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Theta divisors and the geometry of tautological model

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Abstract Let E be a stable vector bundle of rank r and slope $2g-1$ on a smooth irreducible complex projective curve C of genus $g \geq 3$. In this paper we show a relation between theta divisor Θ_E and the geometry of the tautological model P_E of E. In particular, we prove that for $r > g-1$, if C is a Petri curve and E is general in its moduli space then Θ_E defines an irreducible component of the variety parametrizing $(g-2)$-linear spaces which are g-secant to the tautological model P_E. Conversely, for a stable, $(g-2)$-very ample vector bundle E, the existence of an irreducible non special component of dimension $g-1$ of the above variety implies that E admits theta divisor.

Keywords Vector bundles · Theta divisors · Moduli spaces · Tautological map

1 Introduction

Let C be a smooth, irreducible, complex projective curve of genus $g \geq 3$. Theta divisors play a fundamental role in studying the *theta map* of the moduli space which parametrizes semistable vector bundles on C with rank r and trivial determinant, see [5]. In this paper we will consider semistable vector bundles on C with rank $r \geq 2$ and integer slope $2g-1$, they are parametrized by the moduli space $\mathcal{U}_C(r, r(2g-1))$. Let $E \in \mathcal{U}_C(r, r(2g-1))$ be a stable bundle, we say that E *admits theta divisor* if for general line bundles $l \in Pic^{-g}(C)$ we have $h^0(E \otimes l) = 0$, in this case the locus

$$\Theta_E = \{l \in Pic^{-g}(C) \mid h^0(E \otimes l) \geq 1\}$$

has a natural structure of effective divisor in $Pic^{-g}(C)$ and it is said the *theta divisor* of E, see Sect. 3.

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Let $E \in \mathcal{U}_C(r, r(2g-1))$ be a stable bundle. Let $\mathbb{P}_E$ denote the associated projective bundle and let $p\colon \mathbb{P}_E \to C$ be the natural projection. The tautological linear system defines a morphism

$$u_E\colon \mathbb{P}_E \to \mathbb{P}^{rg-1},$$

whose image $P_E \subset \mathbb{P}^{rg-1}$ will be said the *tautological model* of E, see Sect. 7. Let $V_{g,g-2}(P_E)$ be the variety which parametrizes $(g-2)$-linear spaces in $\mathbb{P}^{rg-1}$ which are g-*secant* to the tautological model P_E. This variety is closely related to the theta divisor of E and to k-ampleness properties of E. We recall that E is said $(k-1)$-very ample if for any $d \in C^{(k)}$ the restriction map $\rho_d\colon H^0(E) \to H^0(E_d)$ is surjective, see Sect. 5. Our first result is the following (see Proposition 8.3):

Proposition *Let $[\Pi] \in V_{g,g-2}(P_E)$ and $Z = u_E^*(\Pi)$. If $p_*(Z)$ is not a finite subscheme of C, then E does not admits theta divisor; if E is $(g-2)$-very ample, actually $p_*(Z)$ is a finite subscheme of C.*

Assume that $r > g-1$, in Proposition 6.3, we prove that a general stable bundle $E \in \mathcal{U}_C(r, r(2g-1))$ is $(g-2)$-very ample. This induces on the geometry of the tautological model of E the following property: let $d \in C^{(g)}$ be an effective divisor, $d = x_1 + \cdots + x_g$ with $x_i \neq x_j$ for any $i \neq j$ and assume that $h^0(E(-d)) = 1$, then there is a unique $(g-2)$-linear space $\Pi_d \subset \mathbb{P}^{rg-1}$ such that $Z_d = u_E^*(\Pi_d)_{|p^*(d)}$ is a zero scheme biregular to d. This means that such a divisor d defines a $(g-2)$-linear space Π_d, which is g-secant to the tautological model P_E: that is $[\Pi_d] \in V_{g,g-2}(P_E)$. Our second result is the following (see Theorem 8.6):

Theorem *Let C be a Petri curve of genus $g \geq 3$, let $r > g-1$ and $E \in \mathcal{U}(r, r(2g-1))$ be a general stable bundle. Then there exists an irreducible component $\Sigma_E \subseteq V_{g,g-2}(P_E)$ of dimension $g-1$ which is birational to Θ_E.*

We say that an irreducible component $\Sigma \subseteq V_{g,g-2}(P_E)$ is *non special* if a general $[\Pi] \in \Sigma$ satisfies the following properties: $u_E^*(\Pi)$ is a finite subscheme of $\mathbb{P}_E$ and there is a subscheme $Z \subseteq u_E^*(\Pi)$ of lenght g which is biregular to $p_*(Z)$. Finally, as a consequence of previous results we obtain the following sufficient condition for a vector bundle to admit theta divisor (see Proposition 8.7):

Proposition *Let E be vector bundle with slope $2g-1$ and $h^1(E) = 0$, admitting a tautological model $P_E \subset \mathbb{P}^{rg-1}$. Assume moreover that E is $(g-2)$-very ample. If there exists a non special component $\Sigma \subseteq V_{g,g-2}(P_E)$ of dimension $g-1$, then E admits theta divisor.*

2 Notations

All over the paper, C will be a smooth irreducible, complex projective curve of genus $g \geq 3$ and ω_C will denote its canonical line bundle. Since C will be fixed throughout the paper, we will use $H^0(E)$ (resp. $H^0(L)$) instead of $H^0(C, E)$ (resp. $H^0(C, L)$) for the cohomology groups, for any vector bundle E (resp. line bundle L) on the curve.
For any $2 \leq k \leq g$, $C^{(k)}$ will denote the k-symmetric product of C, parametrizing effective divisors d of degree k on the curve C. It is well known that $C^{(k)}$ is a smooth projective variety of dimension k, see [1]. Let

$$S_k^1 = \{d \in C^{(k)} | h^0(\mathcal{O}_C(d)) \geq 2\}, \quad (1)$$

be the closed subset parametrizing special divisor, i.e. effective divisors d moving in a linear system; it is well known that for $k = g$ actually $S^1_g \subset C^{(g)}$ is an effective divisor. We will denote by Δ^k_1 the multidiagonal divisor of $C^{(k)}$, i.e. the simple coincidence locus:

$$\Delta^k_1 = \{d \in C^{(k)} | \, \exists x \in C \colon d = 2x + x_1 + \cdots + x_{k-2}, \; x_i \in C\}. \tag{2}$$

Moreover, for any $k \geq 3$, we will consider its closed subset Δ^k_2:

$$\Delta^k_2 = \{d \in C^{(k)} | \, \exists x \in C \colon \; d = 3x + x_1 + \cdots + x_{k-3}, \;\; x_i \in C\}, \tag{3}$$

with $\dim \Delta^k_2 = k - 2$.

For any $k \in \mathbb{Z}$, $Pic^k(C)$ will denote the variety parametrizing line bundles of degree k on C. For $2 \leq k \leq g$, we recall the natural Abel map:

$$C^{(k)} \to Pic^k(C), \quad d \to O_C(d);$$

and $W^1_k \subset Pic^k(C)$ will denote the image of S^1_k by the Abel map. Finally, C will be said *a Petri curve* if for any line bundle l on the curve the Petri map, given by multiplication of section:

$$\mu_l \colon H^0(l) \otimes H^0(\omega_C \otimes l^{-1}) \to H^0(\omega_C)$$

is injective, see [2].

3 Theta divisors

For any $r \geq 2$ and $n \in \mathbb{Z}$, let $\mathcal{U}_C(r, n)$ denote the moduli space of semistable vector bundles of rank r and degree n on a curve C of genus g. It is a normal irreducible projective variety of dimension $r^2(g-1)+1$, moreover $\mathcal{U}_C(r, n) \simeq \mathcal{U}_C(r, n')$ whenever $n' - n = kr, \quad k \in \mathbb{Z}$. In particular, when $n = r(g-1)$ a natural Brill–Noether locus is defined as follows:

$$\Theta_r = \{[E] \in \mathcal{U}_C(r, r(g-1)) | \, h^0(gr(E)) \geq 1\}, \tag{4}$$

where $[E]$ denotes the $S-$equivalence class of E and $gr(E)$ is the polystable bundle defined by a Jordan-Holder filtration of E, see [15]. Actually, Θ_r is an integral Cartier divisor, which is said the *theta divisor of* $\mathcal{U}_C(r, r(g-1))$, see [10].

For semistable vector bundles with integer slope we can introduce the notion of theta divisors as follows. Let E be a semistable vector bundle on C with slope $m = \frac{degE}{r} \in \mathbb{Z}$, we set $h = g - 1 - m$. The tensor product defines a morphism:

$$\mu \colon \mathcal{U}_C(r, rm) \times Pic^h(C) \to \mathcal{U}_C(r, r(g-1)), \tag{5}$$

sending $([E], N) \to [E \otimes N]$. We can consider the pull-back $\mu^*\Theta_r$ of the divisor Θ_r. When the intersection $\mu^*\Theta_r \cdot ([E] \times Pic^h(C))$ is proper, it defines an effective divisor Θ_E on $Pic^h(C)$ which is called *the theta divisor* of E, which is set theoretically:

$$\Theta_E = \{N \in Pic^h(C) | h^0(gr(E) \otimes N) \geq 1\}; \tag{6}$$

finally, this occurs for a general $[E] \in \mathcal{U}_C(r, rm)$, see [13].

Moreover, assume that $det E \simeq M^{\otimes r}$ with $M \in Pic^m(C)$, then we have:

$$\mathcal{O}_{[E] \times Pic^h(C)}(\mu^*\Theta_r) \simeq \mathcal{O}_{Pic^h(C)}(r\Theta_M), \tag{7}$$

where

$$\Theta_M = \{N \in Pic^h(C) \mid h^0(M \otimes N) \geq 1\}, \tag{8}$$

is a translate of the canonical theta divisor $\Theta \subset Pic^{g-1}(C)$, parametrizing line bundles N of degree $g-1$ such that $h^0(N) \geq 1$.

Actually, the notion of theta divisor can be extended to any vector bundle E of rank r and integer slope. We will say that E *admits theta divisor* if Θ_E, given as in (6), is a divisor in $Pic^h(C)$. It is easy to see that if E is not semistable, then $\Theta_E = Pic^h(C)$, hence if E admits theta divisor then it is semistable. The converse actually is not true, at least for $r \geq 4$, see [4] for a survey on the matter.

Finally, if E is semistable but not stable and admits theta divisor, then Θ_E is not integral. But there exist examples of stable vector bundles admitting reducible theta divisors, see [3]. Let $[E] \in \mathcal{U}_C(r, rm)$, in order to study its theta divisor, see (6), we will introduce the following closed subscheme:

$$\Theta_E^2 = \{N \in Pic^h(C) \mid h^0(gr(E) \otimes N) \geq 2\} \subseteq \Theta_E. \tag{9}$$

We recall the following result, see [7]:

Proposition 3.1 *Let $r \geq 2$ and $m \in \mathbb{Z}$.*

(1) *Let C be a Petri curve of genus $g \geq 4$. For a general stable vector bundle $[E] \in \mathcal{U}_C(r, rm)$, Θ_E is an irreducible and reduced divisor and its singular locus satisfies the following properties:*

$$\mathrm{Sing}(\Theta_E) = \Theta_E^2, \quad \dim\ \mathrm{Sing}(\Theta_E) = g - 4.$$

(2) *Let C be a non hyperelliptic curve of genus 3. A general stable vector bundle $[E] \in \mathcal{U}_C(r, rm)$ admits a smooth irreducible and reduced theta divisor Θ_E.*

4 Brill–Noether theory

We will introduce some special subvarieties of the moduli space $\mathcal{U}_C(r, n)$, for any $r \geq 2$. Let $\mathcal{U}_C(r, n)^s$ denote the open subset of stable bundles of rank r and degree n. For any $k \geq 1$, let's consider the following Brill–Noether loci:

$$B(r, n, k) = \{[F] \in \mathcal{U}_C(r, n)^s \mid h^0(F) \geq k\}, \tag{10}$$

$$\tilde{B}(r, n, k) = \{[F] \in \mathcal{U}_C(r, n) \mid h^0(gr(F)) \geq k\}, \tag{11}$$

which are closed subschemes of their moduli spaces. We recall some fundamental results on these varieties, see for example [6] and [12]. $B(r, n, k)$ has a natural description as a determinantal locus, the Brill–Noether number is defined as follows:

$$\rho(r, n, k) = r^2(g-1) + 1 - k(k - n + r(g-1)). \tag{12}$$

If $B(r, n, k) \neq \emptyset$ and $B(r, n, k) \neq \mathcal{U}_C(r, n)^s$, then every irreducible component of $B(r, n, k)$ has dimension at least $\rho(r, n, k)$, moreover we have the inclusion $B(r, n, k+1) \subseteq Sing(B(r, n, k))$. Finally, note that when $n = r(g-1)$ the Brill–Noether locus $\tilde{B}(r, r(g-1), 1)$ coincides with the theta divisor Θ_r of $\mathcal{U}(r, r(g-1))$, see (4).

In particular for $k = 1$, we recall the following result which we will use in the sequel, see [6], Theorem 11.7.

Proposition 4.1 *Let $0 \leq n \leq r(g-1)$, the Brill–Noether locus $B(r, n, 1)$ is irreducible of dimension $\rho(r, n, 1)$ and it is smooth outside $B(r, n, 2)$.*

5 *k*-very ample vector bundles

Let E be a vector bundle of rank $r \geq 1$ on a curve C of genus $g \geq 3$. We denote by $\mu(E) = \frac{deg(E)}{r}$ the slope of E. Let $1 \leq k \leq g$, for any effective divisor $d \in C^{(k)}$ we can consider the restriction $E_d = E \otimes O_d$ and the following linear map:

$$\rho_d : H^0(E) \to H^0(E_d), \tag{13}$$

given by restriction of global sections. Following the notations of Beltrametti Sommese, see [9], we have the following:

Definition 5.1 A vector bundle E is said $(k-1)$-very ample if for any $d \in C^{(k)}$ the restriction map ρ_d is surjective.

Note that E is 0-very ample if and only if it is globally generated; it is 1-very ample if and only if it is very ample, finally it is easy to verify that if E is $(k-1)$-very ample then it is $(k-2)$-very ample too. Moreover, if E is $(k-1)$-very ample, then $h^0(E) \geq rk$.

Lemma 5.1 *Let E be a vector bundle with $h^1(E) = 0$. Then E is $(k-1)$-very ample if and only if $h^1(E(-d)) = 0$ for any $d \in C^{(k)}$.*

Proof Let's consider the exact sequence:

$$0 \to E(-d) \to E \to E_d \to 0,$$

passing to cohomology we have:

$$0 \to H^0(E(-d)) \to H^0(E) \to H^0(E_d) \to H^1(E(-d)) \to 0,$$

from which it follows that ρ_d is surjective if and only if $h^1(E(-d)) = 0$. □

Lemma 5.2 *Let E be a $(k-1)$-very ample vector bundle of rank $r \geq 2$ with $h^1(E) = 0$. Then any quotient bundle F of E is $(k-1)$-very ample.*

Proof Let's consider the exact sequence of vector bundles:

$$0 \to G \to E \to F \to 0,$$

$h^1(E) = 0$ implies $h^1(F) = 0$ too. Let $d \in C^{(k)}$, by tensoring with $O_C(-d)$ we have:

$$0 \to G(-d) \to E(-d) \to F(-d) \to 0,$$

$h^1(E(-d)) = 0$ implies $h^1(F(-d)) = 0$ too, so the assertion follows from Lemma 5.1. □

We recall a well known fact about vector bundles that we will frequently use in the sequel, see [16].

Lemma 5.3 *Let E be a semistable vector bundle with slope $\mu(E) > 2g-2$, then $h^1(E) = 0$.*

As a consequence of previous lemmas, we have:

Lemma 5.4 *Let E be a vector bundle of rank $r \geq 1$. If E is semistable and $\mu(E) > 2g-2+k$, then E is $(k-1)$-very ample.*

Proof Let $d \in C^{(k)}$ we have $\mu(E(-d)) = \mu(E) - k > 2g-2$, since $E(-d)$ is semistable too, then by Lemma 5.3 we have $h^1(E(-d)) = 0$ for any $d \in C^{(k)}$, hence E is $(k-1)$-very ample. □

In the sequel we will be interested in $(g-2)$-very ample vector bundles.

Lemma 5.5 *Let E be a $(g-2)$-very ample vector bundle of rank $r \geq 1$ with $h^1(E) = 0$.*

(1) *If $r = 1$, then $deg E > 3g - 4$;*
(2) *if $r \geq 2$, then $\mu(E) > 2g - 2$.*

Proof Let $a = deg(E)$, since $h^1(E) = 0$, by Riemann-Roch formula we have: $h^0(E) = a + r(1-g)$. Since E is $(g-2)$- very ample, then $h^0(E) \geq r(g-1)$, which implies $a \geq (2g-2)r$.

Assume now that $\mu(E) = 2g - 2$, then for any $d \in C^{(g-1)}$, by Riemann-Roch formula we have:

$$h^0(E(-d)) = h^1(E(-d)).$$

If E is $(g-2)$-very ample, from Lemma 5.1, it follows that $h^0(E(-d)) = 0$ for any $d \in C^{(g-1)}$. But this is impossible, since either Θ_E is an effective divisor or Θ_E is all $Pic^{1-g}(C)$, see Sect. 3.
Assume now that L is a $(g-2)$-very ample line bundle with $h^1(L) = 0$. From Lemma 5.1, we have: $h^1(L(-d)) = 0$ for any $d \in C^{(g-1)}$. Note that since $d \in C^{(g-1)}$, then $\omega_C \otimes O_C(-d) = O_C(d')$ with $d' \in C^{(g-1)}$ too, so we have:

$$h^1(L(-d)) = h^0(\omega_C \otimes L^*(d)) = h^0(\omega_C^{\otimes 2} \otimes L^*(-d')).$$

If $\deg(\omega_C^{\otimes 2} \otimes L^*) \geq g$ then $\omega_C^{\otimes 2} \otimes L^* = O_C(B)$, where B is an effective divisor of degree $\geq g$, so for any $d' \in C^{(g-1)}$ with $d' \subset B$ we have:

$$h^0(\omega_C^{\otimes 2} \otimes L^*(-d')) \neq 0.$$

This concludes the proof. □

Proposition 5.6 *Let E be a vector bundle of rank $r \geq 2$ with $\mu(E) = 2g-1$ and $h^1(E) = 0$. If E is $(g-2)$-very ample, then E is semistable.*

Proof Assume that there exists a proper subbundle $G \subset E$ destabilizing E: $\mu(G) > 2g-1$. It gives a quotient bundle:

$$0 \to G \to E \to F \to 0,$$

with $\mu(F) < 2g - 1$. By Lemma 5.2 we have $h^1(F) = 0$ and F is $(g-2)$-very ample too. But this contradicts Lemma 5.5. □

6 Vector bundles with slope $2g - 1$.

In this paper we will consider semistable vector bundles of rank $r \geq 2$ and slope $2g - 1$ on a curve C of genus $g \geq 3$. They are parametrized by the moduli space $\mathcal{U}_C(r, r(2g-1))$.

Lemma 6.1 *Let $[E] \in \mathcal{U}_C(r, r(2g-1))$, then the following properties hold:*

(1) *$h^0(E) = rg$;*
(2) *if E is stable then it is globally generated, i.e. the evaluation map $ev_E : H^0(E) \otimes O_C \to E$ is surjective.*

Proof Let E be a semistable vector bundle with slope $2g-1$, by Riemann-Roch Theorem we have $h^0(E) - h^1(E) = rg$; since $\mu(E) > 2g-2$, then by Lemma 5.3 we have $h^1(E) = 0$. Assume that E is not globally generated at $x \in C$, i.e. $h^1(E \otimes \mathcal{O}_C(-x)) \geq 1$. This implies that $h^0(\omega_C \otimes \mathcal{O}_C(x) \otimes E^*) \neq 0$, so there exists a non zero homomorphism $\phi\colon E \to \omega_C \otimes \mathcal{O}_C(x)$. This implies that E is not stable. □

Lemma 6.2 *Let $r > g-1$ and $[E] \in \mathcal{U}_C(r, r(2g-1))$ be a general stable bundle. We have the following properties:*

(1) $h^0(E \otimes N^{-1}) = 0$ *for any line bundle N of degree $\geq g+1$;*
(2) $h^1(E \otimes N^{-1}) = 0$ *for any line bundle $N \in Pic^{g-1}(C)$.*

Proof (1) For any $k \geq g+1$ we consider the Brill–Noether locus:

$$B(r, r(2g-1-k), 1) \subseteq \mathcal{U}_C(r, r(2g-1-k))^s,$$

and the natural morphism:

$$m\colon B(r, r(2g-1-k), 1) \times Pic^k(C) \to \mathcal{U}_C(r, r(2g-1))^s$$

sending $([F], N) \to [F \otimes N]$. By Proposition 4.1, $B(r, r(2g-1-k), 1)$ is an irreducible scheme of dimension

$$\rho(r, r(2g-1-k), 1) = r^2(g-1) - r(k-g) \leq r^2(g-1) - r.$$

Since $r > g-1$, we can conclude that the image of m is a proper closed subset of $\mathcal{U}_C(r, r(2g-1))^s$.

(2) Now we consider the Brill–Noether locus

$$B(r, rg, r+1) \subseteq \mathcal{U}_C(r, rg)^s,$$

and the natural morphism:

$$m\colon B(r, rg, r+1) \times Pic^{g-1}(C) \to \mathcal{U}_C(r, r(2g-1))^s$$

sending $([F], N) \to [F \otimes N]$. By [6], Theorem 5.11, $B(r, rg, r+1)$ is irreducible of dimension $\rho(r, rg, r+1) = r^2(g-1) - r$. The same argument as above allows us to conclude the proof. □

As a consequence of Lemma 6.2, we have the following result:

Proposition 6.3 *Let $r > g-1$ and $[E] \in \mathcal{U}_C(r, r(2g-1))$ be a general stable bundle: E is $(g-2)$-very ample, but it is not $(g-1)$-very ample.*

Proof By Lemma 5.3 we have $h^1(E) = 0$, so by Lemma 5.1, E is $(g-2)$-very ample if and only if $h^1(E(-d)) = 0$ for any $d \in C^{(g-1)}$. This follows from Lemma 6.2 (2). Finally, if $d \in C^{(g)}$, then ρ_d fails to be surjective whenever $\mathcal{O}_C(-d) \in \Theta_E$. □

In the sequel we will consider the natural map:

$$a\colon C^{(g)} \to Pic^{-g}(C), \tag{14}$$

sending $d \to \mathcal{O}_C(-d)$. It is the composition of the Abel map $C^{(g)} \to Pic^g(C)$ with the natural isomorphism $Pic^g(C) \to Pic^{-g}(C)$, hence the map a is biregular on the open subset $C^{(g)} \backslash S^1_g$.

Let $[E] \in \mathcal{U}_C(r, r(2g-1))$ admitting theta divisor, see (6), then we have:

$$\Theta_E = \{l \in Pic^{-g}(C) \mid h^0(gr(E) \otimes l) \geq 1\}, \tag{15}$$

and consider $\Theta_E^2 \subseteq \Theta_E$ defined in (9). We will be interested in the pull back of theta divisors on the symmetric product $C^{(g)}$ by the map a. For this we will need the following lemma.

Lemma 6.4 *Let $[E] \in \mathcal{U}_C(r, r(2g-1))$ be a general semi-stable bundle admitting theta divisor, then we have:*

(1) *$a(S_g^1) \not\subset \Theta_E$;*
(2) *$a(\Delta_1^g) \not\subset \Theta_E$, if $g \geq 3$ then $a(\Delta_2^g) \not\subset \Theta_E$;*
(3) *if $g \geq 4$, then $\Theta_E^2 \not\subset a(S_g^1)$.*

Proof (1) Let $E = M^{\oplus r}$, with $M \in Pic^{2g-1}(C)$ a base points free line bundle. Then $[E] \in \mathcal{U}_C(r, r(2g-1))$, it admits theta divisor and $\Theta_E = r\Theta_M$. In particular, if we choose $M = N \otimes O_C(d)$, where $N \in Pic^{g-1}(C)$ is a general line bundle with $h^0(N) = 0$ and $d \in S_g^1$, then we have:

$$h^0(M \otimes O_C(-d)) = h^0(N) = 0,$$

hence $a(S_g^1) \not\subset \Theta_M$. Since $\Theta_E \in |r\Theta_M|$, by semicontinuity, the property is satisfied for a general $[E] \in \mathcal{SU}(r, M^{\otimes r})$. Since, up to a finite etale covering, $\mathcal{U}_C(r, r(2g-1))$ is a product of $\mathcal{SU}(r, M^{\otimes r})$ with $Pic^0(C)$, then the assertion follows. (2) can be verified with similar arguments.

(3) Choose a line bundle $A \in Pic^{g-1}(C)$ with $h^0(A) = 2$ and an effective divisor $d \notin S_g^1$. Let $M = A \otimes O_C(d)$: then $d \in \Theta_M^2$ but $d \notin S_g^1$. Let $E = F \oplus M$, with M as above and F a generale stable bundle in the moduli space $\mathcal{U}_C(r-1, (r-1)(2g-1))$. Then $\Theta_E = \Theta_M \cup \Theta_F$ and $d \in \Theta_M^2 \subset \Theta_E^2$ but $d \notin S_g^1$. By semicontinuity the result follows for a general $[E] \in \mathcal{U}_C(r, r(2g-1))$. □

Let $[E] \in \mathcal{U}_C(r, r(2g-1))$, which admits theta divisor Θ_E, we define the following divisor on $C^{(g)}$:

$$D_E = a^*(\Theta_E). \tag{16}$$

Note that set theoretically we have:

$$D_E = \{d \in C^{(g)} \mid h^0(gr(E) \otimes O_C(-d)) \geq 1\}. \tag{17}$$

We have a natural closed subset of D_E as follows:

$$D_E^2 = \{d \in D_E \mid h^0(gr(E) \otimes O_C(-d)) \geq 2\}, \tag{18}$$

and we set $D_E^1 = D_E \backslash D_E^2$. As an immediate consequence of Proposition 3.1, we have the following result:

Proposition 6.5 *Let C be a Petri curve of genus $g \geq 3$. For a general stable bundle $E \in \mathcal{U}_C(r, r(2g-1))$ D_E is an irreducible effective divisor on $C^{(g)}$ which is birational to Θ_E. In particular, if $g = 3$ then $D_E = D_E^1$ is a smooth irreducible surface; if $g \geq 4$, then D_E^1 is a non empty open subset which consists of smooth points, the singular locus $Sing(D_E) \subset D_E^2$ is a closed subset of dimension $g-4$.*

Proof Since the restriction

$$a_{|D_E \backslash D_E \cap S_g^1} : D_E \backslash (D_E \cap S_g^1) \to \Theta_E \backslash (\Theta_E \cap a(S_g^1))$$

is biregular, by Lemma 6.4 we can conclude that D_E is irreducible too and $Sing(D_E) \subset a^*(Sing(\Theta_E))$. The assertion follows from Proposition 3.1 and Lemma 6.4. □

7 Geometry of the tautological model

In this section we will study the geometry of the tautological model of E. Let $E \in \mathcal{U}_C(r, r(2g-1))$ be a stable bundle. Let

$$\mathbb{P}_E = \mathbb{P}E^* \tag{19}$$

be the projective bundle associated to E and let $p \colon \mathbb{P}_E \to C$ be the natural projection map. As usual, on $\mathbb{P}_E$ we define the tautological line bundle $O_{\mathbb{P}_E}(1)$, satisfying the condition $p_* O_{\mathbb{P}_E}(1) = E$, which implies

$$H^0(\mathbb{P}_E, O_{\mathbb{P}_E}(1)) \simeq H^0(C, E).$$

If E is stable then, by Lemma 6.1, it is globally generated and $h^0(E) = rg$. This implies that the map defined by $O_{\mathbb{P}_E}(1)$ is a morphism:

$$u_E \colon \mathbb{P}_E \to \mathbb{P}(H^0(E)^*) = \mathbb{P}^{rg-1}, \tag{20}$$

in particular the restriction of u_E to any fiber $\mathbb{P}_{E,x} = p^{-1}(x)$ of $\mathbb{P}_E$ is a linear embedding:

$$u_{E,x} \colon \mathbb{P}_{E,x} \hookrightarrow \mathbb{P}^{rg-1}. \tag{21}$$

Definition 7.1 We denote by P_E (resp. $P_{E,x}$) the image of u_E (resp. $u_{E,x}$) in $\mathbb{P}^{rg-1}$. We call P_E the tautological model of $\mathbb{P}_E$.

Let $2 \leq k \leq g$, for any effective divisor $d \in C^{(k)}$ let

$$\lambda_d \colon \mathbb{P}H^0(E_d)^* \to \mathbb{P}H^0(E)^* \tag{22}$$

be the projectivization of the dual map of the linear map ρ_d defined in (13). Note that λ_d is injective for any $d \in C^{(k)}$ if and only if E is $(k-1)$-very ample, see Sect. 5.
Let $d \in C^{(k)} \backslash \Delta_1^k$, then $d = x_1 + \cdots + x_k$ with $x_i \neq x_j$ for $i \neq j$. We associate to d a natural subscheme of $\mathbb{P}_E$:

$$p^*(d) = \mathbb{P}_{E,x_1} \cup \mathbb{P}_{E,x_2} \cup \cdots \cup \mathbb{P}_{E,x_k}, \tag{23}$$

which is just the disjoint union of the fibers $\mathbb{P}_{E,x_i}$.

Lemma 7.1 *Let $d \in C^{(k)} \backslash \Delta_1^k$, there is a natural embedding*

$$i_d \colon p^*(d) \hookrightarrow \mathbb{P}H^0(E_d)^* = \mathbb{P}^{rk-1}$$

such that the following diagram commutes:

$$\begin{array}{ccc} i_d \colon p^*(d) & \longrightarrow & \mathbb{P}H^0(E_d)^* \\ & \searrow^{u_{E|p^*(d)}} & \downarrow \lambda_d \\ & & \mathbb{P}H^0(E)^*. \end{array}$$

Let $V_d \subset \mathbb{P}H^0(E)^* = \mathbb{P}^{rg-1}$ be the linear span of the image $u_E(p^*(d))$. If E is $(g-2)$-very ample, then:

(1) $\dim V_d = rk - 1$ *for any* $k \leq g-1$*;*
(2) *for* $k = g$*:* $\dim V_d \leq rg - 2$ *if and only if* $d \in D_E$.

Proof Let $d = x_1 + \cdots + x_k$ with $x_i \neq x_j$ for $i \neq j$. Note that we have a natural isomorphism:

$$H^0(E_d) \simeq E_d = E_{x_1} \oplus \cdots \oplus E_{x_k}.$$

For any $x_i \in Supp(d)$ we have the following commutative diagram:

$$\begin{array}{ccc} H^0(E) & \xrightarrow{\rho_d} & E_{x_1} \oplus \cdots \oplus E_{x_k} \\ {\scriptstyle ev_{E,x_i}} \downarrow & & \downarrow {\scriptstyle \pi_i} \\ E_{x_i} & \xrightarrow{\simeq} & E_{x_i} \end{array}$$

which induces the following natural embedding

$$i_{x_i} : \mathbb{P}_{E,x_i} \hookrightarrow \mathbb{P}(E_{x_1} \oplus \cdots \oplus E_{x_k}),$$

moreover we have $\lambda_d \cdot i_{x_i} = u_{E,x_i}$. Finally, $i_d : p^*(d) \hookrightarrow \mathbb{P}H^0(E_d)^*$ is defined on each fiber as follows: $i_{d|\mathbb{P}_{E,x_i}} = i_{x_i}$.

Let $V_d \subset \mathbb{P}^{rg-1}$ be the linear span of $u_E(p^*(d))$. Note that by definition it follows that the image $i_d(p^*(d))$ spans $\mathbb{P}H^0(E_d)^*$. So from the above commutative diagram we have that $V_d = \lambda_d(\mathbb{P}H^0(E_d)^*)$. If E is $(g-2)$-very ample, then it is $(k-1)$-very ample for any $k \leq g-1$. This implies that λ_d is injective, so $\dim V_d = rk - 1$. Finally, note that if $k = g$, then $\dim V_d \leq rg - 2$ if and only if λ_d is not injective, by Lemma 5.1, this occurs if and only if $d \in D_E$. □

In the sequel we will be interested on 0−dimensional subschemes $Z \subset p^*(d)$ which are biregular to d. The very-ampleness property of E for $k \leq g-1$ gives us the following result on the geometry of the tautological model:

Proposition 7.2 *Let $E \in \mathcal{U}_C(r, r(2g-1))$ be a stable bundle with tautological model $P_E \subset \mathbb{P}^{rg-1}$. Let $d \in C^{(k)} \backslash \Delta_1^k$, with $2 \leq k \leq g$, let $i_d : p^*(d) \hookrightarrow \mathbb{P}H^0(E_d)^*$ be the embedding defined in Lemma 7.1. Assume that E is $(g-2)$-very ample. Then we have the following properties:*

(1) *let $Z \subset p^*(d)$ be a subscheme biregular to d, then $i_d(Z)$ spans a linear space of dimension $k-1$ in $\mathbb{P}H^0(E_d)^*$ whose pull back on $p^*(d)$ is Z; moreover through a general point $q \in \mathbb{P}H^0(E_d)^*$ there exists a unique linear space $\Lambda \subset \mathbb{P}H^0(E_d)^*$ of dimension $k-1$ such that ${i_d}^*\Lambda$ is biregular to d.*
(2) *Let $Z \subset p^*(d)$ be a scheme biregular to d, if $k \leq g-1$, then the linear span of $u_E(Z)$ has dimension $k-1$; when $k = g$, if the linear span of $u_E(Z)$ has dimension $\leq g-2$, then $h^0(E(-d)) \geq 1$.*

Proof (1) We prove by induction on k that the linear span of $i_d(Z)$ has dimension $k-1$. If $k = 2$, then $i_d(Z)$ spans a line since i_d is an embedding, and its pull back is exactly Z. Assume that the property holds for any subcheme of $p^*(d')$ which is biregular to d' of degree $k-1$. Let $d = d' + x_k$, then we have $Z = Z' + z_k$, where Z' is a subcheme of $p^*(d')$ which is biregular to d'. By induction hypothesis the linear span of $i_{d'}(Z')$ is a $k-2$ dimensional subspace of $\mathbb{P}H^0(E_{d'})^*$. We have a natural inclusion

$$\mathbb{P}H^0(E_{d'})^* \hookrightarrow \mathbb{P}H^0(E_d)^*$$

and the restriction $i_{d|Z'} = i_{d'}(Z')$, so the assertion follows immediately since

$$\mathbb{P}H^0(E_{d'})^* \cap i_d(\mathbb{P}_{E,x}) = \emptyset.$$

To prove uniqueness we proceed again by induction on k. Let $k = 2$ and $d = x + y$ with $x \neq y$. Let $q \in \mathbb{P}H^0(E_d)^*$ such that $q \notin i_d(P_{E,x}) \cup i_d(P_{E,y})$. The intersection of the linear spans $< i_d(P_{E,x}), q >$ and $< i_d(P_{E,y}), q >$ is a line through q which intersects $i_d(P_{E,x})$ and $i_d(P_{E,y})$: this is actually Λ.
Let $d \in C^{(k)}$ and let $q \in \mathbb{P}H^0(E_d)^*$ such that $q \notin \mathbb{P}H^0(E_{d'})^*$ for any $d' \subset d$. The intersection of the linear spans $< i_d(P_{E,x}), q >$ and $< \mathbb{P}H^0(E_{d'})^*, q >$ is a line trough the point q, which intersects $i_d(P_{E,x})$ at a point p and $\mathbb{P}H^0(E_{d'})^*$ at a point q'. By induction hypothesis there is a unique linear space $\Lambda' \subset \mathbb{P}H^0(E_{d'})^*$ of dimension $k-2$ trough the point q' such that $i_{d'}{}^*\lambda^*_{d'}(\Lambda')$ is a zero scheme biregular to d'. Then Λ is actually the linear span $< \Lambda', q >$.
(2) Let $Z \subset p^*(d)$ be a $0-$dimensional scheme biregular to d. First of all note that from commutative diagram of Lemma 7.1, the linear span of $u_E(Z)$ is actually the image by λ_d of the linear span of $i_d(Z)$. Let $k \leq g-1$, since E is $(k-1)$-very ample, then λ_d is injective, so the assertion follows from property (1). Let $k = g$: by property (1) the scheme $i_d(Z)$ spans a linear space in $\mathbb{P}H^0(E_d)^*$ of dimension $g-1$, whose image by the map λ_d has dimension $\leq g-2$. Then λ_d is not injective, so by Lemma 5.1 we have: $h^1(E(-d)) = h^0(E(-d)) \geq 1$. □

Remark 7.1 Property (1) of Proposition 7.2 can be suitably extended to any divisor $d \in \Delta_1^2$ and to any divisor $d \in \Delta_1^k \backslash \Delta_2^k$, when $3 \leq k \leq g$.

We prove uniqueness by induction on k. Let $k = 2$ and $d = 2x$, with $x \in C$. For any $z \in P_{E,x}$ let 's consider the tangent space $T_z(P_E)$ of the variety P_E at the point z and let 's consider

$$T = \cup_{z \in P_{E,x}} T_z(P_E),$$

the variety spanned by all the tangent spaces at points $z \in P_{E,x}$. Since E is very ample, then λ_d is an embedding and actually we have:

$$\lambda_d(\mathbb{P}(H^0(E_{2x})^*) = T \simeq \mathbb{P}^{2r-1}.$$

Hence we have a natural embedding:

$$i_d \colon \mathbb{P}_{E,x} \hookrightarrow \mathbb{P}H^0(E_{2x})^* = \mathbb{P}^{2r-1}.$$

Let $q \in \mathbb{P}^{2r-1}$ be a point such that $q \notin i_d(\mathbb{P}_{E,x})$, then there is a unique $z \in \mathbb{P}_{E,x}$ such that $q \in T_z(\mathbb{P}_E)$. Let Λ be the line through q and z, then $i_d^*(\Lambda)$ is biregular to $2x$.

Let $3 \leq k \leq g$ and $d \in \Delta_1^k \backslash \Delta_2^k$, then we have:

$$d = 2x + d', \quad d' \in C^{(k-2)} \backslash \Delta_2^{k-2}, \quad x \notin Supp(d').$$

Note that $H^0(E_d) = H^0(E_{2x}) \oplus H^0(E_{d'})$, hence in the projective space $\mathbb{P}H^0(E_d)^* = \mathbb{P}^{rk-1}$ we have the natural inclusions

$$\mathbb{P}H^0(E_{2x})^* \hookrightarrow \mathbb{P}^{rk-1} \quad \mathbb{P}H^0(E_{d'})^* \hookrightarrow \mathbb{P}^{rk-1}$$

whose images are disjoint linear subspaces $\mathbb{P}^{2r-1}$ and $\mathbb{P}^{r(k-2)-1}$. We define a natural embedding:

$$i_d\colon\, p^*(Supp(d)) \hookrightarrow \mathbb{P}^{rk-1},$$

where $p^*(Supp(d)) = \mathbb{P}_{E,x} \cup \mathbb{P}_{E,x_1} \cup \cdots \cup \mathbb{P}_{E,x_k}$, $x_i \in Supp(d')$.

Let $q \in \mathbb{P}^{rk-1}$ be a point such that $q \notin \mathbb{P}^{2r-1} \cup \mathbb{P}^{r(k-2)-1}$. The intersection of the linear spans $< \mathbb{P}^{2r-1}, q >$ and $< \mathbb{P}^{r(k-2)-1}, q >$ is a line through the point q, which intersects the above linear spaces respectively in q_1 and q_2. It is easy to see that $q_1 \notin \mathbb{P}_{E,x}$, hence through q_1 there is a unique tangent line $l =< z, q_1 >$, with $z \in \mathbb{P}_{E,x}$. This implies that $i_d^*(l)_{|\mathbb{P}_{E,x}}$ is biregular to $2x$. Similarly, q_2 is a general point in $\mathbb{P}^{r(k-2)-1}$, so by induction hypothesis there exist a unique linear space Λ' of dimension $k-3$ satisfying property (1) of Proposition 7.2, that is $i_d^*(\Lambda')_{|p^*(Supp(d'))}$ is biregular to d'. We set $\Lambda =< l, \Lambda' >$, then Λ is a linear space of dimension $k-1$ through the point q satisfying property (1) of Proposition 7.2, that is $i_d^*(\Lambda)$ is biregular to d.

As a consequence of Proposition 7.2 we obtain the following result:

Proposition 7.3 *Let $E \in \mathcal{U}_C(r, r(2g-1))$ be a stable bundle which is $(g-2)$-very ample and let $P_E \subset \mathbb{P}^{rg-1}$ be its tautological model. For any $d \in C^{(g)} \backslash \Delta_1^g$ such that $h^0(E \otimes O_C(-d)) = 1$ there exists a unique linear space*

$$\Pi_d \subset V_d \subset \mathbb{P}^{rg-1}$$

of dimension $g-2$ such that the zero scheme $Z_d = i_d{}^*(\lambda_d{}^*(\Pi_d))$ is biregular to d. In particular,

$$\Pi_d = \cap_{i=1}^g V_{d_i}, \quad d_i \subset d, \quad d_i \in C^{(g-1)}.$$

Proof Since $h^1(E \otimes O_C(-d)) = 1$, the map λ_d is a linear projection from a point $q \in \mathbb{P}H^0(E_d)^*$: its image is the hyperplane

$$V_d \subset \mathbb{P}^{rg-1}$$

which is the linear span of $u_E(p^*(d))$, see Lemma 7.1. Note that, since E is $(g-2)$-very ample, $q \notin \mathbb{P}H^0(E_{d'})^*$ for any $d' \subset d$. By Proposition 7.2 (1), there is unique linear space $\Lambda_d \subset \mathbb{P}H^0(E_d)^*$ of dimension $g-1$ trough the point q such that $i_d{}^*(\Lambda_d)$ is a zero scheme Z_d biregular to d. Let Π_d be the linear projection of Λ_d from the point q, then it satisfies the assertion.

Finally, let $d_i \subset d$, by Lemma 7.1, the linear span of $u_E(p^*(d_i))$ is a linear space $V_{d_i} \subset V_d$ of dimension $rg-r-1$. Let $Z_i \subset Z_d$ be a 0- dimensional subscheme such that Z_i is biregular to d_i, then by Proposition 7.2 (2) Π_d is the linear span of $u_E(Z_i)$. This implies that $\Pi_d \subset V_{d_i}$, for any $i = 1, \dots, g$. We claim that actually we have:

$$\cap_{i=1}^g V_{d_i} = \Pi_d.$$

We prove by induction that $\dim \cap_{i=1}^k V_{d_i} = rg - rk + (k-2)$.

Note that the linear span $< V_{d_1}, V_{d_2} >$ is the hyperplane V_d so we have:

$$\dim(V_{d_1} \cap V_{d_2}) = \dim V_{d_1} + \dim V_{d_2} - \dim V_d = rg - 2r.$$

Assume that $\dim(V_{d_1} \cap \cdots \cap V_{d_{k-1}}) = rg - (k-1)r + (k-3)$. As above, note that the linear span $< V_{d_1} \cap \cdots \cap V_{d_{k-1}}, V_{d_k} >$ is V_d, so we have:

$$\dim(V_{d_1} \cap \cdots \cap V_{d_k}) = \dim(V_{d_1} \cap \cdots \cap V_{d_{k-1}}) + \dim V_{d_k} - \dim V_d = rg - rk + (k-2).$$

□

8 The family of g-secant $(g-2)-$planes parametrized by D_E

Let $E \in \mathcal{U}_C(r, r(2g-1))$ be a stable bundle which is $(g-2)$-very ample. In this section we will assume that E admits theta divisor, whose pull back $D_E = a^*(\Theta_E)$, on the symmetric product $C^{(g)}$, is irreducible and reduced, moreover we will also assume that the following subset

$$D_E^1 = \{d \in D_E \mid h^0(E \otimes O_C(-d)) = 1\}$$

is a non empty open subset of it and $D_E^2 = D_E \backslash D_E^1$ has codimension ≥ 2. If C is a Petri curve of genus $g \geq 3$ and $r > g-1$ then these assumptions hold for a general stable bundle $E \in \mathcal{U}_C(r, r(2g-1))$, see Propositions 6.5 and 6.3.

Let $P_E \subset \mathbb{P}^{rg-1}$ be the tautological model of E. We denote by $\mathbb{G}(g-1, rg)$ the Grassmannian variety which parametrizes $(g-2)$- linear spaces of $\mathbb{P}^{rg-1}$. We will assume that

$$\mathbb{G}(g-1, rg) \hookrightarrow \mathbb{P}^{N-1}, \quad N = \binom{rg}{g-1},$$

embedded by the Pluecker map.

Proposition 7.3 allows us to define a map g_E as follows:

$$g_E(d) = [\Pi_d] \in \mathbb{G}(g-1, rg), \tag{24}$$

where $d \in D_E^1$ and $\Pi_d \subset \mathbb{P}^{rg-1}$ is the unique linear space of dimension $g-2$ such that the scheme $i_d{}^*(\lambda_d{}^*(\Pi_d))$ is biregular to d . We have the following result:

Proposition 8.1 *Let $E \in \mathcal{U}_C(r, r(2g-1))$ be a stable bundle which is $(g-2)$-very ample. Assume that $D_E = a^*(\Theta_E)$ is irreducible and reduced and $codim D_E^2 \geq 2$. The map*

$$g_E \colon D_E \dashrightarrow \mathbb{G}(g-1, rg), \quad d \to [\Pi_d]$$

is a rational map which is defined on the open subset $U_E = D_E^1 \backslash \Delta_2^g$.

Proof By Proposition 7.3, on the open subset $D_E^1 \backslash \Delta_1^g$, the map g_E is actually the map which sends

$$d \to [\Pi_d], \quad \Pi_d = \cap_{i=1}^g V_{d_i},$$

where $d_i \subset d$ with $d_i \in C^{(g-1)}$ and $V_{d_i} \subset \mathbb{P}^{rg-1}$ is the linear span of $u_E(p^*(d_i))$. By Remark 7.1 actually g_E can also be defined at divisors $d \in U_E = D_E^1 \cap (\Delta_1^g \backslash \Delta_2^g)$ and by Lemma 6.4 and Proposition 6.5 we have codim $U_E \geq 2$. □

Remark 8.1 Let E be a stable vector bundle which does not admits theta divisor. Assume that E is $(g-2)$-very ample and the following subset

$$\{d \in C^{(g)} | h^0(E \otimes O_C(-d)) \geq 2\}$$

is a proper closed subset of $C^{(g)}$ of codimension ≥ 2. Then, as in Proposition 8.1, we can define a rational map:

$$g_E \colon C^{(g)} \dashrightarrow \mathbb{G}(g-1, rg), \quad d \to [\Pi_d].$$

We will denote by

$$\mathcal{V}_{g,g-2}(P_E) \subseteq \mathbb{G}(g-1, rg) \tag{25}$$

the variety parametrizing $(g-2)$-linear spaces in $\mathbb{P}^{rg-1}$ which are g-secant to the tautological model P_E of E. It is well known that $\mathcal{V}_{g,g-2}(P_E)$ is the closure in $\mathbb{G}(g-1, rg)$ of the locus of $(g-2)$-planes which contain at least g distinct points of P_E and are spanned by them, see [11]. Its expected dimension can be easily computed as follows, see [14]:

$$exp\dim(\mathcal{V}_{g,g-2}(P_E)) = gr - (rg-(g-1))(g-(g-1)) = g-1,$$

hence each irreducible component of $\mathcal{V}_{g,g-2}(P_E)$ has dimension $\geq g-1$.

Let $d \in U_E$ and $g_E(d) = [\Pi_d]$. Since $i_d^*(\lambda_d{}^*(\Pi_d)) = Z_d$ is a 0-dimensional subscheme of $p^*(d)$ which is biregular to d, then we have:

Corollary 8.2 *For any $d \in U_E$ we have $[g_E(d)] \in \mathcal{V}_{g,g-2}(P_E)$.*

The relation between the variety $\mathcal{V}_{g,g-2}(P_E)$ of the tautological model of E and both its theta divisor and k-ampleness property of E comes out suddendly.

Proposition 8.3 *Let E be a vector bundle of rank r and integer slope $2g-1$ with $h^1(E)=0$. Let $\mathbb{P}_E$ be the projective bundle associated to E with projection $p\colon \mathbb{P}_E \to C$, let $u_E\colon \mathbb{P}_E \to \mathbb{P}^{rg-1}$ be the tautological map and $P_E \subset \mathbb{P}^{rg-1}$ its image. Let $[\Pi] \in \mathcal{V}_{g,g-2}(P_E)$.*

(1) *If $p_*(u_E^*(\Pi)) \subseteq C$ is not a finite subscheme, then E does not admits theta divisor;*
(2) *if E is $(g-2)$-very ample, then $p_*(u_E^*(\Pi)) \subseteq C$ is a finite subscheme.*

Proof Assume that $p_*(u_E^*(\Pi))$ is not a finite subset of C, then this implies that $\dim u_E^*(\Pi) \geq 1$. Let Z be an irreducible and reduced component of $u_E^*(\Pi)$. Since $p\colon \mathbb{P}_E \to C$ is a proper map and Z is a closed subscheme of $\mathbb{P}_E$, then $p(Z)$ is a closed subscheme of C too. Assume that it is not finite, then $p(Z) = C$, hence the restriction

$$p_Z = p_{|Z}\colon Z \to C$$

is surjective. Let $d \in C^{(g)} \backslash \Delta_1^g$, there exists a finite subscheme $Z_d \subset Z$ such that $p_*(Z_d) = d$ and Z_d is biregular to d. Since the linear span of $u_E(Z_d)$ is contained in Π, then by Proposition 7.2 (2), we have $h^0(E \otimes O_C(-d)) \geq 1$. Hence E does not admits theta divisor.

Assume now that E is $(g-2)$- very ample. Let $d' \in C^{(g-1)} \backslash \Delta_1^{g-1}$, since p_Z is surjective, there exists a zero scheme $Z' \subset Z$ such that $p_*(Z') = d'$ and Z' is biregular to d'. Since E is $(g-2)$-very ample, then Π is actually the linear span of $u_E(Z')$. By Proposition 7.2 (1) and Lemma 7.1 we have:

$$i_{d'}^*(\lambda_{d'}^*(\Pi)) = u_E^*(\Pi)_{|p^*(d')} = Z'.$$

Since Z is irreducible and reduced, this implies that the restriction map p_Z is a birational morphism. So there exists a line bundle L on C with the following properties: $p_Z{}^*(L) \simeq O_{\mathbb{P}(E)}(1)_{|Z}$ and $u_E(Z) \subset \Pi \simeq \mathbb{P}^{g-2}$ is defined by a linear system $W \subseteq |L|$ of dimension $g-2$. By Clifford's Theorem we have:

$$deg(L) \geq 2\dim|L| \geq 2g-4.$$

Since $u_{E|Z}$ is a birational map, this implies

$$deg(u_E(Z)) = deg(L) \geq 2g-4 \geq g-1, \quad \forall g \geq 3.$$

Let $H \subset \mathbb{P}^{g-2}$ be a general hyperplane and set $Z_H = u_{E|Z}{}^*(H)$. Then we have $Z_H \subset Z$ is a finite scheme of degree $\geq g-1$ which is biregular to $p_*(Z_H)$ and $u_E(Z_H) \subset H \simeq \mathbb{P}^{g-3}$. This is impossible since E is $(g-2)$-very ample. □

Moreover, we have the following information:

Lemma 8.4 *Let E be as in Proposition 8.3 and assume that it is $(g-2)$-very ample. Let $[\Pi] \in \mathcal{V}_{g,g-2}(P_E)$ and $Z = u_E^*(\Pi)$. Assume that there exist $d \in C^{(g-1)} \backslash \Delta_1^{g-1}$ with $d \subseteq p_*(Z)$. Then Z is a finite scheme of length $\geq g$.*

Proof Since E is $(g-2)$-very ample, by Proposition 8.3 it follows that $p_*(Z)$ is a finite subset of C. So it is enough to prove that for any $x \in p_*(Z)$ the intersection $Z \cap \mathbb{P}_{E,x}$ is a finite subset too.
Let $x \in p_*(Z)$: there exists $\overline{d} \in C^{(g-2)}$ with the following properties:

$$\overline{d} \subset d, \quad \overline{d} + x \in C^{(g-1)} \backslash \Delta_1^{g-1}.$$

Moreover, $\overline{d} + x \subseteq p_*(Z)$, hence there exists a zero subscheme $\overline{Z} + z \subset Z$ such that $p_*(\overline{Z} + z) = \overline{d} + x$ and $\overline{Z} + z$ is biregular to $\overline{d} + x$. Since E is $(g-2)$-very ample, Π is the linear span of $u_E(\overline{Z} + z)$, so by Proposition 7.2 (1), we have:

$$i^*_{\overline{d}+x}(\lambda^*_{\overline{d}+x}(\Pi)) = \overline{Z} + z.$$

This implies that $Z \cap \mathbb{P}_{E,x}$ is finite and moreover $Supp(Z \cap \mathbb{P}_{E,x}) = \{z\}$. □

Let us consider the following set:

$$V_E = \left\{ d \in D_E^1 \mid \exists\, d' \subset Supp(d) \mid d' \in C^{(g-1)} \backslash \Delta_1^{g-1} \right\}, \qquad (26)$$

then V_E is a non empty open subset of D_E, actually $V_E \subset U_E$.
Finally, we have the following result:

Theorem 8.5 *Let $E \in \mathcal{U}_C(r, r(2g-1))$ be a stable bundle. Assume that E is $(g-2)$-very ample, that $D_E = a^*(\Theta_E)$ is irreducible and reduced and that $codim D_E^2 \geq 2$. Then the restriction of g_E to the open subset V_E*

$$g_{E|V_E} : V_E \to g_E(V_E),$$

is a finite morphism onto its image. The closure of the image $g_E(V_E)$ is an irreducible component Σ_E of $\mathcal{V}_{g,g-2}(P_E)$ of the expected dimension.

Proof To prove that $g_{E|V_E}$ is a finite morphism, it is enough to prove that for any $[\Pi] \in g_E(V_E)$ the fiber

$$F_{[\Pi]} = \{d \in V_E \mid g_E(d) = [\Pi]\},$$

is a finite set. Let $Z = u_E^*(\Pi)$. Note that for any $d \in F_{[\Pi]}$ we have that $Z_d = i_d^* \lambda_d^*(\Pi)$ is biregular to d and $Z_d \subseteq Z$. Since by Lemma 8.4 actually Z is a finite set, then $F_{[\Pi]}$ is finite too.
This implies that the closure of the image $g_E(V_E)$ is irreducible of dimension $g-1$. It remains to prove that it is an irreducible component of $\mathcal{V}_{g,g-2}(P_E)$. Let Σ_0 be the irreducible component containing $g_E(V_E)$. A general $[\Pi] \in \Sigma_0$ satisfies the following properties:

(1) the scheme $Z = u_E^*(\Pi)$ is finite of length $\geq g$;
(2) there exists $Z' \subseteq Z$ of length g which is biregular to $p_*(Z')$ and Π is the linear span of $u_E(Z')$.

Let $Z' \subseteq Z$ be as in (2) and let $d = p_*(Z')$. By Proposition 7.2 (2), it follows that $h^1(E \otimes O_C(-d)) \geq 1$. This implies that $[\Pi]$ is an element of the closure of $g_E(V_E)$. □

Definition 8.1 We will call Σ_E the irreducible component of $\mathcal{V}_{g,g-2}(P_E)$ parametrized by Θ_E.

We expect that for a general stable bundle $E \in \mathcal{U}_C(r, r(2g-1))$ and for a general $[\Pi] \in \Sigma_E$ we actually have that $u_E^*(\Pi)$ is a zero scheme of degree g. This is a proved in the following result:

Theorem 8.6 *Let C be a Petri curve of genus $g \geq 3$ and assume that $r > g - 1$. Let $E \in \mathcal{U}_C(r, r(2g-1))$ be a general stable bundle. Then $\Sigma_E \subseteq \mathcal{V}_{g,g-2}(P_E)$ is birational to Θ_E.*

Proof Let C be a Petri curve of genus $g \geq 3$ and let $r > g - 1$. By Proposition 6.5 a general stable bundle $E \in \mathcal{U}_C(r, r(2g-1))$ admits theta divisor and its pull back $D_E = a^*(\Theta_E)$ is birational to Θ_E, so it is irreducible and reduced with $codim D_E^2 = 4$. Since $r > g - 1$, by Proposition 6.3 it follows that E is $(g-2)$-very ample. So by Theorem 8.5, we have a rational map $g_E \colon D_E \dashrightarrow \mathcal{V}_{g,g-2}(P_E)$, whose image Σ_E has dimension $g - 1$. Finally, the restriction $g_{|V_E} \colon V_E \to g_E(V_E)$ is a finite map. So to prove the assertion, it is enough to prove that $g_{E|V_E}$ has degree 1.
Assume that $\deg g_{E|V_E} \geq 2$, for a general stable $E \in \mathcal{U}_C(r, r(2g-1))$. Then for general $[\Pi] \in g_E(V_E)$ we have $\deg u_E^*(\Pi) \geq g + 1$, so there exists a zero scheme $Z \subseteq u_E^*(\Pi)$ such that

$$p_*(Z) = d = x_1 + \cdots + x_{g+1}, \quad x_i \neq x_j, i \neq j,$$

Z biregular to $p_*(Z)$.
Since E is a general stable bundle and $r > g - 1$, then by Lemma 6.2 we have: $h^1(E \otimes O_C(-d)) = r$. We can consider a commutative diagram as in (7.1):

$$\begin{array}{ccc} i_d \colon p^*(d) & \longrightarrow & \mathbb{P}H^0(E_d)^* \\ & \searrow^{u_{E|p^*(d)}} & \downarrow \lambda_d \\ & & \mathbb{P}H^0(E)^* \end{array}$$

where λ_d is a linear projection from $\mathbb{P}(H^1(E \otimes O_C(-d))^*) = \mathbb{P}^{r-1}$. Let Λ be the linear span of $i_d(Z)$, then Λ is a g-dimensional linear space such that $\lambda_d(\Lambda) = \Pi$. This implies that

$$\Lambda \cap \mathbb{P}^{r-1} = \mathbb{P}^1.$$

For any $i = 1, \ldots, g+1$, let $d_i = d - x_i$ and $Z_i \subseteq Z$ such that $p_*(Z_i) = d_i$. Then Π is the linear span of $u_E(Z_i)$ for any $i = 1, \ldots, g+1$, so that $d_i \in D_E$. Actually, for general $[\Pi] \in g_E(D_E)$ we have $d_i \in V_E$, hence $h^1(E \otimes O_C(-d_i)) = 1$. We set $q_i = \mathbb{P}(H^1(E \otimes O_C(-d_i)^*)$, note that $q_i \in \mathbb{P}^1$. This gives us a 2-dimensional linear space V of global sections

$$V \subset H^0(\omega_C \otimes E^* \otimes O_C(d)) \simeq \mathbb{C}^r,$$

and, for any $i = 1 \ldots, g+1$, a non zero global section

$$s_i \in H^0(\omega_C \otimes E^* \otimes O_C(d - x_i)) \simeq \mathbb{C}$$

such that $s_i \in V$. Look at the evaluation map:

$$ev_V \colon V \otimes O_C \to \omega_C \otimes E^* \otimes O_C(d),$$

we claim that ev_V is generically injective.

Assume that the image is a line bundle $L \subset \omega_C \otimes E^* \otimes O_C(d)$ with $h^0(L) \geq 2$. Since E is stable and C is a Petri curve, then $3 \leq \deg L \leq g-1$. The line bundle $A = \omega_C \otimes L^* \otimes O_C(d)$ is a quotient of E. Since E is $(g-2)$-very ample with $h^1(E) = 0$: then A is $(g-2)$-very ample too, see Lemma 5.2. But this is impossible since $\deg A \leq 3g-4$, see Lemma 5.5.
This gives us a pair (E_2, V) where $E_2 \subset \omega_C \otimes E^* \otimes O_C(d)$ is a vector bundle of rank 2 fitting into the following exact sequence:

$$0 \to V \otimes O_C \to E_2 \to O_D \to 0, \tag{27}$$

where D is an effective divisor with $d \subset D$. Since $\omega_C \otimes E^* \otimes O_C(d)$ is stable, then we have:

$$\frac{1}{2}\deg(D) < g, \implies \deg(D) < 2g.$$

We will call D the fundamental divisor of (E_2, V), see [8] for details, and we will denote it as $\mathbb{D}_{(E_2,V)}$.
A general stable $F \in \mathcal{U}_C(r, rg)$ satisfies the following conditions: $h^0(F) = r$ and F is generically globally generated, see [8]; this means that F fit into an exact sequence as follows:

$$0 \to \mathbb{C}^r \to F \to O_{D_F} \to, \tag{28}$$

where D_F is an effective divisor in C^{rg}. The map

$$\Phi\colon \mathcal{U}_C(r, rg) \to C^{rg}$$

sending $F \to D_F$ is actually a rational surjective map, a general fiber is birational to the GIT quotient of $(\mathbb{P}^{r-1})^{rg}$ with respect to the diagonal action of $PGL(r)$, see [8] for details. For any $d \in C^{(g+1)}$ let's consider the following subvariety of $\mathcal{U}_C(r, rg)$:

$$\mathcal{A}_d = \overline{\{F \in \mathcal{U}_C(r, rg)^s \mid d \subset D_F\}},$$

then $\mathcal{A}_d = \overline{\Phi^{-1}(d \times C^{(rg-g-1)})}$, so $\mathcal{A}_d$ is irreducible and we have:

$$\dim \mathcal{A}_d = \dim \mathcal{U}(r, rg) - (g+1) = r^2(g-1) + 1 - (g+1).$$

Finally, let's consider

$$\mathcal{E}_d = \overline{\{F \in \mathcal{A}_d \mid \exists (E_2, V)\colon E_2 \subset F,\ \dim V = 2,\ \ d \subset \mathbb{D}_{(E_2,V)}\}}.$$

Note that the restriction:

$$\Phi_{|\mathcal{E}_d}\colon \mathcal{E}_d \to d \times C^{(rg-g-1)}$$

is surjective too, and a general fiber is birational to the GIT quotient of $(\mathbb{P}^1)^{g+1} \times (\mathbb{P}^{r-1})^{rg-g-1}$ with respect to the diagonal action of of $PGL(r)$. This implies that $\mathcal{E}_d$ is irreducible too and

$$\begin{aligned}\dim \mathcal{E}_d &= (g+1) + (r-1)(rg-g-1) - (r^2-1) + (rg-g-1)\\ &= r^2(g-1) + 1 - (g+1)(r-1).\end{aligned}$$

This allow us the conclude the proof. In fact, consider the variety

$$\mathcal{J} = \{(E, d) \in \mathcal{U}_C(r, r(2g-1)) \times C^{(g+1)} | E^* \otimes \omega_C \otimes O_C(d) \in \mathcal{E}_d\}.$$

Let as usual p_i denote the projections on factors, $i = 1, 2$. Note that we have:

$$p_2^{-1}(d) = \{(F^* \otimes \omega_C \otimes O_C(d), d) \mid F \in \mathcal{E}_d\} \simeq \mathcal{E}_d,$$

so $\mathcal{J}$ is irreducible and

$$\dim \mathcal{J} = r^2(g-1) + 1 - (g+1)(r-2) < r^2(g-1) + 1, \quad \forall r \geq 3.$$

This implies that $p_1(\mathcal{J})$ is a proper subvariety of $\mathcal{U}_C(r, r(2g-1))$ and E is not general.

□

Remark 8.2 Let E be as in Theorem 8.5. Let $d \in D_E \backslash \Delta_1^g$ and assume that $h^0(E \otimes O_C(-d)) = m \geq 2$. Set $\mathbb{P}^{m-1} = \mathbb{P}(H^1(E \otimes O_C(-d)^*)$, it can be proved that there is a finite morphism

$$\sigma_d \colon \mathbb{P}^{m-1} \to \mathcal{V}_{g,g-2}(P_E),$$

sending $q \to [\Pi_{d,q}]$, where $\Pi_{d,q} = \lambda_d(\Lambda_{d,q})$ and $\Lambda_{d,q}$ is the unique linear space of dimension $g-1$ through the point q whose pull back on $p^*(d)$ is biregular to d, see Proposition 7.2 (1).

Remark 8.3 Let $E = F \oplus G$, where F and G are stable bundles of rank, respectively, r_1 and r_2 and slope $2g-1$, which are $(g-2)$-very ample. Then E is semistable (not stable) with slope $2g-1$ and it is $(g-2)$-very ample too, see Proposition 5.2. Assume that F and G admit theta divisors which are irreducible and reduced. So E admits theta divisor too and $D_E = D_F \cup D_G$. It can be proved that, by applying Theorem 8.5, Σ_F and Σ_G are $(g-1)$-dimensional irreducible components of $\mathcal{V}_{g,g-2}(P_E)$. Finally, for any $d \in D_F \cap D_G$ we have: $h^1(E \otimes O_C(-d)) \geq 2$. By Remark 8.2, we get an other irreducible component $\Sigma \subset \mathcal{V}_{g,g-2}(P_E)$ of dimension $\geq g-1$.

Let E be any vector bundle of rank r and integer slope $2g-1$ on the curve C. Assume that $h^1(E) = 0$ and E is $(g-2)$-very ample. Let $\mathbb{P}_E$ be the associated projective bundle and $p \colon \mathbb{P}_E \to C$ the natural projection. Let $u_E \colon \mathbb{P}_E \to \mathbb{P}^{rg-1}$ the tautological map and P_E its image. Let $\mathcal{V}_{g,g-2}(P_E)$ be the variety parametrizing $(g-2)$-linear spaces in $\mathbb{P}^{rg-1}$ which are g-secant P_E.

Definition 8.2 Let $\Sigma \subseteq \mathcal{V}_{g,g-2}(P_E)$ be an irreducible component, we say that Σ is a non special irreducible component if there is a non empty open subset $\Sigma_0 \subseteq \Sigma$ satisfying the following properties:
for any $[\Pi] \in \Sigma_0$ the scheme $u_E^*(\Pi)$ is finite and there exists a subscheme $Z \subseteq u_E^*(\Pi)$ of lenght g which is biregular to $p_*(Z)$.

The following gives a sufficient condition for a vector bundle E to admit theta divisor:

Proposition 8.7 *Let E be any vector bundle of rank r and integer slope $2g-1$ on the curve C, assume that $h^0(E) = rg$ and E is $(g-2)$-very ample. Let $P_E \subset \mathbb{P}^{rg-1}$ be its tautological model and let Σ be a non special irreducible component of $\mathcal{V}_{g,g-2}(P_E)$. If* $\dim \Sigma = g-1$, *then E admits theta divisor.*

Proof Let's consider the following incidence variety:

$$\mathcal{J} = \{(d, [\Pi]) \in C^{(g)} \times \mathcal{V}_{g,g-2}(P_E) \mid \exists Z' \subseteq u_E^*(\Pi)\ p_*(Z') = d, Z' \simeq d\},$$

and let as usual denote by p_1 and p_2 the projections onto factors. We claim that we have:

$$\overline{p_1(\mathcal{J})} = \{d \in C^{(g)} \mid h^0(E \otimes O_C(-d)) \geq 1\}.$$

In fact the inclusion

$$p_1(\mathcal{J}) \subseteq \{d \in C^{(g)} \mid h^0(E \otimes O_C(-d)) \geq 1\},$$

follows from Proposition 7.2 (2). For the other inclusion, let $d \in C^{(g)} \backslash \Delta_2^g$ with $h^0(E \otimes O_C(-d)) \geq 1$. If $h^0(E \otimes O_C(-d)) = 1$, there is a unique linear space $[\Pi_d] = g_E(d)$ of dimension $g-2$ such that $Z_d = i_d^*(\lambda_d^*(\Pi_d))$ is biregular to d, see Proposition 8.1. This implies $(d, [\Pi_d]) \in \mathcal{J}$ and so $d \in p_1(\mathcal{J})$. Finally, assume that $h^0(E \otimes O_C(-d)) = m \geq 2$, then, by Remark 8.2, for any $q \in \mathbb{P}^{m-1}$ there is a linear space $\Pi_{d,q}$ with the above property. This implies that $(d, [\Pi_{d,q}]) \in \mathcal{J}$ for any $q \in \mathbb{P}^{m-1}$, so $d \in p_1(\mathcal{J})$ too.

Let $\mathcal{J}_\Sigma = p_2^{-1}(\Sigma)$, by Definition 8.2, the restriction

$$p_{2|\mathcal{J}_\Sigma} : \mathcal{J}_\Sigma \to \Sigma$$

is generically finite and dominant, hence $\mathcal{J}_\Sigma$ is an irreducible component of $\mathcal{J}$ with $\dim \mathcal{J}_\Sigma = \dim \Sigma = g - 1$.

Assume that E does not admits theta divisor, we prove that each irreducible component of $\mathcal{J}$ has dimension $\geq g$. Since E does not admits theta divisor, we have $p_1(\mathcal{J}) = C^{(g)}$. There exist an integer $m \geq 0$ such that $h^0(E \otimes O_C(-d)) = 1 + m$ for general $d \in C^{(g)}$. Assume that the locus

$$W^{m+2} = \{d \in C^{(g)} | h^0(E(-d)) \geq m + 2\}$$

has codimension at least 2. Since p_1 is a dominant map, we can conclude that $\mathcal{J} = p^{-1}(C^{(g)})$ is irreducible of dimension $g + m$. On the contrary, $p^{-1}(W^{m+2})$ is an other component of $\mathcal{J}$ of dimension $g + m$ too. □

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