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# Norms of the lattice points discrepancy

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# Contents

# Introduction

The main topic of this thesis is discrepancy. We remember that the discrepancy of a body $\Omega$ with respect to the lattice of integer points $\mathbb{Z}^d$ is defined as

$$\mathcal{D}_\Omega(t) = |\Omega| - \mathrm{card}(\mathbb{Z}^d \cap (\Omega + t))$$

where $|\Omega|$ is the area of $\Omega$.

In Chapter 1 we consider an ellipsoid $\Omega$ in $\mathbb{R}^d$ with $d \geq 2$ and we want to estimate the $L^p$ norm of the discrepancy

$$\left\{ \int_R^{R+1} \int_{\mathbb{T}^d} |r^{-(d-1)/2} \mathcal{D}(r\Omega - x)|^p dx d\mu(r) \right\}^{1/p}$$

where $R \geq 2$, $r$ is a dilation and $x$ a traslation of the domain and $d\mu$ is a finite Borel measure.

In Chapter 2 we consider a convex domain $\Omega$ in $\mathbb{R}^d$ with $d \geq 2$ and we want to estimate the $L^p(L^2)$ norm of the discrepancy

$$\left\{ \int_{\mathbb{T}^d} \left[ \frac{1}{H} \int_R^{R+H} \left| r^{-(d-1)/2} \mathcal{D}\left(r\Omega - x\right) \right|^2 d\mu(r) \right]^{p/2} dx \right\}^{1/p}$$

where $0 < H < +\infty$ and $d\mu$ is a finite Borel measure.

In Chapter 3 we consider the discrepancy of a rotated square $\Omega$ in the plane with sides perpendicular to the unit vectors $\sigma = (\cos(\vartheta), \sin(\theta))$ and $\sigma^\perp = (-\sin(\vartheta), \cos(\theta))$. In particular we want to estimate the $L^s(L^p)$ mixed norm

$$\left\{ \int_{\mathbb{SO}(2)} \left[ \int_{\mathbb{T}^2} |\mathcal{D}(R\sigma\Omega - x)|^p dx \right]^{s/p} d\mu(\vartheta) \right\}^{1/s} .$$

As a corollary we study the Hausdorff dimension of the set of rotations which give a discrepancy less than $|n|^\beta$ with $0 < \beta < 1$.

In Chapter 4 we consider a convex set with a point with zero curvature. In particular we consider a set$B_\gamma$ with $\partial B_\gamma$ graph of the function $y = |x|^\gamma$ in a neighborhood of the origin, with $\gamma \geq 2$ and we want to understand the role of dilations, translations and rotations in the estimates of the $L^p$ norm and the $L^p(L^s)$ mixed norm of the discrepancy of this domain in $\mathbb{R}^d$ with $d \geq 2$.

Every chapter is indipendent from the others. Therefore the reader will find repetitions from a chapter to chapter. We hope that this fact will not displease the reader.

# Chapter 1

# $L^p$ norms of the lattice point discrepancy

The discrepancy between the volume and the number of integer points in $r\Omega - x$, a dilated by a factor $r$ and translated by a vector $x$ of a domain $\Omega$ in $\mathbb{R}^d$, is

$$\mathcal{D}\left(r\Omega - x\right) = \sum_{k\in\mathbb{Z}^d} \chi_{r\Omega - x}(k) - r^d \left|\Omega\right| .$$

Here $\chi_{r\Omega-x}(y)$ denotes the characteristic function of $r\Omega - x$ and $|\Omega|$ the measure of $\Omega$. A classical problem is to estimate the size of $\mathcal{D}\left(r\Omega - x\right)$, as $r \to +\infty$. See e.g. the monograph *"Lattice points"* of E. Krätzel [27]. We want to estimate the $L^p$ norm of this discrepancy:

$$\left\{ \int_{\mathbb{T}^d} \frac{1}{H} \int_R^{R+H} \left|\mathcal{D}\left(r\Omega - x\right)\right|^p drdx \right\}^{1/p} .$$

Here is a short non-exhaustive list of previous results on the mean square discrepancy.

As far as we know G.H.Hardy was the first who considered a mean square average of the discrepancy under dilations. In particular, studing the mean value of the arithmetical function $r(n)$, the number of integer pairs $(h,k)$ with $n = h^2 + k^2$, in [13] he proved that for every $\epsilon > 0$,

$$\int_0^T \left| \sum_{n\leq t} r(n) - \pi t \right|^2 dt \leq C T^{3/2+\epsilon}.$$

In our notation $\sum_{n\leq t} r(n) - \pi t$ is nothing but the discrepancy $\mathcal{D}\left(\sqrt{t}\Omega\right)$, where $\Omega$ is the disc $\{|x| < 1\}$ in the plane. In the same paper Hardy also stated that it is not unlikely that the supremum norm has the same size. This is the so called Gauss circle problem.

H. Cramer in [9] removed the $\epsilon$ in the theorem of Hardy and proved the more precise asymptotic estimate

$$\lim_{T\to+\infty} T^{-3/2}\int_0^T \left|\sum_{n\leq t} r(n)-\pi t\right|^2 dt = \frac{1}{3\pi^2}\sum_{n=1}^{+\infty}\frac{r(n)^2}{n^{3/2}}.$$

The distribution and higher power moment in the Gauss circle problem and the related Dirichlet divisor problem have been studied by D.R. Heath-Brown in [16] and by K. M. Tsang in [38].

W. Nowak in [31] proved that if $\Omega$ is a convex set in the plane with smooth boundary with strictly positive curvature, then for every $R\geq 2$,

$$\left\{\frac{1}{R}\int_0^R |\mathcal{D}(r\Omega)|^2\,dr\right\}^{1/2}\leq CR^{1/2}.$$

Indeed, P. Bleher proved in [3] a more precise asymptotic estimate:

$$\lim_{R\to+\infty}\frac{1}{R^{1/2}}\left\{\frac{1}{R}\int_0^R |\mathcal{D}(r\Omega)|^2\,dr\right\}^{1/2} = C.$$

M.Huxley in [20] considered the mean value of the discrepancy over short intervals and proved that if $\Omega$ is a convex set in the plane with smooth boundary with strictly positive curvature, then

$$\left\{\int_R^{R+1} |\mathcal{D}(r\Omega)|^2\,dr\right\}^{1/2}\leq CR^{1/2}\log^{1/2}(R).$$

W.Nowak in [32] proved that the above estimate remains valid for an interval up to a length of order $\log R$, while for $H\leq R$ but $H/\log(R)\to+\infty$ he proved the more precise asymptotic estimate

$$\lim_{R\to+\infty}\frac{1}{R^{1/2}}\left\{\frac{1}{H}\int_R^{R+H} |\mathcal{D}(r\Omega)|^2\,dr\right\}^{1/2} = C.$$

A.Iosevich, E.Sawyer, A.Seeger in [23] extended the above results to convex sets in $\mathbb{R}^3$ with smooth boundary with strictly positive Gaussian curvature,

$$\left\{\frac{1}{R}\int_0^R |\mathcal{D}(r\Omega)|^2\,dr\right\}^{1/2}\leq CR\log(R).$$

D.G.Kendall considered the mean square average of the discrepancy under translations and proved in [26] that if $\Omega$ is a convex set in $\mathbb{R}^d$ with smooth boundary with strictly positive Gaussian curvature,

$$\left\{\int_{\mathbb{T}^d} |\mathcal{D}(R\Omega - x)|^2 \, dx\right\}^{1/2} \leq CR^{(d-1)/2}.$$

The study of the $L^p$ norm of the discrepancy with $p \neq 2$ is more recent and the results are less complete.

In [4] the authors studied the $L^p$ norm of the discrepancy for random polyhedra $\Omega$ in $\mathbb{R}^d$,

$$\left\{\int_{\mathbb{SO}(d)} \int_{\mathbb{T}^d} |\mathcal{D}(\sigma R\Omega - x)|^p \, dx d\sigma\right\}^{1/p} \leq \begin{cases} C \log^d (R) & \text{if } p = 1, \\ CR^{(d-1)(1-1/p)} & \text{if } 1 < p \leq +\infty. \end{cases}$$

In the same paper it was also proved that the above inequalities can be reversed, at least for a simplex and for $p > 1$. In other words, the $L^p$ discrepancy of a polyhedron grows with $p$. On the contrary, here we will show that for certain domains with curvature there exists a range of indices $p$ where the $L^p$ discrepancy is of the same order as the $L^2$ discrepancy, possibly up to a logarithmic transgression. Indeed, M.Huxley in [21] proved that if $\Omega$ is a convex set in the plane with boundary with continuous positive curvature, then

$$\left\{\int_{\mathbb{T}^2} |\mathcal{D}(R\Omega - x)|^4 \, dx\right\}^{1/4} \leq CR^{1/2} \log^{1/4} (R).$$

Here we shall give an alternative proof of this result. In [5] the authors extended the above result to convex sets with smooth boundary with positive Gaussian curvature in higher dimensions,

$$\left\{\int_{\mathbb{T}^d} |\mathcal{D}(R\Omega - x)|^p \, dx\right\}^{1/p} \leq \begin{cases} CR^{(d-1)/2} & \text{if } p < 2d/(d-1), \\ CR^{(d-1)/2} \log^{(d-1)/2d} (R) & \text{if } p = 2d/(d-1). \end{cases}$$

The present chapter continues this line of research, presenting the proofs for some estimates of the $L^p$ norms of the discrepancy for ellipsoids:

$$\left\{\int_R^{R+1} \int_{\mathbb{T}^d} |r^{-(d-1)/2} \mathcal{D}(r\Omega - x)|^p dx dr\right\}^{1/p}.$$

With the same techniques one can also study the integral

$$\left\{\int_R^{R+1} \int_{\mathbb{T}^d} |r^{-(d-1)/2} \mathcal{D}(r\Omega - x)|^p dx d\mu(r)\right\}^{1/p},$$

where $d\mu$ is a finite Borel measure.

The main results of this chapter are

**Theorem 1.** *(A) If $\Omega$ is an ellipse in the plane $\mathbb{R}^2$ and if $p \leq 6$, then there exists $C > 0$ such that for every $R \geq 2$,*

$$\left\{\int_R^{R+1}\int_{\mathbb{T}^2}\left|r^{-1/2}\mathcal{D}(r\Omega - x)\right|^p dxdr\right\}^{1/p} \leq \begin{cases} C & \textit{if } p < 6, \\ C\log^{2/3}(R) & \textit{if } p = 6. \end{cases}$$

*(B) If $\Omega$ is an ellipsoid in the space $\mathbb{R}^d$ with $d \geq 3$ and if $p \leq 2(d-1)/(d-2)$, then there exists $C > 0$ such that for every $R \geq 2$,*

$$\left\{\int_R^{R+1}\int_{\mathbb{T}^d}\left|r^{-(d-1)/2}\mathcal{D}(r\Omega - x)\right|^p dxdr\right\}^{1/p}$$
$$\leq \begin{cases} C & \textit{if } p < 2(d-1)/(d-2), \\ C\log^{1/p}(R) & \textit{if } p = 2(d-1)/(d-2) \textit{ and } d > 3, \\ C\log^{2/p}(R) & \textit{if } p = 2(d-1)/(d-2) \textit{ and } d = 3. \end{cases}$$

## 1.1 Huxley's theorem

As a start-up, in this section we give an alternative proof of the result of M. Huxley [21] on the fourth power mean of the discrepancy of a convex domain in the plane. We first state a number of easy lemmas:

**Lemma 1.** *The number of integer points in $r\Omega - x$, a translated by a vector $x \in \mathbb{R}^d$ and dilated by a factor $r > 0$ of a domain $\Omega$ in the $d$ dimensional Euclidean space is a periodic function of the translation with Fourier expansion*

$$\sum_{k\in\mathbb{Z}^d}\chi_{r\Omega-x}(k) = \sum_{n\in\mathbb{Z}^d} r^d\widehat{\chi}_\Omega(rn)\, e^{2\pi inx}.$$

*In particular,*

$$\mathcal{D}(r\Omega - x) = \sum_{n\in\mathbb{Z}^d\setminus\{0\}} r^d\widehat{\chi}_\Omega(rn)\, e^{2\pi inx}.$$

*Proof.* This is a particular case of the Poisson summation formula. □

**Lemma 2.** *If the domain $\Omega$ is convex and contains the origin, then there exists $\varepsilon > 0$ such that if $\varphi(x)$ is a non negative smooth radial function with support in $\{|x| \leq \varepsilon\}$ and with integral 1, and if $0 < \delta \leq 1$ and $r \geq 1$, then*

$$|\Omega|\left((r-\delta)^d - r^d\right) + (r-\delta)^d \sum_{n\in\mathbb{Z}^d\setminus\{0\}} \widehat{\varphi}(\delta n)\,\widehat{\chi}_\Omega((r-\delta)n)\, e^{2\pi inx}$$

$$\leq \sum_{n\in\mathbb{Z}^d} \chi_{r\Omega}(n+x) - |\Omega|\, r^d$$

$$\leq |\Omega| \left( (r+\delta)^d - r^d \right) + (r+\delta)^d \sum_{n\in\mathbb{Z}^d\setminus\{0\}} \widehat{\varphi}\,(\delta n)\, \widehat{\chi}_\Omega\,((r+\delta)\, n)\, e^{2\pi i n x}.$$

*Proof.* This is a consequence of the inequality

$$\varphi_\delta \star \chi_{(r-\delta)\Omega}(x) \leq \chi_{r\Omega}(x) \leq \varphi_\delta \star \chi_{(r+\delta)\Omega}(x),$$

where $\varphi_\delta(x) = \delta^{-d}\varphi(\delta^{-1}x)$. □

**Lemma 3.** *Assume that $\Omega$ is a convex body in $\mathbb{R}^d$ with smooth boundary having everywhere positive Gaussian curvature. Then*

$$\widehat{\chi}_\Omega\,(\xi) \leq 1 + |\xi|^{-(d+1)/2}.$$

*Proof.* This is a classical result. See e.g. [12], [19], [18], [34]. □

**Theorem 2.** *If $\Omega$ is a convex body in the plane $\mathbb{R}^2$ with smooth boundary with everywhere positive Gaussian curvature, and if $0 < p \leq 4$, then there exists $C > 0$ such that for every $R \geq 2$,*

$$\left\{ \int_{\mathbb{T}^2} |\sum_{k\in\mathbb{Z}^2} \chi_{R\Omega - x}(k) - \pi R^2|^p dx \right\}^{1/p} \leq \begin{cases} CR^{1/2} & \text{if } p < 4, \\ CR^{1/2}\log^{1/4} R & \text{if } p = 4. \end{cases}$$

*Proof.* By the Hausdorff-Young inequality, with $p \geq 2$ and $1/p + 1/q = 1$,

$$\left\{ \int_{\mathbb{T}^2} |R^2 \sum_{n\in\mathbb{Z}^2\setminus\{0\}} \hat{\chi}_\Omega(Rn) e^{2\pi i n x}|^p dx \right\}^{1/p} \leq R^2 \left\{ \sum_{n\in\mathbb{Z}^2\setminus\{0\}} |\hat{\chi}_\Omega(Rn)|^q \right\}^{1/q}.$$

Since $\partial\Omega$ has strictly positive curvature, $|\hat{\chi}_\Omega(\xi)| \leq C|\xi|^{-3/2}$, so that

$$R^2 \left\{ \sum_{n\in\mathbb{Z}^2\setminus\{0\}} |\hat{\chi}_\Omega(Rn)|^q \right\}^{1/q} \leq CR^{1/2} \left\{ \sum_{n\in\mathbb{Z}^2\setminus\{0\}} |n|^{-3q/2} \right\}.$$

The last series converges when $3q/2 > 2$, that is $p < 4$.

If $p = 4$ it suffices to consider the mollified discrepancy. Let $\varphi(x)$ be a smooth function. Then it has a Fourier transform with a fast decay at infinity, $|\hat{\varphi}(\xi)| \leq C(1+|\xi|)^{-j}$ for every $j > 0$. Hence, by Parseval equality,

$$\left\{ \int_{\mathbb{T}^2} |R^2 \sum_{n\in\mathbb{Z}^2\setminus\{0\}} \hat{\varphi}(\delta n)\hat{\chi}_\Omega(Rn) e^{2\pi i n x}|^4 dx \right\}^{1/4}$$

$$\leq CR^{1/2}\left\{\sum_{k\in\mathbb{Z}^2}\left(\sum_{n\neq 0,k}(1+\delta|n|)^{-j}(1+\delta|k-n|)^{-j}|n|^{-3/2}|k-n|^{-3/2}\right)^2\right\}^{1/4}$$
$$+CR\delta+CR^{-1/2}.$$

Observe that

$$\begin{aligned}(1+\delta|n|)^{-j}(1+\delta|k-n|)^{-j} &= \big(1+\delta(|n|+|k-n|)+\delta^2|n||k-n|\big)^{-j}\\ &\leq (1+\delta(|n|+|k-n|))^{-j}\leq(1+\delta|k|)^{-j}\,.\end{aligned}$$

And also observe that

$$\sum_{n\neq 0,k}|n|^{-3/2}|k-n|^{-3/2}\leq C(1+|k|)^{-1}.$$

Indeed for $k=0$ one has

$$\sum_{n\neq 0,k}|n|^{-3/2}|k-n|^{-3/2}=\sum_{n\neq 0}|n|^{-3}\leq C,$$

while for $k\neq 0$ the result easily follows from the inequality

$$\begin{aligned}&\sum_{n\neq 0,k}|n|^{-3/2}|k-n|^{-3/2}\\ &\leq 2^{3/2}|k|^{-3/2}\sum_{0<|n|<|k|/2}|n|^{-3/2}+2^{3/2}|k|^{-3/2}\sum_{0<|k-n|<|k|/2}|k-n|^{-3/2}\\ &+2^3|k|^{-3}\sum_{|k|/2\leq|n|\leq 2|k|}1+2^{3/2}\sum_{|n|>2|k|}|n|^{-3}.\end{aligned}$$

Hence

$$\begin{aligned}&\sum_{k\in\mathbb{Z}^2}\left(\sum_{n\neq 0,k}(1+\delta|n|)^{-j}(1+\delta|k-n|)^{-j}|n|^{-3/2}|k-n|^{-3/2}\right)^2\\ &\leq C\sum_{k\in\mathbb{Z}^{\neq}}(1+\delta|k|)^{-2j}\left(\sum_{n\neq 0,k}|n|^{-3/2}|k-n|^{-3/2}\right)^2\\ &\leq C\sum_{k\in\mathbb{Z}^{\neq}}(1+\delta|k|)^{-2j}(1+|k|)^{-2}\\ &\leq C\log(1+1/\delta).\end{aligned}$$

The choice $\delta=1/R$ gives the theorem. □

A natural question is whether $p=4$ is really a critical index for the two dimensional discrepancy. In the above theorem, for $p<4$ there is no logarithm, but for $p=4$ it appears. We do not know if the logarithm is really necessary. In the next section another candidate for the critical index ($p=6$) will appear. Also we would like to notice that when $\Omega$ is the unit disk in the plane with center in the origin, Tsang [38] and Heath-Brown [16] showed that for every $p\le 9$

$$\left\{\frac{1}{R}\int_0^R |\mathcal{D}(r\Omega)|^p dr\right\}^{1/p} \le CR^{1/2}.$$

Observe that in the above theorem of Huxley the average is on the set of translations, which has dimension two and measure one, while in the last result of Tsang and Heath-Brown the average is over the set of dilations, which has dimension one and large measure. We acknowledge that we do not understand if there is a real difference between averaging over translations and averaging over dilations. In the result that follows we try to investigate this problem by mixing dilations and translations.

## 1.2 $L^p$ norms for ellipsoids

Let $\Sigma=\left\{x\in\mathbb{R}^d : |x|\le 1\right\}$ be the unit sphere and let $\Omega=M\Sigma$, with $M$ a non singular $d\times d$ matrix, that is $\Omega=\left\{x\in\mathbb{R}^d : |M^{-1}x|\le 1\right\}$. Then one has

$$\begin{aligned}\hat{\chi}_\Omega(\xi) &= |\det(M)|\hat{\chi}_\Sigma(M^T\xi)\\ &= |\det(M)||M^T\xi|^{-d/2}J_{d/2}(2\pi|M^T\xi|)\end{aligned}$$

where $J_{d/2}(x)$ is the Bessel function.

Hence for every non negative integer $h$, one has

$$\begin{aligned}\hat{\chi}_\Omega(\xi) &= |\det(M)|\sum_{\ell=0}^{h}\frac{a_\ell(d)}{|M^T\xi|^{(d+4\ell+1)/2}}\cos(2\pi|M^T\xi|-(d+1)\pi/4)\\ &+|\det(M)|\sum_{\ell=0}^{h}\frac{b_\ell(d)}{|M^T\xi|^{(d+4\ell+3)/2}}\sin(2\pi|M^T\xi|-(d+1)\pi/4)+\mathcal{O}\left(|\xi|^{-(d+4h+5)/2}\right)\\ &=|\det(M)|\sum_{\ell=0}^{2h+1}\frac{c_\ell(d)}{|M^T\xi|^{(d+2\ell+1)/2}}e^{2\pi i|M^T\xi|}\\ &+|\det(M)|\sum_{\ell=0}^{2h+1}\frac{\overline{c_\ell(d)}}{|M^T\xi|^{(d+2\ell+1)/2}}e^{-2\pi i|M^T\xi|}+\mathcal{O}\left(|\xi|^{-(d+4h+5)/2}\right),\end{aligned}$$

where $a_\ell(d)$, $b_\ell(d)$, $c_\ell(d)$ are coefficients depending on the dimension.

Let $\varphi(x)$ be a non negative smooth radial function with support in $\{|x| \leq \varepsilon\}$ and with integral 1. Let $z \in \mathbb{C}$ and $\tau = 0, 1$. Let define the tempered distributions:

$$\Phi_\tau^M(z, r, x) = |\det(M)| \sum_{n \in \mathbb{Z}^d \setminus \{0\}} |M^T n|^{-z} e^{(-1)^\tau 2\pi i |M^T n| r} e^{2\pi i n x},$$

$$\Phi_\tau^M(\delta, z, r, x) = |\det(M)| \sum_{n \in \mathbb{Z}^d \setminus \{0\}} \hat{\varphi}(\delta n) |M^T n|^{-z} e^{(-1)^\tau 2\pi i |M^T n| r} e^{2\pi i n x}.$$

One can notice that if $\mathrm{Re}(z) > d/2$, then the Fourier espansion that defines $\Phi_\tau^M(z, r, x)$ converges in the topology of $L^2(\mathbb{T}^d)$, while the Fourier expansion that defines $\Phi_\tau^M(\delta, z, r, x)$ converges absolutely and uniformly for every complex $z$.

Moreover, for every $h > (d-5)/4$ there exists $C$ such that for every $r > 0$

$$\begin{aligned} r^{-(d-1)/2} \mathcal{D}(r\Omega - x) = & \sum_{\ell=0}^{2h+1} c_\ell(d) r^{-\ell} \Phi_0^M((d + 2\ell + 1)/2, r, x) \\ & + \sum_{\ell=0}^{2h+1} \overline{c_\ell(d)} r^{-\ell} \Phi_1^M((d + 2\ell + 1)/2, r, x) + \mathcal{R}_h(r, x) \end{aligned}$$

where

$$|\mathcal{R}_h(r, x)| \leq C r^{-2h-2}.$$

If $r \to +\infty$ and $\delta \to 0^+$, one has

$$\begin{aligned} & r^{-(d-1)/2} |\mathcal{D}(r\Omega - x)| \\ & \leq \sum_{\sigma, \tau = 0, 1} \sum_{\ell=0}^{2h+1} |\Phi_\tau^M(\delta, (d + 2\ell + 1)/2, r + (-1)^\sigma \delta, x)| + \mathcal{R}_h(r, \delta) \end{aligned}$$

where

$$\mathcal{R}_h(r, \delta) \leq C \left( r^{(d-1)/2} \delta + r^{-2h-2} \right).$$

In order to simplify the notation, in the following we shall consider $\tau = 0$ and $M$ the identity matrix.

**Lemma 4.** *Let*

$$\Phi(z, r, x) = \sum_{n \in \mathbb{Z}^d \setminus \{0\}} |n|^{-z} e^{2\pi i |n| r} e^{2\pi i n x},$$

$$\Phi(\delta, z, r, x) = \sum_{n \in \mathbb{Z}^d \setminus \{0\}} \hat{\varphi}(\delta n) |n|^{-z} e^{2\pi i |n| r} e^{2\pi i n x}.$$

*Also, let $N$ be a positive integer, and $\psi(t)$ a nonnegative smooth function with compact support. Then for every $j > 0$ there exists $C > 0$ with the following properties.*

(1) *For every* $-\infty < R < +\infty$,

$$\int_{\mathbb{R}}\int_{\mathbb{T}^d}\psi(r-R)|\Phi(z,r,x)|^{2N}dxdr$$
$$\leq C\int_{\mathbb{R}^d}\int_{\substack{m,n,\dots\in\mathbb{R}^d\\ |m|,|n|,\dots>1\\ m+n+\dots=k}}|m|^{-\mathrm{Re}(z)}|n|^{-\mathrm{Re}(z)}\dots\int_{\substack{u,v,\dots\in\mathbb{R}^d\\ |u|,|v|,\dots>1\\ u+v+\dots=k}}|u|^{-\mathrm{Re}(z)}|v|^{-\mathrm{Re}(z)}\dots$$
$$\times\left(1+\left||m|+|n|+\dots-|u|-|v|-\dots\right|\right)^{-j}dudv\dots dmdn\dots dk.$$

(2) *For every* $-\infty < R < +\infty$ *and* $0<\delta<1/2$,

$$\int_{\mathbb{R}}\int_{\mathbb{T}^d}\psi(r-R)|\Phi(\delta,z,r,x)|^{2N}dxdr$$
$$\leq C\int_{\mathbb{R}^d}(1+\delta|k|)^{-j}\int_{\substack{m,n,\dots\in\mathbb{R}^d\\ |m|,|n|,\dots>1\\ m+n+\dots=k}}(1+\delta|m|)^{-j}(1+\delta|n|)^{-j}|m|^{-\mathrm{Re}(z)}|n|^{-\mathrm{Re}(z)}\dots$$
$$\times\int_{\substack{u,v,\dots\in\mathbb{R}^d\\ |u|,|v|,\dots>1\\ u+v+\dots=k}}(1+\delta|u|)^{-j}(1+\delta|v|)^{-j}|u|^{-\mathrm{Re}(z)}|v|^{-\mathrm{Re}(z)}\dots$$
$$\times\left(1+\left||m|+|n|+\dots-|u|-|v|-\dots\right|\right)^{-j}dudv\dots dmdn\dots dk.$$

*The inner integrals are over the* $(N-1)d$*-dimensional variety of* $N$ *points with sum* $k$.

(3) *The above final expression are decreasing function of* $\mathrm{Re}(z)$.

*Proof.* (1) is the limit of (2) when $\delta\to 0^+$. It then suffices to prove (2). From the Fourier expansion of $\Phi(\delta,z,r,x)$ it follows that for every positive integer $N$,

$$(\Phi(\delta,z,r,x))^N$$
$$=\sum_{k\in\mathbb{Z}^d}\sum_{\substack{m,n,\dots\neq 0\\ m+n+\dots=k}}\hat{\varphi}(\delta m)\hat{\varphi}(\delta n)\dots|m|^{-z}|n|^{-z}\dots e^{2\pi i(|m|+|n|+\dots)r}e^{2\pi ikx}.$$

For a proof, just observe that since $\hat{\varphi}(\xi)$ has a fast decay at infinity, all series involved are absolutely convergent, and one can freely expand the $N$-th power and rearrange the terms. Then, by Parseval equality,

$$\int_{\mathbb{T}^d}|\Phi(\delta,z,r,x)|^{2N}dx$$

$$= \sum_{k\in\mathbb{Z}^d} \Big| \sum_{\substack{m,n,\dots\neq 0\\ m+n+\dots=k}} \hat{\varphi}(\delta m)\hat{\varphi}(\delta n)\dots |m|^{-z}|n|^{-z}\dots e^{(-1)^\tau 2\pi i(|m|+|n|+\dots)r}\Big|^2.$$

Expanding the square and integrating in the variable $r$, one obtains

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{T}^d} \psi(r-R)|\Phi(\delta,z,r,x)|^{2N}dxdr \\
&= \sum_{k\in\mathbb{Z}^d} \sum_{\substack{m,n,\dots\neq 0\\ m+n+\dots=k}} \hat{\varphi}(\delta m)\hat{\varphi}(\delta n)\dots |m|^{-z}|n|^{-z}\cdots \sum_{\substack{u,v,\dots\neq 0\\ u+v+\dots=k}} \hat{\varphi}(\delta u)\hat{\varphi}(\delta v)\dots |u|^{-\overline{z}}|v|^{-\overline{z}}\dots \\
&\times \int_{\mathbb{R}} \psi(r-R)e^{2\pi i(|m|+|n|+\dots-|u|-|v|-\dots)r}dr.
\end{aligned}$$

The last integral is the Fourier transform of the function $\psi(t)$,

$$\begin{aligned}
&\int_{\mathbb{R}} \psi(r-R)e^{2\pi i(|m|+|n|+\dots-|u|-|v|-\dots)r}dr \\
&= e^{2\pi i(|m|+|n|+\dots-|u|-|v|-\dots)R} \\
&\times \int_{\mathbb{R}} \psi(t)e^{2\pi i(|m|+|n|+\dots-|u|-|v|-\dots)t}dt.
\end{aligned}$$

The functions $\varphi(x)$ and $\psi(r)$ are smooth, so that $|\hat{\varphi}(\xi)| \leq C\left(1+|M^T\xi|\right)^{-j}$ and $|\hat{\psi}(\tau)| \leq C\left(1+|\tau|\right)^{-j}$ for every $j$. Hence the above quantity is dominated up to a constant by

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}^d} \sum_{\substack{m,n,\dots\neq 0\\ m+n+\dots=k}} (1+\delta|m|)^{-j}(1+\delta|n|)^{-j}\dots |m|^{-\mathrm{Re}(z)}|n|^{-\mathrm{Re}(z)}\dots \\
&\times \sum_{\substack{u,v,\dots\neq 0\\ u+v+\dots=k}} (1+\delta|u|)^{-j}(1+\delta|v|)^{-j}\dots |u|^{-\mathrm{Re}(z)}|v|^{-\mathrm{Re}(z)}\dots \\
&\times \left(1+||m|+|n|+\dots-|u|-|v|-\dots|\right)^{-j}.
\end{aligned}$$

In this formula there is no cutoff in the variable $k$. In order to obtain a cutoff in $k$, observe that, if $m+n+\dots=k$, then

$$\begin{aligned}
&(1+\delta|m|)^{-s}(1+\delta|n|)^{-s}\dots = \left(1+\delta\left(|m|+|n|+\dots\right)+\delta^2\left(|m||n|+\dots\right)+\dots\right)^{-s} \\
&\leq \left(1+\delta\left(|m|+|n|+\dots\right)\right)^{-s} \leq \left(1+\delta|m+n+\dots|\right)^{-s} = \left(1+\delta|k|\right)^{-s}.
\end{aligned}$$

In particular, some of the cutoff functions $(1+\delta|m|)^{-j}(1+\delta|n|)^{-j}\dots$ can be replaced with $(1+\delta|k|)^{-j}$. Finally, in the above formulas one can replace the sums with integrals. Indeed, there exist positive constants $A$ and $B$ such that for every integer point $m \neq 0$ and every $x \in Q$, the cube centered at the origin with sides parallel to the axes and of length one,

$$A|m| \leq |m+x| \leq B|m|.$$

This implies that the function $|m+x|^{-\mathrm{Re}(z)}$ is slowly varying in the cube $Q$. Moreover, also the function

$$(1 + ||m+x| + |n| + \cdots - |u| - |v| - \dots|)^{-j}$$

is slowly varying. Hence, one can replace a sum over $m$ with an integral over the union of cubes $m+Q$,

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}^d}(1+\delta|k|)^{-j} \sum_{\substack{m,n,\dots\neq 0\\ m+n+\dots=k}} (1+\delta|m|)^{-j}(1+\delta|n|)^{-j}\dots|m|^{-\mathrm{Re}(z)}|n|^{-\mathrm{Re}(z)}\dots\\
&\times \sum_{\substack{u,v,\dots\neq 0\\ u+v+\dots=k}} (1+\delta|u|)^{-j}(1+\delta|v|)^{-j}\dots|u|^{-\mathrm{Re}(z)}|v|^{-\mathrm{Re}(z)}\dots\\
&\times (1+||m|+|n|+\cdots-|u|-|v|-\dots|)^{-j}\\
&\leq C\int_{\mathbb{R}^d}(1+\delta|k|)^{-j}\int_{\substack{m,n,\dots\in\mathbb{R}^d\\ |m|,|n|,\dots>1/2\\ m+n+\dots=k}} (1+\delta|m|)^{-j}(1+\delta|n|)^{-j}\dots|m|^{-\mathrm{Re}(z)}|n|^{-\mathrm{Re}(z)}\dots\\
&\times \int_{\substack{m,n,\dots\in\mathbb{R}^d\\ |m|,|n|,\dots>1/2\\ m+n+\dots=k}} (1+\delta|u|)^{-j}(1+\delta|v|)^{-j}\dots|u|^{-\mathrm{Re}(z)}|v|^{-\mathrm{Re}(z)}\dots\\
&\times (1+||m|+|n|+\cdots-|u|-|v|-\dots|)^{-j}\,dudv\dots dmdn\dots dk.
\end{aligned}$$

Finally, with a change of variables one can transform the domain of integration $\{|x| > 1/2\}$ into $\{|y| > 1\}$, and (3) follows immediately. Indeed, if $|x| > 1$ then $|x|^{-\mathrm{Re}(z)}$ decreases as $\mathrm{Re}(z)$ increases. □

The integrals in the above lemma look like convolutions. It is well known that the convolution of two radial functions homogeneous of degree $-\alpha$ and $-\beta$ is a radial function homogeneous of degree $d-\alpha-\beta$. For later references, we state this result as a lemma.

**Lemma 5.** *(1) If $0 < \alpha < d$ and $0 < \beta < d$ with $\alpha + \beta > d$, then there exists a constant $C$ such that for every $k \in \mathbb{R}^d \setminus \{0\}$,*

$$\int_{\mathbb{R}^d} |x|^{-\alpha}|k-x|^{-\beta}dx = C|k|^{d-\alpha-\beta}.$$

*(2) If $\alpha > 0$ and $\beta > 0$, and $\alpha + \beta > d$, then there exists a constant $C$ such that for every $k \in \mathbb{R}^d$,*

$$\int_{\{|x|>1,\ |k-x|>1\}} |x|^{-\alpha}|k-x|^{-\beta}dx \leq C.$$

*Proof.* (1) follows from the change of variables $k = |k|\vartheta$, $x = |k|y$, $dx = |k|^d dy$. (2) follows from the fact that the integral $\int_{\{|x|>1,\ |k-x|>1\}} |x|^{-\alpha}|k-x|^{-\beta}dx$ is a continuous function of the variable $k$ vanishing at infinity. In particular this function has a maximum. Indeed, this maximum is attained at $k = 0$. □

**Lemma 6.** *Let $d \geq 2$, $-\infty < \beta < +\infty$, $0 \leq \varepsilon < 1$, and let*

$$E(\varepsilon, \beta, d) = \int_0^\pi (1 - 2\varepsilon\cos(\vartheta) + \varepsilon^2)(1 - \varepsilon\cos(\vartheta))^{\beta-d-1}\sin^{d-2}(\vartheta)d\vartheta.$$

*Then, for every $-\infty < j < +\infty$ and $-\infty < \alpha < d$, there exists a constant $C$ such that for every $-\infty < Y < +\infty$ and every $k \in \mathbb{R}^d \setminus \{0\}$,*

$$\begin{aligned}
&\int_{\mathbb{R}^d} |x|^{-\alpha}|k-x|^{-\alpha}\left(1 + |Y - |x| - |k-x||\right)^{-j} dx \\
&\leq C|k|^{-\alpha}\int_0^{2|k|} \left(1 + |Y - |k| - \tau|\right)^{-j} \tau^{d-1-\alpha}E(|k|/(|k|+\tau), \alpha, d)d\tau \\
&+ C\int_{2|k|}^{+\infty} \left(1 + |Y - |k| - \tau|\right)^{-j} \tau^{d-1-2\alpha}E(|k|/(|k|+\tau), 2\alpha, d)d\tau.
\end{aligned}$$

*In particular, if $\alpha \geq 0$, then*

$$\begin{aligned}
&\int_{\mathbb{R}^d} |x|^{-\alpha}|k-x|^{-\alpha}\left(1 + |Y - |x| - |k-x||\right)^{-j} dx \\
&\leq C|k|^{-\alpha}\int_0^{+\infty} \left(1 + |Y - |k| - \tau|\right)^{-j} \tau^{d-1-\alpha}E(|k|/(|k|+\tau), \alpha, d)d\tau.
\end{aligned}$$

*Proof.* The symmetry between 0 and $k$ gives

$$\begin{aligned}
&\int_{\mathbb{R}^d} |x|^{-\alpha}|k-x|^{-\alpha}\left(1 + |Y - |x| - |k-x||\right)^{-j} dx \\
&= 2\int_{\{|x|+|k-x|\leq 3|k|,\ |x|\leq|k-x|\}} |x|^{-\alpha}|k-x|^{-\alpha}\left(1 + |Y - |x| - |k-x||\right)^{-j} dx \\
&+ 2\int_{\{|x|+|k-x|\geq 3|k|,\ |x|\leq|k-x|\}} |x|^{-\alpha}|k-x|^{-\alpha}\left(1 + |Y - |x| - |k-x||\right)^{-j} dx \\
&\leq C|k|^{-\alpha}\int_{\{|x|+|k-x|\leq 3|k|\}} |x|^{-\alpha}\left(1 + |Y - |x| - |k-x||\right)^{-j} dx
\end{aligned}$$

$$+ C \int_{\{|x|+|k-x|\geq 3|k|\}} |x|^{-2\alpha} (1 + |Y - |x| - |k - x||)^{-j} \, dx.$$

The integral is invariant under rotations of $k$, so that one can assume $k = (|k|, 0)$. Write in spherical coordinates $y = (\rho \cos(\vartheta), \rho \sin(\vartheta) \sigma)$, with $0 \leq \rho < +\infty$, $0 \leq \vartheta \leq \pi$, $\sigma \in \mathbb{S}^{d-2}$, the $d-2$ dimensional unit sphere $\{\sigma \in \mathbb{R}^{d-1} : \ |\sigma| = 1\}$. In these spherical coordinates the ellipsoid $\{|x| + |k - x| = \tau\}$ has equation

$$\rho = \frac{\tau^2 - |k|^2}{2(\tau - |k| \cos(\vartheta))}.$$

In the variables $(\tau, \vartheta, \sigma)$, $|k| \leq \tau < +\infty$, $0 \leq \vartheta \leq \pi$, $\sigma \in \mathbb{S}^{d-2}$, one has

$$\frac{d\rho}{d\tau} = \frac{\tau^2 - 2|k|\tau \cos(\vartheta) + |k|^2}{2(\tau - |k| \cos(\vartheta))^2},$$

and

$$\begin{aligned} dy &= \rho^{d-1} \sin^{d-2}(\vartheta) \, d\rho d\vartheta d\sigma \\ &= \left( \frac{\tau^2 - |k|^2}{2(\tau - |k| \cos(\vartheta))} \right)^{d-1} \frac{\tau^2 - 2|k|\tau \cos(\vartheta) + |k|^2}{2(\tau - |k| \cos(\vartheta))^2} \sin^{d-2}(\vartheta) \, d\tau d\vartheta d\sigma. \end{aligned}$$

Hence,

$$\begin{aligned} &\int_{\{|x|+|k-x|\leq 3|k|\}} (1 + |Y - |x| - |k - x||)^{-j} |x|^{-\alpha} dx \\ &= \int_{|k|}^{3|k|} \int_0^{\pi} \int_{\mathbb{S}^{d-2}} (1 + |Y - \tau|)^{-j} \left( \frac{\tau^2 - |k|^2}{2(\tau - |k| \cos(\vartheta))} \right)^{-\alpha} \\ &\times \left( \frac{\tau^2 - |k|^2}{2(\tau - |k| \cos(\vartheta))} \right)^{d-1} \frac{\tau^2 - 2|k|\tau \cos(\vartheta) + |k|^2}{2(\tau - |k| \cos(\vartheta))^2} \sin^{d-2}(\vartheta) \, d\tau d\vartheta d\sigma \\ &= 2^{\alpha-d} |\mathbb{S}^{d-2}| \int_{|k|}^{3|k|} (1 + |Y - \tau|)^{-j} (\tau - |k|)^{d-1-\alpha} (1 + (|k|/\tau))^{d-1-\alpha} \\ &\times \int_0^{\pi} \left(1 - 2(|k|/\tau) \cos(\vartheta) + (|k|/\tau)^2\right) (1 - (|k|/\tau) \cos(\vartheta))^{\alpha-d-1} \sin^{d-2}(\vartheta) \, d\vartheta d\tau. \end{aligned}$$

The term $1 + (|k|/\tau)$ in the last double integral is bounded between 1 and 2, and it is negligible. Hence,

$$\begin{aligned} &\int_{\{|x|+|k-x|\leq 3|k|\}} (1 + |Y - |x| - |k - x||)^{-j} |x|^{-\alpha} dx \\ &\leq C \int_0^{2|k|} (1 + |Y - |k| - \tau|)^{-j} \tau^{d-1-\alpha} E(|k|/(|k| + \tau), \alpha, d) \, d\tau. \end{aligned}$$

The integral over $\{|x|+|k-x|\geq 3|k|\}$ is estimated with the same change of variables. □

**Lemma 7.** *If $-\infty<\beta<+\infty$, $d\geq 2$, there exists $C$ such that for every $0<\varepsilon<1$,*

$$E\left(\varepsilon,\beta,d\right)\int_0^{\pi}\left(1-2\varepsilon\cos\left(\vartheta\right)+\varepsilon^2\right)\left(1-\varepsilon\cos\left(\vartheta\right)\right)^{\beta-d-1}\sin^{d-2}\left(\vartheta\right)d\vartheta$$
$$\leq\begin{cases}C\left(1-\varepsilon\right)^{\beta-(d+1)/2} & \text{if } \beta<(d+1)/2,\\ C\left(1-\log\left(1-\varepsilon\right)\right) & \text{if } \beta=(d+1)/2,\\ C & \text{if } \beta>(d+1)/2.\end{cases}$$

*Proof.* When $\beta>d+1$ there is nothing to prove. Assume that $\beta\leq d+1$. Again, when $0<\varepsilon<1/2$, there is nothing to prove. When $1/2\leq\varepsilon<1$, the integral over $\pi/2\leq\vartheta\leq\pi$ is bounded independently of $\varepsilon$, and when $0\leq\vartheta\leq\pi/2$ one has $\sin\left(\vartheta\right)\leq\vartheta$ and $1-\vartheta^2/2\leq\cos\left(\vartheta\right)\leq 1-4\vartheta^2/\pi^2$. Hence one ends up with the integral

$$\int_0^{\pi/2}\left(1-2\varepsilon\left(1-\vartheta^2/2\right)+\varepsilon^2\right)\left(1-\varepsilon\left(1-4\vartheta^2/\pi^2\right)\right)^{\beta-d-1}\vartheta^{d-2}d\vartheta$$
$$=\int_0^{\pi/2}\left(\left(1-\varepsilon\right)^2+\varepsilon\vartheta^2\right)\left(1-\varepsilon+4\varepsilon\vartheta^2/\pi^2\right)^{\beta-d-1}\vartheta^{d-2}d\vartheta$$
$$\leq C\left(1-\varepsilon\right)^{\beta-d+1}\int_0^{1-\varepsilon}\vartheta^{d-2}d\vartheta+C\left(1-\varepsilon\right)^{\beta-d-1}\int_{1-\varepsilon}^{\sqrt{1-\varepsilon}}\vartheta^{d}d\vartheta+C\int_{\sqrt{1-\varepsilon}}^{\pi/2}\vartheta^{2\beta-d-2}d\vartheta$$
$$\leq C\left(1-\varepsilon\right)^{\beta}+C\left(1-\varepsilon\right)^{\beta-(d+1)/2}+\begin{cases}C\left(1-\varepsilon\right)^{\beta-(d+1)/2} & \text{if } \beta<(d+1)/2,\\ -C\log\left(1-\varepsilon\right) & \text{if } \beta=(d+1)/2,\\ C & \text{if } \beta>(d+1)/2.\end{cases}$$

□

**Lemma 8.** *(1) If $j>0$ and $-1<\beta<j-1$, there exists $C$ such that for every $-\infty<X<+\infty$,*

$$\int_0^{+\infty}\left(1+|X-\tau|\right)^{-j}\tau^{\beta}d\tau\leq\begin{cases}C\left(1+|X|\right)^{\beta+1-j} & \text{if } 0<j<1,\\ C\left(1+|X|\right)^{\beta}\log\left(1+|X|\right) & \text{if } j=1,\\ C\left(1+|X|\right)^{\beta} & \text{if } j>1.\end{cases}$$

*(2) If $\beta>-1$, then for every $j$ there exists $C$ such that for every $-\infty<X<+\infty$,*

$$\int_0^{1}\left(1+|X-\tau|\right)^{-j}\tau^{\beta}\log\left(\tau\right)d\tau\leq C\left(1+|X|\right)^{-j}.$$

*(3) If $0 < j < 1$, there exists $C$ such that for every $0 \leq X < +\infty$ and $2 \leq Y < +\infty$,*

$$\int_0^Y (1 + |X - \tau|)^{-j} \tau^{j-1} d\tau \leq C \log(Y).$$

*(4) If $0 < j < 1$ and $\beta < -1$, there exists $C$ such that for every $0 \leq X < +\infty$ and $2 \leq Y < +\infty$,*

$$\int_Y^{+\infty} (1 + |X - \tau|)^{-j} \tau^{\beta} d\tau \leq C Y^{1+\beta-j}.$$

The proof of the lemma reduces to some elementary and boring computations. We include details for sake of completeness.

*Proof.* (1) If $X \leq 0$, then

$$\int_0^{+\infty} (1 + |X - \tau|)^{-j} \tau^{\beta} d\tau \leq (1 + |X|)^{-j} \int_0^{1+|X|} \tau^{\beta} d\tau + \int_{1+|X|}^{+\infty} \tau^{\beta-j} d\tau \leq C (1 + |X|)^{\beta+1-j}.$$

If $0 \leq X \leq 1$, then

$$\int_0^{+\infty} (1 + |X - \tau|)^{-j} \tau^{\beta} d\tau \leq \int_0^2 \tau^{\beta} d\tau + \int_2^{+\infty} \tau^{\beta-j} d\tau \leq C.$$

If $X \geq 1$, then

$$\begin{aligned} &\int_0^{+\infty} (1 + |X - \tau|)^{-j} \tau^{\beta} d\tau \\ &\leq 2^j X^{-j} \int_0^{X/2} \tau^{\beta} d\tau + \max\left\{2^{-\beta}, 2^{\beta}\right\} X^{\beta} \int_{X/2}^{2X} (1 + |X - \tau|)^{-j} d\tau + 2^j \int_{2X}^{+\infty} \tau^{\beta-j} d\tau \\ &\leq \begin{cases} CX^{\beta+1-j} & \text{if } 0 < j < 1, \\ CX^{\beta} \log(1 + X) & \text{if } j = 1, \\ CX^{\beta} & \text{if } j > 1. \end{cases} \end{aligned}$$

(2) It suffices to observe that there exists $C$ such that for every $X$,

$$\max_{0 \leq \tau \leq 1} \left\{ (1 + |X - \tau|)^{-j} \right\} \leq C (1 + |X|)^{-j}.$$

(3) If $Y \leq 2X$, then

$$\int_0^Y (1 + |X - \tau|)^{-j} \tau^{j-1} d\tau \leq 2^j X^{-j} \int_0^{Y/2} \tau^{j-1} d\tau + 2^{1-j} Y^{j-1} \int_{Y/2}^{2X} (1 + |X - \tau|)^{-j} d\tau \leq C.$$

If $Y \geq 2X$, then

$$\begin{aligned}
&\int_0^Y (1+|X-\tau|)^{-j}\,\tau^{j-1}d\tau \\
&\leq 2^j X^{-j}\int_0^{X/2}\tau^{j-1}d\tau + 2^{1-j}X^{j-1}\int_{X/2}^{2X}(1+|X-\tau|)^{-j}\,d\tau + 2^j\int_{2X}^{Y}\tau^{-1}d\tau \\
&\leq C + C + C\log(Y)\,.
\end{aligned}$$

(4) If $Y \leq X/2$, then

$$\begin{aligned}
&\int_Y^{+\infty}(1+|X-\tau|)^{-j}\,\tau^{\beta}d\tau \\
&\leq 2^j X^{-j}\int_Y^{X/2}\tau^{\beta}d\tau + 2^{\beta}X^{\beta}\int_{X/2}^{2X}(1+|X-\tau|)^{-j}\,d\tau + 2^j\int_{2X}^{+\infty}\tau^{\beta-j}d\tau \\
&\leq CX^{-j}Y^{1+\beta} + CX^{1+\beta-j} + CX^{1+\beta-j} \leq CY^{1+\beta-j}.
\end{aligned}$$

If $X/2 \leq Y \leq 2X$, then

$$\begin{aligned}
&\int_Y^{+\infty}(1+|X-\tau|)^{-j}\,\tau^{\beta}d\tau \\
&\leq Y^{\beta}\int_{X/2}^{2X}(1+|X-\tau|)^{-j}\,d\tau + 2^j\int_{2X}^{+\infty}\tau^{\beta-j}d\tau \leq CY^{1+\beta-j}.
\end{aligned}$$

If $Y \geq 2X$, then

$$\int_Y^{+\infty}(1+|X-\tau|)^{-j}\,\tau^{\beta}d\tau \leq 2^jY^{-j}\int_Y^{+\infty}\tau^{\beta}d\tau \leq CY^{1+\beta-j}.$$

□

**Lemma 9.** *(1) If* $\mathrm{Re}\,(z) > d/2$*, there exists* $C > 0$ *such that for every* $-\infty < R < +\infty$*,*

$$\int_{\mathbb{R}}\psi(r-R)\int_{\mathbb{T}^d}|\Phi(z,r,x)|^2dxdr \leq C.$$

*(2) If* $\mathrm{Re}\,(z) \geq d/2$*, there exists* $C > 0$ *such that for every* $-\infty < R < +\infty$ *and* $0 < \delta < 1/2$*,*

$$\int_{\mathbb{R}}\psi(r-R)\int_{\mathbb{T}^d}|\Phi(\delta,z,r,x)|^2dxdr \leq \begin{cases} C & \text{if } \mathrm{Re}\,(z) > d/2, \\ C\log(1/\delta) & \text{if } \mathrm{Re}\,(z) = d/2. \end{cases}$$

*Proof.* (1) is the limit of (2) when $\delta \to 0+$. It then suffices to prove (2). This follows immediately by Plancherel formula applied to the function$\Phi(\delta, z, r, x)$,

$$\begin{aligned}
&\int_{\mathbb{R}} \int_{\mathbb{T}^d} |\Phi(\delta, z, r, x)|^2 dx dr \\
&= \int_{\mathbb{R}} \psi(r-R) \sum_{m \in \mathbb{Z}^d,\, m \neq 0} |\widehat{\varphi}(\delta m)|^2 |m|^{-2\mathrm{Re}(z)} dr \\
&= \int_{\mathbb{R}} \psi(r)\, dr \sum_{m \in \mathbb{Z}^d,\, m \neq 0} |\widehat{\varphi}(\delta m)|^2 |m|^{-2\mathrm{Re}(z)} \\
&\leq C \sum_{m \in \mathbb{Z}^d,\, m \neq 0} (1 + \delta |m|)^{-j} |m|^{-2\mathrm{Re}(z)}.
\end{aligned}$$

□

The following lemma is an estimate of the $L(p)$ norms of the functions $\Phi(z, r, x)$ and $\Phi(\delta, z, r, x)$ when $p = 4$ and the space dimension $d \geq 3$. In dimension $d = 2$ the relevant exponent is $p = 6$, and this will be considered later.

**Lemma 10.** *(1) If* $\mathrm{Re}(z) > (3d-1)/4$, *there exists* $C > 0$ *such that for every* $-\infty < R < +\infty$,

$$\int_{\mathbb{R}} \psi(r-R) \int_{\mathbb{T}^d} |\Phi(\delta, z, r, x)|^4 dx dr \leq C.$$

*(2) If* $\mathrm{Re}(z) \geq (3d-1)/4$, *there exists* $C > 0$ *such that for every* $-\infty < R < +\infty$ *and* $0 < \delta < 1/2$,

$$\begin{aligned}
&\int_{\mathbb{R}} \psi(r-R) \int_{\mathbb{T}^d} |\Phi(\delta, z, r, x)|^4 dx dr \\
&\leq \begin{cases} C & \text{if } \mathrm{Re}(z) > (3d-1)/4, \\ C \log(1/\delta) & \text{if } \mathrm{Re}(z) = (3d-1)/4, \text{ and if } d > 3, \\ C \log^2(1/\delta) & \text{if } \mathrm{Re}(z) = (3d-1)/4, \text{ and if } d = 3. \end{cases}
\end{aligned}$$

*Proof.* (1) is the limit of (2) when $\delta \to 0+$. It then suffices to prove (2). Set $\alpha = \mathrm{Re}(z)$. By the above Lemma 4 with $N = 2$, it suffices to estimate

$$\begin{aligned}
&\int_{\mathbb{R}^d} (1 + \delta|k|)^{-j} \int_{|m|, |k-m| > 1} |m|^{-\alpha} |k-m|^{-\alpha} \\
&\times \int_{|u|, |k-u| > 1} |u|^{-\alpha} |k-u|^{-\alpha} (1 + ||m| + |k-m| - |u| - |k-u||)^{-j}\, du dm dk.
\end{aligned}$$

Notice that we have canceled all the cutoff functions in the variables $m$, $k-m$, $u$, $k-u$. By Lemma 5 the integral over the set $\{|k|\leq 2\}$ is bounded by

$$\begin{aligned}&\int_{\{|k|\leq 2\}}\int_{|m|,|k-m|>1}|m|^{-\alpha}|k-m|^{-\alpha}\int_{|u|,|k-u|>1}|u|^{-\alpha}|k-u|^{-\alpha}dudmdk\\&=\int_{|k|\leq 2}\left(\int_{|m|,|k-m|>1}|m|^{-\alpha}|k-m|^{-\alpha}dm\right)^2dk\leq C.\end{aligned}$$

Let us now consider the integral over the set $\{|k|\geq 2\}$,

$$\begin{aligned}&\int_{|k|\geq 2}(1+\delta|k|)^{-j}\int_{|m|,|k-m|>1}|m|^{-\alpha}|k-m|^{-\alpha}\\&\times\int_{|u|,|k-u|>1}|u|^{-\alpha}|k-u|^{-\alpha}\left(1+\left||m|+|k-m|-|u|-|k-u|\right|\right)^{-j}dudmdk.\end{aligned}$$

Assume first $d\geq 4$. Since this integral is decreasing in $\alpha$, one can assume without loss of generality that $(3d-1)/4\leq\alpha<d-1$. By Lemma 6, the inner integral

$$\int_{|u|,|k-u|>1}|u|^{-\alpha}|k-u|^{-\alpha}\left(1+\left||m|+|k-m|-|u|-|k-u|\right|\right)^{-j}du$$

is bounded by

$$C|k|^{-\alpha}\int_0^{+\infty}\left(1+\left||m|+|k-m|-|k|-\tau\right|\right)^{-j}\tau^{d-1-\alpha}E\left(|k|/\left(|k|+\tau\right),\alpha,d\right)d\tau.$$

By Lemma 7 and Lemma 8, this is bounded by

$$\begin{aligned}&C|k|^{-\alpha}\int_0^{+\infty}\left(1+\left||m|+|k-m|-|k|-\tau\right|\right)^{-j}\tau^{d-1-\alpha}d\tau\\&\leq C|k|^{-\alpha}\left(1+|m|+|k-m|-|k|\right)^{d-1-\alpha}.\end{aligned}$$

Thus, the goal estimate becomes

$$\begin{aligned}&\int_{|k|>2}(1+\delta|k|)^{-j}|k|^{-\alpha}\int_{\mathbb{R}^d}|m|^{-\alpha}|k-m|^{-\alpha}\left(1+|m|+|k-m|-|k|\right)^{d-1-\alpha}dmdk\\&\leq C\int_{|k|>2}(1+\delta|k|)^{-j}|k|^{-\alpha}\int_{|k|\leq|m|+|k-m|\leq 3|k|}|m|^{-\alpha}|k-m|^{-\alpha}|k|^{d-1-\alpha}dmdk\\&+C\int_{|k|>2}(1+\delta|k|)^{-j}|k|^{-\alpha}\int_{3|k|\leq|m|+|k-m|}|m|^{-\alpha}|k-m|^{-\alpha}\left||m|+|k-m|-|k|\right|^{d-1-\alpha}dmdk.\end{aligned}$$

The change of variables $m=|k|n$, with $w=k/|k|$, in the inner integrals gives

$$\int_{|k|>2}(1+\delta|k|)^{-j}|k|^{2d-1-4\alpha}\int_{1\leq|n|+|w-n|\leq 3}|n|^{-\alpha}|w-n|^{-\alpha}dndk$$

$$+\int_{|k|>2}(1+\delta|k|)^{-j}|k|^{2d-1-4\alpha}\int_{3\le|n|+|w-n|}|n|^{-\alpha}|w-n|^{-\alpha}||n|+|w-n|-1|^{d-1-\alpha}dndk$$
$$\le C\int_{|k|>2}(1+\delta|k|)^{-j}|k|^{2d-1-4\alpha}dk$$
$$\le\begin{cases}C & \text{if } \alpha>(3d-1)/4,\\ C\log(2/\delta) & \text{if } \alpha=(3d-1)/4.\end{cases}$$

Assume now $d=3$ and $\alpha=(3d-1)/4=2$. By Lemma 6, the integral

$$\int_{|u|,|k-u|>1}|u|^{-2}|k-u|^{-2}(1+||m|+|k-m|-|u|-|k-u||)^{-j}du$$

is bounded by

$$C|k|^{-2}\int_0^{+\infty}(1+||m|+|k-m|-|k|-\tau|)^{-j}E(|k|/(|k|+\tau),2,3)\,d\tau.$$

By Lemma 7 and Lemma 8 this is bounded by

$$C|k|^{-2}\int_0^{+\infty}(1+||m|+|k-m|-|k|-\tau|)^{-j}(1+\log(1+|k|/\tau))\,d\tau$$
$$\le C|k|^{-2}(1+\log(1+|k|))\int_0^{+\infty}(1+||m|+|k-m|-|k|-\tau|)^{-j}d\tau$$
$$-C|k|^{-2}\int_0^1(1+||m|+|k-m|-|k|-\tau|)^{-j}\log(\tau)\,d\tau$$
$$\le C|k|^{-2}(1+\log(1+|k|)).$$

We used the inequality

$$1+\log(1+|k|/\tau)\le 1+\log(1+|k|)+\log(1+1/\tau).$$

The goal estimate becomes

$$\int_{|k|>2}(1+\delta|k|)^{-j}|k|^{-2}\log(|k|)\int_{\mathbb{R}^3}|m|^{-2}|k-m|^{-2}dmdk$$
$$\le C\int_{|k|>2}(1+\delta|k|)^{-j}|k|^{-3}\log(|k|)\,dk$$
$$\le C\log^2(1/\delta).$$

Finally, assume $d=3$ and $(3d-1)/4=2<\alpha<3$. By Lemma 6, the integral

$$\int_{|u|,|k-u|>1}|u|^{-\alpha}|k-u|^{-\alpha}(1+||m|+|k-m|-|u|-|k-u||)^{-j}du$$

is bounded by

$$C|k|^{-\alpha}\int_{0}^{+\infty}\left(1+||m|+|k-m|-|k|-\tau|\right)^{-j}\tau^{2-\alpha}E\left(|k|/\left(|k+|\tau\right),\alpha,3\right)d\tau.$$

By Lemma 7 and Lemma 8 this is bounded by

$$\begin{aligned}&C|k|^{-\alpha}\int_{0}^{+\infty}\left(1+||m|+|k-m|-|k|-\tau|\right)^{-j}\tau^{2-\alpha}d\tau\\&\leq C|k|^{-\alpha}\left(1+||m|+|k-m|-|k||\right)^{2-\alpha}.\end{aligned}$$

Thus, the goal estimate becomes

$$\int_{|k|>2}\left(1+\delta|k|\right)^{-j}|k|^{-\alpha}\int_{\mathbb{R}^d}|m|^{-\alpha}|k-m|^{-\alpha}\left(1+||m|+|k-m|-|k||\right)^{2-\alpha}dmdk.$$

Again by Lemma 6 and Lemma 7

$$\begin{aligned}&\int_{\mathbb{R}^d}|m|^{-\alpha}|k-m|^{-\alpha}\left(1+||m|+|k-m|-|k||\right)^{2-\alpha}dm\\&\leq C|k|^{-\alpha}\int_{0}^{2|k|}\left(1+\tau\right)^{2-\alpha}\tau^{2-\alpha}E\left(|k|/\left(|k|+\tau\right),\alpha,3\right)d\tau\\&+C\int_{2|k|}^{+\infty}\left(1+\tau\right)^{2-\alpha}\tau^{2-2\alpha}E\left(|k|/\left(|k|+\tau\right),2\alpha,3\right)d\tau\\&\leq C|k|^{-\alpha}\int_{0}^{2|k|}\left(1+\tau\right)^{2-\alpha}\tau^{2-\alpha}d\tau+C\int_{2|k|}^{+\infty}\left(1+\tau\right)^{2-\alpha}\tau^{2-2\alpha}d\tau\\&\leq C|k|^{5-3\alpha}.\end{aligned}$$

Finally, since $\alpha>2$,

$$\int_{|k|>2}\left(1+\delta|k|\right)^{-j}|k|^{5-4\alpha}dk\leq C.$$

□

In the following lemma the space dimension is $d=2$.

**Lemma 11.** *(1) If* $\mathrm{Re}\,(z)>3/2$*, there exists* $C>0$ *such that for every* $-\infty<R<+\infty$*,*

$$\int_{\mathbb{R}}\psi\left(r-R\right)\int_{\mathbb{T}^2}|\Phi\left(z,r,x\right)|^6dxdr\leq C.$$

*(2) If* $\operatorname{Re}(z) \geq 3/2$*, there exists* $C > 0$ *such that for every* $-\infty < R < +\infty$ *and* $0 < \delta < 1/2$*,*

$$\int_{\mathbb{R}} \psi(r - R) \int_{\mathbb{T}^2} |\Phi_\tau(\delta, z, r, x)|^6 dx dr \leq \begin{cases} C & \text{if } \operatorname{Re}(z) > 3/2, \\ C \log^4(1/\delta) & \text{if } \operatorname{Re}(z) = 3/2. \end{cases}$$

*Proof.* (1) is the limit of (2) when $\delta \to 0+$. It then suffices to prove (2). Set $\alpha = \operatorname{Re}(z)$. By the Lemma 4 with $N = 3$, it suffices to estimate

$$\begin{aligned}
&\int_{\mathbb{R}^2} (1 + \delta|k|)^{-j} \\
&\times \iint_{\{|m|,|n|,|k-m-n|>1\}} (1 + \delta|m|)^{-j} (1 + \delta|n|)^{-j} (1 + \delta|k - m - n|)^{-j} |m|^{-\alpha}|n|^{\alpha}|k - m - n|^{-\alpha} \\
&\times \iint_{\{|u|,|v|,|k-u-v|>1\}} (1 + \delta|u|)^{-j} (1 + \delta|v|)^{-j} (1 + \delta|k - u - v|)^{-j} |u|^{-\alpha}|v|^{-\alpha}|k - u - v|^{-\alpha} \\
&\times (1 + ||m| + |n| + |k - m - n| - |u| - |v| - |k - u - v||)^{-j} \, dudvdmdndk.
\end{aligned}$$

The above expression is decreasing in $\alpha$, so one can assume $3/2 \leq \alpha < 5/3$. Split $\mathbb{R}^2$ as $\{|k| \leq 2\} \cup \{|k| \geq 2\}$. Disregarding the cutoff functions in the variables $k$, $m$, $n$, $k - m - n$, $u$, $v$, $k - u - v$, the integral over the disc $\{|k| \leq 2\}$ is bounded by

$$\begin{aligned}
&\int_{\{|k|\leq 2\}} \left( \int_{\mathbb{R}^2} |m|^{-\alpha} \int_{\mathbb{R}^2} |n|^{-\alpha}|k - m - n|^{-\alpha} dmdn \right)^2 dk \\
&= C \int_{\{|k|\leq 2\}} \left( \int_{\mathbb{R}^2} |m|^{-\alpha}|k - m|^{2-2\alpha} dm \right)^2 dk \\
&= C \int_{\{|k|\leq 2\}} |k|^{8-6\alpha} dk \leq C.
\end{aligned}$$

Consider now the case $\{|k| \geq 2\}$, and assume first $\alpha > 3/2$. Disregarding all cutoff functions, an application of Lemma 6, Lemma 7, Lemma8, gives

$$\begin{aligned}
&\int_{\mathbb{R}^2} |v|^{-\alpha}|k - u - v|^{-\alpha} (1 + ||m| + |n| + |k - m - n| - |u| - |v| - |k - u - v||)^{-j} \, dv \\
&\leq C|k - u|^{-\alpha} \int_0^{+\infty} (1 + ||m| + |n| + |k - m - n| - |u| - |k - u| - \tau|)^{-j} \tau^{1-\alpha} \\
&\times E(|k - u|/(|k - u| + \tau), \alpha, 2) \, d\tau \\
&\leq C|k - u|^{-\alpha} \int_0^{+\infty} (1 + ||m| + |n| + |k - m - n| - |u| - |k - u| - \tau|)^{-j} \tau^{1-\alpha} d\tau
\end{aligned}$$

$$\leq C|k-u|^{-\alpha}\left(1+||m|+|n|+|k-m-n|-|u|-|k-u||\right)^{1-\alpha}.$$

Hence, again by Lemma 6, Lemma 7, Lemma 8,

$$\int_{\mathbb{R}^2}|u|^{-\alpha}\int_{\mathbb{R}^2}|v|^{-\alpha}|k-u-v|^{-\alpha}$$
$$\times\left(1+||m|+|n|+|k-m-n|-|u|-|v|-|k-u-v||\right)^{-j}dvdu$$
$$\leq C\int_{\mathbb{R}^2}|u|^{-\alpha}|k-u|^{-\alpha}\left(1+||m|+|n|+|k-m-n|-|u|-|k-u||\right)^{1-\alpha}du$$
$$\leq C|k|^{-\alpha}\int_0^{+\infty}\left(1+||m|+|n|+|k-m-n|-|k|-\tau|\right)^{1-\alpha}\tau^{1-\alpha}E\left(|k|/\left(|k|+\tau\right),\alpha,2\right)d\tau$$
$$\leq C|k|^{-\alpha}\int_0^{+\infty}\left(1+||m|+|n|+|k-m-n|-|k|-\tau|\right)^{1-\alpha}\tau^{1-\alpha}d\tau$$
$$\leq C|k|^{-\alpha}\left(1+||m|+|n|+|k-m-n|-|k||\right)^{3-2\alpha}$$
$$\leq C|k|^{-\alpha}.$$

Moreover,

$$\int_{\mathbb{R}^2}|m|^{-\alpha}\int_{\mathbb{R}^2}|n|^{-\alpha}|k-m-n|^{-\alpha}dmdn=C\int_{\mathbb{R}^2}|m|^{-\alpha}|k-m|^{2-2\alpha}dm=C|k|^{4-3\alpha}.$$

Finally, the integral over $\{|k|\geq 2\}$ gives

$$\int_{|k|\geq 2}|k|^{4-4\alpha}dk\leq C.$$

Now assume $\alpha=3/2$. Again one can delete the cutoff functions in $v$ and $k-u-v$. An application of Lemma 6, Lemma 7, Lemma 8, gives

$$\int_{\mathbb{R}^2}|v|^{-3/2}|k-u-v|^{-3/2}\left(1+||m|+|n|+|k-m-n|-|u|-|v|-|k-u-v||\right)^{-j}dv$$
$$\leq C|k-u|^{-3/2}\int_0^{+\infty}\left(1+||m|+|n|+|k-m-n|-|u|-|k-u|-\tau|\right)^{-j}\tau^{-1/2}$$
$$\times E\left(|k-u|/\left(|k-u|+\tau\right),3/2,2\right)d\tau$$
$$\leq C|k-u|^{-3/2}\int_0^{+\infty}\left(1+||m|+|n|+|k-m-n|-|u|-|k-u|-\tau|\right)^{-j}\tau^{-1/2}$$
$$\times\left(1+\log\left(1+|k-u|/\tau\right)\right)d\tau$$
$$\leq C|k-u|^{-3/2}\left(1+\log\left(1+|k-u|\right)\right)$$
$$\times\int_0^{+\infty}\left(1+||m|+|n|+|k-m-n|-|u|-|k-u|-\tau|\right)^{-j}\tau^{-1/2}d\tau$$

$$- C|k-u|^{-3/2} \int_0^1 \left(1 + ||m| + |n| + |k-m-n| - |u| - |k-u| - \tau|\right)^{-j} \tau^{-1/2} \log(\tau)\, d\tau$$
$$\leq C|k-u|^{-3/2} \left(1 + \log(1+|k-u|)\right) \left(1 + ||m| + |n| + |k-m-n| - |u| - |k-u||\right)^{-1/2}.$$

Observe that

$$(1+\delta|k|)^{-\varepsilon} (1+\delta|u|)^{-\varepsilon} \log(1+|k-u|)$$
$$\leq (1+\delta|k|)^{-\varepsilon} (1+\delta|u|)^{-\varepsilon} \left(\log(1+|k|) + \log(1+|u|)\right)$$
$$\leq (1+\delta|k|)^{-\varepsilon} \left(\log(1+\delta|k|) + \log(1+1/\delta)\right) + (1+\delta|u|)^{-\varepsilon} \left(\log(1+\delta|u|) + \log(1+1/\delta)\right)$$
$$\leq 2 \left( \sup_{t \geq 0} \left\{ (1+t)^{-\varepsilon} \log(1+t) \right\} + \log(1+1/\delta) \right).$$

Roughly speaking, this inequality allows to replace a variable $\log(1+|x-y|)$ with a constant $\log(1+1/\delta)$. In particular, if $0 < \delta < 1/2$,

$$(1+\delta|k|)^{-\varepsilon} (1+\delta|u|)^{-\varepsilon} \left(1 + \log(1+|k-u|)\right) \leq C \log(1/\delta).$$

By this inequality, and again by Lemma 6, Lemma 7, Lemma 8,

$$\int_{\mathbb{R}^2} (1+\delta|k|)^{-\varepsilon} (1+\delta|u|)^{-\varepsilon} |u|^{-3/2}$$
$$\times \int_{\mathbb{R}^2} |v|^{-3/2} |k-u-v|^{-3/2} \left(1 + ||m| + |n| + |k-m-n| - |u| - |v| - |k-u-v||\right)^{-j} dv du$$
$$\leq C \log(1/\delta) \int_{\mathbb{R}^2} |u|^{-3/2} |k-u|^{-3/2} \left(1 + ||m| + |n| + |k-m-n| - |u| - |k-u||\right)^{-1/2} du$$
$$\leq C \log(1/\delta)\, |k|^{-3/2} \int_0^{2|k|} \left(1 + ||m| + |n| + |k-m-n| - |k| - \tau|\right)^{-1/2} \tau^{-1/2}$$
$$\times E\left(|k|/\left(|k|+\tau\right), 3/2, 2\right) d\tau$$
$$+ C \log(1/\delta) \int_{2|k|}^{+\infty} \left(1 + ||m| + |n| + |k-m-n| - |k| - \tau|\right)^{-1/2} \tau^{-2} E\left(|k|/\left(|k|+\tau\right), 3, 2\right) d\tau$$
$$\leq C \log(1/\delta)\, |k|^{-3/2} \int_0^{2|k|} \left(1 + ||m| + |n| + |k-m-n| - |k| - \tau|\right)^{-1/2} \tau^{-1/2}$$
$$\times \left(1 + \log(1+|k|/\tau)\right) d\tau$$
$$+ C \log(1/\delta) \int_{2|k|}^{+\infty} \left(1 + ||m| + |n| + |k-m-n| - |k| - \tau|\right)^{-1/2} \tau^{-2} d\tau$$
$$\leq C \log(1/\delta)\, |k|^{-3/2} \log(|k|) \int_0^{2|k|} \left(1 + ||m| + |n| + |k-m-n| - |k| - \tau|\right)^{-1/2} \tau^{-1/2} d\tau$$
$$- C \log(1/\delta)\, |k|^{-3/2} \int_0^1 \left(1 + ||m| + |n| + |k-m-n| - |k| - \tau|\right)^{-1/2} \tau^{-1/2} \log(\tau)\, d\tau$$

$$+ C\log(1/\delta)\int_{2|k|}^{+\infty}(1+||m|+|n|+|k-m-n|-|k|-\tau|)^{-1/2}\tau^{-2}d\tau$$
$$\le C\log(1/\delta)|k|^{-3/2}\log^2(|k|)+C\log(1/\delta)|k|^{-3/2}+C\log(1/\delta)|k|^{-3/2}$$
$$\le C\log(1/\delta)|k|^{-3/2}\log^2(|k|).$$

Moreover,

$$\int_{\mathbb{R}^2}|m|^{-3/2}\int_{\mathbb{R}^2}|n|^{-3/2}|k-m-n|^{-3/2}dmdn = C\int_{\mathbb{R}^2}|m|^{-3/2}|k-m|^{-1}dm = C|k|^{-1/2}.$$

Finally, the integral over $\{|k|\ge 2\}$ gives

$$\log(1/\delta)\int_{|k|\ge 2}(1+\delta|k|)^{-j}|k|^{-2}\log^2(|k|)\,dk \le C\log^4(1/\delta).$$

□

**Lemma 12.** *The notation is as in the previous lemmas.*

*(1) Let* $d=2$, $\mathrm{Re}(z)\ge 3/2$, *and* $p<6$. *Then there exists a constant* $C$ *such that for every* $-\infty<R<+\infty$,

$$\left\{\int_{\mathbb{R}}\psi(r-R)\int_{\mathbb{T}^2}|\Phi(z,r,x)|^p dxdr\right\}^{1/p}\le C.$$

*(2) Let* $d=2$, $\mathrm{Re}(z)\ge 3/2$, *and* $p\le 6$. *Then there exists a constant* $C$ *such that for every* $-\infty<R<+\infty$,

$$\left\{\int_{\mathbb{R}}\psi(r-R)\int_{\mathbb{T}^2}|\Phi(\delta,z,r,x)|^p dxdr\right\}^{1/p}\le\begin{cases}C & \text{if } p<6,\\ C\log^{2/3}(1/\delta) & \text{if } p=6.\end{cases}$$

*(3) Let* $d\ge 3$, $\mathrm{Re}(z)\ge(d+1)/2$, *and* $p<2(d-1)/(d-2)$. *Then there exists a constant* $C$ *such that for every* $-\infty<R<+\infty$,

$$\left\{\int_{\mathbb{R}}\psi(r-R)\int_{\mathbb{T}^d}|\Phi(\delta,z,r,x)|^p dxdr\right\}^{1/p}\le C.$$

*(4) Let* $d\ge 3$, $\mathrm{Re}(z)\ge(d+1)/2$, *and* $p\le 2(d-1)/(d-2)$. *Then there exists a constant* $C$ *such that for every* $-\infty<R<+\infty$ *and* $0<\delta<1/2$,

$$\left\{\int_{\mathbb{R}}\psi(r-R)\int_{\mathbb{T}^d}|\Phi(\delta,z,r,x)|^p dxdr\right\}^{1/p}$$
$$\le\begin{cases}C & \text{if } p<2(d-1)/(d-2),\\ C\log^{2/p}(1/\delta) & \text{if } p=2(d-1)/(d-2) \text{ and } d=3,\\ C\log^{1/p}(1/\delta) & \text{if } p=2(d-1)/(d-2) \text{ and } d>3.\end{cases}$$

*Proof.* Assume that $z = (d+1)/2$. The cases $\mathrm{Re}(z) \geq (d+1)/2$ is similar. The case $d = 2$ and $p = 6$ is contained in Lemma 4, and the case $d = 3$ and $p = 4$ is contained in Lemma 10. The other cases follow from these lemmas, via complex interpolation. For the definition of the complex interpolation method and the complex interpolation of $L(p)$ spaces, see for example Chapter 4 and Chapter 5 of Bergh and Lofstrom. Here we recall the relevant result: Let $\mathbb{X}$ be a measure space, $1 \leq a < b \leq +\infty$, $-\infty < A < B < +\infty$, and let $\Phi(z)$ be a function with values in $L^a(\mathbb{X}) + L^b(\mathbb{X})$, continuous and bounded on the closed strip $\{A \leq \mathrm{Re}(z) \leq B\}$ and analytic on the open strip $\{A < \mathrm{Re}(z) < B\}$. Assume that there exist constants $H$ and $K$ such that for every $-\infty < t < +\infty$,

$$\begin{cases} \|\Phi(A+it)\|_{L^a(\mathbb{X})} \leq H, \\ \|\Phi(B+it)\|_{L^a(\mathbb{X})} \leq K. \end{cases}$$

If $1/p = (1-\vartheta)/a + \vartheta/b$, with $0 < \vartheta < 1$, then

$$\|\Phi((1-\vartheta)A + \vartheta B)\|_{L^p(\mathbb{X})} \leq H^{1-\vartheta} K^{\vartheta}.$$

In (4) the analytic function is $\Phi(\delta, z, r, x)$, the measure space is $\mathbb{R} \times \mathbb{T}^d$ with measure $\psi(r-R)\,drdx$, $a = 2$, $b = 4$, $A = d/2 + \varepsilon$, $B = (3d-1)/4 + \varepsilon$, with $\varepsilon \geq 0$. Set

$$\frac{d+1}{2} = (1-\vartheta)A + \vartheta B.$$

This gives

$$\vartheta = \frac{2-4\varepsilon}{d-1},$$

and

$$\frac{1}{p} = \frac{(1-\vartheta)}{a} + \frac{\vartheta}{b} = \frac{d-2+2\varepsilon}{2d-2}.$$

When $\varepsilon > 0$ and $p < (2d-2)/(d-2)$,

$$\left\{ \int_{\mathbb{R}} \psi(r-R) \int_{\mathbb{T}^d} |\Phi(\delta, (d+1)/2, r, x)|^p dxdr \right\}^{1/p} \leq C.$$

When $\varepsilon = 0$ and $p = (2d-2)/(d-2)$ and $d > 3$,

$$\left\{ \int_{\mathbb{R}} \psi(r-R) \int_{\mathbb{T}^d} |\Phi(\delta, (d+1)/2, r, x)|^p dxdr \right\}^{1/p} \leq C \log^{1/p}(1/\delta).$$

This gives (4). The proof of (3) is similar. And also the proof of (1) and (2) is similar, and it follows by complex interpolation with $a = 2$, $b = 6$,$A = 1 + \varepsilon$, $B = 3/2 + \varepsilon$, with $\varepsilon \geq 0$. □

*Proof.* (of the Theorem 1) One has

$$\left\{ \int_{\mathbb{R}} \psi\left(r-R\right) \int_{\mathbb{T}^d} |r^{-\frac{d-1}{2}} \mathcal{D}\left(r\Omega - x\right)|^p dx dr \right\}^{1/p}$$
$$\leq \sum_{\sigma,\tau=0,1} \sum_{\ell=0}^{h} |b_\ell\left(d\right)| \left\{ \int_{\mathbb{R}} \psi\left(r-R\right) \int_{\mathbb{T}^d} r^{-\ell p} |\Phi_\tau^M\left(\delta, \left(d+2\ell+1\right)/2, r+\left(-1\right)^\sigma \delta, x\right)|^p dx dr \right\}^{1/p}$$
$$+ \left\{ \int_{\mathbb{R}} \psi\left(r-R\right) |r^{-(d-1)/2} \mathcal{R}_h\left(r,\delta\right)|^p dr \right\}^{1/p}.$$

If $h \geq \left(d-3\right)/2$ and $\delta = R^{-(d-1)/2}$ the last term is bounded and, with this choice of $\delta$, each term in the double sum is estimated by the previous lemma. □

# Chapter 2

# Mixed $L^p(L^2)$ norms of the lattice point discrepancy

As we have seen in the previous chapter, the discrepancy between the volume and the number of integer points in $r\Omega - x$, a dilated by a factor $r$ and translated by a vector $x$ of a domain $\Omega$ in $\mathbb{R}^d$, is

$$\mathcal{D}(r\Omega - x) = \sum_{k\in\mathbb{Z}^d} \chi_{r\Omega-x}(k) - r^d |\Omega| .$$

Here we want to estimate the mixed norm $L^p(L^2)$ of the discrepancy:

$$\left\{ \int_{\mathbb{T}^d} \left[ \int_{\mathbb{R}} |\mathcal{D}(r\Omega - x)|^2 \, d\mu(r) \right]^{p/2} dx \right\}^{1/p} .$$

First some notation. If $d\mu(r)$ is a finite Borel measure on the line $-\infty < r < +\infty$, and if $0 < H < +\infty$ and $-\infty < R < +\infty$, the dilated and translated measure $d\mu_{H,R}(r)$ is defined by

$$\mu_{H,R}\{\Omega\} = \mu\left\{H^{-1}(\Omega - R)\right\} .$$

Alternatively, by duality with continuous bounded functions,

$$\int_{\mathbb{R}} f(r) \, d\mu_{H,R}(r) = \int_{\mathbb{R}} f(R + Hr) \, d\mu(r) .$$

With this definition, the Fourier transform of $d\mu(r)$ and $d\mu_{H,R}(r)$ are related by the equation

$$\widehat{\mu}_{H,R}(\zeta) = \int_{\mathbb{R}} e^{-2\pi i\zeta r} d\mu_{H,R}(r)$$

$$= \int_{\mathbb{R}} e^{-2\pi i\zeta(R+Hr)} d\mu(r) = e^{-2\pi iR\zeta} \widehat{\mu}(H\zeta).$$

Recall that the Fourier dimension of a measure is the supremum of all $\sigma$ such that there exists $C$ such that $|\widehat{\mu}(\zeta)| \leq C|\zeta|^{-\sigma/2}$ (see 4.4 in [11]). Since the $L^2$ growth of the discrepancy $\mathcal{D}(r\Omega - x)$ is of the order of $r^{(d-1)/2}$, we call $r^{-(d-1)/2}\mathcal{D}(r\Omega - x)$ the normalized discrepancy. Our main result (Theorem 3) can be seen as an estimate of the Fourier dimension of the set where this normalized discrepancy may be large.

The main results of this chapter are the following.

**Theorem 3.** *Assume that $d\mu(r)$ is a Borel probability measure on $\mathbb{R}$, with support in $\varepsilon < r < \delta$, with $\delta > \varepsilon > 0$, and assume that the Fourier transform of $d\mu(r)$ has the decay*

$$|\widehat{\mu}(\zeta)| \leq B(1+|\zeta|)^{-\beta},$$

*for some $\beta \geq 0$ and $B > 0$. Assume that $\Omega$ is a convex set in $\mathbb{R}^d$, with a smooth boundary with strictly positive Gaussian curvature. Finally assume one of the following rows:*

$$\begin{cases} d=2, & 0 \leq \beta < 1, & A = \frac{4}{1-\beta}, & \alpha = \frac{1+\beta}{4}, \\ d=2, & \beta = 1, & A = +\infty, & \alpha = 1, \\ d=2, & \beta > 1, & A = +\infty, & \alpha = 1/2, \end{cases}$$

$$\begin{cases} d=3 & 0 \leq \beta \leq 1/2, & A = \frac{3-2\beta}{1-\beta}, & \alpha = \frac{1-\beta}{3-2\beta}, \\ d=3 & 1/2 \leq \beta < 1, & A = \frac{6}{2-\beta}, & \alpha = \frac{1+\beta}{6}, \\ d=3 & \beta = 1, & A = 6, & \alpha = 5/6, \\ d=3 & \beta > 1 & A = 6, & \alpha = 1/3, \end{cases}$$

$$\begin{cases} d \geq 4, & 0 \leq \beta < 1, & A = \frac{2d-4\beta}{d-1-2\beta}, & \alpha = \frac{d-1-2\beta}{2d-4\beta}, \\ d \geq 4, & \beta = 1, & A = \frac{2d-4}{d-3}, & \alpha = \frac{d-1}{2d-4}, \\ d \geq 4, & \beta > 1, & A = \frac{2d-4}{d-3}, & \alpha = \frac{d-3}{2d-4}. \end{cases}$$

*Then the following hold:*

*(1) If $p < A$, then there exists $C$ such that for every $H, R \geq 1$,*

$$\left\{ \int_{\mathbb{T}^d} \left[ \int_{\mathbb{R}} \left| r^{-(d-1)/2} \mathcal{D}(r\Omega - x) \right|^2 d\mu_{H,R}(r) \right]^{p/2} dx \right\}^{1/p} \leq C \left( \frac{1}{p} - \frac{1}{A} \right)^{-\alpha}.$$

*(2) If $p = A$, then there exists $C$ such that for every $H, R \geq 1$,*

$$\left\{ \int_{\mathbb{T}^d} \left[ \int_{\mathbb{R}} \left| r^{-(d-1)/2} \mathcal{D}(r\Omega - x) \right|^2 d\mu_{H,R}(r) \right]^{p/2} dx \right\}^{1/p} \leq C \log^{\alpha}(1+R).$$

*(3) If $\beta > 0$ and $p < A$, the sequence of functions*

$$\int_{\mathbb{R}} \left| r^{-(d-1)/2} \mathcal{D}\left(r\Omega - x\right) \right|^2 d\mu_{H,R}\left(r\right)$$

*has a limit $\mathcal{G}(x)$ in the norm of $L^{p/2}\left(\mathbb{T}^d\right)$ as $H \to +\infty$. In particular,*

$$\lim_{H \to +\infty} \left\{ \int_{\mathbb{T}^d} \left[ \int_{\mathbb{R}} \left| r^{-(d-1)/2} \mathcal{D}\left(r\Omega - x\right) \right|^2 d\mu_{H,R}\left(r\right) \right]^{p/2} dx \right\}^{1/p}$$
$$= \left\{ \int_{\mathbb{T}^d} \left| \mathcal{G}\left(x\right) \right|^{p/2} dx \right\}^{1/p}.$$

The growth of the norm of the discrepancy in this theorem allows to extrapolate some Orlicz type estimates at the critical indexes $p = A$.

**Corollary 1.** *(1) Assume one of the following rows:*

$$\begin{cases} d = 2, & \beta = 1, & \alpha = 2, & \gamma < 2/e, \\ d = 2, & \beta > 1, & \alpha = 1, & \gamma < 1/e. \end{cases}$$

*Then there exists $C > 0$ such that for every $H, R \geq 1$,*

$$\int_{\mathbb{T}^2} \exp\left( \gamma \left[ \int_{\mathbb{R}} \left| r^{-(d-1)/2} \mathcal{D}\left(r\Omega - x\right) \right|^2 d\mu_{H,R}\left(r\right) \right]^{1/\alpha} \right) dx \leq C.$$

*(2) Assume one of the following rows:*

$$\left\{ \; d = 2, \quad 0 \leq \beta < 1, \quad p = 4/\left(1 - \beta\right), \quad \gamma > 2/\left(1 - \beta\right), \right.$$

$$\begin{cases} d = 3, & 0 \leq \beta \leq 1/2, & p = \left(3 - 2\beta\right)/\left(1 - \beta\right), & \gamma > 2, \\ d = 3, & 1/2 \leq \beta < 1, & p = 6/\left(2 - \beta\right), & \gamma > 3/\left(2 - \beta\right), \\ d = 3, & \beta = 1, & p = 6, & \gamma > 6, \\ d = 3, & \beta > 1, & p = 6, & \gamma > 3, \end{cases}$$

$$\begin{cases} d \geq 4, & 0 \leq \beta < 1, & p = \left(2d - 4\beta\right)/\left(d - 1 - 2\beta\right), & \gamma > 2, \\ d \geq 4, & \beta = 1, & p = \left(2d - 4\right)/\left(d - 3\right), & \gamma > \left(2d - 4\right)/\left(d - 3\right), \\ d \geq 4, & \beta > 1, & p = \left(2d - 4\right)/\left(d - 3\right), & \gamma > 2. \end{cases}$$

*Then there exists $C$ such that for every $H, R \geq 1$,*

$$\int_{\mathbb{T}^d} \left[ \int_{\mathbb{R}} \left| r^{-(d-1)/2} \mathcal{D}\left(r\Omega - x\right) \right|^2 d\mu_{H,R}\left(r\right) \right]^{p/2}$$
$$\times \log^{-\gamma}\left( 2 + \int_{\mathbb{R}} \left| r^{-(d-1)/2} \mathcal{D}\left(r\Omega - x\right) \right|^2 d\mu_{H,R}\left(r\right) \right) dx \leq C.$$

For *"generic"* convex sets, the theorem can be slightly strengthened.

**Corollary 2.** *If the support function of the convex set* $g(x)=\sup_{y\in\Omega}\{x\cdot y\}$ *has the property that there exists* $C$ *such that for every* $m$ *in* $\mathbb{Z}^d$ *the equation* $g(m)=g(n)$ *has at most* $C$ *solutions* $n$ *in* $\mathbb{Z}^d$, *then the limit function* $\mathcal{G}(x)$ *in the Theorem 3 (3) is bounded and continuous in* $\mathbb{T}^d$. *If the support function is injective when restricted to the integers, that is* $g(m)\neq g(n)$ *for every* $m,n\in\mathbb{Z}^d$ *with* $m\neq n$, *then this limit function* $\mathcal{G}(x)$ *is constant.*

The family of compact convex sets endowed with the Hausdorff metric is a complete metric space. We will see that the collection of convex sets with injective support functions is an intersection of a countable family of open dense sets. In particular, it can be shown that the above corollary applies to almost every ellipsoid $\{|M(x-p)|\leq 1\}$, but not to the ball $\{|x|\leq 1\}$.

The results of G.H.Hardy in [13], E.Landau in [28] and A.E. Ingham in [22] imply that in Theorem 3, when $d=2$ and $\beta>1$ the assumption $p<+\infty$ cannot be replaced by the equality $p=+\infty$. In the other cases we do not know if the indexes in the above theorem and corollaries are best possible. Anyhow, the following holds.

**Theorem 4.** *(1) Assume that* $\Sigma=\{|x|\leq 1\}$ *is a ball in* $\mathbb{R}^d$ *with* $d\geq 4$, *and that* $d\mu(r)$ *is a Borel probability measure on* $\mathbb{R}$. *Then for every* $p>2d/(d-3)$,

$$\limsup_{H,R\to+\infty}\left\{\int_{\mathbb{T}^d}\left[\int_{\mathbb{R}}\left|r^{-(d-1)/2}\mathcal{D}(r\Sigma-x)\right|^2 d\mu_{H,R}(r)\right]^{p/2}dx\right\}^{1/p}=+\infty.$$

*(2) If in addition the Fourier transform of* $d\mu(r)$ *vanishes at infinity, that is*

$$\lim_{|\zeta|\to+\infty}\{|\widehat{\mu}(\zeta)|\}=0,$$

*the supremum limit can be replaced by a limit,*

$$\lim_{H,R\to+\infty}\left\{\int_{\mathbb{T}^d}\left[\int_{\mathbb{R}}\left|r^{-(d-1)/2}\mathcal{D}(r\Sigma-x)\right|^2 d\mu_{H,R}(r)\right]^{p/2}dx\right\}^{1/p}=+\infty.$$

The proof of Theorem 4 reduces essentially to an estimate of the norm in $L^{p/2}(\mathbb{T}^d)$ of the function $\mathcal{G}(x)$ which appears as a limit of the discrepancy in Theorem 3. While the limit function associated to the ball $\{|x|\leq 1\}$ is unbounded, for a generic convex the limit function is constant. In particular, we do not know if the statement of the theorem for the ball also applies to all convex sets.

Let us conclude this introduction with a few examples.

**Example.** If $d\mu(r)$ is the uniformly distributed measure in $\{0<r<1\}$, then the $L^p(L^2)$ mixed norm in the theorem is

$$\left\{\int_{\mathbb{T}^d}\left[\frac{1}{H}\int_R^{R+H}\left|r^{-(d-1)/2}\mathcal{D}(r\Omega-x)\right|^2dr\right]^{p/2}\right\}^{1/p}.$$

The Fourier transform of the uniformly distributed measure in $\{0<r<1\}$ has decay $\beta=1$,

$$\widehat{\mu}(\zeta)=\int_0^1 e^{-2\pi i\zeta r}dr=e^{-\pi i\zeta}\frac{\sin(\pi\zeta)}{\pi\zeta}.$$

On the other hand, if $\psi(r)$ is a non negative smooth function with integral one and support in $0\le r\le 1$, one can considers a smoothed average

$$\left\{\int_{\mathbb{T}^d}\left[\int_{\mathbb{R}}\left|r^{-(d-1)/2}\mathcal{D}(r\Omega-x)\right|^2H^{-1}\psi\left(H^{-1}(r-R)\right)dr\right]^{p/2}\right\}^{1/p}.$$

This smoothed average is of equivalent to the uniform average over $\{R<r<R+H\}$, but the decay of the Fourier transform $\widehat{\psi}(\zeta)$ is faster than any power $\beta$. Hence for the uniformly distributed measure in $\{0<r<1\}$ the theorem applies with the indexes corresponding to $\beta>1$:

1) if $d=2$

$$\left\{\int_{\mathbb{T}^d}\left[\frac{1}{H}\int_R^{R+H}\left|r^{-(d-1)/2}\mathcal{D}(r\Omega-x)\right|^2dr\right]^{p/2}\right\}^{1/p}\le\begin{cases}Cp^{1/2} & \text{if } p<+\infty,\\ C\log^{1/2}(1+R) & \text{if } p=+\infty.\end{cases}$$

2) if $d=3$

$$\left\{\int_{\mathbb{T}^d}\left[\frac{1}{H}\int_R^{R+H}\left|r^{-(d-1)/2}\mathcal{D}(r\Omega-x)\right|^2dr\right]^{p/2}\right\}^{1/p}\le\begin{cases}C(6-p)^{-1/3} & \text{if } p<6,\\ C\log^{1/3}(1+R) & \text{if } p=6.\end{cases}$$

3) if $d\ge 4$

$$\left\{\int_{\mathbb{T}^d}\left[\frac{1}{H}\int_R^{R+H}\left|r^{-(d-1)/2}\mathcal{D}(r\Omega-x)\right|^2dr\right]^{p/2}\right\}^{1/p}$$

$$\leq \begin{cases} C\left((2d-4)/(d-3)-p\right)^{-(d-3)/(2d-4)} & \text{if } p<(2d-4)/(d-3)\,, \\ C\log^{(d-3)/(2d-4)}(1+R) & \text{if } p=(2d-4)/(d-3)\,. \end{cases}$$

Observe that the range of indexes in the above theorem and corollaries for which the mixed $L^p(L^2)$ norm remains uniformly bounded is larger than the range of indexes in [21] and [5] and in the previous chapter.

**Example.** If $d\mu(r)$ is the unit mass concentrated at $r=0$, then $\widehat{\mu}(\zeta)=1$, so that $\beta=0$, and the $L^p(L^2)$ mixed norm in the theorem reduce to a pure $L^p$ norm, and one obtains

$$\left\{\int_{\mathbb{T}^d}\left|R^{-(d-1)/2}\mathcal{D}(R\Omega-x)\right|^p dx\right\}^{1/p}$$
$$\leq \begin{cases} C\left(2d/(d-1)-p\right)^{-(d-1)/2d} & \text{if } d\geq 2 \text{ and } p<2d/(d-1)\,, \\ C\log^{(d-1)/2d}(1+R) & \text{if } d\geq 2 \text{ and } p=2d/(d-1)\,. \end{cases}$$

In particular, one recovers some of the results in [21] and [5].

**Example.** If $d\mu(r)$ is $r^{-\alpha}\chi_{\{0<r<1\}}(r)\,dr$, with $0<\alpha<1$, then $|\widehat{\mu}(\zeta)|\leq C(1+|\zeta|)^{\alpha-1}$, that is $\beta=1-\alpha$.

**Example.** A probability measure is a Salem measure if its Fourier dimension $\gamma=\sup\left\{\delta:\ |\widehat{\mu}(\zeta)|\leq C(1+|\zeta|)^{-\delta/2}\right\}$ is equal to the Hausdorff dimension of the support. Such measures exist for every dimension $0<\gamma<1$, and the above theorem and corollary assert that the discrepancy cannot be too large in mean on the supports of translated and dilates of these measures.

The techniques used to prove these theorems are similar to the ones in the previous chapter used to estimate the pure $L^p$ norms of the discrepancy.

## 2.1 Proof of theorems and corollaries

The proofs will be splitted into a number of lemmas, some of them well known. The starting point is the observation of D.G.Kendall that the discrepancy $\mathcal{D}(r\Omega-x)$ is a periodic function of the translation, and it has a Fourier expansion with coefficients that are a sampling of the Fourier transform of $\Omega$,

$$\widehat{\chi}_\Omega(\xi)=\int_\Omega e^{-2\pi i\xi x}dx.$$

**Lemma 13.** *The number of integer points in $r\Omega - x$, a translated by a vector $x \in \mathbb{R}^d$ and dilated by a factor $r > 0$ of a domain $\Omega$ in the $d$ dimensional Euclidean space is a periodic function of the translation with Fourier expansion*

$$\sum_{k\in\mathbb{Z}^d} \chi_{r\Omega-x}(k) = \sum_{n\in\mathbb{Z}^d} r^d \widehat{\chi}_\Omega (rn)\, e^{2\pi inx}.$$

*In particular,*

$$\mathcal{D}\left(r\Omega - x\right) = \sum_{n\in\mathbb{Z}^d\setminus\{0\}} r^d \widehat{\chi}_\Omega (rn)\, e^{2\pi inx}.$$

*Proof.* This is a particular case of the Poisson summation formula. □

**Remark:** We emphasize that the Fourier expansion of the discrepancy converges at least in $L^2\left(\mathbb{T}^d\right)$, but we are not claiming that it converges pointwise. Indeed, the discrepancy is discontinuous, hence the associated Fourier expansion does not converge absolutely or uniformly. To overcome this problem, one could introduce a mollified discrepancy. If the domain $\Omega$ is convex and contains the origin, then there exists $\varepsilon > 0$ such that if $\varphi(x)$ is a non negative smooth radial function with support in $\{|x| \leq \varepsilon\}$ and with integral 1, and if $0 < \delta \leq 1$ and $r \geq 1$, then

$$\begin{aligned} |\Omega| \left( (r-\delta)^d - r^d \right) + (r-\delta)^d \sum_{n\in\mathbb{Z}^d\setminus\{0\}} \widehat{\varphi}(\delta n)\, \widehat{\chi}_\Omega\left((r-\delta)\, n\right) e^{2\pi inx} \\ \leq \sum_{n\in\mathbb{Z}^d} \chi_{r\Omega}(n+x) - |\Omega|\, r^d \\ \leq |\Omega| \left( (r+\delta)^d - r^d \right) + (r+\delta)^d \sum_{n\in\mathbb{Z}^d\setminus\{0\}} \widehat{\varphi}(\delta n)\, \widehat{\chi}_\Omega\left((r+\delta)\, n\right) e^{2\pi inx}. \end{aligned}$$

One has $\left|(r+\delta)^d - r^d\right| \leq Cr^{d-1}\delta$, and one can define the mollified discrepancy

$$(r\pm\delta)^d \sum_{n\in\mathbb{Z}^d\setminus\{0\}} \widehat{\varphi}(\delta n)\, \widehat{\chi}_\Omega\left((r\pm\delta)\, n\right) e^{2\pi inx}.$$

Observe that the discrepancy is the limit of this mollified discrepancy as $\delta \to 0+$. Also observe that since $|\widehat{\varphi}(\zeta)| \leq C\,(1+|\zeta|)^{-\gamma}$ for every $\gamma > 0$, with this mollified Fourier expansion there are no problem of convergence.

**Lemma 14.** *Assume that $\Omega$ is a convex body in $\mathbb{R}^d$ with smooth boundary having everywhere positive Gaussian curvature. Define the support function $g(x) = \sup_{y\in\Omega}\{x\cdot y\}$. Then, there exist functions $\{a_j(\xi)\}_{j=0}^{+\infty}$ and $\{b_j(\xi)\}_{j=0}^{+\infty}$ homogeneous* *of degree* 0 *and smooth in* $\mathbb{R}^d - \{0\}$ *such that the Fourier transform of the characteristic function of* $\Omega$ *for* $|\xi| \to +\infty$ *has the asymptotic expansion*

$$\begin{aligned}
&\widehat{\chi}_\Omega(\xi) = \int_\Omega e^{-2\pi i\xi\cdot x}dx \\
&= e^{-2\pi i g(\xi)} |\xi|^{-(d+1)/2} \sum_{j=0}^{h} a_j(\xi) |\xi|^{-j} + e^{2\pi i g(-\xi)} |\xi|^{-(d+1)/2} \sum_{j=0}^{h} b_j(\xi) |\xi|^{-j} \\
&+ \mathcal{O}\left(|\xi|^{-(d+2h+3)/2}\right).
\end{aligned}$$

*The functions* $a_j(\xi)$ *and* $b_j(\xi)$ *depend on a finite number of derivatives of a parametrization of the boundary of* $\Omega$ *at the points with outward unit normal* $\pm\xi/|\xi|$. *In particular,* $a_0(\xi)$ *and* $b_0(\xi)$ *are, up to some absolute constants, equal to* $K(\pm\xi)^{-1/2}$, *with* $K(\pm\xi)$ *the Gaussian curvature of* $\partial\Omega$ *at the points with outward unit normal* $\pm\xi/|\xi|$.

*Proof.* This is a classical result. See e.g. [12], [19], [18], [34]. Here, as an explicit example, we want just to recall that the Fourier transform of a ball $\left\{x \in \mathbb{R}^d : |x| \le R\right\}$ can be expressed in terms of a Bessel function, and Bessel functions have simple asymptotic expansions in terms of trigonometric functions,

$$\begin{aligned}
&\widehat{\chi}_{\{|x|\le R\}}(\xi) = R^d \widehat{\chi}_{\{|x|\le 1\}}(R\xi) = R^d |R\xi|^{-d/2} J_{d/2}(2\pi |R\xi|) \\
&\approx \pi^{-1} R^{(d-1)/2} |\xi|^{-(d+1)/2} \cos(2\pi R|\xi| - (d+1)\pi/4) \\
&- 2^{-4}\pi^{-2}\left(d^2-1\right) R^{(d-3)/2} |\xi|^{-(d+3)/2} \sin(2\pi R|\xi| - (d+1)\pi/4) + ...
\end{aligned}$$

More generally, also the Fourier transform of an ellipsoid, that is an affine image of a ball, can be expressed in terms of Bessel functions. □

**Lemma 15.** *Assume that* $\Omega$ *is a convex body in* $\mathbb{R}^d$ *with smooth boundary having everywhere positive Gaussian curvature. Let* $z$ *be a complex parameter, and for every* $j = 0, 1, 2, ...$ *and* $r \ge 1$, *with the notation of the previous lemmas, let define the tempered distributions* $\Phi_j(z, r, x)$ *via the Fourier expansion*

$$\begin{aligned}
&\Phi_j(z,r,x) = r^{-j} \sum_{n\in\mathbb{Z}^d\setminus\{0\}} a_j(n) |n|^{-z-j} e^{-2\pi i g(n) r} e^{2\pi i n x} \\
&+ r^{-j} \sum_{n\in\mathbb{Z}^d\setminus\{0\}} b_j(n) |n|^{-z-j} e^{2\pi i g(-n) r} e^{2\pi i n x}.
\end{aligned}$$

(1) *If* $\mathrm{Re}(z)+j > d/2$ *then the Fourier expansion that defines* $\Phi_j(z,r,x)$ *converges in* $L^2\left(\mathbb{T}^d\right)$. *If* $\mathrm{Re}(z)+j > d$ *then the convergence is absolute and uniform.*

*(2) Let*

$$\mathcal{R}_h(r,x) = r^{-(d-1)/2}\mathcal{D}(r\Omega - x) - \sum_{j=0}^{h} \Phi_j((d+1)/2, r, x).$$

*If* $h > (d-3)/2$ *there exists* $C$ *such that if* $r \geq 1$,

$$|\mathcal{R}_h(r,x)| \leq Cr^{-h-1}.$$

*Proof.* This is a simple consequence of the previous lemmas. The terms $\Phi_j((d+1)/2, r, x)$ come from the terms homogeneous of degree $-(d+1)/2 - j$ in the asymptotic expansion of the Fourier transform of $\Omega$, while the remainder $\mathcal{R}_h(r,x)$ is given by an absolutely and uniformly convergent Fourier expansion. □

**Lemma 16.** *Let* $g(x) = \sup_{y \in \Omega}\{x \cdot y\}$ *be the support function of a convex* $\Omega$ *which contains the origin, and with a smooth boundary with everywhere positive Gaussian curvature.*

*(1) This support function is convex, homogeneous of degree one, positive and smooth away from the origin, and it is equivalent to the Euclidean norm, that is there exist* $0 < c < C$ *such that for every* $x$,

$$c|x| \leq g(x) \leq C|x|.$$

*(2) There exists* $C > 0$ *such that for all unit vectors* $\omega$ *and* $\vartheta$ *in* $\mathbb{R}^d$, *there exists a number* $A(\vartheta,\omega)$ *such that for every real* $\tau$ *one has*

$$|g(\vartheta - \tau\omega) - g(\vartheta)| \geq C\frac{|\tau|\,|\tau - A(\vartheta,\omega)|}{1+|\tau|}.$$

*Proof.* (1) The convexity of the support function easily follows from the convexity of $\Omega$, and also the other properties are elementary. In order to prove (2), observe that for $|\omega| = |\vartheta| = 1$ and $-\infty < \tau < +\infty$,

$$|g(\vartheta - \tau\omega) - g(\vartheta)| = \left|\frac{g(\vartheta-\tau\omega)^2 - g(\vartheta)^2}{g(\vartheta-\tau\omega) + g(\vartheta)}\right|$$
$$\geq C\frac{\left|g(\vartheta-\tau\omega)^2 - g(\vartheta)^2\right|}{1+|\tau|}.$$

It then suffices to prove that there exists a $C > 0$ such that for all unit vectors $\vartheta$ and $\omega$, there exists a number $A(\vartheta,\omega)$ such that for every real $\tau$ one has

$$\left|g(\vartheta-\tau\omega)^2 - g(\vartheta)^2\right| \geq C|\tau|\,|\tau - A(\vartheta,\omega)|.$$

Let us show that the function $f(\tau) = g(\vartheta - \tau\omega)^2 - g(\vartheta)^2$ is strictly convex. If $\omega = \pm\vartheta$, then

$$f(\tau) = g((1 \pm \tau)\vartheta)^2 - g(\vartheta)^2 = \left((1 \pm \tau)^2 - 1\right) g(\vartheta)^2 = \left(\tau^2 \pm 2\tau\right) g(\vartheta)^2 .$$

Therefore, if $\omega = \pm\vartheta$,

$$\frac{d^2}{d\tau^2} f(\tau) = 2g(\vartheta)^2 \geq \delta > 0.$$

If $\omega \neq \pm\vartheta$, then $\vartheta - \tau\omega \neq 0$, and

$$\frac{d}{d\tau} f(\tau) = -2g(\vartheta - \tau\omega)\nabla g(\vartheta - \tau\omega) \cdot \omega,$$

$$\frac{d^2}{d\tau^2} f(\tau) = 4(\nabla g(\vartheta - \tau\omega) \cdot \omega)^2 + 2g(\vartheta - \tau\omega)\omega^t \cdot \nabla^2 g(\vartheta - \tau\omega) \cdot \omega.$$

For notational simplicity call $\vartheta - \tau\omega = x$. The Hessian matrix $\nabla^2 g(x)$ is homogeneous of degree $-1$ and positive semidefinite. When $|x| = 1$ one eigenvalue is 0 and the associated eigenvector is the gradient $\nabla g(x)$, while all the other eigenvalues are the reciprocal of the principal curvatures at the point where the normal is $\nabla g(x)$. See [33, Corollary 2.5.2]. Let $\alpha$ be the minimum of $g(x)$ on the sphere $\{|x| = 1\}$, let $\beta > 0$ be the minimum of $|\nabla g(x)|$ on $\{|x| = 1\}$, and let $\gamma > 0$ be the minimum of the non zero eigenvalues of $\nabla^2 g(x)$ on $\{|x| = 1\}$. If one decomposes $\omega$ into $\omega_0 + \omega_1$, where $\omega_0$ is parallel to $\nabla g(x)$ and $\omega_1$ is orthogonal to $\nabla g(x)$, then

$$\begin{aligned}
\frac{d^2}{d\tau^2} f(\tau) &= 4(\nabla g(x) \cdot \omega_0)^2 + 2g(x)\omega_1^t \cdot \nabla^2 g(x) \cdot \omega_1 \\
&= 4|\nabla g(x)|^2 |\omega_0|^2 + 2g(x/|x|)\omega_1^t \cdot \nabla^2 g(x/|x|) \cdot \omega_1 \\
&\geq 4\beta^2 |\omega_0|^2 + 2\alpha\gamma |\omega_1|^2 \geq \delta > 0.
\end{aligned}$$

Therefore, for every $\vartheta$ and $\omega$ the function $f(\tau)$ is strictly convex, and it has exactly two zeros, one is $\tau = 0$ and the other is $\tau = A(\vartheta, \omega)$, possibly the zero is double. By the Lagrange remainder in interpolation, there exists $\varepsilon$ such that

$$f(\tau) = \tau(\tau - A(\omega, \vartheta)) \frac{1}{2} \frac{d^2 f}{d\tau^2}(\varepsilon).$$

And since $d^2 f(\tau)/d\tau^2 \geq \delta > 0$,

$$\left|g(\vartheta - \tau\omega)^2 - g(\vartheta)^2\right| \geq \frac{\delta}{2} |\tau| |\tau - A(\omega, \vartheta)| .$$

**Lemma 17.** *If $g(x)$ is the support function of $\Omega$, then for every $(d+1)/2 \leq \alpha < d$ and $\beta \geq 0$, there exists $C$ such that for every $y \in \mathbb{R}^d - \{0\}$,*

$$\int_{\mathbb{R}^d} |x|^{-\alpha} |x-y|^{-\alpha} (1 + |g(x) - g(x-y)|)^{-\beta} dx$$
$$\leq \begin{cases} C|y|^{d-2\alpha-\beta} & \text{if } 0 \leq \beta < 1, \\ C|y|^{d-2\alpha-1} \log(2+|y|) & \text{if } \beta = 1, \\ C|y|^{d-2\alpha-1} & \text{if } \beta > 1. \end{cases}$$

*Proof.* It is easy to explain the numerology behind the lemma. Assume that there is no cutoff $(1 + |g(x) - g(x-y)|)^{-\beta}$. Then the change of variables $x = |y|z$ and $y = |y|\omega$ gives

$$\int_{\mathbb{R}^d} |x|^{-\alpha} |x-y|^{-\alpha} dx = |y|^{d-2\alpha} \int_{\mathbb{R}^d} |z|^{-\alpha} |z-\omega|^{-\alpha} dx = C|y|^{d-2\alpha}.$$

On the other hand, the cutoff $(1 + |g(x) - g(x-y)|)^{-\beta}$ should give an extra decay. In particular, the integral with the cutoff $(1 + |g(x) - g(x-y)|)^{-\beta}$ with $\beta$ large is essentially over the set $\{g(x) = g(x-y)\}$, that is the cutoff reduces the space dimension by 1. This suggests that, at least when $\beta$ is large, the integral with the cutoff can be seen as the convolution in $\mathbb{R}^{d-1}$ of two homogeneous functions of degree $-\alpha$, and this gives the decay $|y|^{d-1-2\alpha}$. When $\beta = 0$ the decay is $|y|^{d-2\alpha}$, and when $\beta > 1$ the decay is $|y|^{d-1-2\alpha}$. By interpolation, when $0 < \beta < 1$ the decay is $|y|^{d-\beta-2\alpha}$. This is just the numerology, the details of the proof are more delicate. The change of variables $x = |y|z$ and $y = |y|\omega$ gives

$$\int_{\mathbb{R}^d} |x|^{-\alpha} |x-y|^{-\alpha} (1 + |g(x) - g(x-y)|)^{-\beta} dx$$
$$= |y|^{d-2\alpha} \int_{\mathbb{R}^d} |z|^{-\alpha} |z-\omega|^{-\alpha} (1 + |y| |g(z) - g(z-\omega)|)^{-\beta} dz.$$

If $\varepsilon$ is positive and suitably small, there exists $\delta > 0$ such that for every $\omega$ and $z$ with $|\omega| = 1$ and $|z| < \varepsilon$ one has $g(z-\omega) - g(z) > \delta$, and the domain of integration can be split into

$$\{|z| \leq \varepsilon\} \cup \{|z-\omega| \leq \varepsilon\} \cup \{|z| \geq \varepsilon,\ |z-\omega| \geq \varepsilon\}.$$

The integral over the domain $\{|z| \leq \varepsilon\}$ is bounded by

$$\int_{\{|z|\leq\varepsilon\}} |z|^{-\alpha} |z-\omega|^{-\alpha} (1 + |y| |g(z) - g(z-\omega)|)^{-\beta} dz$$
$$\leq (1-\varepsilon)^{-\alpha} (1 + \delta|y|)^{-\beta} \int_{\{|z|\leq\varepsilon\}} |z|^{-\alpha} dz$$

$$\leq C\left(1+|y|\right)^{-\beta}.$$

The integral over the domain $\{|z-\omega|\leq\varepsilon\}$ is bounded similarly,

$$\int_{\{|z-\omega|\leq\varepsilon\}}|z|^{-\alpha}\,|z-\omega|^{-\alpha}\left(1+|y|\,|g\left(z\right)-g\left(z-\omega\right)|\right)^{-\beta}dz\leq C\left(1+|y|\right)^{-\beta}.$$

It remains to estimate the integral over $\{|z|\geq\varepsilon,\ |z-\omega|\geq\varepsilon\}$. First observe that

$$\begin{aligned}&\int_{\{|z|\geq\varepsilon,\ |z-\omega|\geq\varepsilon\}}|z|^{-\alpha}\,|z-\omega|^{-\alpha}\left(1+|y|\,|g\left(z\right)-g\left(z-\omega\right)|\right)^{-\beta}dz\\&\leq C\int_{\{|z|\geq\varepsilon\}}|z|^{-2\alpha}\left(1+|y|\,|g\left(z\right)-g\left(z-\omega\right)|\right)^{-\beta}dz.\end{aligned}$$

In spherical coordinates write $z=\rho\vartheta$, with $\varepsilon\leq\rho<+\infty$ and $|\vartheta|=1$, and $dz=\rho^{d-1}d\rho d\vartheta$, with $d\vartheta$ the surface measure on the $d-1$ dimensional sphere $\mathbb{S}^{d-1}=\{|\vartheta|=1\}$. Then by the above lemma and the change of variables $\rho=1/\tau$, recalling that $|\omega|=1$ and $2\alpha-d-1\geq0$,

$$\begin{aligned}&\int_{\{|z|\geq\varepsilon\}}|z|^{-2\alpha}\left(1+|y|\,|g\left(z\right)-g\left(z-\omega\right)|\right)^{-\beta}dz\\&=\int_{\mathbb{S}^{d-1}}\int_{\varepsilon}^{+\infty}\rho^{d-1-2\alpha}\left(1+|y|\,|g\left(\rho\vartheta\right)-g\left(\rho\vartheta-\omega\right)|\right)^{-\beta}d\rho d\vartheta\\&=\int_{\mathbb{S}^{d-1}}\int_{0}^{1/\varepsilon}\tau^{2\alpha-d-1}\left(1+|y|\left|\frac{g\left(\vartheta\right)-g\left(\vartheta-\tau\omega\right)}{\tau}\right|\right)^{-\beta}d\tau d\vartheta\\&\leq C\int_{\mathbb{S}^{d-1}}\int_{0}^{1/\varepsilon}\left(1+|y|\left|\frac{g\left(\vartheta\right)-g\left(\vartheta-\tau\omega\right)}{\tau}\right|\right)^{-\beta}d\tau d\vartheta\\&\leq C\int_{\mathbb{S}^{d-1}}\int_{0}^{1/\varepsilon}\left(1+|y|\,|\tau-A\left(\vartheta,\omega\right)|\right)^{-\beta}d\tau d\vartheta.\end{aligned}$$

Finally, if $\sup_{|\vartheta|=|\omega|=1}\{|A\left(\vartheta,\omega\right)|\}=\gamma$, then

$$\begin{aligned}&\int_{\mathbb{S}^{d-1}}\int_{0}^{1/\varepsilon}\left(1+|y|\,|\tau-A\left(\vartheta,\omega\right)|\right)^{-\beta}d\tau d\vartheta\\&\leq\left|\mathbb{S}^{d-1}\right|\int_{-\gamma-1/\varepsilon}^{\gamma+1/\varepsilon}\left(1+|y|\,|\tau|\right)^{-\beta}d\tau\\&=\left|\mathbb{S}^{d-1}\right||y|^{-1}\int_{-(\gamma+1/\varepsilon)|y|}^{(\gamma+1/\varepsilon)|y|}\left(1+|\tau|\right)^{-\beta}d\tau\\&\leq\begin{cases}C\left(1+|y|\right)^{-\beta} & \text{if } 0\leq\beta<1,\\ C\left(1+|y|\right)^{-1}\log\left(2+|y|\right) & \text{if } \beta=1,\\ C\left(1+|y|\right)^{-1} & \text{if } \beta>1.\end{cases}\end{aligned}$$

□

Roughly speaking one can describe the strategy of the proof as follows: The normalized discrepancy $r^{-(d-1)/2}\mathcal{D}(r\Omega - x)$ has a Fourier expansion of the form

$$\sum_{n\in\mathbb{Z}^d\setminus\{0\}} |n|^{-(d+1)/2} e^{\pm 2\pi i g(\pm n) r} e^{2\pi i n x}.$$

One can replace the real parameter $(d+1)/2$ which describes the decay of the Fourier transform by a complex parameter $z$, and define the function

$$\Theta(z,r,x) = \sum_{n\in\mathbb{Z}^d\setminus\{0\}} |n|^{-z} e^{\pm 2\pi i g(\pm n) r} e^{2\pi i n x}.$$

Observe that the Fourier coefficients of this function are analytic functions of the complex variable $z$. One can estimate the $L^2(L^2)$ norms of this function via the Parseval equality, and the norm is finite if $\operatorname{Re}(z) > d/2$. Then one can estimate the $L^p(L^2)$ norm with $p \geq 4$ via the Hausdorff Young inequality, and the norm is finite if $\operatorname{Re}(z) > d(1-1/p) - 1/2$. Finally, the result for $z = (d+1)/2$ follows by complex interpolation.

By the above lemma, the normalized discrepancy $r^{-(d-1)/2}\mathcal{D}(r\Omega - x)$ is a sum of terms $\Phi_j((d+1)/2, r, x)$, and the following lemma studies the two terms that appear in the definition of $\Phi_j(z,r,x)$.

**Lemma 18.** *Let $d\mu(r)$ be a Borel probability measure on $\mathbb{R}$ with support in $0 < \varepsilon < r < \delta < +\infty$, and with $|\widehat{\mu}(\zeta)| \leq B(1+|\zeta|)^{-\beta}$. Let $c(\xi)$ be a bounded homogeneous function of degree 0, let $\operatorname{Re}(z) \geq (d+1)/2$, and for $j = 0,1,2,...$ and any choice of $\pm$, define*

$$\Theta_j(z,r,x) = r^{-j} \sum_{n\in\mathbb{Z}^d\setminus\{0\}} c(n)|n|^{-z-j} e^{\pm 2\pi i g(\pm n) r} e^{2\pi i n x}.$$

*Moreover, for $H, R \geq 1$, define*

$$\mathcal{F}_j(z,H,R,x) = \int_{\mathbb{R}} |\Theta_j(z,r,x)|^2 \, d\mu_{H,R}(r).$$

*Expand this last function into a Fourier series in the variable $x$,*

$$\mathcal{F}_j(z,H,R,x) = \sum_{k\in\mathbb{Z}^d} \widehat{\mathcal{F}}_j(z,H,R,k) e^{2\pi i k x}.$$

*(1) If $j = 0$ there exists a constant $C$, which may depend on $d$, $B$, $\beta$, but is independent of the complex parameter $z$, the real parameters $H$ and $R$, and on the measure*

*$d\mu(r)$, such that the Fourier coefficients of $\mathcal{F}_0(z,H,R,x)$ satisfy for every $H,R\geq 1$ and $k\in\mathbb{Z}^d$ the estimates*

$$\left|\widehat{\mathcal{F}}_0(z,H,R,k)\right| \leq \begin{cases} C(1+|k|)^{d-\beta-2\operatorname{Re}(z)} & \text{if } 0\leq\beta<1, \\ C(1+|k|)^{d-1-2\operatorname{Re}(z)}\log(2+|k|) & \text{if } \beta=1, \\ C(1+|k|)^{d-1-2\operatorname{Re}(z)} & \text{if } \beta>1. \end{cases}$$

*(2) If $j\geq 1$ then there exists a constant $C$ such that the Fourier coefficients of $\mathcal{F}_j(z,H,R,x)$ satisfy for every $H,R\geq 1$ and $k\in\mathbb{Z}^d$ the estimates*

$$\left|\widehat{\mathcal{F}}_j(z,H,R,k)\right| \leq C(R+H)^{-2j}(1+|k|)^{d-1-2\operatorname{Re}(z)}.$$

*Proof.* Let us fix a choice of $\pm$,

$$\Theta_j(z,r,x) = r^{-j}\sum_{n\in\mathbb{Z}^d\setminus\{0\}} c(n)|n|^{-z-j}e^{-2\pi i g(n) r}e^{2\pi i n x}.$$

Expanding the product $\Theta_j(z,r,x)\cdot\overline{\Theta_j(z,r,x)}$ and integrating against $d\mu(r)$, one obtains

$$\begin{aligned}
&\mathcal{F}_j(z,H,R,x) = \int_{\mathbb{R}} |\Theta_j(z,r,x)|^2\, d\mu_{H,R}(r) \\
&= \sum_{k\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}^d\setminus\{0,k\}} c(n)\,c(n-k)\,|n|^{-z-j}\,|n-k|^{-\overline{z}-j}\,e^{2\pi i k x} \\
&\times e^{2\pi i(g(n-k)-g(n))R}\int_{\mathbb{R}}(R+Hr)^{-2j}e^{2\pi i H(g(n-k)-g(n))r}d\mu(r).
\end{aligned}$$

The product term by term of the series and the integration term by term can be justified with a suitable summation method, which amounts to introduce a cutoff in the the series that defines $\Theta_j(z,r,x)$. See the Remark after Lemma 13. In particular, the Fourier coefficients of $\mathcal{F}_j(z,H,R,x)$ are

$$\begin{aligned}
&\widehat{\mathcal{F}}_j(z,H,R,k) \\
&= \sum_{n\in\mathbb{Z}^d\setminus\{0,k\}} c(n)\,c(n-k)\,|n|^{-z-j}\,e^{2\pi i(g(n-k)-g(n))R}\int_{\mathbb{R}}(R+Hr)^{-2j}e^{2\pi i H(g(n-k)-g(n))r}d\mu(r).
\end{aligned}$$

Let us first consider the case $j=0$. By the assumption on the Fourier transform of the measure $d\mu(r)$,

$$\left|\widehat{\mathcal{F}}_0(z,H,R,k)\right|$$

$$
\begin{aligned}
&\leq B \sup_{n\in\mathbb{Z}^d}\left\{|c(n)|^2\right\} \sum_{n\in\mathbb{Z}^d\setminus\{0,k\}} |n|^{-\operatorname{Re}(z)} |n-k|^{-\operatorname{Re}(z)} \left(1+|g(n-k)-g(n)|\right)^{-\beta} \\
&\leq C \int_{\mathbb{R}^d} |x|^{-\operatorname{Re}(z)} |x-k|^{-\operatorname{Re}(z)} \left(1+|g(x-k)-g(x)|\right)^{-\beta} dx.
\end{aligned}
$$

It then suffices to apply Lemma 17. The case $j>0$ is simpler.

$$
\begin{aligned}
&\left|\widehat{\mathcal{F}}_j(z,H,R,k)\right| \\
&\leq \sup_{n\in\mathbb{Z}^d}\left\{|c(n)|^2\right\} \sum_{n\in\mathbb{Z}^d\setminus\{0,k\}} |n|^{-\operatorname{Re}(z)-j} |n-k|^{-\operatorname{Re}(z)-j} \int_{\mathbb{R}} (R+Hr)^{-2j} d\mu(r) \\
&\leq C(R+H)^{-2j}(1+|k|)^{d-1-2\operatorname{Re}(z)}.
\end{aligned}
$$

□

**Lemma 19.** *Let* $0\leq\beta<1$ *and* $\mathcal{F}_0(z,H,R,x)$ *be defined as in the previous lemmas. Then there exists* $C$ *such that for every* $H,R\geq 1$ *the following hold.*

*(1) If* $2\leq p\leq 4$ *and* $\operatorname{Re}(z)>d(1-1/p)+2\beta/p-\beta$,

$$
\begin{aligned}
&\left\{\int_{\mathbb{T}^d} |\mathcal{F}_0(z,H,R,x)|^{p/2} dx\right\}^{1/p} \\
&\leq C\left(\operatorname{Re}(z)+\beta-\frac{2\beta}{p}-d\left(1-\frac{1}{p}\right)\right)^{-1/p}
\end{aligned}
$$

*(2) If* $4\leq p\leq+\infty$ *and* $\operatorname{Re}(z)>d(1-1/p)-\beta/2$,

$$
\begin{aligned}
&\left\{\int_{\mathbb{T}^d} |\mathcal{F}_0(z,H,R,x)|^{p/2} dx\right\}^{1/p} \\
&\leq C\left(\operatorname{Re}(z)+\frac{\beta}{2}-d\left(1-\frac{1}{p}\right)\right)^{1/p-1/2}.
\end{aligned}
$$

*Proof.* If $p=2$ and $\operatorname{Re}(z)>d/2$ then, by Parseval equality,

$$
\begin{aligned}
&\left\{\int_{\mathbb{T}^d} |\mathcal{F}_0(z,H,R,x)|^{p/2} dx\right\}^{1/p} \\
&=\left\{\int_{\mathbb{T}^d}\int_{\mathbb{R}} |\Theta_0(z,r,x)|^2 d\mu_{H,R}(r)\, dx\right\}^{1/p} \\
&=\left\{\int_{\mathbb{R}} d\mu(r) \sum_{n\in\mathbb{Z}^d-\{0\}} |c(n)|^2 |n|^{-2\operatorname{Re}(z)}\right\}^{1/2}
\end{aligned}
$$

$$\leq C\left(\operatorname{Re}(z)-\frac{d}{2}\right)^{-1/2}.$$

If $4\leq p\leq+\infty$ and $\operatorname{Re}(z)>d(1-1/p)-\beta/2$ then, by the Hausdorff Young inequality,

$$\begin{aligned}
&\left\{\int_{\mathbb{T}^d}|\mathcal{F}_0(z,H,R,x)|^{p/2}\,dx\right\}^{1/p}\\
&\leq\left\{\sum_{k\in\mathbb{Z}^d}\left|\widehat{\mathcal{F}}_0(z,H,R,k)\right|^{p/(p-2)}\right\}^{(p-2)/2p}\\
&\leq C\left\{\sum_{k\in\mathbb{Z}^d}\left|(1+|k|)^{d-\beta-2\operatorname{Re}(z)}\right|^{p/(p-2)}\right\}^{(p-2)/2p}\\
&\leq C\left((2\operatorname{Re}(z)+\beta-d)\frac{p}{p-2}-d\right)^{-(p-2)/2p}\\
&=C\left(\frac{p-2}{2p}\right)^{(p-2)/2p}\left(\operatorname{Re}(z)+\frac{\beta}{2}-d\left(1-\frac{1}{p}\right)\right)^{-(p-2)/2p}\\
&\leq C\left(\operatorname{Re}(z)+\frac{\beta}{2}-d\left(1-\frac{1}{p}\right)\right)^{1/p-1/2}.
\end{aligned}$$

This proves the cases $p=2$ and $p\geq4$. The cases $2<p<4$ follow from these cases via complex interpolation of vector valued $L(p)$ spaces. For the definition of the complex interpolation method, see for example [1, Chapter 4 and Chapter 5 ]. Here we recall the relevant result: Let $\mathbb{H}$ be a Hilbert space and $\mathbb{X}$ a measure space, let $1\leq a<b\leq+\infty$, $-\infty<A<B<+\infty$, and let $\Theta(z)$ be a function with values in the vector valued space $L^a(\mathbb{X},\mathbb{H})+L^b(\mathbb{X},\mathbb{H})$, continuous and bounded on the closed strip $\{A\leq\operatorname{Re}(z)\leq B\}$ and analytic on the open strip $\{A<\operatorname{Re}(z)<B\}$. Assume that there exist constants $M$ and $N$ such that for every $-\infty<t<+\infty$,

$$\begin{cases}\|\Theta(A+it)\|_{L^a(\mathbb{X},\mathbb{H})}\leq M,\\ \|\Theta(B+it)\|_{L^b(\mathbb{X},\mathbb{H})}\leq N.\end{cases}$$

If $1/p=(1-\vartheta)/a+\vartheta/b$, with $0<\vartheta<1$, then

$$\|\Theta((1-\vartheta)A+\vartheta B)\|_{L^p(\mathbb{X},\mathbb{H})}\leq M^{1-\vartheta}N^{\vartheta}.$$

Here the analytic function is $\Theta_0(z,r,x)$, the Hilbert space is $L^2(\mathbb{R},d\mu_{H,R}(r))$, the measure space is the torus $\mathbb{T}^d$, $a=2$, $A=d/2+\varepsilon$, $b=4$, $B=3d/4-\beta/2+\varepsilon$, with $\varepsilon>0$, $M=C\varepsilon^{-1/2}$ and $N=C\varepsilon^{-1/4}$. By the above computations,

$$\left\{\int_{\mathbb{T}^d}|\mathcal{F}_0(z,H,R,x)|^{p/2}\,dx\right\}^{1/p}$$

$$\leq \begin{cases} C\varepsilon^{-1/2} & \text{if } p = 2 \text{ and } \operatorname{Re}(z) = d/2 + \varepsilon, \\ C\varepsilon^{-1/4} & \text{if } p = 4 \text{ and } \operatorname{Re}(z) = 3d/4 - \beta/2 + \varepsilon. \end{cases}$$

By complex interpolation with

$$\begin{aligned} &1/p = (1-\vartheta)/2 + \vartheta/4, \\ &\operatorname{Re}(z) = (1-\vartheta)(d/2+\varepsilon) + \vartheta(3d/4 - \beta/2 + \varepsilon), \end{aligned}$$

that is, $2 < p < 4$ and $\operatorname{Re}(z) = d(1-1/p) + 2\beta/p - \beta + \varepsilon$,

$$\begin{aligned} &\left\{ \int_{\mathbb{T}^d} |\mathcal{F}_0(z, H, R, x)|^{p/2} \, dx \right\}^{1/p} \\ &\leq C \left( \operatorname{Re}(z) + \beta - \frac{2\beta}{p} - d\left(1 - \frac{1}{p}\right) \right)^{-1/p}. \end{aligned}$$

□

**Lemma 20.** *Let $\beta = 1$ and $\mathcal{F}_0(z, H, R, x)$ be defined as in the previous lemmas. Then there exists $C$ such that for every $H, R \geq 1$ the following hold.*

*(1) If $2 \leq p \leq 4$ and $\operatorname{Re}(z) > d(1-1/p) + 2/p - 1$,*

$$\begin{aligned} &\left\{ \int_{\mathbb{T}^d} |\mathcal{F}_0(z, H, R, x)|^{p/2} \, dx \right\}^{1/p} \\ &\leq C \left( \operatorname{Re}(z) - \left( d\left(1 - \frac{1}{p}\right) + \frac{2}{p} - 1 \right) \right)^{1/p-1} \end{aligned}$$

*(2) If $4 \leq p \leq +\infty$ and $\operatorname{Re}(z) > d(1-1/p) - 1/2$,*

$$\begin{aligned} &\left\{ \int_{\mathbb{T}^d} |\mathcal{F}_0(z, H, R, x)|^{p/2} \, dx \right\}^{1/p} \\ &\leq C \left( \operatorname{Re}(z) - d\left(1 - \frac{1}{p}\right) + \frac{1}{2} \right)^{1/p-1}. \end{aligned}$$

*Proof.* The proof is as in the previous lemma. If $p = 2$ and $\operatorname{Re}(z) > d/2$, then

$$\left\{ \int_{\mathbb{T}^d} |\mathcal{F}_0(z, H, R, x)|^{p/2} \, dx \right\}^{1/p} \leq C \left( \operatorname{Re}(z) - \frac{d}{2} \right)^{-1/2}.$$

If $4 \leq p \leq +\infty$ and $\operatorname{Re}(z) > d(1-1/p) - 1/2$ then, by the Hausdorff Young inequality,

$$\left\{ \int_{\mathbb{T}^d} |\mathcal{F}_0(z, H, R, x)|^{p/2} \, dx \right\}^{1/p}$$

$$\leq \left\{ \sum_{k \in \mathbb{Z}^d} \left| \widehat{\mathcal{F}}_0 (z, H, R, k) \right|^{p/(p-2)} \right\}^{(p-2)/2p}$$

$$\leq C \left\{ \sum_{k \in \mathbb{Z}^d} \left| (1+|k|)^{d-1-2\operatorname{Re}(z)} \log(2+|k|) \right|^{p/(p-2)} \right\}^{(p-2)/2p} .$$

The series can be compared with the integral

$$\left\{ \int_{\mathbb{R}^d} \left| (1+|x|)^{d-1-2\operatorname{Re}(z)} \log(2+|x|) \right|^{p/(p-2)} dx \right\}^{(p-2)/2p}$$

$$= \left\{ |\{|\vartheta|=1\}| \int_0^{+\infty} (1+\rho)^{(d-1-2\operatorname{Re}(z))p/(p-2)} \log^{p/(p-2)}(2+\rho)\, \rho^{d-1} d\rho \right\}^{(p-2)/2p} .$$

The last integral can be compared to another integral,

$$\int_1^{+\infty} t^{-\alpha} \log^{\beta}(t)\, dt = (\alpha-1)^{-(\beta+1)} \int_0^{+\infty} s^{\beta} e^{-s} ds = (\alpha-1)^{-(\beta+1)}\, \Gamma(\beta+1) .$$

Hence,

$$\left\{ \int_0^{+\infty} (1+\rho)^{(d-1-2\operatorname{Re}(z))p/(p-2)+d-1} \log^{p/(p-2)}(2+\rho)\, d\rho \right\}^{(p-2)/2p}$$

$$\leq C \left( (2\operatorname{Re}(z)+1-d) \frac{p}{p-2} - d \right)^{-(1+p/(p-2))(p-2)/2p}$$

$$\leq C \left( \operatorname{Re}(z) - d\left(1-\frac{1}{p}\right) + \frac{1}{2} \right)^{1/p-1} .$$

This proves the cases $p = 2$ and $p \geq 4$. The cases $2 < p < 4$ follow from these cases via complex interpolation. By the above computations,

$$\left\{ \int_{\mathbb{T}^d} |\mathcal{F}_0(z, H, R, x)|^{p/2} dx \right\}^{1/p}$$

$$\leq \begin{cases} C\varepsilon^{-1/2} & \text{if } p = 2 \text{ and } \operatorname{Re}(z) = d/2 + \varepsilon, \\ C\varepsilon^{-3/4} & \text{if } p = 4 \text{ and } \operatorname{Re}(z) = (3d-2)/4 + \varepsilon. \end{cases}$$

By complex interpolation, if $2 < p < 4$ and $\operatorname{Re}(z) = d(1-1/p) + 2/p - 1 + \varepsilon$,

$$\left\{ \int_{\mathbb{T}^d} |\mathcal{F}_0(z, H, R, x)|^{p/2} dx \right\}^{1/p}$$

$$\leq C \left( \operatorname{Re}(z) - \left( d\left(1-\frac{1}{p}\right) + \frac{2}{p} - 1 \right) \right)^{1/p-1} .$$

□

**Lemma 21.** *Let $\beta > 1$ and $\mathcal{F}_0(z, H, R, x)$ be defined as in the previous lemma. Then there exists $C$ such that for every $H, R \geq 1$ the following hold.*

*(1) If $2 \leq p \leq 4$ and $\mathrm{Re}(z) > d(1 - 1/p) + 2/p - 1$,*

$$\left\{ \int_{\mathbb{T}^d} |\mathcal{F}_0(z, H, R, x)|^{p/2} \, dx \right\}^{1/p}$$
$$\leq C \left( \mathrm{Re}(z) - d\left(1 - \frac{1}{p}\right) - \frac{2}{p} + 1 \right)^{-1/p}$$

*(2) If $4 \leq p \leq +\infty$ and $\mathrm{Re}(z) > d(1 - 1/p) - 1/2$,*

$$\left\{ \int_{\mathbb{T}^d} |\mathcal{F}_0(z, H, R, x)|^{p/2} \, dx \right\}^{1/p}$$
$$\leq C \left( \mathrm{Re}(z) - d\left(1 - \frac{1}{p}\right) + \frac{1}{2} \right)^{1/p - 1/2}.$$

*Proof.* The proof is as in the previous lemma. If $p = 2$ and $\mathrm{Re}(z) > d/2$ then, by Parseval equality,

$$\left\{ \int_{\mathbb{T}^d} |\mathcal{F}_0(z, H, R, x)|^{p/2} \, dx \right\}^{1/p} \leq C \left( \mathrm{Re}(z) - \frac{d}{2} \right)^{-1/2}.$$

If $4 \leq p \leq +\infty$ and $\mathrm{Re}(z) > d(1 - 1/p) - 1/2$ then, by the Hausdorff Young inequality,

$$\left\{ \int_{\mathbb{T}^d} |\mathcal{F}_0(z, H, R, x)|^{p/2} \, dx \right\}^{1/p}$$
$$\leq C \left( \mathrm{Re}(z) - d\left(1 - \frac{1}{p}\right) + \frac{1}{2} \right)^{1/p - 1/2}.$$

The cases $2 < p < 4$ follow from these cases via complex interpolation. By the above computations,

$$\left\{ \int_{\mathbb{T}^d} |\mathcal{F}_0(z, H, R, x)|^{p/2} \, dx \right\}^{1/p}$$
$$\leq \begin{cases} C\varepsilon^{-1/2} & \text{if } p = 2 \text{ and } \mathrm{Re}(z) = d/2 + \varepsilon, \\ C\varepsilon^{-1/4} & \text{if } p = 4 \text{ and } \mathrm{Re}(z) = (3d - 2)/4 + \varepsilon. \end{cases}$$

By complex interpolation, with $2 < p < 4$ and $\mathrm{Re}(z) = d(1 - 1/p) + 2/p - 1 + \varepsilon$,

$$\left\{ \int_{\mathbb{T}^d} |\mathcal{F}_0(z, H, R, x)|^{p/2} \, dx \right\}^{1/p}$$

$$\leq C\left(\operatorname{Re}(z)-\left(d\left(1-\frac{1}{p}\right)+\frac{2}{p}-1\right)\right)^{-1/p}.$$

□

**Lemma 22.** *Let $\beta \geq 0$ and $j \geq 1$, and let $\mathcal{F}_j(z,H,R,x)$ with be defined as in the previous lemma. Then there exists $C$ such that for every $H,R \geq 1$ the following hold.*

*(1) If $2 \leq p \leq 4$ and $\operatorname{Re}(z) > d(1-1/p)+2/p-1$,*

$$\left\{\int_{\mathbb{T}^d}\left|\mathcal{F}_j(z,H,R,x)\right|^{p/2}dx\right\}^{1/p}$$
$$\leq C(H+R)^{-j}\left(\operatorname{Re}(z)-d\left(1-\frac{1}{p}\right)-\frac{2}{p}+1\right)^{-1/p}$$

*(2) If $4 \leq p \leq +\infty$ and $\operatorname{Re}(z) > d(1-1/p)-1/2$,*

$$\left\{\int_{\mathbb{T}^d}\left|\mathcal{F}_j(z,H,R,x)\right|^{p/2}dx\right\}^{1/p}$$
$$\leq C(H+R)^{-j}\left(\operatorname{Re}(z)-d\left(1-\frac{1}{p}\right)+\frac{1}{2}\right)^{1/p-1/2}.$$

*Proof.* The proof is as in the previous lemma. If $p=2$ and $\operatorname{Re}(z) > d/2$ then, by Parseval equality,

$$\left\{\int_{\mathbb{T}^d}\left|\mathcal{F}_j(z,H,R,x)\right|^{p/2}dx\right\}^{1/p}$$
$$=\left\{\int_{\mathbb{R}}(R+Hr)^{-2j}\,d\mu(r)\sum_{n\in\mathbb{Z}^d\setminus\{0\}}|c(n)|^2\,|n|^{-2\operatorname{Re}(z)-2j}\right\}^{1/2}$$
$$\leq C(H+R)^{-j}\left(\operatorname{Re}(z)-\frac{d}{2}\right)^{-1/2}.$$

If $4 \leq p \leq +\infty$ and $\operatorname{Re}(z) > d(1-1/p)-1/2$ then, by the Hausdorff Young inequality,

$$\left\{\int_{\mathbb{T}^d}\left|\mathcal{F}_j(z,H,R,x)\right|^{p/2}dx\right\}^{1/p}$$

$$\leq \left\{ \sum_{k\in\mathbb{Z}^d} \left| \widehat{\mathcal{F}}_j(z,H,R,k) \right|^{p/(p-2)} \right\}^{(p-2)/2p}$$

$$\leq C(H+R)^{-j} \left\{ \sum_{k\in\mathbb{Z}^d} \left| (1+|k|)^{d-1-2\operatorname{Re}(z)} \right|^{p/(p-2)} \right\}^{(p-2)/2p}$$

$$\leq C(H+R)^{-j} \left( \operatorname{Re}(z) - d\left(1-\frac{1}{p}\right) + \frac{1}{2} \right)^{1/p-1/2}.$$

By complex interpolation, if $2<p<4$ and $\operatorname{Re}(z) = d(1-1/p)+2/p-1+\varepsilon$,

$$\left\{ \int_{\mathbb{T}^d} |\mathcal{F}_j(z,H,R,x)|^{p/2}\,dx \right\}^{1/p}$$

$$\leq C(H+R)^{-j} \left( \operatorname{Re}(z) - \left( d\left(1-\frac{1}{p}\right) + \frac{2}{p} - 1 \right) \right)^{1/p-1}.$$

□

The estimates in the previous lemmas blow up when $\operatorname{Re}(z) \to \mathit{critical}(z)^+$,

$$\operatorname{Re}(z) \to \begin{cases} \left[ d\left(1-\frac{1}{p}\right) + \frac{2\beta}{p} - \beta \right]^+ & \text{if } \beta\leq 1 \text{ and } p\leq 4, \\ \left[ d\left(1-\frac{1}{p}\right) - \frac{\beta}{2} \right]^+ & \text{if } \beta\leq 1 \text{ and } p\geq 4, \\ \left[ d\left(1-\frac{1}{p}\right) + \frac{2}{p} - 1 \right]^+ & \text{if } \beta\geq 1 \text{ and } p\leq 4, \\ \left[ d\left(1-\frac{1}{p}\right) - \frac{1}{2} \right]^+ & \text{if } \beta\geq 1 \text{ and } p\geq 4, \end{cases}$$

which is the same as $p \to \mathit{critical}(p)^-$,

$$p \to \begin{cases} \left( \dfrac{d-2\beta}{d-\beta-\operatorname{Re}(z)} \right)^- & \text{if } \beta\leq 1 \text{ and } p\leq 4, \\ \left( \dfrac{d}{d-\beta/2-\operatorname{Re}(z)} \right)^- & \text{if } \beta\leq 1 \text{ and } p\geq 4, \\ \left( \dfrac{d-2}{d-1-\operatorname{Re}(z)} \right)^- & \text{if } \beta\geq 1 \text{ and } p\leq 4, \\ \left( \dfrac{d}{d-1/2-\operatorname{Re}(z)} \right)^- & \text{if } \beta\geq 1 \text{ and } p\geq 4. \end{cases}$$

The following lemmas are essentially a rewriting of the previous ones, with $\operatorname{Re}(z) - \mathit{critical}(z)$ replaced by $\mathit{critical}(p) - p$.

**Lemma 23.** *Let $0 \leq \beta < 1$ and let $\mathcal{F}_0(z,H,R,x)$ be as in Lemma 19. Then there exists $C$ such that for every $H,R \geq 1$ the following hold.*

*(1) If $d/2 < \mathrm{Re}(z) \leq (3d-2\beta)/4$ and $2 \leq p < (d-2\beta)/(d-\beta-\mathrm{Re}(z))$, then*

$$\left\{\int_{\mathbb{T}^d} |\mathcal{F}_0(z,H,R,x)|^{p/2}\,dx\right\}^{1/p} \leq C\left(\frac{d-2\beta}{d-\beta-\mathrm{Re}(z)}-p\right)^{-(d-\beta-\mathrm{Re}(z))/(d-2\beta)}.$$

*(2) If $(3d-2\beta)/4 < \mathrm{Re}(z) < d-\beta/2$ and $4 \leq p < d/(d-\beta/2-\mathrm{Re}(z))$, then*

$$\left\{\int_{\mathbb{T}^d} |\mathcal{F}_0(z,H,R,x)|^{p/2}\,dx\right\}^{1/p} \leq C\left(\frac{d}{d-\beta/2-\mathrm{Re}(z)}-p\right)^{(d-\beta/2-\mathrm{Re}(z))/d-1/2}.$$

*(3) If $\mathrm{Re}(z) = d-\beta/2$ and $p < +\infty$, then*

$$\left\{\int_{\mathbb{T}^d} |\mathcal{F}_0(z,H,R,x)|^{p/2}\,dx\right\}^{1/p} \leq Cp^{1/2}.$$

*Proof.* This is a rewriting of Lemma 19. (1) If $d/2 < \mathrm{Re}(z) \leq (3d-2\beta)/4$ and $2 \leq p < (d-2\beta)/(d-\beta-\mathrm{Re}(z))$, then $2 \leq p \leq 4$ and $\mathrm{Re}(z) > d(1-1/p)+2\beta/p-\beta$. The estimate (1) of Lemma 19 applies, and

$$\begin{aligned}
&\left(\mathrm{Re}(z)+\beta-\frac{2\beta}{p}-d\left(1-\frac{1}{p}\right)\right)^{-1/p} \\
&= p^{1/p}(d-\beta-\mathrm{Re}(z))^{-1/p}\left(\frac{d-2\beta}{d-\beta-\mathrm{Re}(z)}-p\right)^{(d-\beta-\mathrm{Re}(z))/(d-2\beta)-1/p} \\
&\times\left(\frac{d-2\beta}{d-\beta-\mathrm{Re}(z)}-p\right)^{-(d-\beta-\mathrm{Re}(z))/(d-2\beta)} \\
&\leq C\left(\frac{d-2\beta}{d-\beta-\mathrm{Re}(z)}-p\right)^{-(d-\beta-\mathrm{Re}(z))/(d-2\beta)}.
\end{aligned}$$

We have used the inequalities $x^{1/x} \leq e^{1/e}$ for every $x > 0$, and $(x-y)^{1/x-1/y} = \left((x-y)^{-(x-y)}\right)^{1/xy} \leq \left(e^{1/e}\right)^{1/xy} \leq e^{1/e}$ for every $x > y \geq 1$. Observe that the above constant $C$ may depend on $d$, $\beta$, $\mathrm{Re}(z)$, but it is independent of $p$.

(2) If $(3d-2\beta)/4 < \mathrm{Re}(z) < d-\beta/2$ and $4 \leq p < d/(d-\beta/2-\mathrm{Re}(z))$ then $\mathrm{Re}(z) > d(1-1/p)-\beta/2$. The estimate (2) of Lemma 19 applies, and

$$\left(\mathrm{Re}(z)+\frac{\beta}{2}-d\left(1-\frac{1}{p}\right)\right)^{1/p-1/2}$$

$$
\begin{aligned}
&= p^{1/2-1/p}\left(d-\left(d-\beta/2-\operatorname{Re}(z)\right)p\right)^{1/p-1/2} \\
&= p^{1/2-1/p}\left(d-\beta/2-\operatorname{Re}(z)\right)^{1/p-1/2}\left(\frac{d}{d-\beta/2-\operatorname{Re}(z)}-p\right)^{1/p-(d-\beta/2-\operatorname{Re}(z))/d} \\
&\times\left(\frac{d}{d-\beta/2-\operatorname{Re}(z)}-p\right)^{(d-\beta/2-\operatorname{Re}(z))/d-1/2} \\
&\leq C\left(\frac{d}{d-\beta/2-\operatorname{Re}(z)}-p\right)^{(d-\beta/2-\operatorname{Re}(z))/d-1/2}.
\end{aligned}
$$

(3) If $\operatorname{Re}(z)=d-\beta/2$ and $p<+\infty$, the estimate (3) of Lemma 19 applies, and

$$
\left(\operatorname{Re}(z)+\frac{\beta}{2}-d\left(1-\frac{1}{p}\right)\right)^{1/p-1/2}=d^{1/p-1/2}p^{-1/p}p^{1/2}\leq Cp^{1/2}.
$$

□

**Lemma 24.** *Let $\beta=1$ and let $\mathcal{F}_0(z,H,R,x)$ be as in Lemma 20. Then there exists $C$ such that for every $H,R\geq 1$ the following hold.*

*(1) If $d/2<\operatorname{Re}(z)\leq(3d-2)/4$ and $2\leq p<\dfrac{d-2}{d-1-\operatorname{Re}(z)}$ then*

$$
\begin{aligned}
&\left\{\int_{\mathbb{T}^d}\left|\mathcal{F}_0(z,H,R,x)\right|^{p/2}dx\right\}^{1/p} \\
&\leq C\left(\frac{d-2}{d-1-\operatorname{Re}(z)}-p\right)^{-(\operatorname{Re}(z)-1)/(d-2)}.
\end{aligned}
$$

*(2) If $(3d-2)/4<\operatorname{Re}(z)<d-1/2$ and $4\leq p<d/\left(d-\operatorname{Re}(z)-1/2\right)$ then*

$$
\begin{aligned}
&\left\{\int_{\mathbb{T}^d}\left|\mathcal{F}_0(z,H,R,x)\right|^{p/2}dx\right\}^{1/p} \\
&\leq C\left(\frac{d}{d-1/2-\operatorname{Re}(z)}-p\right)^{-(2\operatorname{Re}(z)+1)/2d}.
\end{aligned}
$$

*(3) If $\operatorname{Re}(z)=d-1/2$ and $p<+\infty$ then*

$$
\left\{\int_{\mathbb{T}^d}\left|\mathcal{F}_0(z,H,R,x)\right|^{p/2}dx\right\}^{1/p}\leq Cp.
$$

*Proof.* This is a rewriting of Lemma 20, and the proof is as in the previous lemma.

□

**Lemma 25.** *Let $\beta > 1$ and let $\mathcal{F}_0(z,H,R,x)$ be as in the Lemma 21. Then there exists $C$ such that for every $H,R \geq 1$ the following hold.*

*(1) If $d/2 < \mathrm{Re}(z) \leq (3d-2)/4$ and $2 \leq p < \dfrac{d-2}{d-1-\mathrm{Re}(z)}$ then*

$$\left\{\int_{\mathbb{T}^d} |\mathcal{F}_0(z,H,R,x)|^{p/2}\,dx\right\}^{1/p} \leq C\left(\frac{d-2}{d-1-\mathrm{Re}(z)} - p\right)^{-(d-1-\mathrm{Re}(z))/(d-2)}.$$

*(2) If $(3d-2)/4 < \mathrm{Re}(z) < d-1/2$ and $4 \leq p < d/(d-\mathrm{Re}(z)-1/2)$ then*

$$\left\{\int_{\mathbb{T}^d} |\mathcal{F}_0(z,H,R,x)|^{p/2}\,dx\right\}^{1/p} \leq C\left(\frac{d}{d-1/2-\mathrm{Re}(z)} - p\right)^{(d-1/2-\mathrm{Re}(z))/d-1/2}.$$

*(3) If $\mathrm{Re}(z) = d-1/2$ and $p < +\infty$ then*

$$\left\{\int_{\mathbb{T}^d} |\mathcal{F}_0(z,H,R,x)|^{p/2}\,dx\right\}^{1/p} \leq Cp^{1/2}.$$

*Proof.* This is a rewriting of Lemma 21, and the proof is as in the previous lemma. □

**Lemma 26.** *Let $\beta \geq 0$ and $j \geq 1$, and let $\mathcal{F}_j(z,H,R,x)$ be as in the Lemma 22. Then there exists $C$ such that for every $H,R \geq 1$ the following hold.*

*(1) If $d/2 < \mathrm{Re}(z) \leq (3d-2)/4$ and $2 \leq p < \dfrac{d-2}{d-1-\mathrm{Re}(z)}$ then*

$$\left\{\int_{\mathbb{T}^d} |\mathcal{F}_j(z,H,R,x)|^{p/2}\,dx\right\}^{1/p} \leq C(H+R)^{-j}\left(\frac{d-2}{d-1-\mathrm{Re}(z)} - p\right)^{-(d-1-\mathrm{Re}(z))/(d-2)}.$$

*(2) If $(3d-2)/4 < \mathrm{Re}(z) < d-1/2$ and $4 \leq p < d/(d-\mathrm{Re}(z)-1/2)$ then*

$$\left\{\int_{\mathbb{T}^d} |\mathcal{F}_j(z,H,R,x)|^{p/2}\,dx\right\}^{1/p} \leq C(H+R)^{-j}\left(\frac{d}{d-1/2-\mathrm{Re}(z)} - p\right)^{(d-1/2-\mathrm{Re}(z))/d-1/2}.$$

(3) *If* $\mathrm{Re}(z) = d - 1/2$ *and* $p < +\infty$ *then*

$$\left\{ \int_{\mathbb{T}^d} \left| \mathcal{F}_j(z, H, R, x) \right|^{p/2} dx \right\}^{1/p} \leq C (H + R)^{-j} p^{1/2}.$$

*Proof.* This is a rewriting of Lemma 22, and the proof is as in the previous lemma. □

The above lemmas are enough for an upper bound for the norms of the discrepancy. In order to prove the asymptotics of the norms as $H \to +\infty$, one has to work a bit more. Recall that $\Phi_0(z, r, x)$, the main term in the asymptotic expansion of the discrepancy, is defined by

$$\Phi_0(z, r, x) = \sum_{n \in \mathbb{Z}^d \setminus \{0\}} a_0(n) |n|^{-z} e^{-2\pi i g(n) r} e^{2\pi i n x}$$
$$+ \sum_{n \in \mathbb{Z}^d \setminus \{0\}} b_0(n) |n|^{-z} e^{2\pi i g(-n) r} e^{2\pi i n x}.$$

The following lemma is similar to the previous ones, just observe that one integrates the square of this function, and not the square of the modulus.

**Lemma 27.** *Define* $\mathcal{G}(z, x)$ *and* $\mathcal{B}(z, H, R, x)$ *by*

$$\mathcal{G}(z, x) = \sum_{k \in \mathbb{Z}^d} \left( 2 \sum_{n \in \mathbb{Z}^d \setminus \{0, k\},\ g(n-k) = g(n)} a_0(n) b_0(k - n) |n|^{-z} |k - n|^{-z} \right) e^{2\pi i k x},$$

$$\int_{\mathbb{R}} \Phi_0(z, r, x)^2 \, d\mu_{H,R}(r) = \mathcal{G}(z, x) + \mathcal{B}(z, H, R, x).$$

(1) *The function* $\mathcal{G}(z, x)$ *does not depend on* $H$ *and* $R$ *and it is in* $L^{p/2}(\mathbb{T}^d)$*, under the relations between* $p$ *and* $\beta$ *in Lemma 19 or the equivalent Lemma 22,*

$$\left\{ \int_{\mathbb{T}^d} \left| \mathcal{G}(z, x) \right|^{p/2} dx \right\}^{1/p} \leq C.$$

(2) *Under the relations between* $p$ *and* $\beta$ *in the lemmas 19, 20, 21, or the equivalent lemmas 22, 23, 24, also the function* $\mathcal{B}(z, H, R, x)$ *is in* $L^{p/2}(\mathbb{T}^d)$*, and there exists* $C$ *such that for every* $H, R \geq 1$*,*

$$\left\{ \int_{\mathbb{T}^d} \left| \mathcal{B}(z, H, R, x) \right|^{p/2} dx \right\}^{1/p} \leq C.$$

*Moreover, if $\beta > 0$ then this function vanishes as $H \to +\infty$, uniformly in $R \geq 1$,*

$$\lim_{H\to+\infty} \left\{ \int_{\mathbb{T}^d} |\mathcal{B}(z,H,R,x)|^{p/2} \, dx \right\}^{1/p} = 0.$$

*Proof.* Expanding the product $\Phi_0(z,r,x) \cdot \Phi_0(z,r,x)$ and integrating, one obtains

$$\begin{aligned}&\int_{\mathbb{R}} \Phi_0(z,r,x)^2 \, d\mu_{H,R}(r) \\ &= \mathcal{G}(z,x) + \mathcal{B}_1(z,H,R,x) + \mathcal{B}_2(z,H,R,x) + \mathcal{B}_3(z,H,R,x),\end{aligned}$$

where

$$\mathcal{G}(z,x) = 2 \sum_{k\in\mathbb{Z}^d} \sum_{n\in\mathbb{Z}^d\setminus\{0,k\},\ g(n-k)=g(n)} a_0(n)\, b_0(k-n)\, |n|^{-z} \, |k-n|^{-z} \, e^{2\pi ikx},$$

$$\begin{aligned}&\mathcal{B}_1(z,H,R,x) \\ &= 2 \sum_{k\in\mathbb{Z}^d} \sum_{n\in\mathbb{Z}^d\setminus\{0,k\},\ g(n-k)\neq g(n)} a_0(n)\, b_0(k-n)\, |n|^{-z} \, |k-n|^{-z} \, e^{2\pi ikx} \\ &\times e^{2\pi i(g(n-k)-g(n))R} \int_{\mathbb{R}} e^{2\pi iH(g(n-k)-g(n))r} d\mu(r),\end{aligned}$$

$$\begin{aligned}&\mathcal{B}_2(z,H,R,x) \\ &= \sum_{k\in\mathbb{Z}^d} \sum_{n\in\mathbb{Z}^d\setminus\{0,k\}} a_0(n)\, a_0(k-n)\, |n|^{-z} \, |k-n|^{-z} \, e^{2\pi ikx} \\ &\times e^{-2\pi i(g(n)+g(k-n))R} \int_{\mathbb{R}} e^{-2\pi iH(g(n)+g(k-n))r} d\mu(r),\end{aligned}$$

$$\begin{aligned}&\mathcal{B}_3(z,H,R,x) \\ &= \sum_{k\in\mathbb{Z}^d} \sum_{n\in\mathbb{Z}^d\setminus\{0,k\}} b_0(n)\, b_0(k-n)\, |n|^{-z} \, |k-n|^{-z} \, e^{2\pi ikx} \\ &\times e^{2\pi i(g(-n)+g(n-k))R} \int_{\mathbb{R}} e^{2\pi iH(g(-n)+g(n-k))r} d\mu(r).\end{aligned}$$

Observe that $\mathcal{G}(z,x)$ does not depend on $H$ and $R$, and let us consider the Fourier coefficients of this function. Since $a_0(n)$ and $b_0(-n)$ are bounded, the Fourier coefficient with $k = 0$ is bounded by

$$\left|\widehat{\mathcal{G}}(z,0)\right| = \left| 2 \sum_{n\in\mathbb{Z}^d\setminus\{0\}} a_0(n) b_0(-n) \, |n|^{-2z} \right| \leq C \sum_{n\in\mathbb{Z}^d\setminus\{0\}} |n|^{-2\,\mathrm{Re}(z)} \leq C.$$

By the Lemma 17 with an arbitrary $\gamma > 1$, the Fourier coefficients with $k \neq 0$ can be bounded by

$$\begin{aligned}
&\left|\widehat{\mathcal{G}}(z,k)\right| = \left|2 \sum_{n\in\mathbb{Z}^d\setminus\{0,k\},\ g(n-k)=g(n)} a_0(n)\, b_0(k-n)\, |n|^{-z}\, |k-n|^{-z}\right| \\
&\leq C \sum_{n\in\mathbb{Z}^d\setminus\{0,k\}} |n|^{-\operatorname{Re}(z)}\, |k-n|^{-\operatorname{Re}(z)}\, (1+|g(n-k)-g(n)|)^{-\gamma} \\
&\leq C \int_{\mathbb{R}^d} |x|^{-\operatorname{Re}(z)}\, |k-x|^{-\operatorname{Re}(z)}\, (1+|g(x-k)-g(x)|)^{-\gamma}\, dx \\
&\leq C\, |k|^{d-1-2\operatorname{Re}(z)}.
\end{aligned}$$

The estimates of the Fourier coefficients of $\mathcal{B}_1(z,H,R,x)$ are similar to the ones of $\mathcal{G}(z,x)$. First observe that $\widehat{\mathcal{B}_1}(z,H,R,0) = 0$. Then, by the assumption on the measure $d\mu(r)$, if $k \neq 0$ there exists $C$ such that for every $H \geq 1$,

$$\begin{aligned}
&\left|\widehat{\mathcal{B}_1}(z,H,R,k)\right| \\
&= \left|2 \sum_{n\in\mathbb{Z}^d\setminus\{0,k\},\ g(n-k)\neq g(n)} a_0(n)\, b_0(k-n)\, |n|^{-z}\, |k-n|^{-z}\right. \\
&\left.\times e^{2\pi i(g(n-k)-g(n))R} \int_{\mathbb{R}} e^{2\pi i H(g(n-k)-g(n))r} d\mu(r)\right| \\
&\leq C \sum_{n\in\mathbb{Z}^d\setminus\{0,k\}} |n|^{-\operatorname{Re}(z)}\, |k-n|^{-\operatorname{Re}(z)}\, (1+H\,|g(n-k)-g(n)|)^{-\beta} \\
&\leq C \int_{\mathbb{R}^d} |x|^{-\operatorname{Re}(z)}\, |k-x|^{-\operatorname{Re}(z)}\, (1+H\,|g(x-k)-g(x)|)^{-\beta}\, dx.
\end{aligned}$$

By Lemma 17, the last integral is bounded by

$$\begin{cases} C\,|k|^{d-2\alpha-\beta} & \text{if } 0 \leq \beta < 1, \\ C\,|k|^{d-2\alpha-1} \log(2+|k|) & \text{if } \beta = 1, \\ C\,|k|^{d-2\alpha-1} & \text{if } \beta > 1. \end{cases}$$

These estimates are independent of $H, R \geq 1$. Hence, by dominated convergence applied to the sum that defines $\widehat{\mathcal{B}_1}(z,H,R,k)$, if $\beta > 0$ then

$$\lim_{H\to+\infty} \left\{\widehat{\mathcal{B}_1}(z,H,R,k)\right\} = 0.$$

The estimates of the Fourier coefficients of $\mathcal{B}_2(z,H,R,x)$ and $\mathcal{B}_3(z,H,R,x)$ are easier. Since $g(x) \geq c\,|x|$ with $c > 0$,

$$\left|\widehat{\mathcal{B}_2}(z,H,R,k)\right|$$

$$
\begin{aligned}
&= \Bigg| \sum_{n \in \mathbb{Z}^d \setminus \{0,k\}} a_0(n)\, a_0(k-n)\, |n|^{-z} |k-n|^{-z} \\
&\times e^{-2\pi i (g(n)+g(k-n))R} \int_{\mathbb{R}} e^{-2\pi i H (g(n)+g(k-n)) r} d\mu(r) \Bigg| \\
&\le C H^{-\beta} \sum_{n \in \mathbb{Z}^d \setminus \{0,k\}} |n|^{-\operatorname{Re}(z)} |k-n|^{-\operatorname{Re}(z)} \left(|n| + |k-n|\right)^{-\beta} \\
&\le C H^{-\beta} \left(1+|k|\right)^{-\beta} \sum_{n \in \mathbb{Z}^d \setminus \{0,k\}} |n|^{-\operatorname{Re}(z)} |k-n|^{-\operatorname{Re}(z)} \\
&\le C H^{-\beta} \left(1+|k|\right)^{d-\beta-2\operatorname{Re}(z)} .
\end{aligned}
$$

Moreover, by this estimate, if $\beta > 0$ then

$$
\lim_{H \to +\infty} \left\{ \widehat{\mathcal{B}}_2(z,H,R,k) \right\} = 0.
$$

The estimates of the Fourier coefficients of $\mathcal{B}_3(z,H,R,x)$ are analogous to the ones of $\mathcal{B}_2(z,H,R,x)$. The estimates of the norms in $L^{p/2}\left(\mathbb{T}^d\right)$ of these functions in the cases $p=2$ and $p \ge 4$ follow from the estimates of the Fourier coefficients of the functions involved, the Parseval or Hausdorff Young inequality, and dominated convergence. Finally, the cases $2 < p < 4$ follow by complex interpolation. The details are as in the proof of the lemmas 19, 20, 21. □

The above lemmas are sufficient to prove the theorem for norms $p$ less than the critical index $A$. In order to reach the critical index, one need an easy lemma suggested by the Yano extrapolation theorem. See [39] or [40](chapter XII-4.41).

**Lemma 28.** *Let $\alpha \ge 0$, $A \ge 1$, $K \ge 2$, and assume that*

$$
\begin{aligned}
&\sup_{x \in \mathbb{T}^d} \left\{ |\mathcal{F}(x)| \right\} \le K, \\
&\left\{ \int_{\mathbb{T}^d} |\mathcal{F}(x)|^p \, dx \right\}^{1/p} \le (A-p)^{-\alpha} \quad \text{for every } 0 < p < A.
\end{aligned}
$$

*Then, there exists $C$ independent of $\mathcal{F}(x)$ and $K$ such that*

$$
\left\{ \int_{\mathbb{T}^d} |\mathcal{F}(x)|^A \, dx \right\}^{1/A} \le C \log^{\alpha}(K) .
$$

*Proof.* If $\alpha \ge 0$ and $A \ge 1$ and $0 < p < A$,

$$
\begin{aligned}
&\left\{ \int_{\mathbb{T}^d} |\mathcal{F}(x)|^A \, dx \right\}^{1/A} \le \sup_{x \in \mathbb{T}^d} \left\{ |\mathcal{F}(x)|^{(A-p)/A} \right\} \left\{ \int_{\mathbb{T}^d} |\mathcal{F}(x)|^p \, dx \right\}^{1/A} \\
&\le K^{(A-p)/A} (A-p)^{-\alpha p/A} = A^{-\alpha p/A} (1-p/A)^{\alpha(1-p/A)} K^{1-p/A} (1-p/A)^{-\alpha}
\end{aligned}
$$

$$\leq K^{1-p/A}\left(1-p/A\right)^{-\alpha}.$$

Then, with $1-p/A=t$,

$$\left\{\int_{\mathbb{T}^d}\left|\mathcal{F}\left(x\right)\right|^A dx\right\}^{1/A}\leq\inf_{0<t<1}\left\{K^t t^{-\alpha}\right\}=e^{\alpha}\alpha^{-\alpha}\log^{\alpha}\left(K\right).$$

□

*Proof.* (of Theorem 3) By Lemma 15,

$$\left\{\int_{\mathbb{T}^d}\left[\int_{\mathbb{R}}\left|r^{-(d-1)/2}\mathcal{D}\left(r\Omega-x\right)\right|^2 d\mu_{H,R}\left(r\right)\right]^{p/2}dx\right\}^{1/p}$$
$$\leq\sum_{j=0}^{h}\left\{\int_{\mathbb{T}^d}\left[\int_{\mathbb{R}}\left|\Phi_j\left(\left(d+1\right)/2,r,x\right)\right|^2 d\mu_{H,R}\left(r\right)\right]^{p/2}dx\right\}^{1/p}$$
$$+\left\{\int_{\mathbb{T}^d}\left[\int_{\mathbb{R}}\left|\mathcal{R}_h\left(r,x\right)\right|^2 d\mu_{H,R}\left(r\right)\right]^{p/2}dx\right\}^{1/p}.$$

Since the $\Phi_j\left(\left(d+1\right)/2,r,x\right)$ are a sum of two $\Theta_j\left(\left(d+1\right)/2,r,x\right)$ to which the above lemmas apply, under appropriate relations between $p$ and $\beta$ the mixed norm of the discrepancy is uniformly bounded, and (1) follows from the estimates in Lemma 23, 24, 25, 26.

(2) follows from (1) via Lemma 28. The cases $p<+\infty$ follow from the Lemma 28. One has just to recall that the discrepancy satisfy the trivial bound $\left|\mathcal{D}\left(r\Omega-x\right)\right|\leq Cr^d$ for every $r\geq 1$. The case $d=2$ and $p=+\infty$ and $d\mu\left(x\right)=\chi_{\{0<r<1\}}\left(r\right)$ is proved in [20]. An alternative proof of all cases can also be obtained via the mollified discrepancy. For example, when $d=2$, with the techniques in the above lemmas, one can prove that if $1\leq H\leq R$, and $\delta\leq 1/R$,

$$\sup_{x\in\mathbb{T}^2}\left\{\int_{\mathbb{R}}\left|\left(r\pm\delta\right)^{-3/2}\sum_{n\in\mathbb{Z}^2-\{0\}}\widehat{\varphi}\left(\delta n\right)\widehat{\chi}_{\Omega}\left(\left(r\pm\delta\right)n\right)e^{2\pi inx}\right|^2 d\mu_{H,R}\left(r\right)\right\}^{1/2}$$
$$\leq\begin{cases}C\left[\sum_{n\in\mathbb{Z}^2}\left(1+\left|\delta n\right|\right)^{-\gamma}\left(1+\left|k\right|\right)^{-\beta-1}\right]^{1/2} & \text{if } 0\leq\beta<1,\\ C\left[\sum_{n\in\mathbb{Z}^2}\left(1+\left|\delta n\right|\right)^{-\gamma}\left(1+\left|k\right|\right)^{-2}\log\left(2+\left|k\right|\right)\right]^{1/2} & \text{if } \beta=1,\\ C\left[\sum_{n\in\mathbb{Z}^2}\left(1+\left|\delta n\right|\right)^{-\gamma}\left(1+\left|k\right|\right)^{-2}\right]^{1/2} & \text{if } \beta>1,\end{cases}$$

$$\leq \begin{cases} C\delta^{(1-\beta)/2} & \text{if } 0 \leq \beta < 1, \\ C\log(1/\delta) & \text{if } \beta = 1, \\ C\log^{1/2}(1/\delta) & \text{if } \beta > 1, \end{cases}$$
$$\leq \begin{cases} CR^{(1-\beta)/2} & \text{if } 0 \leq \beta < 1, \\ C\log(R) & \text{if } \beta = 1, \\ C\log^{1/2}(R) & \text{if } \beta > 1. \end{cases}$$

In order to prove the asymptotic estimate for the norm in (3), with the notation of the previous lemmas, since the discrepancy is real one can write

$$\left\{\int_{\mathbb{T}^d}\left[\int_{\mathbb{R}}\left|r^{-(d-1)/2}\mathcal{D}\left(r\Omega - x\right)\right|^2 d\mu_{H,R}(r)\right]^{p/2} dx\right\}^{1/p}$$
$$= \left\{\int_{\mathbb{T}^d}\left[\int_{\mathbb{R}}\left(\sum_{j=0}^{h}\Phi_j\left((d+1)/2, r, x\right) + \mathcal{R}_h(r,x)\right)^2 d\mu_{H,R}(r)\right]^{p/2} dx\right\}^{1/p}.$$

The inner integral is equal to

$$\int_{\mathbb{R}}\left(\sum_{j=0}^{h}\Phi_j\left((d+1)/2, r, x\right) + \mathcal{R}_h(r,x)\right)^2 d\mu_{H,R}(r)$$
$$= \mathcal{G}\left((d+1)/2, x\right) + \mathcal{B}\left((d+1)/2, H, R, x\right)$$
$$+ \sum_{0\leq i,j\leq h,\ i+j>0}\int_{\mathbb{R}}\Phi_i\left((d+1)/2, r, x\right)\Phi_j\left((d+1)/2, r, x\right) d\mu_{H,R}(r)$$
$$+ 2\sum_{0\leq j\leq h}\int_{\mathbb{R}}\Phi_j\left((d+1)/2, r, x\right)\mathcal{R}_h(r,x)\, d\mu_{H,R}(r)$$
$$+ \int_{\mathbb{R}}\mathcal{R}_h(r,x)^2\, d\mu_{H,R}(r).$$

By the above lemmas, all these terms give a bounded contribution. The main term is $\mathcal{G}\left((d+1)/2, x\right)$, and it is independent of $H$ and $R$. The contributions of the other terms is negligible when $H \to +\infty$. For example, let us estimate the integral with the mixed product $\Phi_i\left((d+1)/2, r, x\right)\Phi_j\left((d+1)/2, r, x\right)$. A repeated application of the Cauchy Schwarz inequality gives

$$\int_{\mathbb{T}^d}\left|\int_{\mathbb{R}}\Phi_i\left((d+1)/2, r, x\right)\Phi_j\left((d+1)/2, r, x\right) d\mu_{H,R}(r)\right|^{p/2} dx$$
$$\leq \int_{\mathbb{T}^d}\left[\int_{\mathbb{R}}\left|\Phi_i\left((d+1)/2, r, x\right)\right|^2 d\mu_{H,R}(r)\right]^{p/4}$$

$$\times \left[ \int_{\mathbb{R}} |\Phi_j ((d+1)/2, r, x)|^2 \, d\mu_{H,R}(r) \right]^{p/4} dx$$

$$\leq \left\{ \int_{\mathbb{T}^d} \left[ \int_{\mathbb{R}} |\Phi_i ((d+1)/2, r, x)|^2 \, d\mu_{H,R}(r) \right]^{p/2} \right\}^{1/2}$$

$$\times \left\{ \int_{\mathbb{T}^d} \left[ \int_{\mathbb{R}} |\Phi_j ((d+1)/2, r, x)|^2 \, d\mu_{H,R}(r) \right]^{p/2} \right\}^{1/2}.$$

By Lemma 19, if $i + j > 0$ this converges to 0 when $H + R \to +\infty$. □

*Proof.* (of Corollary 1) The corollary is an immediate consequence of part (1) of the theorem, and another easy lemma suggested by the Yano extrapolation theorem. See [39] or [40] (chapter XII-4.41). □

**Lemma 29.** *(1) Assume that* $\alpha > 0$ *and that for every* $p < A < +\infty$*,*

$$\left\{ \int_{\mathbb{T}^d} |\mathcal{F}(x)|^p \, dx \right\}^{1/p} \leq (A - p)^{-\alpha}.$$

*Then, for every* $\gamma > 1 + \alpha A$ *there exists* $C$ *independent of* $\mathcal{F}(x)$ *such that*

$$\int_{\mathbb{T}^d} |\mathcal{F}(x)|^A \log^{-\gamma} (2 + |\mathcal{F}(x)|) \, dx \leq C.$$

*(2) Assume that* $\alpha > 0$ *and that for every* $p < +\infty$*,*

$$\left\{ \int_{\mathbb{T}^d} |\mathcal{F}(x)|^p \, dx \right\}^{1/p} \leq p^\alpha.$$

*Then, for every* $\gamma < \alpha/e$ *there exists* $C > 0$ *independent of* $f(x)$ *such that*

$$\int_{\mathbb{T}^d} \exp\left( \gamma |\mathcal{F}(x)|^{1/\alpha} \right) dx \leq C.$$

*Proof.* (1) Let $\mathcal{F}_0(x) = \mathcal{F}(x) \chi_{\{|\mathcal{F}(x)|<2\}}(x)$ and $\mathcal{F}_j(x) = \mathcal{F}(x) \chi_{\{2^j \leq |\mathcal{F}(x)| < 2^{j+1}\}}(x)$ if $j \geq 1$, and let $\varepsilon_j$ the measure of the set where $\mathcal{F}_j(x) \neq 0$. Then, if $j \geq 1$ and $p < A$,

$$2^{jp} \varepsilon_j \leq \int_{\mathbb{T}^d} |\mathcal{F}_j(x)|^p \, dx \leq \int_{\mathbb{T}^d} |\mathcal{F}(x)|^p \, dx \leq (A - p)^{-\alpha p}.$$

Hence, $\varepsilon_j \leq 2^{-jp} (A - p)^{-\alpha p} = 2^{-Aj} 2^{j(A-p)} (A - p)^{-\alpha A}$, and the minimum of this expression is when $p = A - \alpha A / j \log(2)$. This gives

$$\varepsilon_j \leq C 2^{-Aj} j^{\alpha A}.$$

Hence, if $\gamma > 1 + \alpha A$,

$$\begin{aligned}
&\int_{\mathbb{T}^d} |\mathcal{F}(x)|^A \log^{-\gamma}(2 + |\mathcal{F}(x)|)\,dx \\
&= \sum_{j=0}^{+\infty} \int_{\mathbb{T}^d} |\mathcal{F}_j(x)|^A \log^{-\gamma}(2 + |\mathcal{F}_j(x)|)\,dx \\
&\leq 2^A \log^{-\gamma}(2) + \sum_{j=1}^{+\infty} 2^{(j+1)A} \log^{-\gamma}\left(2 + 2^j\right)\varepsilon_j \\
&\leq C + C\sum_{j=1}^{+\infty} j^{\alpha A - \gamma} \leq C.
\end{aligned}$$

(2) Let $\mathcal{F}_j(x) = \mathcal{F}(x)\,\chi_{\{j \leq |\mathcal{F}(x)| < j+1\}}(x)$, and let $\varepsilon_j$ the measure of the set where $\mathcal{F}_j(x) \neq 0$. Then, if $j \geq 1$ and $p < A$,

$$j^p \varepsilon_j \leq \int_{\mathbb{T}^d} |\mathcal{F}_j(x)|^p\,dx \leq \int_{\mathbb{T}^d} |\mathcal{F}(x)|^p\,dx \leq p^{\alpha p}.$$

Hence, $\varepsilon_j \leq j^{-p} p^{\alpha p}$. The minimum of this expression is when $p = e^{-1} j^{1/\alpha}$, and this gives

$$\varepsilon_j \leq \exp\left(-(\alpha/e)\,j^{1/\alpha}\right).$$

Hence, if $\gamma < \alpha/e$,

$$\begin{aligned}
&\int_{\mathbb{T}^d} \exp\left(\gamma\,|\mathcal{F}(x)|^{1/\alpha}\right)dx = \sum_{j=0}^{+\infty} \int_{\mathbb{T}^d} \exp\left(\gamma\,|\mathcal{F}_j(x)|^{1/\alpha}\right)dx \\
&\leq \sum_{j=0}^{+\infty} \varepsilon_j \exp\left(\gamma\,(j+1)^{1/\alpha}\right) \leq e^{\gamma} + \sum_{j=1}^{+\infty} \exp\left(-\left(\alpha/e - \gamma\,(1+1/j)^{1/\alpha}\right) j^{1/\alpha}\right) \leq C.
\end{aligned}$$

□

*Proof.* (of Corollary 2) By the Lemma 15,

$$\mathcal{G}(x) = \sum_{k \in \mathbb{Z}^d} \left( 2 \sum_{\substack{n \in \mathbb{Z}^d \setminus \{0,k\},\\ g(n-k) = g(n)}} a_0(n)\,b_0(k-n)\,|n|^{-(d+1)/2}\,|k-n|^{-(d+1)/2} \right) e^{2\pi i k x}.$$

Since $c\,|x| \leq g(x) \leq C\,|x|$, if $g(n-k) = g(n)$ then $|k| \leq C\,|n|$ and

$$a_0(n)\,b_0(k-n)\,|n|^{-(d+1)/2}\,|k-n|^{-(d+1)/2} \leq C\,|n|^{-d-1} \leq C\,|k|^{-d-1}.$$

Hence, under the assumption that that for every $m$ in $\mathbb{Z}^d$ the equation $g(m) = g(n)$ has at most $C$ solutions $n$ in $\mathbb{Z}^d$, the Fourier coefficients of $\mathcal{G}(x)$ are bounded by

$$2 \sum_{n\in\mathbb{Z}^d\setminus\{0,k\},\ g(n-k)=g(n)} a_0(n)\, b_0(k-n)\, |n|^{-(d+1)/2}\, |k-n|^{-(d+1)/2} \le C\, |k|^{-d-1}.$$

This implies that the Fourier expansion that defines $\mathcal{G}(x)$ is absolutely and uniformly and convergent, and $\mathcal{G}(x)$ is bounded and continuous. In particular, under the additional assumption that $g(m) \neq g(n)$ for every $m, n \in \mathbb{Z}^d$ with $m \neq n$, all Fourier coefficients with $k \neq 0$ vanish, and this function reduces to the constant

$$2 \sum_{n\in\mathbb{Z}^d\setminus\{0\}} a_0(n)\, b_0(-n)\, |n|^{-d-1}.$$

In order to prove theorem we need an easy algebraic lemma. □

**Lemma 30.** *If $(A, B, C, D, ...)$ is a vector with integers coordinates, then the integer vectors $(x, y, z, w, ...)$ which are solutions to the equation $Ax + By + Cz + Dw + ... = 0$ are a lattice. Assume that $A$ and $B$ are coprimes, so that there exist integers there exist $u$ and $v$ such that $Au + Bv = 1$. Then a basis of the lattice $\{Ax + By + Cz + Dw + ... = 0\}$ is*

$$\{(B, -A, 0, 0, ...), (uC, vC, -1, 0, ...), (uD, vD, 0, -1, ...), ...\}.$$

*The area of a fundamental domain of this lattice is the length of the vector $(A, B, C, D, ...)$,*

$$\sqrt{A^2 + B^2 + C^2 + D^2 + ....}$$

*Proof.* The solutions to the equation $Ax + By + Cz + Dw + ... = 0$ are a sum of a particular solution to the non homogeneous equation $Ax + By = -Cz - Dw - ...$, plus all solutions to the homogeneous equation $Ax + By = 0$. The solutions to the homogeneous equation $Ax + By = 0$ are $x = Br$ and $y = -Ar$, and a particular solution to the equation $Ax + By = -Cz - Dw - ...$ is $x = -u(Cz + Dw + ...)$ and $y = -v(Cz + Dw + ...)$. Hence, all integral solutions to $Ax+By+Cz+Dw+... = 0$ are

$$(x, y, z, w, ...) = r(B, -A, 0, 0, ...) + s(uC, vC, -1, 0, ...) + t(uD, vD, 0, -1, ...) + ....$$

The area of a fundamental domain of the lattice $\{Ax + By + Cz + Dw + ... = 0\}$ is the length of the vector

$$\left\| \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 & ... \\ B & -A & 0 & 0 & ... \\ uC & vC & -1 & 0 & ... \\ uD & vD & 0 & -1 & ... \\ ... & ... & ... & ... & ... \end{bmatrix} \right\|$$

$$= \pm A\mathbf{e}_1 \pm B\mathbf{e}_2 \pm C\left(Au+Bv\right)\mathbf{e}_3 \pm D\left(Au+Bv\right)\mathbf{e}_4 \pm ...$$
$$= \left(\pm A, \pm B, \pm C, \pm D, ...\right).$$

□

*Proof.* (of Theorem 4) Let us first prove (2). Set

$$\mathcal{G}\left(H,R,x\right) = \int_{\mathbb{R}} \left|r^{-(d-1)/2}\mathcal{D}\left(r\Omega - x\right)\right|^2 d\mu_{H,R}\left(r\right).$$

If the statement of the theorem does not apply to $\Omega$, then there exist $2d/\left(d-3\right) < p < +\infty$ and sequences $\{R_n\} \to +\infty$ and $\{H_n\} \to +\infty$ such that

$$\limsup_{n\to+\infty} \left\{\int_{\mathbb{T}^d} \left|\mathcal{G}\left(H_n,R_n,x\right)\right|^{p/2} dx\right\}^{2/p} < +\infty.$$

Then a suitable subsequence converges weakly in $L^{p/2}\left(\mathbb{T}^d\right)$.
Since weak convergence implies the convergence of Fourier coefficients, by the assumption that $\lim_{|\zeta|\to+\infty}\{|\widehat{\mu}\left(\zeta\right)|\} = 0$ the subsequence converges weakly to the function $\mathcal{G}\left(x\right)$ defined by the Fourier expansion

$$\mathcal{G}\left(x\right) = \sum_{k\in\mathbb{Z}^d}\left(2\sum_{\substack{n\in\mathbb{Z}^d\setminus\{0,k\},\\ g(n-k)=g(n)}} a_0\left(n\right)b_0\left(k-n\right)\left|n\right|^{-(d+1)/2}\left|k-n\right|^{-(d+1)/2}\right)e^{2\pi ikx}.$$

This follows from from Lemma 18 and Lemma 27. By Theorem 3 this function $\mathcal{G}\left(x\right)$ is in $L^{p/2}\left(\mathbb{T}^d\right)$ for every $p < \left(2d-4\right)/\left(d-3\right)$. In order to prove the theorem, it suffices to show that when $\Sigma = \{|x| \le 1\}$ this function is not in $L^{p/2}\left(\mathbb{T}^d\right)$ if $p > 2d/\left(d-3\right)$. In order to give an estimate of the norm from below, one can test this function against a Bessel potential of order $\alpha > 0$,

$$\mathcal{B}\left(x\right) = \sum_{k\in\mathbb{Z}^d}\left(1+4\pi^2\left|k\right|^2\right)^{-\alpha/2}e^{2\pi ikx}.$$

This Bessel potential is a positive integrable function, which blows up as $x \to 0$ with an asymptotic expansion

$$\mathcal{B}\left(x\right) \approx \begin{cases} C\left|x\right|^{\alpha-d} & \text{if } 0<\alpha<d,\\ C\log\left(1/\left|x\right|\right) & \text{if } \alpha=d,\\ C & \text{if } \alpha>d.\end{cases}$$

This follows from the Poisson summation formula, and the asymptotic estimate of the Bessel potentials in $\mathbb{R}^d$. See [35] (chapter V 3.1). It follows that if $1 \le r \le +\infty$ and $\alpha > d(1-1/r)$, then

$$\left\{\int_{\mathbb{T}^d} |\mathcal{B}(x)|^r \, dx\right\}^{1/r} < +\infty.$$

By the way, when $2 \le r \le +\infty$ and $1/r + 1/s = 1$ and $\alpha > d(1-1/r) = d/s$, this also follows via the Hausdorff Young inequality:

$$\left\{\int_{\mathbb{T}^d} \left|\sum_{k\in\mathbb{Z}^d} \left(1+4\pi^2|k|^2\right)^{-\alpha/2} e^{2\pi ikx}\right|^r dx\right\}^{1/r} \le \left\{\sum_{k\in\mathbb{Z}^d} \left|\left(1+4\pi^2|k|^2\right)^{-\alpha/2}\right|^s\right\}^{1/s} < +\infty.$$

If $1/r + 1/s = 1$, then

$$2\sum_{k\in\mathbb{Z}^d} \left(1+4\pi^2|k|^2\right)^{-\alpha/2} \sum_{\substack{n\in\mathbb{Z}^d\setminus\{0,k\},\\ g(n-k)=g(n)}} a_0(n)\, b_0(k-n)\, |n|^{-(d+1)/2} |k-n|^{-(d+1)/2}$$
$$= \int_{\mathbb{T}^d} \mathcal{B}(x)\,\mathcal{G}(x)\, dx \le \left\{\int_{\mathbb{T}^d} |\mathcal{B}(x)|^r dx\right\}^{1/r} \left\{\int_{\mathbb{T}^d} |\mathcal{G}(x)|^s dx\right\}^{1/s}.$$

Recalling that $g(n) \approx |n|$, for every $\alpha > d(1-1/r) = d/s$, one obtains

$$\left\{\int_{\mathbb{T}^d} |\mathcal{G}(x)|^s dx\right\}^{1/s} \ge C \sum_{k\in\mathbb{Z}^d\setminus\{0\}} |k|^{-\alpha} \sum_{n\in\mathbb{Z}^d\setminus\{0\},\ g(n-k)=g(n)} |n|^{-d-1}.$$

In particular, if $\Sigma = \{|x| \le 1\}$ is a ball, then $g(n) = |n|$, and the above inequality takes the form

$$\left\{\int_{\mathbb{T}^d} |\mathcal{G}(x)|^s dx\right\}^{1/s} \ge C \sum_{k\in\mathbb{Z}^d\setminus\{0\}} |k|^{-\alpha} \sum_{|n-k|=|n|} |n|^{-d-1}.$$

In order to bound this expression from below, one can restrict the sum to the $k$ even,

$$\sum_{k\in\mathbb{Z}^d\setminus\{0\}} |k|^{-\alpha} \sum_{|n-k|=|n|} |n|^{-d-1} \ge \sum_{k\in\mathbb{Z}^d\setminus\{0\}} |2k|^{-\alpha} \sum_{|n-2k|=|n|} |n|^{-d-1}.$$

The equation $|m-2k|=|m|$ is the same as $k\cdot m=k\cdot k$, and with the change of variables $m=k+n$ one obtains $k\cdot n=0$, so that for every $\alpha>d/s$,

$$\left\{\int_{\mathbb{T}^d}|\mathcal{G}(x)|^s\,dx\right\}^{1/s}\geq C\sum_{k\in\mathbb{Z}^d\setminus\{0\}}|k|^{-\alpha}\sum_{k\cdot n=0}\left(|k|^2+|n|^2\right)^{-(d+1)/2}.$$

By the above lemma, when two entries of the vector $k$ are coprimes, the area of a fundamental domain of the $(d-1)$-dimensional lattice $\{k\cdot n=0\}$ is $|k|$, and the density of the lattice is $|k|^{-1}$. For such a $k$,

$$\sum_{k\cdot n=0}\left(|k|^2+|n|^2\right)^{-(d+1)/2}\geq\left(2\,|k|^2\right)^{-(d+1)/2}|\{k\cdot n=0,\ |n|\leq|k|\}|\geq C\,|k|^{-3}.$$

Since, by a theorem of E.Cesàro, the probability that two random non negative integers are coprime is $6/\pi^2$, the probability that two entries of the vector $k$ are coprimes is positive. Hence, if $\alpha\leq d-3$,

$$\sum_{k\in\mathbb{Z}^d\setminus\{0\}}|k|^{-\alpha}\sum_{k\cdot n=0}\left(|k|^2+|n|^2\right)^{-(d+1)/2}\geq C\sum_{k\in\mathbb{Z}^d\setminus\{0\}}|k|^{-\alpha-3}=+\infty.$$

In particular, recalling that $s=p/2$ and $\alpha>d/s=2d/p$, if $p>2d/(d-3)$ then

$$\left\{\int_{\mathbb{T}^d}|\mathcal{G}(x)|^{p/2}\,dx\right\}^{1/p}=+\infty.$$

This proves (2). Finally, (1) follows from (2) by replacing the measure $d\mu(r)$ with a convolution $\varphi*\mu(r)\,dr$, with $\varphi(r)$ a non negative smooth function on $\mathbb{R}$ with integral one. This convolution is a probability measure with Fourier transform that vanishes at infinity. Observe that

$$\left\{\int_{\mathbb{T}^d}\left[\int_{\mathbb{R}}\left|r^{-(d-1)/2}\mathcal{D}(r\Omega-x)\right|^2d(\varphi*\mu)_{H,R}(r)\right]^{p/2}dx\right\}^{2/p}$$
$$=\left\{\int_{\mathbb{T}^d}\left[\int_{\mathbb{R}}\int_{\mathbb{R}}\left|(R+H(r+t))^{-(d-1)/2}\mathcal{D}((R+H(r+t))\Omega-x)\right|^2d\mu(r)\,\varphi(t)\,dt\right]^{p/2}dx\right\}^{2/p}$$
$$\leq\int_{\mathbb{R}}\left\{\int_{\mathbb{T}^d}\left[\int_{\mathbb{R}}\left|(R+H(r+t))^{-(d-1)/2}\mathcal{D}((R+H(r+t))\Omega-x)\right|^2d\mu(r)\right]^{p/2}dx\right\}^{2/p}\varphi(t)\,dt.$$

Hence, if

$$\left\{\int_{\mathbb{T}^d}\left[\int_{\mathbb{R}}\left|r^{-(d-1)/2}\mathcal{D}(r\Omega-x)\right|^2d\mu_{H,R}(r)\right]^{p/2}dx\right\}^{2/p}\leq C<+\infty,$$

then also

$$\left\{\int_{\mathbb{T}^d}\left[\int_{\mathbb{R}}\left|r^{-(d-1)/2}\mathcal{D}\left(r\Omega-x\right)\right|^2 d\left(\varphi*\mu\right)_{H,R}(r)\right]^{p/2} dx\right\}^{2/p} \leq C<+\infty,$$

and the argument used to prove (2) applies. □

We conclude with some remarks.

**Remark:** The ellipsoid $\Omega=\{|M(x-p)|\leq 1\}$, with $M$ a non singular $d\times d$ matrix and $p$ a fixed point in $\mathbb{R}^d$, has support function $g(x)=\left|(M^t)^{-1}x\right|+x\cdot p$,

$$g(x)=\sup_{y\in\Omega}\{x\cdot y\}=\sup_{\{|M(y-p)|\leq 1\}}\left\{M^t\left(M^t\right)^{-1}x\cdot y\right\}$$
$$=\sup_{\{|M(y-p)|\leq 1\}}\left\{\left(M^t\right)^{-1}x\cdot M(y-p)\right\}+\left(M^t\right)^{-1}x\cdot Mp=\left|\left(M^t\right)^{-1}x\right|+x\cdot p.$$

The equality $g(m)=g(n)$ is a non trivial algebraic relation between the coordinates of $p=(p_1,p_2,...,p_d)$,

$$(m-n)\cdot p=\left|\left(M^t\right)^{-1}n\right|-\left|\left(M^t\right)^{-1}m\right|.$$

If $\{1,p_1,p_2,...,p_d\}$ are linearly independent over the algebraic closure of the field generated by the entries of the matrix $M$, then this relation holds only if $m=n$. Hence, under these assumptions, the support function is injective when restricted to the integers. In the case $p=0$, then the equality $g(m)=g(n)$ when squared gives

$$Am_1^2+Bm_2^2+...+Cm_1m_2+...=An_1^2+Bn_2^2+...+Cn_1n_2+...,$$
$$A\left(m_1^2-n_1^2\right)+B\left(m_2^2-n_2^2\right)+...+C\left(m_1m_2-n_1n_2\right)+...=0.$$

Here $m=(m_1,m_2,...)$, $n=(n_1,n_2,...)$, and $A$, $B$, $C$,... are homogeneous second degree polynomials in the entries of the matrix $M$. This equation has has at least the solutions $n=\pm m$. On the other hand, if the entries of the matrix $M$ are algebraically independent, then $n=\pm m$ are the only solutions. A cardinality argument shows that for a fixed $M$, then almost every $p$ has the property that there exist no algebraic relation between its coordinates. Similarly, almost every matrix $M$ has the property that its entries are algebraically independent.

**Remark:** In Corollary 2 we defined a convex set *"generic"* if its support function is injective when restricted to the integers. Not only *"generic"* convex set exists, but they are the majority, they are of second category in space of compact convex sets endowed with the Hausdorff metric. If $A+\Omega$ is the Minkowski sum of $A$ and $\Omega$, then $g_{rA+\Omega}(x) = rg_A(x) + g_\Omega(x)$. For a fixed $x$ in $\mathbb{R}^d$, the function $\Omega \to g_\Omega(x)$ is continuous in the Hausdorff metric. For fixed $m, n \in \mathbb{Z}^d$ with $m \neq n$, the collection of convex sets $\Omega$ with $g_\Omega(m) \neq g_\Omega(n)$ is open in the Hausdorff metric. On the other hand, if $g_\Omega(m) = g_\Omega(n)$, and if $A$ is a convex set with $g_A(m) \neq g_A(n)$, as in the previous remark, then $rA + \Omega \to \Omega$ as $r \to 0+$, and $g_{rA+\Omega}(m) \neq g_{rA+\Omega}(n)$. This implies that the set of $\Omega$ with $g_\Omega(m) \neq g_\Omega(n)$ is open and dense. Hence the set of $\Omega$ with $g_\Omega(m) \neq g_\Omega(n)$ for every $m, n \in \mathbb{Z}^d$ with $m \neq n$ is the intersection of a countable family of open dense sets.

**Remark:** For the ball centered at the origin $\Sigma = \{|x| \leq 1\}$ the function $\mathcal{G}(x)$ defined in Lemma 27 and studied in Theorem 4 is not constant. For almost every $p$, the function $\mathcal{G}(x)$ associated to the shifted ball $\Omega = \{|x-p| \leq 1\} = \Sigma + p$ is constant. This may seem contradictory, but observe that in the case of $\Sigma$ the function $\mathcal{G}(x)$ is an $r$ average of the discrepancy of $r\Sigma - x$, while if $\Omega = \Sigma + p$ the function $\mathcal{G}(x)$ is an $r$ average of the discrepancy of $r\Omega - x = r\Sigma + (rp - x)$. These averages are different. In particular, in the averages of $r\Sigma + (rp - x)$ are a mix of an average over the dilations $r\Sigma$ together with an average over the translations $rp - x$. Observe that for irrational choices of $p$, these translations $rp - x$ are dense in the set of all translations. Hence it is not completely surprising that in this case $\mathcal{G}(x)$ is constant.

**Remark:** For every convex $\Omega$ with a boundary with strictly positive Gaussian curvature, if the Fourier transform of the probability measure $d\mu(r)$ has a good decay at infinity, then for every $p < (2d-4)/(d-3)$,

$$\sup_{R,H} \left\{ \int_{\mathbb{T}^d} \left[ \int_{\mathbb{R}} \left| r^{-(d-1)/2} \mathcal{D}(r\Omega - x) \right|^2 d\mu_{H,R}(r) \right]^{p/2} dx \right\}^{2/p} < +\infty.$$

On the other hand, if $\Omega = \{|x| \leq 1\}$, for every probability measure $d\mu(r)$ and every $p > 2d/(d-3)$,

$$\sup_{R,H} \left\{ \int_{\mathbb{T}^d} \left[ \int_{\mathbb{R}} \left| r^{-(d-1)/2} \mathcal{D}(r\Omega - x) \right|^2 d\mu_{H,R}(r) \right]^{p/2} dx \right\}^{2/p} = +\infty.$$

Both these indexes $(2d-4)/(d-3)$ and $2d/(d-3)$ are asymptotic to 2 as $d \to +\infty$.

# Chapter 3

# Discrepancy and Hausdorff dimension

The origin of the problem considered in this chapter can be traced to the work of Hardy and Littlewood in [15] and to the work of Jarnik (see [24], [25]) and Besicovitch (see [2]) about diophantine approximation (see also theHardy's book about Ramanujan [14])-

Hardy and Littlewood consider two positive numbers $\omega$ and $\omega'$. Let $\Delta$ be the triangle whose sides are the coordinate axes and the line $\omega x + \omega' y = n > 0$. The main thing really relevant to count integer points in $\Delta$ and on its boundary (half counted) is the ratio $\theta = \omega/\omega'$. The simplest case is when $\theta$ is rational: in this case the hypotenuse of the triangle may contain a number of integer points comparable to the perimeter and the discrepancy $\mathcal{D}(\Delta)$ between the number of integer points in $\Delta$ and its area is bounded by $n$. The problem is more difficult when $\theta$ is irrational. It is again true that the discrepancy is bounded by $n$ for all irrational $\theta$, but there are sharper results for special classes of slopes $\theta$. These results depend on the approximation $|\theta - p/q|$ of $\theta$ by the rationals $p/q$, hence on the partial quotients of the expansion $\theta$ in continued fractions. In particular Hardy and Littlewood prove that if the quotients are bounded, then the discrepancy $\mathcal{D}(\Delta)$ is bounded by $\log n$ (e. g. in the case of quadratic $\theta$, which have a periodic expansion); if the quotients do not increase rapidly, then the discrepancy is bounded by $n^\epsilon$ for every $\epsilon > 0$ (e.g. in the case of algebraic numbers). Hence a fundamental topic in what follows is the diophantine approximation of a given number by rationals (see [24], [25], [2] [11]).

A classical theorem by Lagrange and Dirichlet states that for every real number $x$, there are infinitely many positive integers $q$ such that

$$\left| x - \frac{p}{q} \right| \leq \frac{1}{q^2}$$

for some integer $p$; such $p/q$ are good rational approximations to $x$. Written in another way,

$$\|qx\| \leq q^{-1}$$

for infinitely many $q$, where $\|y\| = \min_{m\in\mathbb{Z}} |y - m|$ denotes the distance from $y$ to the nearest integer.

Motivated by the above results, one can define a $\alpha$-well-approximable number $x$ when

$$\|qx\| \leq q^{1-\alpha}$$

for infinitely many positive integers $q$.

A classical theorem of Jarnik, relevant in what follows, states that the set of $\alpha$-well-approximable numbers has Hausdorff dimension $2/\alpha$.

In this chapter, using the above quoted result of Jarnik, we consider a rotated square $\Omega$ in the plane with sides perpendicular to the unit vectors $\sigma = (\cos\theta, \sin\theta)$ and $\sigma^{\perp} = (-\sin\theta, \cos\theta)$ and we study properties and Hausdorff dimension of the set of rotations $\theta$ which give a discrepancy less than $|n|^{\beta}$ with $0 < \beta < 1$. The estimates of the Hausdorff dimension will be a corollary of the estimate of the $L^s(L^p)$ mixed norm

$$\left\{ \int_{\mathbb{SO}(2)} \left[ \int_{\mathbb{T}^2} |\mathcal{D}(R\sigma\Omega - x)|^p dx \right]^{s/p} d\mu(\sigma) \right\}^{1/s},$$

where $d\mu$ is a suitable Borel measure on the set of rotations $\sigma$.

## 3.1 The square in the plane

Let $\Omega$ be a square in the plane. In what follows the rotations in $\mathbb{SO}(2)$ are identified with the unit vectors $\{|\sigma| = 1\}$ in $\mathbb{R}^2$.

**Lemma 31.** *(1) If $\sigma\Omega$ is the unit square in the plane centered at the origin and with sides perpendicular to $\sigma$ and $\sigma^{\perp}$, then*

$$\sum_{k\in\mathbb{Z}^2} \chi_{R\sigma\Omega - x}(k) = \sum_{n\in\mathbb{Z}^2} \frac{\sin(\pi Rn\cdot\sigma)}{\pi n\cdot\sigma} \frac{\sin\left(\pi Rn\cdot\sigma^{\perp}\right)}{\pi n\cdot\sigma^{\perp}} \exp(2\pi i n x).$$

*(2) There exists $\varepsilon > 0$ such that if $\zeta(x)$ is a smooth non negative radial function with support in $\{|x| < \varepsilon\}$ and with integral 1, and if $0 < \delta \leq 1$ and $R \geq 1$, then*

$$\left| \sum_{n\in\mathbb{Z}^2} \chi_{R\Omega}(n + x) - |\Omega| R^2 \right| \leq 2R\delta$$

$$+\left|\sum_{n\in\mathbb{Z}^2\setminus\{0\}}\widehat{\zeta}(\delta n)\frac{\sin(\pi(R-\delta)n\cdot\sigma)}{\pi n\cdot\sigma}\frac{\sin\left(\pi(R-\delta)n\cdot\sigma^{\perp}\right)}{\pi n\cdot\sigma^{\perp}}\exp(2\pi inx)\right|$$
$$+\left|\sum_{n\in\mathbb{Z}^2\setminus\{0\}}\widehat{\zeta}(\delta n)\frac{\sin(\pi(R+\delta)n\cdot\sigma)}{\pi n\cdot\sigma}\frac{\sin\left(\pi(R+\delta)n\cdot\sigma^{\perp}\right)}{\pi n\cdot\sigma^{\perp}}\exp(2\pi inx)\right|.$$

*Proof.* This follows from the Poisson summation formula. □

**Theorem 5.** *Fix $0<\delta<1/2$, and let $\psi_0(t),\psi_1(t),\psi_2(t),\ldots$ be positive functions in $\{t>0\}$ decreasing to zero, and let $\varphi_0(t),\varphi_1(t),\varphi_2(t),\ldots$ be defined in $\{|\sigma|=1\}$ by the conditions*

$$\varphi_j(\sigma)=\inf_{0<|n|\leq 2^j/\delta}\left\{\frac{|n\cdot\sigma|}{\psi_j(|n|)}\right\}.$$

*In particular, $|n\cdot\sigma|\geq\psi_j(|n|)\varphi_j(\sigma)$ if $0<|n|\leq 2^j/\delta$. If $\sigma\Omega$ is the unit square in $\mathbb{R}^2$ with sides perpendicular to $\sigma$ and $\sigma^{\perp}$, and if $2\leq p\leq+\infty$ and $1/p+1/q=1$, then for every $a>0$ there exists a constant $C$ such that for every $R\geq 2$, the following hold:*

*(1) If $2\leq p<+\infty$, then*

$$\left\{\int_{\mathbb{T}^2}|\mathcal{D}(R\sigma\Omega-x)|^p\,dx\right\}^{1/p}\leq C(1+R\delta)$$
$$+CR\sum_{j=0}^{+\infty}2^{-aj}\left\{\sum_{k=0}^{[\log_2(R)]}2^{-kq}\min\left\{1,\left(\psi_j^{-1}\left(2^k/(R\varphi_j(\sigma))\right)\right)^{-q}\right\}\right\}^{1/q}$$

*(2) If $p=+\infty$, then*

$$\sup_{x\in\mathbb{T}^2}\{|\mathcal{D}(R\sigma\Omega-x)|\,dx\}\leq C\left(R\delta+\log^2(1/\delta)\right)$$
$$+CR\log(1/\delta)\sum_{j=0}^{+\infty}2^{-aj}\left\{\sum_{k=0}^{[\log_2(R)]}2^{-k}\min\left\{1,\left(\psi_j^{-1}\left(2^k/(R\varphi_j(\sigma))\right)\right)^{-1}\right\}\right\}.$$

*Proof.* By the above lemma, it suffices to estimate the norm in $L^p(\mathbb{T}^2)$ of the Fourier series

$$\sum_{n\in\mathbb{Z}^2\setminus\{0\}}\widehat{\zeta}(\delta n)\frac{\sin(\pi Rn\cdot\sigma)}{\pi n\cdot\sigma}\frac{\sin\left(\pi Rn\cdot\sigma^{\perp}\right)}{\pi n\cdot\sigma^{\perp}}\exp(2\pi inx)$$

$$= \sum_{0<|n|\leq 1/\delta} \widehat{\zeta}(\delta n) \frac{\sin(\pi R n\cdot\sigma)}{\pi n\cdot\sigma} \frac{\sin\left(\pi R n\cdot\sigma^{\perp}\right)}{\pi n\cdot\sigma^{\perp}} \exp(2\pi i n x)$$
$$+ \sum_{j=0}^{+\infty} \left( \sum_{2^{j-1}/\delta<|n|\leq 2^{j}/\delta} \widehat{\zeta}(\delta n) \frac{\sin(\pi R n\cdot\sigma)}{\pi n\cdot\sigma} \frac{\sin\left(\pi R n\cdot\sigma^{\perp}\right)}{\pi n\cdot\sigma^{\perp}} \exp(2\pi i n x) \right).$$

First consider the Fourier series with the sum over $\{0 < |n| \leq 1/\delta\}$. By the Hausdorff Young inequality, if $2 \leq p \leq +\infty$ and $1/p + 1/q = 1$, since $\left|\widehat{\zeta}(\delta n)\right| \leq 1$,

$$\left\{ \int_{\mathbb{T}^2} \left| \sum_{0<|n|\leq 1/\delta} \widehat{\zeta}(\delta n) \frac{\sin(\pi R n\cdot\sigma)}{\pi n\cdot\sigma} \frac{\sin\left(\pi R n\cdot\sigma^{\perp}\right)}{\pi n\cdot\sigma^{\perp}} \exp(2\pi i n x) \right|^{p} dx \right\}^{1/p}$$
$$\leq \left\{ \sum_{0<|n|\leq 1/\delta} \left| \frac{\sin(\pi R n\cdot\sigma)}{\pi n\cdot\sigma} \frac{\sin\left(\pi R n\cdot\sigma^{\perp}\right)}{\pi n\cdot\sigma^{\perp}} \right|^{q} \right\}^{1/q}.$$

The proofs of the cases $q = 1$ and $q > 1$ are slightly different. The sum over $\{0 < |n| \leq 1/\delta\}$ can be splitted into four quadrants

$$\left\{0 < |n| \leq 1/\delta,\ n\cdot\sigma \geq 0,\ n\cdot\sigma^{\perp} \geq 0\right\},$$
$$\left\{0 < |n| \leq 1/\delta,\ n\cdot\sigma < 0,\ n\cdot\sigma^{\perp} \geq 0\right\},$$
$$\left\{0 < |n| \leq 1/\delta,\ n\cdot\sigma < 0,\ n\cdot\sigma^{\perp} < 0\right\},$$
$$\left\{0 < |n| \leq 1/\delta,\ n\cdot\sigma \geq 0,\ n\cdot\sigma^{\perp} < 0\right\}.$$

The sums over these quadrants are similar, and it suffices to consider the first. This first quadrant can be further splitted into

$$\{0 < |n| \leq 1/\delta,\ 0 \leq n\cdot\sigma \leq 1/R\},$$
$$\left\{\left\{0 < |n| \leq 1/\delta,\ 2^{k-1}/R < n\cdot\sigma \leq 2^{k}/R\right\}\right\}_{k=1}^{[\log_2(R)]},$$
$$\left\{0 < |n| \leq 1/\delta,\ 0 \leq n\cdot\sigma^{\perp} \leq 1/R\right\},$$
$$\left\{\left\{0 < |n| \leq 1/\delta,\ 2^{k-1}/R < n\cdot\sigma^{\perp} \leq 2^{k}/R\right\}\right\}_{k=1}^{[\log_2(R)]},$$
$$\left\{0 < |n| \leq 1/\delta,\ n\cdot\sigma > 2^{[\log_2(R)]}/R,\ n\cdot\sigma^{\perp} > 2^{[\log_2(R)]}/R\right\}.$$

Since $|n|^2 = |n\cdot\sigma|^2 + \left|n\cdot\sigma^{\perp}\right|^2$, if $|n\cdot\sigma|$ is close to 0, then $\left|n\cdot\sigma^{\perp}\right|$ is close to $|n|$. This implies that the sum over the strip $\{0 < |n| \leq 1/\delta,\ 0 \leq n\cdot\sigma \leq 1/R\}$ is dominated by

$$\sum_{\substack{0<|n|\leq 1/\delta \\ 0\leq n\cdot\sigma\leq 1/R}} \left| \frac{\sin(\pi R n\cdot\sigma)}{\pi n\cdot\sigma} \frac{\sin\left(\pi R n\cdot\sigma^{\perp}\right)}{\pi n\cdot\sigma^{\perp}} \right|^{q} \leq CR^{q} \sum_{\substack{0<|n|\leq 1/\delta \\ 0\leq n\cdot\sigma\leq 1/R}} |n|^{-q}.$$

If $m$ and $n$ are in this strip, $0<|m|,|n|\leq 1/\delta$, $0\leq m\cdot\sigma\leq 1/R$ and $0\leq n\cdot\sigma\leq 1/R$, then

$$\begin{aligned}\psi_0\left(|m-n|\right)\varphi_0\left(\sigma\right)\leq\left|\left(m-n\right)\cdot\sigma\right|\leq 1/R,\\ |m-n|\geq\psi_0^{-1}\left(1/\left(R\varphi_0\left(\sigma\right)\right)\right).\end{aligned}$$

Hence the integer points in the strip $\{0<|n|\leq 1/\delta,\ 0\leq n\cdot\sigma\leq 1/R\}$ are spaced at least by $\max\left\{1,\psi_0^{-1}\left(1/\left(R\varphi_0\left(\sigma\right)\right)\right)\right\}$. This leads to the estimate

$$\begin{aligned}&R^q\sum_{\substack{0<|n|\leq 1/\delta\\ 0\leq n\cdot\sigma\leq 1/R}}|n|^{-q}\\ &\leq CR^q\sum_{i=1}^{+\infty}\left(i\max\left\{1,\psi_0^{-1}\left(1/\left(R\varphi_0\left(\sigma\right)\right)\right)\right\}\right)^{-q}\\ &\leq CR^q\min\left\{1,\left(\psi_0^{-1}\left(1/\left(R\varphi_0\left(\sigma\right)\right)\right)\right)^{-q}\right\}.\end{aligned}$$

Similarly,

$$\begin{aligned}&\sum_{k=1}^{[\log_2(R)]}\left(\sum_{\substack{0<|n|\leq 1/\delta\\ 2^{k-1}/R<n\cdot\sigma\leq 2^k/R}}\left|\frac{\sin\left(\pi Rn\cdot\sigma\right)}{\pi n\cdot\sigma}\frac{\sin\left(\pi Rn\cdot\sigma^{\perp}\right)}{\pi n\cdot\sigma^{\perp}}\right|^q\right)\\ &\leq C\sum_{k=1}^{[\log_2(R)]}\left(\sum_{\substack{0<|n|\leq 1/\delta\\ 2^{k-1}/R<n\cdot\sigma\leq 2^k/R}}|n\cdot\sigma|^{-q}|n|^{-q}\right)\\ &\leq CR^q\sum_{k=1}^{[\log_2(R)]}\left(2^{-kq}\sum_{\substack{0<|n|\leq 1/\delta\\ 2^{k-1}/R<n\cdot\sigma\leq 2^k/R}}|n|^{-q}\right)\\ &\leq CR^q\sum_{k=1}^{[\log_2(R)]}2^{-kq}\min\left\{1,\left(\psi_0^{-1}\left(2^k/\left(R\varphi_0\left(\sigma\right)\right)\right)\right)^{-q}\right\}.\end{aligned}$$

The sums over the strips defined by $n\cdot\sigma^{\perp}$ can be estimated similarly. Finally, the sum over the quadrant $\left\{0<|n|\leq 1/\delta,\ n\cdot\sigma>2^{[\log_2(R)]},\ n\cdot\sigma^{\perp}>2^{[\log_2(R)]}\right\}$ is uniformly bounded,

$$\sum_{\substack{0<|n|\leq 1/\delta\\ n\cdot\sigma>2^{[\log_2(R)]}/R\\ n\cdot\sigma^{\perp}>2^{[\log_2(R)]}/R}}\left|\frac{\sin\left(\pi Rn\cdot\sigma\right)}{\pi n\cdot\sigma}\frac{\sin\left(\pi Rn\cdot\sigma^{\perp}\right)}{\pi n\cdot\sigma^{\perp}}\right|^q$$

$$\leq C \sum_{\substack{0<|n|\leq 1/\delta \\ n\cdot\sigma > 2^{[\log_2(R)]}/R \\ n\cdot\sigma^\perp > 2^{[\log_2(R)]}/R}} |n\cdot\sigma|^{-q} \left|n\cdot\sigma^\perp\right|^{-q} \leq C.$$

Let us now consider the case $q = 1$.

$$\sum_{\substack{0<|n|\leq 1/\delta \\ 0\leq n\cdot\sigma\leq 1/R}} \left| \frac{\sin(\pi R n\cdot\sigma)}{\pi n\cdot\sigma} \frac{\sin\left(\pi R n\cdot\sigma^\perp\right)}{\pi n\cdot\sigma^\perp} \right|$$

$$\leq CR \sum_{\substack{0<|n|\leq 1/\delta \\ 0\leq n\cdot\sigma\leq 1/R}} |n|^{-1}$$

$$\leq CR \sum_{j<1/\left(\delta\max\left\{1,\psi_0^{-1}(1/(R\varphi_0(\sigma)))\right\}\right)} \left(j\max\left\{1,\psi_0^{-1}\left(1/\left(R\varphi_0(\sigma)\right)\right)\right\}\right)^{-1}$$

$$\leq C\log(1/\delta)\, R\min\left\{1, \left(\psi_0^{-1}\left(1/\left(R\varphi_0(\sigma)\right)\right)\right)^{-1}\right\}.$$

Similarly,

$$\sum_{k=1}^{[\log_2(R)]} \left( \sum_{\substack{0<|n|\leq 1/\delta \\ 2^{k-1}/R<n\cdot\sigma\leq 2^k/R}} \left| \frac{\sin(\pi R n\cdot\sigma)}{\pi n\cdot\sigma} \frac{\sin\left(\pi R n\cdot\sigma^\perp\right)}{\pi n\cdot\sigma^\perp} \right| \right)$$

$$\leq C \sum_{k=1}^{[\log_2(R)]} \left( \sum_{\substack{0<|n|\leq 1/\delta \\ 2^{k-1}/R<n\cdot\sigma\leq 2^k/R}} |n\cdot\sigma|^{-1} |n|^{-1} \right)$$

$$\leq CR \sum_{k=1}^{[\log_2(R)]} \left( 2^{-k} \sum_{\substack{0<|n|\leq 1/\delta \\ 2^{k-1}/R<n\cdot\sigma\leq 2^k/R}} |n|^{-1} \right)$$

$$\leq C\log(1/\delta)\, R \sum_{k=1}^{[\log_2(R)]} 2^{-k} \min\left\{1, \left(\psi_0^{-1}\left(2^k/\left(R\varphi_0(\sigma)\right)\right)\right)^{-1}\right\}.$$

Finally, the sum over the quadrant $\left\{n\cdot\sigma > 2^{[\log_2(R)]},\ n\cdot\sigma^\perp > 2^{[\log_2(R)]}\right\}$ is bounded by $\log^2(1/\delta)$,

$$\sum_{\substack{0<|n|\leq 1/\delta \\ n\cdot\sigma > 2^{[\log_2(R)]}/R \\ n\cdot\sigma^\perp > 2^{[\log_2(R)]}/R}} \left| \frac{\sin(\pi R n\cdot\sigma)}{\pi n\cdot\sigma} \frac{\sin\left(\pi R n\cdot\sigma^\perp\right)}{\pi n\cdot\sigma^\perp} \right|$$

$$\sum_{\substack{0<|n|\leq 1/\delta \\ n\cdot\sigma>2^{[\log_2(R)]}/R \\ n\cdot\sigma^\perp>2^{[\log_2(R)]}/R}} |n\cdot\sigma|^{-1}\left|n\cdot\sigma^\perp\right|^{-1}$$
$$\leq C\log^2(1/\delta).$$

This takes care of the Fourier series over $\{0<|n|\leq 1/\delta\}$. The series over $\{2^{j-1}/\delta<|n|\leq 2^j/\delta\}$ can be treated in the same way. Just observe that since $\zeta(x)$ is smooth with compact support, $\left|\widehat{\zeta}(\delta n)\right| \leq C(1+\delta|n|)^{-a}$ for every $a$, and the factor $2^{-aj}$ appears. □

It is a classical result in Diophantine approximation that for every $\sigma$ there exists a positive constant $C$ and infinite $n$ with $|n\cdot\sigma|\leq C/|n|$. In particular, the condition $|n\cdot\sigma|\geq\varphi(\sigma)|n|^{-\alpha}$ makes sense only for $\alpha\geq 1$.

**Corollary 3.** *Let $\alpha\geq 1$, and let*

$$\varphi_j(\sigma)=\inf_{0<|n|\leq 2^j/\delta}\{|n|^\alpha|n\cdot\sigma|\}.$$

*If $\sigma\Omega$ is the unit square in $\mathbb{R}^2$ with sides perpendicular to $\sigma$ and $\sigma^\perp$, and if $2\leq p\leq +\infty$, there exists a constant $C$ such that for every $R\geq 2$, the following hold:*

*(1) if $2\leq p<+\infty$ and $\alpha=1$,*

$$\left\{\int_{\mathbb{T}^2}|\mathcal{D}(R\sigma\Omega-x)|^p\,dx\right\}^{1/p}\leq CR\delta+C\log^{1-1/p}(R)\sum_{j=0}^{+\infty}2^{-aj}\varphi_j(\sigma)^{-1}.$$

*(2) if $2\leq p<+\infty$ and $\alpha>1$,*

$$\left\{\int_{\mathbb{T}^2}|\mathcal{D}(R\sigma\Omega-x)|^p\,dx\right\}^{1/p}\leq CR\delta+CR^{(\alpha-1)/\alpha}\sum_{j=0}^{+\infty}2^{-aj}\varphi_j(\sigma)^{-1/\alpha}.$$

*(3) if $p=+\infty$ and $\alpha=1$,*

$$\sup_{x\in\mathbb{T}^2}\{|\mathcal{D}(R\sigma\Omega-x)|\,dx\}$$
$$\leq CR\delta+C\log^2(1/\delta)+C\log(R)\log(1/\delta)\sum_{j=0}^{+\infty}2^{-aj}\varphi_j(\sigma)^{-1}.$$

*(4) if* $p = +\infty$ *and* $\alpha > 1$,

$$\sup_{x\in\mathbb{T}^2} \{|\mathcal{D}(R\sigma\Omega - x)|\, dx\}$$
$$\leq CR\delta + C\log^2(1/\delta) + CR^{(\alpha-1)/\alpha}\log(1/\delta)\sum_{j=0}^{+\infty} 2^{-aj}\varphi_j(\sigma)^{-1/\alpha}.$$

*Proof.* By the theorem, if $2 \leq p < +\infty$ and $1/p + 1/q = 1$,

$$\left\{\int_{\mathbb{T}^2} |\mathcal{D}(R\sigma\Omega - x)|^p\, dx\right\}^{1/p} \leq C(1 + R\delta)$$
$$+ CR\sum_{j=0}^{+\infty} 2^{-aj}\left\{\sum_{k=0}^{[\log_2(R)]} 2^{-kq}\min\left\{1, 2^{kq/\alpha}R^{-q/\alpha}\varphi_j(\sigma)^{-q/\alpha}\right\}\right\}^{1/q}$$
$$\leq C(1 + R\delta) + CR^{1-1/\alpha}\sum_{j=0}^{+\infty} 2^{-aj}\left\{\varphi_j(\sigma)^{-q/\alpha}\sum_{k=0}^{[\log_2(R)]} 2^{kq(1-\alpha)/\alpha}\right\}^{1/q}$$
$$\leq \begin{cases} C(1 + R\delta) + C\log^{1/q}(R)\sum_{j=0}^{+\infty} 2^{-aj}\varphi_j(\sigma)^{-1} & \text{if } \alpha = 1, \\ C(1 + R\delta) + R^{1-1/\alpha}\sum_{j=0}^{+\infty} 2^{-aj}\varphi_j(\sigma)^{-1/\alpha} & \text{if } \alpha > 1 \end{cases}$$

Similarly, if $p = +\infty$,

$$\sup_{x\in\mathbb{T}^2} \{|\mathcal{D}(R\sigma\Omega - x)|\, dx\} \leq C\left(R\delta + \log^2(1/\delta)\right)$$
$$+ CR\log(1/\delta)\sum_{j=0}^{+\infty} 2^{-aj}\left\{\sum_{k=0}^{[\log_2(R)]} 2^{-k}\min\left\{1, 2^{k/\alpha}R^{-1/\alpha}\varphi_j(\sigma)^{-1/\alpha}\right\}\right\}$$
$$\leq CR\delta + C\log^2(1/\delta) + CR^{1-1/\alpha}\log(1/\delta)\sum_{j=0}^{+\infty} 2^{-aj}\left\{\varphi_j(\sigma)^{-1/\alpha}\sum_{k=0}^{[\log_2(R)]} 2^{-k(1-1/\alpha)}\right\}$$
$$\leq \begin{cases} CR\delta + C\log^2(1/\delta) + C\log(R)\log(1/\delta)\sum_{j=0}^{+\infty} 2^{-aj}\varphi_j(\sigma)^{-1} & \text{if } \alpha = 1, \\ CR\delta + C\log^2(1/\delta) + CR^{(\alpha-1)/\alpha}\log(1/\delta)\sum_{j=0}^{+\infty} 2^{-aj}\varphi_j(\sigma)^{-1/\alpha} & \text{if } \alpha > 1. \end{cases}$$

□

By the above theorem and corollary, one can control the discrepancy $\mathcal{D}(R\sigma\Omega - x)$ via a function $\varphi(\sigma) = \inf_{|n|\geq 0}\{|n\cdot\sigma|/\psi(|n|)\}$. We want to estimate the size and dimension of the set of rotations $\sigma$ where $\varphi(\sigma) = 0$. The lemma of Frostman is a tool for estimating the Hausdorff dimension of Borel sets in Euclidean spaces. This lemma states that a Borel set $\mathbb{X}$ has positive $\alpha$-dimensional Hausdorff measure if and only if there is a positive Borel measure $\mu$ on $\mathbb{X}$ which assigns to every ball of center $p$ and radius $r$ a measure $\mu\{|x-p|<r\} \leq r^{\alpha}$. The following is nothing but a variant of classical results of Khintchine and Jarník on metric Diophantine approximation.

**Theorem 6.** *Let $\rho(t)$ be a positive increasing function in $\{t>0\}$, and let $d\mu(\vartheta)$ be a non negative measure on $\mathbb{R}\setminus 2\pi\mathbb{Z}$ with the property that for every interval $I$ on $\mathbb{R}\setminus 2\pi\mathbb{Z}$ of length $|I|$,*

$$\mu\{I\} \leq \rho(|I|).$$

*Fix $0<T\leq+\infty$. Let $\psi(t)$ be a positive function in $\{t>0\}$ decreasing to zero, and define the function $\varphi(\vartheta)$ on $\mathbb{R}\setminus 2\pi\mathbb{Z}$ by the condition*

$$\varphi(\vartheta) = \inf_{0<\sqrt{h^2+k^2}<T} \left\{ \frac{|-h\cos(\vartheta)+k\sin(\vartheta)|}{\psi\left(\sqrt{h^2+k^2}\right)} \right\}.$$

*Then for every $t>0$,*

$$\mu\{\varphi(\vartheta)<t\} \leq \min\left\{\rho(2\pi),\ 8\sum_{1\leq k<T}(k+1)\,\rho\left(\frac{2\sqrt{2}\psi(k)\,t}{k}\right)\right\}.$$

*Proof.* The following estimate holds for every $t>0$:

$$\mu\{\varphi(\vartheta)<t\} \leq \mu\{0\leq\vartheta<2\pi\} \leq \rho(2\pi).$$

This estimate is essentially best possible when $t$ is large. Recall that $\varphi(\vartheta)<t$ if and only if there exists $(-h,k)\neq(0,0)$ with

$$|-h\cos(\vartheta)+k\sin(\vartheta)| < t\psi\left(\sqrt{h^2+k^2}\right).$$

By symmetry one can assume that $0\leq\vartheta\leq\pi/4$, and also that $0\leq h\leq k$. Indeed, if $0\leq\sin(\vartheta)\leq\cos(\vartheta)$ and $k>0$, the minimum of $|-h\cos(\vartheta)+k\sin(\vartheta)|$ is achieved in the interval $0\leq h\leq k$. Then, if $\varphi(\vartheta)<t$, there exist $0\leq h\leq k$ such that

$$\begin{aligned} t\psi(k) \geq t\psi\left(\sqrt{h^2+k^2}\right) &> |-h\cos(\vartheta)+k\sin(\vartheta)| \\ &= k\cos(\vartheta)\,|\tan(\vartheta)-\tan(\arctan(h/k))| \\ &\geq \frac{k}{\sqrt{2}}\,|\vartheta-\arctan(h/k)|. \end{aligned}$$

Hence,

$$\{0\leq\vartheta\leq\pi/4:\ \varphi(\vartheta)<t\}$$

$$\subseteq \bigcup_{1\le k<T}\bigcup_{0\le h\le k}\left(\arctan(h/k)-\frac{\sqrt{2}\psi(k)\,t}{k},\arctan(h/k)+\frac{\sqrt{2}\psi(k)\,t}{k}\right).$$

Finally, the measure of this set is dominated by

$$\begin{aligned}&\mu\{0\le\vartheta\le\pi/4:\ \varphi(\vartheta)<t\}\\&\le\sum_{1\le k<T}\sum_{0\le h\le k}\mu\left\{\arctan(h/k)-\frac{\sqrt{2}\psi(k)\,t}{k}<\vartheta<\arctan(h/k)+\frac{\sqrt{2}\psi(k)\,t}{k}\right\}\\&\le\sum_{1\le k<T}(k+1)\,\rho\left(\frac{2\sqrt{2}\psi(k)\,t}{k}\right).\end{aligned}$$

□

**Corollary 4.** *Let $d\mu(\vartheta)$ be a non negative measure on $\mathbb{R}\setminus 2\pi\mathbb{Z}$ with the property that for every interval $I$ on $\mathbb{R}\setminus 2\pi\mathbb{Z}$ of length $|I|$,*

$$\mu\{I\}\le|I|^{\beta}.$$

*Finally, let*

$$\varphi(\vartheta)=\inf_{0<\sqrt{h^2+k^2}<T}\left\{\left(h^2+k^2\right)^{\alpha/2}\left|-h\cos(\vartheta)+k\sin(\vartheta)\right|\right\}.$$

*Then for every $t>0$,*

$$\mu\{\varphi(\vartheta)<t\}\le\begin{cases}C\min\left\{1,\ t^{\beta}T^{2-\beta(1+\alpha)}\right\} & \text{if } \beta<2/(1+\alpha),\\ C\min\left\{1,\ t^{\beta}\log(T)\right\} & \text{if } \beta=2/(1+\alpha),\\ C\min\left\{1,\ t^{\beta}\right\} & \text{if } \beta>2/(1+\alpha).\end{cases}$$

*In particular, if $0<s<\alpha\beta$, then*

$$\left\{\int_0^{2\pi}(\varphi(\vartheta))^{-s/\alpha}d\mu(\vartheta)\right\}^{1/s}\le\begin{cases}CT^{(2-\beta-\alpha\beta)/\alpha\beta} & \text{if } \beta<2/(1+\alpha),\\ C\log^{1/\alpha\beta}(T) & \text{if } \beta=2/(1+\alpha),\\ C & \text{if } \beta>2/(1+\alpha).\end{cases}$$

*Proof.* By the theorem,

$$\mu\{\varphi(\vartheta)<t\}\le\min\left\{(2\pi)^{\beta},\ 8\sum_{1\le k<T}(k+1)\left(2\sqrt{2}k^{-\alpha-1}t\right)^{\beta}\right\}.$$

By the way, remember that the definition of $\varphi(\vartheta)$ when $T=+\infty$ forces $\alpha\geq 1$, while the assumption $\mu\{I\}\leq |I|^{\beta}$ forces $\beta\leq 1$. Observe that when $\beta>2/(\alpha+1)$ then the above estimates can be made independent of $T$. Finally, by the above corollary,

$$
\begin{aligned}
&\left\{\int_0^{2\pi}(\varphi(\vartheta))^{-s/\alpha}\,d\mu(\vartheta)\right\}^{1/s}\\
&=\left\{\int_0^{+\infty}\mu\left\{(\varphi(\vartheta))^{-s/\alpha}>u\right\}du\right\}^{1/s}\\
&=\left\{s/\alpha\int_0^{+\infty}t^{-s/\alpha-1}\mu\{\varphi(\vartheta)<t\}\,dt\right\}^{1/s}\\
&\leq\begin{cases}C\left(\int_0^{+\infty}t^{-s/\alpha-1}\min\left\{1,\ t^{\beta}T^{2-\beta(1+\alpha)}\right\}dt\right)^{1/s} & \text{if } \beta<2/(1+\alpha),\\ C\left(\int_0^{+\infty}t^{-s/\alpha-1}\min\left\{1,\ t^{\beta}\log(T)\right\}dt\right)^{1/s} & \text{if } \beta=2/(1+\alpha),\\ C\left(\int_0^{+\infty}t^{-s/\alpha-1}\min\left\{1,\ t^{\beta}\right\}dt\right)^{1/s} & \text{if } \beta>2/(1+\alpha),\end{cases}\\
&\leq\begin{cases}CT^{(2-\beta-\alpha\beta)/\alpha\beta} & \text{if } \beta<2/(1+\alpha),\\ C\log^{1/\alpha\beta}(T) & \text{if } \beta=2/(1+\alpha),\\ C & \text{if } \beta>2/(1+\alpha).\end{cases}
\end{aligned}
$$

□

**Corollary 5.** *Let $2\leq p\leq+\infty$, also let $\alpha\geq 1$, $0<\beta\leq 1$, and $0<s<\alpha\beta$. Finally, let $d\mu(\vartheta)$ be a non negative measure on $\mathbb{R}\setminus 2\pi\mathbb{Z}$ with the property that for every interval $I$ on $\mathbb{R}\setminus 2\pi\mathbb{Z}$ of length $|I|$,*

$$\mu\{I\}\leq |I|^{\beta}.$$

*If $\sigma\Omega$ is the unit square in $\mathbb{R}^2$ with sides perpendicular to $\sigma$ and $\sigma^{\perp}$, then there exists a constant $C$ such that for every $R\geq 2$,*

*(1) if $\beta<2/(\alpha+1)$*

$$
\begin{aligned}
&\left\{\int_{\mathbb{SO}(2)}\left[\int_{\mathbb{T}^2}|\mathcal{D}(R\sigma\Omega-x)|^p\,dx\right]^{s/p}d\mu(\vartheta)\right\}^{1/s}\\
&\leq\begin{cases}CR^{(2-2\beta)/(2-\beta)}\log^{1-1/p}(R) & \text{if } p<+\infty,\alpha=1,\\ CR^{(2-2\beta)/(2-\beta)}\log^{2}(R) & \text{if } p=+\infty,\alpha=1,\\ CR^{(2-2\beta)/(2-\beta)} & \text{if } p<+\infty,\alpha>1,\\ CR^{(2-2\beta)/(2-\beta)}\log(R) & \text{if } p=+\infty,\alpha>1.\end{cases}
\end{aligned}
$$

*(2) if* $\beta = 2/(\alpha+1)$

$$\left\{\int_{\mathbb{SO}(2)}\left[\int_{\mathbb{T}^2}\left|\mathcal{D}\left(R\sigma\Omega - x\right)\right|^p dx\right]^{s/p} d\mu\left(\vartheta\right)\right\}^{1/s}$$
$$\leq \begin{cases} C\log^{2-1/p}(R) & \text{if } p<+\infty, \alpha=1, \\ C\log^3(R) & \text{if }, p=+\infty, \alpha=1, \\ CR^{(\alpha-1)/\alpha}\log^{1/\alpha\beta}(R) & \text{if } p<+\infty, \alpha>1, \\ CR^{(\alpha-1)/\alpha}\log^{1+1/\alpha\beta}(R) & \text{if } p=+\infty, \alpha>1. \end{cases}$$

*(3) if* $\beta > 2/(\alpha+1)$

$$\left\{\int_{\mathbb{SO}(2)}\left[\int_{\mathbb{T}^2}\left|\mathcal{D}\left(R\sigma\Omega - x\right)\right|^p dx\right]^{s/p} d\mu\left(\vartheta\right)\right\}^{1/s}$$
$$\leq \begin{cases} CR^{(\alpha-1)/\alpha} & \text{if }, p<+\infty, \alpha>1, \\ CR^{(\alpha-1)/\alpha}\log(R) & \text{if }, p=+\infty, \alpha>1. \end{cases}$$

*Observe that the the estimates with* $p = +\infty$ *are the limit of the estimates with* $p < +\infty$*, with an extra factor* $\log(R)$*.*

*Proof.* By the above corollary, with $T = 2^j/\delta$,

$$\left\{\int_0^{2\pi}\left(\varphi_j\left(\vartheta\right)\right)^{-s/\alpha} d\mu\left(\vartheta\right)\right\}^{1/s} \leq \begin{cases} C\left(2^j/\delta\right)^{(2-\beta-\alpha\beta)/\alpha\beta} & \text{if } \beta<2/(1+\alpha), \\ C\log^{1/\alpha\beta}\left(2^j/\delta\right) & \text{if } \beta=2/(1+\alpha), \\ C & \text{if } \beta>2/(1+\alpha). \end{cases}$$

By the previous corollaries, if $2 \leq p < +\infty$ and $\alpha = 1$,

$$\left\{\int_{\mathbb{SO}(2)}\left[\int_{\mathbb{T}^2}\left|\mathcal{D}\left(R\sigma\Omega - x\right)\right|^p dx\right]^{s/p} d\mu\left(\vartheta\right)\right\}^{1/s}$$
$$\leq C\left(R\delta + \log^{1-1/p}(R)\sum_{j=0}^{+\infty}2^{-aj}\left\{\int_{\mathbb{SO}(2)}\varphi_j\left(\vartheta\right)^{-s}d\mu\left(\vartheta\right)\right\}^{1/s}\right),$$

if $2 \leq p < +\infty$ and $\alpha > 1$,

$$\left\{\int_{\mathbb{SO}(2)}\left[\int_{\mathbb{T}^2}\left|\mathcal{D}\left(R\sigma\Omega - x\right)\right|^p dx\right]^{s/p} d\mu\left(\vartheta\right)\right\}^{1/s}$$

$$\leq C\left(R\delta + R^{(\alpha-1)/\alpha}\sum_{j=0}^{+\infty}2^{-aj}\left\{\int_{\mathbb{SO}(2)}\varphi_j(\vartheta)^{-s/\alpha}\,d\mu(\vartheta)\right\}^{1/s}\right),$$

if $p=+\infty$ and $\alpha=1$,

$$\left\{\int_{\mathbb{SO}(2)}\left[\int_{\mathbb{T}^2}|\mathcal{D}(R\sigma\Omega-x)|^p\,dx\right]^{s/p}d\mu(\vartheta)\right\}^{1/s}$$
$$\leq C\left(R\delta+\log^2(1/\delta)+\log(R)\log(1/\delta)\sum_{j=0}^{+\infty}2^{-aj}\left\{\int_{\mathbb{SO}(2)}\varphi_j(\vartheta)^{-s}\,d\mu(\vartheta)\right\}^{1/s}\right),$$

if $p=+\infty$ and $\alpha>1$,

$$\left\{\int_{\mathbb{SO}(2)}\left[\int_{\mathbb{T}^2}|\mathcal{D}(R\sigma\Omega-x)|^p\,dx\right]^{s/p}d\mu(\vartheta)\right\}^{1/s}$$
$$\leq C\left(R\delta+\log^2(1/\delta)+R^{(\alpha-1)/\alpha}\log(1/\delta)\sum_{j=0}^{+\infty}2^{-aj}\left\{\int_{\mathbb{SO}(2)}\varphi_j(\vartheta)^{-s/\alpha}\,d\mu(\vartheta)\right\}^{1/s}\right).$$

If $2\leq p<+\infty$ and $\alpha=1$ and $\delta=R^{-\beta/(2-\beta)}$,

$$R\delta+\log^{1-1/p}(R)\sum_{j=0}^{+\infty}2^{-aj}\left\{\int_{\mathbb{SO}(2)}\varphi_j(\vartheta)^{-s}\,d\mu(\vartheta)\right\}^{1/s}$$
$$\leq R\delta+\log^{1-1/p}(R)\sum_{j=0}^{+\infty}2^{-aj}\begin{cases}C\left(2^j/\delta\right)^{(2-2\beta)/\beta} & \textit{if } \beta<1,\\ C\log\left(2^j/\delta\right) & \textit{if } \beta=1,\end{cases}$$
$$\leq\begin{cases}CR^{(2-2\beta)/(2-\beta)}\log^{1-1/p}(R) & \textit{if } \beta<1 \textit{ and } \delta=R^{-\beta/(2-\beta)},\\ C\log^{2-1/p}(R) & \textit{if } \beta=1 \textit{ and } \delta=R^{-1}.\end{cases}$$

If $2\leq p<+\infty$ and $\alpha>1$,

$$R\delta+R^{(\alpha-1)/\alpha}\sum_{j=0}^{+\infty}2^{-aj}\left\{\int_{\mathbb{SO}(2)}\varphi_j(\vartheta)^{-s/\alpha}\,d\mu(\vartheta)\right\}^{1/s}$$
$$\leq R\delta+R^{(\alpha-1)/\alpha}\sum_{j=0}^{+\infty}2^{-aj}\begin{cases}C\left(2^j/\delta\right)^{(2-\beta-\alpha\beta)/\alpha\beta} & \textit{if } \beta<2/(1+\alpha),\\ C\log^{1/\alpha\beta}\left(2^j/\delta\right) & \textit{if } \beta=2/(1+\alpha),\\ C & \textit{if } \beta>2/(1+\alpha).\end{cases}$$
$$\leq\begin{cases}CR^{(2-2\beta)/(2-\beta)} & \textit{if } \beta<2/(1+\alpha),\\ CR^{(\alpha-1)/\alpha}\log^{1/\alpha\beta}(R) & \textit{if } \beta=2/(1+\alpha),\\ CR^{(\alpha-1)/\alpha} & \textit{if } \beta>2/(1+\alpha).\end{cases}$$

If $p=+\infty$ and $\alpha=1$ and $\delta=R^{-\beta/(2-\beta)}$,

$$\begin{aligned}
&R\delta+\log^2(1/\delta)+\log(R)\log(1/\delta)\sum_{j=0}^{+\infty}2^{-aj}\left\{\int_{\mathbb{SO}(2)}\varphi_j(\vartheta)^{-s}\,d\mu(\vartheta)\right\}^{1/s}\\
&\le R\delta+\log^2(1/\delta)+\log(R)\log(1/\delta)\sum_{j=0}^{+\infty}2^{-aj}\begin{cases}C\left(2^j/\delta\right)^{(2-2\beta)/\beta} & \textit{if } \beta<1,\\ C\log\left(2^j/\delta\right) & \textit{if } \beta=1,\end{cases}\\
&\le\begin{cases}CR^{(2-2\beta)/(2-\beta)}\log^2(R) & \textit{if } \beta<1,\\ C\log^3(R) & \textit{if } \beta=1.\end{cases}
\end{aligned}$$

If $p=+\infty$ and $\alpha>1$ and $\delta=R^{-\beta/(2-\beta)}$,

$$\begin{aligned}
&R\delta+\log^2(1/\delta)+R^{(\alpha-1)/\alpha}\log(1/\delta)\sum_{j=0}^{+\infty}2^{-aj}\left\{\int_{\mathbb{SO}(2)}\varphi_j(\vartheta)^{-s/\alpha}\,d\mu(\vartheta)\right\}^{1/s}\\
&\le R\delta+\log^2(1/\delta)+R^{(\alpha-1)/\alpha}\log(1/\delta)\times\\
&\times\sum_{j=0}^{+\infty}2^{-aj}\begin{cases}C\left(2^j/\delta\right)^{(2-\beta-\alpha\beta)/\alpha\beta} & \textit{if } \beta<2/(1+\alpha),\\ C\log^{1/\alpha\beta}\left(2^j/\delta\right) & \textit{if } \beta=2/(1+\alpha),\\ C & \textit{if } \beta>2/(1+\alpha).\end{cases}\\
&\le\begin{cases}CR^{(2-2\beta)/(2-\beta)}\log(R) & \textit{if } \beta<2/(1+\alpha),\\ CR^{(\alpha-1)/\alpha}\log^{1+1/\alpha\beta}(R) & \textit{if } \beta=2/(1+\alpha),\\ CR^{(\alpha-1)/\alpha}\log(R) & \textit{if } \beta>2/(1+\alpha).\end{cases}
\end{aligned}$$

□

# Chapter 4

# Convex sets with zero curvature at a point

As before, let $\Omega \subset \mathbb{R}^d$ be a convex body with measure $|\Omega|$ and let

$$D_{R\Omega} = \mathrm{card}(\mathbb{Z}^d \cap R\Omega) - R^d|\Omega|.$$

In this chapter we introduce dilations, translations and rotations of this set $\Omega$, and we will try to understand the role of each one of these trasformations in the estimates of the discrepancy. We will consider a convex set with everywhere positive curvature except at a single point. Let $B_\gamma$ be a such convex domain with boundary $\partial B_\gamma$ coinciding with the graph of the function $y = |x|^\gamma$ in a neighborhood of the origin, with $\gamma \geq 2$. In particular if $\gamma = 2$ the curvature in the origin is positive, while if $\gamma > 2$ the curvature is zero.

Some results about this matter can be found in [8], where the author proves that in dimension $d = 2$

$$D_{RB_\gamma} = \begin{cases} R^{2/3} & \text{if } \gamma \leq 3, \\ R^{1-\frac{1}{\gamma}} & \text{if } \gamma > 3. \end{cases}$$

Furthermore, he extends the results to domains in $\mathbb{R}^d$, with $d \leq 7$:

$$D_{R\Omega} \leq C R^{d-2+\frac{2}{d+1}}.$$

The interest of this estimate is that the index $d - 2 + \frac{2}{d+1}$ is the same for domains with positive curvature, as prove in [17] and in [19].

Here we consider the mixed norm $L^s(SO(d), L^p(T^d))$ of the discrepancy, also in the case $s = p$:

$$\left\{ \int_{SO(d)} \left\{ \int_{T^d} |D_{RB_\gamma}(\sigma, t)|^p dt \right\}^{\frac{s}{p}} d\sigma \right\}^{\frac{1}{s}}$$

studying first the bidimensional case, and then the multidimensional one.

The main result is related to the norm $L^p$ with $p = +\infty$:

**Theorem 7.** *Let $\gamma > 2$ and let $B_\gamma \subset [0,1)^2$ be a convex body. Assume that its boundary $\partial B_\gamma$ satisfies the following conditions:*

*i) $\partial B_\gamma$ passes through the origin and it is of class $C^\infty$ in any other points;*

*ii) $\partial B_\gamma$ coincides with the graph of the function $y = |x|^\gamma$ in a neighborhood of the origin;*

*iii) $\partial B_\gamma$ has strictly positive curvature out of this neighborhood.*

*Then for $\gamma \le 3$ it's true that*

$$|D_{RB_\gamma}| \le CR^{2/3}.$$

**Theorem 8.** *Let $\gamma > 2$ and let $B_\gamma \subset [0,1)^2$ be a convex body with a boundary that satisfies the conditions of the previous theorem. Then for $\gamma \le d+1$, it is true that*

$$|D_{RB_\gamma}| \le C_{\gamma,d} R^{d-2+\frac{2}{d+1}}.$$

## 4.1 $L^p(SO(d), L^p(T^d))$ estimates

### 4.1.1 Bidimensional case

Let $\gamma > 2$ and let $B_\gamma \subset [0,1)^2$ be a convex body. Assume that its boundary $\partial B_\gamma$ satisfies the following conditions:

i) $\partial B_\gamma$ passes through the origin and it is of class $C^\infty$ in any other points;

ii) $\partial B_\gamma$ coincides with the graph of the function $y = |x|^\gamma$ in a neighborhood of the origin;

iii) $\partial B_\gamma$ has strictly positive curvature out of this neighborhood.

Let $R \ge 1$, $t \in T^2$ and $\sigma \in SO(2)$. We remember that the discrepancy of a body $\Omega$ is defined as

$$D_\Omega(t) = |\Omega| - \text{card}\left(\mathbb{Z}^2 \cap (\Omega + t)\right)$$

where $|\Omega|$ is the area of $\Omega$. We want to study the discrepancy of the body $B_\gamma$ rotated by $\sigma$, traslated by $t$ and dilated by $R$:

$$\begin{aligned} D_{RB_\gamma}(\sigma,t) &= R^2|B_\gamma| - \operatorname{card}\left(\mathbb{Z}^2 \cap (R\sigma(B_\gamma)+t)\right) \\ &= R^2|B_\gamma| - \sum_{n\in\mathbb{Z}^2} \hat{\chi}_{RB_\gamma}(\sigma(n))e^{2\pi int} \\ &= R^2|B_\gamma| - R^2\sum_{n\in\mathbb{Z}^2} \hat{\chi}_{B_\gamma}(R\sigma(n))e^{2\pi int}. \end{aligned} \tag{4.1}$$

From [7] we know that

$$\left\{\int_0^{2\pi} |\hat{\chi}_{B_\gamma}(\rho\Theta)|^p d\theta\right\}^{\frac{1}{p}} \approx \begin{cases} C\rho^{-3/2} & \text{if } p < (2\gamma-2)/(\gamma-2), \\ C\rho^{-3/2}\log^{(\gamma-2)/(2\gamma-2)}(\rho) & \text{if } p = (2\gamma-2)/(\gamma-2), \\ C\rho^{-1-1/p-1/(q\gamma)} & \text{if } p > (2\gamma-2)/(\gamma-2). \end{cases} \tag{4.2}$$

where $\rho \geq 0$ and $\Theta = (\cos\theta, \sin\theta)$. We will use this result to calculate the norm $L^p(SO(2), L^p(T^2))$ of the discrepancy $D_{RB_\gamma}(\sigma,t)$:

$$\left\{\int_{SO(2)}\int_{T^2} |D_{RB_\gamma}(\sigma,t)|^p dt d\sigma\right\}^{\frac{1}{p}}.$$

one has three results, presented in three different theorems, depending on the value of $p$ and $\gamma$.

**Theorem 9.** *Let* $4 < (2\gamma-2)/(\gamma-2)$ *(and so* $\gamma < 3$*) and* $1/p + 1/q = 1$*. Then*

$$\left\{\int_{SO(2)}\int_{T^2} |D_{RB_\gamma}(\sigma,t)|^p dt d\sigma\right\}^{\frac{1}{p}} \leq \begin{cases} CR^{1/2} & \text{if } p < 4, \\ CR^{1/2}\log)^{3/4}(R) & \text{if } p = 4, \\ CR^{(2q-4)/(q-4)} & \text{if } 4 < p \leq (2\gamma-2)/(\gamma-2), \\ CR^{2\left(q-2+\frac{1}{\gamma}\right)/\left(q-3+\frac{1}{\gamma}\right)} & \text{if } p > (2\gamma-2)/(\gamma-2). \end{cases}$$

*Proof.* We start considering the first case: $p < 4$. For Hausdorff-Young and Minkowski inequalities, one has

$$\begin{aligned} &\left\{\int_{SO(2)}\int_{T^2} |D_{RB_\gamma}(\sigma,t)|^p dt d\sigma\right\}^{\frac{1}{p}} \\ &= \left\{\int_{SO(2)} \left\{\left(\int_{T^2} |D_{RB_\gamma}(\sigma,t)|^p dt\right)^{\frac{1}{p}}\right\}^p d\sigma\right\}^{\frac{1}{p}} \\ &\leq R^2 \left\{\int_{SO(2)} \left(\sum_{n\neq 0\in\mathbb{Z}^2} |\hat{\chi}_{B_\gamma}(R\sigma(n))|^q\right)^{\frac{p}{q}} d\sigma\right\}^{\frac{1}{p}} \end{aligned}$$

$$= R^2 \left\{ \| \sum_{n\neq 0} |\hat{\chi}_{B_\gamma}(R\sigma(n))|^q \|_{L^{\frac{p}{q}}(SO(2))} \right\}^{\frac{1}{q}}$$

$$\leq R^2 \left\{ \sum_{n\neq 0} \| |\hat{\chi}_{B_\gamma}(R\sigma(n))|^q \|_{L^{\frac{p}{q}}(SO(2))} \right\}^{\frac{1}{q}}$$

$$= R^2 \left\{ \sum_{n\neq 0} \left\{ \int_{SO(2)} |\hat{\chi}_{B_\gamma}(R\sigma(n))|^p d\sigma \right\}^{\frac{q}{p}} \right\}^{\frac{1}{q}}.$$

At this point we can use (4.2):

$$R^2 \left\{ \sum_{n\neq 0} \left\{ \int_{SO(2)} |\hat{\chi}_{B_\gamma}(R\sigma(n))|^p d\sigma \right\}^{\frac{q}{p}} \right\}^{\frac{1}{q}}$$

$$\leq R^2 \left\{ \sum_{n\neq 0} |Rn|^{\left(-\frac{3}{2}\right)q} \right\}^{\frac{1}{q}}$$

$$= R^{\frac{1}{2}} \left\{ \sum_{n\neq 0} |n|^{-\frac{3}{2}q} \right\}^{\frac{1}{q}}.$$

This series converges when $\frac{3}{2}q > 2$, that it means $p < 4$. Therefore one has

$$\left\{ \int_{SO(2)} \int_{T^2} |D_{RB_\gamma}(\sigma,t)|^p dt d\sigma \right\}^{\frac{1}{p}} \leq CR^{1/2}.$$

Now let $4 < p < (2\gamma-2)/(\gamma-2)$ (the case $p = (2\gamma-2)/(\gamma-2)$ will be analyzed later). In this case the final series doesn't converge because $p > 4$, so one has to introduce the cut-off $\varphi$. Hence, one has

$$|D_{RB_\gamma}(\sigma,t)| \leq |(R\pm\varepsilon)^2 - R^2||B_\gamma| + |\sum_{n\neq 0} \hat{\varphi}_\varepsilon(\sigma(n))\hat{\chi}_{RB_\gamma}(\sigma(n))|$$

$$\leq CR\varepsilon + R^2 |\sum_{n\neq 0} \hat{\varphi}(\varepsilon\sigma(n))\hat{\chi}_{B_\gamma}(R\sigma(n))|.$$

Considering the last series, one has

$$R^2 \left\{ \int_{SO(2)} \int_{T^2} |\sum_{n\neq 0} \hat{\varphi}(\varepsilon\sigma(n))\hat{\chi}_{B_\gamma}(R\sigma(n))|^p dt d\sigma \right\}^{\frac{1}{p}}$$

$$\leq R^2 \left\{ \int_{SO(2)} \left( \sum_{n \neq 0} |\hat{\varphi}(\varepsilon\sigma(n)) \hat{\chi}_{B_\gamma}(R\sigma(n))|^q \right)^{\frac{p}{q}} d\sigma \right\}^{\frac{1}{p}}$$

$$\leq R^2 \left\{ \sum_{n \neq 0} \left\{ \int_{SO(2)} |\hat{\varphi}(\varepsilon\sigma(n)) \hat{\chi}_{B_\gamma}(R\sigma(n))|^p d\sigma \right\}^{\frac{q}{p}} \right\}^{\frac{1}{q}}$$

$$= CR^2 \left\{ \sum_{n \neq 0} \frac{1}{1 + |\varepsilon n|^{K_1}} \left\{ \int_{SO(2)} |\hat{\chi}_{B_\gamma}(R\sigma(n))|^p d\sigma \right\}^{\frac{q}{p}} \right\}^{\frac{1}{q}}$$

At this point one can use (4.2) for $p < (2\gamma - 2)/(\gamma - 2)$. Therefore we get

$$R^2 \left\{ \sum_{n \neq 0} \frac{1}{1 + |\varepsilon n|^K} \|\hat{\chi}_{B_\gamma}(R\sigma(n))\|^q_{L^p(SO(2))} \right\}^{\frac{1}{q}}$$

$$\leq R^2 \left\{ \sum_{n \neq 0} \frac{1}{1 + |\varepsilon n|^K} |Rn|^{-\frac{3}{2}q} \right\}^{\frac{1}{q}}$$

$$\leq R^2 \left\{ \int_{|\xi| \geq 1} \frac{1}{1 + (\varepsilon|\xi|)^K} (R|\xi|)^{-\frac{3}{2}q} d\xi \right\}^{\frac{1}{q}}$$

$$\leq R^{\frac{1}{2}} \left\{ \int_1^\infty \frac{1}{1 + (\varepsilon t)^K} t^{-\frac{3}{2}q + 1} dt \right\}^{\frac{1}{q}}$$

$$\leq R^{\frac{1}{2}} \varepsilon^{\frac{3}{2} - \frac{2}{q}} \left\{ \int_0^\infty \frac{s^{-\frac{3}{2}q + 1}}{1 + s^K} ds \right\}^{\frac{1}{q}} = A.$$

Choose $K$ large enough so that the last integral has no problems at $\infty$. Then one has that the integral converges in 0 for $-\frac{3}{2}q + 1 > -1 \Rightarrow q < \frac{4}{3}$, that it means $p > 4$. Therefore

$$A \leq CR^{\frac{1}{2}} \varepsilon^{\frac{3}{2} - \frac{2}{q}}.$$

Then

$$\left\{ \int_{SO(2)} \int_{T^2} |D_{RB_\gamma}(\sigma, t)|^p dt d\sigma \right\}^{\frac{1}{p}} \leq C(R\varepsilon + R^{\frac{1}{2}} \varepsilon^{\frac{3}{2} - \frac{2}{q}})$$

and choosing $\varepsilon = R^{q/(q-4}$ we can conclude that for $4 < p < (2\gamma - 2)/(\gamma - 2)$

$$\left\{ \int_{SO(2)} \int_{T^2} |D_{RB_\gamma}(\sigma, t)|^p dt d\sigma \right\}^{\frac{1}{p}} \leq CR^{(2q-4)/(q-4)}.$$

Considering the last integral of the inequality, one can find a result also for $p = 4$. One has to analyze

$$A = R^{\frac{1}{2}} \varepsilon^{\frac{3}{2} - \frac{2}{q}} \left\{ \int_{\varepsilon}^{\infty} \frac{s^{-\frac{3}{2}q+1}}{1+s^K} ds \right\}^{\frac{1}{q}}$$

with $K$ large to not have problems at $\infty$. For $-\frac{3}{2}q + 1 = -1 \Rightarrow q = \frac{4}{3}$ (i.e. $p = 4$), one has

$$A \le C R^{\frac{1}{2}} (\log(1/\varepsilon))^{\frac{3}{4}}$$

and choosing $\varepsilon = 1/R$ we obtain

$$\left\{ \int_{SO(2)} \int_{T^2} |D_{RB_\gamma}(\sigma, t)|^p dt d\sigma \right\}^{\frac{1}{p}} \le C R^{1/2} \log^{3/4}(R).$$

Now let $p = (2\gamma - 2)/(\gamma - 2)$ (i. e. $q = (2\gamma - 2)/(\gamma)$). We can resume the computation above using the right norm in (4.2). Therefore one has

$$\begin{aligned}
& R^2 \left\{ \sum_{n \neq 0} \frac{1}{1 + |\varepsilon n|^K} \|\hat{\chi}_{B_\gamma}(R\sigma(n))\|^q_{L^p(SO(2))} \right\}^{\frac{1}{q}} \\
& \le R^2 \left\{ \sum_{n \neq 0} \frac{1}{1 + |\varepsilon n|^K} |Rn|^{-\frac{3}{2}q} (\log |Rn|)^{\frac{\gamma-2}{2\gamma-2}q} \right\}^{\frac{1}{q}} \\
& \le R^2 \left\{ \int_{|\xi| \ge 1} \frac{1}{1 + |\varepsilon n|^K} (R|\xi|)^{-\frac{3}{2}q} (\log (R|\xi|))^{\frac{\gamma-2}{2\gamma-2}q} d\xi \right\}^{\frac{1}{q}} \\
& = R^{\frac{1}{2}} \left\{ \int_1^{\infty} \frac{1}{1 + (\varepsilon t)^K} t^{-\frac{3}{2}q+1} (\log (Rt))^{\frac{\gamma-2}{\gamma}} dt \right\}^{\frac{1}{q}} \\
& \le R^{\frac{1}{2}} \varepsilon^{\frac{3}{2} - \frac{2}{q}} \left\{ \int_0^{\infty} \frac{1}{1 + s^K} s^{-\frac{3}{2}q+1} \left( \log \left( \frac{R}{\varepsilon} s \right) \right)^{\frac{\gamma-2}{\gamma}} ds \right\}^{\frac{1}{q}}.
\end{aligned}$$

Considering only the integral, one has that

$$\int_0^{\infty} \frac{1}{1 + s^K} s^{-\frac{3}{2}q+1} \left( \log \left( \frac{R}{\varepsilon} s \right) \right)^{\frac{\gamma-2}{\gamma}} ds = \int_0^{\infty} \frac{1}{1 + s^K} s^{-\frac{3}{2}q+1} (C + \log (s))^{\frac{\gamma-2}{\gamma}} ds$$

and the integral $\int_0^{\infty} \frac{1}{1+s^K} s^{-\frac{3}{2}q+1} (\log (s))^{\frac{\gamma-2}{\gamma}} ds$ converges for $-\frac{3}{2}q + 1 > -1 \Rightarrow -\frac{3(\gamma-1)}{\gamma} > -2 \Rightarrow \gamma < 3$.

Therefore

$$\left\{\int_{SO(2)}\int_{T^2}|D_{RB_\gamma}(\sigma,t)|^p dtd\sigma\right\}^{\frac{1}{p}} \le C(R\varepsilon + R^{\frac{1}{2}}\varepsilon^{\frac{3}{2}-\frac{2}{q}})$$

and choosing $\varepsilon = R^{q/(q-4)}$ we can conclude that

$$\left\{\int_{SO(2)}\int_{T^2}|D_{RB_\gamma}(\sigma,t)|^p dtd\sigma\right\}^{\frac{1}{p}} \le CR^{(2q-4)/(q-4)}.$$

We can now consider the last case: $p > (2\gamma-2)/(\gamma-2)$ (i. e. $q < (2\gamma-2)/(\gamma)$). Also in this case we can resume the computation above using the right norm in (4.2). One has

$$\begin{aligned}
&R^2\left\{\sum_{n\neq 0}\frac{1}{1+|\varepsilon n|^K}\|\hat{\chi}_{B_\gamma}(R\sigma(n))\|^q_{L^p(SO(2))}\right\}^{\frac{1}{q}} \\
&\le R^2\left\{\sum_{n\neq 0}\frac{1}{1+|\varepsilon n|^K}|Rn|^{-q-\frac{q}{p}-\frac{1}{\gamma}}\right\}^{\frac{1}{q}} \\
&\le R^2\left\{\int_{|\xi|\ge 1}\frac{1}{1+(\varepsilon|\xi|)^K}(R|\xi|)^{-q-\frac{q}{p}-\frac{1}{\gamma}}d\xi\right\}^{\frac{1}{q}} \\
&\le R^{\frac{1}{q}\left(1-\frac{1}{\gamma}\right)}\varepsilon^{1+\frac{1}{p}+\frac{1}{\gamma q}-\frac{2}{q}}\left\{\int_0^\infty\frac{s^{-q-\frac{q}{p}-\frac{1}{\gamma}+1}}{1+s^K}ds\right\}^{\frac{1}{q}} = A.
\end{aligned}$$

Choose $K$ large enough so that the last integral has no problems at $\infty$. Then one has that the integral converges in 0 for $-q-\frac{q}{p}-\frac{1}{\gamma}+1 > -1 \Rightarrow q < (3\gamma-1)/(2\gamma)$. Notice that $(3\gamma-1)/(2\gamma) > (2\gamma-2)/\gamma$ and so it is satisfied the condition $q < (2\gamma-2)/\gamma$. Therefore

$$A \le CR^{\frac{1}{q}\left(1-\frac{1}{\gamma}\right)\varepsilon^{1+\frac{1}{p}+\frac{1}{\gamma q}-\frac{2}{q}}}.$$

Then

$$\left\{\int_{SO(2)}\int_{T^2}|D_{RB_\gamma}(\sigma,t)|^p dtd\sigma\right\}^{\frac{1}{p}} \le C(R\varepsilon + R^{\frac{1}{q}\left(1-\frac{1}{\gamma}\right)}\varepsilon^{1+\frac{1}{p}+\frac{1}{\gamma q}-\frac{2}{q}})$$

and choosing $\varepsilon = R^{\left(q-1+\frac{1}{\gamma}\right)/\left(q-3+\frac{1}{\gamma}\right)}$ we can conclude that

$$\left\{\int_{SO(2)}\int_{T^2}|D_{RB_\gamma}(\sigma,t)|^p dtd\sigma\right\}^{\frac{1}{p}} \le CR^{2\left(q-2+\frac{1}{\gamma}\right)/\left(q-3+\frac{1}{\gamma}\right)}.$$

□

For $p = 4$ we can reason also in this way. One has to estimate this integral:

$$\left\{ \int_{SO(2)} \| D_{RB_\gamma}(\sigma, t) \|^p_{L^4(T^2)} d\sigma \right\}^{\frac{1}{p}}.$$

We know from [5] that

$$\| D_{RB_\gamma}(\sigma, t) \|_{L^4(T^2)} \leq CR^2 \log^{\frac{1}{4}}(R) \| \left\{ \hat{\chi}_{B_\gamma}(R\sigma n) \right\}_{n \neq 0} \|_{L^{q,\infty}(\mathbb{Z}^2)}$$

where $q = 4/3$. Noticing that $4/3 < (2\gamma - 2)/(\gamma - 2)$ $\forall \gamma > 2$, one has

$$\left\{ \sum_{n \neq 0} |\hat{\chi}_{B_\gamma}(R\sigma n)|^q \right\}^{\frac{1}{q}} \leq R^{-\frac{3}{2}} \left\{ \sum_{n \neq 0} |n|^{-\frac{3}{2}q} \right\}^{\frac{1}{q}}$$

and we know that this series is in $L^{q,\infty}(\mathbb{Z}^2)$ if and only if $\frac{3}{2}q \geq 2 \Rightarrow q \geq \frac{4}{3}$. Therefore

$$\| D_{RB_\gamma}(\sigma, t) \|_{L^4(T^2)} \leq CR^{\frac{1}{2}} \log^{\frac{1}{4}}(R)$$

and

$$\| D_{RB_\gamma}(\sigma, t) \|_{L^4(SO(2), L^4(T^2))} \leq CR^{\frac{1}{2}} \log^{\frac{1}{4}}(R).$$

**Theorem 10.** *Let* $(2\gamma - 2)/(\gamma - 2) < 4$ *(and so* $\gamma > 3$*) and* $1/p + 1/q = 1$*. Then*

$$\left\{ \int_{SO(2)} \int_{T^2} |D_{RB_\gamma}(\sigma, t)|^p dt d\sigma \right\}^{\frac{1}{p}} \leq \begin{cases} CR^{1/2} & \text{if } p \leq (2\gamma - 2)/(\gamma - 2), \\ CR^{\frac{1}{q}\left(1 - \frac{1}{\gamma}\right)} & \text{if } (2\gamma - 2)/(\gamma - 2) < p < (3\gamma - 1)/(\gamma - 1), \\ CR^{\frac{1}{q}\left(1 - \frac{1}{\gamma}\right)} \log^{1/q}(R) & \text{if } p = (3\gamma - 1)/(\gamma - 1), \\ CR^{2\left(q - 2 + \frac{1}{\gamma}\right)/\left(q - 3 + \frac{1}{\gamma}\right)} & \text{if } p > (3\gamma - 1)/(\gamma - 1). \end{cases}$$

*Proof.* For the case $p < (2\gamma - 2)/(\gamma - 2)$ the proof is the same of the first case of the Theorem 9, because we use the same norm for the computation and the convergence condition $p \leq 4$ is still satisfied. Let $p = (2\gamma - 2)/(\gamma - 2)$ (i. e. $q = (2\gamma - 2)/gamma$). Reasoning as before, one has the following inequalities:

$$\left\{ \int_{SO(2)} \int_{T^2} |D_{RB_\gamma}(\sigma, t)|^p dt d\sigma \right\}^{\frac{1}{p}}$$
$$= \left\{ \int_{SO(2)} \left\{ \left( \int_{T^2} |D_{RB_\gamma}(\sigma, t)|^p dt \right)^{\frac{1}{p}} \right\}^p d\sigma \right\}^{\frac{1}{p}}$$

$$\leq R^2 \left\{ \sum_{n\neq 0} \left\{ \int_{SO(2)} |\hat{\chi}_{B_\gamma}(R\sigma(n))|^p d\sigma \right\}^{\frac{q}{p}} \right\}^{\frac{1}{q}}$$

$$\leq R^2 \left\{ \sum_{n\neq 0} |Rn|^{-\frac{3}{2}q} (\log|Rn|)^{\frac{\gamma-2}{2\gamma-2}q} \right\}^{\frac{1}{q}}$$

$$\leq R^{\frac{1}{2}} \left\{ \int_1^{\infty} t^{-\frac{3}{2}q+1} (\log(Rt))^{\frac{\gamma-2}{\gamma}} dt \right\}^{\frac{1}{q}}$$

Considering only the integral, one has that

$$\int_1^{\infty} t^{-\frac{3}{2}q+1} (\log(Rt))^{\frac{\gamma-2}{\gamma}} dt = \int_1^{\infty} t^{-\frac{3}{2}q+1} (C + \log(t))^{\frac{\gamma-2}{\gamma}} dt$$

and the integral $\int_1^{\infty} t^{-\frac{3}{2}q+1} (\log(t))^{\frac{\gamma-2}{\gamma}} dt$ converges for $-\frac{3}{2}q + 1 < -1 \Rightarrow \frac{3(\gamma-1)}{\gamma} > 2 \Rightarrow \gamma > 3$. Then

$$\left\{ \int_{SO(2)} \int_{T^2} |D_{RB_\gamma}(\sigma, t)|^p dt d\sigma \right\}^{\frac{1}{p}} \leq C R^{1/2}.$$

Let $(2\gamma - 2)/(\gamma - 2) < p \leq (3\gamma - 1)/(\gamma - 1)$. Reasoning in the same way of the previous cases but changing the norm, one has

$$\left\{ \int_{SO(2)} \int_{T^2} |D_{RB_\gamma}(\sigma, t)|^p dt d\sigma \right\}^{\frac{1}{p}}$$

$$\leq R^2 \left\{ \sum_{n\neq 0} \left\{ \int_{SO(2)} |\hat{\chi}_{B_\gamma}(R\sigma(n))|^p d\sigma \right\}^{\frac{q}{p}} \right\}^{\frac{1}{q}}$$

$$\leq R^2 \left\{ \sum_{n\neq 0} |Rn|^{\left(-1-\frac{1}{p}-\frac{1}{\gamma}\frac{1}{q}\right)q} \right\}^{\frac{1}{q}}$$

$$= R^{\frac{1}{q}\left(1-\frac{1}{\gamma}\right)} \left\{ \sum |n|^{-q-\frac{q}{p}-\frac{1}{\gamma}} \right\}^{\frac{1}{q}} \leq C R^{\frac{1}{q}\left(1-\frac{1}{\gamma}\right)}$$

where the last inequality is possible because the series converges when $q > (3\gamma - 1)/(2\gamma)$, that it means $p < (3\gamma - 1)/(\gamma - 1)$.

Now consider $p \geq (3\gamma - 1)/(\gamma - 1)$ resuming the proof of the last case of the first theorem. In particular for $p > (3\gamma - 1)/(\gamma - 1)$ one has

$$R^2 \left\{ \sum_{n \neq 0} \frac{1}{1 + |\varepsilon n|^K} \|\hat{\chi}_{B_\gamma}(R\sigma(n))\|^q_{L^p(SO(2))} \right\}^{\frac{1}{q}}$$

$$\leq R^2 \left\{ \sum_{n \neq 0} \frac{1}{1 + |\varepsilon n|^K} |Rn|^{-q - \frac{q}{p} - \frac{1}{\gamma}} \right\}^{\frac{1}{q}}$$

$$\leq R^2 \left\{ \int_{|\xi| \geq 1} \frac{1}{1 + (\varepsilon|\xi|)^K} (R|\xi|)^{-q - \frac{q}{p} - \frac{1}{\gamma}} d\xi \right\}^{\frac{1}{q}}$$

$$\leq R^{\frac{1}{q}\left(1 - \frac{1}{\gamma}\right)} \varepsilon^{1 + \frac{1}{p} + \frac{1}{\gamma q} - \frac{2}{q}} \left\{ \int_0^\infty \frac{s^{-q - \frac{q}{p} - \frac{1}{\gamma} + 1}}{1 + s^K} ds \right\}^{\frac{1}{q}} = A.$$

Choose $K$ large enough so that the last integral has no problems at $\infty$. The integral converges for $-q - \frac{q}{p} - \frac{1}{\gamma} + 1 > -1 \Rightarrow q < \frac{3\gamma - 1}{2\gamma}$. Then one has

$$A \leq C R^{\frac{1}{q}\left(1 - \frac{1}{\gamma}\right)} \varepsilon^{1 + \frac{1}{p} + \frac{1}{\gamma q} - \frac{2}{q}}.$$

Therefore

$$\left\{ \int_{SO(2)} \int_{T^2} |D_{RB_\gamma}(\sigma, t)|^p dt d\sigma \right\}^{\frac{1}{p}} \leq C(R\varepsilon + R^{\frac{1}{q}\left(1 - \frac{1}{\gamma}\right)} \varepsilon^{1 + \frac{1}{p} + \frac{1}{\gamma q} - \frac{2}{q}})$$

and choosing $\varepsilon = R^{\left(q - 1 + \frac{1}{\gamma}\right)/\left(q - 3 + \frac{1}{\gamma}\right)}$ we can conclude that

$$\left\{ \int_{SO(2)} \int_{T^2} |D_{RB_\gamma}(\sigma, t)|^p dt d\sigma \right\}^{\frac{1}{p}} \leq C R^{2\left(q - 2 + \frac{1}{\gamma}\right)/\left(q - 3 + \frac{1}{\gamma}\right)}.$$

For $p = (3\gamma - 1)/(\gamma - 1)$ one has to consider

$$A = R^{\frac{1}{q}\left(1 - \frac{1}{\gamma}\right)} \varepsilon^{1 + \frac{1}{p} + \frac{1}{\gamma q} - \frac{2}{q}} \left\{ \int_0^\infty \frac{s^{-q - \frac{q}{p} - \frac{1}{\gamma} + 1}}{1 + s^K} ds \right\}^{\frac{1}{q}}$$

with $K$ large enough to not have problems at $\infty$. For $-q - \frac{q}{p} - \frac{1}{\gamma} + 1 = -1 \Rightarrow q = \frac{3\gamma - 1}{2\gamma}$ (i.e. $p = \frac{3\gamma - 1}{\gamma - 1}$), one has

$$A \leq C R^{\frac{1}{q}\left(1 - \frac{1}{\gamma}\right)} (\log(1/\varepsilon))^{\frac{1}{q}}$$

and choosing $\varepsilon = 1/R$ we obtain

$$\left\{ \int_{SO(2)} \int_{T^2} |D_{RB_\gamma}(\sigma, t)|^p dt d\sigma \right\}^{\frac{1}{p}} \leq C R^{\frac{1}{q}\left(1 - \frac{1}{\gamma}\right)} (\log R)^{\frac{1}{q}}.$$

□

**Theorem 11.** *Let* $(2\gamma-2)/(\gamma-2)=4$ *(and so* $\gamma=3$*) and* $1/p+1/q=1$*. Then*

$$\left\{\int_{SO(2)}\int_{T^2}|D_{RB_\gamma}(\sigma,t)|^p dt d\sigma\right\}^{\frac{1}{p}} \leq \begin{cases} CR^{1/2} & \text{if } p<(2\gamma-2)/(\gamma-2),\\ CR^{1/2}\log^{1/4}(R) & \text{if } p=(2\gamma-2)/(\gamma-2),\\ CR^{2(3q-5)/(3q-8)} & \text{if } p>(2\gamma-2)/(\gamma-2).\end{cases}$$

*Proof.* For $p<(2\gamma-2)/(\gamma-2)$ the proof is the same of the first case of the Theorem 1, because we use the same norm for the computation and the convergence condition $p<4$ is still satisfied.

The case $p=4$ is the same of the first theorem.

For $p>(2\gamma-2)/(\gamma-2)$ we consider the proof of the last cases of both previous theorems with $\gamma=3$. □

**Estimates from below**

One has that $\forall k\neq 0$

$$\begin{aligned}&\|D_{RB_\gamma}(\sigma,t)\|_{L^p(SO(2),L^p(T^2))}\\ &= R^2\left\{\int_{SO(2)}\int_{T^2}|\sum_{n\neq 0}\hat{\chi}_{B_\gamma}(R\sigma n)e^{2\pi int}|^p dt d\sigma\right\}^{\frac{1}{p}}\\ &\geq R^2\left\{\int_{SO(2)}|\hat{\chi}_{B_\gamma}(R\sigma k)|^p d\sigma\right\}^{\frac{1}{p}}\end{aligned}$$

Let $\gamma<3$. Using [7] one has

for $p<4$

$$cR^{1/2}\leq\|D_{RB_\gamma}(\sigma,t)\|_{L^p(SO(2),L^p(T^2))}\leq CR^{1/2};$$

for $4<p<(2\gamma-2)/(\gamma-2)$

$$cR^{1/2}\leq\|D_{RB_\gamma}(\sigma,t)\|_{L^p(SO(2),L^p(T^2))}\leq CR^{(2q-4)/(q-4)};$$

for $p>(2\gamma-2)/(\gamma-2)$

$$cR^{\frac{1}{q}(1-\frac{1}{\gamma})}\leq\|D_{RB_\gamma}(\sigma,t)\|_{L^p(SO(2),L^p(T^2))}\leq CR^{\frac{2(q-2+\frac{1}{\gamma})}{q-3+\frac{1}{\gamma}}}.$$

Let $\gamma>3$. One has

for $p < (2\gamma - 2)/(\gamma - 2)$

$$cR^{\frac{1}{2}} \leq \|D_{RB_\gamma}(\sigma, t)\|_{L^p(SO(2),L^p(T^2))} \leq CR^{\frac{1}{2}};$$

for $(2\gamma - 2)/(\gamma - 2) < p < (3\gamma - 1)/(\gamma - 1)$

$$cR^{\frac{1}{q}(1-\frac{1}{\gamma})} \leq \|D_{RB_\gamma}(\sigma, t)\|_{L^p(SO(2),L^p(T^2))} \leq CR^{\frac{1}{q}(1-\frac{1}{\gamma})};$$

for $p > (3\gamma - 1)/(\gamma - 1)$

$$cR^{\frac{1}{q}(1-\frac{1}{\gamma})} \leq \|D_{RB_\gamma}(\sigma, t)\|_{L^p(SO(2),L^p(T^2))} \leq CR^{\frac{2(q-2+\frac{1}{\gamma})}{q-3+\frac{1}{\gamma}}}.$$

**Main result**

For $p = \infty$ the norm che be improved:

**Theorem 12.** *Let $\gamma > 2$ and let $B_\gamma \subset [0,1)^2$ be a convex body. Assume that its boundary $\partial B_\gamma$ satisfies the following conditions:*

*i) $\partial B_\gamma$ passes through the origin and it is of class $C^\infty$ in any other points;*

*ii) $\partial B_\gamma$ coincides with the graph of the function $y = |x|^\gamma$ in a neighborhood of the origin;*

*iii) $\partial B_\gamma$ has strictly positive curvature out of this neighborhood.*

*Then for $\gamma < 3$ we have*

$$|D_{RB_\gamma}(\sigma, t)| \leq CR^{2/3}.$$

Simmetry enables us to consider only half $B_\gamma$. We can divide it in three parts: the first part $A$ with the origin, the second one $B$ behaving as $|x|^\gamma$ and the third one $C$ with positive curvature. In particular, we can define better the angles limiting these parts: for $A$ one has $0 \leq \theta \leq c\rho^{-1+\frac{1}{\gamma}}$, for $B$ one has $c\rho^{-1+\frac{1}{\gamma}} < \theta \leq \delta$ and for $C$ one has $\delta < \theta \leq \pi$.

At this point we can consider the $L^\infty$-norm remembering that

$$|D_{RB_\gamma}(\sigma, t)| \leq CR\varepsilon + R^2 |\sum_{n \neq 0} \hat{\varphi}(\varepsilon\sigma(n))\hat{\chi}_{B_\gamma}(R\sigma(n))|.$$

Let

$$S = R^2 \sum_{n \neq 0} |\hat{\chi}_{B_\gamma}(R\sigma(n))| \frac{1}{1 + |\varepsilon n|^K}.$$

We divide $S$ in three pieces:

$$\begin{aligned} S = R^2 & \sum_{0 \le \arg\left(\frac{\sigma(n)}{|n|}\right) \le c\rho^{-1+\frac{1}{\gamma}}} |\hat{\chi}_{B_\gamma}(R\sigma(n))| \frac{1}{1+|\varepsilon n|^K} \\ & + R^2 \sum_{c\rho^{-1+\frac{1}{\gamma}} < \arg\left(\frac{\sigma(n)}{|n|}\right) \le \varepsilon} |\hat{\chi}_{B_\gamma}(R\sigma(n))| \frac{1}{1+|\varepsilon n|^K} \\ & + R^2 \sum_{\varepsilon < \arg\left(\frac{\sigma(n)}{|n|}\right) \le \pi} |\hat{\chi}_{B_\gamma}(R\sigma(n))| \frac{1}{1+|\varepsilon n|^K} \end{aligned}$$

and we analyze the pieces one by one using for each region the estimates of $|\hat{\chi}_{B_\gamma}(\rho\Theta)|$ present in [7].

For the first piece, we start considering the part of the region:

$$\begin{aligned} R^2 \sum_{1 \le |n| \le \rho^{1-\frac{1}{\gamma}}} |\hat{\chi}_{B_\gamma}(R\sigma(n))| &= R^2 \sum_{1 \le |n| \le \rho^{1-\frac{1}{\gamma}}} (R|n|)^{-1-\frac{1}{\gamma}} \\ &= R^{1-\frac{1}{\gamma}} \sum_{1 \le |n| \le \rho^{1-\frac{1}{\gamma}}} |n|^{-1-\frac{1}{\gamma}} \end{aligned}$$

and this is finite. Therefore we can estimate the first series with an integral and one has

$$\begin{aligned} & R^2 \sum_{0 \le \arg\left(\frac{n}{|n|}\right) \le c\rho^{-1+\frac{1}{\gamma}}} |\hat{\chi}_{B_\gamma}(R\sigma(n))| \frac{1}{1+|\varepsilon n|^K} \\ & \le R^2 \int_1^\infty \frac{1}{1+\varepsilon\rho^K} \rho (R\rho)^{-1-\frac{1}{\gamma}} \rho^{-1+\frac{1}{\gamma}} d\rho \\ & = R^{1-\frac{1}{\gamma}} \int_1^\infty \frac{1}{1+\varepsilon\rho^K} \rho^{-1} d\rho \\ & = R^{1-\frac{1}{\gamma}} \int_\varepsilon^\infty \frac{1}{1+t^K} (t)^{-1} dt \\ & = R^{1-\frac{1}{\gamma}} \log\left(\frac{1}{\varepsilon}\right) \end{aligned}$$

and choosing $\varepsilon = R^{-1/3}$ one has that

$$R^2 \sum_{0 \le \arg\left(\frac{n}{|n|}\right) \le c\rho^{-1+\frac{1}{\gamma}}} |\hat{\chi}_{B_\gamma}(R\sigma(n))| \frac{1}{1+|\varepsilon n|^K} \le CR^{1-\frac{1}{\gamma}} \log(R).$$

For the second part one has

$$
\begin{aligned}
&R^2 \sum_{c\rho^{-1+\frac{1}{\gamma}} < \arg\left(\frac{n}{|n|}\right) \le \delta} |\hat{\chi}_{B_\gamma}(R\sigma(n))| \frac{1}{1+|\varepsilon n|^K} \\
&\le R^2 \int_1^\infty \frac{1}{1+(\varepsilon\rho)^K} \rho (R\rho)^{-\frac{3}{2}} \int_{\rho^{-1+\frac{1}{\gamma}}}^{\delta} \theta^{\frac{2-\gamma}{2\gamma-2}} d\theta d\rho \\
&\le R^{\frac{1}{2}} \int_1^\infty \frac{1}{1+(\varepsilon\rho)^K} \rho^{-\frac{1}{2}} d\rho \\
&= R^{\frac{1}{2}} \varepsilon^{-\frac{1}{2}} \int_\varepsilon^\infty \frac{1}{1+t^K} t^{-\frac{1}{2}} dt \\
&\le C R^{\frac{1}{2}} \varepsilon^{-\frac{1}{2}}
\end{aligned}
$$

because $(2-\gamma)/(2\gamma-2) > -1$ and so the integral in $\theta$ converges. For $\varepsilon = R^{-1/3}$ we get that

$$
R^2 \sum_{c\rho^{-1+\frac{1}{\gamma}} < \arg\left(\frac{n}{|n|}\right) \le \varepsilon} |\hat{\chi}_{B_\gamma}(R\sigma(n))| \frac{1}{1+|\varepsilon n|^K} \le CR^{2/3}.
$$

For the third part we can use the classical result:

$$
R^2 \sum_{\varepsilon < \arg\left(\frac{n}{|n|}\right) \le \pi} |\hat{\chi}_{B_\gamma}(R\sigma(n))| \frac{1}{1+|\varepsilon n|^K} \le CR^{2/3}.
$$

Therefore

$$
S \le CR^{1-\frac{1}{\gamma}} \log R + CR^{2/3}.
$$

If $\gamma < 3$, one has $1 - \frac{1}{\gamma} < \frac{2}{3}$ and so

$$
|D_{RB_\gamma}(\sigma, t)| \le CR^{2/3}.
$$

We can now compare this result with the one in Theorem 9. In this theorem one has that for $p = \infty$

$$
\left\{ \int_{SO(2)} \int_{T^2} |D_{RB_\gamma}(\sigma, t)|^p dt d\sigma \right\}^{\frac{1}{p}} \le CR^{1-\frac{1}{2\gamma-1}}.
$$

But for $2 < \gamma < 3$ one has $1 - \frac{1}{2\gamma-1} > \frac{2}{3}$ and so one has improved the estimate. Look at 4.1.1.

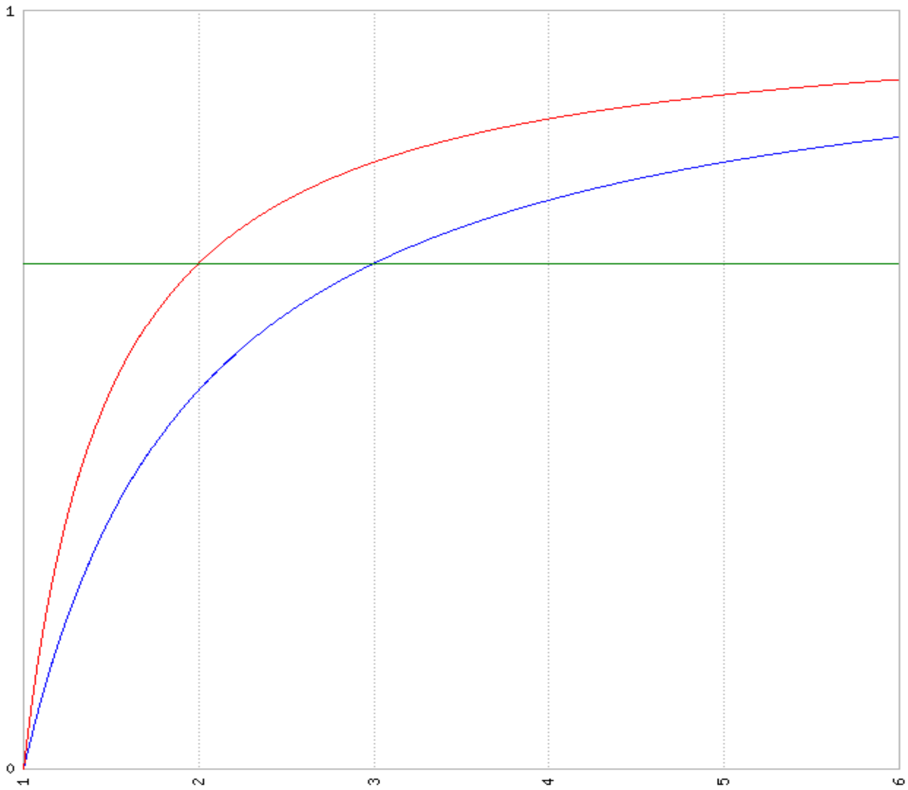


Figure 4.1: red= $1 - \frac{1}{2\gamma - 1}$, blue= $1 - \frac{1}{\gamma}$, green= $2/3$

### 4.1.2 Multidimensional case

Let

$$\partial B_\gamma = \{(x', x_d) \in \mathbb{R}^d : x_d = |x'|^\gamma\}$$

for values of $\gamma > 2$ near the origin. We are interested at the decay at infinity of the Fourier transform of $\chi_{B_\gamma}$:

$$\begin{aligned}
\hat{\chi}_{B_\gamma}(\xi) &= \int_{\mathbb{R}^n} e^{-2\pi i(\xi \cdot x)} \chi_{B_\gamma}(x) dx \\
&= \int_{B_\gamma} e^{-2\pi i(\xi \cdot x)} dx \\
&= \int_{\partial B_\gamma} e^{-2\pi i(\xi \cdot x)} \cdot dS \\
&= \int_{\mathbb{R}^{d-1}} e^{-2\pi i(\xi' \cdot x' + \xi_n |x'|^\gamma)} |x'|^\gamma dx'.
\end{aligned}$$

**Proposition 1.** *One has*

$$|\hat{\chi}_{B_\gamma}(\xi)| \le |\xi'|^{-\frac{(d-1)(\gamma-2)}{2(\gamma-1)}} |\xi_d|^{-\frac{d-1}{2(\gamma-1)}} |\xi|^{-1}$$

*and*

$$|\hat{\chi}_{B_\gamma}(\xi)| \le |\xi|^{-\frac{d-1}{\gamma}-1}.$$

*Proof.* It follows from [6].

□

**Proposition 2.** *One has the following estimates:*

$$\left\{ \int_{S^{d-1}} |\hat{\chi}_{B_\gamma}(\rho\omega)|^p d\omega \right\}^{\frac{1}{p}} \le \begin{cases} \rho^{-(d+1)/2}, & \text{if } p < 2(\gamma-1)/(\gamma-2), \\ \rho^{-(d+1)/2} \log^{(\gamma-2)(d-1)/2(\gamma-1)}(\rho), & \text{if } p = 2(\gamma-1)/(\gamma-2), \\ \rho^{-(d-1)\left(\frac{1}{p}+\frac{1}{\gamma}-\frac{1}{p\gamma}\right)-1}, & \text{if } p > 2(\gamma-1)/(\gamma-2. \end{cases}$$

*Proof.* Let $\xi = (\xi', \xi_d) = (\rho\omega' \sin\theta, \rho\cos\theta)$ with $\omega' \in S^{d-2}$ and $0 \le \theta \le \pi$. When $\varepsilon < \theta < \pi - \varepsilon$ one has the estimate

$$|\hat{\chi}_{B_\gamma}(\rho\omega' \sin\theta, \rho\cos\theta)| \le \rho^{-\frac{d+1}{2}}.$$

Therefore for $p > \frac{2(\gamma-1)}{\gamma-2}$ one has

$$\begin{aligned}
&\int_0^\pi \int_{S^{d-2}} |\hat{\chi}_{B_\gamma}(\rho\omega' \sin\theta, \rho\cos\theta)|^p \sin^{d-2}\theta d\omega' d\theta \\
&\le \rho^{-\frac{d+1}{2}p} + \int_0^\varepsilon \left( \min\left( \rho^{-\frac{d-1}{\gamma}-1}, \rho^{-\frac{d+1}{2}} (\sin\theta)^{-\frac{(d-1)(\gamma-2)}{2(\gamma-1)}} (\cos\theta)^{-\frac{d-1}{2(\gamma-1)}} \right) \right)^p (\sin\theta)^{d-2} d\theta \\
&\le \rho^{-\frac{d+1}{2}p} + \int_0^\varepsilon \left( \min\left( \rho^{-\frac{d-1}{\gamma}-1}, \rho^{-\frac{d+1}{2}} \theta^{-\frac{(d-1)(\gamma-2)}{2(\gamma-1)}} \right) \right)^p \theta^{d-2} d\theta \\
&\le \rho^{-\frac{d+1}{2}p} + \int_0^{\rho^{-1+1/\gamma}} \rho^{\left(-\frac{d-1}{\gamma}-1\right)p} \theta^{d-2} d\theta + \int_{\rho^{-1+1/\gamma}}^\varepsilon \rho^{-\frac{d+1}{2}p} \theta^{-p\frac{(d-1)(\gamma-2)}{2(\gamma-1)}} \theta^{d-2} d\theta \\
&\le \rho^{-\frac{d+1}{2}p} + \rho^{-(d-1)\frac{\gamma+p-1}{\gamma}p-p} + \rho^{-(d-1)\frac{\gamma+p-1}{\gamma}p-p} \le \rho^{-(d-1)\frac{\gamma+p-1}{\gamma}p-p}.
\end{aligned}$$

In the same way you can obtain the other two results. □

At this point one can consider the convex body $B_\gamma \subset [0,1)^d$ with $\gamma > 2$, remembering that

$$D_{RB_\gamma} = R^d |B_\gamma| - R^d \sum_{n\in\mathbb{Z}} \hat{\chi}_{B_\gamma}(R\sigma n) e^{2\pi i n t}$$

and we can estimate the norm $L^p(SO(d), L^p(T^d))$ of the discrepancy $D_{RB_\gamma}(\sigma, t)$. Like for $d = 2$, one has different cases, depending on the value of $p$ and $\gamma$.

**Theorem 13.** *Let* $2d/(d-1) < (2\gamma-2)/(\gamma-2)$ *(so* $\gamma < d+1$*) and* $1/p + 1/q = 1$. *Then*

$$\left\{\int_{SO(d)}\int_{T^d} |D_{RB_\gamma}(\sigma,t)|^p dt d\sigma\right\}^{\frac{1}{p}}$$

$$\le \begin{cases} CR^{(d-1)/2} & \text{if } p < 2d/(d-1), \\ CR^{d(d-1)(q-2)/[q(d-1)-2d]} & \text{if } 2d/(d-1) < p < 2(\gamma-1)/(\gamma-2), \\ R^{d(d-1)(q-2+\frac{1}{\gamma})/[(d-1)(q-2+\frac{1}{\gamma})-1]} & \text{if } p > 2(\gamma-1)/(\gamma-2). \end{cases}$$

*Proof.* As before, one has

$$\left\{\int_{SO(d)}\int_{T^d} |D_{RB_\gamma}(\sigma,t)|^p dt d\sigma\right\}^{\frac{1}{p}} \le R^d \left\{\sum_{n\neq 0}\left\{\int_{SO(d)} |\hat{\chi}_{B_\gamma}(R\sigma(n))|^p d\sigma\right\}^{\frac{q}{p}}\right\}^{\frac{1}{q}}.$$

Hence for $p < 2d/(d-1)$

$$R^d \left\{\sum_{n\neq 0}\left\{\int_{SO(d)} |\hat{\chi}_{B_\gamma}(R\sigma(n))|^p d\sigma\right\}^{\frac{q}{p}}\right\}^{\frac{1}{q}}$$

$$\le R^d \left\{\sum_{n\neq 0} |Rn|^{-\frac{d+1}{2}q}\right\}^{\frac{1}{q}}$$

$$= R^{\frac{d-1}{2}} \left\{\sum_{n\neq 0} |n|^{-\frac{d+1}{2}q}\right\}^{\frac{1}{q}}.$$

This series converges when $\frac{d+1}{2}q > d$, that it means $p < \frac{2d}{d-1}$. Therefore

$$\left\{\int_{SO(d)}\int_{T^d} |D_{RB_\gamma}(\sigma,t)|^p dt d\sigma\right\}^{\frac{1}{p}} \le CR^{(d-1)/2}.$$

Now let $2d/(d-1) < p < 2(\gamma-1)/(\gamma-2)$. In this case, the final series doesn't converge because $p > 2d/(d-1)$. So we can use the smoothing-trick on $\mathbb{R}^d$ to solve the problem. Therefore

$$|D_{RB_\gamma}(\sigma,t)| \le CR^{d-1}\varepsilon + R^d \left|\sum_{n\neq 0} \hat{\varphi}(\varepsilon\sigma n)\hat{\chi}_{B_\gamma}(R\sigma n)\right|.$$

Considering the last series, one has

$$R^d \left\{\int_{SO(d)}\int_{T^d} \left|\sum_{n\neq 0} \hat{\varphi}(\varepsilon\sigma n)\hat{\chi}_{B_\gamma}(R\sigma n)\right|^p dt d\sigma\right\}^{\frac{1}{p}}$$

$$\leq CR^d \left\{ \sum_{n\neq 0} \frac{1}{1+|\varepsilon n|^K} \left\{ \int_{SO(d)} |\hat{\chi}_{B_\gamma}(R\sigma n)|^p d\sigma \right\}^{\frac{p}{q}} \right\}^{\frac{1}{q}}$$

and so using the estimate for $p < (2\gamma - 2)/(\gamma - 2)$ we get

$$R^d \left\{ \sum_{n\neq 0} \frac{1}{1+|\varepsilon n|^K} \|\hat{\chi}_{B_\gamma}(R\sigma n)\|^q_{L^p(SO(d))} \right\}^{\frac{1}{q}}$$

$$\leq CR^d \left\{ \sum_{n\neq 0} \frac{1}{1+|\varepsilon n|^K} |Rn|^{-\frac{d+1}{2}q} \right\}^{\frac{1}{q}}$$

$$= CR^d \left\{ \int_{|\xi|\geq 1} \frac{1}{1+|\varepsilon|\xi||^K} (R|\xi|)^{-\frac{d+1}{2}q} d\xi \right\}^{\frac{1}{q}}$$

$$= CR^{\frac{d-1}{2}} \left\{ \int_1^{+\infty} \frac{1}{1+(\varepsilon t)^K} t^{-\frac{d+1}{2}q+d-1} dt \right\}^{\frac{1}{q}}$$

$$= CR^{\frac{d-1}{2}} \left\{ \int_0^{+\infty} \frac{1}{1+s^K} s^{-\frac{d+1}{2}q+d-1} \varepsilon^{\frac{d+1}{2}q-d+1-1} ds \right\}^{\frac{1}{q}}$$

$$= CR^{\frac{d-1}{2}} \varepsilon^{\frac{d+1}{2}-\frac{d}{q}} \left\{ \int_0^{+\infty} \frac{1}{1+s^K} s^{-\frac{d+1}{2}q+d-1} ds \right\}^{\frac{1}{q}} = A.$$

This integral converges in 0 for $-\frac{d+1}{2}q + d - 1 > -1 \Rightarrow q < \frac{2d}{d+1}$. Therefore

$$A \leq CR^{\frac{d-1}{2}} \varepsilon^{\frac{d+1}{2}-\frac{d}{q}}.$$

Then

$$\left\{ \int_{SO(d)} \int_{T^d} |D_{RB_\gamma}(\sigma, t)|^p dt d\sigma \right\}^{\frac{1}{p}} \leq C(R^{d-1}\varepsilon + R^{\frac{d-1}{2}} \varepsilon^{\frac{d+1}{2}-\frac{d}{q}})$$

and choosing $\varepsilon = R^{q(d-1)/[q(d-1)-2d]}$ we can say that for $2d/(d-1) < p < 2(\gamma - 1)/(\gamma - 2)$

$$\left\{ \int_{SO(d)} \int_{T^d} |D_{RB_\gamma}(\sigma, t)|^p dt d\sigma \right\}^{\frac{1}{p}} \leq CR^{\frac{d(d-1)(q-2)}{q(d-1)-2d}}.$$

Using the right estimate, we can find an estimate also for $p > (2\gamma - 2)/(\gamma - 2)$. So one has

$$R^d \left\{ \sum_{n\neq 0} \frac{1}{1+|\varepsilon n|^K} \|\hat{\chi}_{B_\gamma}(R\sigma n)\|^q_{L^p(SO(d))} \right\}^{\frac{1}{q}}$$

$$\leq CR^d \left\{ \sum_{n\neq 0} \frac{1}{1+|\varepsilon n|^K} |Rn|^{-q-(d-1)(\frac{q}{p}+\frac{1}{\gamma})} \right\}^{\frac{1}{q}}$$

$$\leq R^{\frac{1}{q}(d-1)(1-\frac{1}{\gamma})} \varepsilon^{1+(d-1)(\frac{1}{p}+\frac{1}{q\gamma}-\frac{1}{q})-\frac{1}{q}} \left\{ \int_0^{+\infty} \frac{s^{-q-(d-1)(\frac{q}{p}+\frac{1}{\gamma}-1)}}{1+s^K} \right\}^{\frac{1}{q}}.$$

This integral converges for $-q-(d-1)(\frac{q}{p}+\frac{1}{\gamma}-1) > -1 \Rightarrow q < \frac{\gamma(2d-1)-(d-1)}{d\gamma}$. Notice that $\frac{\gamma(2d-1)-(d-1)}{d\gamma} > \frac{2\gamma-2}{\gamma}$ and so we satisfy the condition $p > \frac{2\gamma-2}{\gamma-2}$. Therefore

$$\left\{ \int_{SO(d)} \int_{T^d} |D_{RB_\gamma}(\sigma,t)|^p dt d\sigma \right\}^{\frac{1}{p}} \leq C(R^{d-1}\varepsilon + R^{\frac{1}{q}(d-1)(1-\frac{1}{\gamma})} \varepsilon^{1+(d-1)(\frac{1}{p}+\frac{1}{q\gamma}-\frac{1}{q})-\frac{1}{q}})$$

and choosing $\varepsilon = R^{\frac{(d-1)(\gamma q-\gamma+1)}{(d-1)(q\gamma-2\gamma+1)-\gamma}}$ one has

$$\left\{ \int_{SO(d)} \int_{T^d} |D_{RB_\gamma}(\sigma,t)|^p dt d\sigma \right\}^{\frac{1}{p}} \leq R^{\frac{d(d-1)(q-2+\frac{1}{\gamma})}{(d-1)(q-2+\frac{1}{\gamma})-1}}.$$

□

**Theorem 14.** *Let* $(2\gamma-2)/(\gamma-2) < 2d/(d-1)$ *(so* $\gamma > d+1$*) and* $1/p+1/q=1$*. Then*

$$\left\{ \int_{SO(d)} \int_{T^d} |D_{RB_\gamma}(\sigma,t)|^p dt d\sigma \right\}^{\frac{1}{p}}$$

$$\leq \begin{cases} CR^{(d-1)/2} & \text{if } p < (2\gamma-2)/(\gamma-2), \\ CR^{\frac{1}{q}(d-1)(1-\frac{1}{\gamma})} & \text{if } (2\gamma-2)/(\gamma-2) < p < [\gamma(2d-1)-(d-1)]/d\gamma, \\ CR^{\frac{d(d-1)(q-2+\frac{1}{\gamma})}{(d-1)(q-2+\frac{1}{\gamma})-1}} & \text{if } p > [\gamma(2d-1)-(d-1)]/d\gamma. \end{cases}$$

*Proof.* For $p < (2\gamma-2)/(\gamma-2)$ the proof is as before.
Let $p > (2\gamma-2)/(\gamma-2)$. One has

$$R^d \left\{ \sum_{n\neq 0} \|\hat{\chi}_{B_\gamma}(R\sigma n)\|^q_{L^p(SO(d))} \right\}^{\frac{1}{q}}$$

$$\leq R^d \left\{ \sum |Rn|^{-q(d-1)(\frac{1}{p}+\frac{1}{\gamma}-\frac{1}{p\gamma})-q} \right\}^{\frac{1}{q}}$$

$$= R^{\frac{1}{q}(d-1)(1-\frac{1}{\gamma})} \left\{ \sum |n|^{-q(d-1)(\frac{1}{p}+\frac{1}{\gamma}-\frac{1}{p\gamma})-q} \right\}^{\frac{1}{q}}.$$

This series converges when $q > \frac{\gamma(2d-1)-(d-1)}{d\gamma} \Rightarrow p < \frac{\gamma(2d-1)-(d-1)}{(d-1)(\gamma-1)}$.Therefore

$$\left\{ \int_{SO(d)} \int_{T^d} |D_{RB_\gamma}(\sigma, t)|^p dt d\sigma \right\}^{\frac{1}{p}} \leq C R^{\frac{1}{q}(d-1)(1-\frac{1}{\gamma})}.$$

For $p > \frac{\gamma(2d-1)-(d-1)}{(d-1)(\gamma-1)}$ one has to use the smoothing-trick:

$$R^d \left\{ \sum_{n \neq 0} \frac{1}{1+|\varepsilon n|^K} \|\hat{\chi}_{B_\gamma}(R\sigma n)\|^q_{L^p(SO(d))} \right\}^{\frac{1}{q}}$$
$$\leq R^d \left\{ \sum \frac{1}{1+|\varepsilon n|^K} |Rn|^{-q(d-1)(\frac{1}{p}+\frac{1}{\gamma}-\frac{1}{p\gamma})-q} \right\}^{\frac{1}{q}}$$
$$= R^{\frac{1}{q}(d-1)(1-\frac{1}{\gamma})} \varepsilon^{1+(d-1)(\frac{1}{p}+\frac{1}{q\gamma}-\frac{1}{q})-\frac{1}{q}} \left\{ \int_0^{+\infty} \frac{s^{-q-(d-1)(\frac{q}{p}+\frac{1}{\gamma}-1)}}{1+s^K} ds \right\}^{\frac{1}{q}}$$

and this converges for $p > \frac{\gamma(2d-1)-(d-1)}{(d-1)(\gamma-1)}$. Hence choosing $\varepsilon = R^{\frac{(d-1)(\gamma q-\gamma+1)}{(d-1)(q\gamma-2\gamma+1)-\gamma}}$ we get

$$\left\{ \int_{SO(d)} \int_{T^d} |D_{RB_\gamma}(\sigma, t)|^p dt d\sigma \right\}^{\frac{1}{p}} \leq R^{\frac{d(d-1)(q-2+\frac{1}{\gamma})}{(d-1)(q-2+\frac{1}{\gamma})-1}}.$$

□

**Main result**

One can improve the $L^\infty$-norm:

**Theorem 15.** *Let $\gamma > 2$ and let $B_\gamma \subset [0,1)^2$ be a convex body as before. If $\gamma \leq d+1$, it is true that*

$$|D_{RB_\gamma}(\sigma, t)| \leq C_{\gamma,d} R^{d-2+\frac{2}{d+1}}.$$

*Proof.* Let $\varphi(t)$ be a cut-off function as before. One has

$$|D_R| := |D_{RB_\gamma}(\sigma, t)| \leq R^{d-1}\epsilon + R^d \sum_{n \neq 0} |\hat{\chi}_{B_\gamma}(R\sigma n)| \frac{1}{1+|\varepsilon n|^K}. \tag{4.3}$$

Let $\xi = (\xi', \xi_d)$ with $\xi' \in \mathbb{R}^{-\not{1}}$ and $\xi_d \in \mathbb{R}$. One has

$$(\xi', \xi_d) = (\rho\omega' \sin\theta, \rho\cos\theta)$$

with $\rho > 0$, $\omega' \in S^{d-2}$ and $0 \leq \theta \leq \pi$. We want to replace the series in 4.3 with the following integral:

$$\int_{|\xi|\geq 1/2} |\hat{\chi}_{B_\gamma}(Rn)| \frac{1}{1+|\varepsilon n|^K} d\xi.$$

This means that one has to consider the set of the points near the origin and the set of the line $\{n_d = 0\}$ in a different way. Hence, remembering that $\partial B_\gamma$ has a strictly positive curvature out of a neighborhood of the origin, one has

$$\begin{aligned}
|D_R| &\leq R^{d-1}\varepsilon + R^d \max_{1\leq |n| \leq c} |\hat{\chi}_{B_\gamma}(Rn)| + R^d \sum_{n_d=1}^{+\infty} |\hat{\chi}_{B_\gamma}(R(0,\cdots,n_d))| \\
&+ R^d \int_{1/2}^{+\infty} \int_0^\pi \int_{S^{d-2}} |\hat{\chi}_{B_\gamma}(R\rho(\omega' \sin\theta, \cos\theta))| d\omega' \sin^{d-2}(\theta) d\theta \frac{1}{1+(\varepsilon\rho)^K} \rho^{d-1} d\rho \\
&\leq R^{d-1} + R^{(d-1)\left(1-\frac{1}{\gamma}\right)} + R^{(d-1)\left(1-\frac{1}{\gamma}\right)} \sum_{n_d=1}^{+\infty} n_d^{-\frac{d-1}{\gamma}-1} + R^{\frac{d(d-1)}{d+1}} \\
&+ R^d \int_{1/2}^{+\infty} \int_0^{\pi/4} \min\left((R\rho)^{-\frac{d-1}{\gamma}-1}, (R\rho\sin\theta)^{-\frac{(d-1)(\gamma-2)}{2(\gamma-1)}} (R\rho\cos\theta)^{-\frac{d-1}{2(\gamma-1)}} (R\rho)^{-1}\right) \\
&\times \sin^{d-2}\theta d\theta \frac{1}{1+(\varepsilon\rho)^K} \rho^{d-1} d\rho.
\end{aligned}$$

If $0 \leq \theta \leq \pi/4$, one has

$$(R\rho)^{-\frac{d-1}{\gamma}-1} = (R\rho\sin\theta)^{-\frac{(d-1)(\gamma-2)}{2(\gamma-1)}} (R\rho\cos\theta)^{-\frac{d-1}{2(\gamma-1)}} (R\rho)^{-1}$$

$$\Rightarrow \theta \approx (R\rho)^{\frac{1}{\gamma}-1}.$$

Therefore one has to consider the following integral:

$$\begin{aligned}
&R^d \int_{1/2}^{+\infty} \int_0^{c(R\rho)^{\frac{1}{\gamma}-1}} \theta^{d-2} d\theta (R\rho)^{-\frac{d-1}{\gamma}-1} \frac{1}{1+(\varepsilon\rho)^K} \rho^{d-1} d\rho \\
&+ R^d \int_{1/2}^{+\infty} \int_{c(R\rho)^{\frac{1}{\gamma}-1}}^{\pi/4} (R\rho\theta)^{-\frac{(d-1)(\gamma-2)}{2(\gamma-1)}} (R\rho)^{-\frac{d-1}{2(\gamma-1)}-1} \theta^{d-2} d\theta \\
&\times \frac{1}{1+(\varepsilon\rho)^K} \rho^{d-1} d\rho \\
&:= A + B.
\end{aligned}$$

One has

$$A = R^d \int_{1/2}^{+\infty} \int_0^{c(R\rho)^{\frac{1}{\gamma}-1}} \theta^{d-2}\ d\theta\ \ (R\rho)^{-\frac{d-1}{\gamma}-1} \frac{1}{(1+\varepsilon\rho)^K} \rho^{d-1}\ d\rho$$

$$
\begin{aligned}
&= R^{(d-1)\left(1-\frac{1}{\gamma}\right)} \int_{1/2}^{+\infty} \int_{0}^{c(R\rho)^{\frac{1}{\gamma}-1}} \theta^{d-2}\, d\theta\ \rho^{-\frac{d-1}{\gamma}+d-2} \frac{1}{(1+\varepsilon\rho)^K}\, d\rho \\
&\lesssim R^{(d-1)\left(1-\frac{1}{\gamma}\right)} \int_{1/2}^{+\infty} (R\rho)^{-\frac{(\gamma-1)(d-1)}{\gamma}}\ \rho^{-\frac{d-1}{\gamma}+d-2} \frac{1}{(1+\varepsilon\rho)^K}\, d\rho \\
&= \int_{1/2}^{+\infty} \rho^{-1} \frac{1}{(1+\varepsilon\rho)^K}\, d\rho \lesssim \int_{\varepsilon/2}^{+\infty} t^{-1} \frac{1}{(1+t)^K}\, dt \lesssim \log\left(\frac{1}{\varepsilon}\right) .
\end{aligned}
$$

and

$$
\begin{aligned}
B &= R^d \int_{1/2}^{+\infty} \int_{c(R\rho)^{\frac{1}{\gamma}-1}}^{\pi/4} (R\rho\theta)^{-\frac{(d-1)(\gamma-2)}{2(\gamma-1)}} (R\rho)^{-\frac{d-1}{2(\gamma-1)}-1}\, \theta^{d-2}\, d\theta \frac{1}{(1+\varepsilon\rho)^K} \rho^{d-1}\, d\rho \\
&= R^{\frac{d-1}{2}} \int_{1/2}^{+\infty} \int_{c(R\rho)^{\frac{1}{\gamma}-1}}^{\pi/4} \theta^{\frac{d\gamma+2-3\gamma}{2(\gamma-1)}}\, d\theta\ \rho^{\frac{d-3}{2}} \frac{1}{(1+\varepsilon\rho)^K}\, d\rho \\
&\lesssim R^{\frac{d-1}{2}} \int_{1/2}^{+\infty} \rho^{\frac{d-3}{2}} \frac{1}{(1+\varepsilon\rho)^K}\, d\rho = R^{\frac{d-1}{2}} \int_{\varepsilon}^{+\infty} t^{\frac{d-3}{2}} \varepsilon^{-\frac{d-3}{2}} \frac{1}{\varepsilon\,(1+t)^K}\, dt \\
&\lesssim R^{\frac{d-1}{2}} \varepsilon^{-\frac{d-1}{2}} \int_{0}^{+\infty} t^{\frac{d-3}{2}} \frac{1}{\varepsilon\,(1+t)^K}\, dt \lesssim R^{\frac{d-1}{2}} \varepsilon^{-\frac{d-1}{2}} ,
\end{aligned}
$$

because $\theta^{\frac{d\gamma+2-3\gamma}{2(\gamma-1)}}$ is convergent in $0^+$. Therefore one has

$$
|D_R| \lesssim R^{d-1}\varepsilon + R^{(d-1)\left(1-\frac{1}{\gamma}\right)} + R^{\frac{d(d-1)}{d+1}} + \log\left(\frac{1}{\varepsilon}\right) + R^{\frac{d-1}{2}} \varepsilon^{-\frac{d-1}{2}} ,
$$

and, if $R^{d-1}\varepsilon = R^{\frac{d-1}{2}}\varepsilon^{-\frac{d-1}{2}}$, we have $\varepsilon = R^{-\frac{d-1}{d+1}}$ and

$$
|D_R| \lesssim R^{(d-1)\left(1-\frac{1}{\gamma}\right)} + \log(R) + R^{\frac{d(d-1)}{d+1}} \lesssim R^{\frac{d(d-1)}{d+1}} = C_{\gamma,d} R^{d-2+\frac{2}{d+1}}
$$

because $\gamma \leq d+1$. □

## 4.2 $L^s(SO(2), L^p(T^2))$ estimates

We want now to calculate the norm $L^s(SO(2), L^p(T^2))$ of the discrepancy $D_{RB_\gamma}(\sigma, t)$:

$$
\left\{ \int_{SO(2)} \left\{ \int_{T^2} |D_{RB_\gamma}(\sigma, t)|^p dt \right\}^{\frac{s}{p}} d\sigma \right\}^{\frac{1}{s}} .
$$

One can consider two cases: $1 < p \leq 2$ and $p > 2$. Let $1 < p \leq 2$.

**Theorem 16.** *Let $1 < p \leq 2$ and $s < 2$.*
*For $s < 2$ one has*

$$\left\{ \int_{SO(2)} \left\{ \int_{T^2} |D_{RB_\gamma}(\sigma, t)|^p dt \right\}^{\frac{s}{p}} d\sigma \right\}^{\frac{1}{s}} \leq C R^{\frac{1}{2}}.$$

*Proof.* For $1 < p \leq 2$ we know it's true that $\|f\|_p \leq \|f\|_2 = \|\hat{f}\|_2$. So one has

$$\begin{aligned}
&\left\{ \int_{SO(2)} \left\{ \int_{T^2} |D_{RB_\gamma}(\sigma, t)|^p dt \right\}^{\frac{s}{p}} d\sigma \right\}^{\frac{1}{s}} \\
&\leq \left\{ \int_{SO(2)} \left\{ \int_{T^2} |D_{RB_\gamma}(\sigma, t)|^2 dt \right\}^{\frac{s}{2}} d\sigma \right\}^{\frac{1}{s}} \\
&= R^2 \left\{ \int_{SO(2)} \left\{ \sum_{n \neq 0} |\hat{\chi}_{B_\gamma}(R\sigma n)|^2 \right\}^{\frac{s}{2}} d\sigma \right\}^{\frac{1}{s}}
\end{aligned}$$

We know also that for $0 < p \leq 1$ it's true that $\|f+g\|_p^p \leq \|f\|_p^p + \|g\|_p^p$. Therefore, since $\frac{s}{2} < 1$,

$$\begin{aligned}
&R^2 \left\{ \int_{SO(2)} \left\{ \sum_{n \neq 0} |\hat{\chi}_{B_\gamma}(R\sigma n)|^2 \right\}^{\frac{s}{2}} d\sigma \right\}^{\frac{1}{s}} \\
&= R^2 \left\{ \| \sum_{n \neq 0} |\hat{\chi}_{B_\gamma}(R\sigma n)|^2 \|_{L^{\frac{s}{2}}(SO(2))}^{\frac{s}{2}} \right\}^{\frac{1}{s}} \\
&= R^2 \left\{ \| \sum_{n \neq 0} |\hat{\chi}_{B_\gamma}(R\sigma n)|^2 \|_{L^{\frac{s}{2}}(SO(2))} \right\}^{\frac{1}{2}} \\
&\leq R^2 \left\{ \sum_{n \neq 0} \| |\hat{\chi}_{B_\gamma}(R\sigma n)|^2 \|_{L^{\frac{s}{2}}(SO(2))} \right\}^{\frac{1}{2}} \\
&= R^2 \left\{ \sum_{n \neq 0} \left( \int_{SO(2)} |\hat{\chi}_{B_\gamma}(R\sigma n)|^s d\sigma \right)^{\frac{2}{s}} \right\}^{\frac{1}{2}} \\
&= R^2 \left\{ \sum_{n \neq 0} \|\hat{\chi}_{B_\gamma}(R\sigma n)\|_{L^s(SO(2))}^2 \right\}^{\frac{1}{2}}
\end{aligned}$$

Now one has to use a result in [7] for the $L^s(SO(2))$-norm of $\hat{\chi}_{B_\gamma}$. Noticing that $2 < (2\gamma - 2)/(\gamma - 2)$ for all $\gamma$, one has

$$R^2 \left\{ \sum_{n \neq 0} \| |\hat{\chi}_{B_\gamma}(R\sigma n)| \|^2_{L^s(SO(2))} \right\}^{\frac{1}{2}}$$
$$\leq CR^2 \left\{ \sum_{n \neq 0} |Rn|^{-3} \right\}^{\frac{1}{2}}$$
$$= CR^{\frac{1}{2}} \left\{ \sum_{n \neq 0} |n|^{-3} \right\}^{\frac{1}{2}}$$

This series converges. Therefore for $s < 2$ one has

$$\left\{ \int_{SO(2)} \left\{ \int_{T^2} |D_{RB_\gamma}(\sigma, t)|^p dt \right\}^{\frac{s}{p}} d\sigma \right\}^{\frac{1}{s}} \leq CR^{1/2}.$$

□

**Theorem 17.** *Let* $1 < p \leq 2$, $s > 2$ *and* $1/s + 1/r = 1$. *Then*

$$\left\{ \int_{SO(2)} \left\{ \int_{T^2} |D_{RB_\gamma}(\sigma, t)|^p dt \right\}^{\frac{s}{p}} d\sigma \right\}^{\frac{1}{s}} \leq \begin{cases} CR^{1/2} & \text{if } 2 < s \leq (2\gamma - 2)/(\gamma - 2), \\ CR^{\frac{r - 2(1 - \frac{1}{\gamma})}{\frac{1}{\gamma} - 1}} & \text{if } s > (2\gamma - 2)/(\gamma - 2). \end{cases}$$

*Proof.* Reasoning as before and remembering that $s < (2\gamma - 2)/(\gamma - 2)$, one has

$$\left\{ \int_{SO(2)} \left\{ \int_{T^2} |D_{RB_\gamma}(\sigma, t)|^p dt \right\}^{\frac{s}{p}} d\sigma \right\}^{\frac{1}{s}} \leq CR^{\frac{1}{2}} \left\{ \sum_{n \neq 0} |n|^{-3} \right\}^{\frac{1}{2}}.$$

The last series is finite. Therefore we can conclude that for $2 < s < (2\gamma - 2)/(\gamma - 2)$

$$\left\{ \int_{SO(2)} \left\{ \int_{T^2} |D_{RB_\gamma}(\sigma, t)|^p dt \right\}^{\frac{s}{p}} d\sigma \right\}^{\frac{1}{s}} \leq CR^{1/2}.$$

For $s = (2\gamma - 2)/(\gamma - 2)$ one has

$$R^2 \left\{ \sum_{n \neq 0} \| \hat{\chi}_{B_\gamma}(R\sigma n) \|^2_{L^s(SO(2))} \right\}^{\frac{1}{2}}$$

$$\leq CR^2 \left\{ \sum_{n\neq 0} |Rn|^{-3} (\log{(Rn)})^{\frac{\gamma-2}{\gamma-1}} \right\}^{\frac{1}{2}}$$

$$\leq CR^{\frac{1}{2}} \left\{ \sum_{n\neq 0} |n|^{-3} (\log{(Rn)})^{\frac{\gamma-2}{\gamma-1}} \right\}^{\frac{1}{2}}.$$

Also this series is finite and so for $s = (2\gamma - 2)/(\gamma - 2)$

$$\left\{ \int_{SO(2)} \left\{ \int_{T^2} |D_{RB_\gamma}(\sigma, t)|^p dt \right\}^{\frac{s}{p}} d\sigma \right\}^{\frac{1}{s}} \leq CR^{1/2}.$$

Let $s > (2\gamma - 2)/(\gamma - 2)$. Like in the first theorem, we can use the function $\varphi$ to get a convergent integral:

$$R^2 \left\{ \int_{SO(2)} \left( \int_{T^2} |\sum_{n\neq 0} \hat{\varphi}(\varepsilon\sigma n) \hat{\chi}_{B_\gamma}(R\sigma n)|^p dt \right)^{\frac{s}{p}} d\sigma \right\}^{\frac{1}{s}}$$

$$\leq R^2 \left\{ \int_{SO(2)} \left( \int_{T^2} |\sum_{n\neq 0} \hat{\varphi}(\varepsilon\sigma n) \hat{\chi}_{B_\gamma}(R\sigma n)|^2 dt \right)^{\frac{s}{2}} d\sigma \right\}^{\frac{1}{s}}$$

$$= R^2 \left\{ \int_{SO(2)} \left( \sum_{n\neq 0} |\hat{\varphi}(\varepsilon\sigma n) \hat{\chi}_{B_\gamma}(R\sigma n)|^2 \right)^{\frac{s}{2}} d\sigma \right\}^{\frac{1}{s}}$$

$$= R^2 \left\{ \| \sum_{n\neq 0} |\hat{\varphi}(\varepsilon\sigma n) \hat{\chi}_{B_\gamma}(R\sigma n)|^2 \|_{L^{\frac{s}{2}}(SO(2))} \right\}^{\frac{1}{2}}$$

$$\leq R^2 \left\{ \sum_{n\neq 0} \| |\hat{\varphi}(\varepsilon\sigma n) \hat{\chi}_{B_\gamma}(R\sigma n)|^2 \|_{L^{\frac{s}{2}}(SO(2))} \right\}^{\frac{1}{2}}$$

$$= R^2 \left\{ \sum_{n\neq 0} \left( \int_{SO(2)} |\hat{\varphi}(\varepsilon\sigma n) \hat{\chi}_{B_\gamma}(R\sigma n)|^s d\sigma \right)^{\frac{2}{s}} \right\}^{\frac{1}{2}}$$

$$\leq CR^2 \left\{ \sum_{n\neq 0} \frac{1}{1 + |\varepsilon n|^K} \left( \int_{SO(2)} |\hat{\chi}_{B_\gamma}(R\sigma n)|^s d\sigma \right)^{\frac{2}{s}} \right\}^{\frac{1}{2}}$$

$$\leq CR^2 \left\{ \sum_{n\neq 0} \frac{1}{1 + |\varepsilon n|^K} |Rn|^{-2-\frac{2}{s}-\frac{2}{r\gamma}} \right\}^{\frac{1}{2}}$$

$$= CR^2 \left\{ \int_{|\xi|\le 1} \frac{1}{1+(\varepsilon|\xi|)^K} (R|\xi|)^{-2-\frac{2}{s}-\frac{2}{r\gamma}} d\xi \right\}^{\frac{1}{2}}$$
$$= CR^{1-\frac{1}{s}-\frac{1}{r\gamma}} \left\{ \int_1^\infty \frac{1}{1+(\varepsilon t)^K} (t)^{-1-\frac{2}{s}-\frac{2}{r\gamma}} dt \right\}^{\frac{1}{2}}$$
$$\le CR^{1-\frac{1}{s}-\frac{1}{r\gamma}} \left\{ \int_0^\infty \frac{1}{1+x^K} \left(\frac{x}{\varepsilon}\right)^{-1-\frac{2}{s}-\frac{2}{r\gamma}} \frac{1}{\varepsilon} dx \right\}^{\frac{1}{2}}$$
$$= CR^{\frac{1}{r}(1-\frac{1}{\gamma})} \varepsilon^{\frac{1}{s}+\frac{1}{r\gamma}} \left\{ \int_0^\infty \frac{1}{1+x^K} x^{-1-\frac{2}{s}-\frac{2}{r\gamma}} dx \right\}^{\frac{1}{2}} = A.$$

We choose $K$ large enough to not have problems at $\infty$. Therefore

$$A \le CR^{\frac{1}{r}(1-\frac{1}{\gamma})} \varepsilon^{1-\frac{1}{r}(1-\frac{1}{\gamma})}$$

and

$$\left\{ \int_{SO(2)} \left\{ \int_{T^2} |D_{RB_\gamma}(\sigma,t)|^p dt \right\}^{\frac{s}{p}} d\sigma \right\}^{\frac{1}{s}} \le C(R\varepsilon + R^{\frac{1}{r}(1-\frac{1}{\gamma})} \varepsilon^{1-\frac{1}{r}(1-\frac{1}{\gamma})}).$$

Choosing $\varepsilon = R^{\frac{1-\frac{1}{r}(1-\frac{1}{\gamma})}{\frac{1}{r}(\frac{1}{\gamma}-1)}}$, we get

$$\left\{ \int_{SO(2)} \left\{ \int_{T^2} |D_{RB_\gamma}(\sigma,t)|^p dt \right\}^{\frac{s}{p}} d\sigma \right\}^{\frac{1}{s}} \le CR^{\frac{r-2(1-\frac{1}{\gamma})}{\frac{1}{\gamma}-1}}.$$

□

Now let $p > 2$.

**Theorem 18.** *Let $s \le (2\gamma-2)/(\gamma-2)$. Then*

$$\left\{ \int_{SO(2)} \left\{ \int_{T^2} |D_{RB_\gamma}(\sigma,t)|^p dt \right\}^{\frac{s}{p}} d\sigma \right\}^{\frac{1}{s}} \le \begin{cases} CR^{1/2} & \text{if } 2<p<4, \\ CR^{1/2}\log^{1/4}(R) & \text{if } p=4, \\ CR^{(2q-4)/(q-4)} & \text{if } p>4. \end{cases}$$

*Proof.* Let $2 < p < 4$. One has

$$\left\{ \int_{SO(2)} \left\{ \int_{T^2} |D_{RB_\gamma}(\sigma,t)|^p dt \right\}^{\frac{s}{p}} d\sigma \right\}^{\frac{1}{s}}$$

$$
\begin{aligned}
&= \left\{ \int_{SO(2)} \left\{ \| |D_{RB_\gamma}(\sigma,t)| \|_{L^p(T^2)} \right\}^s d\sigma \right\}^{\frac{1}{s}} \\
&\le R^2 \left\{ \int_{SO(2)} \left( \sum_{n\neq 0} |\hat{\chi}_{B_\gamma}(R\sigma n)|^q \right)^{\frac{s}{q}} d\sigma \right\}^{\frac{1}{s}} \\
&= R^2 \left\{ \| \sum_{n\neq 0} |\hat{\chi}_{B_\gamma}(R\sigma n)|^q \|_{L^{\frac{s}{q}}(SO(2))} \right\}^{\frac{1}{q}} \\
&\le R^2 \left\{ \sum_{n\neq 0} \| |\hat{\chi}_{B_\gamma}(R\sigma n)|^q \|_{L^{\frac{s}{q}}(SO(2))} \right\}^{\frac{1}{q}} \\
&= R^2 \left\{ \sum_{n\neq 0} \left( \int_{SO(2)} |\hat{\chi}_{B_\gamma}(R\sigma n)|^s d\sigma \right)^{\frac{q}{s}} \right\}^{\frac{1}{q}} \\
&= R^2 \left\{ \sum_{n\neq 0} \|\hat{\chi}_{B_\gamma}(R\sigma n)\|^q_{L^s(SO(2))} \right\}^{\frac{1}{q}}
\end{aligned}
$$

For $s < (2\gamma - 2)/(\gamma - 1)$, using [7] we get

$$
\begin{aligned}
&R^2 \left\{ \sum_{n\neq 0} \|\hat{\chi}_{B_\gamma}(R\sigma n)\|^q_{L^s(SO(2))} \right\}^{\frac{1}{q}} \\
&\le R^2 \left\{ \sum_{n\neq 0} |Rn|^{-\frac{3}{2}q} \right\}^{\frac{1}{q}} \\
&= R^{\frac{1}{2}} \left\{ \sum_{n\neq 0} |n|^{-\frac{3}{2}q} \right\}^{\frac{1}{q}}
\end{aligned}
$$

This series is finite when $q > 4/3$ and therefore

$$
\left\{ \int_{SO(2)} \left\{ \int_{T^2} |D_{RB_\gamma}(\sigma,t)|^p dt \right\}^{\frac{s}{p}} d\sigma \right\}^{\frac{1}{s}} \le CR^{1/2}.
$$

For $s = (2\gamma - 2)/(\gamma - 2)$ one has

$$R^2 \left\{ \sum_{n \neq 0} \|\hat{\chi}_{B_\gamma}(R\sigma n)\|^q_{L^s(SO(2))} \right\}^{\frac{1}{q}}$$
$$\leq R^2 \left\{ \sum_{n \neq 0} |Rn|^{-\frac{3}{2}q} (\log |Rn|)^{\frac{\gamma-2}{2\gamma-2}q} \right\}^{\frac{1}{q}}$$
$$= R^{\frac{1}{2}} \left\{ \sum_{n \neq 0} |n|^{-\frac{3}{2}q} (\log |Rn|)^{\frac{\gamma-2}{2\gamma-2}q} \right\}^{\frac{1}{q}}$$

This series is finite when $q > 4/3$ and therefore

$$\left\{ \int_{SO(2)} \left\{ \int_{T^2} |D_{RB_\gamma}(\sigma, t)|^p dt \right\}^{\frac{s}{p}} d\sigma \right\}^{\frac{1}{s}} \leq CR^{1/2}.$$

Let $p > 4$. We can use the smoothing-trick with the function $\varphi$ to get a convergent final integral. One has

$$R^2 \left\{ \int_{SO(2)} \left( \int_{T^2} |\sum_{n \neq 0} \hat{\varphi}(\varepsilon\sigma n) \hat{\chi}_{B_\gamma}(R\sigma n)|^p dt \right)^{\frac{s}{p}} d\sigma \right\}^{\frac{1}{s}}$$
$$\leq R^2 \left\{ \int_{SO(2)} \left( \sum_{n \neq 0} |\hat{\varphi}(\varepsilon\sigma n) \hat{\chi}_{B_\gamma}(R\sigma n)|^q \right)^{\frac{s}{q}} d\sigma \right\}^{\frac{1}{s}}$$
$$= R^2 \left\{ \| \sum_{n \neq 0} |\hat{\varphi}(\varepsilon\sigma n) \hat{\chi}_{B_\gamma}(R\sigma n)|^q \|_{L^{\frac{s}{q}}(SO(2))} \right\}^{\frac{1}{q}}$$
$$\leq R^2 \left\{ \sum_{n \neq 0} \| |\hat{\varphi}(\varepsilon\sigma n) \hat{\chi}_{B_\gamma}(R\sigma n)|^q \|_{L^{\frac{s}{q}}(SO(2))} \right\}^{\frac{1}{q}}$$
$$= R^2 \left\{ \sum_{n \neq 0} \left( \int_{SO(2)} |\hat{\varphi}(\varepsilon\sigma n) \hat{\chi}_{B_\gamma}(R\sigma n)|^s d\sigma \right)^{\frac{q}{s}} \right\}^{\frac{1}{q}}$$
$$\leq CR^2 \left\{ \sum_{n \neq 0} \frac{1}{1 + |\varepsilon n|^K} \left( \int_{SO(2)} |\hat{\chi}_{B_\gamma}(R\sigma n)|^s d\sigma \right)^{\frac{q}{s}} \right\}^{\frac{1}{q}}$$

$$= CR^2 \left\{ \sum_{n\neq 0} \frac{1}{1+|\varepsilon n|^K} \|\hat{\chi}_{B_\gamma}(R\sigma n)\|^q_{L^s(SO(2))} \right\}^{\frac{1}{q}}$$

For $s < (2\gamma - 2)/(\gamma - 2)$ we get

$$\begin{aligned}
& R^2 \left\{ \sum_{n\neq 0} \frac{1}{1+|\varepsilon n|^K} \|\hat{\chi}_{B_\gamma}(R\sigma(n))\|^q_{L^s(SO(2))} \right\}^{\frac{1}{q}} \\
& \leq R^2 \left\{ \sum_{n\neq 0} \frac{1}{1+|\varepsilon n|^K} |Rn|^{-\frac{3}{2}q} \right\}^{\frac{1}{q}} \\
& \leq R^2 \left\{ \int_{|\xi|\geq 1} \frac{1}{1+(\varepsilon|\xi|)^K} (R|\xi|)^{-\frac{3}{2}q} d\xi \right\}^{\frac{1}{q}} \\
& \leq R^{\frac{1}{2}} \left\{ \int_1^\infty \frac{1}{1+(\varepsilon t)^K} t^{-\frac{3}{2}q+1} dt \right\}^{\frac{1}{q}} \\
& \leq R^{\frac{1}{2}} \left\{ \int_0^\infty \frac{1}{1+x^K} \left(\frac{x}{\varepsilon}\right)^{-\frac{3}{2}q+1} \frac{1}{\varepsilon} dx \right\}^{\frac{1}{q}} \\
& \leq R^{\frac{1}{2}} \varepsilon^{\frac{3}{2}-\frac{2}{q}} \left\{ \int_0^\infty \frac{x^{-\frac{3}{2}q+1}}{1+x^K} dx \right\}^{\frac{1}{q}} = A.
\end{aligned}$$

Choose $K$ large enough so that the last integral has no problems at $\infty$. Then one has that the integral converges in 0 for $-\frac{3}{2}q + 1 > -1 \Rightarrow q < \frac{4}{3}$, that it means $p > 4$. Therefore

$$A \leq CR^{\frac{1}{2}} \varepsilon^{\frac{3}{2}-\frac{2}{q}}.$$

Then

$$\left\{ \int_{SO(2)} \left\{ \int_{T^2} |D_{RB_\gamma}(\sigma,t)|^p dt \right\}^{\frac{s}{p}} d\sigma \right\}^{\frac{1}{s}} \leq C(R\varepsilon + R^{\frac{1}{2}} \varepsilon^{\frac{3}{2}-\frac{2}{q}})$$

and choosing $\varepsilon = R^{q/(q-4)}$ we can conclude that for $s < \frac{2\gamma-2}{\gamma-2}$ and $p > 4$

$$\left\{ \int_{SO(2)} \left\{ \int_{T^2} |D_{RB_\gamma}(\sigma,t)|^p dt \right\}^{\frac{s}{p}} d\sigma \right\}^{\frac{1}{s}} \leq CR^{(2q-4)/(q-4)}.$$

For $s = (2\gamma - 2)/(\gamma - 2)$, one has

$$R^2 \left\{ \sum_{n\neq 0} \frac{1}{1+|\varepsilon n|^K} \|\hat{\chi}_{B_\gamma}(R\sigma(n))\|^q_{L^s(SO(2))} \right\}^{\frac{1}{q}}$$

$$\le R^2 \left\{ \sum_{n\neq 0} \frac{1}{1+|\varepsilon n|^K} |Rn|^{-\frac{3}{2}q} (\log|Rn|)^{\frac{\gamma-2}{\gamma-2}q} \right\}^{\frac{1}{q}}$$

$$\le R^2 \left\{ \int_{|\xi|\ge 1} \frac{1}{1+(\varepsilon|\xi|)^K} (R|\xi|)^{-\frac{3}{2}q} (\log(R|\xi|))^{\frac{\gamma-2}{2\gamma-2}q} d\xi \right\}^{\frac{1}{q}}$$

$$\le R^{\frac{1}{2}} \left\{ \int_1^\infty \frac{1}{1+(\varepsilon t)^K} t^{-\frac{3}{2}q+1} (\log(Rt))^{\frac{\gamma-2}{2\gamma-2}q} dt \right\}^{\frac{1}{q}}$$

$$\le R^{\frac{1}{2}} \left\{ \int_0^\infty \frac{1}{1+x^K} \left(\frac{x}{\varepsilon}\right)^{-\frac{3}{2}q+1} (\log\left(R\frac{x}{\varepsilon}\right))^{\frac{\gamma-2}{2\gamma-2}q} \frac{1}{\varepsilon} dx \right\}^{\frac{1}{q}}$$

$$\le R^{\frac{1}{2}} \varepsilon^{\frac{3}{2}-\frac{2}{q}} \left\{ \int_0^\infty \frac{x^{-\frac{3}{2}q+1}}{1+x^K} (\log(R\frac{x}{\varepsilon}))^{\frac{\gamma-2}{2\gamma-2}q} dx \right\}^{\frac{1}{q}}.$$

This integral converges for $q < 4/3 \Rightarrow p > 4$. Therefore

$$\left\{ \int_{SO(2)} \left\{ \int_{T^2} |D_{RB_\gamma}(\sigma,t)|^p dt \right\}^{\frac{s}{p}} d\sigma \right\}^{\frac{1}{s}} \le C(R\varepsilon + R^{\frac{1}{2}}\varepsilon^{\frac{3}{2}-\frac{2}{q}})$$

and choosing $\varepsilon = R^{q/(q-4)}$ we can conclude that for $s = \frac{2\gamma-2}{\gamma-2}$ and $p > 4$

$$\left\{ \int_{SO(2)} \left\{ \int_{T^2} |D_{RB_\gamma}(\sigma,t)|^p dt \right\}^{\frac{s}{p}} d\sigma \right\}^{\frac{1}{s}} \le CR^{(2q-4)/(q-4)}.$$

The case $p = 4$ is the critic one for $s \le (2\gamma-2)/(\gamma-2)$ for every $\gamma$. One can resume the computation done in the previous proofs to get the thesis.

For $s = (2\gamma-2)/(\gamma-2)$ one can think in the same way using the right $L^s(SO(2))$-norm to get the same result. □

One can notice that these results are independent of $\gamma$. So for $s < (2\gamma-2)/(\gamma-2)$, if $p \le 4$ the discrepancy norm doesn't depend on $p$ or $s$. For $p > 4$ the value of $s$ isn't important: what matters is $p$ and its coniugated $q$.
Now let $s > (2\gamma-2)/(\gamma-2)$.

**Theorem 19.** *Let $s > (2\gamma-2)/(\gamma-2)$. Then*

$$\left\{ \int_{SO(2)} \left\{ \int_{T^2} |D_{RB_\gamma}(\sigma,t)|^p dt \right\}^{\frac{s}{p}} d\sigma \right\}^{\frac{1}{s}}$$

$$\leq \begin{cases} CR^{2+\frac{2r}{rq-q(1-\frac{1}{\gamma})-2r}} & \textit{if } p < (2\gamma)/(\gamma-1) \textit{ and } p > 4, \\ CR^{\frac{1}{r}(1-\frac{1}{\gamma})} & \textit{if } 2\gamma/(\gamma-1) < p < 4. \end{cases}$$

*Proof.* One has

$$\begin{aligned} &\left\{\int_{SO(2)} \left\{\int_{T^2} |D_{RB_\gamma}(\sigma,t)|^p dt\right\}^{\frac{s}{p}} d\sigma\right\}^{\frac{1}{s}} \\ &= \left\{\int_{SO(2)} \left\{\| |D_{RB_\gamma}(\sigma,t)| \|_{L^p(T^2)}\right\}^s d\sigma\right\}^{\frac{1}{s}} \\ &\leq R^2 \left\{\sum_{n\neq 0} \|\hat{\chi}_{B_\gamma}(R\sigma n)\|^q_{L^s(SO(2))}\right\}^{\frac{1}{q}} \\ &\leq R^2 \left\{\sum_{n\neq 0} |Rn|^{(-1-\frac{1}{s}-\frac{1}{\gamma}+\frac{1}{\gamma s})q}\right\}^{\frac{1}{q}} \\ &\leq R^{\frac{1}{r}(1-\frac{1}{\gamma})} \left\{\sum_{n\neq 0} |n|^{(-1-\frac{1}{s}-\frac{1}{\gamma}+\frac{1}{\gamma s})q}\right\}^{\frac{1}{q}} \end{aligned}$$

This series is finite for $q + \frac{q}{s} + \frac{q}{\gamma} - \frac{q}{\gamma s} > 2 \Rightarrow s < \frac{q(\gamma-1)}{2\gamma-q(\gamma+1)}$. Because of $s > \frac{2\gamma-2}{\gamma-2}$, if $\frac{q(\gamma-1)}{2\gamma-q(\gamma+1)} > \frac{2\gamma-2}{\gamma-2}$ we get the result. This inequality is verified when

$$q(\gamma-2) - 2(2\gamma - q\gamma - q) > 0 \Rightarrow q > \frac{4}{3} \text{ and } 2\gamma - q(\gamma+1) > 0 \Rightarrow q < \frac{2\gamma}{\gamma+1}.$$

Notice that for $\gamma > 2$ it's $\frac{4}{3} < \frac{2\gamma}{\gamma+1}$ and so for $\frac{2\gamma}{\gamma-1} < p < 4$ one has

$$\left\{\int_{SO(2)} \left\{\int_{T^2} |D_{RB_\gamma}(\sigma,t)|^p dt\right\}^{\frac{s}{p}} d\sigma\right\}^{\frac{1}{s}} \leq CR^{\frac{1}{r}(1-\frac{1}{\gamma})}.$$

For the other results we can use the function $\varphi$ to get a convergent integral. One has

$$R^2 \left\{\int_{SO(2)} \left(\int_{T^2} |\sum_{n\neq 0} \hat{\varphi}(\varepsilon\sigma n)\hat{\chi}_{B_\gamma}(R\sigma n)|^p dt\right)^{\frac{s}{p}} d\sigma\right\}^{\frac{1}{s}}$$

$$\leq CR^2\left\{\sum_{n\neq 0}\frac{1}{1+|\varepsilon n|^K}\left(\int_{SO(2)}|\hat{\chi}_{B_\gamma}(R\sigma n)|^s d\sigma\right)^{\frac{q}{s}}\right\}^{\frac{1}{q}}$$

$$= CR^2\left\{\sum_{n\neq 0}\frac{1}{1+|\varepsilon n|^K}\|\hat{\chi}_{B_\gamma}(R\sigma n)\|^q_{L^s(SO(2))}\right\}^{\frac{1}{q}}$$

$$\leq R^2\left\{\sum_{n\neq 0}\frac{1}{1+|\varepsilon n|^K}|Rn|^{(-1-\frac{1}{s}-\frac{1}{\gamma}+\frac{1}{\gamma s})q}\right\}^{\frac{1}{q}}$$

$$\leq R^{\frac{1}{r}(1-\frac{1}{\gamma})}\varepsilon^{1+\frac{1}{s}+\frac{1}{\gamma}-\frac{1}{\gamma s}-\frac{2}{q}}\left\{\int_0^\infty\frac{x^{-q-\frac{q}{s}-\frac{q}{\gamma}+\frac{q}{\gamma s}+1}}{1+x^K}dx\right\}^{\frac{1}{q}} = A.$$

With the right $K$, the integral is finite for $-q-\frac{q}{s}-\frac{q}{\gamma}+\frac{q}{\gamma s}+1>-1 \Rightarrow s > \frac{q(\gamma-1)}{2\gamma-q(\gamma+1)}$. If $\frac{q(\gamma-1)}{2\gamma-q(\gamma+1)} < \frac{2\gamma-2}{\gamma-2}$ we can conclude the proof. This is verified when

$$q(\gamma-2)-2(2\gamma-q\gamma-q)<0 \Rightarrow q<\frac{4}{3} \text{ and } 2\gamma-q(\gamma+1)>0 \Rightarrow q<\frac{2\gamma}{\gamma+1}$$

and when

$$q(\gamma-2)-2(2\gamma-q\gamma-q)>0 \Rightarrow q>\frac{4}{3} \text{ and } 2\gamma-q(\gamma+1)<0 \Rightarrow q>\frac{2\gamma}{\gamma+1}.$$

So one has that for $q<\frac{4}{3}$ and for $q>\frac{2\gamma}{\gamma+1}$ the integral converges. Therefore

$$A \leq R^{\frac{1}{r}(1-\frac{1}{\gamma})}\varepsilon^{1+\frac{1}{s}+\frac{1}{\gamma}-\frac{1}{\gamma s}-\frac{2}{q}}$$

and

$$\left\{\int_{SO(2)}\left\{\int_{T^2}|D_{RB_\gamma}(\sigma,t)|^p dt\right\}^{\frac{s}{p}}d\sigma\right\}^{\frac{1}{s}} \leq C(R\varepsilon + R^{\frac{1}{r}(1-\frac{1}{\gamma})}\varepsilon^{1+\frac{1}{s}+\frac{1}{\gamma}-\frac{1}{\gamma s}-\frac{2}{q}}).$$

Choosing $\varepsilon = R^{1+\frac{2r}{rq-q(1-\frac{1}{\gamma})-2r}}$, one has

$$\left\{\int_{SO(2)}\left\{\int_{T^2}|D_{RB_\gamma}(\sigma,t)|^p dt\right\}^{\frac{s}{p}}d\sigma\right\}^{\frac{1}{s}} \leq CR^{2+\frac{2r}{rq-q(1-\frac{1}{\gamma})-2r}}.$$

$\square$

Let $s = \infty$. We want to estimate $\|D_{RB_\gamma}(\sigma,t)\|_{L^\infty(SO(2),L^p(T^2))}$, that it means the $\sup \|D_{RB_\gamma}(\sigma,t)\|_{L^p(T^2)}$. One has

$$\begin{aligned}\|D_{RB_\gamma}(\sigma,t)\|_{L^p(T^2)} &= \left\{\int_{T^2} |D_{RB_\gamma}(\sigma,t)|^p dt\right\}^{\frac{1}{p}} \\ &\leq R^2 \|\{\hat{\chi}_{B_\gamma}(R\sigma n)\}_{n\neq 0}\|_{L^q(\mathbb{Z}^2)} \\ &= R^2 \left(\sum_{n\neq 0} |\hat{\chi}_{B_\gamma}(R\sigma n)|^q\right)^{\frac{1}{q}}\end{aligned}$$

Remembering that for $s = \infty$

$$\left\{\int_0^{2\pi} |\hat{\chi}_{B_\gamma}(\rho\Theta)|^s d\theta\right\}^{\frac{1}{s}} \leq c\rho^{-1-\frac{1}{\gamma}},$$

one gets

$$\begin{aligned}R^2 \left(\sum_{n\neq 0} |\hat{\chi}_{B_\gamma}(R\sigma n)|^q\right)^{\frac{1}{q}} &\leq R^2 \left(\sum_{n\neq 0} |Rn|^{(-1-\frac{1}{\gamma})q}\right)^{\frac{1}{q}} \\ &= R^{1-\frac{1}{\gamma}} \left(\sum_{n\neq 0} |n|^{(-1-\frac{1}{\gamma})q}\right)^{\frac{1}{q}}\end{aligned}$$

This series converges for $(1+\frac{1}{\gamma})q > 2 \Rightarrow q > \frac{2\gamma}{\gamma+1}$. Therefore for $p < \frac{2\gamma}{\gamma-1}$ one gets

$$\|D_{RB_\gamma}(\sigma,t)\|_{L^\infty(SO(2),L^p(T^2))} \leq CR^{1-\frac{1}{\gamma}}.$$

One can notice that this estimate is better than the previous one, because $1 - \frac{1}{\gamma} < \frac{2q-2\gamma}{q-2\gamma}$ for $q > \frac{2\gamma}{\gamma+1}$.

One can try with the smoothing-trick to obtain another result. One has

$$\begin{aligned}&R^2 \left(\sum_{n\neq 0} |\hat{\chi}_{B_\gamma}(R\sigma n)|^q \frac{1}{1+|\varepsilon n|^K}\right)^{\frac{1}{q}} \\ &\leq R^2 \left(\sum_{n\neq 0} |Rn|^{(-1-\frac{1}{\gamma})q} \frac{1}{1+|\varepsilon n|^K}\right)^{\frac{1}{q}}\end{aligned}$$

$$= R^{1-\frac{1}{\gamma}} \left( \sum_{n \neq 0} |n|^{(-1-\frac{1}{\gamma})q} \frac{1}{1+|\varepsilon n|^K} \right)^{\frac{1}{q}}$$
$$\leq R^{1-\frac{1}{\gamma}} \left( \int_{|\xi|\geq 1} \frac{1}{1+(\varepsilon|\xi|)^K} |\xi|^{(-1-\frac{1}{\gamma})q} d\xi \right)^{\frac{1}{q}}$$
$$= R^{1-\frac{1}{\gamma}} \left( \int_1^\infty \frac{1}{1+(\varepsilon t)^K} t^{(-1-\frac{1}{\gamma})q+1} dt \right)^{\frac{1}{q}}$$
$$\leq R^{1-\frac{1}{\gamma}} \varepsilon^{(1+\frac{1}{\gamma})-\frac{2}{q}} \left( \int_0^\infty \frac{1}{1+(s)^K} s^{(-1-\frac{1}{\gamma})q+1} ds \right)^{\frac{1}{q}}$$

With the right $K$, this integral converges for $q < \frac{2\gamma}{1+\gamma}$. So it is

$$\|D_{RB_\gamma}(\sigma,t)\|_{L^\infty(SO(2),L^p(T^2))} \leq C(R\varepsilon + R^{1-\frac{1}{\gamma}} \varepsilon^{(1+\frac{1}{\gamma})-\frac{2}{q}})$$

and choosing $\varepsilon = R^{\frac{q}{q-2\gamma}}$ one has

$$\|D_{RB_\gamma}(\sigma,t)\|_{L^\infty(SO(2),L^p(T^2))} \leq CR^{1-\frac{q}{2\gamma-q}}.$$

Comparing this result with the estimate in the theorem for $s = \infty$, we can conclude that for $\frac{2\gamma}{\gamma-1} < p < 4$

$$\|D_{RB_\gamma}(\sigma,t)\|_{L^\infty(SO(2),L^p(T^2))} \leq CR^{1-\frac{1}{\gamma}}$$

because $1 - \frac{q}{2\gamma-q} > 1 - \frac{1}{\gamma}$ for $q < \frac{2\gamma}{\gamma+1}$, and for $p > 4$

$$\|D_{RB_\gamma}(\sigma,t)\|_{L^\infty(SO(2),L^p(T^2))} \leq CR^{1-\frac{q}{2\gamma-q}}$$

**Theorem 20.** *Let $s = \infty$. Then*

$$\left\{ \int_{SO(2)} \left\{ \int_{T^2} |D_{RB_\gamma}(\sigma,t)|^p dt \right\}^{\frac{s}{p}} d\sigma \right\}^{\frac{1}{s}} \leq \begin{cases} CR^{1-\frac{q}{2\gamma-q}} & \text{if } p > 4, \\ CR^{1-\frac{1}{\gamma}} & \text{if } p < 4. \end{cases}$$

### Estimates from below

One can do as before with the $L^s(SO(2), L^p(T^2))$-norm. Therefore $\forall k \neq 0$

$$\|D_{RB_\gamma}(\sigma,t)\|_{L^s(SO(2),L^p(T^2))}$$
$$= \left\{ \int_{SO(2)} \left( \int_{T^2} |D_{RB_\gamma}(\sigma,t)|^p dt \right)^{\frac{1}{p}\cdot s} d\sigma \right\}$$

$$= R^2 \left\{ \int_{SO(2)} \left( \int_{T^2} |\sum_{n\neq 0} \hat{\chi}_{B_\gamma}(R\sigma n) e^{2\pi i n t}|^p dt \right)^{\frac{s}{p}} d\sigma \right\}^{\frac{1}{s}}$$
$$\geq R^2 \left\{ \int_{SO(2)} |\hat{\chi}_{B_\gamma}(R\sigma k)|^s d\sigma \right\}^{\frac{1}{s}}.$$

One can notice that this estimate depends only on the value of $s$: for $s < (2\gamma-2)/(\gamma-2)$ it's

$$\|D_{RB_\gamma}(\sigma,t)\|_{L^s(SO(2),L^p(T^2))} \geq cR^{1/2}$$

and for $s > (2\gamma-2)/(\gamma-2)$ it's

$$\|D_{RB_\gamma}(\sigma,t)\|_{L^s(SO(2),L^p(T^2))} \geq cR^{\frac{1}{r}(1-\frac{1}{\gamma})}.$$

**Summary**

In summary one has that for $s < \frac{2\gamma-2}{\gamma-2}$

$$\left\{ \int_{SO(2)} \left\{ \int_{T^2} |D_{RB_\gamma}(\sigma,t)|^p dt \right\}^{\frac{s}{p}} d\sigma \right\}^{\frac{1}{s}} \leq \begin{cases} CR^{\frac{1}{2}} & \text{for } p < 4 \\ CR^{\frac{2q-4}{q-4}} & \text{for } p > 4. \end{cases}$$

and for $s > \frac{2\gamma-2}{\gamma-2}$

$$\left\{ \int_{SO(2)} \left\{ \int_{T^2} |D_{RB_\gamma}(\sigma,t)|^p dt \right\}^{\frac{s}{p}} d\sigma \right\}^{\frac{1}{s}} \leq \begin{cases} CR^{\frac{r-2(1-\frac{1}{\gamma})}{\frac{1}{\gamma}-1}} & \text{for } p < 2 \\ CR^{2+\frac{2r}{rq-q(1-\frac{1}{\gamma})-2r}} & \text{for } 2 < p < \frac{2\gamma}{\gamma-1} \\ CR^{\frac{1}{r}(1-\frac{1}{\gamma})} & \text{for } \frac{2\gamma}{\gamma-1} < p < 4 \\ CR^{2+\frac{2r}{rq-q(1-\frac{1}{\gamma})-2r}} & \text{for } p > 4. \end{cases}$$

We can control that these results agree with the ones for $p = s$. For $p < \frac{2\gamma-2}{\gamma-2}$ one has

$$\left\{ \int_{SO(2)} \int_{T^2} |D_{RB_\gamma}(\sigma,t)|^p dt d\sigma \right\}^{\frac{1}{p}} \leq \begin{cases} CR^{\frac{1}{2}} & \text{for } p < 4 < \frac{2\gamma-2}{\gamma-2}; \\ CR^{\frac{2q-4}{q-4}} & \text{for } 4 < p < \frac{2\gamma-2}{\gamma-2}; \\ CR^{\frac{1}{2}} & \text{for } p < \frac{2\gamma-2}{\gamma-2} < 4 \end{cases}$$

and it agrees with what said before. For $p > \frac{2\gamma-2}{\gamma-2}$ one has

$$\left\{\int_{SO(2)}\int_{T^2}|D_{RB_\gamma}(\sigma,t)|^p dt d\sigma\right\}^{\frac{1}{p}} \le \begin{cases} CR^{\frac{2(q-2+1/\gamma)}{q-3+1/\gamma}} & \textit{for } 4 < \frac{2\gamma-2}{\gamma-2} < p; \\ CR^{\frac{1}{q}(1-\frac{1}{\gamma})} & \textit{for } \frac{2\gamma-2}{\gamma-2} < p < \frac{3\gamma-1}{\gamma-1} < 4; \\ CR^{\frac{2(q-2+1/\gamma)}{q-3+1/\gamma}} & \textit{for } \frac{2\gamma-2}{\gamma-2} < \frac{3\gamma-1}{\gamma-1} < p < 4; \\ CR^{\frac{2(q-2+1/\gamma)}{q-3+1/\gamma}} & \textit{for } \frac{2\gamma-2}{\gamma-2} < \frac{3\gamma-1}{\gamma-1} < 4 < p. \end{cases}$$

The only different thing is the case $\frac{3\gamma-1}{\gamma-1} < p < 4$, where we obtain a better estimate studying $s \neq p$. So one has:

$$\left\{\int_{SO(2)}\int_{T^2}|D_{RB_\gamma}(\sigma,t)|^p dt d\sigma\right\}^{\frac{1}{p}} \le \begin{cases} CR^{\frac{2(q-2+1/\gamma)}{q-3+1/\gamma}} & \textit{for } 4 < \frac{2\gamma-2}{\gamma-2} < p; \\ CR^{\frac{1}{q}(1-\frac{1}{\gamma})} & \textit{for } \frac{2\gamma-2}{\gamma-2} < p < \frac{3\gamma-1}{\gamma-1} < 4; \\ CR^{\frac{1}{q}(1-\frac{1}{\gamma})} & \textit{for } \frac{2\gamma-2}{\gamma-2} < \frac{3\gamma-1}{\gamma-1} < p < 4; \\ CR^{\frac{2(q-2+1/\gamma)}{q-3+1/\gamma}} & \textit{for } \frac{2\gamma-2}{\gamma-2} < \frac{3\gamma-1}{\gamma-1} < 4 < p. \end{cases}$$

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