

Intrinsic torsion in quaternionic contact geometry

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To my father, Franco Conti

Abstract

We investigate quaternionic contact (qc) manifolds from the point of view of intrinsic torsion. We argue that the natural structure group for this geometry is a non-compact Lie group K containing $\mathrm{Sp}(n)\mathbb{H}^*$, and show that any qc structure gives rise to a canonical K -structure with constant intrinsic torsion, except in seven dimensions, when this condition is equivalent to integrability in the sense of Duchemin.

We prove that the choice of a reduction to $\mathrm{Sp}(n)\mathbb{H}^*$ (or equivalently, a complement of the qc distribution) yields a unique K -connection satisfying natural conditions on torsion and curvature.

We show that the choice of a compatible metric on the qc distribution determines a canonical reduction to $\mathrm{Sp}(n)\mathrm{Sp}(1)$ and a canonical $\mathrm{Sp}(n)\mathrm{Sp}(1)$ -connection whose curvature is almost entirely determined by its torsion. We show that its Ricci tensor, as well as the Ricci tensor of the Biquard connection, has an interpretation in terms of intrinsic torsion.

Quaternionic contact geometry was introduced by Biquard in [2]; its model is the sphere, viewed as the conformal infinity of quaternionic hyperbolic space. A quaternionic contact (qc) structure is canonically defined on any 3-Sasakian manifold, and on more general classes of hypersurfaces in quaternionic manifolds; see [5, 14]. Explicit examples on Lie groups are also known (see [4, 3]). Aside from the link with quaternionic-kähler geometry, a motivating aspect of qc geometry is the presence of a conformal class of subriemannian metrics for which the Yamabe problem can be studied (see [10]).

A qc structure on a manifold of dimension $4n + 3$ is defined as a distribution of rank $4n$ that can locally be written as $\ker \eta_1 \cap \ker \eta_2 \cap \ker \eta_3$, where the 2-forms $d\eta_s$ define an almost quaternion Hermitian metric on the distribution at each point. Whilst the metric on the distribution is not fixed, its conformal class is determined. Qc geometry is generally studied by fixing a metric in this class; when $n > 1$, this determines a unique Riemannian metric and a metric connection, called the Biquard connection [2]. For $n = 1$, a similar result holds; however, the resulting connection only has the same features as the Biquard connection when the qc structure is integrable in the sense of Duchemin [6].

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These connections are defined by torsion conditions which make them unique. The definition of the Riemannian metric involves the choice of a complement to the qc distribution, which is characterized by the existence of a compatible connection with the required torsion conditions. One is led to wonder to which extent the choice of these conditions is canonical. The literature shows the geometric significance of the curvature of the Biquard connection, and in particular of its Ricci tensor: for instance, it was shown in [14, 9] that the traceless Ricci is zero precisely when the qc structure is 3-Sasakian up to local homothety, and a Lichnerowicz-type result involving the Ricci tensor was obtained in [11]. Further, the scalar curvature used in the study of the Yamabe problem is the trace of this Ricci tensor. It is natural to ask why and whether the Biquard connection and its Ricci tensor are canonical objects of qc geometry.

This paper uses the language of special geometries: namely, of G -structures whose intrinsic torsion is partially prescribed. The intrinsic torsion of a G -structure is a tensor representing the first order obstruction to its flatness; it is obtained from the torsion of any connection via a projection to Coker ∂_G , where

$$\partial_G: (\mathbb{R}^{4n+3})^* \otimes \mathfrak{g} \rightarrow \Lambda^2(\mathbb{R}^{4n+3})^* \otimes \mathbb{R}^{4n+3}$$

is induced by the inclusion $\mathfrak{g} \subset (\mathbb{R}^{4n+3})^* \otimes \mathbb{R}^{4n+3}$. A qc structure cannot be flat in the sense of G -structures, for this would make the distribution integrable in the sense of the Frobenius theorem; the type of condition that we will consider is that the intrinsic torsion take values in a fixed G -invariant subspace of Coker ∂_G .

Understanding what the group G should be is one of the goals of this paper. As a first step, letting Q be the stabilizer of a point in the Grassmannian of $4n$ -planes in \mathbb{R}^{4n+3} , we show that a qc distribution consists in a Q -structure with intrinsic torsion taking values in the orbit $Q \cdot \Theta_0^Q$, where Θ_0^Q is a distinguished element in Coker ∂_Q . The fact that we are dealing with a single Q -orbit suggests that a qc Q -structure has a canonical reduction, determined by the intrinsic torsion itself. We are led to consider a smaller structure group B , namely the stabilizer in Q of Θ_0^Q , obtaining a notion of qc B -structure.

The definition gives obvious constraints on the intrinsic torsion of a qc B -structure, which we refine using the Bianchi identity and a calculation involving certain equivariant linear maps. We show that a qc B -structure takes values in $B \cdot \Theta_0^B$ when $n > 1$; if $n = 1$, this is only true up to a 12-dimensional, irreducible representation of $\mathrm{SO}(4)$, denoted by $S^{5,1}$ in [6]. In fact, in the course of the paper we prove that this component of the intrinsic torsion is zero if and only if the structure is integrable in the sense of [6] (an empty condition for $n > 1$).

Restricting now to the integrable case, we can repeat the argument and further reduce the structure group to the stabilizer of Θ_0^B , which has the form

$$K = \mathrm{Sp}(n)\mathbb{H}^* \ltimes (\mathbb{R}^{4n})^*,$$

where $(\mathbb{R}^{4n})^*$ is the canonical representation of $\mathrm{Sp}(n)\mathbb{H}^*$ considered in quaternionic geometry, acting as a subgroup of $\mathrm{Hom}(\mathbb{R}^3, \mathbb{R}^{4n})$. Using the Bianchi identity again, we prove that the intrinsic torsion of an integrable qc K -structure is constant. In other words, the corresponding K -orbit of Coker ∂_K contains a single point, and we cannot repeat the usual procedure. We take this as evidence of the fact that K is the “natural” structure group of qc geometry. We emphasize that a K -structure does not involve either the choice of a metric on the qc distribution nor of a complement.

The fact that the intrinsic torsion is constant suggests that there could be a canonical connection with constant torsion, obtained by inverting the map

$$\partial_K: (\mathbb{R}^{4n+3})^* \otimes \mathfrak{k} \rightarrow \text{Im } \partial_K.$$

This cannot be done for two reasons: ∂_K is not injective, so the torsion condition does not define a unique connection, and secondly K is not reductive, so ∂_K does not even have a K -equivariant right inverse. Nonetheless, we are able to show that a canonical connection exists on any $\text{Sp}(n)\mathbb{H}^*$ -reduction of a qc K -structure; it is characterized by having constant torsion and satisfying natural conditions on the curvature. It will be called the qc connection.

The torsion condition gives strong restrictions on the curvature via the Bianchi identity. A long computation with highest weight vectors allows us to compute the space in which the curvature takes values. This curvature is a stronger invariant than the qc conformal curvature tensor introduced in [14], since it obstructs the existence of a local diffeomorphism with the Heisenberg group that preserves not only the qc distribution, but also the choice of a complement.

In the last part of the paper, we consider integrable qc structures with a fixed compatible metric on the associated distribution. These can be characterized as $\text{Sp}(n)\text{Sp}(1)$ -structures satisfying an intrinsic torsion condition; we refer to them as quaternionic-contact metric (qcm) structures.

The qc connection has a metric analogue that we call the qcm connection; the two are related via a projection, and the results on the curvature of the former carry over to the latter. In fact, the curvature is entirely determined by the intrinsic torsion and its covariant derivative, except for the component $S^4E \subset \mathfrak{sp}(n) \otimes \mathfrak{sp}(n)$, which corresponds to the curvature space of hyperkähler manifolds of dimension $4n$. This leads to a new proof of a result of [12], relating closedness of the fundamental four-form to the vanishing of the traceless Ricci tensor for $n > 1$.

We show that the intrinsic torsion of a qcm structure consists of three components: one is a symmetric tensor which can be identified with the Ricci tensor, one obstructs the integrability of the complement (or “vertical” distribution), and one is trivial, determined by the definitions.

Comparing our results with the literature, we recover the existence of both the Biquard and Duchemin connection on integrable qcm manifolds of arbitrary dimensions. We show that the “horizontal” part of the curvature of the Biquard connection is determined linearly by the qcm curvature and torsion; this indicates that formulae using this part of the curvature can be equally expressed in terms of the qcm curvature. In particular, we prove that the Ricci tensor of the Biquard connection can also be identified with the symmetric part of the intrinsic torsion.

1 Representations of $\text{Sp}(n)\text{Sp}(1)$

The structure group $\text{Sp}(n)\text{Sp}(1)$ plays a central rôle in qc geometry; whilst its representation theory is well understood (see [20, 18]), it will be useful to write down some explicit formulae for use in subsequent computations.

Consider the usual inclusion $\text{Sp}(n) \subset \text{Sp}(2n, \mathbb{C})$ obtained by identifying \mathbb{H}^n

Table 1: Roots of $\mathfrak{sp}(n)$

$\mathfrak{sp}(2n, \mathbb{C})$	$\mathbb{C} \otimes \mathfrak{sp}(n)$	root	> 0
$E_{i,n+j} + E_{j,n+i}$	$-\frac{1}{2}(E_{ij} + E_{ji})j + \frac{1}{2}i \otimes (E_{ij} + E_{ji})k$	$L_i + L_j, i \neq j$	yes
$E_{i,n+i}$	$-\frac{1}{2}E_{ii}j + \frac{1}{2}i \otimes E_{ii}k$	$2L_i$	yes
$E_{i,j} - E_{n+j,n+i}$	$\frac{1}{2}(E_{ij} - E_{ji}) - \frac{1}{2}i \otimes (E_{ij} + E_{ji})i$	$L_i - L_j$	$i < j$
$E_{n+i,j} + E_{n+j,i}$	$\frac{1}{2}(E_{ij} + E_{ji})j + \frac{1}{2}i \otimes (E_{ij} + E_{ji})k$	$-L_i - L_j, i \neq j$	no
$E_{n+i,i}$	$\frac{1}{2}E_{ii}j + \frac{1}{2}i \otimes E_{ii}k$	$-2L_i$	no

with \mathbb{C}^{2n} in such a way that multiplication on the left by j is \mathbb{C} -linear. This inclusion induces an identification of $\mathfrak{sp}(2n, \mathbb{C})$ with the complexification of $\mathfrak{sp}(n)$.

Denoting by E_{ij} the elementary matrix with 1 at the entry (i, j) , the Cartan subalgebra of $\mathfrak{sp}(n)$ is given as

$$\text{Span}\{H_1, \dots, H_n\}, \quad H_k = iE_{kk}$$

and it maps to the standard Cartan subalgebra of $\mathfrak{sp}(2n, \mathbb{C})$ by

$$H_k \rightarrow i(E_{kk} - E_{n+k, n+k}).$$

The weight lattice of the latter is generated by

$$L_k: \mathfrak{h}^* \rightarrow \mathbb{C}, \quad L_k(H_j) = i\delta_{kj}.$$

We shall denote by E the standard representation

$$E = \mathbb{C}^{2n} = \text{Span}\{v_1, \dots, v_{2n}\}$$

of $\mathfrak{sp}(2n, \mathbb{C})$, so that v_i has weight L_i and v_{n+i} has weight $-L_i$.

The second factor of the product $\text{Sp}(n)\text{Sp}(1)$ has $\mathfrak{sp}(1)$ as its Lie algebra. We shall fix a generator M of its weight lattice with $M(H_1) = -i$, and denote by

$$H = \mathbb{C}^2 = \text{Span}\{h_1, h_2\}$$

the standard representation, so that h_1 has weight $-M$ and h_2 has weight M . We can think of H as the $\mathfrak{sp}(1)$ -representation given by left multiplication on the quaternions, the identification being given by

$$a + jb \rightarrow ah_1 + bh_2, \quad a, b \in \mathbb{C}.$$

We fix the standard ordering for the roots of $\mathfrak{sp}(n)$, and declare M to be positive. This is summarized in Tables 1 and 2, which also contain a generator for each root space. The isomorphism $\mathbb{R}^4 = \mathbb{H}$ determined by the standard basis $\{1, i, j, k\}$ can be extended to an identification $\mathbb{R}^{4n} = \mathbb{C}^{2n} = \mathbb{H}^n$ via

$$e_{4(j-1)+1} = v_j, \quad e_{4(j-1)+2} = iv_j, \quad e_{4(j-1)+3} = v_{n+j}, \quad e_{4(j-1)+4} = -iv_{n+j}.$$

Here $\{e_1, \dots, e_{4n}\}$ is the standard basis of \mathbb{R}^{4n} ; the dual basis will be denoted by e^1, \dots, e^{4n} . This induces a representation of $\text{Sp}(n)\mathbb{H}^*$, and hence of its subgroup $\text{Sp}(n)\text{Sp}(1)$, via

$$(\text{Sp}(n) \times \mathbb{H}^*) \times \mathbb{H}^n \rightarrow \mathbb{H}^n, \quad (g, p) \cdot v = gvp^{-1}.$$

Table 2: Roots of $\mathfrak{sp}(1)$

$\mathfrak{sp}(2, \mathbb{C})$	$\mathbb{C} \otimes \mathfrak{sp}(1)$	root	> 0
$E_{1,2}$	$-\frac{1}{2}j + \frac{1}{2}i \otimes k$	$-2M$	no
$E_{2,1}$	$\frac{1}{2}j + \frac{1}{2}i \otimes k$	$2M$	yes

Table 3: Isomorphism of $\mathbb{H}^n \otimes \mathbb{C}$ with EH

$\mathbb{R}^{4n} \otimes \mathbb{C}$	EH	weight
$e_{4(j-1)+1} - ie_{4(j-1)+2}$	$v_j h_2$	$L_j + M$
$e_{4(j-1)j+1} + ie_{4(j-1)+2}$	$-v_{n+j} h_1$	$-L_j - M$
$e_{4(j-1)+3} - ie_{4(j-1)+4}$	$v_j h_1$	$L_j - M$
$e_{4(j-1)+3} + ie_{4(j-1)+4}$	$v_{n+j} h_2$	$-L_j + M$

This is a real representation whose complexification is well known to be isomorphic to $E \otimes H$, or EH (we shall omit tensor products signs in this context). By choosing a highest weight vector in both representations and applying subsequently negative roots to both sides, one obtains the explicit isomorphism of Table 3. Since \mathbb{R}^{4n} is isomorphic to its dual via $e_i \rightarrow e^i$, we shall also identify EH with $(\mathbb{R}^{4n})^*$ as an $\mathrm{Sp}(n)\mathrm{Sp}(1)$ -module. Thus, we will represent both $e_1 - ie_2$ and $e^1 - ie^2$ by $v_1 h_2$; this has the consequence that $v_1 h_2 \lrcorner v_1 h_2 = 0$, but $v_1 h_2 \lrcorner v_{n+1} h_1 = -2$.

We also identify $\Lambda^2(\mathbb{R}^{4n})^* \cong \mathfrak{so}(4n)$ by making $\Lambda^2(\mathbb{R}^{4n})^*$ act on \mathbb{R}^{4n} as

$$\Lambda^2(\mathbb{R}^{4n})^* \times \mathbb{R}^{4n} \rightarrow \mathbb{R}^{4n}, \quad (\alpha, v) \rightarrow v \lrcorner \alpha.$$

In other words, $e^i \wedge e^j$ is identified with $e^i \otimes e_j - e^j \otimes e_i = E_{ji} - E_{ij}$. Then the action of \mathbb{H}^* on \mathbb{R}^{4n} induces the Lie algebra homomorphism

$$\begin{aligned} \mathrm{Lie}(\mathbb{H}^*) &= \mathfrak{sp}(1) \oplus \mathbb{R} \rightarrow \mathrm{Span}\{\mathrm{Id}\} \oplus \mathfrak{so}(4n), \\ 1 &\rightarrow -\mathrm{Id}, \quad i \rightarrow -\omega_1, \quad j \rightarrow -\omega_2, \quad k \rightarrow -\omega_3, \end{aligned} \tag{1}$$

where (as in [19]) the ω_s satisfy

$$\begin{aligned} \omega_1 &= \frac{1}{2}i(v_j h_2 \wedge v_{n+j} h_1 + v_j h_1 \wedge v_{n+j} h_2), \\ \omega_2 + i\omega_3 &= v_j h_2 \wedge v_{n+j} h_2, \quad \omega_2 - i\omega_3 = v_j h_1 \wedge v_{n+j} h_1. \end{aligned}$$

Here and in the sequel, e^{ij} or $e^{i,j}$ stands for the wedge product $e^i \wedge e^j$, and summation over double indices is implied.

It is easy to deduce that the subspace

$$\mathrm{Span}\{\omega_1, \omega_2, \omega_3\} \subset \Lambda^2(\mathbb{R}^{4k})^*$$

is fixed under \mathbb{H}^* action; in fact, it is isomorphic to $\mathrm{Im}\mathbb{H} = \mathfrak{sp}(1)$ as an \mathbb{H}^* -module via (1), where \mathbb{H}^* acts on $\mathrm{Im}\mathbb{H}$ via

$$\widetilde{\mathrm{Ad}}: \mathbb{H}^* \rightarrow \mathrm{End}_{\mathbb{R}}(\mathrm{Im}\mathbb{H}), \quad \widetilde{\mathrm{Ad}}(p)q = pq\bar{p}.$$

The tensor product of two complex representations of $\mathrm{Sp}(n)$ and $\mathrm{Sp}(1)$ has a real structure when both factors have a real or quaternionic structure; all real representations of $\mathrm{Sp}(n)\mathrm{Sp}(1)$ can be written in this way. We denote by $\Lambda_0^k E$ the irreducible representation of $\mathrm{Sp}(n)$ with highest weight $L_1 + \dots + L_k$, and more generally let V_{l_1, \dots, l_k} be the irreducible representation with highest weight $l_1 L_1 + \dots + l_k L_k$. It will be understood that $\Lambda_0^k E$ and V_{l_1, \dots, l_k} represent the zero vector space when $n < k$. We have the following decompositions (see [21]):

$$S^2 E \otimes E = S^3 E \oplus E \oplus V_{21}, \quad \Lambda_0^2 E \otimes E = \Lambda_0^3 E \oplus E \oplus V_{21},$$

$$\Lambda^3(EH) = \begin{cases} \Lambda_0^3 E S^3 H + V_{21} H + E(S^3 H + H) & n > 1 \\ EH & n = 1 \end{cases}.$$

We shall also need:

Lemma 1. *The following isomorphisms of $\mathrm{Sp}(n)$ -modules hold:*

$$\Lambda_0^2 E \otimes S^2 E = V_{31} + V_{211} + \Lambda_0^2 E + S^2 E,$$

$$S^2 E \otimes S^2 E = S^4 E + V_{31} + V_{22} + S^2 E + \Lambda_0^2 E + \mathbb{R}.$$

Proof. The fact that each module on the right hand side appears in the tensor product can be shown by exhibiting a highest weight vector. Moreover, the Weyl Character Formula (see (24.19) in [8]) gives

$$\dim V_{211} = \frac{1}{2}(n+1)(2n+1)(2n-1)(n-2), \quad \dim V_{l,1} = \frac{l(2n-2)}{l+2n-1} \binom{l+2n}{l+1},$$

$$\dim V_{l,2} = \frac{(l^2 + 2ln - 2n - 1)}{2} \binom{l+2n-2}{l+1}.$$

It is now a matter of showing that dimensions add up. \square

2 Distributions as G -structures

In this section we show that qc structures can be characterized in terms of intrinsic torsion using the language of G -structures. The structure group considered in this section, denoted by Q , is inherent in the definition of qc structures, but we will see in later sections that smaller structure groups capture the geometry more completely.

A *qc structure* on a manifold of dimension $4n+3$ is a distribution \mathcal{D} of rank $4n$ which can locally be defined as

$$\mathcal{D} = \ker \eta_1 \cap \ker \eta_2 \cap \ker \eta_3,$$

where the η_s are one-forms such that $(d\eta_1, d\eta_2, d\eta_3)$ restricted to \mathcal{D} are compatible with an almost quaternion Hermitian metric. The latter condition can be rephrased by requiring the existence at each point x of a frame

$$u: \mathbb{R}^{4n} \rightarrow \mathcal{D}_x, \quad d\eta_s(u(e_a), u(e_b)) = \omega_s(e_a, e_b),$$

where the ω_s are as in Section 1. The qc structure is said to be *integrable* (in the sense of Duchemin) if in addition at each point there are vectors R_s such that

$$(R_s \lrcorner d\eta_r)|_{\mathcal{D}} + (R_r \lrcorner d\eta_s)|_{\mathcal{D}} = 0;$$

this condition turns out to be automatic for $n > 1$ (see [6]). Notice that this is not related to integrability of the distribution \mathcal{D} , nor to integrability in the sense of G -structures.

Whether qc or not, a codimension three distribution can be viewed as a Q -structure, where

$$Q = \mathrm{GL}(4n, \mathbb{R}) \times \mathrm{GL}(3, \mathbb{R}) \times \mathrm{Hom}(\mathbb{R}^3, \mathbb{R}^{4n})$$

is the stabilizer of a point in the Grassmannian of $4n$ -planes in \mathbb{R}^{4n+3} . We shall denote by T the Q -module obtained by letting Q act on \mathbb{R}^{4n+3} via

$$(g, p, h): \begin{pmatrix} v \\ w \end{pmatrix} \rightarrow \begin{pmatrix} gv + h(p(w)) \\ p(w) \end{pmatrix},$$

denoting by e_1, \dots, e_{4n+3} the standard basis of T , and by e^1, \dots, e^{4n+3} the dual basis of T^* .

We shall write

$$T = \mathbb{R}^{4n} \oplus \mathbb{R}^3 = V \oplus W;$$

notice that V and W^* are Q -submodules of T and T^* respectively, but W and V^* , despite having a natural Q -module structure, are only $\mathrm{GL}(4n, \mathbb{R}) \times \mathrm{GL}(3, \mathbb{R})$ -submodules. On the other hand, T/V is a Q -module isomorphic to W^* . It will be convenient to denote by w_s the image in T/V of e_{4n+s} , and the e^{4n+s} by w^s . We shall use the contracted notation w^{rs} , w_{rs} for the wedge product of these elements as well.

The intrinsic torsion of a Q -structure takes values in a Q -module defined as the cokernel of the map

$$\partial_Q: T^* \otimes \mathfrak{q} \rightarrow \Lambda^2 T^* \otimes T$$

obtained by restriction from the map

$$\partial: T^* \otimes \mathfrak{gl}(T) \rightarrow \Lambda^2 T^* \otimes T, \quad e^k \otimes (e^i \otimes e_j) \rightarrow e^{ki} \otimes e_j.$$

Lemma 2. *The alternating map ∂_Q fits into the exact sequence of Q -modules*

$$0 \rightarrow S^2 T^* \otimes V + S^2 W^* \otimes T \rightarrow T^* \otimes \mathfrak{q} \xrightarrow{\partial_Q} \Lambda^2 T^* \otimes T \rightarrow \Lambda^2 V^* \otimes \frac{T}{V} \rightarrow 0.$$

Proof. Decomposing $T^* \otimes \mathfrak{q}$ into $\mathrm{GL}(4k, \mathbb{R}) \times \mathrm{GL}(3, \mathbb{R})$ -submodules, ∂_Q determines three isomorphisms

$$\begin{aligned} W^* \otimes \mathfrak{gl}(4n, \mathbb{R}) &\xrightarrow{\cong} W^* \otimes V^* \otimes V, & V^* \otimes \mathfrak{gl}(3, \mathbb{R}) &\xrightarrow{\cong} W^* \otimes V^* \otimes W, \\ V^* \otimes \mathrm{Hom}(\mathbb{R}^3, \mathbb{R}^{4n}) &\xrightarrow{\cong} W^* \otimes V^* \otimes V, \end{aligned}$$

and three exact sequences

$$\begin{aligned} 0 \rightarrow S^2 V^* \otimes V &\rightarrow V^* \otimes \mathfrak{gl}(4k, \mathbb{R}) \xrightarrow{\partial_Q} \Lambda^2 V^* \otimes V \rightarrow 0 \\ 0 \rightarrow S^2 W^* \otimes W &\rightarrow W^* \otimes \mathfrak{gl}(3, \mathbb{R}) \xrightarrow{\partial_Q} \Lambda^2 W^* \otimes W \rightarrow 0 \\ 0 \rightarrow S^2 W^* \otimes V &\rightarrow W^* \otimes \mathrm{Hom}(\mathbb{R}^3, \mathbb{R}^{4k}) \xrightarrow{\partial_Q} \Lambda^2 W^* \otimes V \rightarrow 0 \end{aligned}$$

Thus, the cokernel of ∂_Q is $\Lambda^2 V^* \otimes W$, and the kernel is

$$S^2 V^* \otimes V + S^2 W^* \otimes T + W^* \otimes V^* \otimes V,$$

which is a Q -submodule of $S^2 T^* \otimes T$ that can be written as the sum of the two submodules $S^2 T^* \otimes V$ and $S^2 W^* \otimes T$ (intersecting non-trivially). \square

This shows that the intrinsic torsion is a map

$$\Theta^Q: P \rightarrow \Lambda^2 V^* \otimes \frac{T}{V}.$$

We can now characterize qc geometry as follows:

Proposition 3. *The distribution associated to a Q -structure is integrable if and only if Θ^Q is identically zero. It is qc if and only if Θ^Q takes values in the Q -orbit of*

$$\omega_1 \otimes w^{23} + \omega_2 \otimes w^{31} + \omega_3 \otimes w^{12}.$$

Proof. Let P be a Q -structure on M . We can represent \mathfrak{q} as a space of block matrices, and decompose connection form, tautological form and torsion as

$$\omega = \begin{pmatrix} \omega_V & * \\ 0 & \omega_W \end{pmatrix}, \quad \theta = \begin{pmatrix} \theta_V \\ \theta_W \end{pmatrix}, \quad \Theta = \begin{pmatrix} \Theta_V \\ \Theta_W \end{pmatrix}.$$

The Q -structure P determines at each point $x \in M$ a projection

$$h: \Lambda^2 T_x^* M \rightarrow \Lambda^2(\mathcal{D}_x)^*,$$

where \mathcal{D} is the distribution determined by Q . Then, working with a local section s of P , we can identify the intrinsic torsion with

$$h(s^* \Theta_W) = h(s^*(d\theta_W + \omega_W \wedge \theta_W)) = h(s^* d\theta_W).$$

By the Frobenius theorem, \mathcal{D} is integrable if and only if the ideal generated by $s^* \theta_{4n+1}$, $s^* \theta_{4n+2}$ and $s^* \theta_{4n+3}$ is a differential ideal; this is equivalent to

$$h(s^*(d\theta_W)) = 0.$$

On the other hand, $h(s^* \Theta_W)$ is zero if and only if Θ^Q is zero.

Similarly, for the second part of the statement, the qc condition is equivalent to

$$h(s^* \Theta_W) = \omega_1 \otimes w_1 + \omega_2 \otimes w_2 + \omega_3 \otimes w_3$$

for an appropriately chosen section s . \square

A recurring phenomenon in the study of G -structures is that the intrinsic torsion is determined by the exterior derivative of some invariant forms. The structure group Q has no invariant forms; there is, however, an invariant *vector-bundle-valued* differential form whose exterior covariant derivative determines the intrinsic torsion.

This relation is best expressed in the language of tensorial forms (see e.g. [15]). If P is a Q -structure and S a Q -module, a differential form in $\Omega^k(P, S)$ is called pseudotensorial if it is invariant under the natural action of Q on

$\Omega^k(P, S) \cong \Omega^k(P) \otimes S$. It is called tensorial if in addition it is horizontal, i.e. the interior product with any fundamental vector field is identically zero.

Given a connection and a pseudotensorial form α in $\Omega^k(P, S)$, we denote by $D\alpha$ its exterior covariant derivative, as a tensorial $k+1$ -form. In particular, if θ is the tautological form, $\Theta = D\theta$ is the torsion. To any tensorial form α in $\Omega^k(P, S)$ one can associate an equivariant map

$$\alpha_\theta: P \rightarrow \Lambda^k T^* \otimes S, \quad \langle \alpha_\theta, \frac{1}{k!} \theta \wedge \cdots \wedge \theta \rangle = \alpha,$$

where the angle brackets represent the standard contraction

$$\Lambda^k T^* \otimes \Lambda^k T \rightarrow \mathbb{R}, \quad \langle \eta^1 \wedge \cdots \wedge \eta^k, X_1 \wedge \cdots \wedge X_k \rangle = \det(\eta^i(X_j)).$$

With this choice of constants, if G is the trivial group and $\alpha = \theta_1 \wedge \cdots \wedge \theta_k$, then α_θ is the constant map $\alpha_\theta \equiv e^{1, \dots, k}$.

We shall denote by $\nabla\alpha$ the covariant derivative of α , i.e.

$$\nabla\alpha = D(\alpha_\theta) \in \Omega^1(P, \Lambda^k T^* \otimes S).$$

Given two tensorial forms $\alpha \in \Omega^h(P, T)$, $\beta \in \Omega^k(P, S)$, where $\alpha = \alpha^i \otimes e_i$, one can define the interior product $\alpha \lrcorner \beta$ as the tensorial, S -valued $h+k-1$ -form

$$(\alpha \lrcorner \beta)_u = \alpha_u^i \wedge (X_i \lrcorner \beta_u), \quad \pi_{*u}(X_i) = u(e_i);$$

for $h=0$, this is the usual interior product. In this notation, if α is a tensorial k -form, then

$$D\alpha = \langle \nabla\alpha, \frac{1}{k!} \theta \wedge \cdots \wedge \theta \rangle + \Theta \lrcorner \alpha. \quad (2)$$

The structure group Q fixes a tensor

$$w^{123} \otimes w_{123} \in \Lambda^3 W^* \otimes \Lambda^3(T/V) \subset \Lambda^3 T^* \otimes \kappa^*,$$

where we have set $\kappa = \Lambda^3 W^*$. Accordingly, on a Q -structure P the associated tensorial 3-form

$$\sigma \in \Omega^3(P, \kappa^*), \quad \sigma_\theta \equiv w^{123} \otimes w_{123}$$

is Q -invariant, hence parallel.

Proposition 4. *Fix a connection on a Q -structure P . Then the intrinsic torsion Θ^Q is given by the composition*

$$P \xrightarrow{(D\sigma)_\theta} \Lambda^2 T^* \wedge \Lambda^2 W^* \otimes \kappa^* \xrightarrow{p} \Lambda^2 V^* \otimes \Lambda^2 W^* \otimes \kappa^* \xrightarrow{c} \Lambda^2 V^* \otimes \frac{T}{V}$$

where p is induced by the restriction map $\Lambda^2 T^* \rightarrow \Lambda^2 V^*$ and c is the contraction induced by interior product.

Proof. By (2), since σ is parallel,

$$\Theta \lrcorner \sigma = D\sigma;$$

on the other hand $(\Theta \lrcorner \sigma)_\theta$ is obtained from $\Theta_\theta \otimes \sigma_\theta$ via a contraction

$$(\Lambda^2 T^* \otimes W) \otimes \Lambda^3 W^* \otimes \kappa^* \rightarrow \Lambda^2 T^* \wedge \Lambda^2 W^* \otimes \kappa^*.$$

This shows that $(D\sigma)_\theta$ takes values in $\Lambda^2 T^* \wedge \Lambda^2 W^* \otimes \kappa^*$, so the composition appearing in the statement is well defined.

It is now straightforward to verify that $c \circ p \circ (D\sigma)_\theta$ coincides with the projection to $\text{Coker } \partial_B$ of the torsion Θ . \square

This link between intrinsic torsion and tensorial forms is a recurrent feature of qc geometry; it will be used in Sections 5 and 7 to prove vanishing conditions on the intrinsic torsion via the Bianchi identity.

3 Examples

In this section we recall three explicit examples of qc structures, which will be used in the rest of the paper for reference. They can be seen as the “space forms” of qc geometry, corresponding to the case of positive, negative and zero scalar curvature.

Example 5. Consider the sphere as a homogeneous space G/H , where

$$G = \mathrm{Sp}(n+1)\mathrm{Sp}(1), \quad H = \mathrm{Sp}(n)\mathrm{Sp}(1).$$

The Lie algebra of G is

$$\mathfrak{g} = \left\{ \left(\begin{pmatrix} a & b \\ -\bar{b}^T & d \end{pmatrix}, q \right) \mid a \in \mathfrak{sp}(n), b \in \mathbb{H}^n, d, q \in \mathrm{Im} \mathbb{H} \right\},$$

and \mathfrak{h} is defined by $b = 0, d = q$.

We set

$$\begin{aligned} w_1 &= (iE_{nn}, -i) & w_2 &= (jE_{nn}, -j) & w_3 &= (kE_{nn}, -k) \\ e_{4l+1} &= -E_{n,l+1} + E_{l+1,n}, & e_{4l+2} &= i(E_{n,l+1} + E_{l+1,n}), \\ e_{4l+3} &= j(E_{n,l+1} + E_{l+1,n}), & e_{4l+4} &= k(E_{n,l+1} + E_{l+1,n}), \quad 0 \leq l \leq n-1. \end{aligned}$$

Then $e_1, \dots, e_{4n}, w_1, w_2, w_3$ define a frame on a complement \mathfrak{m} of \mathfrak{h} in \mathfrak{g} , hence an $\mathrm{Sp}(n)\mathrm{Sp}(1)$ -structure; this is invariant under H , so it defines a global structure on G/H . We compute

$$de^a|_{\Lambda^2 \mathfrak{m}} = (e_a \lrcorner \omega_s) \wedge w^s, \quad dw^s|_{\Lambda^2 \mathfrak{m}} = \omega_s.$$

The projection to \mathfrak{h} defines a connection; its torsion is

$$\Theta = (e_a \lrcorner \omega_s) \wedge w^s \otimes e_a + \omega_s \otimes w_s.$$

This shows immediately that $\Theta^Q = \omega_s \otimes w_s$, so by Proposition 3 this is a qc structure. Consistently with Proposition 4,

$$D\sigma = (\omega_1 \wedge w^{23} + \omega_2 \wedge w^{31} + \omega_3 \wedge w^{12}) \otimes w_{123}.$$

We note for future reference that the curvature of this connection is

$$\Omega = - \sum_{a < b} e^{ab} \otimes e^a \wedge e^b - \sum_{a < b, s} e^{ab} \otimes e_a \lrcorner \omega_s \wedge e_b \lrcorner \omega_s - (\omega_s - 2w_s \lrcorner w^{123}) \otimes \omega_s.$$

Example 6. A similar example is the homogeneous space G/H , where

$$G = \mathrm{Sp}(n, 1)\mathrm{Sp}(1), \quad H = \mathrm{Sp}(n)\mathrm{Sp}(1).$$

In this case we choose a complement \mathfrak{m} spanned by

$$\begin{aligned} w_1 &= (-iE_{nn}, i), & w_2 &= (-jE_{nn}, j), & w_3 &= (-kE_{nn}, k), \\ e_{4l+1} &= E_{n,l+1} + E_{l+1,n}, & e_{4l+2} &= i(-E_{n,l+1} + E_{l+1,n}), \\ e_{4l+3} &= j(-E_{n,l+1} + E_{l+1,n}), & e_{4l+4} &= k(-E_{n,l+1} + E_{l+1,n}). \end{aligned}$$

The connection defined by the projection has torsion

$$\Theta = -(e_a \lrcorner \omega_s) \wedge w^s \otimes e_a + \omega_s \otimes w_s$$

and curvature

$$\Omega = \sum_{a < b} e^{ab} \otimes e^a \wedge e^b + \sum_{a < b, s} e^{ab} \otimes e_a \lrcorner \omega_s \wedge e_b \lrcorner \omega_s + (\omega_s + 2w_s \lrcorner w^{123}) \otimes \omega_s.$$

Example 7. The remaining example to consider is the Heisenberg group, which is characterized by the existence of a left-invariant basis of one-forms e^1, \dots, e^{4n+3} with

$$de^a = 0, \quad de^{4n+s} = \omega_s,$$

where the ω_s are defined in terms of the e^a in the usual way. In this case we can use the $(-)$ connection, i.e. the connection for which the indicated frame is parallel. It has torsion Θ_0 and curvature zero.

A common feature of these examples is the presence of a natural connection, which will play a rôle in Section 9. Other homogeneous examples appear in [4, 3]. Hypersurfaces in quaternionic manifolds also give rise to qc structures under certain conditions, as shown in [5, 14].

4 Qc dialectics

In the language of Proposition 3, a qc structure is a Q -structure whose intrinsic torsion is in the Q -orbit of Θ_0^Q , which we define as the image in $\Lambda^2 V^* \otimes \frac{T}{V}$ of

$$\Theta_0 = \omega_1 \otimes e_{4n+1} + \omega_2 \otimes e_{4n+2} + \omega_3 \otimes e_{4n+3} \in \Lambda^2 T^* \otimes T.$$

This means that any qc structure has a natural reduction to B , where B denotes the stabilizer in Q of Θ_0^Q . In this section we determine B and study the space Coker ∂_B in which the intrinsic torsion of B -structures takes values.

The forms ω_s are fixed by the action of $\mathrm{Sp}(n)$. More generally, we say three elements $\gamma_1, \gamma_2, \gamma_3$ of $\Lambda^2 \mathbb{R}^{4n}$ are compatible with an $\mathrm{Sp}(n)$ -structure if some linear isomorphism of \mathbb{R}^{4n} maps each γ_s in ω_s .

Lemma 8. *Let $\Gamma \subset \Lambda^2 \mathbb{R}^{4n}$ be the space spanned by three 2-forms compatible with an $\mathrm{Sp}(n)$ -structure. Then the structure is uniquely determined by Γ , up to \mathbb{H}^* action.*

Proof. Let $\gamma_1, \gamma_2, \gamma_3$ be compatible with an $\mathrm{Sp}(n)$ -structure. Each two-form γ_s defines an isomorphism

$$\gamma_s: \mathbb{R}^{4k} \rightarrow (\mathbb{R}^{4k})^*, \quad X \rightarrow X \lrcorner \gamma_s.$$

Three complex structures are induced on \mathbb{R}^{4n} by

$$J_3 = -\gamma_2^{-1}\gamma_1 = \gamma_1^{-1}\gamma_2$$

and cyclic permutations. It follows that

$$(a\gamma_1 + b\gamma_2 + c\gamma_3)^{-1} = \frac{1}{a^2 + b^2 + c^2} a\gamma_1^{-1} + b\gamma_2^{-1} + c\gamma_3^{-1}.$$

We can therefore define a linear map on the space Γ spanned by the γ_s ,

$$\Gamma \rightarrow \text{Hom}((\mathbb{R}^{4n})^*, \mathbb{R}^{4n}), \quad \gamma \rightarrow \gamma^\dagger = \|\gamma\|^2 \gamma^{-1}, \quad \gamma \neq 0.$$

By construction

$$\alpha^\dagger \beta + \beta^\dagger \alpha = \langle \alpha, \beta \rangle \text{Id}, \quad \alpha, \beta \in \Gamma.$$

Take three elements α, β, γ in Γ and assume they are also compatible with an $\text{Sp}(n)$ -structure. Then

$$\alpha^{-1}\beta + \beta^{-1}\alpha = \|\alpha\|^{-2} \alpha^\dagger \beta + \|\beta\|^{-2} \beta^\dagger \alpha = 0,$$

leading to

$$\|\alpha\|^{-2} \alpha^\dagger \beta - \|\beta\|^{-2} \alpha^\dagger \beta + \|\beta\|^{-2} \langle \alpha, \beta \rangle \text{Id} = 0.$$

Now observe that $\alpha^\dagger \beta$ is not a multiple of the identity whenever α, β are linearly independent. Thus, α, β are orthogonal with the same norm.

Summing up, α, β and γ form an orthogonal basis of Γ of elements with the same norm; this basis is positively oriented by construction, and so uniquely determined up to \mathbb{H}^* action. \square

Recall from Section 1 that both \mathbb{H}^n and $\text{Im } \mathbb{H}$ (and therefore its dual $(\text{Im } \mathbb{H})^*$) are equipped with a left \mathbb{H}^* -action. The identification $\mathbb{R}^{4n} = \mathbb{H}^n$ induces

$$\rho: \text{GL}(n, \mathbb{H}) \times \mathbb{H}^* \rightarrow \text{GL}(4n, \mathbb{R}), \quad \rho(g, p)(v) = gvp^{-1}.$$

Proposition 9. *The stabilizer in Q of Θ_0^Q is the group*

$$B = \text{Sp}(n)\mathbb{H}^* \times \text{Hom}(W, V),$$

where the first factor represents the image of the homomorphism

$$\iota: \text{Sp}(n) \times \mathbb{H}^* \rightarrow \text{GL}(V) \times \text{GL}(W), \quad (g, p) \rightarrow (\rho(g, p), \widetilde{\text{Ad}}(p))$$

having implicitly identified W with $(\text{Im } \mathbb{H})^*$.

Proof. It is clear that $\text{Sp}(n) \times \text{Hom}(W, V)$ fixes Θ_0^Q . As for \mathbb{H}^* , observe that ι makes

$$\text{Span}\{\omega_1, \omega_2, \omega_3\} \subset \Lambda^2 V^*$$

isomorphic to $\text{Im } \mathbb{H}$ as a representation of \mathbb{H}^* (see Section 1). Thus, identifying W with $(\text{Im } \mathbb{H})^*$, Θ_0^Q is in the trivial submodule of $\text{Im } \mathbb{H} \otimes (\text{Im } \mathbb{H})^*$.

Conversely, we must show that the stabilizer of Θ_0^Q in $\text{GL}(V) \times \text{GL}(W)$ is $\iota(\text{Sp}(n) \times \mathbb{H}^*)$. In fact, the stabilizer of $\text{Span}\{\omega_1, \omega_2, \omega_3\}$ in $\text{GL}(V) \times \text{GL}(W)$ is $\text{Sp}(n)\mathbb{H}^* \times \text{GL}(W)$ by Lemma 8. On the other hand Θ_0^Q determines an isomorphism

$$W^* \rightarrow \text{Span}\{\omega_1, \omega_2, \omega_3\}$$

and the subgroup of $\text{Sp}(n)\mathbb{H}^* \times \text{GL}(W)$ that fixes this isomorphism is precisely the image of ι . \square

Remark. The identification of W with $(\text{Im } \mathbb{H})^*$ has the consequence that scalars $\lambda \in \mathbb{R}^* \subset B$ act on T as $\lambda^{-2}\text{Id}_W + \lambda^{-1}\text{Id}_V$.

We can now refine the second part of Proposition 3 in the following way:

Corollary 10. *Every qc Q -structure has a unique B -reduction P such that*

$$\Theta^Q(u) = \Theta_0, \quad u \in P.$$

We shall refer to such a structure as a *qc B -structure*. This leads us to consider the intrinsic torsion of B -structures. Consider the diagram

$$\begin{array}{ccccccc} T^* \otimes \mathfrak{b} & \xrightarrow{\partial_B} & \Lambda^2 T^* \otimes T & \longrightarrow & \text{Coker } \partial_B & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow r & & \\ T^* \otimes \mathfrak{q} & \xrightarrow{\partial_Q} & \Lambda^2 T^* \otimes T & \longrightarrow & \Lambda^2 V^* \otimes \frac{T}{V} & \longrightarrow & 0 \end{array}$$

By construction r is B -equivariant, and maps the intrinsic torsion of a B -structure to its Q -intrinsic torsion, i.e. the intrinsic torsion of the induced Q -structure. Accordingly, a qc B -structure has intrinsic torsion in $r^{-1}(\Theta_0^Q)$. In fact, we will see in Section 5 that the intrinsic torsion of a qc B -structure is forced to lie in a much smaller space. For the moment, we use the above diagram to study $\text{Coker } \partial_B$.

Lemma 11. *The projection $\Lambda^2 T^* \otimes T \rightarrow \text{Coker } \partial_B$ induces by restriction B -equivariant maps*

$$p_{VVV}: \Lambda^2 V^* \otimes V \rightarrow \text{Coker } \partial_B \quad p_{VWW}: V^* \otimes W^* \otimes W \rightarrow \text{Coker } \partial_B$$

such that

$$\ker r = \text{Im } p_{VVV} + \text{Im } p_{VWW} \quad (\text{not a direct sum}).$$

Proof. We first prove that

$$\text{Im } \partial_B \supset \Lambda^2 W^* \otimes T + V^* \otimes W^* \otimes V. \quad (3)$$

Indeed, the maps

$$\partial_B: V^* \otimes \text{Hom}(W, V) \rightarrow \Lambda^{1,1} \otimes V, \quad \partial_B: W^* \otimes \text{Hom}(W, V) \rightarrow \Lambda^{0,2} \otimes V$$

are obviously surjective; moreover, the composition map

$$W \otimes \mathfrak{sp}(1) \xrightarrow{\partial_B} \Lambda^2 W^* \otimes W + (V \otimes W) \otimes V \rightarrow \Lambda^2 W^* \otimes W$$

is an isomorphism. This proves (3).

The projection $\Lambda^2 T^* \otimes T \rightarrow \text{Coker } \partial_B$ induces a B -equivariant map

$$p: \Lambda^2 T^* \otimes V \rightarrow \text{Coker } \partial_B;$$

by (3), p factors through

$$\Lambda^2 T^* \otimes V \rightarrow \Lambda^2 V^* \otimes V \rightarrow \text{Coker } \partial_B.$$

Composing on the left with the inclusion $\Lambda^2 V^* \otimes V \rightarrow \Lambda^2 T^* \otimes V$ (which is not B -equivariant) gives an equivariant map.

Similarly, the map

$$T^* \wedge W^* \otimes T \rightarrow \text{Coker } \partial_B$$

factors through

$$T^* \wedge W^* \otimes T \rightarrow V^* \otimes W^* \otimes W \rightarrow \text{Coker } \partial_B.$$

Finally, (3) implies that the map

$$\Lambda^2 V^* \otimes T + V^* \otimes W^* \otimes W \rightarrow \text{Coker } \partial_B$$

is surjective, yielding the final part of the statement. \square

Remark. It is not possible to construct an analogous B -equivariant map

$$p_{VW}: \Lambda^2 V^* \otimes W \rightarrow \text{Coker } \partial_B.$$

Indeed, the smallest B -module in $\Lambda^2 T^* \otimes T$ containing $\Lambda^2 V^* \otimes W$ is $\Lambda^2 V^* \otimes T$, and the image of $\Lambda^2 V^* \otimes T$ in the cokernel is bigger than the image of $\Lambda^2 V^* \otimes W$.

It follows from the above remark that we cannot think of Θ_0^Q as a ‘‘component’’ of the B -intrinsic torsion: we have to express the relation in terms of a short exact sequence.

Proposition 12. *There is an exact sequence of B -modules*

$$0 \rightarrow W_1 \oplus W_2 \xrightarrow{i} \text{Coker } \partial_B \xrightarrow{r} \text{Coker } \partial_Q \rightarrow 0 \quad (4)$$

where

$$W_1 = V^* \otimes \mathfrak{sp}(n)^\perp \cong \begin{cases} (V_{21} + \Lambda_0^3 E + 2E)(S^3 H + H), & n > 1, \\ ES^3 H + EH, & n = 1, \end{cases}$$

$$W_2 = V^* \otimes S_0^2(W) \cong ES^3 H + ES^5 H,$$

and the restriction of i to each component is given by restricting the B -equivariant alternating maps

$$\partial_1: V^* \otimes \mathfrak{gl}(V) \rightarrow \Lambda^2 V^* \otimes V, \quad \partial_2: V^* \otimes \mathfrak{gl}(W) \rightarrow V^* \otimes W^* \otimes W.$$

In this statement, $\mathfrak{sp}(n)^\perp$ denotes the orthogonal complement of $\mathfrak{sp}(n)$ in $\mathfrak{so}(4n)$.

Proof. As an $\text{Sp}(n)\mathbb{H}^*$ -module, the Lie algebra of B decomposes as

$$\mathfrak{b} = \mathfrak{sp}(n) \oplus \mathfrak{sp}(1) \oplus \mathbb{R} \oplus \text{Hom}(W, V),$$

where the inclusion of $\mathfrak{sp}(1) \oplus \mathbb{R}$ in $\mathfrak{gl}(T)$ is given by

$$p \rightarrow (R_{-p}, \widetilde{\text{ad}}(p)), \quad p \in \mathfrak{sp}(1); \quad \lambda \rightarrow (-\lambda \text{Id}, -2\lambda \text{Id}), \quad \lambda \in \mathbb{R}.$$

By Lemma 11

$$\ker r = \frac{\Lambda^2 V^* \otimes V + V^* \otimes W^* \otimes W}{\text{Im } \partial_B \cap (\Lambda^2 V^* \otimes V + V^* \otimes W^* \otimes W)} = \frac{\Lambda^2 V^* \otimes V + V^* \otimes W^* \otimes W}{\partial_B(V^* \otimes \mathfrak{sp}(n)\mathbb{H})};$$

using the fact that ∂_2 is injective, the snake lemma applied to

$$\begin{array}{ccccccc} 0 & \longrightarrow & V^* \otimes \mathfrak{sp}(n) & \longrightarrow & V^* \otimes \mathfrak{sp}(n)\mathbb{H} & \longrightarrow & V^* \otimes \mathbb{H} \longrightarrow 0 \\ & & \downarrow \partial_1 & & \downarrow \partial & & \downarrow \partial_2 \\ 0 & \longrightarrow & \Lambda^2 V^* \otimes V & \longrightarrow & \Lambda^2 V^* \otimes V + V^* \otimes W^* \otimes W & \longrightarrow & V^* \otimes W^* \otimes W \longrightarrow 0 \end{array}$$

yields

$$0 \rightarrow W_1 \rightarrow \ker r \rightarrow W_2 \rightarrow 0.$$

This sequence splits by Lemma 11. Now observe that $\text{Hom}(W, V)$ acts trivially on W_1 and W_2 , and the component \mathbb{R} acts as a multiple of the identity on $\Lambda^2 V^* \otimes V + V^* \otimes W^* \otimes W$. The decomposition of W_1 and W_2 into irreducible B -modules is therefore the same as the decomposition into $\text{Sp}(n)\text{Sp}(1)$ -modules. \square

Remark. An alternative description can be obtained by applying the snake lemma to

$$\begin{array}{ccccccc} 0 & \longrightarrow & T^* \otimes \mathfrak{b} & \longrightarrow & T^* \otimes \mathfrak{q} & \longrightarrow & T^* \otimes \frac{\mathfrak{q}}{\mathfrak{b}} \longrightarrow 0 \\ & & \downarrow \partial_{\mathfrak{b}} & & \downarrow \partial_{\mathfrak{q}} & & \downarrow \\ 0 & \longrightarrow & \Lambda^2 T^* \otimes T & \longrightarrow & \Lambda^2 T^* \otimes T & \longrightarrow & 0 \end{array}$$

giving an exact sequence

$$\ker \partial_Q \xrightarrow{\alpha} T^* \otimes \frac{\mathfrak{q}}{\mathfrak{b}} \rightarrow \text{Coker } \partial_B \rightarrow \text{Coker } \partial_Q \rightarrow 0$$

This means that

$$\text{Coker } \alpha = W_1 + W_2,$$

i.e. we can think of the W_i as the components of $\text{Coker } \alpha$.

Corollary 13. *The kernel of ∂_B is $S^2 W^* \otimes V + (S^2 E S^2 H + S^2 H)$, where the second summand lies diagonally in $W^* \otimes (\mathfrak{sp}(n) + \mathbb{H}) + V^* \otimes \text{Hom}(W, V)$.*

Proof. By Proposition 12, the kernel of ∂_B is an $\text{Sp}(n)\text{Sp}(1)$ -module of the same dimension as $S^2 W^* \otimes V + S^2 H(S^2 E \oplus \mathbb{R})$, and clearly it contains $S^2 W \otimes V$. The restriction of ∂ to $V^* \otimes \text{Hom}(W, V)$ is injective with image $V^* \otimes W^* \otimes V$, so it contains $\partial_1(W^* \otimes (\mathfrak{sp}(n) + \mathbb{H}))$. Taking the kernel of ∂_2 in $W^* \otimes (\mathfrak{sp}(n) + \mathbb{H})$, we find $S^2 W^* \otimes V + (S^2 E S^2 H + S^2 H)$. \square

In later sections we shall have to work with certain invariant maps. Since the decomposition of $W_1 + W_2$ into $\text{Sp}(n)\text{Sp}(1)$ -modules contains some modules with multiplicity greater than one, it is clear that Schur's lemma will not be sufficient in order to study these maps. Thus, we shall have to be more explicit. Since we refer to $\text{Sp}(n)\text{Sp}(1)$ -modules, we can identify T, V and W with their duals through the metric here.

Assume first $n > 1$. The space $V \otimes \Lambda^2 V$ contains three copies of EH , corresponding to the highest weight vectors

$$\begin{aligned} \alpha_1 &= v_1 h_2 \otimes (v_{n+j} h_1 \wedge v_j h_2 + v_{n+j} h_2 \wedge v_j h_1) + 2v_1 h_1 \otimes v_j h_2 \wedge v_{n+j} h_2 \\ \alpha_2 &= v_{n+j} h_2 \otimes (v_j h_2 \wedge v_1 h_1 + v_1 h_2 \wedge v_j h_1) - v_j h_2 \otimes (v_{n+j} h_2 \wedge v_1 h_1 + v_1 h_2 \wedge v_{n+j} h_1) \\ \alpha_3 &= v_{n+j} h_2 \otimes (v_1 h_1 \wedge v_j h_2 + v_1 h_2 \wedge v_j h_1) - v_j h_2 \otimes (v_1 h_1 \wedge v_{n+j} h_2 + v_1 h_2 \wedge v_{n+j} h_1) \\ &\quad - 2v_{n+j} h_1 \otimes (v_1 h_2 \wedge v_j h_2) + 2v_j h_1 \otimes (v_1 h_2 \wedge v_{n+j} h_2). \end{aligned}$$

Similarly, $V \otimes \Lambda^2 V$ contains the linearly independent highest weight vectors

$$\beta_1 = v_1 h_2 \otimes (v_j h_2 \wedge v_{n+j} h_2), \quad \beta_2 = v_j h_2 \otimes (v_1 h_2 \wedge v_{n+j} h_2) - v_{n+j} h_2 \otimes (v_1 h_2 \wedge v_j h_2),$$

each generating a submodule isomorphic to $ES^3 H$.

We shall denote by $\tilde{\beta}_i, \tilde{\alpha}_i$ the images of these vectors under the isomorphism

$$\tilde{\cdot}: V \otimes \Lambda^2 V \rightarrow \Lambda^2 V \otimes V, \quad \widetilde{v \otimes \eta} = \eta \otimes v.$$

Each of $V \otimes S_0^2 W$ and $V \otimes \Lambda^2 W$ contains an $ES^3 H$, with highest weight vectors

$$\begin{aligned} \beta_3 &= v_1 h_2 \otimes (w_1 \otimes (w_2 + iw_3) + (w_2 + iw_3) \otimes w_1) - 2iv_1 h_1 \otimes (w_2 + iw_3) \otimes (w_2 + iw_3), \\ \beta_4 &= v_1 h_2 \otimes w_1 \otimes (w_2 + iw_3) - v_1 h_2 \otimes (w_2 + iw_3) \otimes w_1. \end{aligned}$$

Finally, $V \otimes W \otimes W$ contains two copies of EH , generated by

$$\begin{aligned} \alpha_4 &= v_1 h_2 \otimes (w^1 \otimes w_1 + w^2 \otimes w_2 + w^3 \otimes w_3), \\ \alpha_5 &= v_1 h_2 \otimes (w^2 \otimes w_3 - w^3 \otimes w_2) + v_1 h_1 \otimes (w^1 \otimes (w_2 + iw_3) - (w^2 + iw^3) \otimes w_1). \end{aligned}$$

If $n = 1$, all these highest weight vectors remain well defined, although they do not generate distinct modules, as $\alpha_3 = \alpha_1$ and $\beta_2 = \beta_1$. Accordingly, we can drop the assumption on n for the rest of the section.

Lemma 14. *We have*

$$\begin{aligned} \partial(\alpha_1) &= \frac{1}{2}\tilde{\alpha}_3 - \frac{3}{2}\tilde{\alpha}_2, & \partial(\alpha_2) &= -\tilde{\alpha}_1 - \frac{1}{2}\tilde{\alpha}_3 + \frac{1}{2}\tilde{\alpha}_2, & \partial(\alpha_3) &= \tilde{\alpha}_1 - \frac{1}{2}\tilde{\alpha}_3 - \frac{3}{2}\tilde{\alpha}_2, \\ \partial\beta_1 &= -\tilde{\beta}_2, & \partial\beta_2 &= \tilde{\beta}_2 - 2\tilde{\beta}_1, \end{aligned}$$

and $\partial(V^* \otimes (\mathfrak{sp}(n) + \mathbb{H})) \subset \text{Im } \partial_B$ contains

$$\partial(\alpha_2), \quad \tilde{\alpha}_2 + \tilde{\alpha}_3 + 8\alpha_4, \quad 8i\alpha_5 - \tilde{\alpha}_3 + 3\tilde{\alpha}_2, \quad \tilde{\beta}_2 + 2i\beta_4.$$

Proof. The first part is a straightforward computation. For the second part, observe that by Section 1 $\omega_2 + i\omega_3 \in \mathfrak{sp}(1) \otimes \mathbb{C}$ acts on $W \cong \text{Im } \mathbb{H}$ as

$$2(-w^3 \otimes w_1 + w^1 \otimes w_3 - iw^1 \otimes w_2 + iw^2 \otimes w_1) = -2i(w^1 \otimes (w_2 + iw_3) - (w^2 + iw^3) \otimes w_1),$$

and ω_1 acts as

$$-2w^2 \otimes w_3 + 2w^3 \otimes w_2.$$

It follows that $4i\alpha_5 - \alpha_1, \beta_1 - 2i\beta_4$ lie in $V \otimes \mathfrak{b}$.

Now recall that $-\text{Id}_V - 2\text{Id}_W$ lies in \mathfrak{b} , and

$$\text{Id}_V = \frac{1}{2}(-v_j h_2 \otimes v_{n+j} h_1 - v_{n+j} h_1 \otimes v_j h_2 + v_j h_1 \otimes v_{n+j} h_2 + v_{n+j} h_2 \otimes v_j h_1),$$

giving

$$\text{Im } \partial_B \ni \partial(v_1 h_2 \otimes (-2\text{Id}_V - 4\text{Id}_W)) = -\frac{1}{2}\tilde{\alpha}_2 - \frac{1}{2}\tilde{\alpha}_3 - 4\alpha_4. \quad \square$$

Proposition 15. *The components isomorphic to EH and ES^3H inside W_1 , $\partial_1(W_1)$ and W_2 are identified by*

$$W_1 \ni \beta_1, \beta_2, \alpha_1, \alpha_3, \quad \partial_1(W_1) \ni \tilde{\beta}_1, \tilde{\beta}_2, \tilde{\alpha}_1 - \tilde{\alpha}_3, \tilde{\alpha}_1 - 3\tilde{\alpha}_2, \quad W_2 \ni \beta_3;$$

moreover the following equivalences modulo $\text{Im } \partial_B$ hold:

$$\begin{aligned} \tilde{\alpha}_1 &\equiv \frac{3}{8}(\tilde{\alpha}_1 - \tilde{\alpha}_3) - \frac{1}{8}(\tilde{\alpha}_1 - 3\tilde{\alpha}_2), & \tilde{\alpha}_2 &\equiv \frac{1}{8}(\tilde{\alpha}_1 - \tilde{\alpha}_3) - \frac{3}{8}(\tilde{\alpha}_1 - 3\tilde{\alpha}_2), \\ \tilde{\alpha}_3 &\equiv -\frac{5}{8}(\tilde{\alpha}_1 - \tilde{\alpha}_3) - \frac{1}{8}(\tilde{\alpha}_1 - 3\tilde{\alpha}_2), & \alpha_4 &\equiv \frac{1}{16}(\tilde{\alpha}_1 - \tilde{\alpha}_3) + \frac{1}{16}(\tilde{\alpha}_1 - 3\tilde{\alpha}_2), \\ \alpha_5 &\equiv \frac{i}{8}(\tilde{\alpha}_1 - \tilde{\alpha}_3) - \frac{i}{8}(\tilde{\alpha}_1 - 3\tilde{\alpha}_2). \end{aligned}$$

Proof. With respect to the splitting

$$V \otimes \Lambda^2 V = V \otimes \mathfrak{sp}(1) + V \otimes S^2 E + V \otimes \Lambda_0^2 ES^2 H,$$

α_1, β_1 lie in the first component, α_2 in the second, $n\alpha_3 - \alpha_1$, $n\beta_2 - \beta_1$ in the third. Hence, by Lemma 14, $2\tilde{\alpha}_1 + \tilde{\alpha}_3 - \tilde{\alpha}_2$ is in the image of ∂_B . The rest of the statement is now a straightforward computation. \square

5 Intrinsic torsion conditions

In this section we study the intrinsic torsion of B -structures, establishing formulae to compute the ‘‘components’’ of the intrinsic torsion in terms of the exterior covariant derivative of suitable invariant tensorial forms. Then we specialize to the qc case, showing that the intrinsic torsion lies in a specific invariant subspace.

The first problem is that $\text{Coker } \partial_B$ is not completely reducible as a B -module, i.e. the sequence (4) does not split. To work around this problem, we shall employ a reduction to $\text{Sp}(n)\mathbb{H}^*$, which amounts to choosing an arbitrary complement of \mathcal{D} at each point. Nonetheless, we are still thinking of B as the structure group of qc geometry, and the main result of this section is stated in terms of B -structures.

Under $\text{Sp}(n)\mathbb{H}^*$, we have the decomposition

$$\Lambda^2 T^* \otimes T = \text{Im}(\partial_B) \oplus \Lambda^2 V^* \otimes W \oplus \partial_1(W_1) \oplus \partial_2(W_2).$$

Accordingly, the torsion of a connection splits into components as

$$\Theta = \Theta_* + \Theta^Q + \Theta_1 + \Theta_2, \tag{5}$$

and the qc condition reads

$$\Theta^Q = \Theta_0.$$

Moreover, the reduction makes ΛT^* into a bigraded vector space,

$$\Lambda T^* = \bigoplus_{p,q} \Lambda^{p,q}, \quad \Lambda^{p,q} = \Lambda^p V^* \otimes \Lambda^q W^*.$$

Given $\alpha \in \Lambda T^* \otimes S$, we shall denote by $\alpha^{p,q}$ its component in $\Lambda^{p,q} \otimes S$. This notation carries over to tensorial forms, i.e.

$$(\alpha_\theta)^{p,q} = (\alpha^{p,q})_\theta, \quad \alpha \in \Omega^{p+q}(P, S).$$

We shall also need to consider the projection

$$\pi_{-1}: \text{Im } \partial_B \rightarrow \Lambda^{1,1} \otimes W,$$

and set

$$\Theta_{-1} = \pi_{-1}(\Theta_*).$$

We can now prove a result analogous to Proposition 4. The first step is choosing two B -invariant tensorial forms, namely

$$\begin{aligned} \eta &\in \Omega^1(P, T/V), & \eta_\theta &\equiv w^1 \otimes w_1 + w^2 \otimes w_2 + w^3 \otimes w_3, \\ \gamma &\in \Omega^5(P, T/V \otimes \kappa^*), & \gamma_\theta &\equiv \omega_s \wedge w^{123} \otimes (w_s \otimes w_{123}), \end{aligned}$$

where as usual $\kappa = \Lambda^3 W^*$. A direct computation with highest weight vectors, together with Lemma 14, give:

Lemma 16. *The kernel of the map*

$$z: \Lambda^2 T^* \otimes T \rightarrow \Lambda^3 T^* \otimes W, \quad \alpha \rightarrow \alpha \lrcorner (\omega_s \wedge w^{123}) \otimes w_s$$

contains (and equals when $n > 1$)

$$\partial_B \left(T \otimes (\mathfrak{sp}(n) + \text{Hom}(W, V)) + W^* \otimes \mathbb{H} \right) + \partial_2(W_2) + 2EH + ES^3H,$$

where $2EH + ES^3H \subset \partial_B(V^* \otimes \mathbb{H}) + \partial_1(W_1)$ contains the highest weight vectors

$$\begin{aligned} &(\tilde{\alpha}_2 + \tilde{\alpha}_3 + 8\alpha_4) + 2(\tilde{\alpha}_1 - \tilde{\alpha}_3) + 2(\tilde{\alpha}_1 - 3\tilde{\alpha}_2), \\ &(8i\alpha_5 - \tilde{\alpha}_3 + 3\tilde{\alpha}_2) - (\tilde{\alpha}_1 - \tilde{\alpha}_3) + (\tilde{\alpha}_1 - 3\tilde{\alpha}_2), \quad (\tilde{\beta}_2 + 2i\beta_4) - \tilde{\beta}_2. \end{aligned}$$

Regardless of n , the restriction $z|_{\partial_1(W_1)}$ is injective, and $z|_{\text{Im } \partial_B} = g \circ \pi_{-1}$, where

$$g(\alpha) = \alpha \wedge (\Theta_0 \lrcorner \sigma) + \Theta_0 \wedge (\alpha \lrcorner \sigma).$$

Proposition 17. *Let P be a B -structure; for every connection on P and every reduction to $\text{Sp}(n)\mathbb{H}^*$,*

$$\begin{aligned} \Theta_2 + \Theta_{-1} &= (D\eta)^{1,1}, & \Theta^Q &= (D\eta)^{2,0}, \\ (\Theta_2 + \Theta_{-1}) \lrcorner \sigma &= (D\sigma)^{1,3}, & \Theta^Q \lrcorner \sigma &= (D\sigma)^{2,2}, \\ \Theta_1 \lrcorner \gamma &= (D\gamma)^{3,3} - g(\Theta_{-1}), & \Theta^Q \lrcorner \gamma &= (D\gamma)^{4,2}. \end{aligned}$$

These equations determine the intrinsic torsion in the sense that in each equation (except the last one for $n = 1$) the components of the intrinsic torsion appearing on the left hand side are determined by the right hand side.

Proof. By (2) $D\eta = \Theta \lrcorner \eta$; more precisely,

$$(D\eta)^{2,0} = \Theta^{2,0} \lrcorner \eta, \quad (D\eta)^{1,1} = \Theta^{1,1} \lrcorner \eta.$$

These interior products correspond respectively to the contractions

$$(\Lambda^{2,0} \otimes W) \otimes (W^* \otimes W) \rightarrow \Lambda^{2,0} \otimes W, \quad (\Lambda^{1,1} \otimes W) \otimes (W^* \otimes W) \rightarrow \Lambda^{1,1} \otimes W.$$

Moreover, since η is the identity in $W^* \otimes W = \text{Hom}(W, W)$, contraction with η gives rise to two isomorphisms. Thus, $(D\eta)^{1,1}$ determines the component $\Lambda^{1,1} \otimes W$ of the torsion, which equals $\Theta_2 + \Theta_{-1}$. Similarly, the component $\Lambda^2 V^* \otimes W$ can be read off $(D\eta)^{2,0}$, and the same arguments apply to $D\sigma$.

By the same token,

$$D\gamma = \Theta_{\lrcorner}\gamma;$$

since $(\Theta_{\lrcorner}\gamma)_{\theta} = z(\Theta_{\theta}) \otimes w_{123}$, and by Lemma 16 the restriction of z to

$$\Lambda^{2,0} \otimes W \rightarrow \Lambda^{4,2} \otimes W$$

is injective, it follows that $(D\gamma)^{4,2} = \Theta^Q_{\lrcorner}\gamma$ determines Θ^Q if $n > 1$.

Again by Lemma 16,

$$(D\gamma)^{3,3} = \Theta_{1\lrcorner}\gamma + g(\Theta_{-1}),$$

and this equation determines Θ_1 because z is injective on $\partial_1(W_1)$. \square

Example 18. Going back to the example $\text{Sp}(n+1)/\text{Sp}(n)\text{Sp}(1)$ of Section 3, observe that D is the horizontal part of d ; therefore,

$$\begin{aligned} D\sigma &= (\omega_1 \wedge w^{23} + \omega_2 \wedge e^{31} + \omega_3 \wedge e^{12}) \otimes w_{123}, \\ D\eta &= \omega_s \otimes w_s - 2w_{s\lrcorner}w^{123} \otimes w_s, \\ D\gamma &= \omega_s \wedge (\omega_1 \wedge w^{23} + \omega_2 \wedge e^{31} + \omega_3 \wedge e^{12}) \otimes (w_s \otimes w_{123}). \end{aligned}$$

Working with the reduction to $\text{Sp}(n)\mathbb{H}^*$ introduced in Section 3, Proposition 17 gives

$$\Theta_1 = \Theta_2 = \Theta_{-1} = 0, \quad \Theta^Q = \Theta_0.$$

We now turn to qc geometry. Let us consider the map

$$h: V^* \otimes W^* \otimes W \rightarrow \Lambda^{3,0} \otimes W, \quad e^i \otimes w^j \otimes w_k \rightarrow e^i \wedge \omega_j \otimes w_k.$$

Lemma 19. *The kernel of*

$$f: \partial_1(W_1) + \partial_2(W_2) \rightarrow \Lambda^{3,0} \otimes W, \quad f(\Theta_1, \Theta_2) = z(\Theta_1) - h(\Theta_2)$$

is isomorphic to

$$\begin{cases} ES^3H + ES^5H, & n = 1 \\ ES^3H, & n > 1 \end{cases};$$

the component ES^3H contains the highest weight vector

$$4\tilde{\beta}_1 + \tilde{\beta}_2 + 2i\beta_3 = \partial_1(-2\beta_2 - 3\beta_1) + 2i\partial_2(\beta_3).$$

Notice that $W_1 \oplus W_2$ has a unique submodule isomorphic to ES^5H ; from now on, ES^5H will indicate this submodule unless otherwise specified. The component isomorphic to ES^3H identified in the lemma will be denoted by $\widetilde{ES^3H}$, and we will denote by Θ_0^B the image of Θ_0 in $\text{Coker } \partial_B$.

Theorem 20. *A B-structure is qc if and only if the intrinsic torsion takes values in*

$$\begin{cases} \Theta_0^B + \widetilde{ES^3H}, & n > 1 \\ \Theta_0^B + \widetilde{ES^3H} + ES^5H, & n = 1 \end{cases}$$

Proof. Choose an arbitrary connection. The qc condition $\Theta^Q = \Theta_0$ implies

$$\gamma = \Theta_0 \wedge \sigma = D\eta \wedge \sigma,$$

giving

$$D\gamma = D^2\eta \wedge \sigma + D\eta \wedge D\sigma;$$

however, if Ω is the curvature, $D^2\eta \wedge \sigma = \Omega \wedge \eta \wedge \sigma = 0$, so

$$D\gamma - D\eta \wedge D\sigma = 0.$$

Decomposing into components, we get

$$(D\gamma)^{3,3} - (D\eta)^{1,1} \wedge (D\sigma)^{2,2} - (D\eta)^{2,0} \wedge (D\sigma)^{1,3} = 0,$$

whence, by Proposition 17,

$$(D\gamma)^{3,3} = \Theta_2 \wedge (\Theta_0 \lrcorner \sigma) + \Theta_0 \wedge (\Theta_2 \lrcorner \sigma) + \Theta_{-1} \wedge (\Theta_0 \lrcorner \sigma) + \Theta_0 \wedge (\Theta_{-1} \lrcorner \sigma).$$

Up to a contraction $\Lambda^3 W^* \otimes \kappa^* \cong \mathbb{R}$, the map $\Theta_2 \mapsto \Theta_2 \lrcorner \sigma$ corresponds to the trace $V^* \otimes W^* \otimes W \rightarrow V^*$; by construction this is zero. It follows that

$$(D\gamma)^{3,3} = \Theta_2 \wedge (\Theta_0 \lrcorner \sigma) + g(\Theta_{-1}),$$

so by Proposition 17 $z(\Theta_1) = h(\Theta_2)$, and the statement follows from Lemma 19. \square

6 A further reduction

Theorem 20 relies on the decomposition (5), which depends in turn on the choice of a reduction of the structure group to $\mathrm{Sp}(n)\mathbb{H}^*$. In this section we illustrate how the choice of this reduction affects the torsion, and show that it is canonical *in part*; in other words, we obtain a canonical reduction to an intermediate group K ,

$$\mathrm{Sp}(n)\mathbb{H}^* \subset K \subset B.$$

More precisely, the splitting under $\mathrm{Sp}(n)\mathbb{H}^*$

$$\mathrm{Hom}(W, V) = EH \times ES^3H,$$

is also a product of abelian Lie groups, so that EH and ES^3H appear as subgroups of B . Notice that this EH is isomorphic to V^* , rather than V , as an $\mathrm{Sp}(n)\mathbb{H}^*$ -module; as an $\mathrm{Sp}(n)\mathrm{Sp}(1)$ -module, we can write

$$EH = \{w^s \otimes v \lrcorner \omega_s \mid v \in V\}. \quad (6)$$

We set

$$K = \mathrm{Sp}(n)\mathbb{H}^* \ltimes EH.$$

We will see that an arbitrary B -structure has a canonical K -reduction induced by the choice of a complement of $\widetilde{ES^3H}$ in $W_1 + W_2$, but in the qc case, thanks to Theorem 20, the reduction is independent of the choice of complement of $\widetilde{ES^3H}$.

The key fact is that the action of the subgroup $\mathrm{Hom}(W, V) \subset B$ on Θ_0 is “linearized” when taking the quotient by $\mathrm{Im} \partial_B$, as shown in the following lemma.

Lemma 21. *The Lie group $\text{Hom}(W, V)$ acts on Θ_0^B with stabilizer equal to EH ; the orbit is $\Theta_0^B + \widetilde{ES^3H}$.*

Proof. The action of the Lie group $\text{Hom}(W, V)$ induces an infinitesimal action of its Lie algebra, which coincides with $\text{Hom}(W, V)$ itself. Denoting the former action by juxtaposition and the latter by \cdot , we see that

$$w^i \otimes e_j \cdot \alpha = -w^i \wedge e_j \lrcorner \alpha, \quad \alpha \in \Lambda^2 V^*.$$

Therefore

$$g\Theta_0 = \Theta_0 + g \cdot \Theta_0 \pmod{\Lambda^{0,2} \otimes T + \Lambda^{1,1} \otimes V} \subset \text{Im } \partial_B,$$

i.e. $g\Theta_0^B = \Theta_0^B + g \cdot \Theta_0^B$. The Lie algebra action gives an $\text{Sp}(n)\text{Sp}(1)$ -invariant map

$$\text{Hom}(W, V) \otimes \Lambda^2 T^* \otimes T \rightarrow \Lambda^2 T^* \otimes T;$$

since Θ_0 is invariant, the action on Θ_0 gives an invariant map

$$\text{Hom}(W, V) \rightarrow \Lambda^2 T^* \otimes T.$$

Under this map, the highest weight vector

$$(w^2 + iw^3) \otimes v_1 h_2 \in ES^3H \subset \text{Hom}(W, V)$$

has image

$$\tilde{\beta}_1 + \frac{i}{2}\beta_3 - \frac{i}{2}\beta_4 \equiv \tilde{\beta}_1 + \frac{i}{2}\beta_3 + \frac{1}{4}\tilde{\beta}_2,$$

which by definition lies in $\widetilde{ES^3H}$.

On the other hand the highest weight vector

$$(w^2 + iw^3) \otimes v_1 h_1 + iw^1 \otimes v_1 h_2 \in EH \subset \text{Hom}(W, V)$$

has image

$$\begin{aligned} & (w_2 + iw_3) \otimes v_1 h_1 + iw_1 \otimes v_1 h_2 - v_1 h_2 \wedge w^s \otimes w_s \\ & + iv_1 h_1 \wedge (w^1 \wedge (w_2 + iw_3)) + iv_1 h_2 \wedge (w_2 \wedge w_3) = \frac{1}{2}\tilde{\alpha}_1 - \alpha_4 + i\alpha_5, \end{aligned}$$

which by Proposition 15 lies in $\text{Im } \partial_B$. \square

Theorem 22. *For any fixed $\text{Sp}(n)\mathbb{H}^*$ -invariant complement $\widetilde{ES^3H}^\perp$ of $\widetilde{ES^3H}$ in $W_1 + W_2$, every B -structure has a unique K -reduction such that the restriction of the B -intrinsic torsion takes values in $\Theta_0^B + \widetilde{ES^3H}^\perp$.*

Proof. Let $\Theta^B: P \rightarrow \text{Coker } \partial_B$ be the intrinsic torsion. By Lemma 21, Θ_0^B has stabilizer K ; consider the K -equivariant map

$$f: P \rightarrow \text{Coker } \partial_B, \quad f(u) = \Theta^B(u) - \Theta_0^B.$$

Since $\Theta^Q(u) = r(\Theta^B(u)) = \Theta_0^Q$, the map f takes values in $\ker r$, which by Proposition 12 equals $W_1 + W_2$.

Now set

$$\tilde{P} = \{u \in P \mid f(u) \in \widetilde{ES^3H}^\perp\}.$$

By construction, \tilde{P} is closed under the action of K . Conversely, given any $u \in P$, $g \in \text{Hom}(W, V)$,

$$f(ug) = \Theta^B(ug) - \Theta_0^B = g^{-1}(\Theta^B(u) - g\Theta_0^B)$$

lies in $\widetilde{ES^3H}^\perp$ if and only if so does $\Theta^B(u) - g\Theta_0^B$. By Lemma 21, this condition is satisfied for exactly one $g \in ES^3H$. Thus, \tilde{P} is a K -structure. \square

Combining this result with Theorem 20, and specializing to the qc case, we find

Corollary 23. *Every qc B -structure has a unique K -reduction P such that*

$$\begin{cases} \Theta^B(u) = \Theta_0^B, & n > 1 \\ \Theta^B(u) \in \Theta_0^B + ES^5H, & n = 1 \end{cases}$$

for all u in P .

We shall refer to such a K -structure as a *qc K -structure*. We shall say a qc K -structure P is *integrable* if $\Theta^B = \Theta_0^B$ identically on P . This condition is automatic when $n > 1$. We shall see in Corollary 37 that this definition agrees with that of [6].

Remark. An alternative approach more akin to Biquard's would be to fix a compatible metric on \mathcal{D} ; in other words, by choosing an arbitrary reduction from B to $\tilde{B} = \text{Sp}(n)\text{Sp}(1) \ltimes \text{Hom}(W, V)$. Proceeding as in Lemma 21, we would see that $\text{Im } \partial_{\tilde{B}}$ does not contain $\frac{1}{2}\tilde{\alpha}_1 - \alpha_4 + i\alpha_5$, and so the construction of Corollary 23 would give a ‘‘canonical’’ reduction to $\text{Sp}(n)\text{Sp}(1)$, depending only on the choice of the metric. However, as we are mainly interested in the intrinsic geometry of qc structures, we will refrain from fixing a metric until Section 9.

Remark. Since the projection of $\widetilde{ES^3H}$ onto W_2 is injective, the K -reduction is characterized by the condition $\Theta_2 = 0$, or $\Theta_2 \in ES^5H$ when $n = 1$. In practice, if one finds a connection which satisfies this condition with respect to some frame, then the frame is compatible with the K -reduction of Corollary 23.

7 K -intrinsic torsion

Corollary 23 motivates us to study the intrinsic torsion of qc K -structure. In fact, we have defined integrable qc K -structures by the condition that the restriction of the B -intrinsic torsion be constant, equal to Θ_0^B . In this section we show that the K -intrinsic torsion of an integrable qc K -structure is also constant.

It follows that a qc K -structure admits a family of connections whose torsion equals Θ_0 ; this family is parametrized by sections of a bundle with fibre $\ker \partial_K$, which is identified by the following:

Lemma 24. *The kernel of ∂_K is the S^2H containing*

$$\begin{aligned} & v_j h_2 \otimes ((w^2 + iw^3) \otimes v_{n+j} h_1 + iw^1 \otimes v_{n+j} h_2) - v_{n+j} h_2 \otimes ((w^2 + iw^3) \otimes v_j h_1 + iw^1 \otimes v_j h_2) \\ & - (w^2 + iw^3) \otimes (\text{Id}_V + 2\text{Id}_W) + iw^1 \otimes (\omega_2 + i\omega_3) - i(w^2 + iw^3) \otimes \omega_1, \end{aligned} \quad (7)$$

and the inclusion $V^* \otimes \frac{\mathfrak{b}}{\mathfrak{k}} \rightarrow T^* \otimes \frac{\mathfrak{b}}{\mathfrak{k}}$ induces an exact sequence of K -modules

$$0 \rightarrow S^2 E S^2 H \rightarrow V^* \otimes \frac{\mathfrak{b}}{\mathfrak{k}} \rightarrow \text{Coker } \partial_K \rightarrow \text{Coker } \partial_B \rightarrow 0 \quad (8)$$

Proof. If we set

$$T_1 = W^* \otimes \text{Hom}(W, V), \quad T_2 = W^* \otimes \text{Hom}(\mathfrak{sp}(n) + \mathbb{H}) + V^* \otimes \text{Hom}(W, V),$$

Corollary 13 implies $\ker \partial_B$ decomposes as the direct sum of $S^2 W^* \otimes V \subset T_1$ and $S^2 H(S^2 E + \mathbb{R}) \subset T_2$. Since $\partial(T_1)$ and $\partial(T_2)$ intersect trivially, $\ker \partial_K$ is also a direct sum, i.e.

$$\ker \partial_K = (S^2 W^* \otimes V \cap (T^* \otimes \mathfrak{k})) + (S^2 H(S^2 E + \mathbb{R}) \cap (T^* \otimes \mathfrak{k})).$$

A direct computation shows that the first summand is trivial.

The components of type $S^2 H S^2 E$ inside $T_2 \cap (T^* \otimes \mathfrak{k})$ are identified by

$$(w^2 + iw^3) \otimes (v_1 h_2 \wedge v_1 h_1), \quad v_1 h_2 \otimes ((w^2 + iw^3) \otimes v_1 h_1 + iw^1 \otimes v_1 h_2),$$

so $\ker \partial_K$ contains no $S^2 H S^2 E$. The $S^2 H$ components inside $T_2 \cap (T^* \otimes \mathfrak{k})$ contain

$$\begin{aligned} & (w^2 + iw^3) \otimes (\text{Id}_V + 2\text{Id}_W), \quad w_1 \otimes (\omega_2 + i\omega_3) - (w^2 + iw^3) \otimes \omega_1, \\ & v_j h_2 \otimes ((w^2 + iw^3) \otimes v_{n+j} h_1 + iw^1 \otimes v_{n+j} h_2) - v_{n+j} h_2 \otimes ((w^2 + iw^3) \otimes v_j h_1 + iw^1 \otimes v_j h_2). \end{aligned}$$

It is straightforward to verify that there is only one linear combination that goes to zero, up to multiple, namely (7).

The commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & T^* \otimes \mathfrak{k} & \longrightarrow & T^* \otimes \mathfrak{b} & \longrightarrow & T^* \otimes \frac{\mathfrak{b}}{\mathfrak{k}} \longrightarrow 0 \\ & & \downarrow \partial_K & & \downarrow \partial_B & & \downarrow \\ 0 & \longrightarrow & \Lambda^2 T^* \otimes T & \longrightarrow & \Lambda^2 T^* \otimes T & \longrightarrow & 0 \end{array}$$

determines an exact sequence

$$0 \rightarrow \ker \partial_K \rightarrow \ker \partial_B \rightarrow T^* \otimes \frac{\mathfrak{b}}{\mathfrak{k}} \xrightarrow{f} \text{Coker } \partial_K \rightarrow \text{Coker } \partial_B \rightarrow 0.$$

By the exactness of the top row in the diagram, $S^2 W^* \otimes V$ maps injectively into $T^* \otimes \frac{\mathfrak{b}}{\mathfrak{k}}$; the image equals $W^* \otimes \frac{\mathfrak{b}}{\mathfrak{k}}$ by a dimension count. This gives exactness of (8) as a sequence of vector spaces. Moreover, we have a diagram

$$\begin{array}{ccccc} V^* \otimes \frac{\mathfrak{b}}{\mathfrak{k}} & \xrightarrow{\iota} & T^* \otimes \frac{\mathfrak{b}}{\mathfrak{k}} & \xrightarrow{f} & \text{Coker } \partial_K \\ & \searrow \text{Id} & \downarrow & \nearrow \tilde{f} & \\ & & V^* \otimes \frac{\mathfrak{b}}{\mathfrak{k}} & & \end{array}$$

where \tilde{f} is K -equivariant. Therefore, $f \circ \iota$ is also equivariant.

Since EH acts trivially on $V^* \otimes \frac{\mathfrak{k}}{\mathfrak{b}}$, all maps in (8) are equivariant. \square

Working with $\mathrm{Sp}(n)\mathrm{Sp}(1)$ -modules, we can define a complement of S^2ES^2H in $V^* \otimes ES^3H \subset V^* \otimes \mathfrak{b}$, namely

$$W_3 = \begin{cases} S^4H(S^2E + \Lambda_0^2E + \mathbb{R}) + S^2H\Lambda_0^2E + S^2H, & n > 1 \\ S^4H(S^2E + \mathbb{R}) + S^2H, & n = 1 \end{cases},$$

and by the above lemma we have a decomposition into $\mathrm{Sp}(n)\mathrm{Sp}(1)$ -modules

$$\Lambda^2T^* \otimes T = \mathrm{Im}(\partial_K) \oplus \Lambda^2V^* \otimes W \oplus \partial_1(W_1) \oplus \partial_2(W_2) \oplus \partial(W_3).$$

Accordingly, the torsion of a connection splits into components as

$$\Theta = \Theta_* + \Theta^Q + \Theta_1 + \Theta_2 + \Theta_3.$$

This decomposition is not a decomposition of K -modules, i.e. it depends on the pointwise choice of a complement of \mathcal{D} . But restricting to the qc case, we find the following:

Lemma 25. *The intrinsic torsion of a qc K -structure takes values in*

$$\begin{cases} \mathrm{Span}\{\Theta_0^K\} \oplus \partial(W_3), & n > 1 \\ \mathrm{Span}\{\Theta_0^K\} \oplus \partial(W_3) \oplus ES^5H, & n = 1 \end{cases} \quad (9)$$

which is a direct sum of K -submodules of $\mathrm{Coker}\partial_K$.

Proof. Having defined qc K -structures by the condition of Corollary 23, it suffices to show that (9) is a K -module. In (8), EH acts trivially on $V^* \otimes \frac{\mathfrak{k}}{\mathfrak{b}}$, so $\partial(W_3)$ is a K -submodule. The same applies to $ES^5H \subset \Lambda^{1,1} \otimes W$. It remains to be seen how EH acts on Θ_0^K . The group action of $\mathrm{Hom}(W, V)$ on ΛV^* is given on simple elements by

$$(w^i \otimes e_j)\alpha = \alpha - w^i \wedge e_j \lrcorner \alpha.$$

Recalling (6),

$$\begin{aligned} (w^s \otimes v \lrcorner \omega_s)\omega_r \otimes w_r &= \Theta_0 + (w^s \otimes v \lrcorner \omega_s) \cdot \Theta_0 - \|v\|^2 \eta \lrcorner \sigma \\ &\quad - w^s \wedge ((v \lrcorner \omega_s) \lrcorner \omega_r) \otimes (v \lrcorner \omega_r) - \|v\|^2 w_r \lrcorner w^{123} \otimes (v \lrcorner \omega_r). \end{aligned}$$

However $(w^s \otimes v \lrcorner \omega_s) \cdot \Theta_0$ was seen in Lemma 21 to be in some $EH \subset \mathrm{Im}\partial_B$, but it is really in $\mathrm{Im}\partial_K$, since $V^* \otimes ES^3H$ contains no EH . Similarly, $\eta \lrcorner \sigma$ lies in a trivial submodule of $\Lambda^{0,2} \otimes W$, which is contained in $\mathrm{Im}\partial_K$, and

$$w_r \lrcorner w^{123} \otimes (v \lrcorner \omega_r) \in \Lambda^{0,2} \otimes V \subset \mathrm{Im}\partial_K.$$

Thus,

$$(w^s \otimes v \lrcorner \omega_s)\omega_r \otimes w_r - \Theta_0 = -w^s \wedge ((v \lrcorner \omega_s) \lrcorner \omega_r) \otimes (v \lrcorner \omega_r) \pmod{\mathrm{Im}\partial_K},$$

which is in the image of $S^2(EH)$ under the equivariant map $V \otimes V \rightarrow \Lambda^{1,1} \otimes V$

$$u \otimes v \rightarrow -(w^1 \wedge ((u \lrcorner \omega_1) \lrcorner \omega_r) + w^2 \wedge ((u \lrcorner \omega_2) \lrcorner \omega_r) + w^3 \wedge ((u \lrcorner \omega_3) \lrcorner \omega_r)) \otimes (v \lrcorner \omega_r).$$

The target space $\Lambda^{1,1} \otimes V$ is contained in $\partial(W_3) + \mathrm{Im}\partial_K$; however, $\partial(W_3)$ has no irreducible component in common with $S^2(EH) = S^2ES^2H + \Lambda_0^2E + \mathbb{R}$. We conclude that the action of EH on Θ_0^K is trivial. \square

A computation with highest weight vectors yields:

Lemma 26. *Consider the map*

$$q: \Lambda^{2,0} \otimes \mathbb{H} \rightarrow \Lambda^{2,1} \otimes W, \quad q(\alpha \otimes p) = \alpha \wedge w^s \otimes \widetilde{\text{ad}}(p)w_s;$$

then

$$\{v \in W_3 \mid \partial(v) \lrcorner \Theta_0 \in \text{Im } q\} = 0.$$

We can now prove the main result of this section.

Theorem 27. *The intrinsic torsion of an integrable qc K -structure is constant, i.e.*

$$\Theta^K \equiv \Theta_0^K.$$

Proof. Choose an arbitrary reduction to $\text{Sp}(n)\text{Sp}(1)$ and a connection such that the torsion takes values in $\partial(W_3) + \Theta_0$. In particular, Θ_{-1} and Θ_2 vanish, as does the component $\Theta^{0,2}$. Then the Bianchi identity and Proposition 17 give

$$\Omega \wedge \eta = D^2\eta = D\Theta_0 = \Theta \lrcorner \Theta_0.$$

In particular,

$$\Omega^{2,0} \wedge \eta = \Theta^{1,1} \lrcorner \Theta_0.$$

By Lemma 26, this implies that $\Theta^{1,1}$ is zero. \square

Since the intrinsic torsion is trivial, we are motivated to consider specific connections. As Θ_0 is not invariant under K , we cannot conclude that there exists a connection with torsion equal to Θ_0 : we must consider the stabilizer of Θ_0 in $\Lambda^2 T^* \otimes T$ and take a corresponding reduction.

Given a K -structure P and a reduction \tilde{P} to $\text{Sp}(n)\mathbb{H}^*$, we define a K -connection on \tilde{P} as a one-form $\omega \in \Omega^1(\tilde{P}, \mathfrak{k})$ which is the restriction of a (uniquely determined) connection form on P .

Corollary 28. *Any reduction to $\text{Sp}(n)\mathbb{H}^*$ of an integrable qc K -structure admits a K -connection with torsion equal to Θ_0 .*

Proof. Fix an arbitrary connection ω , and let Θ be its torsion. By Theorem 27, $\Theta_\theta - \Theta_0$ defines an $\text{Sp}(n)\mathbb{H}^*$ -equivariant map taking values in $\text{Im } \partial_K$. Since $\partial_K: T^* \otimes \mathfrak{k} \rightarrow \text{Im } \partial_K$ admits an $\text{Sp}(n)\mathbb{H}^*$ -invariant right inverse, we obtain a tensorial \mathfrak{k} -valued 1-form A such that the connection $\omega + A$ has torsion Θ_0 . \square

This result only applies to integrable qc structures. For this reason, in the rest of the paper we shall only consider integrable qc structures.

Remark. Given a reduction to $\text{Sp}(n)\mathbb{H}^*$, all other reductions are parametrized by forms of type $(1, 0)$, since $K/\text{Sp}(n)\mathbb{H}^* \cong V^*$ as an $\text{Sp}(n)\mathbb{H}^*$ -module. Such a reduction is determined canonically once one fixes a compatible metric on \mathcal{D} (see [2], or Proposition 36); replacing the metric g with a different metric fg in the conformal class affects the reduction to $\text{Sp}(n)\mathbb{H}^*$ via $df^{1,0}$.

8 Connections and curvature

We saw in Corollary 28 that given an integrable qc K -structure, any reduction to $\mathrm{Sp}(n)\mathbb{H}^*$ has a K -connection with torsion equal to Θ_0 , though this is not unique due to the fact that ∂_K has a kernel. In this section we find a natural condition that can be imposed on the curvature of such a connection that makes it unique, given the choice of the reduction. As a byproduct, we compute the K -module in which the curvature lies. Notice that this is not a metric connection, unlike the Biquard connection.

The curvature of a connection with torsion Θ_0 takes values in $\Lambda^2 T^* \otimes \mathfrak{k}$; this is a space of dimension $16n^4 + O(n^3)$ which, as an $\mathrm{Sp}(n)\mathrm{Sp}(1)$ -module, decomposes as

$$\begin{aligned} S^4 E + V_{31}(\mathbb{R} + S^2 H) + V_{22} + V_{211} S^2 H + S^3 E(2H + S^3 H) + V_{21}(3H + 2S^3 H) \\ + \Lambda_0^3 E(H + S^3 H) + S^2 E(3\mathbb{R} + 6S^2 H + S^4 H) + \Lambda_0^2 E(3\mathbb{R} + 5S^2 H + 2S^4 H) \\ + E(8H + 7S^3 H + S^5 H) + 4\mathbb{R} + 6S^2 H + 3S^4 H \end{aligned}$$

However, the Bianchi identity implies that the curvature lies in a module of dimension $\frac{2}{3}n^4 + O(n^3)$, identified by the following:

Lemma 29. *Let*

$$\delta: \Lambda^2 T^* \otimes \mathfrak{k} \rightarrow \Lambda^3 \otimes T$$

be the restriction of the skew-symmetrization map $\Lambda^2 T^ \otimes \mathfrak{gl}(T) \rightarrow \Lambda^3 \otimes T$, and set*

$$s: T^* \otimes (\Lambda^2 T^* \otimes T) \rightarrow \Lambda^3 T^* \otimes T, \quad x \otimes y \otimes z \rightarrow x \wedge y \otimes z.$$

If we denote by \widetilde{EH} the submodule of $\Lambda^2 T^ \otimes T$ containing*

$$\frac{1}{2}\tilde{\alpha}_1 - \alpha_4 + i\alpha_5 \in \Lambda^{2,0} \otimes V + \Lambda^{1,1} \otimes W, \quad (10)$$

the preimage of $s(T^ \otimes \widetilde{EH})$ under δ is the direct sum of*

$$\begin{aligned} R_1 &\cong S^4 E + (S^2 E + \Lambda_0^2 E + \mathbb{R})(S^2 H + \mathbb{R}) && \subset \Lambda^{2,0} \otimes (\mathfrak{sp}(n) + \mathbb{H}) \\ R_2 &\cong S^3 E H + 2E S^3 H + 2E H && \subset \Lambda^{1,1} \otimes (\mathfrak{sp}(n) + \mathbb{H}) + \Lambda^{2,0} \otimes E H \\ R_3 &\cong S^2 E S^2 H + S^4 H + S^2 H + \mathbb{R} && \subset \Lambda^{0,2} \otimes (\mathfrak{sp}(n) + \mathbb{H}) + \Lambda^{1,1} \otimes E H \\ R_4 &\cong E S^3 H && \subset \Lambda^{0,2} \otimes E H. \end{aligned}$$

Moreover, denoting by $\mathrm{tr}: \Lambda^2 T^ \otimes \mathfrak{k} \rightarrow \Lambda^2 T^*$ the trace,*

$$\begin{aligned} \tilde{R}_1 &= R_1 \cap \ker \mathrm{tr} \cong S^4 E + S^2 E S^2 H + \Lambda_0^2 E + \mathbb{R}, \\ \tilde{R}_2 &= R_2 \cap \ker \mathrm{tr} \cong S^3 E H + E S^3 H + E H, \\ \tilde{R}_3 &= R_3 \cap \ker \mathrm{tr} \cong S^2 E S^2 H + S^4 H + \mathbb{R}. \end{aligned}$$

Proof. To begin with, we determine the kernel of δ , which we decompose as the direct sum of the kernels of the following maps, obtained by restriction:

$$\begin{aligned} \delta_1: \Lambda^{2,0} \otimes (\mathfrak{sp}(n) + \mathbb{H}) &\rightarrow \Lambda^{3,0} \otimes V + \Lambda^{2,1} \otimes W \\ \delta_2: \Lambda^{1,1} \otimes (\mathfrak{sp}(n) + \mathbb{H}) + \Lambda^{2,0} \otimes E H &\rightarrow \Lambda^{2,1} \otimes V + \Lambda^{1,2} \otimes W \\ \delta_3: \Lambda^{0,2} \otimes (\mathfrak{sp}(n) + \mathbb{H}) + \Lambda^{1,1} \otimes E H &\rightarrow \Lambda^{1,2} \otimes V + \Lambda^{0,3} \otimes W \\ \delta_4: \Lambda^{0,2} \otimes E H &\rightarrow \Lambda^{0,3} \otimes V \end{aligned}$$

The kernel of δ_1 is contained in $\Lambda^{2,0} \otimes (\mathfrak{sp}(n))$; by construction, it coincides with the space of curvature tensors of metrics with holonomy $\mathrm{Sp}(n)$, i.e. the kernel of

$$S^2(\mathfrak{sp}(n)) \rightarrow \Lambda^4 T^*, \quad a \odot b \rightarrow a \wedge b,$$

which is known to equal $S^4 E$ (see [17]).

The restriction of δ_3 to $\Lambda^{1,1} \otimes EH + \Lambda^{0,2} \otimes \mathbb{R}$ is an isomorphism: in fact, $\delta_3(\Lambda^{1,1} \otimes EH)$ coincides with $\Lambda^{1,0} \otimes \partial(W^* \otimes EH)$, and we know from Lemma 24 that ∂_K is injective. Thus, $\ker \delta_3 \cong S^2 ES^2 H + \mathbb{R} + S^2 H + S^4 H$.

The kernel of δ_4 is $ES^3 H$, because it is clearly surjective.

Considering the composition of δ_2 with the projection on $\Lambda^{1,2} \otimes W$, we see that δ_2 has the same kernel as its restriction to

$$\Lambda^{1,1} \otimes \mathfrak{sp}(n) + \Lambda^{1,0} \wedge S^2 H + \Lambda^{2,0} \otimes EH,$$

where $S^2 H$ contains

$$w^1 \otimes (\omega_2 + i\omega_3) - (w^2 + iw^3) \otimes \omega_1 + i(w^2 + iw^3) \otimes (\mathrm{Id}_V + 2\mathrm{Id}_W).$$

Moreover the restrictions of δ_2 to $\Lambda^{1,1} \otimes \mathfrak{sp}(n) + \Lambda^{1,0} \wedge S^2 H$ and $\Lambda^{2,0} \otimes EH$ are injective. Thus, we must investigate the common components in

$$\begin{cases} S^3 E(S^3 H + H) + V_{21}(S^3 H + H) + 2E(S^3 H + H), & n > 1 \\ S^3 E(S^3 H + H) + 2E(S^3 H + H), & n = 1 \end{cases}$$

and

$$\begin{cases} S^3 EH + V_{21}(2H + S^3 H) + \Lambda_0^3 E(H + S^3 H) + E(3H + 2S^3 H), & n > 1 \\ S^3 EH + E(2H + S^3 H), & n = 1 \end{cases}$$

Decomposing the target space $\Lambda^{2,1} \otimes V$ as

$$\begin{cases} S^3 E(S^3 H + H) + V_{21}(3H + 3S^3 H + S^5 H) & n > 1 \\ \quad + \Lambda_0^3 E(2H + 2S^3 H + S^5 H) + E(5S^3 H + 5H + 2S^5 H) & \\ S^3 E(S^3 H + H) + E(3S^3 H + 3H + S^5 H) & n = 1 \end{cases}$$

we see that the kernel contains exactly one copy of $S^3 EH$.

Considering the components isomorphic to $ES^3 H$ in $\Lambda^{1,1} \otimes \mathfrak{sp}(n)$ and $\Lambda^{1,0} \wedge S^2 H$, one verifies that only the second among $(w^2 + iw^3) \wedge \alpha_2$ and

$$v_1 h_2 \wedge w^1 \otimes (\omega_2 + i\omega_3) - v_1 h_2 \wedge (w^2 + iw^3) \otimes \omega_1 + i v_1 h_2 \wedge (w^2 + iw^3) \otimes (\mathrm{Id}_V + 2\mathrm{Id}_W) \quad (11)$$

has image in $\delta_2(\Lambda^{2,0} \otimes EH)$. Thus, the kernel contains exactly one copy of $ES^3 H$. Similarly, only one of the two components isomorphic to EH in $\Lambda^{1,1} \otimes \mathfrak{sp}(n) + \Lambda^{1,0} \wedge S^2 H$ has image contained in $\delta_2(\Lambda^{2,0} \otimes EH)$; this component is contained in $\Lambda^{1,0} \wedge S^2 H$ and is identified by the highest weight vector

$$\begin{aligned} & v_1 h_1 \wedge (w^1 \otimes (\omega_2 + i\omega_3) + (w^2 + iw^3) \otimes (-\omega_1 + i(\mathrm{Id}_V + 2\mathrm{Id}_W))) \\ & + v_1 h_2 \wedge \left(\frac{1}{2} i(w_2 + iw_3) \otimes (\omega_2 - i\omega_3) - w^1 \otimes (\mathrm{Id}_V + 2\mathrm{Id}_W) \right. \\ & \quad \left. - \frac{1}{2} i(w_2 - iw_3) \otimes (\omega_2 + i\omega_3) \right). \end{aligned} \quad (12)$$

A long yet straightforward computation shows that for the remaining modules, δ_2 has the greatest rank that Schur's lemma allows, so

$$\ker \delta_2 = S^3 EH + ES^3 H + EH.$$

It follows from Lemma 14 that \widetilde{EH} is contained in $\text{Im } \partial_K$; this implies that the image of s is contained in the image of δ . Moreover, s is injective, and the image of $W^* \otimes \widetilde{EH} \cong EH + ES^3 H$ is contained in $\Lambda^{2,1} \otimes V + \Lambda^{1,2} \otimes W$, giving

$$R_2 = \ker \delta_2 + EH + ES^3 H.$$

Similarly, we see that

$$R_1 = \ker \delta_1 + S^2 E(S^2 H + \mathbb{R}) + \Lambda_0^2 E(S^2 H + \mathbb{R}) + (S^2 H + \mathbb{R}).$$

Finally, consider the commutative diagram

$$\begin{array}{ccc} R_1 & \xrightarrow{\text{tr}} & \Lambda^{2,0} \\ & \searrow \delta_1 & \nearrow f \\ V \otimes \widetilde{EH} & \xrightarrow{s} & \Lambda^{3,0} \otimes V + \Lambda^{2,1} \otimes W \end{array}$$

where f is induced by the contraction $W \otimes W \rightarrow \mathbb{R}$. Since $f \circ s$ is surjective and $\text{Im } \delta_1$ contains $\text{Im } s$, it follows that $\text{tr} = f \circ \delta_1$ is also surjective, proving that R_1 is isomorphic to $\widetilde{R}_1 + \Lambda^{2,0}$.

Similarly, the trace map

$$\text{tr}: \Lambda^{1,1} \otimes (\mathfrak{sp}(n) + \mathbb{H}) \rightarrow \Lambda^{1,1} \cong EH + ES^3 H,$$

does not kill either (11) or (12), so the restriction to R_2 is surjective.

In the same way, the trace maps R_3 to $\Lambda^{0,2} \cong S^2 H$, so \widetilde{R}_3 differs from R_3 at most by an $S^2 H$. A non-zero element

$$x = x_1 + x_2, \quad x_1 \in \Lambda^{0,2} \otimes \mathfrak{sp}(1), \quad x_2 \in \Lambda^{1,1} \otimes EH,$$

can only be in the kernel of δ if x_1 is not zero; however, if x_1 is in the submodule isomorphic to $S^2 H$, then $\delta(x_1)^{0,3}$ is not zero. It follows that there is no $S^2 H$ in \widetilde{R}_3 . \square

Proposition 30. *On an integrable qc manifold, the curvature of a connection with torsion Θ_0 takes values in $R_1 + R_2 + R_3 + R_4$.*

Proof. Writing Θ_0 for $\langle \Theta_0, \frac{1}{2}\theta \wedge \theta \rangle$, the Bianchi identity gives

$$D\Theta_0 = \Omega \wedge \theta.$$

By (2),

$$D\Theta_0 = \langle \nabla \Theta_0, \frac{1}{6}\theta \wedge \theta \wedge \theta \rangle + \Theta_0 \lrcorner \Theta_0.$$

The calculations of Lemma 21 show that the infinitesimal action of \mathfrak{k} on Θ_0 gives an EH containing (10). Thus $D\Theta_0$ lies in the image of $T^* \otimes \widetilde{EH}$ under the map s of Lemma 29.

It now follows from the Bianchi identity that Ω_θ is in the preimage of $s(T^* \otimes \widetilde{EH})$ under δ ; Lemma 29 concludes the proof. \square

We can now ask whether among the connections with torsion Θ_0 there is one with “minimal” curvature. Since the connection is well defined up to a section of a bundle with fibre S^2H , minimality should be taken to mean that the component in one of the two S^2H appearing in $R_1 + R_2 + R_3 + R_4$ is zero. The key observation is that taking the interior product with Θ_0 of a highest weight vector in $\ker \partial_K$ gives

$$-(\omega^2 + i\omega^3) \otimes (\text{Id}_V + 2\text{Id}_W) + i\omega^1 \otimes (\omega_2 + i\omega_3) - i(\omega^2 + i\omega^3) \otimes \omega_1,$$

which is an element of $S^2H \subset R_1$. This is therefore the natural candidate as a component of curvature to kill.

In fact, it turns out that this component behaves like torsion, i.e. it depends on the choice of connection via a pointwise isomorphism $\ker \partial_K \cong S^2H$, although of course the full curvature tensor depends on the connection in a more complicated way. This enables us to prove:

Theorem 31. *Any $\text{Sp}(n)\mathbb{H}^*$ -reduction of an integrable qc K -structure has a unique K -connection such that*

1. *the torsion is Θ_0 ;*
2. *the curvature has no component in $S^2H \subset R_1$.*

Condition (2) can be replaced with

- 2'. *the curvature Ω satisfies $(\text{tr } \Omega^{2,0})_\theta \in S^2E + \Lambda_0^2 S^2E$.*

Proof. Let ω be any connection with torsion Θ_0 . By Proposition 30, the curvature lies in $R_1 + R_2 + R_3 + R_4$. By Lemma 29, the component of R_1 isomorphic to S^2H is not contained in \tilde{R}_1 , so the conditions (2) and (2') are equivalent.

The generic connection with torsion Θ_0 has the form $\omega_A = \omega + A$, where A_θ takes values in $\ker \partial_K$. The curvature of ω_A is

$$\Omega_A = \Omega + DA + \frac{1}{2}[A, A],$$

where Ω denotes the curvature of ω and D is the exterior covariant derivative with respect to ω .

By Lemma 24, $A^{1,0}$ takes values in the abelian subalgebra EH , so

$$(\Omega_A)^{2,0} = \Omega^{2,0} + (DA)^{2,0},$$

and, by (2),

$$DA = \frac{1}{2}\langle \theta \wedge \theta, \nabla A \rangle + \Theta_0 \lrcorner A.$$

However, the infinitesimal action of \mathfrak{k} on $T^* \otimes \mathfrak{k}$ takes $\ker \partial_K = S^2H$ into

$$\ker \partial_K + W^* \otimes \mathfrak{k};$$

it follows that $\langle \theta \wedge \theta, \nabla A \rangle^{2,0}$ is trace-free, and

$$\text{tr}(\Omega_A)^{2,0} - \text{tr } \Omega^{2,0} = \text{tr}(\Theta_0 \lrcorner A).$$

By Proposition 30, the right-hand side lies in $S^2H \subset \Lambda^{2,0}$. Since the map

$$\ker \partial_K \ni t \rightarrow \text{tr}(\Theta_0 \lrcorner t) \in S^2H \subset \Lambda^{2,0}$$

is easily seen to be an isomorphism, there is a unique ω_A such as in the statement. \square

We shall refer to the connection of Theorem 31 as the *qc connection*.

Example 32. Let us consider the sphere, with the connection ω considered in Section 3. Then the connection

$$\omega_{qc} = \omega + A, \quad A = -e^a \otimes w^s \otimes (e_a \lrcorner \omega_s)$$

has torsion Θ_0 and curvature

$$\Omega_{qc} = d\omega + \frac{1}{2}[\omega, \omega] + DA + \frac{1}{2}[A, A].$$

For X, Y in \mathfrak{m} , we find

$$\Omega_{qc}(X, Y) = \Omega(X, Y) + dA(X, Y) + \frac{1}{2}[A, A](X, Y);$$

since EH is abelian $[A, A] = 0$, and we compute

$$\begin{aligned} \Omega_{qc} = & - \sum_{a < b} e^{ab} \otimes e^a \wedge e^b - \sum_{a < b, s} e^{ab} \otimes e_a \lrcorner \omega_s \wedge e_b \lrcorner \omega_s - (\omega_s - 2w_s \lrcorner w^{123}) \otimes \omega_s \\ & - \sum_{a, s, r} e_a \lrcorner \omega_r \wedge w^r \otimes w^s \otimes (e_a \lrcorner \omega_s). \end{aligned}$$

In particular $\Omega_{qc}^{2,0}$ is traceless, and ω_{qc} is the qc connection. In fact, the curvature is contained in the trivial submodules of R_1 and R_3 . This can also be seen as a consequence of the fact that the curvature is both G -invariant and H -equivariant as a map

$$\Omega_\theta: G \rightarrow \Lambda^2 T^* \otimes \mathfrak{k},$$

and so must take values in an invariant space.

Example 33. Similarly, the qc connection on the homogeneous space

$$\mathrm{Sp}(n, 1)\mathrm{Sp}(1)/\mathrm{Sp}(n)\mathrm{Sp}(1)$$

is related to the connection ω considered in Section 3 via

$$\omega_{qc} = \omega + A, \quad A = e^a \otimes w^s \otimes (e_a \lrcorner \omega_s);$$

in this case

$$\begin{aligned} \Omega_{qc} = & \sum_{a < b} e^{ab} \otimes e^a \wedge e^b + \sum_{a < b, s} e^{ab} \otimes e_a \lrcorner \omega_s \wedge e_b \lrcorner \omega_s + (\omega_s + 2w_s \lrcorner w^{123}) \otimes \omega_s \\ & - \sum_{a, s, r} e_a \lrcorner \omega_r \wedge w^r \otimes w^s \otimes (e_a \lrcorner \omega_s). \end{aligned}$$

Thus, the scalar component in R_3 is the same as in the case of the sphere, whilst the component in R_1 has the opposite sign.

Example 34. Consider now the solvable Lie group G of [3] characterized by the existence of left-invariant one-forms e^1, \dots, e^7 such that

$$\begin{aligned} (de^1, \dots, de^7) = & (0, e^{15} + e^{34} - e^{46}, -e^{24} + e^{16} + e^{45}, -2e^{14}, \\ & e^{12} - e^{34} + e^{46}, e^{13} - e^{42} - e^{45}, e^{14} - e^{23} + e^{56}) \end{aligned} \quad (13)$$

The coframe e^1, \dots, e^7 defines a K -structure. Applying Proposition 17 to the $(-)$ connection, we obtain:

$$\Theta_{-1} = e^{46} \otimes w_1 - e^{45} \otimes w_2, \quad g(\Theta_{-1}) = e^{134567} \otimes w_1 - e^{124567} \otimes w_2 = (D\gamma)^{3,3}.$$

Thus $\Theta_2 = \Theta_1 = 0$ and the structure is integrable. The qc connection is given by

$$\begin{aligned} \omega_{qc} = & \frac{1}{4}e^5 \otimes \omega_1 + \frac{1}{4}e^6 \otimes \omega_2 - \left(\frac{1}{2}e^4 + \frac{1}{4}e^7\right) \otimes \omega_3 - \left(\frac{3}{2}e^4 + \frac{1}{2}e^7\right) \otimes (e^{14} + e^{23}) \\ & - \frac{3}{4}e^1 \otimes w^s \otimes (e_{1 \lrcorner} \omega_s) + \frac{1}{4}e^2 \otimes w^s \otimes (e_{2 \lrcorner} \omega_s) + \frac{1}{4}e^3 \otimes w^s \otimes (e_{3 \lrcorner} \omega_s) \\ & - \frac{3}{4}e^4 \otimes w^s \otimes (e_{4 \lrcorner} \omega_s). \end{aligned}$$

In terms of the modules of Proposition 30, we see that the curvature is contained in $\tilde{R}_1 + \tilde{R}_3$, and has a non-zero component in each irreducible submodule of $\tilde{R}_1 + \tilde{R}_3$.

Recall that two qc manifolds (M, \mathcal{D}) , (M', \mathcal{D}') are said to be *qc conformal* if there exists a diffeomorphism $\phi: M \rightarrow M'$ such that $d\phi$ maps \mathcal{D} into \mathcal{D}' . It was proved in [12] that a qc manifold is locally qc conformal to the Heisenberg group if and only if a tensor called qc conformal curvature is zero (see also [16, 1]).

The existence of such a local diffeomorphism that also preserves the $\mathrm{Sp}(n)\mathbb{H}^*$ -structure can be characterized by the flatness of the qc connection:

Corollary 35. *A qc $\mathrm{Sp}(n)\mathbb{H}^*$ -structure is locally equivalent to the standard structure on the Heisenberg group if and only if the qc connection is flat.*

Proof. On the Heisenberg group, the $(-)$ connection has torsion Θ_0 and zero curvature, so it satisfies the conditions of Theorem 31; consequently, the qc connection is flat, and one implication is proved.

Conversely, assume that ω_{qc} is flat. Then every point has a neighbourhood U on which an adapted parallel coframe e^1, \dots, e^{4n+3} is defined; since the torsion is Θ_0 , it follows that

$$de^1 = 0 = \dots = de^{4n}, \quad de^{4n+s} = \omega_s, \quad s = 1, 2, 3.$$

In other words, the dual basis of vector fields defines a subalgebra of $\mathfrak{X}(U)$ isomorphic to the Lie algebra of the Heisenberg group; by a standard result (see [7, Theorem 1.8.3]), this shows that U is locally equivalent to the Heisenberg group. \square

Since the qc connection on the sphere is not flat, it follows that the standard $\mathrm{Sp}(n)\mathbb{H}^*$ -structures on the sphere and the Heisenberg group are not equivalent. However, it was shown in [14] that these qc manifolds are locally conformally equivalent via the Cayley transform; this is not an equivalence of $\mathrm{Sp}(n)\mathbb{H}^*$ -structures, because it does not preserve the complement.

9 Qcm structures

There is no natural choice of metric on a qc K -structure, although there is a conformal class of metrics on the horizontal distribution. In this section we

begin by fixing a metric in this class, partly to compare our results with those of [2, 6]. Indeed, we recover the known fact that the choice of metric determines the complement in full. In addition we construct a canonical metric connection whose torsion lies in an affine space parallel to $S^2(EH) + W \otimes EH$. Its curvature lies in a “small” submodule which we identify, and is almost entirely determined by the torsion and its derivative. This connection differs from the Biquard connection; it should be regarded as the metric version of the qc connection of Section 8.

In our language, the choice of a metric amounts to considering an arbitrary reduction of an integrable qc K -structure to $G = \mathrm{Sp}(n)\mathrm{Sp}(1) \ltimes EH$. Since ∂_G is injective, the argument of the remark on p. 15 shows that the sequence

$$0 \rightarrow \ker \partial_K \rightarrow T^* \otimes \mathbb{R} \xrightarrow{\partial} \mathrm{Coker} \partial_G \rightarrow \mathrm{Coker} \partial_K \rightarrow 0.$$

is exact. Thus, the G -intrinsic torsion takes values in $\Theta_0^G + \partial(T^* \otimes \mathbb{R})$. In analogy with Theorem 22, this suggests that some reduction to $\mathrm{Sp}(n)\mathrm{Sp}(1)$ kills a component of the intrinsic torsion isomorphic to EH .

It turns out that such a reduction exists and is unique; in other words, a qc distribution and a compatible metric on the distribution determine a canonical $\mathrm{Sp}(n)\mathrm{Sp}(1)$ -structure:

Proposition 36. *Given an integrable qc structure and a choice of metric on the horizontal distribution compatible with the structure, there is a unique reduction P to $\mathrm{Sp}(n)\mathrm{Sp}(1)$ such that $\Theta^G|_P \equiv \Theta_0^G$ and the metric is compatible with P .*

Proof. Corollary 23 gives a canonical K -reduction; by Theorem 31, its K -intrinsic torsion is Θ_0^K . The choice of metric gives a reduction to G .

In terms of the structure group $\mathrm{Sp}(n)\mathrm{Sp}(1)$, the calculations of Lemma 25 give

$$(w^s \otimes v \lrcorner \omega_s) \omega_r \otimes w_r - \Theta_0 = (w^s \otimes v \lrcorner \omega_s) \cdot \Theta_0 \pmod{\mathrm{Im} \partial_G}.$$

By Lemma 21, the right hand side lies in the EH containing $\frac{1}{2}\tilde{\alpha}_1 - \alpha_4 + i\alpha_5$, which is not in $\mathrm{Im} \partial_G$. Thus, the argument of Theorem 22 applies, and we find a unique reduction to $\mathrm{Sp}(n)\mathrm{Sp}(1)$ that satisfies the required condition.

The uniqueness follows from the fact that the torsion condition in the definition implies that the B -intrinsic torsion is Θ_0^B , which makes the reduction to K unique. \square

We can now prove that the two definitions of integrability for seven-dimensional qc structures agree:

Corollary 37. *A qc structure is integrable in the sense of Duchemin if and only if it is compatible with an integrable qc K -structure.*

Proof. An integrable qc structure in the sense of Duchemin is a qc B -structure that about each point admits a section s such that (with obvious notation)

$$(s^* d\eta)^{1,1} \in \Lambda^{1,1} \otimes W \cong V \otimes W \otimes W$$

is skew-symmetric in the last two indices. Fix a B -connection ω on such a structure; Proposition 17 gives

$$(d\eta + \omega \wedge \eta)^{1,1} = (D\eta)^{1,1} = \Theta_2 + \Theta_{-1}.$$

By construction, both $s^*(\omega \wedge \eta)$ and Θ_{-1} have no component in $V \otimes S_0^2 W$. Thus, if $s^*(d\eta)^{1,1}$ is skew-symmetric then Θ_2 is zero.

Conversely, given an integrable qc K -structure, take a reduction P such as in Proposition 36, and choose a G -connection on P with torsion $\Theta = \Theta_0$. Then

$$(d\eta + \omega \wedge \eta)^{1,1} = (D\eta)^{1,1} = 0;$$

because of how G is defined, in this case $s^*(\omega \wedge \eta)$ has no component in $V \otimes S^2 W$, so $s^*(d\eta)^{1,1}$ is skew-symmetric for any section of P . \square

Proposition 36 motivates the following:

Definition 38. A quaternionic-contact metric (qcm) structure on a $4n + 3$ -dimensional manifold is an $\mathrm{Sp}(n)\mathrm{Sp}(1)$ -structure with G -intrinsic torsion equal to Θ_0^G .

In this language, Proposition 36 asserts that an integrable qc structure determines a family of qcm structures, one for each choice of compatible metric on the horizontal distribution. Notice that the qc structure underlying a qcm structure is always integrable.

Having reduced the structure group, we refine the usual decomposition as

$$\begin{aligned} \Lambda^2 T^* \otimes T = \mathrm{Im}(\partial_{\mathrm{Sp}(n)\mathrm{Sp}(1)}) \oplus \Lambda^2 V^* \otimes W \oplus \partial_1(W_1) \oplus \partial_2(W_2) \oplus \partial(W_3) \\ \oplus \partial(V^* \otimes \mathbb{R}) \oplus \partial(T^* \otimes EH), \end{aligned}$$

and decompose the torsion of any $\mathrm{Sp}(n)\mathrm{Sp}(1)$ -connection as

$$\Theta = \Theta_* + \Theta^Q + \Theta_1 + \cdots + \Theta_5. \quad (14)$$

By construction, any connection on a qcm structure satisfies

$$\Theta^Q = \Theta_0, \quad \Theta_1 = \cdots = \Theta_4 = 0.$$

We shall need the following lemma in order to characterize the curvature of the qcm connection. The final part of the lemma will also be used in the proof of Corollary 41.

Lemma 39. *Consider the projections*

$$\begin{aligned} p_1^{\mathrm{sp}(1)}: \tilde{R}_1 \rightarrow \Lambda^{2,0} \otimes \mathfrak{sp}(1), \quad p_2^{\mathrm{sp}(1)}: \tilde{R}_2 \rightarrow \Lambda^{1,1} \otimes \mathfrak{sp}(1), \\ p_2^{EH}: \tilde{R}_2 \rightarrow \Lambda^{2,0} \otimes EH, \quad p_3^{EH}: \tilde{R}_3 \rightarrow \Lambda^{1,1} \otimes EH. \end{aligned}$$

Then

$$\mathrm{Im} p_1^{\mathrm{sp}(1)} \cong S^2(EH), \quad \mathrm{Im} p_2^{\mathrm{sp}(1)} \cong EH + ES^3H, \quad \ker p_2^{EH} = 0 = \ker p_3^{EH}.$$

Moreover, if $n > 1$, and \mathbb{R} denotes the trivial module in $\Lambda^{1,0} \otimes EH$,

$$\mathrm{Im} p_2^{EH} \cap (\mathfrak{sp}(1) \otimes EH + \Lambda^{1,0} \wedge \mathbb{R}) = 0.$$

If $n = 1$, $\mathrm{Im} p_2^{EH}$ is the direct sum of $ES^3H \subset \mathfrak{sp}(1) \otimes EH$ and a diagonal EH in $\mathfrak{sp}(1) \otimes EH + \Lambda^{1,0} \wedge \mathbb{R}$.

Proof. By construction $\ker p_1^{\mathfrak{sp}(1)}$ is the preimage of $s(V^* \otimes \widetilde{EH})$ in $\Lambda^{2,0} \otimes \mathfrak{sp}(n)$ under δ . It is easy to check that the submodule of $s(V^* \otimes \widetilde{EH})$ isomorphic to $S^2(EH)$ is transverse to $\Lambda^{3,0} \otimes V$, which contains $\delta(\Lambda^{2,0} \otimes \mathfrak{sp}(n))$. Thus, $\ker p_1^{\mathfrak{sp}(1)}$ is the kernel of δ in $\Lambda^{2,0} \otimes \mathfrak{sp}(n)$.

The fact that the image of $p_2^{\mathfrak{sp}(1)}$ contains $EH + ES^3H$ follows immediately from (11) and (12). By Schur's lemma, equality holds.

The projection p_3^{EH} is injective because otherwise δ would not be injective on $\Lambda^{0,2} \otimes (\mathfrak{sp}(n) + \mathfrak{sp}(1))$, which is absurd.

Similarly, if p_2^{EH} were not injective then $\delta(\Lambda^{1,1} \otimes (\mathfrak{sp}(n) + \mathfrak{sp}(1)))$ would intersect $s(W^* \otimes \widetilde{EH})$ non-trivially; then $\partial(\Lambda^{1,0} \otimes (\mathfrak{sp}(n) + \mathfrak{sp}(1)))$ would intersect \widetilde{EH} , which is absurd because by Lemma 14 $\tilde{\alpha}_1$ is not a linear combination of $\partial(\alpha_1)$ and $\partial(\alpha_2)$.

The last part of the statement amounts to proving that the intersection

$$\delta(\Lambda^{2,0} \otimes (\mathfrak{sp}(n) + \mathfrak{sp}(1)) + \mathfrak{sp}(1) \otimes EH + \Lambda^{1,0} \wedge \mathbb{R}) \cap s(W^* \otimes \widetilde{EH})$$

is $EH + ES^3H$ when $n = 1$ and zero otherwise. Computing with highest weight vectors, we see that for $n = 1$

$$\begin{aligned} & s((w^2 + iw^3) \otimes (\frac{1}{2}\tilde{\alpha}_1 - \alpha_4 + i\alpha_5)) \\ &= -\frac{1}{2}\delta\left(\left((w^2 + iw^3) \wedge v_1h_1 + iw^1 \wedge v_1h_2\right) \otimes (\omega_2 + i\omega_3) + (w^2 + iw^3) \wedge \alpha_2 \right. \\ & \quad \left. + v_jh_2 \wedge v_{n+j}h_2 \wedge (iw^1 \otimes v_1h_2 + (w^2 + iw^3) \otimes v_1h_1)\right), \end{aligned}$$

showing that the intersection contains ES^3H ; no such equality holds for $n > 1$. Similarly, one verifies that the intersection only contains EH when $n = 1$, but the relevant EH projects non-trivially to both $\mathfrak{sp}(1) \otimes EH$ and $\Lambda^{1,0} \wedge \mathbb{R}$. \square

Theorem 40. *The intrinsic torsion of a qcm structure lies in*

$$\text{Span}\{\Theta_0\} + S^2ES^2H + \Lambda_0^2E + \mathbb{R} + EH + ES^3H.$$

In particular any qcm structure has a unique connection with torsion

$$\Theta_0 + \partial(\chi_V) + \partial(\chi_W), \quad \chi_V \in S^2(EH) \subset V^* \otimes EH, \quad \chi_W \in W^* \otimes EH;$$

its curvature satisfies

$$\begin{aligned} \Omega^{2,0} &\in S^2(EH) + S^4E, \quad \Omega^{1,1} \in S^3EH + ES^3H + EH, \\ \Omega^{0,2} &\in S^2ES^2H + S^4H + \mathbb{R}. \end{aligned}$$

Moreover there are linear equivariant maps

$$\begin{aligned} f_1: S^2(EH) &\rightarrow \Lambda^{2,0} \otimes (\mathfrak{sp}(n) + \mathfrak{sp}(1)), & f_2: W^* \otimes EH &\rightarrow \Lambda^{1,1} \otimes (\mathfrak{sp}(n) + \mathfrak{sp}(1)) \\ f_3: \Lambda^{2,0} \otimes EH &\rightarrow \Lambda^{1,1} \otimes (\mathfrak{sp}(n) + \mathfrak{sp}(1)), & f_4: \Lambda^{1,1} \otimes EH &\rightarrow \Lambda^{0,2} \otimes (\mathfrak{sp}(n) + \mathfrak{sp}(1)) \end{aligned}$$

such that f_1 and f_2 are injective,

$$\Omega^{2,0} - f_1(\chi_V) \in S^4E, \quad \Omega^{1,1} - f_2(\chi_W) \in S^3EH, \quad \Omega^{0,2} = f_4(D(\chi_V + \chi_W)^{1,1}),$$

and $\Omega^{1,1} - f_3((D\chi_V)^{2,0})$ is in ES^3H (and zero when $n > 1$).

Proof. Consider an $\mathrm{Sp}(n)\mathrm{Sp}(1)$ connection ω with torsion $\Theta_0 + \Theta_5$, where

$$\Theta_5 = \partial(\chi_V) + \partial(\chi_W), \quad \chi_V \in V^* \otimes EH, \quad \chi_W \in W^* \otimes EH.$$

Then

$$\omega_{qc} = \omega - \chi_V - \chi_W$$

is a K -connection with curvature

$$\Omega_{qc} = \Omega - D(\chi_V + \chi_W);$$

in particular, $\mathrm{tr} \Omega_C$ is zero, so ω_{qc} is the qc connection and, in the notation of Lemma 29,

$$\Omega_{qc} \in \tilde{R}_1 + \tilde{R}_2 + \tilde{R}_3 + R_4.$$

In particular

$$[\Omega_{qc}^{2,0}]_{\mathrm{sp}(1)} \in S^2 ES^2 H + \Lambda_0^2 E + \mathbb{R}, \quad [\Omega_{qc}^{1,1}]_{\mathrm{sp}(1)} \in ES^3 H + EH.$$

Since η and Θ_0 are parallel under ω , we compute

$$D\eta = \Theta_0 \lrcorner \eta = \Theta_0, \quad \Omega \wedge \eta = D\Theta_0 = \Theta_5 \lrcorner \Theta_0.$$

Now

$$(\Theta_5 \lrcorner \Theta_0)^{2,1} = \partial(\chi_V) \lrcorner \Theta_0, \quad (\Theta_5 \lrcorner \Theta_0)^{1,2} = \partial(\chi_W) \lrcorner \Theta_0.$$

Notice that applying this argument to the $(0, 3)$ part we obtain no information, as both $[\Omega_{qc}^{0,2}]_{\mathrm{sp}(1)} \wedge \eta$ and $(\Theta_5 \lrcorner \Theta_0)^{0,3}$ are zero.

Using the fact that $\Omega = [\Omega_{qc}]_{\mathrm{sp}(n)+\mathrm{sp}(1)}$, we deduce

$$[\Omega_{qc}^{2,0}]_{\mathrm{sp}(1)} \wedge \eta = \partial(\chi_V) \lrcorner \Theta_0, \quad [\Omega_{qc}^{1,1}]_{\mathrm{sp}(1)} \wedge \eta = \partial(\chi_W) \lrcorner \Theta_0. \quad (15)$$

It is easy to verify that the map

$$T^* \otimes EH \rightarrow \Lambda^{2,1} \otimes W, \quad v \rightarrow \partial(v) \lrcorner \Theta_0 \quad (16)$$

is injective; thus, (15) implies that χ_V is in $S^2(EH)$. By Lemma 39, χ_V and χ_W determine part of the curvature, and the dependence can be expressed by linear maps f_1, f_2 as in the statement. Notice that f_1 and f_2 are necessarily injective because so is the map (16).

We can now write

$$D\chi_V \in T^* \wedge S^2(EH) + \Theta \lrcorner \chi_V, \quad D\chi_W \in (\Lambda^{1,1} + \Lambda^{0,2}) \otimes EH + \Theta \lrcorner \chi_W,$$

where

$$\Theta \lrcorner \chi_V = \partial(\chi_V + \chi_W) \lrcorner \chi_V, \quad \Theta \lrcorner \chi_W = \Theta_0 \lrcorner \chi_W;$$

therefore,

$$\begin{aligned} \Omega_{qc}^{2,0} &= \Omega^{2,0} - \Theta_0 \lrcorner \chi_W - (D\chi_V)^{2,0}, \quad \Omega_{qc}^{1,1} = \Omega^{1,1} - (D\chi_V)^{1,1} - (D\chi_W)^{1,1}, \\ \Omega_{qc}^{0,2} &= \Omega^{0,2} - \partial(\chi_W) \lrcorner \chi_V - (D\chi_W)^{0,2}. \end{aligned}$$

In consequence,

$$\begin{aligned} \Omega^{2,0} &\in \tilde{R}_1, \quad \Omega^{1,1} - \Theta_0 \lrcorner \chi_W - (D\chi_V)^{2,0} \in \tilde{R}_2, \\ \Omega^{0,2} - (D\chi_V)^{1,1} - (D\chi_W)^{1,1} &\in \tilde{R}_3. \end{aligned}$$

Taking p_2^{EH} we see that $\Omega^{1,1}$ is determined by $\Theta_0 \lrcorner \chi_W + (D\chi_V)^{2,0}$, where the first summand lies in $S^2H \otimes EH$. It also follows from Lemma 39 that $(D\chi_V)^{2,0}$ alone determines all of $\Omega^{1,1}$ when $n > 1$, and its component $S^3EH + EH$ when $n = 1$.

Similarly, the fact that p_3^{EH} is injective shows that $(D\chi_V + D\chi_W)^{1,1}$ determines $\Omega^{0,2}$. \square

We shall refer to the connection of Theorem 40 as the *qcm connection*; it can be regarded as a canonical object of qcm geometry in the same way that the qc connection is canonical in qc geometry, since the two are related by the obvious projection from \mathfrak{k} to $\mathfrak{sp}(n) + \mathfrak{sp}(1)$. The qcm connection is also natural in that it coincides with the natural connection that appears in the three fundamental examples of Section 3.

Remark. The curvature of the qcm connection has the remarkable property that the torsion (together with its covariant derivative) determines all of its curvature except for $S^4E \subset \tilde{R}_1$. This component of the curvature is the same whether one considers the qc, qcm or Biquard connection (see Corollary 47), and can be identified with the qc conformal curvature tensor constructed in [12].

Remark. We could have considered the curvature of the qc connection instead, but this would have changed little, because its curvature, as shown in the course of the proof, can be identified with the curvature of the qcm connection by means of a projection to $\mathfrak{sp}(n)\mathfrak{sp}(1)$. The fact that the modules appearing in Theorem 40 are smaller than those of Proposition 30 is a consequence of the fact that we are now working with a qcm structure rather than an arbitrary $\mathrm{Sp}(n)\mathrm{Sp}(1)$ reduction of a qc structure.

Recall that the (horizontal) Ricci tensor is defined as

$$\mathrm{Ric}(X, Y) = \sum_{a=1}^{4n} \langle e^a, \Omega(e_a, X)Y \rangle, \quad X, Y \in V;$$

notice that the contraction is only taken on indices along V . Its trace is called the qc scalar curvature. The Ricci tensor arises from an equivariant map $\Lambda^{2,0} \otimes (\mathfrak{sp}(n) + \mathfrak{sp}(1)) \rightarrow V^* \otimes V^*$. Restricting to \tilde{R}_1 , it follows by Schur's lemma that the image is contained in $S^2(V^*)$: so, the Ricci tensor is symmetric, as proved in [2] for the Biquard connection.

Remark. With a bit of work one can show that the image is exactly $S^2(V^*)$, so the Ricci of the qcm connection can be identified with χ_V . In particular, the qc scalar curvature is the scalar part of χ_V . The remaining part of the intrinsic torsion, χ_W , is easily seen to be the obstruction to the complement distribution being integrable.

A qc manifold is called *qc-Einstein* if the Biquard connection has zero traceless Ricci. This is a very strong condition, due to results of [13]. We can now prove an analogous result which uses the qcm connection instead.

Corollary 41. *On a qcm manifold of dimension greater than seven, the following conditions are equivalent:*

1. the four-form $\sum_s \omega_s^2$ is closed;
2. χ_V is a constant scalar and $\chi_W = 0$;

3. the qcm connection has curvature in $S^4E + 2\mathbb{R} \subset R_1 + R_3$;
4. the horizontal traceless Ricci of the qcm connection is zero;
5. the traceless part of χ_V is zero.

Proof. The fundamental form $\sum_s \omega_s^2$ is parallel under ω_{qcm} ; therefore, it is closed if $\Theta \lrcorner \sum \omega_s^2 = 0$, or

$$\partial(\chi_V) \lrcorner \sum_s \omega_s^2 = 0, \quad \partial(\chi_W) \lrcorner \sum_s \omega_s^2 = 0.$$

The second equation is equivalent to $\chi_W = 0$, because interior product with the fundamental form is an injection of V into $\Lambda^{3,0}$. The first equation means that χ_V is in the kernel of a map $S^2(EH) \rightarrow \Lambda^{4,1}$ which is easily checked to have kernel \mathbb{R} (since we assume $n > 1$). Thus, (1) is equivalent to (2).

The fact that (2) implies (3) follows from Theorem 40; on the other hand, (3) obviously implies (4).

The fact that (4) is equivalent to (5) follows from

$$\Omega^{2,0} - f_1(\chi_V) \in S^4E,$$

and the fact that the Ricci contraction from \tilde{R}_1 to $S^2(EH)$ has kernel S^4E , as noted in a remark above.

Now assume (5) holds; then $(D\chi_V)^{2,0} \in V^* \wedge \mathbb{R}$. By Lemma 39,

$$p_2^{EH}(\Omega^{1,1} - \Theta_0 \lrcorner \chi_W - (D\chi_V)^{2,0}) = 0.$$

This implies that both χ_W and $(D\chi_V)^{2,0}$ are zero.

Similarly, $(D\chi_V)^{1,1}$ lies in a module isomorphic to S^2H , so

$$\Omega^{0,2} - (D\chi_V)^{1,1} - (D\chi_W)^{1,1} \in \tilde{R}_3$$

implies that $(D\chi_V)^{1,1}$ is zero. Summing up, $\nabla\chi_V$ is zero, i.e. χ_V is constant. This establishes the equivalence of (2) and (5), completing the proof. \square

Remark. One can rephrase Corollary 41 in terms of the Biquard connection, for condition (5) is equivalent to $T_\xi = 0$ (Corollary 45), and condition (4) is equivalent to qc-Einstein (Corollary 48). This version of the statement was proved in [13].

Remark. There are two points in the proof where the assumption on the dimension is used. First, the map $S^2(EH) \rightarrow \Lambda^{4,1}$ whose kernel contains χ_V is zero when $n = 1$, so (1) does not imply (2) for $n = 1$. In fact, this implication has a known counterexample [3].

Secondly, the fact that χ_V is a scalar does not apparently force χ_W to vanish for $n = 1$, because of the form that Lemma 39 takes in this instance. A very recent result [9] shows however that even when the dimension is seven, χ_V is a scalar only when the vertical distribution is integrable, hence $\chi_W = 0$.

We conclude this section by noting that the natural qcm structure in each example of Section 3 satisfies the conditions of Corollary 41, with χ_V a positive, negative or zero constant. In these examples, the S^4E component of the qcm curvature is also zero. More generally, 3-Sasakian manifolds have a natural qc-Einstein structure; the converse also holds up to local homothety (see [13]).

10 The Biquard connection

In this section we compare our results with those of [2, 6]. In fact, we recover the existence of the Biquard and Duchemin connection, and express them in terms of the qcm connection, showing that all three connections exist in all dimensions. We show that the component of the torsion of the Biquard connection usually denoted by T_ξ can be identified with a traceless symmetric endomorphism of \mathcal{D} , which can be identified with the traceless Ricci of either the Biquard or the qcm connection.

We shall decompose any $\eta \in \Lambda^2 T^* \otimes T$ as

$$[\eta]_V + [\eta]_W, \quad [\eta]_V \in \Lambda^2 T^* \otimes V, \quad [\eta]_W \in \Lambda^2 T^* \otimes W.$$

Recall the following:

Theorem 42 (Biquard [2]). *If $n > 1$, given a qc structure with a fixed compatible metric on the distribution \mathcal{D} , there is a unique complement W_B and a unique connection which:*

1. *preserves both \mathcal{D} and W_B , as well as the $\mathrm{Sp}(n)\mathrm{Sp}(1)$ -structure on \mathcal{D} , and acts on W_B as on the subbundle of $\mathrm{End}(V)$ determined by the almost complex structures;*
2. *satisfies the torsion conditions*

$$[\Theta^{2,0}]_V = 0, \quad [\Theta^{1,1}]_V \in \partial(W^* \otimes (\mathfrak{sp}(n)\mathfrak{sp}(1))^\perp), \quad (17)$$

where the orthogonal complement is taken in $\mathfrak{gl}(V)$.

In seven dimensions, we have the following similar result:

Theorem 43 (Duchemin [6]). *If $n = 1$, given a qc structure with a fixed compatible metric on the distribution \mathcal{D} , then there is a unique complement W_D and a unique connection which:*

1. *preserves both \mathcal{D} and W_D , and the metrics on them;*
2. *satisfies the torsion conditions*

$$[\Theta^{2,0}]_V = 0, \quad [\Theta^{0,2}]_W = 0, \quad \Theta^{1,1} \in ES^5H + W^* \otimes S^2V. \quad (18)$$

The ES^5H component is zero if and only if the qc structure is integrable.

We shall refer to these connections as the Biquard connection and the Duchemin connection. Notice that the Duchemin connection has holonomy contained in $\mathrm{SO}(4) \times \mathrm{SO}(3)$, or $\mathrm{Sp}(n)\mathrm{Sp}(1) \times \mathrm{SO}(W)$.

We introduce two equivariant maps

$$T_B: S^2(EH) \rightarrow W^* \otimes (\mathfrak{sp}(n) + \mathfrak{sp}(1)), \quad T_D: S^2(EH) \rightarrow W^* \otimes \mathfrak{so}(W);$$

equivariance implies that T_B is zero on $\Lambda_0^2 E$ and T_D is zero on $\Lambda_0^2 E + S^2 ES^2 H$: on the remaining components, we set

$$T_D(e_a \otimes w^s \otimes e_a \lrcorner \omega_s) = 4w^s \otimes (w_s \lrcorner w^{123}), \quad T_B(e_a \otimes w^s \otimes e_a \lrcorner \omega_s) = 2w^s \otimes \omega_s, \\ T_B(v_1 h_2 \otimes ((w^2 + iw^3) \otimes v_1 h_1 + iw^1 \otimes v_1 h_2)) = \frac{1}{2}(w^2 + iw^3) \otimes v_1 h_2 \wedge v_1 h_1.$$

We can recover the existence of the Biquard and Duchemin connections as a consequence of what we have proved so far; in particular, we show that our choice of complement coincides with W_B and W_D .

Theorem 44. *On a qcm structure:*

- there is a unique connection ω_B whose torsion Θ_B satisfies (17); it is related to the qcm connection via

$$\omega_B = \omega_{qcm} + T_B(\chi_V),$$

and satisfies

$$\Theta_B^{2,0} = \Theta_0, \quad \Theta_B^{1,1} \in S^2ES^2H + \Lambda_0^2E, \quad \Theta_B^{0,2} \in EH + ES^3H + \mathbb{R};$$

- there is a unique $\mathrm{Sp}(n)\mathrm{Sp}(1) \times \mathrm{SO}(W)$ -connection ω_D whose torsion Θ_D satisfies (18), given by

$$\omega_D = \omega_{qcm} + T_B(\chi_V) + T_D(\chi_W);$$

moreover $\Theta_D = \Theta_B - [\Theta_B^{0,2}]_W$.

Given a qc structure and a metric on the horizontal distribution:

- there is a unique $\mathrm{Sp}(n)\mathrm{Sp}(1)$ -structure compatible with structure and metric that admits a connection satisfying (17);
- there is a unique $\mathrm{Sp}(n)\mathrm{Sp}(1)$ -structure compatible with structure and metric that admits an $\mathrm{Sp}(n)\mathrm{Sp}(1) \times \mathrm{SO}(W)$ -connection satisfying (18).

These two $\mathrm{Sp}(n)\mathrm{Sp}(1)$ -structures coincide, they are a qcm structure P in the sense of Definition 38, and the natural complement

$$P \times_{\mathrm{Sp}(n)\mathrm{Sp}(1)} W \subset P \times_{\mathrm{Sp}(n)\mathrm{Sp}(1)} T$$

coincides with the complement W_B (resp. W_D).

Proof. Let ω_{qcm} be the qcm connection; its torsion is determined by χ_V, χ_W as defined in Theorem 40. Consider the connection

$$\omega_B = \omega_{qcm} + \eta_V + \eta_W, \quad \eta_V \in V^* \otimes (\mathfrak{sp}(n) + \mathfrak{sp}(1)), \quad \eta_W \in W^* \otimes (\mathfrak{sp}(n) + \mathfrak{sp}(1));$$

its torsion Θ_B satisfies

$$\begin{aligned} (\Theta_B)^{2,0} &= \Theta_0 + \partial(\eta_V)^{2,0}, & (\Theta_B)^{0,2} &= \partial(\chi_W) + \partial(\eta_W)^{0,2}, \\ (\Theta_B)^{1,1} &= \partial(\chi_V) + \partial(\eta_V)^{1,1} + \partial(\eta_W)^{1,1}. \end{aligned}$$

The first equation in (17) is equivalent to $\eta_V = 0$, whereas the second equation means that

$$\partial(\chi_V) + \partial(\eta_W)^{1,1} \in \partial(W^* \otimes (\mathfrak{sp}(n) + \mathfrak{sp}(1))^\perp).$$

There is a unique solution in η_W , proving existence and uniqueness of ∇_B . Since T_B satisfies

$$\partial(T_B(v)) + \partial(v) \in \partial(W^* \otimes (\mathfrak{sp}(n) + \mathfrak{sp}(1))^\perp), \quad v \in S^2(EH),$$

the solution is given by $\eta_W = T_B(\chi_V)$. Notice that $\partial \circ T_B + \partial$ is zero on the component isomorphic to \mathbb{R} , so $(\Theta_B)^{1,1}$ lies in $S^2ES^2H + \Lambda_0^2E$.

To determine ω_D , we think of the Lie algebra of $\mathrm{Sp}(n)\mathrm{Sp}(1) \times \mathrm{SO}(W)$ as $(\mathfrak{sp}(n) + \mathfrak{sp}(1)) + \mathfrak{so}(W)$, with $\mathfrak{sp}(1)$ contained diagonally in $\mathfrak{so}(V) + \mathfrak{so}(W)$ as usual. Accordingly, we can write an arbitrary $\mathrm{Sp}(n)\mathrm{Sp}(1) \times \mathrm{SO}(W)$ -connection as $\omega_D = \omega_{qcm} + \eta_V + \eta_W + A$, where η_V and η_W are as in the first part of the proof and A is an $\mathfrak{so}(W)$ -valued tensorial 1-form. The torsion is then

$$\begin{aligned} (\Theta_D)^{2,0} &= \Theta_0 + \partial(\eta_V)^{2,0}, & (\Theta_D)^{0,2} &= \partial(\chi_W) + \partial(\eta_W)^{0,2} + \partial(A)^{0,2}, \\ (\Theta_D)^{1,1} &= \partial(\chi_V) + \partial(\eta_V)^{1,1} + \partial(\eta_W)^{1,1} + \partial(A)^{1,1}. \end{aligned}$$

The condition on $(\Theta_D)^{2,0}$ immediately implies that $\eta_V = 0$; so, the condition on $(\Theta_D)^{1,1}$ reads

$$\partial(A)^{1,1} = 0, \quad \partial(\chi_V) + \partial(\eta_W)^{1,1} \in W^* \otimes (\mathfrak{sp}(n) + \mathfrak{sp}(1))^\perp.$$

Therefore $\eta_W = T_B(\chi_V)$. Imposing

$$0 = [(\Theta_D)^{0,2}]_W = \partial(T_B(\chi_V))^{0,2} + (\partial A)^{0,2}$$

gives $A = T_D(\chi_V)$. It is now clear that the torsion Θ_D differs from Θ_B only in that the \mathbb{R} component is zero.

In order to prove the uniqueness of the qcm structure, observe that a qc structure and a metric determine a qc $\mathrm{Sp}(n)\mathrm{Sp}(1) \times \mathrm{Hom}(W, V)$ -structure. Take an arbitrary reduction to $\mathrm{Sp}(n)\mathrm{Sp}(1)$, and assume it has a connection satisfying (17). Decomposing its torsion according to (14), we find

$$[\Theta^{2,0}]_V = [(\Theta_*)^{2,0}]_V + [(\Theta_1)^{2,0}]_V + [(\Theta_4)^{2,0}]_V.$$

Write

$$\Theta_* = \partial(\eta_V + \eta_W), \quad \Theta_1 = \partial_1(\eta_1), \quad \Theta_4 = \partial(\delta_V + \delta_W),$$

with obvious notation; then $(\eta_V, \eta_1, \delta_V)$ is in the kernel of

$$V^* \otimes (\mathfrak{sp}(n) + \mathfrak{sp}(1)) \oplus W_1 \oplus V^* \otimes \mathbb{R} \rightarrow \Lambda^2 V^* \otimes V, \quad v \rightarrow [\partial(v)^{2,0}]_V.$$

This map is surjective with kernel $2EH + ES^3H$. To identify these subspaces more precisely, observe that $V^* \otimes (\mathfrak{sp}(n) + \mathfrak{sp}(1))$ intersects W_1 in $V^* \otimes \mathfrak{sp}(1)$; thus, the kernel contains $(\alpha_1, -\alpha_1, 0)$ and $(\beta_1, -\beta_1, 0)$. The calculations of Lemma 14 show that the kernel also contains

$$(\alpha_2, \alpha_3, v_1 h_2 \otimes (4\mathrm{Id}_V + 8\mathrm{Id}_W));$$

in particular, the kernel projects injectively on W_1 .

By Theorem 20, η_1 lies in the ES^3H containing $2\beta_2 + 3\beta_1$, which intersects $V^* \otimes \mathfrak{sp}(1)$ trivially; hence, $\eta_1 = 0$. Thus, $\Theta_1 = 0$; by Theorem 20 and integrability, this implies that $\Theta_2 = 0$, so $\Theta^B = \Theta_0^B$. In addition, δ_V is also forced to be zero, so P is a qcm structure. \square

This result shows that both the Duchemin and Biquard connections exist in all dimensions; moreover, they only differ by a component in $\mathbb{R} \subset W^* \otimes \mathfrak{so}(W)$, which has the effect of killing $[\Theta^{0,2}]_W$.

The tensor T_ξ that appears in the literature can be identified with $\Theta^{1,1}$. It is customary to decompose T_ξ as

$$T_\xi = T_\xi^0 + b_\xi, \quad b_\xi \in W^* \otimes \mathfrak{so}(V), \quad T_\xi^0 \in W^* \otimes S^2V.$$

Decomposing into irreducible modules,

$$b_\xi \in \Lambda_0^2 E(S^4H + S^2H + \mathbb{R}), \quad T_\xi^0 \in S^2E(S^4H + S^2H + \mathbb{R}) + \Lambda_0^2 ES^2H + S^2H;$$

however, it follows from Theorem 44 that b_ξ is really contained in $\Lambda_0^2 E$ and T_ξ^0 in S^2ES^2H . This is consistent with the results of [2] (see also [14, Proposition 2.4] and [5, Proposition 3.1]).

As an element of $S^2ES^2H + \Lambda_0^2 E$, T_ξ can be viewed as a traceless symmetric tensor; in fact, it can be identified with the traceless part of χ_V :

Corollary 45. *There is a linear map $S^2(EH) \rightarrow W^* \otimes (\mathfrak{sp}(n) + \mathfrak{sp}(1))^\perp$ with kernel \mathbb{R} that maps χ_V to T_ξ .*

Proof. It suffices to check that the map

$$S^2(EH) \rightarrow \Lambda^2 T^* \otimes T, \quad \chi_V \rightarrow \partial(\chi_V) + T_B(\chi_V)$$

has kernel \mathbb{R} . □

Example 46. Theorem 44 also tells us how to compute the torsion of the Biquard connection from that of the qcm connection. For instance, the qcm connection on the Lie group (13) satisfies

$$\chi_V = \frac{1}{2}(e^1 \otimes e^1 + e^4 \otimes e^4 - e^2 \otimes e^2 - e^3 \otimes e^3) + \frac{1}{4}e^a \otimes e_a, \quad \chi_W = 0;$$

therefore the S^2ES^2H component of torsion of the Biquard connection is the projection to $W \otimes S^2V \subset \Lambda^{1,1} \otimes V$ of

$$\partial\left(\frac{1}{2}(e^1 \otimes e^1 + e^4 \otimes e^4 - e^2 \otimes e^2 - e^3 \otimes e^3)\right)$$

i.e.

$$(\Theta_B)^{1,1} = -\frac{1}{2}e^5 \wedge (e^1 \odot e^2 - e^3 \odot e^4) - \frac{1}{2}e^6 \wedge (e^1 \odot e^3 - e^4 \odot e^2).$$

The \mathbb{R} component is

$$\partial\left(\frac{1}{2}w^s \otimes \omega_s\right)^{0,2} = w_s \lrcorner w^{123} \otimes w_s;$$

finally, the $EH + ES^3H$ component is zero because χ_W is zero, so

$$(\Theta_B)^{0,2} = \sum_s w_s \lrcorner w^{123} \otimes w_s.$$

The torsion of the Duchemin connection only differs in that $\Theta_D^{0,2} = 0$.

It is not surprising that the curvatures of the Biquard and qcm connections are related by a formula involving the torsion. The remarkable fact is that the $(2, 0)$ part of this curvature only differs by a term that depends linearly on χ_V :

Corollary 47. *On a qcm structure, the Biquard connection has curvature*

$$\Omega_B = \Omega_{qcm} + D_{qcm}T_B(\chi_V) + \frac{1}{2}[T_B(\chi_V), T_B(\chi_V)].$$

In particular,

$$(\Omega_B)^{2,0} = (\Omega_{qcm})^{2,0} + \Theta_0 \lrcorner T_B(\chi_V) \in \tilde{R}_1 + (S^2ES^2H + \mathbb{R}),$$

where the two summands intersect trivially.

Proof. The first formula is obvious. Now $[T_B(\chi_V), T_B(\chi_V)]$ has type $(0, 2)$, and

$$(D_{qcm}T_B)^{2,0} = \Theta^{2,0} \lrcorner T_B(\chi_V) = \Theta_0 \lrcorner T_B(\chi_V)$$

takes values in $S^2ES^2H + \mathbb{R}$; the relevant modules contain

$$(\omega_2 + i\omega_3) \otimes v_1h_2 \wedge v_1h_1, \quad \omega_s \otimes \omega_s.$$

In order to prove that S^2ES^2H is not contained in \tilde{R}_1 , it is sufficient to show that

$$\delta((\omega_2 + i\omega_3) \otimes v_1h_2 \wedge v_1h_1) = v_jh_2 \wedge v_{n+j}h_2 \wedge (v_1h_2 \otimes v_1h_1 - v_1h_1 \otimes v_1h_2)$$

does not lie in $s(V^* \otimes \widetilde{EH})$, which follows from

$$\begin{aligned} s(v_1h_2 \otimes v_1h_2) &= \frac{1}{2}v_1h_2 \wedge (v_{n+j}h_1 \wedge v_jh_2 + v_{n+j}h_2 \wedge v_jh_1) \otimes v_1h_2 \\ &+ v_1h_2 \wedge v_jh_2 \wedge v_{n+j}h_2 \otimes v_1h_1 + iv_1h_2 \wedge v_1h_1 \wedge (w^1 \otimes (w_2 + iw_3) - (w^2 + iw^3) \otimes w_1). \end{aligned}$$

A similar computation shows that $\delta(\omega_s \otimes \omega_s)$ does not belong to the image of s . \square

We can now use results from [14] to give a geometric characterization of the Ricci tensors.

Corollary 48. *On a qcm manifold, the component χ_V of the intrinsic torsion, the Ricci tensor of the Biquard connection and the Ricci tensor of the qcm connection coincide up to linear equivariant automorphisms of $S^2(V)$.*

Proof. By Theorem 40 and subsequent remarks, the Ricci of the qcm connection can be identified with χ_V . By Corollary 47, there is a linear equivariant endomorphism f of $S^2(EH)$ that maps χ_V to the Ricci of the Biquard connection.

To prove that f is an isomorphism, observe that by [14] the traceless Ricci of the Biquard connection can be identified with the component of T_ξ in $S^2ES^2H + \Lambda_0^2E$, which can in turn be identified with χ_V by Corollary 45. This shows that f is injective on $S^2ES^2H + \Lambda_0^2E$. Suppose that, for some n , f is zero on the component \mathbb{R} of $S^2(EH)$. This implies that on any integrable qc manifold of dimension $4n + 3$, the Biquard connection has qc scalar curvature equal to zero. The example of the sphere (see Section 3 or [2]), which exists in all dimensions, shows that is not true. \square

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