THE EXISTENCE OF SOLUTIONS TO VARIATIONAL PROBLEMS OF SLOW GROWTH.

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Abstract. We consider the existence of solutions, in the space $W^{1,1}(\Omega)$, to the problem

$$\text{minimize} \int_\Omega L(\nabla v(x))dx \quad \text{on} \quad \phi + W_0^{1,1}(\Omega)$$

where L is of slow (linear or at most quadratic) growth. We present a necessary and sufficient condition in order that, for any smooth boundary datum ϕ and for any bounded Ω with smooth boundary, the minimum problem be solvable.

1. Introduction

We consider the existence of solutions, in the space $W^{1,1}(\Omega)$, to the problem

$$\text{minimize} \int_\Omega L(\nabla v(x))dx \quad \text{on} \quad \phi + W_0^{1,1}(\Omega) \tag{1}$$

where L is smooth and of slow (linear or at most quadratic) growth. More precisely, the class $\mathbb{L}$ of Lagrangians L we shall consider is

$$\mathbb{L} = \big\{ L(\xi) = l(|\xi|) : l : \mathbb{R} \to \mathbb{R}^+ \text{ is strictly convex },\ l(t) = l(-t), \\ l \in C^2 \ \text{ and } \ l'' \text{ is non-increasing} \big\}$$

Lagrangians defined by smooth strictly convex functions l that, for $|t|$ large, grow like $\frac{1}{2}t^2$, or $|t| - \sqrt{|t|} + \gamma$, or $\sqrt{1+t^2}$, all belong to $\mathbb{L}$. The theory of existence of solutions for different Lagrangians, in particular, for some Lagrangians belonging to the class $\mathbb{L}$, is based on different arguments. For Lagrangians of superlinear growth as $L(\xi) = \frac{1}{2}|\xi|^2$, the direct method yields existence of solutions, based on lower semicontinuity and weak compactnesss, with no mention of the properties neither of ϕ nor of the boundary of Ω. On the other hand of the spectrum, for the non-parametric minimal surface problem, i.e. for $L(\xi) = \sqrt{1+|\xi|^2}$, in order that the minimum problem be solvable for any smooth datum ϕ, a necessary and sufficient condition [7] is that the mean curvature of $\partial\Omega$ be non-positive.

A condition for the existence of solutions (intermediate growth condition) that does not imply superlinear growth was introduced in [4]; the same condition was used in [1] and [2] to prove existence and regularity (lipschitzianity) of solutions to the problem

$$\text{minimize} \int_a^b L(x(t), x'(t))dt;\ x(a) = \alpha,\ x(b) = \beta.$$

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The results of the above mentioned papers are based on reparametrizations, an argument specific to one-dimensional integration set and, to these authors' knowledge, this intermediate growth condition has not yet been used for problems on a multi-dimensional integration set Ω.

In this paper we show that this condition (Assumption 1 below) is *necessary and sufficient* in order that, for any smooth boundary datum ϕ and for any bounded Ω with smooth boundary, the minimization problem (1) admits a solution.

In particular, this condition is able to divide Lagrangians of linear growth in two separate classes: those like the non-parametric minimal area problem, $L(\xi) = \sqrt{1+|\xi|^2}$, where the above statement is not true, from those Lagrangians growing, for $|\xi|$ large, like $L(\xi) = |\xi| - \sqrt{|\xi|} + \gamma$, for which we shall prove existence of solutions.

Models with Lagrangians growing linearly are important in elasticity [5]; our u is a scalar and not a vector and we make no claims at solving these problems; still, finding connections might be of some interest.

2. Notation and preliminary results

In what follows we assume that $\mathbb{R}^N$ is endowed with the Euclidian norm $|.|$ and the inner product $\langle .,. \rangle$. For $u \in C^2(\Omega)$ we denote by $u_{x_i} = \frac{\partial u}{\partial x_i}$ and $u_{x_i,x_j} = \frac{\partial^2 u}{\partial x_i \partial x_j}$ the first and the second order partial derivatives of u, respectively. The gradient ∇u and the Hessian H_u of u are given by $\nabla u = (u_{x_1}, ..., u_{x_n})$ and $H_u = \left[u_{x_i,x_j}\right]_{1 \le i,j \le n}$, respectively. The transpose of a is a^T. We shall set

$$d(x) = d(x, \partial\Omega) = \inf_{y \in \partial\Omega} \{|x-y|\},$$

the distance of a point $x \in \Omega$ from the boundary of Ω. The measure of the unit ball $B \subset \mathbb{R}^n$ is ω_n, and the $(n-1)$ Hausdorff measure of ∂B is $n\omega_n$. By I_A we mean the indicator function of the set A. We shall also set

$$\mathbb{F} = \{\phi \in C^2(\overline{\Omega})\} \text{ and } \mathbb{O} = \{\Omega \text{ open and bounded: } \partial\Omega \in C^2\}.$$

For a function $l : \mathbb{R} \to \mathbb{R}^+$, by l^* we mean its *polar*, or *Legendre-Fenchel transform*, defined by

$$l^*(p) = \sup_{t \in \mathbb{R}} pt - l(t)$$

[8]. l^* is a convex, possibly extended-valued function, i.e., with values in $\mathbb{R} \cup \{+\infty\}$. We set Dom$l^*$ to be the effective domain of l^*, i.e. the set of points where l^* takes finite values. When l is strictly convex, l^* is differentiable on its effective domain. By its very definition, the map l^*, as a map from $\mathbb{R}$ to $\mathbb{R} \cup +\infty$, is lower semicontinuous. In this paper we shall discuss the implications of the following assumption.

Assumption 1. *The map l^* is continuous as a map from $\mathbb{R}$ to $\mathbb{R} \cup +\infty$.*

In particular, under Assumption 1, it cannot happen that Dom$l^* = [-p^*, +p^*]$ with p^* finite, since l^* would be discontinuous at p^* and at $-p^*$.

The polar to the map $t \to \frac{1}{2}t^2$ is $p \to \frac{1}{2}p^2$, a continuous map on $\mathbb{R}$, so that Assumption 1 is satisfied.

The map

$$l(t) = \begin{cases} |t| - \sqrt{|t|} + \frac{1}{8}\alpha + \frac{1}{2}\beta + \gamma & \text{for } |t| \ge \frac{1}{2} \\ \frac{1}{4}\alpha|t|^4 + \frac{1}{2}\beta|t|^2 & \text{for } |t| \le \frac{1}{2} \end{cases}$$

where $\alpha = \frac{3}{\sqrt{2}} - 2$, $\beta = \frac{3+\sqrt{2}}{2} - \frac{9}{4\sqrt{2}}$ and $\gamma = \sqrt{2} - 1$ is strictly convex, C^2 and of linear growth; again, Assumption 1 is satisfied. In fact, for this l we have (we compute l^* only for $|p| \geq 1 - \frac{1}{\sqrt{2}}$),

$$l^*(p) = \begin{cases} \frac{1}{4}\frac{1}{1-|p|}+ & \text{for } 1 > |p| \geq 1 - \frac{1}{\sqrt{2}} \\ +\infty & \text{for } |p| \geq 1 \end{cases},$$

so that l^* is continuous as a map to $\mathbb{R} \cup \{+\infty\}$.

Consider our third example, the Lagrangian of the non-parametric minimal area problem, i.e, $l(t) = \sqrt{1+t^2} - 1$; L belongs to $\mathbb{L}$ but it does not satisfy Assumption 1; in fact, in this case we have that $l^*(p) = 1 - \sqrt{1-p^2}$ for $-1 \leq p \leq 1$, and $l^*(p) = +\infty$ for $|p| > 1$, so that l^* is lower semicontinuous and not continuous.

The following is our main result.

Theorem 1. *Let $L \in \mathbb{L}$. Then, for every $\phi \in \mathbb{F}$, for every $\Omega \in \mathbb{O}$, problem* (1) *admits a (unique) solution $\tilde{u} \in W^{1,1}(\Omega)$ if and only if Assumption 1 holds.*

For the non-parametric minimal area problem Assumption 1 is not satisfied and it is *not* true that for every $\phi \in \mathbb{F}$, for every $\Omega \in \mathbb{O}$, problem (1) admits a solution.

Proposition 1. *Let $l \in \mathbb{L}$ satisfy Assumption 1; let* $\mathrm{Dom} l^*$ *be $(-p^*, +p^*)$, p^* possibly $+\infty$. Then, $\lim_{p\uparrow p^*} l^*(p) = +\infty$ and $\lim_{p\uparrow p^*} (l^*(p))' = +\infty$.*

Proof. When p^* is finite, the assumption of continuity implies that $\lim_{p\uparrow p^*} l^*(p) = +\infty$; when $p^* = +\infty$, it can be that $\lim_{p\uparrow p^*} l^*(p)$ is finite only if l^* is a constant c; in this case, $l = l^{**}$ would be $I_{\{0\}} - c$, a contradiction to the properties of l.

Since l is strictly convex, $(l^*)'$ exists on $(-p^*, +p^*)$ and, for $p \geq 0$, it is an increasing function, so that $\lim_{p\uparrow p^*} (l^*(p))'$ exists. Assume that $(l^*)'$ is bounded above by K; then, from $(l^*)'(l'(t)) \equiv t$ we would obtain that t must be bounded, a contradiction to $\mathrm{Dom} l = \mathbb{R}$. □

Proposition 2. *Let $l \in \mathbb{L}$; then, there exist τ and $\tilde{\lambda}$: $t \geq \tau$ implies*

$$l''(t) \leq \tilde{\lambda} l'(t)$$

Proof. We have that $\lim_{t\to+\infty} l'(t)$ is either $+\infty$ or $p^* > 0$. Since $l''(0) \geq l''(t)$, for a suitable τ we have, for $t \geq \tau$, in the first case $l''(t) \leq l'(t)$ and, in the second case (choosing τ so that $t \geq \tau$ implies $l'(t) \geq \frac{p^*}{2}$), $l''(t) \leq l''(0)\frac{2l'(t)}{p^*}$. □

Proposition 3. *Let B be a ball about 0; let $u \in W^{1,1}(B)$. For $x \in B$ set*

$$\tilde{u}(x) = \frac{1}{n\omega_n} \int_{|\omega|=1} u(\omega|x|) d\omega.$$

Then,

$$\nabla\tilde{u}(x) = \begin{cases} \frac{1}{n\omega_n} \frac{x}{|x|} \int_{|\omega|=1} \langle \nabla u(\omega|x|), \omega \rangle d\omega & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Proof. It is proved in [3]. □

We also recall a definition and an useful result.

Definition 1. *A function $w \in C^2(\Omega)$ with $w|_{\partial\Omega} = \phi$ is called supersolution (resp. subsolution) of* (1) *if for every $v \in C^2(\Omega)$ with $w|_{\partial\Omega} = v|_{\partial\Omega}$ and $v \geq w$ on Ω, (resp. $v \leq w$ on Ω) we have*

$$\int_\Omega L(\nabla v(x))\, dx \geq \int_\Omega L(\nabla w(x))\, dx$$

Lemma 1. *If $w \in C^2(\Omega)$ is such that div $\nabla L(\nabla w(x)) \leq 0$ on Ω and $w\,|_{\partial\Omega}= \phi$ then w is a supersolution of* (1).

3. Proof of Theorem 1

Proof of sufficiency. Let Φ be such that $-\Phi \leq \phi \leq \Phi$, so that for any solution $\tilde{u}$ we must have $-\Phi \leq \tilde{u} \leq \Phi$. The Proof is based on constructing a neighborhood of $\partial\Omega$ in Ω, $\Omega_\delta = \{x \in \Omega : d(x) < \delta\}$ and, on Ω_δ, a super-solution u^+ and a sub-solution u^-, both satisfying the same boundary condition ϕ on $\partial\Omega$ while satisfying respectively $u^+ \geq \Phi$ and $u^- \leq -\Phi$ on that part of $\partial(\Omega_\delta)$ that is contained in Ω. The construction then guarantees that a solution $\tilde{u}$ is constrained between two Lipschitzian functions, each coinciding with ϕ at $\partial\Omega$. Then, the results implying that the supremum of the absolute value of the differential quotients is attained at the boundary of Ω, apply and we obtain an a-priori upper bound for the Lipschitz constant of a possible solution $\tilde{u}$. In turn (see e.g. [6]), this fact and the strict convexity of L imply the existence of a Lipschitzian solution $\tilde{u}$ to problem (1). Then, the sufficiency part of the proof consists in showing that, for $L \in \mathbb{L}$ and $\phi \in \mathbb{F}$, the validity of Assumption 1 permits the construction of u^+ and u^-. Hence, this part of the proof follows the same pattern as the proof for the non-parametric minimal area problem, the essential difference being that in our case we have no control whatsoever on the sign of Δd so that the discussion of the properties of the solutiuons to the basic differential equation for ψ' is totally different.

a) In the paper we shall build a super solution u^+, the construction for u^- being similar.

For a generic smooth function u we have

$$\text{div}\nabla L(\nabla u(x)) = \frac{\nabla u^T}{|\nabla u|} H_u \frac{\nabla u}{|\nabla u|}(l'' - \frac{l'}{|\nabla u|}) + \frac{l'}{|\nabla u|}\Delta u; \tag{2}$$

In particular, We shall define u^+ through the distance function d and a function ψ, to be defined, such that $\psi(0) = 0$, setting

$$u(x) = \psi(d(x)) + \phi(x).$$

This function u is such that, for $x \in \partial\Omega$, we have $u(x) = \phi(x)$ and it satisfies

$$\nabla u = \psi'\nabla d + \nabla\phi\ ;\ |\nabla u|^2 = (\psi')^2 + |\nabla\phi|^2 + 2\langle\nabla\phi, \nabla d\rangle\psi', \tag{3}$$

so that (taking $\psi' > 0$)

$$|\nabla u| = \psi' + e(\psi') \tag{4}$$

with $e(\psi') = \sqrt{(\psi')^2 + |\nabla\phi|^2 + 2\langle\nabla\phi, \nabla d\rangle\psi'} - \psi'$ such that $\frac{e(\psi')}{\psi'} \to 0$ as $\psi' \to \infty$. Moreover,

$$u_{x_i x_j} = u_{ij} = \psi'' d_i d_j + \psi' d_{ij} + \phi_{ij},$$

hence

$$H_u = \psi''\nabla d \otimes \nabla d + \psi' H_d + H_\phi$$

and, in particular,

$$\Delta u = \psi'(d)\Delta d + \psi''(d) + \Delta\phi.$$

Then, from (2) we have

$$\text{div}\nabla L(\nabla u(x))$$
$$= \frac{\nabla u}{|\nabla u|}^T (\psi''\nabla d \otimes \nabla d + \psi' H_d + H_\phi)\frac{\nabla u}{|\nabla u|}(l'' - \frac{l'}{|\nabla u|}) + \frac{l'}{|\nabla u|}(\psi'(d)\Delta d + \psi''(d) + \Delta\phi).$$

Since $H_d \nabla d = 0$ and $\langle d, d\rangle = 1$, recalling (3) we obtain

$$\begin{aligned}
\mathrm{div} \nabla L(\nabla u(x)) = & \frac{(l'' - \frac{l'}{|\nabla u|})}{|\nabla u|^2} (\psi' \nabla d + \nabla \phi)^T (\psi'' \nabla d \otimes \nabla d + \psi' H_d + H_\phi)(\psi' \nabla d + \nabla \phi) \\
& + \frac{l'}{|\nabla u|} (\psi' \Delta d + \psi'' + \Delta \phi) \\
= & \frac{(\psi')^2}{|\nabla u|^2} \psi'' (l'' - \frac{l'}{|\nabla u|}) + \frac{\psi''}{|\nabla u|^2} (\langle \nabla \phi, \nabla d\rangle^2 + 2\psi' \langle \nabla \phi, \nabla d\rangle)(l'' - \frac{l'}{|\nabla u|}) \\
& + \psi'' \frac{l'}{|\nabla u|} \frac{(\psi')^2}{|\nabla u|^2} \frac{|\nabla u|^2}{(\psi')^2} + \frac{l'[\psi' \Delta d + \Delta \phi]}{|\nabla u|} \\
+ & \frac{(l'' - \frac{l'}{|\nabla u|})}{|\nabla u|^2} \left(\psi' \nabla \phi^T H_d \nabla \phi + (\nabla \phi)^T H_\phi \nabla \phi + 2\psi' (\nabla \phi)^T H_\phi \nabla d + (\psi')^2 (\nabla d)^T H_\phi \nabla d \right).
\end{aligned}$$

We have, from (3),

$$\begin{aligned}
& \frac{(\psi')^2}{|\nabla u|^2} \psi'' (l'' - \frac{l'}{|\nabla u|}) + \frac{\psi''}{|\nabla u|^2} (\langle \nabla \phi, \nabla d\rangle^2 + 2\psi' \langle \nabla \phi, \nabla d\rangle)(l'' - \frac{l'}{|\nabla u|}) \\
& + \psi'' \frac{l'}{|\nabla u|} \frac{(\psi')^2}{|\nabla u|^2} \frac{|\nabla u|^2}{(\psi')^2} \\
= & \frac{(\psi')^2}{|\nabla u|^2} \psi'' l'' + \frac{(\psi')^2}{|\nabla u|^2} \psi'' \frac{l'}{|\nabla u|} (-1 + \frac{(\psi')^2 + |\nabla \phi|^2 + 2\langle \nabla \phi, \nabla d\rangle \psi'}{(\psi')^2}) \\
& + \psi'' \frac{1}{|\nabla u|^2} (\langle \nabla \phi, \nabla d\rangle^2 + 2\psi' \langle \nabla \phi, \nabla d\rangle)(l'' - \frac{l'}{|\nabla u|}) \\
= & \frac{(\psi')^2}{|\nabla u|^2} \psi'' \left\{ l'' + \frac{l'}{|\nabla u|} \left[\frac{|\nabla \phi|^2 - \langle \nabla \phi, \nabla d\rangle^2}{(\psi')^2} \right] + \frac{l''}{(\psi')^2} (\langle \nabla \phi, \nabla d\rangle^2 + 2\psi' \langle \nabla \phi, \nabla d\rangle) \right\} \\
\leq & \frac{(\psi')^2}{|\nabla u|^2} \psi'' l'' \left\{ 1 + \frac{1}{(\psi')^2} (\langle \nabla \phi, \nabla d\rangle^2 + 2\psi' \langle \nabla \phi, \nabla d\rangle) \right\}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathrm{div} \nabla L(\nabla u(x)) \leq & \frac{(\psi')^2}{|\nabla u|^2} \psi'' l'' \left\{ 1 + \frac{1}{(\psi')^2} (\langle \nabla \phi, \nabla d\rangle^2 + 2\psi' \langle \nabla \phi, \nabla d\rangle) \right\} \\
& + \frac{l'[\psi' \Delta d + \Delta \phi]}{|\nabla u|} \\
+ & \frac{\left(\psi' (\nabla \phi)^T H_\phi \nabla \phi + \psi' (\nabla \phi)^T H_\phi \nabla d + (\psi')^2 (\nabla d)^T H_\phi \nabla d \right) (l'' - \frac{l'}{|\nabla u|})}{|\nabla u|^2}
\end{aligned}$$

and, in order to have $\mathrm{div} \nabla L(\nabla u(x)) \leq 0$, it is enough to have

$$\begin{aligned}
0 \geq & \psi'' l'' \left\{ 1 + \frac{1}{(\psi')^2} (\langle \nabla \phi, \nabla d\rangle^2 + 2\psi' \langle \nabla \phi, \nabla d\rangle) \right\} \\
& + \frac{l' |\nabla u| \Delta d}{\psi'} + \frac{l' |\nabla u| \Delta \phi}{(\psi')^2} \\
& + (l'' - \frac{l'}{|\nabla u|}) \left((\nabla d)^T H_\phi \nabla d + \frac{1}{\psi'} [(\nabla \phi)^T H_\phi \nabla \phi + (\nabla \phi)^T H_\phi \nabla d] \right)
\end{aligned}$$

i.e.,

$$0 \geq \psi'' l'' \left\{ 1 + \frac{1}{(\psi')^2} (\langle \nabla \phi \nabla d\rangle^2 + 2\psi' \langle \nabla \phi \nabla d\rangle) \right\} \tag{5}$$

$$+l'\left\{\frac{|\nabla u|\Delta d}{\psi'}+\frac{|\nabla u|\Delta\phi}{(\psi')^2}-\frac{1}{\psi'|\nabla u|}(\nabla\phi)^T H_d\nabla\phi-\frac{1}{(\psi')^2|\nabla u|}(\nabla\phi)^T H_\phi\nabla\phi\right.$$
$$\left.-\frac{2}{\psi'|\nabla u|}(\nabla d)^T H_\phi\nabla\phi)-\frac{1}{|\nabla u|}((\nabla d)^T H_\phi\nabla d\right\}$$
$$+l''\left(\frac{1}{\psi'}\nabla\phi^T H_d\nabla\phi+\frac{1}{(\psi')^2}(\nabla\phi)^T H_\phi\nabla\phi+\frac{2}{\psi'}(\nabla d)^T H_\phi\nabla\phi+(\nabla d)^T H_\phi\nabla d\right)$$
$$=\psi''l''\left\{1+\frac{1}{(\psi')^2}(\langle\nabla\phi,\nabla d\rangle^2+2\psi'\langle\nabla\phi,\nabla d\rangle)\right\}+l'A+l''B,$$

where

$$A=\frac{|\nabla u|\Delta d}{\psi'}+\frac{|\nabla u|\Delta\phi}{(\psi')^2}-\frac{1}{\psi'|\nabla u|}(\nabla\phi)^T H_d\nabla\phi-\frac{1}{(\psi')^2|\nabla u|}(\nabla\phi)^T H_\phi\nabla\phi$$
$$-\frac{2}{\psi'|\nabla u|}(\nabla d)^T H_\phi\nabla\phi)-\frac{1}{|\nabla u|}((\nabla d)^T H_\phi\nabla d$$

and

$$B=\frac{1}{\psi'}\nabla\phi^T H_d\nabla\phi+\frac{1}{(\psi')^2}(\nabla\phi)^T H_\phi\nabla\phi+\frac{2}{\psi'}(\nabla d)^T H_\phi\nabla\phi+(\nabla d)^T H_\phi\nabla d.$$

b) Let P be such that $\psi'\geq P$ implies that $\frac{1}{2}\leq 1+\frac{1}{(\psi')^2}(\langle\nabla\phi,\nabla d\rangle^2+2\psi'\langle\nabla\phi,\nabla d\rangle)$ so that (since $l''>0$ and $\psi''<0$) to have the r.h.s. of (5) non positive it is enough to have $\psi'\geq P$ and

$$\frac{1}{2}\psi''l''+l'A+l''B\leq 0.$$

Consider A. The assumption that the boundary of Ω is C^2 implies that there exists $\delta^*>0$ such that, on Ω_{δ^*}, H_d and Δd exist and, for some h_Δ, we have $|\Delta d|\leq h_\Delta$ and $|H_d|\leq h_\Delta$. Hence, by (4) and since, for some h_ϕ, we also have $|H_\phi|\leq h_\phi$ and $|\nabla\phi|\leq h_\phi$, it follows that, for ψ' large, $A\leq 2h_\Delta$.

Consider B; we have $|(\nabla d)^T H_\phi\nabla d|\leq h_\phi$; hence, possibly by increasing P, we can assume that $\psi'\geq P$ implies that $B\leq 2h_\phi$ and, to have (5), it is enough to have

$$\psi'\geq P \text{ and } \frac{1}{2}\psi''l''(|\nabla u|)+l'(|\nabla u|)2h_\Delta+l''(|\nabla u|)2h_\phi\leq 0.$$

c) We have to show that $\delta>0$ can be found so that $v(\delta)\geq\tilde{P}$ and such that the condition

$$u(x)=\psi(d(x))+\phi(x)\geq\Phi$$

be satisfied for x such that $d(x)=\delta$. Since $\phi(x)\geq-\Phi$, it will be enough to require that $u(x)\geq 2\Phi$ for all x such that $d(x)=\delta$.

We wish to prove the validity of the following statement: for any P, there exist $\delta>0$ and a function ψ such that,

$$\psi(0)=0;\psi(\delta)\geq 2\Phi \text{ and, for } d\in[0,\delta],\ \psi'(d)\geq P$$

and such that, for $u(x):=\psi(d(x))+\phi(x)$, we have

$$\psi''(d(x))l''(|\nabla u(x)|)+l'(|\nabla u(x)|)4h_\Delta+l''(d(x))(|\nabla u(x)|)4h_\phi\leq 0$$

for any $x\in\Omega_\delta$. If this statement is proved, in particular, the function $u=\psi\circ d+\phi$ on Ω_δ will be such that

$$\operatorname{div}\nabla L(\nabla u(x))\leq 0. \tag{6}$$

To prove the statement, notice that, by Proposition 2, there exists τ and $\tilde{\lambda}$ such that $t \geq \tau$ implies $l''(t) \leq \tilde{\lambda} l'(t)$ so that, for $|\nabla u| \geq \tau$, we have $4h_\phi l''(|\nabla u|) \leq \lambda 4h_\Delta l'(|\nabla u|)$.

Hence, in order to prove the validity of the statement in c), possibly by increasing P to $\tilde{P}$ if required, it is enough to prove that: for any $\tilde{P}$, there exist $\delta > 0$ and a solution ψ to the problem

$$\psi(0) = 0; \psi(\delta) \geq 2\Phi \text{ and, on the interval } [0, \delta], \psi'(d) \geq \tilde{P},$$

such that, for $u = \psi \circ d + \phi$,

$$\psi''(d(x)) l''(|\nabla u(x)|) + C l'(|\nabla u(x)|) \leq 0,$$

where he have set $C = \max\{4h_\phi, \lambda 4h_\Delta\}$.

d) Consider a solution $\psi'(d)$ to the differential equation

$$\psi''(d) l''(2\psi'(d)) + C l'(2\psi'(d)) = 0 \tag{7}$$

such that $\psi' > 0$; then, since $l''(2\psi')$ and $l'(2\psi')$ are positive, ψ'' is negative. For any $|\nabla u(x)|$ such that $|\nabla u(x)| \leq 2\psi'(d(x))$, setting $\psi'(d) = v(d)$ and recalling that l'' is non-increasing while l' is non-decreasing, we have the following inequalities

$$(-v'(d(x)) l''(|\nabla u(x)|) \geq (-v'(d(x)) l''(2v(d(x)) = C l'(2v(d(x))) \geq C l'(|\nabla u(x)|)$$

and hence, the condition on $v = \psi'$ needed to give (6) simplifies to

$$v'(d(x) l''(|\nabla u(x)|) + C l'(|\nabla u(x)|) \leq 0. \tag{8}$$

Moreover, since $\psi''(d) \leq 0$, then ψ' is non-increasing and, in order to have $\psi'(d) \geq \tilde{P}$ on the interval $[0, \delta]$, it is enough to have $\psi'(\delta) \geq \tilde{P}$.

From (7) we have

$$\frac{l''(2v(d))}{l'(2v(d))} 2v'(d) = -2C$$

so that

$$l'(2v(d)) = l'(2v(0)) e^{-2Cd}.$$

Consider $\tilde{v}$, a solution to

$$l'(2\tilde{v}(d)) = l'(2v(0))(1 - 2Cd), \tag{9}$$

so that, in particular, $\tilde{v}(0) = v(0)$; since $e^{-2Cd} \geq 1 - 2Cd$, we have $l'(2\tilde{v}(d)) \leq l'(2v(d))$ and, from the monotonicity of l', we infer that

$$\tilde{v}(d) \leq v(d). \tag{10}$$

To show that $\delta > 0$ can be found so that $v(\delta) \geq \tilde{P}$ and such that the condition

$$u(x) = \psi(d(x)) + \phi(x) \geq \Phi$$

be satisfied for x such that $d(x) = \delta$ it is enough to require that

$$\int_0^\delta \psi'(t) dt = \int_0^\delta v(t) dt \geq 2\Phi.$$

Then, by (10), it will be enough to have

$$\int_0^\delta \tilde{v}(t) dt \geq 2\Phi \text{ and } \tilde{v}(\delta) \geq \tilde{P}. \tag{11}$$

e) Solve (9) setting

$$\tilde{v}(t) = \frac{1}{2} (l^*)'(l'(2v(0))(1 - 2Ct)); \tag{12}$$

the change of variables $\lambda = l'(2v(0))(1-2Ct)$ yields that

$$\int_0^\delta \tilde{v}(t)dt = -\frac{1}{l'(2v(0))2C}\int_{l'(2v(0))}^{l'(2v(0))(1-2C\delta)} \frac{1}{2}(l^*)'(\lambda)d\lambda. \tag{13}$$

$$= \frac{1}{l'(2v(0))4C} l^*(\lambda)\Big|_{l'(2v(0))(1-2C\delta)}^{l'(2v(0))}$$

$$= \frac{1}{l'(2v(0))4C}[l^*(l'(2v(0)) - l^*(l'(2v(0))(1-2C\delta))].$$

From Proposition 1 we infer that $l^*(p)$ is strictly increasing for $0 < p < p^*$, and hence invertible; then, $(l^*)^{-1}(y)$ exists for $0 < y < \infty$, it is strictly increasing and $\lim_{y\to\infty}(l^*)^{-1}(y) = p^*$.

It is convenient to set $p^0 = l'(2v(0))$: we have $p^0 \in \mathrm{Dom} l^*$; set also

$$\delta(p^0) = \frac{p^0 - (l^*)^{-1}(\frac{1}{2}l^*(p^0))}{2Cp^0} \tag{14}$$

so that $l^*(p^0(1-2C\delta)) = \frac{1}{2}l^*(p^0)$. Since $(l^*)^{-1}$ is strictly increasing, we have $(l^*)^{-1}(\frac{1}{2}l^*(p^0)) < (l^*)^{-1}(l^*(p^0)) = p^0$ and hence

$$1 - 2C\delta(p^0) = \frac{(l^*)^{-1}(\frac{1}{2}l^*(p^0))}{p^0} < 1$$

so that $\delta(p^0) > 0$.

Then, (13) becomes

$$\int_0^{\delta(p^0)} \tilde{v}(t)dt = \frac{1}{p^0 4C}[l^*(p^0) - \frac{1}{2}l^*(p^0)] = \frac{1}{p^0 8C}l^*(p^0).$$

We claim that we can choose p^0 sufficiently close to p^* so as to have at once $\frac{1}{p^0 8C}l^*(p^0) \geq 2\Phi$ and $\tilde{v}(\delta(p^0)) \geq \tilde{P}$.

In fact, by (12) we have

$$\tilde{v}(\delta) = \frac{1}{2}(l^*)'(l'(2v(0))(1-2C\delta)) = \frac{1}{2}(l^*)'(p^0(1-2C\delta)) = \frac{1}{2}(l^*)'((l^*)^{-1}(\frac{1}{2}l^*(p^0)));$$

by Proposition 1, $\frac{1}{2}l^*(p^0) \to \infty$ as $p^0 \to p^*$, so that we have $(l^*)^{-1}(\frac{1}{2}l^*(p^0)) \to p^*$ and hence, again by Proposition 1, $\frac{1}{2}(l^*)'((l^*)^{-1}(\frac{1}{2}l^*(p^0))) \to \infty$. Moreover, since, by Proposition 1, we have $\lim_{p^0\to p^*}(l^*)'(p^0) = +\infty$, we obtain $\lim_{p^0\to p^*}\frac{1}{p^0}l^*(p^0) = +\infty$. This proves the Claim.

g) Having thus defined p^0, and hence $v(0)$ and $\delta(p^0)$, set $\tilde{\psi}$ to be the solution to the differential equation (7) such that $\tilde{\psi}(0) = 0$ and $\tilde{\psi}'(0) = v(0)$, and define the function u^+ setting

$$u^+(x) = \tilde{\psi}(d(x)) + \phi(x);$$

the results in d) and e) above imply that, for $x \in \partial\Omega_{\delta(p^0)}$ and u any possible solution to problem (1), we have

$$u^+(x) \geq u(x)$$

while, for $x \in \Omega_\delta$,

$$\mathrm{div}\nabla L(\nabla u^+(x)) \leq 0.$$

Hence, u^+ is the sought super-solution. The analogous construction for u^- proves the sufficiency part.

□

Proof of necessity. Let L be in $\mathbb{L}$ but not satisfying Assumption 1; we shall define $\Omega \in \mathbb{O}$ and $\phi \in \mathbb{F}$ such that the corresponding Problem (1) will admit no solution in $W^{1,1}(\Omega)$.

Set $\Omega = \{x \in \mathbb{R}^n : \frac{1}{2} < |x| < 1\}$ and $\phi(x) = \beta(|x| - \frac{1}{2})$, with $\beta \geq \frac{1}{2}$.

a) We shall first show that β large will imply that problem (1) admits no radial solution. Since L does not satisfy Assumption 1, the domain of l^* cannot coincide with $\mathbb{R}$, and hence it is a compact interval $[-p^*, p^*]$. Moreover, since $l^*(p^*)$ is bounded, there exists M such that $p \in [-p^*, p^*]$ implies $l^*(p) \leq M$.

Assume that we have a radial solution $\tilde{u}$; then, the Euler-Lagrange equation implies that

$$\frac{d}{dr}\left(l'(\tilde{u}'(r))r^{n-1}\right) = l''(\tilde{u}'(r))\tilde{u}''(r)r^{n-1} + (n-1)l'\tilde{(}u'(r))r^{n-2} = 0$$

i.e., that

$$l'(\tilde{u}'(r)) = l'(\tilde{u}'(\frac{1}{2}))(\frac{1}{2r})^{n-1}. \tag{15}$$

Since $\tilde{u}(1) - \tilde{u}(\frac{1}{2}) = \beta$, for some $r \in (\frac{1}{2}, 1)$ we must have $\tilde{u}'(r) \geq 2\beta$; since the right hand side of (15) is non-increasing, $l'(\tilde{u}'(\frac{1}{2}) \geq l'(2\beta) \geq l'(1)$.

Solve (15) as $\tilde{u}'(r)) = (l^*)'(l'(\tilde{u}'(\frac{1}{2}))(\frac{1}{2r})^{n-1})$. Since $r \to (\frac{1}{2r})^{n-1}$ is convex, on the interval $\frac{1}{2} \leq r \leq 1$ we have $(\frac{1}{2r})^{n-1} \leq 2\left((\frac{1}{2})^{n-1} - 1\right)(r - \frac{1}{2}) + 1$ and, since $(l^*)'$ is non-decreasing, $(l^*)'(l'(\tilde{u}'(\frac{1}{2}))(\frac{1}{2r})^{n-1}) \leq (l^*)'(l'(\tilde{u}'(\frac{1}{2}))(2\left((\frac{1}{2})^{n-1} - 1\right)(r - \frac{1}{2}) + 1))$. Hence

$$\int_{\frac{1}{2}}^{1} \tilde{u}'(r)dr = \int_{\frac{1}{2}}^{1} (l^*)'(l'(\tilde{u}'(\frac{1}{2}))(\frac{1}{2r})^{n-1})dr$$
$$\leq \int_{\frac{1}{2}}^{1} (l^*)'(l'(\tilde{u}'(\frac{1}{2}))(2\left((\frac{1}{2})^{n-1} - 1\right)(r - \frac{1}{2}) + 1))dr$$
$$= \frac{1}{l'(\tilde{u}'(\frac{1}{2}))2\left((\frac{1}{2})^{n-1} - 1\right)} \int_{y(\frac{1}{2})}^{y(1)} (l^*)'(y)dy$$
$$= \frac{1}{l'(\tilde{u}'(\frac{1}{2}))2\left((\frac{1}{2})^{n-1} - 1\right)}[l^*(y(1)) - l^*(y(\frac{1}{2}))]$$
$$\leq \frac{1}{l'(1)2\left((\frac{1}{2})^{n-1} - 1\right)}[l^*((\frac{1}{2})^{n-1}l'(\tilde{u}'(\frac{1}{2}))) - l^*(l'(\tilde{u}'(\frac{1}{2}))]$$
$$\leq 2\frac{1}{l'(1)2\left((\frac{1}{2})^{n-1} - 1\right)}M.$$

Then, whenever β is larger than $2\frac{1}{l'(1)2\left((\frac{1}{2})^{n-1}-1\right)}M$, problem (1) admits no radial solution.

b) Let $u \in W^{1,1}(\Omega)$ be a (non-radial) solution to problem (1) with Ω and ϕ defined above and β larger than $2\frac{1}{l'(1)2\left((\frac{1}{2})^{n-1}-1\right)}M$; set $\tilde{u}$ be defined as in the statement of Proposition 3. Then, $\tilde{u}$ satisfies the same (radial) boundary conditions as u. We wish to show that

$$\int_\Omega l(|\nabla\tilde{u}(x)|)dx \leq \int_\Omega l(|\nabla u(x)|)dx.$$

From Proposition 3, we infer that

$$|\nabla\tilde{u}(x)| \leq \frac{1}{n\omega_n}\int_{|\omega|=1} |\nabla u(\omega|x|)|d\omega$$

and, since the map $t \to l(t)$ is increasing for $t \geq 0$, we obtain that

$$\int_\Omega l(|\nabla \tilde{u}(x)|)dx \leq \int_\Omega l(\frac{1}{n\omega_n}\int_{|\omega|=1} |\nabla u(\omega|x|)|d\omega)dx.$$

Jensen's inequality gives

$$l(|\nabla \tilde{u}(x)|) \leq \frac{1}{n\omega_n}\int_{|\omega|=1} l(|\nabla u(\omega|x|)|)d\omega$$

and hence

$$\int_\Omega l(|\nabla \tilde{u}(x)|)dx \leq \int_\Omega \frac{1}{n\omega_n}\int_{|\omega|=1} l(|\nabla u(\omega|x|)|)d\omega\, dx.$$

Being the last integrand a radial function, passing to spherical coordinates we obtain

$$\int_\Omega l(|\nabla \tilde{u}(x)|)dx \leq n\omega_n \int_{\frac{1}{2}}^1 \frac{1}{n\omega_n}\int_{|\omega|=1} l(|\nabla u(\omega r)|)d\omega\, r^{n-1}dr = \int_\Omega l(|\nabla u(x)|)dx$$

so that $\tilde{u}$ is a radial solution to Problem (1). This contradicts point a). □

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