $G(\cdot, s)$ is measurable for every $s \in \mathbb{R}^n$, $G(t, 0) = 0$ for a.e. $t \in]0, T[$ and there exists $\check{\alpha} \in L^1(0, T)$ satisfying*

$$|G(t, s) - G(t, \sigma)| \leq \check{\alpha}(t)|s - \sigma|$$

for a.e. $t \in]0, T[$ and every $s, \sigma \in \mathbb{R}^n$;

(L) *there exists a function*

$$G^\infty :]0, T[\times \mathbb{R}^n \to \mathbb{R}$$

such that

$$\lim_{\tau \to +\infty} \frac{G(t, \tau s)}{\tau} = G^\infty(t, s) \qquad \text{for a.e. } t \in]0, T[\text{ and every } s \in \mathbb{R}^n$$

and, for a.e. $t \in]0, T[$, for every $\tau_k \to +\infty$, $s_k \to s$ and $\sigma_k \to \sigma$,

$$\limsup_k G^0(t, \tau_k s_k; \sigma_k) \leq (G^\infty)^0(t, s; \sigma) .$$

If $G(t, \cdot)$ is of class C^1 in a neighborhood of each $\tau_k s_k$ and $G^\infty(t, \cdot)$ is of class C^1 in a neighborhood of s, then the last condition is equivalent to

$$\lim_k \nabla_s G(t, \tau_k s_k) = \nabla_s G^\infty(t, s) .$$

Remark 4.2.1. *Since*

$$\left| \frac{G(t, \tau s)}{\tau} - \frac{G(t, \tau \sigma)}{\tau} \right| \leq \check{\alpha}(t)|s - \sigma| ,$$

it is easily seen that G^∞ also satisfies (G).

Let us state the main result.

Theorem 4.2.2. *Assume (Ψ), (G), (L), that*

$$\Psi^\infty(-\xi) = \Psi^\infty(\xi) \qquad \text{for every } \xi \in \mathbb{R}^n ,$$
$$G^\infty(t, -s) = G^\infty(t, s) \qquad \text{for a.e. } t \in]0, T[\text{ and every } s \in \mathbb{R}^n ,$$

and that $u = 0$ is the unique solution of

$$(4.2.3) \qquad f_0^\infty(v) + \int_0^T (G^\infty)^0(t, u; u - v)\, dt \geq f_0^\infty(u) \qquad \text{for every } v \in BV(0, T; \mathbb{R}^n) .$$

Then there exists a solution $u \in BV(0, T; \mathbb{R}^n)$ of (HI).

Remark 4.2.4. *Since Ψ^∞ and $G^\infty(t,\cdot)$ are even, it is easily seen that $u=0$ is a solution of (4.2.3). The assumption that $u=0$ is the unique solution is a form of nonresonance condition at infinity.*

The assumption that Ψ^∞ and $G^\infty(t,\cdot)$ are even also avoids a "jumping behavior" (see [5]), which would not be compatible with the assertion of the theorem.

As usual, for the proof of the theorem, we need some lemmas.
First of all, let us define, whenever $\tau \geq 1$,

$$\Psi^\tau(\xi) = \frac{\Psi(\tau\xi)}{\tau}\,, \qquad G^\tau(t,s) = \frac{G(t,\tau s)}{\tau}\,.$$

Then let us introduce f_0^τ, f_1^τ, f^τ and $\mathscr{F}^\tau$ accordingly.

First of all, we have

$$(G^\tau)^0(t,s;\sigma) = G^0(t,\tau s;\sigma)\,,$$

whence, if $\varrho \geq 1$,

$$\frac{\Psi^\tau(\varrho\xi)}{\varrho} = \Psi^{\tau\varrho}(\xi)\,, \qquad \frac{G^\tau(t,\varrho s)}{\varrho} = G^{\tau\varrho}(t,s)\,, \qquad (G^\tau)^0(t,\varrho s;\sigma) = (G^{\tau\varrho})^0(t,s;\sigma)\,.$$

Moreover, it is easily seen that

$$\begin{aligned}
&\Psi^\tau(\xi) \geq \nu\,|\xi| - \frac{1}{\tau\nu} && \text{for every } \tau \in [1,\infty[\text{ and } \xi \in \mathbb{R}^n\,,\\
&\Psi^\infty(\xi) \geq \nu\,|\xi| && \text{for every } \xi \in \mathbb{R}^n\,,\\
&|\Psi^\tau(\xi) - \Psi^\tau(\eta)| \leq \frac{1}{\nu}\,|\xi-\eta| && \text{for every } \tau \in [1,\infty] \text{ and } \xi,\eta \in \mathbb{R}^n\,,\\
&|G^\tau(t,s) - G^\tau(t,\sigma)| \leq \check{\alpha}(t)|s-\sigma| && \text{for every } \tau \in [1,\infty],\\
& && \quad \text{a.e. } t \in]0,T[\text{ and every } s,\sigma \in \mathbb{R}^n\,,\\
&|(G^\tau)^0(t,s;\sigma)| \leq \check{\alpha}(t)|\sigma| && \text{for every } \tau \in [1,\infty],\\
& && \quad \text{a.e. } t \in]0,T[\text{ and every } s,\sigma \in \mathbb{R}^n\,.
\end{aligned}$$

Finally, if $\tau_k \to \tau$ in $[1,\infty]$, $\xi_k \to \xi$, $s_k \to s$ and $\sigma_k \to \sigma$ in $\mathbb{R}^n$, then

$$\begin{aligned}
&\lim_k \Psi^{\tau_k}(\xi_k) = \Psi^\tau(\xi)\,,\\
&\lim_k G^{\tau_k}(t,s_k) = G^\tau(t,s) && \text{for a.e. } t \in]0,T[\,,\\
&\limsup_k\, (G^{\tau_k})^0(t,s_k;\sigma_k) \leq (G^\tau)^0(t,s;\sigma)\,. && \text{for a.e. } t \in]0,T[\,.
\end{aligned}$$

2.1 Uniform Palais-Smale condition and convergence

Lemma 4.2.5. *If $\tau_k \to \tau$ in $[1, \infty]$ and*

$$(u_k, \lambda_k) \in \bigcup_{1 \leq \eta \leq \infty} \mathrm{epi}\,(f_0^\eta)$$

with

$$\sup_k \left(\|u_k\|_1^2 + \lambda_k^2\right) < +\infty\,,$$

we have

$$\lim_k |f_1^{\tau_k}(u_k) - f_1^\tau(u_k)| = 0\,.$$

Proof. Since

$$\Psi^\tau(\xi) \geq \nu\,|\xi| - \frac{1}{\nu}\,,$$

the sequence (u_k) is bounded in $L^\infty(0, T; \mathbb{R}^n)$ and precompact in $L^1(0, T; \mathbb{R}^n)$. Then the assertion easily follows. □

Lemma 4.2.6. *If $\tau_k \to \tau$ in $[1, \infty]$ and $u_k, u \in BV(0, T; \mathbb{R}^n)$ with $\|u_k - u\|_1 \to 0$, then we have*

$$\lim_k f_0^{\tau_k}(u) = f_0^\tau(u)\,,$$
$$\liminf_k f_0^{\tau_k}(u_k) \geq f_0^\tau(u)\,.$$

Proof. Since

$$f_0^\tau(u) = \int_0^T \frac{\Psi(\tau u_a')}{\tau}\,dt + \int_{]0,T[} \Psi^\infty\left(\frac{u_s'}{|u_s'|}\right)\,d|u_s'| + \Psi^\infty\big(u(0_+) - u(T_-)\big)\,,$$

from Lebesgue's theorem it follows that

$$\lim_k f_0^{\tau_k}(u) = f_0^\tau(u)\,.$$

If $\tau < \infty$, from

$$\frac{\tau}{\tau_k}\,\frac{\Psi(\tau\xi)}{\tau} \leq \frac{\Psi(\tau_k\xi)}{\tau_k} + \frac{1}{\nu}\left|\frac{\tau}{\tau_k} - 1\right|\,|\xi|$$

it follows that

$$f_0^\tau(u) \leq \liminf_k f_0^{\tau_k}(u_k)\,.$$

If $\tau = \infty$, we have

$$f_0^{\overline{\tau}}(u_k) \leq f_0^{\tau_k}(u_k) \qquad \text{whenever } \overline{\tau} \leq \tau_k\,.$$

It follows

$$f_0^{\overline{\tau}}(u) \leq \liminf_k f_0^{\overline{\tau}}(u_k) \leq \liminf_k f_0^{\tau_k}(u_k)\,,$$

hence

$$f_0^{\infty}(u) \leq \liminf_k f_0^{\tau_k}(u_k)\,.$$

□

Lemma 4.2.7. *If $c \in \mathbb{R}$, (τ_k) is a sequence in $[1,\infty]$ and (u_k,λ_k) a sequence in $L^1(0,T;\mathbb{R}^n)\times\mathbb{R}$, with $(u_k,\lambda_k)\in \mathrm{epi}\,(f_0^{\tau_k})$ for any k, such that*

$$\mathscr{F}^{\tau_k}(u_k,\lambda_k)\to c\,,\qquad |d\mathscr{F}^{\tau_k}|\,(u_k,\lambda_k)\to 0\,,$$

then (u_k,λ_k) admits a convergent subsequence in $L^1(0,T;\mathbb{R}^n)\times\mathbb{R}$.

Moreover, if

$$\lim_k \tau_k = +\infty\,,$$

then $c=0$ and (u_k,λ_k) is convergent to 0 in $L^1(0,T;\mathbb{R}^n)\times\mathbb{R}$.

Proof. Up to a subsequence, we also have $\tau_k\to\tau$ in $[1,\infty]$. From Corollary 1.2.9 we infer that $\lambda_k = f_0^{\tau_k}(u_k)$ eventually as $k\to\infty$, so that $f^{\tau_k}(u_k)\to c$, and that there exist $w_k\in L^\infty(0,T;\mathbb{R}^n)$ and $\mu_k\leq 1$ such that

$$\|w_k\|_\infty^2 + {\mu_k}^2 \leq \Big(|d\mathscr{F}^{\tau_k}|\big(u_k,f_0(u_k)\big)\Big)^2\,,$$

$$\begin{aligned}(1-\mu_k)f_0^{\tau_k}(v) + \int_0^T (G^{\tau_k})^0(t,u_k;u_k-v)\,dt \\ \geq (1-\mu_k)f_0^{\tau_k}(\varrho_k v_k) + \int_0^T w_k\cdot(v-u_k)\,dt \qquad \forall v\in BV(0,T;\mathbb{R}^n)\,.\end{aligned}$$

We claim that $(u_k, f_0^{\tau_k}(u_k))$ is bounded in $L^1(0,T;\mathbb{R}^n)\times\mathbb{R}$. Assume, for a contradiction, that $u_k=\varrho_k v_k$ with

$$\varrho_k\to+\infty\,,\qquad \|v_k\|_1^2 + (f_0^{\tau_k\varrho_k}(v_k))^2 = 1\,.$$

Since

$$\nu\,\hat f_0(v_k) - \frac{T}{\nu} \leq f_0^{\tau_k\varrho_k}(v_k)\leq 1\,,$$

up to a subsequence (v_k) is bounded in $L^\infty(0,T;\mathbb{R}^n)$ and convergent to some $v\in BV(0,T;\mathbb{R}^n)$ a.e. in $]0,T[$.

For any $z \in BV(0,T;\mathbb{R}^n)$, we have

$$(1-\mu_k)\,\frac{f_0^{\tau_k}(\varrho_k z)}{\varrho_k} + \int_0^T (G^{\tau_k})^0(t,\varrho_k v_k; v_k - z)\,dt \geq (1-\mu_k)\,\frac{f_0^{\tau_k}(\varrho_k v_k)}{\varrho_k} + \int_0^T w_k\cdot(z-v_k)\,dt\,,$$

namely

$$(1-\mu_k)\,f_0^{\tau_k\varrho_k}(z) + \int_0^T (G^{\tau_k\varrho_k})^0(t,v_k; v_k - z)\,dt \geq (1-\mu_k)\,f_0^{\tau_k\varrho_k}(v_k) + \int_0^T w_k\cdot(z-v_k)\,dt\,.$$

Since $\tau_k\varrho_k \to +\infty$, by Lemma 4.2.6 we have

$$\lim_k\, f_0^{\tau_k\varrho_k}(z) = f_0^{\infty}(z)\,, \qquad \liminf_k\, f_0^{\tau_k\varrho_k}(v_k) \geq f_0^{\infty}(v)\,,$$

while Fatou's lemma yields

$$\limsup_k \int_0^T (G^{\tau_k\varrho_k})^0(t,v_k;v_k-z)\,dt \leq \int_0^T (G^{\infty})^0(t,v;v-z)\,dt\,.$$

Therefore

$$f_0^{\infty}(z) + \int_0^T (G^{\infty})^0(t,v;v-z)\,dt \geq f_0^{\infty}(v) \qquad \text{for any } z \in BV(0,T;\mathbb{R}^n)\,,$$

namely v is a solution of (4.2.3).

It follows $v=0$, hence

$$\lim_k\,(f_0^{\tau_k\varrho_k}(v_k))^2 = 1\,.$$

On the other hand, the choice $z=0$ in the previous argument yields

$$\limsup_k \int_0^T (G^{\tau_k\varrho_k})^0(t,v_k;v_k)\,dt \leq 0\,,$$

hence

$$\limsup_k f_0^{\tau_k\varrho_k}(v_k) \leq 0\,.$$

A contradiction follows and therefore it is proved that $(u_k, f_0^{\tau_k}(u_k))$ is bounded in $L^1(0,T;\mathbb{R}^n)\times\mathbb{R}$.

Since

$$\nu\,\hat{f}_0(u_k) - \frac{T}{\nu} \leq f_0^{\tau_k}(u_k)\,,$$

it follows that (u_k) admits a convergent subsequence in $L^1(0,T;\mathbb{R}^n)$.

If now

$$\lim_k \tau_k = +\infty\,,$$

then we can repeat the previous argument with $\tau_k \varrho_k$ and v_k replaced by τ_k and u_k. It follows that $\|u_k\|_1 \to 0$ and $f_0^{\tau_k}(u_k) \to 0$. □

2.2 The recession functional

This subsection is devoted to a study of $\mathscr{F}^\infty$. Let us set

$$C\colon = \{\,(u,\lambda) \in \mathrm{epi}\,(f_0^\infty) : \ \mathscr{F}^\infty(u,\lambda) \le 0\,\} \setminus \{(0,0)\}\,.$$

Lemma 4.2.8. *There exists $m \ge 0$ such that $H^m\left(\mathrm{epi}\,(f_0^\infty)\,, C\right) \ne \{0\}$.*

Proof. The set $\mathrm{epi}\,(f_0^\infty)$ is convex and nonempty, in particular contractible. If $C = \emptyset$, we have $H^0\left(\mathrm{epi}\,(f_0^\infty)\,, C\right) \ne \{0\}$ and the assertion follows.

Therefore assume that $C \ne \emptyset$ and consider

$$K\colon = \left\{\,(u,\lambda) \in \mathrm{epi}\,(f_0^\infty) : \ \|u\|_1^{\,2} + \lambda^2 = 1 \text{ and } \mathscr{F}^\infty(u,\lambda) \le 0\,\right\}\,,$$

which is, by Theorem 1.2.7, a nonempty compact subset of $\mathrm{epi}\,(f_0^\infty)$, and the continuous map $\varrho\colon C \longrightarrow K$ defined as

$$\varrho(u,\lambda) = \left(\frac{u}{\sqrt{\|u\|_1^{\,2} + \lambda^2}}, \frac{\lambda}{\sqrt{\|u\|_1^{\,2} + \lambda^2}}\right).$$

Finally, given the canonical projection

$$\pi_1 : L^1(0,T;\mathbb{R}^n) \times \mathbb{R} \to L^1(0,T;\mathbb{R}^n)\,,$$

consider $\pi_1(K)$, which is a nonempty, compact and symmetric subset of $L^1(0,T;\mathbb{R}^n)$ with $0 \notin \pi_1(K)$, as $f_0^\infty(0) = f_1^\infty(0) = 0$. Let $\hat{u}_0 \in \pi_1(K)$.

From Theorem 4.1.1 we infer that there exists $m \ge 0$ such that

$$H^m\left(L^1(0,T;\mathbb{R}^n), \pi_1(K)\right) \ne \{0\}\,.$$

Since $H^*\left(L^1(0,T;\mathbb{R}^n), \{\hat{u}_0\}\right)$ is trivial, from the exact sequence of the triple

$$\left(L^1(0,T;\mathbb{R}^n), \pi_1(K), \{\hat{u}_0\}\right)$$

we deduce that

$$H^m\left(\pi_1(K), \{\hat{u}_0\}\right) \neq \{0\}\,.$$

On the other hand, we have the continuous map $\pi_1 \circ \varrho : C \to \pi_1(K)$ and we can define a continuous map $\varphi : \pi_1(K) \to K \subseteq C$ by

$$\varphi(u) = \left(u, \sqrt{1 - \|u\|_1^2}\right)\,.$$

Moreover, $(\pi_1 \circ \varrho) \circ \varphi$ is the identity of $\pi_1(K)$. If we set $(u_0, \lambda_0) = \varphi(\hat{u}_0)$, it follows that

$$H^m\left(C, \{(u_0, \lambda_0)\}\right) \neq \{0\}\,.$$

Since $H^m\left(\mathrm{epi}\left(f_0^\infty\right), \{(u_0, \lambda_0)\}\right)$ is trivial, by the exact sequence of the triple

$$\left(\mathrm{epi}\left(f_0^\infty\right), C, \{(u_0, \lambda_0)\}\right)$$

we conclude that $H^m\left(\mathrm{epi}\left(f_0^\infty\right), C\right) \neq \{0\}$. □

Now let

$$\begin{aligned} D_\infty &= \{\,(u, \lambda) \in \mathrm{epi}\left(f_0^\infty\right) : \quad -2 \leq \mathscr{F}^\infty(u, \lambda) \leq 1\,\}\,, \\ E_\infty &= \{\,(u, \lambda) \in \mathrm{epi}\left(f_0^\infty\right) : \quad -2 \leq \mathscr{F}^\infty(u, \lambda) \leq -1\,\}\,. \end{aligned}$$

We can prove the main result of this subsection.

Lemma 4.2.9. *There exists $m \geq 0$ such that*

$$H^m(D_\infty, E_\infty) \neq \{0\}\,.$$

Proof. The function $\mathscr{F}^\infty : \mathrm{epi}\left(f_0^\infty\right) \to \mathbb{R}$ is continuous and satisfies $(PS)_c$ for any $c \in \mathbb{R}$, by Lemma 4.2.7. Since there are no critical points (u, λ) of $\mathscr{F}^\infty$ with $\mathscr{F}^\infty(u, \lambda) > 0$, if we set

$$\widetilde{D} : = \{\,(u, \lambda) \in \mathrm{epi}\left(f_0^\infty\right) : \ \mathscr{F}^\infty(u, \lambda) \leq 1\,\}\,,$$

it follows from Theorem 4.1.2 that

$$H^*\left(\mathrm{epi}\left(f_0^\infty\right), \widetilde{D}\right)$$

is trivial. From the exact sequence of the triple

$$\left(\mathrm{epi}\left(f_0^\infty\right), \widetilde{D}, C\right)$$

and Lemma 4.2.8 we infer that

$$H^m\left(\widetilde{D}, C\right) \neq \{0\}\,.$$

Now define

$$\widetilde{E}\colon\ = \{\,(u,\lambda) \in \mathrm{epi}\,(f_0^\infty):\ \mathscr{F}^\infty(u,\lambda) \leq -1\,\}$$

and consider the restriction of $\mathscr{F}^\infty$ to C. It is continuous, satisfies $(PS)_c$ for any $c < 0$, and has no critical point (u,λ) with $\mathscr{F}^\infty(u,\lambda) \leq 0$ by Proposition 1.1.11. Again from Theorem 4.1.2 we infer that

$$H^*\left(C, \widetilde{E}\right)$$

is trivial. From the exact sequence of the triple

$$\left(\tilde{D}, C, \widetilde{E}\right)$$

it follows that

$$H^m\left(\widetilde{D}, \widetilde{E}\right) \neq \{0\}\,.$$

Finally, by excision we have

$$H^m\left(\widetilde{D}, \widetilde{E}\right) \approx H^m\left(D_\infty, E_\infty\right)$$

and the assertion follows. ☐

2.3 Proof of the main result

Lemma 4.2.10. *The functional $\mathscr{F}^\tau : \mathrm{epi}\,(f_0^\tau) \to \mathbb{R}$ satisfies $(PS)_c$ for any $\tau \in [1,\infty]$ and any $c \in \mathbb{R}$.*

Proof. It follows from Lemma 4.2.7. ☐

Lemma 4.2.11. *There exist $\sigma > 0$, $R > 0$ and $\overline{\tau} < \infty$ such that:*

(a) *$|d\mathscr{F}^\tau|(u,\lambda) \geq \sigma$ whenever $\tau \in [1,\infty]$, $(u,\lambda) \in \mathrm{epi}\,(f_0^\tau)$, $-3 \leq \mathscr{F}^\tau(u,\lambda) \leq 1$ and $\sqrt{\|u\|_1^2 + \lambda^2} > R^2$;*

(b) *$|d\mathscr{F}^\tau|(u,\lambda) \geq \sigma$ whenever $\tau \in [\overline{\tau},\infty]$, $(u,\lambda) \in \mathrm{epi}\,(f_0^\tau)$ and $\mathscr{F}^\tau(u,\lambda) \in [-3,-1] \cup [3/5, 1]\,.$*

Proof. By Lemma 4.2.7 there exists $\sigma > 0$ such that the set

$$\bigcup_{1 \leq \tau \leq \infty} \{ (u,\lambda) \in \mathrm{epi}\,(f_0^\tau) : \ -3 \leq \mathscr{F}^\tau(u,\lambda) \leq 1 \text{ and } |d\mathscr{F}^\tau|(u,\lambda) < \sigma \}$$

is bounded in $L^1(0,T;\mathbb{R}^n) \times \mathbb{R}$. Then assertion (a) follows. Assertion (b) also follows from Lemma 4.2.7. □

Let us define

$$\beta : L^1(0,T;\mathbb{R}^n) \times \mathbb{R} \to [-3,1]$$

as

$$\beta(u,\lambda) = 1 - \min\left\{ \frac{\sigma}{2} \left(\sqrt{\|u\|_1^2 + \lambda^2} - R \right)^+, 4 \right\},$$

which is Lipschitz continuous of constant $\frac{\sigma}{2}$, and set, for every $\tau \in [1,\infty]$,

$$\begin{aligned}
D_\tau &= \{ (u,\lambda) \in \mathrm{epi}\,(f_0^\tau) : \ -2 \leq \mathscr{F}^\tau(u,\lambda) \leq 1 \}, \\
E_\tau &= \{ (u,\lambda) \in \mathrm{epi}\,(f_0^\tau) : \ -2 \leq \mathscr{F}^\tau(u,\lambda) \leq -1 \}, \\
\Sigma_\tau &= \{ (u,\lambda) \in \mathrm{epi}\,(f_0^\tau) : \ -2 \leq \mathscr{F}^\tau(u,\lambda) \leq \beta(u,\lambda) \}, \\
A_\tau &= \{ (u,\lambda) \in \Sigma_\tau : \ \mathscr{F}^\tau(u,\lambda) \leq -1 \}.
\end{aligned}$$

Lemma 4.2.12. *For every $\tau \in [\overline{\tau},\infty]$, we have $H^*(D_\tau,E_\tau) \approx H^*(\Sigma_\tau,A_\tau)$.*

Proof. If we set

$$\mathscr{G}(u,\lambda) = \mathscr{F}^\tau(u,\lambda) - \beta(u,\xi),$$

$$\begin{aligned}
\widetilde{D}_\tau &= \{ (u,\lambda) \in \mathrm{epi}\,(f_0^\tau) : \ \mathscr{F}^\tau(u,\lambda) \leq 1 \}, \\
\widetilde{E}_\tau &= \{ (u,\lambda) \in \mathrm{epi}\,(f_0^\tau) : \ \mathscr{F}^\tau(u,\lambda) \leq -1 \}, \\
\widetilde{\Sigma}_\tau &= \{ (u,\lambda) \in \mathrm{epi}\,(f_0^\tau) : \ \mathscr{F}^\tau(u,\lambda) \leq \beta(u,\lambda) \}, \\
\widetilde{A}_\tau &= \{ (u,\lambda) \in \widetilde{\Sigma}_\tau : \ \mathscr{F}^\tau(u,\lambda) \leq -1 \},
\end{aligned}$$

we have that $\widetilde{E}_\tau$ is a complete metric space, $\mathscr{G}$ is continuous and

$$\widetilde{A}_\tau = \{ (u,\lambda) \in \widetilde{E}_\tau : \ \mathscr{G}(u,\lambda) \leq 0 \}.$$

If $(u,\lambda) \in \widetilde{E}_\tau$ with $\mathscr{G}(u,\lambda) \geq 0$, we have $-3 \leq \mathscr{F}^\tau(u,\lambda) \leq -1$. From Propositions 1.1.11 and 1.1.12 and Lemma 4.2.11 it follows that

$$|d\mathscr{G}|\,(u,\lambda) \geq |d\mathscr{F}^\tau|\,(u,\lambda) - \frac{\sigma}{2} \geq \frac{\sigma}{2}.$$

From Theorem 4.1.2 we infer that $H^*(\widetilde{E}_\tau, \widetilde{A}_\tau)$ is trivial. From the exact sequence of the triple

$$\left(\widetilde{D}_\tau, \widetilde{E}_\tau, \widetilde{A}_\tau\right)$$

we deduce that

$$H^*\left(\widetilde{D}_\tau, \widetilde{E}_\tau\right) \approx H^*\left(\widetilde{D}_\tau, \widetilde{A}_\tau\right) .$$

Now observe that $\widetilde{D}_\tau$ also is a complete metric space and

$$\widetilde{\Sigma}_\tau = \{ (u,\lambda) \in \widetilde{D}_\tau : \; \widehat{\mathscr{G}}(u,\lambda) \leq 0 \} .$$

If $(u,\lambda) \in \widetilde{D}_\tau$ with $\widehat{\mathscr{G}}(u,\lambda) \geq 0$, we have either $\mathscr{F}^\tau(u,\lambda) = 1$ or $-3 \leq \mathscr{F}^\tau(u,\lambda) < 1$ with $\sqrt{\|u\|_1^2 + \lambda^2} > R$. From Propositions 1.1.11 and 1.1.12 and Lemma 4.2.11 it follows that

$$\left|d\widehat{\mathscr{G}}\right|(u,\lambda) \geq |d\mathscr{F}^\tau|(u,\lambda) - \frac{\sigma}{2} \geq \frac{\sigma}{2} .$$

From Theorem 4.1.2 we infer that $H^*(\widetilde{D}_\tau, \widetilde{\Sigma}_\tau)$ is trivial. From the exact sequence of the triple

$$\left(\widetilde{D}_\tau, \widetilde{\Sigma}_\tau, \widetilde{A}_\tau\right)$$

and the previous step we deduce that

$$H^*\left(\widetilde{D}_\tau, \widetilde{E}_\tau\right) \approx H^*\left(\widetilde{D}_\tau, \widetilde{A}_\tau\right) \approx H^*\left(\widetilde{\Sigma}_\tau, \widetilde{A}_\tau\right) .$$

Finally, by excision we have

$$H^*(D_\tau, E_\tau) \approx H^*\left(\widetilde{D}_\tau, \widetilde{E}_\tau\right) , \qquad H^*\left(\widetilde{\Sigma}_\tau, \widetilde{A}_\tau\right) \approx H^*(\Sigma_\tau, A_\tau) ,$$

and the assertion follows. □

Lemma 4.2.13. *There exists* $\tilde{\tau} < \infty$ *such that, for every* $\tau \in [\tilde{\tau}, \infty]$, *we have* $H^*(\Sigma_\tau, A_\tau) \approx H^*(\Sigma_\infty, A_\infty)$.

Proof. Since we aim to apply Theorem 4.1.3 with $a = -2$ and $\varepsilon = 1$, let us set

$$\begin{aligned}
\Sigma'_\tau &= \left\{(u,\lambda) \in \mathrm{epi}\,(f_0^\tau) : \; -\frac{9}{5} \leq \mathscr{F}^\tau(u,\lambda) \leq \beta(u,\lambda) - \frac{1}{5}\right\} , \\
A'_\tau &= \left\{(u,\lambda) \in \Sigma'_\tau : \; \mathscr{F}^\tau(u,\lambda) \leq -\frac{6}{5}\right\} , \\
\Sigma''_\tau &= \left\{(u,\lambda) \in \mathrm{epi}\,(f_0^\tau) : \; -\frac{8}{5} \leq \mathscr{F}^\tau(u,\lambda) \leq \beta(u,\lambda) - \frac{2}{5}\right\} , \\
A''_\tau &= \left\{(u,\lambda) \in \Sigma''_\tau : \; \mathscr{F}^\tau(u,\lambda) \leq -\frac{7}{5}\right\} .
\end{aligned}$$

Define also

$$\psi : L^1(0,T;\mathbb{R}^n) \times \mathbb{R} \to L^1(0,T;\mathbb{R}^n) \times \mathbb{R}$$

by $\psi(u,\lambda) = \left(u, \lambda + \frac{1}{10}\right)$ and observe that

$$\|u\|_1^2 + \lambda^2 \leq \left(R + \frac{6}{\sigma}\right)^2 \qquad \text{whenever } (u,\lambda) \in \bigcup_{1\leq\tau\leq\infty} \Sigma_\tau\,.$$

Let $\varepsilon \in]0,1[$ be such that

$$\frac{\lambda}{1-\varepsilon} \leq \lambda + \frac{1}{20} \qquad \text{whenever } |\lambda| \leq R + \frac{6}{\sigma}$$

and let M_ε be such that

$$(1-\varepsilon) f_0^\infty(u) - TM_\varepsilon \leq f_0(u) \leq f_0^\infty(u) \qquad \text{for any } u \in BV(0,T;\mathbb{R}^n)\,,$$

according to Proposition 1.2.3.

Then, by Lemma 4.2.5, there exists $\tilde{\tau} \geq \overline{\tau}$ such that

$$\frac{TM_\varepsilon}{(1-\varepsilon)\tilde{\tau}} \leq \frac{1}{20}\,,$$

$$\text{(4.2.14)} \quad |f_1^\tau(u) - f_1^\infty(u)| \leq \frac{1}{10} \qquad \text{whenever } \tau \geq \tilde{\tau},$$
$$(u,\lambda) \in \bigcup_{1\leq\eta\leq\infty} \text{epi}\,(f_0^\eta) \text{ and } \|u\|_1^2 + \lambda^2 \leq \left(R + \frac{6}{\sigma}\right)^2.$$

Then, for every $\tau \geq \tilde{\tau}$, we have

$$\begin{array}{llll} \Sigma''_\infty \subseteq \Sigma'_\tau\,, & A''_\infty \subseteq A'_\tau\,, & \Sigma'_\infty \subseteq \Sigma_\tau\,, & A'_\infty \subseteq A_\tau\,, \\ \psi(\Sigma''_\tau) \subseteq \Sigma'_\infty\,, & \psi(A''_\tau) \subseteq A'_\infty\,, & \psi(\Sigma'_\tau) \subseteq \Sigma_\infty\,, & \psi(A'_\tau) \subseteq A_\infty\,. \end{array}$$

Actually, we have $\text{epi}\,(f_0^\infty) \subseteq \text{epi}\,(f_0^\tau)$. On the other hand, if $f_0^\tau(u) \leq \lambda$, we have

$$f_0^\infty(u) \leq \frac{1}{1-\varepsilon} f_0^\tau(u) + \frac{TM_\varepsilon}{(1-\varepsilon)\tau} \leq \frac{1}{1-\varepsilon}\lambda + \frac{TM_\varepsilon}{(1-\varepsilon)\tau} \leq \lambda + \frac{1}{10}\,,$$

whence

$$\psi\left(\text{epi}\,(f_0^\tau)\right) \subseteq \text{epi}\,(f_0^\infty)\,.$$

Then the inclusions easily follow from (4.2.14).

Since the map

$$\psi : (\Sigma''_\tau, A''_\tau) \to (\Sigma_\tau, A_\tau)$$

is homotopic to the inclusion, from Theorem 4.1.3 we infer that the homomorphism induced by inclusion

$$H^*(\Sigma_\tau, A_\tau) \to H^*(\Sigma'_\infty, A'_\infty)$$

is injective.

Since the map

$$\psi : (\Sigma''_\infty, A''_\infty) \to (\Sigma_\infty, A_\infty)$$

is homotopic to the inclusion, from Theorem 4.1.3 we infer that the homomorphism induced by inclusion

$$H^*(\Sigma'_\tau, A'_\tau) \to H^*(\Sigma''_\infty, A''_\infty)$$

is surjective. From Theorem 4.1.3 we infer that the homomorphism induced by inclusion

$$H^*(\Sigma_\tau, A_\tau) \to H^*(\Sigma''_\infty, A''_\infty)$$

is bijective and the assertion follows. □

Proof of Theorem 4.2.2.

From Lemmas 4.2.12, 4.2.13 and 4.2.9, we infer that there exist $m \geq 0$ and $\tau < +\infty$ such that $H^m(D_\tau, E_\tau) \neq \{0\}$.

From Lemma 4.2.10 and Theorem 4.1.2 it follows that there exists a critical point $(u, \lambda) \in \text{epi}\,(f_0^\tau)$ of $\mathscr{F}^\tau$.

By Corollary 1.2.9, we have

$$f_0^\tau(v) + \int_0^T (G^\tau)^0\,(t, u; u - v)\; dt \geq f_0^\tau(u) \qquad \text{for any } v \in BV(0, T; \mathbb{R}^n)\,,$$

namely

$$f_0(v) + \int_0^T G^0(t, \tau u; \tau u - v)\, dt \geq f_0(\tau u) \qquad \text{for any } v \in BV(0, T; \mathbb{R}^n)\,.$$

Therefore, τu is a solution of (HI). □

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