PERTURBATION RESULTS OF CRITICAL ELLIPTIC EQUATIONS OF CAFFARELLI-KOHN-NIRENBERG TYPE

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Abstract. We find for small $\varepsilon$ positive solutions to the equation
\[-\text{div} \left( |x|^{-2a} \nabla u \right) - \frac{\lambda}{|x|^{2(1+a)}} u = \left( 1 + \varepsilon k(x) \right) \frac{u^{p-1}}{|x|^p} \]
in $\mathbb{R}^N$, which branch off from the manifold of minimizers in the class of radial functions of the corresponding Caffarelli-Kohn-Nirenberg type inequality. Moreover, our analysis highlights the symmetry-breaking phenomenon in these inequalities, namely the existence of non-radial minimizers.

1. Introduction

We will consider the following elliptic equation in $\mathbb{R}^N$ in dimension $N \geq 3$
\[-\text{div} \left( |x|^{-2a} \nabla u \right) - \frac{\lambda}{|x|^{2(1+a)}} u = K(x) \frac{u^{p-1}}{|x|^p}, \quad x \in \mathbb{R}^N \tag{1.1} \]
where
\[-\infty < a < \frac{N - 2}{2}, \quad -\infty < \lambda < \left( \frac{N - 2a - 2}{2} \right)^2 \tag{1.2} \]
\[p = p(a, b) = \frac{2N}{N - 2(1 + a - b)} \quad \text{and} \quad a \leq b < a + 1. \]
For $\lambda = 0$ equation (1.1) is related to a family of inequalities given by Caffarelli, Kohn and Nirenberg [6],
\[\|u\|_{p,b}^2 := \left( \int_{\mathbb{R}^N} |x|^{-2b} |u|^p \, dx \right)^{2/p} \leq C_{a,b} \int_{\mathbb{R}^N} |x|^{-2a} |\nabla u|^2 \, dx \quad \forall u \in C_0^\infty(\mathbb{R}^N). \tag{1.3} \]
For sharp constants and extremal functions we refer to Catrina and Wang [7].
The natural functional space to study (1.1) is $D_a^{1,2}(\mathbb{R}^N)$ defined as the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm
\[\|\nabla u\|_a := \|u\|_* = \left[ \int_{\mathbb{R}^N} |x|^{-2a} |\nabla u|^2 \, dx \right]^{1/2}. \]
We will mainly deal with the perturbative case $K(x) = 1 + \varepsilon k(x)$, namely with the problem
\[\begin{cases} -\text{div} \left( |x|^{-2a} \nabla u \right) - \frac{\lambda}{|x|^{2(1+a)}} u = \left( 1 + \varepsilon k(x) \right) \frac{u^{p-1}}{|x|^p} \\ u \in D_a^{1,2}(\mathbb{R}^N), \quad u > 0 \text{ in } \mathbb{R}^N \setminus \{0\}. \end{cases} \tag{Pa,b,\lambda} \]
Concerning the perturbation $k$, we assume
\begin{equation}
    k \in L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N).
\end{equation}

Our approach is based on an abstract perturbative variational method discussed by Ambrosetti and Badiale [2], which splits our procedure in three main steps. First we consider the unperturbed problem, i.e. $\varepsilon = 0$, and find a one dimensional manifold of radial solutions. If this manifold is non-degenerate (see Theorem 1.1 below) a one dimensional reduction of the perturbed variational problem in $D^{1,2}_a(\mathbb{R}^N)$ is possible. Finally we have to find a critical point of a functional defined on the real line.

Solutions of $(P_{a,b,\lambda})$ are critical points in $D^{1,2}_a(\mathbb{R}^N)$ of
\begin{equation}
    f_\varepsilon(u) := \frac{1}{2} \int_{\mathbb{R}^N} |x|^{-2a} |\nabla u|^2 \, dx - \frac{\lambda}{2} \int_{\mathbb{R}^N} \frac{u^2}{|x|^{2(1+a)}} \, dx - \frac{1}{p} \int_{\mathbb{R}^N} \left(1 + \varepsilon k(x)\right) \frac{u^p}{|x|^b} \, dx,
\end{equation}
where $u_+ := \max\{u, 0\}$. For $\varepsilon = 0$ we show that $f_0$ has a one dimensional manifold of critical points
\begin{equation}
    Z_{a,b,\lambda} := \left\{ z_{a,b,\lambda}^{\mu} := \mu \frac{N-2-2a}{2} \frac{x}{\mu} \mid \mu > 0 \right\},
\end{equation}
where $z_{a,b,\lambda}^{\mu}$ is explicitly given in (2.5) below. These radial solutions were computed for $\lambda = 0$ in [7], the case $a = b = 0$ and $-\infty < \lambda < (N-2)^2/4$ was done by Terracini [12]. The exact knowledge of the critical manifold enables us to clarify the question of non-degeneracy.

**Theorem 1.1.** Suppose $a, b, \lambda, p$ satisfy (1.2). Then the critical manifold $Z_{a,b,\lambda}$ is non-degenerate, i.e.
\begin{equation}
    T_z Z_{a,b,\lambda} = \ker D^2 f_0(z) \quad \forall z \in Z_{a,b,\lambda},
\end{equation}
if and only if
\begin{equation}
    b \neq h_j(a, \lambda) := \frac{N}{2} \left[1 + \frac{4j(N+j-1)}{(N-2-2a)^2 - 4\lambda}\right]^{-1/2} - \frac{N-2-2a}{2} \quad \forall j \in \mathbb{N} \setminus \{0\}.
\end{equation}
the symmetry breaking phenomenon of the unperturbed problem observed in [7], i.e. the existence of non-radial minimizers of

$$C_{a,b} := \inf_{u \in D^1_2(\mathbb{R}^N) \setminus \{0\}} \int |x|^{-2a} |\nabla u|^2 = \inf_{u \in D^1_2(\mathbb{R}^N) \setminus \{0\}} \frac{\|\nabla u\|_a^2}{\|u\|_{p,b}^2}. \quad (1.7)$$

In fact we improve [7, Thm 1.3], where it is shown that there is an open subset $H \subset \mathbb{R}^2$ containing $\{(a,a) | a < 0\}$, a real number $a_0 \leq 0$ and a function $h : (-\infty, a_0] \to \mathbb{R}$ satisfying $h(a_0) = a_0$ and $a < h(a) < a + 1$ for all $a < a_0$, such that for every $(a,b) \in H \cup \{(a,b) \in \mathbb{R}^2 | a < a_0, a < b < h(a)\}$ the minimizer in (1.7) is non-radial (see figure 2 below). We show that one may choose $a_0 = 0$ and $h = h_1(\cdot, 0)$ and obtain, as a consequence of Theorem 1.1 for $\lambda = 0$,

**Corollary 1.2.** Suppose $a, b, p$ satisfy (1.2). If $b < h_1(a, 0)$, then $C_{a,b}$ in (1.7) is attained by a non-radially symmetric function.

Concerning step two, the one-dimensional reduction, we follow closely the abstract scheme in [2] and construct a manifold $Z_{a,b,\lambda}^{\varepsilon} = \{z_{a,b,\lambda}^{\varepsilon} + w(\varepsilon, \mu) | \mu > 0\}$, such that any critical point of $f_{\varepsilon}$ restricted to $Z_{a,b,\lambda}^{\varepsilon}$ is a solution to $(P_{a,b,\lambda})$. We emphasize that in contrast to the local approach in [2] we construct a manifold which is globally diffeomorphic to the unperturbed one such that we may estimate the difference $\|w(\varepsilon, \mu)\|$ when $\mu \to \infty$ or $\mu \to 0$ (see also [4, 5]). More precisely we show under assumption (1.8) below that $\|w(\varepsilon, \mu)\|$ vanishes as $\mu \to \infty$ or $\mu \to 0$.

We will prove the following existence results.

**Theorem 1.3.** Suppose $a, b, p, \lambda$ satisfy (1.2), (1.4) and (1.6) holds. Then problem $(P_{a,b,\lambda})$ has a solution for all $|\varepsilon|$ sufficiently small if

$$k(\infty) := \lim_{|x| \to \infty} k(x) \text{ exists and } k(\infty) = k(0) = 0. \quad (1.8)$$

**Theorem 1.4.** Assume (1.2), (1.4), (1.6) and

$$k \in C^2(\mathbb{R}^N), \ |\nabla k| \in L^\infty(\mathbb{R}^N) \text{ and } |D^2k| \in L^\infty(\mathbb{R}^N). \quad (1.9)$$

Then $(P_{a,b,\lambda})$ is solvable for all small $|\varepsilon|$ under each of the following conditions

$$\limsup_{|x| \to \infty} k(x) \leq k(0) \text{ and } \Delta k(0) > 0, \quad (1.10)$$

$$\liminf_{|x| \to \infty} k(x) \geq k(0) \text{ and } \Delta k(0) < 0. \quad (1.11)$$
Remark 1.5. Our analysis of the unperturbed problem allows to consider more general perturbation, for instance it is possible to treat equations like

\[
\begin{aligned}
&-\text{div}(|x|^{2a}\nabla u) - \frac{\lambda + \varepsilon_1 V(x)}{|x|^{2(1+a)}} u = (1 + \varepsilon_2 k(x))^\frac{p-1}{p} |x|^{\frac{1}{p}} \\
&u \in D_{a,\lambda}^{1,2}(\mathbb{R}^N), \quad u > 0 \text{ in } \mathbb{R}^N \setminus \{0\}.
\end{aligned}
\]

Existence results in this direction are given by Abdellaoui and Peral [1], where the case \(a = 0\) and \(b = 0\) and \((N-2)^2/4N < \lambda < (N-2)^2/4\) is studied. We generalize some existence results obtained there to arbitrary \(a, b\) and \(\lambda\) satisfying (1.2) and (1.6).

Problem (1.1), the non-perturbative version of \((P_{a,b,\lambda})\), was studied by Smets [11] in the case \(a = b = 0\) and \(0 < \lambda < (N - 2)^2/4\). A variational minimax method combined with a careful analysis and construction of Palais-Smale sequences shows that in dimension \(N = 4\) equation (1.1) has a positive solution \(u \in D_{a,\lambda}^{1,2}(\mathbb{R}^N)\) if \(K \in C^2\) is positive and satisfies an analogous condition to (1.8), namely \(K(0) = \lim_{|x| \to \infty} K(x)\). In our perturbative approach we need not to impose any condition on the space dimension \(N\). Theorem 1.3 gives the perspective to relax the restriction \(N = 4\) on the space dimension also in the nonperturbative case.

Acknowledgements

The authors would like to thank Prof. A. Ambrosetti for his interest in their work and for helpful suggestions.

Preliminaries

Catrina and Wang [7] proved that for \(b = a + 1\)

\[
C_{a+1}^{-1} = S_{a,a+1} = \inf_{D_{a,\lambda}^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |x|^{-2a} |\nabla u|^2 \, dx}{\left( \frac{\int_{\mathbb{R}^N} |x|^{-2(1+a)} |u|^2 \, dx}{\|u\|_\infty^2} \right)^{1/2}} = \left( \frac{N - 2 - 2a}{2} \right)^2.
\]

Hence we obtain for \(-\infty < \lambda < \left( \frac{N - 2 - 2a}{2} \right)^2\) a norm, equivalent to \(\| \cdot \|_\ast\), given by

\[
\|u\| = \left( \int_{\mathbb{R}^N} |x|^{-2a} |\nabla u|^2 \, dx - \lambda \int_{\mathbb{R}^N} \frac{u^2}{|x|^{2(1+a)}} \, dx \right)^{1/2}.
\]

We denote by \(D_{a,\lambda}^{1,2}(\mathbb{R}^N)\) the Hilbert space equipped with the scalar product induced by \(\| \cdot \|\)

\[
(u, v) = \int_{\mathbb{R}^N} |x|^{-2a} \nabla u \cdot \nabla v \, dx - \lambda \int_{\mathbb{R}^N} \frac{u v}{|x|^{2(1+a)}} \, dx.
\]

We will mainly work in this space. Moreover, we define by \(\mathcal{C}\) the cylinder \(\mathbb{R} \times S^{N-1}\). It is shown in [7, Prop. 2.2] that the transformation

\[
u(x) = |x|^{-\frac{N-2-2a}{2}} v \left( - \ln |x|, \frac{x}{|x|} \right)
\]

induces a Hilbert space isomorphism from \(D_{a,\lambda}^{1,2}(\mathbb{R}^N)\) to \(H_{\lambda}^{1,2}(\mathcal{C})\), where the scalar product in \(H_{\lambda}^{1,2}(\mathcal{C})\) is defined by

\[
(v_1, v_2)_{H_{\lambda}^{1,2}(\mathcal{C})} := \int_{\mathcal{C}} \nabla v_1 \cdot \nabla v_2 + \left( \frac{N - 2 - 2a}{2} \right)^2 - \lambda \right) v_1 v_2.
\]
Using the canonical identification of the Hilbert space $D^{1,2}_{a,\lambda}(\mathbb{R}^N)$ with its dual induced by the scalar product and denoted by $\mathcal{K}$, i.e.

$$\mathcal{K} : (D^{1,2}_{a,\lambda}(\mathbb{R}^N))^\prime \rightarrow D^{1,2}_{a,\lambda}(\mathbb{R}^N), \quad (\mathcal{K}(\varphi), u) = \varphi(u) \quad \forall (\varphi, u) \in (D^{1,2}_{a,\lambda}(\mathbb{R}^N))^\prime \times D^{1,2}_{a,\lambda}(\mathbb{R}^N),$$

we shall consider $f'_e(u)$ as an element of $D^{1,2}_{a,\lambda}(\mathbb{R}^N)$ and $f''_e(u)$ as one of $\mathcal{L}(D^{1,2}_{a,\lambda}(\mathbb{R}^N))$. If we test $f'_e(u)$ with $-\varphi = \max\{-u, 0\}$ we get

$$(f'_e(u), u_-) = \int_{\mathbb{R}^N} |x|^{-2a} \nabla u \cdot \nabla u_- - \lambda \int_{\mathbb{R}^N} \frac{u u_-}{|x|^{2(1+a)}} - \int_{\mathbb{R}^N} (1 + \varepsilon k(x)) \frac{u_-^{p-1} - u_-}{|x|^{bp}} = -\|u_-\|^2$$

and see that any critical point of $f_e$ is nonnegative. The maximum principle applied in $\mathbb{R}^N \setminus \{0\}$ shows that any nontrivial critical point is positive in that region. We cannot expect more since the radial solutions to the unperturbed problem ($\varepsilon = 0$) vanish at the origin if $\lambda < 0$ (see (2.5) below). Moreover from standard elliptic regularity theory, solutions to $(P_{a,b,\lambda})$ are $C^{1,\alpha}(\mathbb{R}^N \setminus \{0\}), \alpha > 0$.

The unperturbed functional $f_0$ is given by

$$f_0(u) := \frac{1}{2} \int_{\mathbb{R}^N} |x|^{-2a} |\nabla u|^2 \, dx - \frac{\lambda}{2} \int_{\mathbb{R}^N} \frac{u^2}{|x|^{2(1+a)}} \, dx - \frac{1}{p} \int_{\mathbb{R}^N} \frac{u^p}{|x|^{bp}} \, dx, \quad u \in D^{1,2}_{a,\lambda}(\mathbb{R}^N)$$

and we may write $f_\varepsilon(u) = f_0(u) + \varepsilon G(u)$, where

$$G(u) := \frac{1}{p} \int_{\mathbb{R}^N} k(x) \frac{u^p}{|x|^{bp}}. \quad \text{(1.14)}$$

2. The unperturbed problem

Critical points of the unperturbed functional $f_0$ solve the equation

$$\begin{cases} -\text{div} (|x|^{-2a} \nabla u) - \frac{\lambda}{|x|^{2(1+a)}} u = \frac{1}{|x|^{bp}} u^{p-1} \\ u \in D^{1,2}_{a,\lambda}(\mathbb{R}^N), \quad u > 0 \text{ in } \mathbb{R}^N \setminus \{0\}. \end{cases} \quad (2.1)$$

To find all radially symmetric solutions $u$ of (2.1), i.e. $u(x) = u(r)$, where $r = |x|$, we follow [7] and note that if $u$ is radial, then equation (2.1) can be written as

$$-\frac{u''}{r^{2a}} - \frac{N - 2a - 1}{r^{2a+1}} u' - \frac{\lambda}{r^{2(a+1)}} u = \frac{1}{r^{bp}} u^{p-1}. \quad (2.2)$$

Making now the change of variable

$$u(r) = r^{\frac{-N - 2a}{2}} \varphi(\ln r), \quad \text{(2.3)}$$

we come to the equation

$$-\varphi'' + \left( \frac{N - 2a}{2} \right)^2 - \lambda \varphi - \varphi^{p-1} = 0. \quad \text{(2.4)}$$

All positive solutions of (2.4) in $H^{1,2}(\mathbb{R})$ are the translates of

$$\varphi_1(t) = \left[ \frac{N(N - 2 - 2a)\sqrt{(N - 2 - 2a)^2 - 4\lambda}}{4(N - 2(1 + a - b))} \right]^{\frac{N - 2(1 + a - b)}{4(1 + a - b)}} \cdot \left( \cosh \left( \frac{(1 + a - b)\sqrt{(N - 2 - 2a)^2 - 4\lambda}}{N - 2(1 + a - b)} \right) \right)^{-\frac{N - 2(1 + a - b)}{2(1 + a - b)}} t, \quad \text{(2.5)}$$
namely $\varphi_{i}(t) = \varphi_{1}(t - \ln \mu)$ for some $\mu > 0$ (see [7]). Consequently all radial solutions of (2.1) are dilations of
\[
z_{a,b,\lambda}^{1}(x) = \left[\frac{N(N - 2 - 2a)\sqrt{(N - 2 - 2a)^{2} - 4\lambda}}{N - 2(1 + a - b)}\right]^{\frac{N - 2(1 + a - b)}{2(1 + a - b)}} \cdot \left[\frac{1}{|x|} \left(1 - \frac{\sqrt{(N - 2 - 2a)^{2} - 4\lambda}}{N - 2(1 + a - b)} \right)^{\frac{|N - 2(1 + a - b)|}{(N - 2)(1 + a - b)}} \left[1 + \frac{|x|^{2(1 + a - b)}(N - 2 - 2a)^{2} - 4\lambda}{(N - 2)(1 + a - b)} \right]^{\frac{-N - 2(1 + a - b)}{2(1 + a - b)}} \right]^{\frac{-N - 2(1 + a - b)}{2(1 + a - b)}}.
\]
and given by
\[
z_{a,b,\lambda}^{\mu}(x) = \mu^{-\frac{N - 2 - 2a}{2}} z_{a,b,\lambda}^{1}(\mu), \quad \mu > 0.
\]
Using the change of coordinates in (2.3), respectively (1.13), and the exponential decay of $z_{a,b,\lambda}^{\mu}$ in these coordinates it is easy to see that the map $\mu \mapsto z_{a,b,\lambda}^{\mu}$ is at least twice continuously differentiable from $(0, \infty)$ to $D^{1,2}_{a,b,\lambda}(\mathbb{R}^{N})$ and we obtain

**Lemma 2.1.** Suppose $a, b, \lambda, p$ satisfy (1.2). Then the unperturbed functional $f_{0}$ has a one dimensional $C^{2}$-manifold of critical points $Z_{a,b,\lambda}$ given by \(\{z_{a,b,\lambda}^{\mu} \mid \mu > 0\}\). Moreover, $Z_{a,b,\lambda}$ is exactly the set of all radially symmetric, positive solutions of (2.1) in $D^{1,2}_{a,b,\lambda}(\mathbb{R}^{N})$.

In order to apply the abstract perturbation method we need to show that the manifold $Z_{a,b,\lambda}$ satisfy a non-degeneracy condition. This is the content of Theorem 1.1.

**Proof of Theorem 1.1.** The inclusion $T_{a,b,\lambda} Z_{a,b,\lambda} \subseteq \ker D^{2}f_{0}(z_{a,b,\lambda}^{\mu})$ always holds and is a consequence of the fact that $Z_{a,b,\lambda}$ is a manifold of critical points of $f_{0}$. Consequently, we have only to show that $\ker D^{2}f_{0}(z_{a,b,\lambda}^{\mu})$ is one dimensional. Fix $u \in \ker D^{2}f_{0}(z_{a,b,\lambda}^{\mu})$. The function $u$ is a solution of the linearized problem
\[
-\text{div}(|x|^{-2a}\nabla u) - \frac{\lambda}{|x|^{2(a+1)}} u = \frac{p - 1}{|x|^{bp}} (z_{a,b,\lambda}^{\mu})^{p-2} u.
\]
We expand $u$ in spherical harmonics
\[
u(r) = \sum_{i=0}^{\infty} \bar{v}_{i}(r) \bar{Y}_{i}(\vartheta), \quad r \in \mathbb{R}^{+}, \quad \vartheta \in S^{N-1},
\]
where $\bar{v}_{i}(r) = \int_{S^{N-1}} u(r\vartheta) \bar{Y}_{i}(\vartheta) \, d\vartheta$ and $\bar{Y}_{i}$ denotes the orthogonal $i$-th spherical harmonic jet satisfying for all $i \in \mathbb{N}_{0}$
\[
-\Delta_{S^{N-1}} \bar{Y}_{i} = i(N + i - 2) \bar{Y}_{i}.
\]
Since $u$ solves (2.6) the functions $\bar{v}_{i}$ satisfy for all $i \geq 0$
\[
-\frac{\bar{v}_{i}''}{r^{2a}} \bar{Y}_{i} - \frac{N - 1 - 2a}{r^{2a+1}} \bar{v}_{i}' \bar{Y}_{i} - \frac{\bar{v}_{i}}{r^{2(a+1)}} \Delta_{\mathbb{S}} \bar{Y}_{i} - \frac{\lambda}{r^{2(a+1)}} \bar{v}_{i} \bar{Y}_{i} = \frac{p - 1}{r^{bp}} (z_{a,b,\lambda}^{\mu})^{p-2} \bar{v}_{i} \bar{Y}_{i}
\]
and hence, in view of (2.7),
\[
-\frac{\bar{v}_{i}''}{r^{2a}} - \frac{N - 1 - 2a}{r^{2a+1}} \bar{v}_{i}' + \frac{i(N + i - 2)}{r^{2(a+1)}} \bar{v}_{i} - \frac{\lambda}{r^{2(a+1)}} \bar{v}_{i} = \frac{p - 1}{r^{bp}} (z_{a,b,\lambda}^{\mu})^{p-2} \bar{v}_{i}.
\]
Making in (2.8) the transformation (2.3) we obtain the equations
\[-\bar{\varphi}_{i}'' - \beta \cosh^{-2} \left(\gamma(t - \ln \mu)\right) \bar{\varphi}_{i} = \left(\lambda - \frac{(N - 2 - 2a)}{2} - i(N + i - 2)\right) \bar{\varphi}_{i}, \quad i \in \mathbb{N}_{0},
\]
where
\[ \beta = \frac{N(N + 2(1 + a - b))((N - 2 - 2a)^2 - 4\lambda)}{4(N - 2(1 + a - b))^2} \]
and
\[ \gamma = \frac{(1 + a - b)\sqrt{(N - 2 - 2a)^2 - 4\lambda}}{N - 2(1 + a - b)}, \]
which is equivalent, through the change of variable \( \zeta(s) = \varphi(s + \ln \mu) \), to
\[ -\zeta'' - \beta \cosh^{-2}(\gamma s)\zeta = \left( \lambda - \left( \frac{N - 2 - 2a}{2} \right)^2 - i(N + i - 2) \right) \zeta, \quad i \in \mathbb{N}_0. \tag{2.9} \]

It is known (see [8], [10, p. 74]) that the negative part of the spectrum of the problem
\[ -\zeta'' - \beta \cosh^{-2}(\gamma s)\zeta = \nu \zeta \]
is discrete, consists of simple eigenvalues and is given by
\[ \nu_j = -\frac{\gamma^2}{4} \left( -(1 + 2j) + \sqrt{1 + 4\beta \gamma^{-2}} \right)^2, \quad j \in \mathbb{N}_0, \quad 0 \leq j < \frac{1}{2} \left( -1 + \sqrt{1 + 4\beta \gamma^{-2}} \right). \]

Thus we have for all \( i \geq 0 \) that zero is the only solution to (2.9) if and only if
\[ A_i(a, \lambda) \neq B_j(a, b, \lambda) \quad \text{for all} \quad 0 \leq j < \frac{N}{2(1 + a - b)}, \tag{2.10} \]
where
\[ A_i(a, \lambda) = \lambda - \left( \frac{N - 2 - 2a}{2} \right)^2 - i(N + i - 2) \]
and
\[ B_j(a, b, \lambda) = -\frac{(N - 2 - 2a)^2 - 4\lambda(1 + a - b)^2}{4(N - 2(1 + a - b))^2} \left[ -2j + \frac{N}{1 + a - b} \right]^2. \]

Note that \( A_0(a, \lambda) = B_1(a, b, \lambda) \), \( A_i(a, \lambda) \geq A_{i+1}(a, \lambda) \) and \( B_j(a, b, \lambda) \leq B_{j+1}(a, b, \lambda) \), which is shown in figure 3 below.

Figure 3

Hence (2.10) is satisfied for \( i \geq 1 \) if and only if \( B_0(a, b, \lambda) \neq A_i(a, b, \lambda) \), which is equivalent to \( b \neq h_i(a, \lambda) \). On the other hand for \( i = 0 \) equation (2.9) has a one dimensional space of nonzero solutions. Hence, \( \ker D^2 f_0(z_{a,b,\lambda}) \) is one dimensional if and only if \( b \neq h_i(a, \lambda) \) for any \( i \geq 1 \), which proves the claim. \( \square \)

**Proof of Corollary 1.2.** We define \( I \) on \( D_a^{1,2}(\mathbb{R}^N) \setminus \{0\} \) by the right hand side of (1.7), i.e.
\[ I(u) := \frac{\|\nabla u\|_{a,b}^2}{\|u\|_{p,b}^2}. \]
$I$ is twice continuously differentiable and
\[
(I'(u), \varphi) = \frac{2}{\|u\|^2_{p,b}} \left( \int_{\mathbb{R}^N} |x|^{-2a} \nabla u \nabla \varphi - \frac{\|\nabla u\|^2_{a,b}}{\|u\|^2_{p,b}} \int_{\mathbb{R}^N} |x|^{-bp}|u|^{p-2}u \varphi \right).
\]
Moreover, for positive critical points $u$ of $I$ a short computation leads to
\[
(I''(u)\varphi_1, \varphi_2) = \frac{2}{\|u\|^2_{p,b}} \left( \int_{\mathbb{R}^N} |x|^{-2a} \nabla \varphi_1 \nabla \varphi_2 - \frac{\|\nabla u\|^2_{a,b}}{\|u\|^2_{p,b}} (p-1) \int_{\mathbb{R}^N} |x|^{-bp}|u|^{p-2}\varphi_1 \varphi_2 \right)
+ (p-2) \frac{2\|\nabla u\|^2_{p,b}}{\|u\|^{2p+2}_{p,b}} \left( \int_{\mathbb{R}^N} |x|^{-bp}|u|^{p-2}\varphi_1 \right) \left( \int_{\mathbb{R}^N} |x|^{-bp}|u|^{p-2}\varphi_2 \right).
\]

Obviously $I$ is constant on $Z_{a,b,0}$ and we obtain for $z_1 := z_{1,a,b,0}$ and all $\varphi_1, \varphi_2 \in D^{1,2}_{a,\lambda}(\mathbb{R}^N)$
\[
(I'(z_1), \varphi_1) = \frac{2}{\|z_1\|^2_{p,b}} (f'_0(z_1) \varphi_1) = 0,
\]
\[
(I''(z_1)\varphi_1, \varphi_2) = \frac{2}{\|u\|^2_{p,b}} (f''_0(z_1) \varphi_1 \varphi_2)
+ (p-2) \frac{2}{\|z_1\|^{p+2}_{p,b}} \left( \int_{\mathbb{R}^N} |x|^{-bp}z_1^{p-1} \varphi_1 \right) \left( \int_{\mathbb{R}^N} |x|^{-bp}z_1^{p-1} \varphi_2 \right).
\]

From the proof of Theorem 1.1 we know that for $b < h_1(a,0)$ there exist functions $\check{\varphi} \in D^{1,2}_{a,\lambda}(\mathbb{R}^N)$ of the form $\check{\varphi}(x) = \varphi(|x|)Y_1(x/|x|)$, where $Y_1$ denotes one of the first spherical harmonics, such that $(f''_0(z_1) \check{\varphi}, \check{\varphi}) < 0$. By (2.11) we get $(I''(z_1)\check{\varphi}, \check{\varphi}) < 0$ because the integral $\int |x|^{-bp}z_1^{p-1} \check{\varphi} = 0$. Consequently $C_{a,b}$ is strictly smaller than $I(z_1) = I(z_{a,b,0})$.

As a particular case of Theorem 1.1 we can state

**Corollary 2.2.**

(i) If $0 < a < \frac{N-2}{2}$ and $0 \leq \lambda < \left( \frac{N-2-2a}{2} \right)^2$ then $Z_{a,b,\lambda}$ is non-degenerate for any $b$ between $a$ and $a + 1$.

(ii) If $a = 0$ and $0 \leq \lambda < \left( \frac{N-2-2a}{2} \right)^2$, then $Z_{0,b,\lambda}$ is degenerate if and only if $b = \lambda = 0$.

**Remark 2.3.** If $a = b = \lambda = 0$, equation (2.1) is invariant not only by dilations but also by translations. The manifold of critical points is in this case $N + 1$-dimensional and given by the translations and dilations of $z_{1,0,0}$. Hence the one dimensional manifold $Z_{0,0,0}$ is degenerate. However, the full $N + 1$-dimensional critical manifold is non-degenerate in the case $a = b = \lambda = 0$ (see [3]).

3. The finite dimensional reduction

We follow the perturbative method developed in [2] and show that a finite dimensional reduction of our problem is possible whenever the critical manifold is non-degenerated. For simplicity of notation we write $\hat{z}_\mu$ instead of $z_{\mu,a,b,\lambda}$ and $Z$ instead of $Z_{a,b,\lambda}$ if there is no possibility of confusion.

**Lemma 3.1.** Suppose $a, b, \lambda, p$ satisfy (1.2) and $v$ is a measurable function such that the integral $\int |v|^{-p+2} |x|^{-bp}$ is finite. Then the operator $J_v : D_{a,\lambda}^{1,2}(\mathbb{R}^N) \to D_{a,\lambda}^{1,2}(\mathbb{R}^N)$, defined by
\[
J_v(u) := \mathcal{K} \left( \int_{\mathbb{R}^N} |x|^{-p_b} vu \cdot \right),
\]
is compact.

**Proof.** Fix a sequence \((u_n)_{n\in\mathbb{N}}\) converging weakly to zero in \(D^{1,2}_{a,\lambda}(\mathbb{R}^N)\). To prove the assertion it is sufficient to show that up to a subsequence \(J_n(u_n) \to 0\) as \(n \to \infty\). Using the Hilbert space isomorphism given in (1.13) we see that the corresponding sequence \((v_n)_{n\in\mathbb{N}}\) converges weakly to zero in \(H^{1,2}_\lambda(C)\). Since \((v_n)_{n\in\mathbb{N}}\) converges strongly in \(L^2(\Omega)\) for all bounded domains \(\Omega\) in \(C\), we may extract a subsequence that converges to zero pointwise almost everywhere. Going back to \(D^{1,2}_{a,\lambda}(\mathbb{R}^N)\) we may assume that this also holds for \((u_n)_{n\in\mathbb{N}}\). By Hölder’s inequality and (1.3)

\[
\|J_n(u_n)\| \leq \sup_{\|h\|_{D^{1,2}_{a,\lambda}(\mathbb{R}^N)} \leq 1} \int_{\mathbb{R}^N} |x|^{-pb}|v| |u_n| |h| \\
\leq \sup_{\|h\|_{D^{1,2}_{a,\lambda}} \leq 1} \left( \int_{\mathbb{R}^N} |x|^{-pb}|h|^p \right)^{1/p} \left( \int_{\mathbb{R}^N} |x|^{-pb}|v| |u_n| |h| \right)^{(p-1)/p} \\
\leq C \left( \int_{\mathbb{R}^N} |x|^{-pb}|v| |u_n| |h| \right)^{(p-1)/p}.
\]

To show that the latter integral converges to zero we use Vitali’s convergence theorem given for instance in [9, 13.38]. Obviously the functions \(|\cdot|^{-pb}|v| |u_n| |h|\) converge pointwise almost everywhere to zero. For any measurable \(\Omega \subset \mathbb{R}^N\) we may estimate using Hölder’s inequality

\[
\int_{\Omega} |x|^{-pb}|v| |u_n| |h| \leq \left( \int_{\Omega} |x|^{-pb}|v| |u_n| \right)^{p/(p-2)} \left( \int_{\Omega} |x|^{-pb}|h|^p \right)^{(2-p)/(p-2)} \left( \int |x|^{-pb}|u_n|^p \right)^{1/(p-1)} \\
\leq C \left( \int_{\Omega} |x|^{-pb}|v| |u_n| \right)^{(p-2)/(p-1)}.
\]

for some positive constant \(C\). Taking \(\Omega\) a set of small measure or the complement of a large ball and the use of Vitali’s convergence theorem prove the assertion. \(\square\)

**Lemma 3.1** immediately leads to

**Corollary 3.2.** For all \(z \in Z\) the operator \(f''_0(z) : D^{1,2}_{a,\lambda}(\mathbb{R}^N) \to D^{1,2}_{a,\lambda}(\mathbb{R}^N)\) may be written as \(f'_0(z) = id - J_{|z|^{-p-2}}\) and is consequently a self-adjoint Fredholm operator of index zero.

Define for \(\mu > 0\) the map \(U_\mu : D^{1,2}_{a,\lambda}(\mathbb{R}^N) \to D^{1,2}_{a,\lambda}(\mathbb{R}^N)\) by

\[
U_\mu(u) := \mu^{-\frac{N-2-2\alpha}{2}} u \left( \frac{x}{\mu} \right).
\]

It is easy to check that \(U_\mu\) conserves the norms \(\|\cdot\|\) and \(\|\cdot\|_{p,h}\), thus for every \(\mu > 0\)

\[
(U_\mu)^{-1} = (U_\mu)^t = U_{\mu^{-1}} \text{ and } f_0 = f_0 \circ U_\mu
\]

(3.2)

where \((U_\mu)^t\) denotes the adjoint of \(U_\mu\). Twice differentiating the identity \(f_0 = f_0 \circ U_\mu\) yields for all \(h_1, h_2, v \in D^{1,2}_{a,\lambda}(\mathbb{R}^N)\)

\[
(f''_0(v)h_1, h_2) = (f''_0(U_\mu(v))U_\mu(h_1), U_\mu(h_2)),
\]

that is

\[
f''_0(v) = (U_\mu)^{-1} \circ f''_0(U_\mu(v)) \circ U_\mu \quad \forall v \in D^{1,2}_{a,\lambda}(\mathbb{R}^N).
\]

(3.3)
Differentiating (3.2) we see that $U(\mu, z) := U_\mu(z)$ maps $(0, \infty) \times Z$ into $Z$, hence

$$
\frac{\partial U}{\partial z}(\mu, z) = U_\mu : T_z Z \to T_{U_\mu(z)} Z \text{ and } U_\mu : (T_z Z)^\perp \to (T_{U_\mu(z)} Z)^\perp.
$$

(3.4)

If the manifold $Z$ is non-degenerated the self-adjoint Fredholm operator $f''_0(z_1)$ maps the space $D^{1,2}_{a,\lambda}(\mathbb{R}^N)$ into $T_{z_1} Z^\perp$ and $f_0'(z_1) \in L(T_{z_1} Z^\perp)$ is invertible. Consequently, using (3.3) and (3.4), we obtain in this case

$$
\left\| (f''_0(z_1))^{-1} \right\|_{L(T_{z_1} Z^\perp)} = \left\| (f'_0(z_1))^{-1} \right\|_{L(T_z Z^\perp)} \quad \forall z \in Z.
$$

(3.5)

**Lemma 3.3.** Suppose $a, b, p, \lambda$ satisfy (1.2) and (1.4) holds. Then there exists a constant $C_1 = C_1(||k||_\infty, a, b, \lambda) > 0$ such that for any $\mu > 0$ and for any $w \in D^{1,2}_{a,\lambda}(\mathbb{R}^N)$

$$
|G(z_\mu + w)| \leq C_1(||k||^1/p z_\mu||_{p,b} + \|w\|^p) \tag{3.6}
$$

$$
||G'(z_\mu + w)|| \leq C_1(||k||^1/p z_\mu||_{p,b}^1 + \|w\|^{p-1}) \tag{3.7}
$$

$$
||G''(z_\mu + w)|| \leq C_1(||k||^1/p z_\mu||_{p,b}^2 + \|w\|^{p-2}). \tag{3.8}
$$

Moreover, if $\lim_{|x| \to \infty} k(x) =: k(\infty) = 0 = k(0)$ then

$$
||k||^1/p z_\mu||_{p,b} \to 0 \text{ as } \mu \to \infty \text{ or } \mu \to 0. \tag{3.9}
$$

**Proof.** (3.6)-(3.8) are consequences of (1.3) and Hölder’s inequality. We will only show (3.8) as (3.6)-(3.7) follow analogously. By Hölder’s inequality and (1.3)

$$
||G''(z_\mu + w)|| \leq (p-1) \sup_{||h_1||_{\infty}, ||h_2||_{\infty} \leq 1} \int_{\mathbb{R}^N} \frac{|k(x)|}{|x|^{bp}} |z_\mu + w|^{p-2} |h_1| |h_2|
$$

$$
\leq (p-1)||k||^1/p_{\infty}^2 \sup_{||h_1||_{\infty}, ||h_2||_{\infty} \leq 1} \||k||^1/p(z_\mu + w)||^{p-2}_{p,b} |h_1| |h_2||_{p,b}
$$

$$
\leq c(||k||_\infty, a, b, \lambda) \||k||^1/p(z_\mu + w)||^{p-2}_{p,b}.
$$

Using the triangle inequality and again (1.3) we obtain (3.8).

Under the additional assumption $k(0) = k(\infty) = 0$ estimate (3.9) follows by the dominated convergence theorem and

$$
\int_{\mathbb{R}^N} \frac{|k(x)|}{|x|^{bp}} z_\mu^p = \int_{\mathbb{R}^N} \frac{|k(\mu x)|}{|x|^{bp}} z_1^p.
$$

\[\square\]

**Lemma 3.4.** Suppose $a, b, p, \lambda$ satisfy (1.2) and (1.4) and (1.5) hold. Then there exist constants $\varepsilon_0, C > 0$ and a smooth function

$$
w = w(\mu, \varepsilon) : (0, +\infty) \times (-\varepsilon_0, \varepsilon_0) \to D^{1,2}_{a,\lambda}(\mathbb{R}^N)
$$

such that for any $\mu > 0$ and $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$$

$$
w(\mu, \varepsilon) \text{ is orthogonal to } T_{z_\mu} Z \tag{3.10}
$$

$$
f'_\varepsilon(z_\mu + w(\mu, \varepsilon)) \in T_{z_\mu} Z \tag{3.11}
$$

$$
\|w(\mu, \varepsilon)\| \leq C |\varepsilon|. \tag{3.12}
$$

Moreover, if (1.8) holds then

$$
\|w(\mu, \varepsilon)\| \to 0 \text{ as } \mu \to 0 \text{ or } \mu \to \infty. \tag{3.13}
$$
Proof. Define $H : (0, \infty) \times D_{\alpha, \lambda}^{1,2}(\mathbb{R}^N) \times \mathbb{R} \times \mathbb{R} \to D_{\alpha, \lambda}^{1,2}(\mathbb{R}^N) \times \mathbb{R}$

$$H(\mu, w, \alpha, \varepsilon) := (f'(z_\mu + w) - \alpha \xi_\mu, (w, \xi_\mu)),$$

where $\xi_\mu$ denotes the normalized tangent vector $\frac{d}{dz_\mu} z_\mu$. If $H(\mu, w, \alpha, \varepsilon) = (0, 0)$ then $w$ satisfies (3.10)-(3.11) and $H(\mu, w, \alpha, \varepsilon) = (0, 0)$ if and only if $(w, \alpha) = F_{\mu, \varepsilon}(w, \alpha)$, where

$$F_{\mu, \varepsilon}(w, \alpha) := -\left(\frac{\partial H}{\partial (w, \alpha)}(\mu, 0, 0, 0)\right)^{-1} H(\mu, w, \alpha, \varepsilon) + (w, \alpha).$$

We prove that $F_{\mu, \varepsilon}(w, \alpha)$ is a contraction in some ball $B_\rho(0)$, where we may choose the radius $\rho = \rho(\varepsilon) > 0$ independent of $z \in Z$. To this end we observe

$$\left(\left(\frac{\partial H}{\partial (w, \alpha)}(\mu, 0, 0, 0)\right)(w, \beta), (f''_0(z_\mu)w - \beta \xi_\mu, (w, \xi_\mu))\right) = \|f''_0(z_\mu)w\|^2 + \beta^2 + \|(w, \xi_\mu)\|^2,$$

where

$$\left(\frac{\partial H}{\partial (w, \alpha)}(\mu, 0, 0, 0)\right)(w, \beta) = (f''_0(z_\mu)w - \beta \xi_\mu, (w, \xi_\mu)).$$

From Corollary 3.2 and (3.14) we infer that $\left(\frac{\partial H}{\partial (w, \alpha)}(\mu, 0, 0, 0)\right)$ is an injective Fredholm operator of index zero, hence invertible and by (3.5) and (3.14) we obtain

$$\left\|\left(\frac{\partial H}{\partial (w, \alpha)}(\mu, 0, 0, 0)\right)^{-1}\right\| \leq \max(1, \|(f''_0(z_\mu))^{-1}\|) = \max(1, \|(f''_0(z_1))^{-1}\|) =: C_*. \quad (3.15)$$

Suppose $(w, \alpha) \in B_\rho(0)$. We use (3.3) and (3.15) to see

$$\|F_{\mu, \varepsilon}(w, \alpha)\| \leq C_\varepsilon \left\|H(\mu, w, \alpha, \varepsilon) - \left(\frac{\partial H}{\partial (w, \alpha)}(\mu, 0, 0, 0)\right)(w, \alpha)\right\|$$

$$\leq C_\varepsilon \|f''_0(z_\mu + w) - f''_0(z_\mu)w\|$$

$$\leq \int_0^1 \|f''_0(z_\mu + tw) - f''_0(z_\mu)\| dt \|w\| + C_\varepsilon \|\varepsilon\| \|G'(z_\mu + w)\|$$

$$\leq \int_0^1 \|f''_0(z_1 + tU_{\mu-1}(w)) - f''_0(z_1)\| dt \|w\| + C_\varepsilon \|\varepsilon\| \|G'(z_\mu + w)\|$$

$$\leq C_\varepsilon \rho \sup_{\|w\| \leq \rho} \|f''_0(z_1 + w) - f''_0(z_1)\| + C_\varepsilon \|\varepsilon\| \sup_{\|w\| \leq \rho} \|G'(z_\mu + w)\|. \quad (3.16)$$

Analogously we get for $(w_1, \alpha_1), (w_2, \alpha_2) \in B_\rho(0)$

$$\|F_{\mu, \varepsilon}(w_1, \alpha_1) - F_{\mu, \varepsilon}(w_2, \alpha_2)\| \leq \frac{\|f''_0(z_\mu + w_1) - f''_0(z_\mu + w_2) - f''_0(z_\mu)(w_1 - w_2)\|}{\|w_1 - w_2\|}$$

$$\leq \int_0^1 \|f''_0(z_\mu + w_2 + t(w_1 - w_2)) - f''_0(z_\mu)\| dt$$

$$\leq \int_0^1 \|f''_0(z_\mu + w_2 + t(w_1 - w_2)) - f''_0(z_\mu)\| dt$$

$$+ \|\varepsilon\| \int_0^1 \|G''(z_\mu + w_2 + t(w_1 - w_2))\| dt$$

$$\leq \sup_{\|w\| \leq 3\rho} \|f''_0(z_1 + w) - f''_0(z_1)\| + \|\varepsilon\| \sup_{\|w\| \leq 3\rho} \|G''(z_\mu + w)\|.$$
We may choose $\rho_0 > 0$ such that
\[ C_* \sup_{\|w\| \leq \rho_0} \| f''_0(z_1 + w) - f''_0(z_1) \| < \frac{1}{2} \]
and $\varepsilon_0 > 0$ such that
\[ 2\varepsilon_0 < \left( \sup_{z \in Z, \|w\| \leq \rho_0} \| G''(z + w) \| \right)^{-1} C_*^{-1} \text{ and } 3\varepsilon_0 < \left( \sup_{z \in Z, \|w\| \leq \rho_0} \| G'(z + w) \| \right)^{-1} C_*^{-1} \rho_0. \]

With these choices and the above estimates it is easy to see that for every $z_\mu \in Z$ and $|\varepsilon| < \varepsilon_0$ the map $F_{\mu,\varepsilon}$ maps $B_{\rho_0}(0)$ in itself and is a contraction there. Thus $F_{\mu,\varepsilon}$ has a unique fix-point $(w(\mu, \varepsilon), \alpha(\mu, \varepsilon))$ in $B_{\rho_0}(0)$ and it is a consequence of the implicit function theorem that $w$ and $\alpha$ are continuously differentiable.

From (3.16) we also infer that $F_{z,\varepsilon}$ maps $B_{\rho}(0)$ into $B_{\rho}(0)$, whenever $\rho \leq \rho_0$ and
\[ \rho > 2|\varepsilon| \left( \sup_{\|w\| \leq \rho} \| G'(z + w) \| \right) C_* \]

Consequently due to the uniqueness of the fix-point we have
\[ \|(w(\varepsilon), \alpha(\varepsilon))\| \leq 3|\varepsilon| \left( \sup_{\|w\| \leq \rho_0} \| G'(z + w) \| \right) C_* \]
which gives (3.12). Let us now prove (3.13). Set
\[ \rho_{\mu} := \min \left\{ 4\varepsilon_0 C_* C_1 \| k \|^{1/p} \mu \| p_{\mu}, \rho_0, \left( \frac{1}{8\varepsilon_0 C_1 C_*} \right)^{\frac{1}{p-1}} \right\} \]
where $C_1$ is given in Lemma 3.3. In view of (3.7) we have that for any $|\varepsilon| < \varepsilon_0$ and $\mu > 0$
\[ 2|\varepsilon| C_* \sup_{\|w\| \leq \rho_{\mu}} \| G'(z_{\mu} + w) \| \leq 2|\varepsilon| C_* C_1 \| k \|^{1/p} \mu \| p_{\mu} + 2|\varepsilon| C_* C_1 \rho_{\mu}^2 - \rho_{\mu}. \]

Since $\rho_{\mu}^2 \leq \frac{1}{8\varepsilon_0 C_1 C_*}$ we have,
\[ 2|\varepsilon| C_* \sup_{\|w\| \leq \rho_{\mu}} \| G'(z_{\mu} + w) \| \leq 2|\varepsilon| C_* C_1 \| k \|^{1/p} \mu \| p_{\mu} + \frac{1}{2} \rho_{\mu} \leq \rho_{\mu}, \]
so that, by the above argument, we can conclude that $F_{\mu,\varepsilon}$ maps $B_{\rho_{\mu}}(0)$ into $B_{\rho_{\mu}}(0)$. Consequently due to the uniqueness of the fix-point we have
\[ \| w(\mu, \varepsilon) \| \leq \rho_{\mu}. \]

Since by (3.9) we have that $\rho_{\mu} \to 0$ for $\mu \to 0$ and for $\mu \to +\infty$, we get (3.13).

Under the assumptions of Lemma 3.4 we may define for $|\varepsilon| < \varepsilon_0$
\[ Z^\varepsilon_{a,b,\lambda} := \{ u \in D^{1,2}_{a,\lambda}(\mathbb{R}^N) \mid u = z_{a,\lambda} + w(\mu, \varepsilon), \mu \in (0, \infty) \}. \]

Note that $Z^\varepsilon$ is a one dimensional manifold.

**Lemma 3.5.** Under the assumptions of Lemma 3.4 we may choose $\varepsilon_0 > 0$ such that for every $|\varepsilon| < \varepsilon_0$ the manifold $Z^\varepsilon$ is a natural constraint for $f_\varepsilon$, i.e. every critical point of $f_\varepsilon|_{Z^\varepsilon}$ is a critical point of $f_\varepsilon$.

**Proof.** Fix $u \in Z^\varepsilon$ such that $f_\varepsilon|_{Z^\varepsilon}(u) = 0$. In the following we use a dot for the derivation with respect to $\mu$. Since $(\dot{z}_\mu, w(\mu, \varepsilon)) = 0$ for all $\mu > 0$ we obtain
\[ (\ddot{z}_\mu, w(\mu, \varepsilon)) + (\dot{z}_\mu, \dot{w}(\mu, \varepsilon)) = 0. \]
Moreover differentiating the identity $z_\mu = U_\sigma z_{\mu/\sigma}$ with respect to $\mu$ we obtain

$$\dot{z}_\sigma = \frac{1}{\sigma} U_\sigma \dot{z}_1 \text{ and } \ddot{z}_\sigma = \frac{1}{\sigma^2} U_\sigma \ddot{z}_1.$$  

(3.19)

From (3.11) we get that $f'_\epsilon(u) = c_1 \dot{z}_{\mu}$ for some $\mu > 0$. By (3.18) and (3.19)

$$0 = (f'_\epsilon(u), \dot{z}_\mu + \dot{w}(\mu, \epsilon)) = c_1(\dot{z}_\mu, \dot{z}_\mu + \dot{w}(\mu, \epsilon))$$

$$= c_1 \mu^{-2}(\|\dot{z}_1\|^2 - (\ddot{z}_1, U_{\mu-1}(\mu, \epsilon))) = c_1 \mu^{-2}(\|\dot{z}_1\|^2 - (\ddot{z}_1)(O(1)\epsilon)).$$

Finally we see that for small $\epsilon > 0$ the number $c_1$ must be zero and the assertion follows. □

In view of the above result we end up facing a finite dimensional problem as it is enough to find critical points of the functional $\Phi_\epsilon : (0, \infty) \to \mathbb{R}$ given by $f_z|_{Z^\epsilon}$.

4. STUDY OF $\Phi_\epsilon$

In this section we will assume that the critical manifold is non-degenerate, i.e. (1.5), such that the functional $\Phi_\epsilon$ is defined. To find critical points of $\Phi_\epsilon = f_z|_{Z^\epsilon}$ it is convenient to introduce the functional $\Gamma$ given below.

Lemma 4.1. Suppose $a, b, p, \lambda$ satisfy (1.2) and (1.4) holds. Then

$$\Gamma(\mu) = f_0(z_1) - \epsilon \Gamma(\mu) + o(\epsilon),$$  

(4.1)

where $\Gamma(\mu) = G(z_\mu)$. In particular, there is $C > 0$, independent of $\mu$ and $\epsilon$, such that

$$|\Phi_\epsilon(\mu) - (f_0(z_1) - \epsilon \Gamma(\mu))| \leq C\|w(\epsilon, \mu)\|^2 + (1 + |\epsilon|)\|w(\epsilon, \mu)\|^p.$$  

(4.2)

Consequently, if there exist $0 < \mu_1 < \mu_2 < \mu_3 < \infty$ such that

$$\Gamma(\mu_2) > \max(\Gamma(\mu_1), \Gamma(\mu_3)) \text{ or } \Gamma(\mu_2) < \min(\Gamma(\mu_1), \Gamma(\mu_3))$$

(4.3)

then $\Phi_\epsilon$ will have a critical point, if $\epsilon > 0$ is sufficiently small.

Proof. Note that for all $\mu > 0$ we have $f_0(z_\mu) = f_0(z_1)$,

$$\|z_\mu\|^2 = \int_{\mathbb{R}^N} \frac{z_\mu^p}{|x|^{bp}} \text{ and } (z_\mu, w(\epsilon, \mu)) = \int_{\mathbb{R}^N} \frac{z_\mu^{p-1}w(\epsilon, \mu)}{|x|^{bp}}.$$  

(4.4)

From (4.4) we infer

$$\Phi_\epsilon(\mu) = \frac{1}{2} \int_{\mathbb{R}^N} \frac{z_\mu^p}{|x|^{bp}} + \frac{1}{2}\|w(\epsilon, \mu)\|^2 + \int_{\mathbb{R}^N} \frac{z_\mu^{p-1}w(\epsilon, \mu)}{|x|^{bp}} - \frac{1}{p} \int_{\mathbb{R}^N} \frac{(1 + \epsilon k)(z_\mu + w(\epsilon, \mu))^p}{|x|^{bp}}$$

and

$$f_0(z_1) = f_0(z_\mu) = \frac{1}{2}\|z_\mu\|^2 - \frac{1}{p} \int_{\mathbb{R}^N} \frac{z_\mu^p}{|x|^{bp}} = \left(1 - \frac{1}{p}\right) \int_{\mathbb{R}^N} \frac{z_\mu^p}{|x|^{bp}}.$$

Hence

$$\Phi_\epsilon(\mu) = f_0(z_1) - \epsilon \Gamma(\mu) + \frac{1}{2}\|w(\epsilon, \mu)\|^2 - \frac{1}{p} H_\epsilon(\mu),$$  

(4.5)

where

$$H_\epsilon(\mu) = \int_{\mathbb{R}^N} \frac{(z_\mu + w(\epsilon, \mu))^p - z_\mu^p - p z_\mu^{p-1}w(\epsilon, \mu) + \epsilon k((z_\mu + w(\epsilon, \mu))^p - z_\mu^p)}{|x|^{bp}}.$$

Using the inequality

$$(z + w)^{s-1} - z^{s-1} - (p-1)z^{s-2}w \leq \begin{cases} C(z^{s-3}w^2 + w^{s-1}) & \text{if } s \geq 3 \\ C w^{s-1} & \text{if } 2 < s < 3, \end{cases}$$

(4.6)
where \(C = C(s) > 0\), with \(s = p + 1\) and Hölder’s inequality we have for some \(c_2, c_3 > 0\)

\[
|H_\varepsilon(\mu)| \leq \int_{\mathbb{R}^N} \left| \frac{(z_\mu + w(\varepsilon, \mu))^p - z_\mu^p - p z_\mu^{p-1} w(\varepsilon, \mu)}{|x|^p} \right| + |\varepsilon| \int_{\mathbb{R}^N} \frac{|k| ((z_\mu + w(\varepsilon, \mu))^p - z_\mu^p)}{|x|^p} \\
\leq c_2 \left[ \int_{\mathbb{R}^N} \frac{z_\mu^{p-2} w^2(\varepsilon, \mu)}{|x|^p} + \int_{\mathbb{R}^N} \frac{|w(\varepsilon, \mu)|^p}{|x|^p} + |\varepsilon| \int_{\mathbb{R}^N} \frac{z_\mu^{p-1} |w(\varepsilon, \mu)|}{|x|^p} + |\varepsilon| \int_{\mathbb{R}^N} \frac{|w(\varepsilon, \mu)|^p}{|x|^p} \right] \\
\leq c_3 \left[ \|w(\varepsilon, \mu)\|^2 + (1 + |\varepsilon|) \|w(\varepsilon, \mu)\|^p + |\varepsilon| \|w(\varepsilon, \mu)\| \right]
\]

and the claim follows. \(\square\)

Although it is convenient to study only the reduced functional \(\Gamma\) instead of \(\Phi_\varepsilon\), it may lead in some cases to a loss of information, i.e. \(\Gamma\) may be constant even if \(k\) is a non-constant function. This is due to the fact that the critical manifold consists of radially symmetric functions. Thus \(\Gamma\) is constant for every \(k\) that has constant mean-value over spheres, i.e.

\[
\frac{1}{r^{N-1}} \int_{\partial B_r(0)} k(x) \, dS(x) \equiv \text{const} \quad \forall r > 0.
\]

In this case we have to study the functional \(\Phi_\varepsilon(\mu)\) directly.

**Proof of Theorem 1.3.** By (1.8), (3.9), (3.13) and (4.2)

\[
\lim_{\mu \to 0^+} \Phi_\varepsilon(\mu) = \lim_{\mu \to +\infty} \Phi_\varepsilon(\mu) = f_0(z_1).
\]

Hence, either the functional \(\Phi_\varepsilon \equiv f_0(z_1)\), and we obtain infinitely many critical points, or \(\Phi_\varepsilon \not\equiv f_0(z_1)\) and \(\Phi_\varepsilon\) has at least a global maximum or minimum. In any case \(\Phi_\varepsilon\) has a critical point that provides a solution of \((P_{a,b,\lambda}).\) \(\square\)

The next lemma shows that it is possible (and convenient) to extend the \(C^2-\) functional \(\Gamma\) by continuity to \(\mu = 0\). The proof of this fact is analogous to the one in [3, Lem. 3.4] and we omit it here.

**Lemma 4.2.** Under the assumptions of Lemma 4.1

\[
\Gamma(0) := \lim_{\mu \to 0} \Gamma(\mu) = k(0) \frac{1}{p} \|z_1\|^p_{p,b} \quad \text{and} \quad (4.6)
\]

\[
\frac{1}{p} \liminf_{|x| \to \infty} k(x) \|z_1\|^p_{p,b} \leq \liminf_{\mu \to \infty} \Gamma(\mu) \leq \limsup_{\mu \to \infty} \Gamma(\mu) \leq \frac{1}{p} \limsup_{|x| \to \infty} k(x) \|z_1\|^p_{p,b} . \quad (4.7)
\]

If, moreover, (1.9) holds we obtain

\[
\Gamma'(0) = 0 \quad \text{and} \quad \Gamma''(0) = \frac{\Delta k(0)}{Np} \int |x|^2 \frac{z_1(x)^p}{|x|^p} . \quad (4.8)
\]

**Proof of Theorem 1.4.** To see that assumptions (1.10) and (1.11) give rise to a critical point we use the functional \(\Gamma\). Condition (1.10) and Lemma 4.2 imply that \(\Gamma\) has a global maximum strictly bigger than \(\Gamma(0)\) and \(\limsup_{\mu \to \infty} \Gamma(\mu)\). Consequently \(\Phi_\varepsilon\) has a critical point in view of Lemma 4.1. The same reasoning yields a critical point under condition (1.11). \(\square\)
REFERENCES


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