UNIVERSITÀ DEGLI STUDI DI MILANO-BICOCCA

Dottorato di Ricerca in Matematica Pura e Applicata

Riesz transforms, spectral multipliers and Hardy spaces on graphs

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Abstract

In this thesis we consider a connected locally finite graph $\mathscr{G}$ that possesses the Cheeger isoperimetric property.

We define a decreasing one-parameter family $\{\mathfrak{X}^\gamma(\mathscr{G}) : \gamma > 0\}$ of Hardy-type spaces on $\mathscr{G}$ associated to the standard nearest neighbour Laplacian $\mathscr{L}$ on $\mathscr{G}$. We show that $\mathfrak{X}^{1/2}(\mathscr{G})$ is the space of all functions in $L^1(\mathscr{G})$ whose Riesz transform is in $L^1(\mathscr{G})$. We show that if $\mathscr{G}$ has bounded geometry and γ is a positive integer, then $\mathfrak{X}^\gamma(\mathscr{G})$ admits an atomic decomposition. We also show that if $\mathscr{G}$ is a homogeneous tree and γ is not an integer, then $\mathfrak{X}^\gamma(\mathscr{G})$ does not admit an atomic decomposition. Furthermore, we consider the Hardy-type spaces $H^1_{\mathscr{H}}(\mathscr{G})$ and $H^1_{\mathscr{P}}(\mathscr{G})$, defined in terms of the heat and the Poisson maximal operators, and analyse their relationships with the spaces $\mathfrak{X}^\gamma(\mathscr{G})$. We also show that $H^1_{\mathscr{H}}(\mathscr{G})$ is properly contained in $H^1_{\mathscr{P}}(\mathscr{G})$, a phenomenon which has no counterpart in the Euclidean setting. Applications to the boundedness of the imaginary powers $\mathscr{L}^{iu}$ are also given.

Finally, we characterise, for each p in $[1, \infty)\backslash\{2\}$, the class of L^p spherical multipliers on homogeneous trees in terms of L^p Fourier multipliers on the torus. Furthermore, we give a sharp sufficient condition on L^p spherical multipliers on the product of homogeneous trees.

Acknowledgements

First of all I would like to express my deepest gratitude to my advisor, Stefano Meda, for his patience, invaluable advice and continuous support during these years.

I would also like to thank Błażej Wrobél for the fruitful collaboration and for his kind hospitality during my visit in Bonn.

I am indebted to Giancarlo Mauceri, Maria Vallarino and Alessio Martini, as well as to many other professors and researchers, for the valuable conversations we had throughout my PhD studies.

Finally I would like to thank all my collegues at the University of Milano-Bicocca for their friendship: these years wouldn't have been the same without you.

Contents

Introduction

The Hardy space $H^1(\mathbb{R}^n)$ may be defined as the subspace of $L^1(\mathbb{R}^n)$ consisting of all functions f such that the Euclidean norm of the vector Riesz transform $\left|\mathscr{R}f\right|$ is in $L^1(\mathbb{R}^n)$. Here $\mathscr{R}$ is the operator $\nabla(-\Delta)^{-1/2}$, where ∇ and Δ denote the standard gradient and Laplacian on $\mathbb{R}^n$, respectively. It is well known that $H^1(\mathbb{R}^n)$ has several characterisations, both in terms of atoms and of various maximal operators (see [St2, Ch. 3 and 4]). In particular, consider the heat and Poisson semigroups $\mathscr{H}_t$ and $\mathscr{P}_t$ on $\mathbb{R}^n$ and the corresponding heat and Poisson maximal operators $\mathscr{H}_*$ and $\mathscr{P}_*$ defined by

$$\mathscr{H}_* f = \sup_{t>0} \left|\mathscr{H}_t f\right| \qquad \text{and} \qquad \mathscr{P}_* f = \sup_{t>0} \left|\mathscr{P}_t f\right|.$$

A celebrated result [FS] states that $H^1(\mathbb{R}^n)$ agrees with the space of all functions in $L^1(\mathbb{R}^n)$ such that either $\mathscr{H}_* f$ or $\mathscr{P}_* f$ are in $L^1(\mathbb{R}^n)$. Another characterisation of $H^1(\mathbb{R}^n)$, due to R.R. Coifman and R. Latter [Co, La], is the following. We say that a function a in $L^2(\mathbb{R}^n)$ is an $H^1(\mathbb{R}^n)$-atom if the support of a is contained in a Euclidean ball, its integral vanishes, and its $L^2(\mathbb{R}^n)$ norm is suitably normalised (see Section 2 for more on atoms). Then f is in $H^1(\mathbb{R}^n)$ if and only if f admits a decomposition of the form $\sum_j c_j\, a_j$, where the a_j's are atoms, and $\sum_j \left|c_j\right| < \infty$.

We recall the following important additional feature of $H^1(\mathbb{R}^n)$: besides the Riesz transform, some interesting operators, which are bounded on $L^p(\mathbb{R}^n)$ for all p in $(1,\infty)$, but unbounded on $L^1(\mathbb{R}^n)$, turn out to be bounded from $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$. Furthermore, an operator which is bounded on $L^2(\mathbb{R}^n)$ and from $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ is automatically bounded on $L^p(\mathbb{R}^n)$ for all p in $(1,2)$.

For similar results concerning $H^1(\mathbb{Z}^n)$, see [BC].

The purpose of this thesis is to develop a theory of Hardy-type spaces on locally finite connected graphs $\mathscr{G}$ that possess the *Cheeger isoperimetric property*, (see Definition 1.3). For each vertex x, denote by $\nu(x)$ the number of neighbours of x. For some, but not all, of our results, we need also to assume that $\mathscr{G}$ has *bounded geometry*, i.e., the function ν is bounded. We endow $\mathscr{G}$ with the measure that associates to the singleton $\{x\}$ the measure $\nu(x)$. Note that μ is nondoubling (for $\mathscr{G}$ has exponential volume growth). In particular, $\mathscr{G}$ is not a space of homogeneous type in the sense of Coifman–Weiss. This fact will have long-range consequences on the theory of Hardy-type spaces on $\mathscr{G}$ that we shall develop.

We follow the approach developed by G. Mauceri, S. Meda and M. Vallarino [MMV1, MMV2, MMV3, MMV4] and S. Meda and S. Volpi [MVo], with various modifications. One reason for considering the class of graphs above is that they are discrete analogues of Riemannian manifolds with bounded geometry and spectral gap. Homogeneous trees are typical examples of graphs satisfying the assumptions above, and may be regarded as discrete analogues of Riemannian symmetric spaces of the noncompact type and real rank one. We strongly believe that our results on trees will pave the way to further developments on Riemannian manifolds with bounded geometry and spectral gap.

It is not hard to see that the nearest neighbour Laplacian $\mathscr{L}$ (see formula (1.1)) is a sectorial operator on $L^1(\mathscr{G})$ and that $\mathscr{L}^\gamma$ is a bounded injective sectorial operator on $L^1(\mathscr{G})$ for all $\gamma > 0$. We then define, for each $\gamma > 0$, the Hardy-type space $\mathfrak{X}^\gamma(\mathscr{G})$ to be the vector space $\mathscr{L}^\gamma\big(L^1(\mathscr{G})\big)$, endowed with the norm which makes $\mathscr{L}^\gamma$ an isometry between $L^1(\mathscr{G})$ and $\mathfrak{X}^\gamma(\mathscr{G})$. We shall prove that $\{\mathfrak{X}^\gamma(\mathscr{G}) : \gamma > 0\}$ is a decreasing family of subspaces of $L^1(\mathscr{G})$, that the Calderón complex interpolation space $\big(\mathfrak{X}^\gamma(\mathscr{G}), L^2(\mathscr{G})\big)_{[\theta]}$ between $\mathfrak{X}^\gamma(\mathscr{G})$ and $L^2(\mathscr{G})$ is $L^{p_\theta}(\mathscr{G})$, where θ is in $(0,1)$ and $p_\theta = 2/(2-\theta)$, and that if γ is a positive integer, then $\mathfrak{X}^\gamma(\mathscr{G})$ admits an atomic decomposition in terms of atoms satisfying a strong cancellation condition.

Furthermore, we shall prove that the *purely imaginary powers* of $\mathscr{L}$, i.e. the operators $\mathscr{L}^{iu}$ where u is in $\mathbb{R}\setminus\{0\}$ are bounded from $\mathfrak{X}^\gamma(\mathscr{G})$ to $L^1(\mathscr{G})$ for all $\gamma > 0$. Observe that if $\mathscr{G}$ is a homogeneous tree, then $\mathscr{L}^{iu}$ is unbounded on $L^1(\mathscr{G})$, for otherwise its convolution kernel $k_{\mathscr{L}^{iu}}$ would be in $L^1(\mathscr{G})$, and this is false, as can be easily derived from the asymptotics of $k_{\mathscr{L}^{iu}}$ (see [CMS3]).

By analogy with the Euclidean case, it is natural to consider the space $H^1_{\mathscr{R}}(\mathscr{G})$, defined by

$$H^1_{\mathscr{R}}(\mathscr{G}) = \big\{ f \in L^1(\mathscr{G}) : \big|\mathscr{R} f\big| \in L^1(\mathscr{G}) \big\},$$

where $\mathscr{R}$ is the *discrete Riesz transform*, briefly Riesz transform, defined by $\nabla \mathscr{L}^{-1/2}$, ∇ denotes the discrete gradient on $\mathscr{G}$, and $\mathscr{L}^{-1/2}$ is defined via the spectral theorem. We shall show that $H^1_{\mathscr{R}}(\mathscr{G}) = \mathfrak{X}^{1/2}(\mathscr{G})$ in great generality, i.e. without assumng that $\mathscr{G}$ has bounded geometry. As a consequence, $\mathscr{R}$ is bounded from $\mathfrak{X}^{\gamma}(\mathscr{G})$ to $L^1(\mathscr{G})$ if and only if $\gamma \geq 1/2$. The analogue on trees of the Helgason–Fourier transform on noncompact symmetric spaces allows us to prove that if $\mathscr{G}$ is a homogeneous tree, then $\mathfrak{X}^{1/2}(\mathscr{G})$, hence $H^1_{\mathscr{R}}(\mathscr{G})$, does not admit an atomic decomposition. Specifically, we prove that functions with compact support are not dense in $\mathfrak{X}^{1/2}(\mathscr{G})$. This is also a new phenomenon, which has no counterpart in the Euclidean setting.

We also consider the problem of relating the Hardy-type spaces $\mathfrak{X}^{\gamma}(\mathscr{G})$ to Hardy-type spaces defined in terms of maximal operators. Our analysis will require rather precise estimates of the size of the kernels of the heat and Poisson semigroups. We are not able to establish such estimates on a generic graph of bounded geometry with the isoperimetric property, and we restrict to homogeneous trees $\mathscr{T}$, where spherical Fourier analysis is available (see [CMS3]). By analogy with the Euclidean case, it is natural to consider the heat semigroup $\{\mathrm{e}^{-t\mathscr{L}} : t \geq 0\}$ and the Poisson semigroup $\{\mathrm{e}^{-t\mathscr{L}^{1/2}} : t \geq 0\}$ (we shall often write $\mathscr{P}_t$ instead of $\mathrm{e}^{-t\mathscr{L}^{1/2}}$), and the associated maximal operators

$$\mathscr{H}_* f := \sup_{t\geq 1} \big|\mathscr{H}_t f\big| \qquad \text{and} \qquad \mathscr{P}_* f := \sup_{t\geq 1} \big|\mathscr{P}_t f\big|.$$

We then define $H^1_{\mathscr{H}}(\mathscr{G})$ and $H^1_{\mathscr{P}}(\mathscr{G})$ by

$$H^1_{\mathscr{H}}(\mathscr{G}) = \{ f \in L^1(\mathscr{G}) : \mathscr{H}_* f \in L^1(\mathscr{G}) \}$$

and

$$H^1_{\mathscr{P}}(\mathscr{G}) = \{ f \in L^1(\mathscr{G}) : \mathscr{P}_* f \in L^1(\mathscr{G}) \}.$$

Perhaps surprisingly, we find that $H^1_{\mathscr{H}}(\mathscr{G}) \subsetneq H^1_{\mathscr{P}}(\mathscr{G})$, and that $H^1_{\mathscr{R}}(\mathscr{G}) \subsetneq H^1_{\mathscr{P}}(\mathscr{G})$. Our analysis hinges on precise estimates of the heat and the Poisson kernel obtained via spherical Fourier analysis. Our analysis requires the understanding of the following two families of maximal operators. For each (possibly negative) real number c, we consider the *Poisson maximal operator* $\mathscr{P}_*^c$ *with parameter* c by

$$\mathscr{P}_*^c f = \sup_{t \geq 1} \, t^c \left| \mathscr{P}_t f \right|.$$

We shall often write $\mathscr{P}_* f$, instead of $\mathscr{P}_*^0$. Similarly, for every real number c the maximal operator $\mathscr{H}_*^c$ is defined by

$$\mathscr{H}_*^c f = \sup_{t \geq 1} \, t^c \left| \mathscr{H} f \right|.$$

A consequence of the statement above is that the heat maximal operator $\mathscr{H}_*$ is unbounded on $H^1_{\mathscr{R}}(\mathscr{T})$. Notice that the situation is quite different from that described above for Euclidean spaces.

In the last chapter of the thesis, we consider spherical Fourier multipliers on homogeneous trees. We treat both the case of a single tree and the case of the product of two trees with possibly different degrees of homogeneity. We remark that our methods extend trivially to the product of a finite number of homogeneous trees. However, to avoid exceeding notational complexity, we give details only in the case of the product of two trees.

First we consider the case of a single tree. The analogue on trees of a celebrated result of J.L. Clerc and E.M. Stein [CSt] states that if k is in $Cv_p(\mathscr{T})$, then its spherical Fourier transform $\widetilde{k}$ extends to a bounded holomorphic function on the strip $\mathbf{S}_{\delta(p)}$ (see Section 1.6 for the definition of $\mathbf{S}_{\delta(p)}$). This necessary condition was sharpened by M. Cowling, S. Meda and A.G. Setti [CMS2, Theorem 2.1], who proved that if k is in $Cv_p(\mathscr{T})$, then the boundary values $\widetilde{k}_{\delta(p)}$ of $\widetilde{k}$ on the strip $\mathbf{S}_{\delta(p)}$ is a multiplier of $L^p(\mathbb{T})$. We prove that this condition is indeed sufficient, thus giving a characterisation of radial convolutors of $L^p(\mathscr{T})$ (see Theorem 5.11). The proof of this result combines techniques from [CMS1] and involves a generalisation of transference results of A.D. Ionescu [Io] for rank one noncompact symmetric spaces.

Next we consider the product of two trees. In this case, we prove that the two-variables analogue of the condition on multipliers cited above is sufficient to induce a bounded convolution operator on $L^p(\mathscr{T}_1 \times \mathscr{T}_2)$.

We shall use the "variable constant convention", and denote by C, possibly with sub- or superscripts, a constant that may vary from place to place and may depend on any factor quantified (implicitly or explicitly) before its occurrence, but not on factors quantified afterwards.

If A is a bounded linear operator from the Banach spaces $\mathscr{X}$ to the Banach space $\mathscr{Y}$, we denote by $|||A|||_{\mathscr{X};\mathscr{Y}}$ its operator norm. If $\mathscr{X} = \mathscr{Y}$, we write $|||A|||_{\mathscr{X}}$ instead of $|||A|||_{\mathscr{X};\mathscr{X}}$.

Chapter 1

Background material and preliminary results

1.1 Graphs

Denote by $\mathscr{G}$ an infinite connected unoriented graph. We say that two points x and y in $\mathscr{G}$ are neighbours, and write $x \sim y$, if they are connected by an edge. We assume that $\mathscr{G}$ is *locally finite*, i.e. every point x in $\mathscr{G}$ has a finite number $\nu(x)$ of neighbours. If

$$\sup_{x \in \mathscr{G}} \nu(x) < \infty,$$

then we say that $\mathscr{G}$ has *bounded geometry*.

A path in $\mathscr{G}$ is a finite number of points $[x_0, \ldots, x_J]$ with the property that x_j and x_{j+1} are neighbours for $j = 0, \ldots, J-1$. The *length* of $[x_0, \ldots, x_J]$ is defined to be J. We endow $\mathscr{G}$ with the so called *combinatorial distance*: $d(x, y)$ is the length of the shortest path joining x and y. Since $\mathscr{G}$ is connected, $d(x, y) > 0$ when $x \neq y$, and $d(x, y) = 1$ if and only if x and y are neighbours. The ball $B_r(x)$ with centre x and radius r is the set of all points y in $\mathscr{G}$ such that $d(x, y) \leq r$, and $S_r(x)$ denotes the set of all y such that $d(x, y) = r$. We call $S_r(x)$ the sphere with centre x and radius r. Notice that $S_r(x)$ is nonempty if and only if r is a nonnegative integer.

We endow $\mathscr{G}$ with the measure μ, defined by

$$\mu\big(\{x\}\big) := \nu(x) \qquad \forall x \in \mathscr{G}.$$

Notice that if the function $x \mapsto \nu(x)$ is constant (this happens, for instance, when $\mathscr{G}$ is a homogeneous tree), then μ is a constant multiple of the counting measure. Lebesgue spaces will be taken with respect to the measure μ, unless ν is constant, in which case usage of the counting measure leads to cleaner formulae. Thus,

$$\big\|f\big\|_p := \Big[\sum_{x\in\mathscr{G}} \big|f(x)\big|^p\,\nu(x)\Big]^{1/p}$$

when p is in $[1,\infty)$, and

$$\big\|f\big\|_\infty := \sup_{x\in\mathscr{G}} \big|f(x)\big|.$$

1.2 The Laplacian

Denote by $\mathscr{E}$ the set of the *oriented* edges of $\mathscr{G}$: an element of $\mathscr{E}$ is of the form $e = (x, y)$, where x and y are neighbours in $\mathscr{G}$; x and y are called the *initial* and the *final* point of e, respectively. We put the counting measure on $\mathscr{E}$, and define the operator $\mathrm{d} : L^2(\mathscr{G}) \to L^2(\mathscr{E})$ by

$$\mathrm{d}f(e) := f(y) - f(x) \qquad \forall e = (x, y) \in \mathscr{E}\ :$$

d may be thought of as the differential operator on $\mathscr{G}$. We shall also write ∇f instead of $\mathrm{d}f$.

Given an oriented edge e, we denote by e_- and e_+ the initial and the final point of e, respectively. A straightforward calculation shows that the Hilbert space adjoint of d is the operator $\mathrm{d}^* : L^2(\mathscr{E}) \to L^2(\mathscr{G})$, defined by

$$\mathrm{d}^* g(x) = \frac{1}{\nu(x)} \sum_{e_+=x} g(e).$$

A natural operator acting on complex-valued functions on $\mathscr{G}$ is the nearest neighbour Laplacian $\mathscr{L}$, defined by $\mathscr{L} = \mathrm{d}^*\,\mathrm{d}$. Thus,

$$\mathscr{L} f(x) := \frac{1}{\nu(x)} \sum_{y\sim x} \big[f(x) - f(y)\big] \qquad \forall x \in \mathscr{G}. \tag{1.1}$$

The associated Dirichlet form Q is given by

$$\begin{aligned}
Q(f) &= (\mathscr{L} f, f) \\
&= \sum_{x\in\mathscr{G}} \nu(x)\, \mathscr{L} f(x)\, \overline{f(x)} \\
&= \sum_{x\in\mathscr{G}} \sum_{y\sim x} \big[f(x) - f(y)\big]\, \overline{f(x)} \\
&= \sum_{x\in\mathscr{G}} \nu(x)\, \big|f(x)\big|^2 - \sum_{x\in\mathscr{G}} \sum_{y\sim x} f(y)\, \overline{f(x)} \\
&= \frac{1}{2}\Big[\sum_{x\in\mathscr{G}} \nu(x)\, \big|f(x)\big|^2 + \sum_{y\in\mathscr{G}} \nu(y)\, \big|f(y)\big|^2 - \sum_{(x,y)\in\mathscr{E}} \big[f(x)\overline{f(y)} + f(y)\, \overline{f(x)}\big]\Big] \\
&= \frac{1}{2} \sum_{e\in\mathscr{E}} \big|f(y) - f(x)\big|^2 \\
&= (\,\mathrm{d}f,\, \mathrm{d}f) \qquad \forall f \in L^2(\mathscr{G}).
\end{aligned}$$

Note that the length $\big|\nabla f\big|$ of ∇f, i.e.,

$$\big|\nabla f(x)\big| = \Big[\frac{1}{2} \sum_{y\sim x} \big|f(x) - f(y)\big|^2\Big]^{1/2} \qquad \forall x \in \mathscr{G},$$

is a function on $\mathscr{G}$. Note that

$$\begin{aligned}
0 \le Q(f) &\le \frac{1}{2} \sum_{e\in\mathscr{E}} \Big[\big|f(x)\big|^2 - 2\,\mathrm{Re}\,\big[f(x)\, \overline{f(y)}\big] + \big|f(y)\big|^2\Big] \\
&\le \sum_{e\in\mathscr{E}} \Big[\big|f(x)\big|^2 + \big|f(y)\big|^2\Big] \\
&= 4 \sum_{x\in\mathscr{G}} \nu(x)\, \big|f(x)\big|^2 \\
&= 4\, \big\|f\big\|_2^2 .
\end{aligned}$$

In particular, this implies that $\sigma_2(\mathscr{L}) \subseteq [0, 2]$. It is straighforward to check that $\mathscr{L}$ is symmetric on $L^2(\mathscr{G})$, hence it is self adjoint.

Define the *adjacency matrix* A by

$$A(x, y) = \begin{cases} \dfrac{1}{\nu(x)} & \text{if } \; y \sim x \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\mathscr{L} = \mathscr{I} - \mathscr{A},$$

where $\mathscr{A}$ is the operator associated to the kernel A, i.e.

$$\mathscr{A} f(x) = \sum_{y \in \mathscr{G}} A(x, y)\, f(y) = \frac{1}{\nu(x)} \sum_{y \sim x} f(y).$$

Notice that $\mathscr{A} f(x)$ is simply the mean value of f over the sphere with centre x and radius 1.

Notice that $\mathscr{L}$ is bounded on $L^p(\mathscr{G})$ for every p in $[1, \infty]$, no matter whether $\sup_{x \in \mathscr{G}} \nu(x)$ is finite or not, i.e., whether $\mathscr{G}$ has bounded geometry or not. This makes some aspects of harmonic analysis on $\mathscr{G}$ simpler than the corresponding issues on noncompact Riemannian manifolds.

Proposition 1.1. *Suppose that $\mathscr{G}$ is a locally finite connected graph. For every p in $[1, \infty]$ the operator $\mathscr{L}$ is bounded on $L^p(\mathscr{G})$, and*

$$\left\| \mathscr{L} f \right\|_p \leq 2 \left\| f \right\|_p \qquad \forall f \in L^p(\mathscr{G}).$$

Proof. Observe that the operator $\mathscr{L}$ is bounded on $L^1(\mathscr{G})$. Indeed,

$$\begin{aligned} \left\| \mathscr{L} f \right\|_1 &\leq \sum_{x \in \mathscr{G}} \nu(x) \left| f(x) \right| + \sum_{x \in \mathscr{G}} \sum_{y \sim x} \left| f(y) \right| \\ &= \left\| f \right\|_1 + \sum_{y \in \mathscr{G}} \nu(y) \left| f(y) \right| \\ &= 2 \left\| f \right\|_1 \qquad \forall f \in L^p(\mathscr{G}). \end{aligned}$$

Furthermore, $\mathscr{L}$ is bounded on $L^\infty(\mathscr{G})$, because

$$\begin{aligned} \left\| \mathscr{L} f \right\|_\infty &\leq \left\| f \right\|_\infty + \sup_{x \in \mathscr{G}} \frac{1}{\nu(x)} \sum_{y \sim x} |f(y)| \\ &\leq 2 \left\| f \right\|_\infty \qquad \forall f \in L^\infty(\mathscr{G}). \end{aligned}$$

The required result follows from Riesz–Thorin's theorem. □

It is well known that the bottom b of the $L^2(\mathscr{G})$ spectrum of $\mathscr{L}$ is given by the variational formula

$$b = \inf \frac{(\mathscr{L} f, f)}{\|f\|_2^2},$$

where the infimum is taken over all not identically vanishing functions f with compact support in $\mathscr{G}$. See, for instance, [Wj] for a proof of this fact. Since $\mathscr{L}$ is a positive operator on $L^2(\mathscr{G})$, $b \geq 0$.

1.3 The heat and Poisson semigroups

By the spectral theorem, the operator $\mathrm{e}^{-t\mathscr{L}}$ is contractive on $L^2(\mathscr{G})$ for all positive t. We refer to the family $\{\mathrm{e}^{-t\mathscr{L}} : t \geq 0\}$ as to the *heat semigroup*, and often we write $\mathscr{H}_t$ instead of $\mathrm{e}^{-t\mathscr{L}}$.

Proposition 1.2. *Suppose that $\mathscr{G}$ is a locally finite connected graph. Then $\{\mathscr{H}_t : t \geq 0\}$ is a Markovian semigroup.*

Proof. We need to prove that $\mathscr{H}_t$ is a positive symmetric contractive operator on $L^p(\mathscr{G})$ for all p in $[1, \infty]$, and that $\mathscr{H}_t \mathbf{1} = \mathbf{1}$.

First recall that $\mathscr{L} = \mathscr{I} - \mathscr{A}$, where $\mathscr{A}$ is the operator naturally associated to the adjacency matrix A defined above. Clearly $\mathscr{I}$ and $\mathscr{A}$ commute, so that

$$\mathrm{e}^{-t\mathscr{L}} = \mathrm{e}^{-t} \sum_{k=0}^{\infty} \frac{(t\mathscr{A})^k}{k!} \tag{1.2}$$

Now observe that $\mathscr{A}$ is a contraction on $L^p(\mathscr{G})$. Indeed, by arguing much as in the proof of Proposition 1.1 above, we see that $\mathscr{A}$ is a contraction on $L^1(\mathscr{G})$ and $L^\infty(\mathscr{G})$, and the required contractivity property on $L^p(\mathscr{G})$ follows from the Riesz–

Thorin interpolation theorem. Therefore

$$\begin{aligned}
\left|\!\left|\!\left| \mathrm{e}^{-t\mathscr{L}} \right|\!\right|\!\right|_p &\le \mathrm{e}^{-t} \sum_{k=0}^{\infty} \frac{\left(t \left|\!\left|\!\left| \mathscr{A} \right|\!\right|\!\right|_p\right)^k}{k!} \\
&\le \mathrm{e}^{-t} \sum_{k=0}^{\infty} \frac{t^k}{k!} \\
&= 1.
\end{aligned}$$

Furthermore, observe that $\mathscr{A}$ is positivity preserving, hence so is $\mathscr{A}^k$ for every k. Formula (1.2) then implies that $\mathrm{e}^{-t\mathscr{L}}$ is positivity preserving.

Finally, denote by $\mathbf{1}$ the function identically equal to 1 on $\mathscr{G}$, and notice that $\mathscr{A}\mathbf{1} = \mathbf{1}$, whence $\mathrm{e}^{-t\mathscr{L}}\mathbf{1} = \mathbf{1}$. □

A well known consequence of Proposition 1.2 is that for each p in $[1, \infty]$ the $L^p(\mathscr{G})$ spectrum of $\mathscr{L}$ is contained in the closure of the right half plane.

In addition to the heat kernel, it is natural to consider the subordinated semigroups $\{\mathrm{e}^{-t\mathscr{L}^\alpha} : t \ge 0\}$, for $0 < \alpha < 1$. In particular, if $\alpha = 1/2$, then the semigroup is called the *Poisson semigroup*. We refer the reader to [Y] for more on subordinated semigroups.

1.4 The isoperimetric property

An important role in our theory is played by the so called Cheeger isoperimetric property, which we now introduce.

Definition 1.3. For any finite subgraph $\mathscr{G}_0$ of $\mathscr{G}$, its boundary $\partial\mathscr{G}_0$ is defined by

$$\partial\mathscr{G}_0 := \{y \in \mathscr{G}_0 : d(y, \mathscr{G}_0^c) = 1\}.$$

Note that $\partial\mathscr{G}_0$ is contained in $\mathscr{G}_0$. We set

$$\beta := \inf \frac{L\big(\partial\mathscr{G}_0\big)}{\mu(\mathscr{G}_0)},$$

where $L(\partial\mathscr{G}_0)$ denotes the *length* of $\mathscr{G}_0$, defined as the cardinality of the set of all points in $\mathscr{G}_0^c$ that admit a neighbour in $\partial\mathscr{G}_0$, and the infimum is taken over all (nonempty) finite subgraphs $\mathscr{G}_0$ of $\mathscr{G}$: β is called the *Cheeger constant* of $\mathscr{G}$. If $\beta > 0$, then we say that $\mathscr{G}$ possesses the Cheeger isoperimetric property.

We shall use the following characterisation of graphs that possess the Cheeger isoperimetric property.

Proposition 1.4. *Suppose that $\mathscr{G}$ is a locally finite connected graph. The following are equivalent:*

(i) $\mathscr{G}$ possesses the Cheeger isoperimetric inequality;

(ii) the bottom b of the $L^2(\mathscr{G})$ spectrum of the standard nearest neighbour Laplacian is strictly positive.

Furthermore,

$$\frac{1}{2}\,\beta^2 \leq b \leq \beta.$$

Proof. A proof of the inequality $(1/2)\beta^2 \leq b$ may be found in [DK, Section 2]. We omit the details. Clearly this proves that (i) implies (ii).

Next we prove that $b \leq \beta$ for every locally finite connected graph. Suppose that $\mathscr{G}_0$ is a nonempty connected finite subgraph of $\mathscr{G}$. Observe that

$$(\mathbf{1}_{\mathscr{G}_0}, \mathbf{1}_{\mathscr{G}_0}) = \mu(\mathscr{G}_0).$$

Furthermore,

$$(\mathscr{L}\mathbf{1}_{\mathscr{G}_0}, \mathbf{1}_{\mathscr{G}_0}) = \sum_{x \in \mathscr{G}_0} \nu(x)\,\mathscr{L}\mathbf{1}_{\mathscr{G}_0}(x),$$

because $\mathbf{1}_{\mathscr{G}_0}$ vanishes off $\mathscr{G}_0$. Observe that $\nu(x)\,\mathscr{L}\mathbf{1}_{\mathscr{G}_0}(x)$ vanishes whenever x is in $\mathscr{G}_0 \setminus \partial\mathscr{G}_0$ for $\mathbf{1}_{\mathscr{G}_0}$ is constant on $\mathscr{G}_0$. Now, if x is in $\partial\mathscr{G}_0$, then

$$\nu(x)\,\mathscr{L}\mathbf{1}_{\mathscr{G}_0}(x) = \#\big\{y \sim x : y \notin \mathscr{G}_0\big\}.$$

Thus,

$$(\mathscr{L}\mathbf{1}_{\mathscr{G}_0}, \mathbf{1}_{\mathscr{G}_0}) = L(\partial\mathscr{G}_0),$$

and we may conclude that

$$b \leq \frac{(\mathscr{L}\mathbf{1}_{\mathscr{G}_0}, \mathbf{1}_{\mathscr{G}_0})}{(\mathbf{1}_{\mathscr{G}_0}, \mathbf{1}_{\mathscr{G}_0})} = \frac{L(\partial\mathscr{G}_0)}{\mu(\mathscr{G}_0)},$$

and the required estimate follows by taking the infimum of both sides with respect to all finite nonempty subgraphs $\mathscr{G}_0$ of $\mathscr{G}$.
Clearly this proves that (ii) implies (i), and conludes the proof of the proposition. □

A noteworthy consequence of Cheeger's isoperimetric property is the so called Federer–Fleming inequality, which we now state.

Theorem 1.5. *Suppose that $\mathscr{G}$ is a connected locally finite graph, which possesses the Cheeger property. Then there exists a positive constant c_{FF} such that*

$$\big\||\nabla g|\big\|_1 \geq c_{FF} \, \big\|g\big\|_1$$

for every integrable function g.

Proof. The proof, which hinges on the discrete co-area formula, may be found in [Ch, Section VI.4]. □

It would be desirable to establish geometric criteria that identify classes of graphs possessing Cheeger's isoperimetric inequality. Here we content ourselves to recall a criterion established by R.K. Wojciechowski in his thesis [Wj].

Given a graph $\mathscr{G}$, and a vertex x_0 in $\mathscr{G}$, write $r(x)$ instead of $d(x, x_0)$. For a vertex x, define

$$\begin{aligned} \nu_0(x) &:= \big\{y \sim x : r(y) = r(x)\big\} \\ \nu_{+1}(x) &:= \big\{y \sim x : r(y) = r(x) + 1\big\} \\ \nu_{-1}(x) &:= \big\{y \sim x : r(y) = r(x) - 1\big\}. \end{aligned} \tag{1.3}$$

Theorem 1.6 (Wojciechowski)**.** *Suppose that $\mathscr{G}$ is a locally finite graph and that there exists a positive constant c such that*

$$\frac{\nu_{+1}(x) - \nu_{-1}(x)}{\nu(x)} \geq c \qquad \forall x \in \mathscr{G}.$$

Then the bottom b of the $L^2(\mathscr{G})$ spectrum of $\mathscr{G}$ satisfies $b \geq c^2/2$.

Recall that a graph with no loops is called a *tree*. For instance, $\mathbb{Z}$ is a tree, but $\mathbb{Z}^2$ is not. Observe that if $\mathscr{G}$ is a tree, then $\nu_0(x) = 0$ and $\nu_{-1}(x) = 1$ for each vertex x. Therefore $\nu_{+1}(x) = \nu(x) - 1$, and

$$\frac{\nu_{+1}(x) - \nu_{-1}(x)}{\nu(x)} = 1 - \frac{2}{\nu(x)}.$$

Clearly the right hand side is bounded from below by a positive constant if and only if $\nu(x) \geq 3$ for every vertex x. Thus, we may state the following corollary.

Corollary 1.7. *Suppose that $\mathscr{G}$ is a locally finite tree. If $\nu(x) \geq 3$ for every vertex x, then $\mathscr{G}$ possesses the Cheeger isoperimetric inequality, equivalently the bottom of the $L^2(\mathscr{G})$ spectrum of $\mathscr{L}$ is strictly positive.*

1.5 Locally compact groups

Denote by Γ an arbitrary locally compact group. We denote by λ and ρ a left and a right Haar measure on Γ, respectively. Integration will be with respect to λ, unless otherwise specified. We denote by Δ_Γ the modular function on Γ, i.e. the Radon–Nykodim derivative $\mathrm{d}\lambda/\,\mathrm{d}\rho$. We denote by $*_\Gamma$ the convolution on Γ, defined by

$$f *_\Gamma g(x) = \int_\Gamma f(xy)\, g(y^{-1})\,\mathrm{d}\lambda(y) = \int_\Gamma f(y)\, g(y^{-1}x)\,\mathrm{d}\lambda(y),$$

for "nice" functions f and g on Γ. We recall the following basic convolution inequalities (see e.g. [HR, Corollary 20.14 (ii) and (iv)]),

$$\big\|k *_\Gamma f\big\|_{L^p(\Gamma,\,\mathrm{d}\lambda)} \leq \big\|f\big\|_{L^p(\Gamma,\,\mathrm{d}\lambda)}\, \big\|k\big\|_{L^1(\Gamma,\,\mathrm{d}\lambda)} \tag{1.4}$$

$$\left\|f *_\Gamma k\right\|_{L^p(\Gamma,\,\mathrm{d}\lambda)} \leq \left\|f\right\|_{L^p(\Gamma,\,\mathrm{d}\lambda)} \left\|\Delta_\Gamma^{-1/p'} k\right\|_{L^1(\Gamma,\,\mathrm{d}\lambda)} . \tag{1.5}$$

We denote by $Cv_p(\Gamma)$ the space of bounded *right* convolutors of $L^p(\Gamma)$. This space is equipped with the norm

$$\left\|k\right\|_{Cv_p(\Gamma)} = \sup_{\|f\|_{L^p(\Gamma)}=1} \left\|f *_\Gamma k\right\|_{L^p(\Gamma)} .$$

1.6 Homogeneous trees

An important subclass of graphs with bounded geometry and Cheeger's property is that of homogeneous trees. A homogeneous tree of degree q is a connected graph $\mathscr{T}$ with no loops such that any point x of $\mathscr{T}$ has exactly $q+1$ neighbours. In this case it is convenient to endow $\mathscr{T}$ with the counting measure $\widetilde{\mu}$. Note that $\widetilde{\mu} = (q+1)\,\mu$, so that it is just a matter of convenience to use $\widetilde{\mu}$ rather than μ.

If $q = 1$, then $\mathscr{T}$ is just the graph associated to the integers $\mathbb{Z}$. In this case many aspects of harmonic analysis on $\mathscr{T}$ resemble their analogues on $\mathbb{R}$, and the techniques employed are reminiscent of those typical in Euclidean harmonic analysis. In this thesis we do not pursue this analysis any further, and refer the interested reader to [BC] and the references therein.

We assume henceforth that $q \geq 2$. Standard references concerning harmonic analysis on trees are the books [FTP, FTN]. The reader is also referred to the papers [CMS1, CMS2, CMS3, CS, MS1, MS2, Se1, Se2] for various aspects of harmonic analysis on homogeneous trees. Some of the ideas and results contained in these papers corroborate the idea that harmonic analysis on homogeneous trees is very much related to harmonic analysis on symmetric spaces of the noncompact type.

Fix an arbitrary reference point o in $\mathscr{T}$, denote by G the group of isometries of $\mathscr{T}$ (endowed with the natural distance) and denote by G_o the stabiliser of o in G. The group G_0 is a maximal compact subgroup of G. The map $g \mapsto g \cdot o$ identifies $\mathscr{T}$ with the coset space G/G_o; thus, a function f on $\mathscr{T}$ gives rise to a G_o-invariant function f' on G by the formula $f'(g) = f(g \cdot o)$, and every G_o-invariant function arises in this way. The distance of x from o will be denoted by $|x|$. A function f on

$\mathscr{T}$ is called radial if $f(x)$ depends only on $|x|$, or equivalently if f is G_o-invariant, or if f' is G_o-bi-invariant. We endow the totally disconnected group G with the Haar measure such that the mass of the open subgroup G_o is 1. Thus

$$\int_G f'(g \cdot o) \, \mathrm{d}g = \sum_{x \in \mathscr{T}} f(x),$$

for all finitely supported functions on $\mathscr{T}$. The reader can find much more on the group G in the book of A. Figà-Talamanca and C. Nebbia [FTN].

Suppose that $\mathscr{K}$ is an invariant continuous linear operator from $L^1(\mathscr{T})$ to $L^\infty(\mathscr{T})$. Denote by $K : \mathfrak{X} \times \mathfrak{X} \to \mathbb{C}$ the kernel of $\mathscr{K}$, defined by

$$K(x, y) = \mathscr{K} \delta_y(x) \qquad \forall x, y \in \mathscr{T}.$$

Then

$$\mathscr{K} f(x) = \sum_{y \in \mathfrak{X}} K(x, y) \, f(y) \qquad \forall x \in \mathfrak{X} \quad \forall f \in L^p(\mathscr{T}).$$

We shall be particularly interested in the invariant operators, i. e., those which commute with the action of the isometry group G of $\mathfrak{X}$. It is easy to see that the condition $\mathscr{K}(f \circ g) = (\mathscr{K} f) \circ g$ for all g in G is equivalent to the condition that $K(g \cdot x, g \cdot y) = K(x, y)$ for all x and y in $\mathfrak{X}$ and g in G, or the condition that $K(x, y)$ depends only on $d(x, y)$. We write k' for the function on G such that

$$k'(g) = K(g \cdot o, o) \qquad \forall g \in G.$$

Then $k'(g_1 g g_2) = k'(g)$ for all g in G and g_1, g_2 in G_o, and so there exists a radial function k on $\mathscr{T}$ such that $k'(g) = k(g \cdot o)$. Further, for f in $L^1(\mathscr{T})$,

$$\begin{aligned} (\mathscr{K} f)'(g) &= \mathscr{K} f(g \cdot o) \\ &= \sum_{y \in \mathscr{T}} K(g \cdot o, y) \, f(y) \\ &= \int_G K(g \cdot o, h \cdot o) \, f(h \cdot o) \, \mathrm{d}h \qquad \forall g \in G. \end{aligned}$$

Now, by the invariance of the kernel K,

$$K(g \cdot o, h \cdot o) = K(h^{-1} g \cdot o, o),$$

so that

$$\begin{aligned}(\mathscr{K} f)'(g) &= \int_G K(h^{-1}g \cdot o, o)\, f(h \cdot o)\,\mathrm{d}h \\ &= \int_G f'(h)\, k'(h^{-1}g)\,\mathrm{d}h \\ &= f' * k'(g) \qquad \forall g \in G.\end{aligned} \tag{1.6}$$

The study of invariant operators on $\mathscr{T}$ is thus essentially a part of the harmonic analysis of G, namely the study of operators from $L^p(G/G_o)$ to $L^r(G/G_o)$ given by convolution on the right by G_o-bi-invariant functions.

We denote by $Cv_p(\mathscr{T})$ the space of *radial* functions k on $\mathscr{T}$ associated to all these G_o–bi-invariant kernels. The norm of an element k in $Cv_p(\mathscr{T})$ is equal to the norm of its G_o–bi-invariant extension k' to G in $Cv_p(G)$.

Since the identification of G_o–right-invariant and G_o–bi-invariant functions on G with functions and with radial functions on $\mathscr{T}$ are standard, we shall henceforth usually not distinguish between these, and omit primes.

Notice that $\mathscr{T}$ is a nondoubling measured metric space. To see this, fix a reference point o in $\mathscr{T}$, and consider, for each positive integer n, the sphere $S_n(o)$ and the ball $B_n(o)$ with centre o and radius n. It is straightforward to check that

$$\mu\big(S_n(o)\big) = (q+1)\, q^{n-1} \qquad \text{and} \qquad \mu\big(B_n(o)\big) = \frac{q^{n+1}+q^n-2}{q-1}.$$

Thus, the ratio $\mu\big(B_{2n}(o)\big)/\mu\big(B_n(o)\big)$ has order of magnitude q^n as n tends to infinity.

The standard nearest neighbour Laplacian $\mathscr{L}$ on $\mathscr{T}$ becomes

$$\mathscr{L} f(x) = f(x) - \frac{1}{q+1}\sum_{y\sim x} f(y).$$

Set

$$\tau := 2\pi/\log q, \tag{1.7}$$

and, for every p in $[1,\infty]$, write $\delta(p)$ for $\big|1/p - 1/2\big|$ and p' for the conjugate index $p/(p-1)$. For any nonnegative real number t, we denote by $\mathbf{S}_t$ and $\overline{\mathbf{S}}_t$ the strip $\{z \in \mathbb{C} : |\operatorname{Im} z| < t\}$ and its closure, respectively. Denote by γ the entire function defined by the formula

$$\gamma(z) = \frac{q^{1/2}}{q+1}\left(q^{iz} + q^{-iz}\right).$$

Then

$$\gamma(z) = \frac{2q^{1/2}}{q+1}\cos(z\log q) = \gamma(0)\cos(z\log q). \tag{1.8}$$

The spherical Fourier transform of $\delta_o - \nu$ is $1-\gamma$, and using this one may show that the L^p spectrum $\sigma_p(\mathscr{L})$ of $\mathscr{L}$ is the image of $\overline{\mathbf{S}}_p$ under the map $1-\gamma$ (see Chapter 2 of [FTN]). The $L^p(\mathscr{T})$ spectrum of the Laplacian $\sigma_p(\mathscr{L})$ is well known (see [CMS3]). Indeed, $\sigma_p(\mathscr{L})$ is the region of all w in $\mathbb{C}$ such that

$$\Big(\frac{1-\mathrm{Re}(w)}{\gamma(0)\cosh(\delta(p)\log q)}\Big)^2 + \Big(\frac{\mathrm{Im}(w)}{\gamma(0)\sinh(\delta(p)\log q)}\Big)^2 \le 1. \tag{1.9}$$

In particular $\sigma_2(\mathscr{L})$ degenerates to the real segment $[1-\gamma(0), 1+\gamma(0)]$. As a consequence, $\mathscr{L}$ is invertible on $L^p(\mathscr{T})$ for $1 < p < \infty$. A straightforward computation leads to

$$\mathscr{L}^{-1}\,\delta_o(x) = q^{1-|x|}.$$

We denote by b_p the infimum of $\mathrm{Re}(\sigma_p(\mathscr{L}))$, so $b_2 = 1-\gamma(0)$ and $b_1 = 0$.

We now summarise the main features of spherical harmonic analysis on $\mathscr{T}$. The spherical functions are the radial eigenfunctions of the Laplace operator $\mathscr{L}$ satisfying the normalisation condition $\phi(o) = 1$, and are given by the formula

$$\phi_z(x) = \begin{cases} \Big(1 + \dfrac{q-1}{q+1}|x|\Big)\,q^{-|x|/2} & \forall z \in \tau\mathbb{Z} \\ \Big(1 + \dfrac{q-1}{q+1}|x|\Big)\,q^{-|x|/2}(-1)^{|x|} & \forall z \in \tau/2 + \tau\mathbb{Z} \\ \mathbf{c}(z)\,q^{(iz-1/2)|x|} + \mathbf{c}(-z)\,q^{(-iz-1/2)|x|} & \forall z \in \mathbb{C}\setminus(\tau/2)\mathbb{Z}, \end{cases}$$

where $\mathbf{c}$ is the meromorphic function defined by the rule

$$\mathbf{c}(z) = \frac{q^{1/2}}{q+1}\,\frac{q^{1/2+iz} - q^{-1/2-iz}}{q^{iz} - q^{-iz}} \qquad \forall z \in \mathbb{C}\setminus(\tau/2)\mathbb{Z}. \tag{1.10}$$

It can be shown (see, e.g., [CMS3]) that $z \mapsto \phi_z(x)$ is an entire function for each x in $\mathscr{T}$, and

$$\big|\phi_z(x)\big| \le 1 \qquad \forall x \in \mathscr{T} \quad \forall z \in \overline{\mathbf{S}}_{1/2}. \tag{1.11}$$

It should perhaps be noted that we use a different parametrisation of the spherical functions from Figà-Talamanca and his collaborators (see, e. g., [FTP] and [FTN]); our ϕ_z corresponds to their $\phi_{1/2+iz}$, and $\mathbf{c}$ is reparametrised similarly.

For any function space $E(\mathscr{T})$ on $\mathscr{T}$, we denote by $E(\mathscr{T})^\sharp$ the (usually closed) subspace of $E(\mathscr{T})$ of radial functions.

The spherical Fourier transform $\widetilde{f}$ of a function f in $L^1(\mathscr{T})^\sharp$ is given by the formula

$$\widetilde{f}(z) = \sum_{x\in\mathscr{T}} f(x)\,\phi_z(x) \qquad \forall z \in \overline{\mathbf{S}}_{1/2}. \tag{1.12}$$

The symmetry properties of the spherical functions imply that $\widetilde{f}$ is even and τ-periodic in the strip $\mathbf{S}_{1/2}$. We denote the torus $\mathbb{R}/\tau\mathbb{Z}$ by $\mathbb{T}$, and usually identify it with $[-\tau/2, \tau/2)$. Set

$$c_{_G} = \frac{q\log q}{4\pi(q+1)}. \tag{1.13}$$

The following theorems are well known.

Theorem 1.8. *The spherical Fourier transformation extends to an isometry of $L^2(\mathscr{T})^\sharp$ onto $L^2(\mathbb{T},\mu)$, and corresponding Plancherel and inversion formulae hold:*

$$\|f\|_2 = c_{_G}\Big[\int_{\mathbb{T}} |\widetilde{f}(s)|^2\,|\mathbf{c}(s)|^{-2}\,\mathrm{d}s\Big]^{1/2} \qquad \forall f\in L^2(\mathscr{T})^\sharp,$$

and

$$f(x) = c_{_G}\int_{\mathbb{T}} \widetilde{f}(s)\,\phi_s(x)|\mathbf{c}(s)|^{-2}\,\mathrm{d}s \qquad \forall x\in\mathscr{T}$$

for "nice" radial functions f on $\mathscr{T}$.

Proof. See, for instance, Chapter 2 of [FTN]. □

We note that the relation $\overline{\mathbf{c}(z)} = \mathbf{c}(-\overline{z})$ and the symmetry properties of spherical functions imply that

$$c_{_G}\,\phi_s(x)\,|\mathbf{c}(s)|^{-2} = c_{_G}\,\mathbf{c}(-s)^{-1}\,q^{(is-1/2)|x|} + c_{_G}\,\mathbf{c}(s)^{-1}\,q^{(-is-1/2)|x|},$$

for all x in $\mathscr{T}$ and s in $\mathbb{T}$. Therefore, if the function $m : \mathbb{R} \to \mathbb{C}$ is even and τ–periodic, then

$$\begin{aligned} &c_G \int_{\mathbb{T}} m(s)\,\phi_s(x)|\mathbf{c}(s)|^{-2}\,\mathrm{d}s \\ &= c_G \int_{\mathbb{T}} m(s)\,\mathbf{c}(-s)^{-1} q^{(is-1/2)|x|}\,\mathrm{d}s + c_G \int_{\mathbb{T}} m(s)\,\mathbf{c}(s)^{-1} q^{(-is-1/2)|x|}\,\mathrm{d}s, \end{aligned}$$

and by making the change of variables $s \mapsto -s$, we see that the two integrals on the right hand side of the equality above are equal. In particular, we may rewrite the inversion formula as follows:

$$\begin{aligned} f(x) &= 2\,c_G \int_{\mathbb{T}} \widetilde{f}(s)\,\mathbf{c}(s)^{-1}\,q^{(is-1/2)|x|}\,\mathrm{d}s \\ &= 2\,c_G\,q^{-|x|/2} \int_{\mathbb{T}} \widetilde{f}(s)\,\mathbf{c}(-s)^{-1}\,q^{is|x|}\,\mathrm{d}s. \end{aligned} \tag{1.14}$$

1.7 The boundary of a tree

A geodesic ray γ in $\mathscr{T}$ is a one-sided sequence $\{\gamma_n : n \in \mathbb{N}\}$ of points of $\mathscr{T}$ such that $d(\gamma_i, \gamma_j) = |i-j|$ for all nonnegative integers i and j. We say that x lies on γ if $x = \gamma_n$ for some n in $\mathbb{N}$. Geodesic rays $\{\gamma_n : n \in \mathbb{N}\}$ and $\{\gamma'_n : n \in \mathbb{N}\}$ are identified if there exist integers i and j such that $\gamma_n = \gamma'_{n+i}$ for all n greater than j; this identification is an equivalence relation. We denote by Ω the set of the equivalence classes, which we call boundary of $\mathscr{T}$, and by Ω_x the set of all geodesic rays starting at x. Note that for every element ω in Ω there exists a unique representative geodesic ray in Ω_x: we denote this geodesic ray by $[x, \omega)$. Given two geodesic rays $\gamma^+ = [x, \omega^+)$ and $\gamma^- = [x, \omega^-)$ with intersection $\gamma^+ \cap \gamma^- = \{x\}$ we define the doubly infinite geodesic $\gamma = \{\gamma_j : j \in \mathbb{Z}\}$ as

$$\gamma_j = \begin{cases} \gamma_j^+ & \text{if } j \geq 0 \\ \gamma_j^- & \text{if } j < 0. \end{cases}$$

If ω^+ and ω^- are two elements of Ω there exists a unique (up to renumbering) geodesic $\{\gamma_j : j \in \mathbb{Z}\}$ such that ω^+ and ω^- are the equivalence classes of $\{\gamma_j : j \in \mathbb{N}\}$

and $\{\gamma_{-j}\colon j \in \mathbb{N}\}$ respectively. For brevity, we denote this geodesic by (ω^+, ω^-) disregarding the labels.

If γ is a geodesic ray and g is an element of G, we define $g \cdot \gamma$ to be the geodesic ray $\{g \cdot \gamma_n \colon n \in \mathbb{N}\}$. It is easily verified that if γ and γ' are equivalent geodesic rays, then also $g \cdot \gamma$ and $g \cdot \gamma'$ are equivalent. This defines an action of G on Ω, which is transitive.

We fix a reference geodesic $\gamma = (\omega^-, \omega^+)$ such that o lies on γ, and assume that γ is indexed so that $\gamma_0 = o$. Define the height function h (associated to ω^+) by the rule

$$h_{\omega^+}(x) := \lim_{i\to\infty} (i - d(x, \gamma_i)) \qquad \forall x \in \mathscr{T}. \tag{1.15}$$

The level sets of the height function are called *horocycles* of $\mathscr{T}$.

Fix a point γ_0 in Ω, and denote by G_{o,γ_0} its stabiliser in G_o. It is easy to see that G_o acts transitively on Ω, so Ω may be identified with $G_o/G_{o,\gamma_0}$. It is easily verified that, if $g \in G$ and γ and γ' are equivalent geodesic rays, then $g \cdot \gamma$ and $g \cdot \gamma'$ are equivalent. This defines a transitive action of G on $[\Omega]$, so that, denoting by G_{γ_0} the stabiliser of $[\gamma_0]$ in G, we may also identify $[\Omega]$ with G/G_{γ_0}. Given that Ω may be identified with $[\Omega]$, we have a G-action on Ω, which may be written formally $(g, \gamma) \mapsto [g \cdot \gamma]_o$.

Because G_o acts transitively on the boundary, there is a unique G_o-invariant probability measure υ on Ω, which is clearly also G-quasi-invariant. By definition the Poisson kernel $P(g, \omega)$ is the Radon–Nikodym derivative $\,\mathrm{d}\upsilon(g^{-1} \cdot \omega) / \,\mathrm{d}\upsilon(\omega)$, so

$$\int_\Omega \xi(g \cdot \omega) \,\mathrm{d}\upsilon(\omega) = \int_\Omega \xi(\omega)\, P(g, \omega) \,\mathrm{d}\upsilon(\omega) \qquad \forall \xi \in C(\Omega) \quad \forall g \in G.$$

Since υ is rotation invariant, $P(gk, \omega) = P(g, \omega)$ for all g in G, k in G_o, and ω in Ω. We therefore sometimes write $P(gG_o, \omega)$ instead of $P(g, \omega)$.

Given f in $\mathscr{D}(\mathscr{T})$, we may define its full Fourier transform, written $\mathscr{H} f$ or $\widehat{f}$, by

$$\widehat{f}(\omega, z) = \sum_{x\in\mathscr{T}} f(x)\, P^{1/2+iz}(x, \omega) \qquad \forall (\omega, z) \in \Omega \times \mathbb{C}.$$

These definitions may be extended to more general classes of functions; see, e.g., [CMS1].

A straightforward consequence of Plancherel's formula is the following

$$\sum_{x\in\mathscr{T}} \int_0^\infty t\,\big|\partial_t \mathscr{P}_t \xi(x)\big|^2\,\mathrm{d}t = \frac{1}{2}\sum_{x\in\mathscr{T}} \big|\xi(x)\big|^2 \qquad \forall \xi \in L^2(\mathscr{T}). \tag{1.16}$$

Indeed, first notice that $\partial_t \mathscr{P}_t \xi = \xi * \partial_t p_t$, whence

$$\begin{aligned}(\partial_t \mathscr{P}_t \xi)\widehat{\ }(\omega, s) &= (\partial_t p_t)\widehat{\ }(s)\,\widehat{\xi}(\omega,s)\\ &= -\big(1-\gamma(s)\big)^{1/2}\,\mathrm{e}^{-t(1-\gamma(s))^{1/2}}\,\widehat{\xi}(\omega,s).\end{aligned}$$

This, Plancherel's formula, and Tonelli's theorem imply that the left hand side of (1.16) is equal to

$$\begin{aligned}&c_G \int_\Omega \mathrm{d}\omega \int_{-\tau/2}^{\tau/2} \mathrm{d}s\,\big|\mathbf{c}(s)\big|^{-2}\,\big(1-\gamma(s)\big)\,\big|\widehat{\xi}(\omega,s)\big|^2 \int_0^\infty \mathrm{e}^{-2t(1-\gamma(s))^{1/2}}\,t\,\mathrm{d}t\\ &= \frac{c_G}{2}\int_\Omega \mathrm{d}\omega \int_{-\tau/2}^{\tau/2} \big|\widehat{\xi}(\omega,s)\big|^2\,\big|\mathbf{c}(s)\big|^{-2}\,\mathrm{d}s\\ &= \frac{1}{2}\,\big\|\xi\big\|_2^2,\end{aligned}$$

as required.

A straightforward polarization argument leads to the following important formula

$$\sum_{x\in\mathscr{T}} \int_0^\infty t\,\partial_t \mathscr{P}_t \xi(x)\,\overline{\partial_t \mathscr{P}_t \eta(x)}\,\mathrm{d}t = \frac{1}{2}\sum_{x\in\mathscr{T}} \xi(x)\,\overline{\eta(x)} \qquad \forall \xi,\eta \in L^2(\mathscr{T}). \tag{1.17}$$

If f is radial, it is possible to prove (see e.g. Section 2 of [CMS1]) that $\widehat{f}(\omega,s)$ is independent of ω. In this case, by the Poisson representation of spherical functions we obtain

$$\widehat{f}(\omega,s) = \int_\Omega \widehat{f}(\omega,s)\,\mathrm{d}\omega = \sum_{x\in\mathscr{T}} f(x)\int_\Omega P^{1/2+is}(x,\omega)\,\mathrm{d}\omega = \widetilde{f}(s),$$

showing that the full Fourier transformation reduces to the spherical Fourier transformation when restricted to radial functions.

We recall the following analogue for trees of the inversion formula for the Helgason–Fourier transform on noncompact symmetric spaces.

Theorem 1.9. *For every finitely supported function f the following inversion and Plancherel formulae hold:*

$$f(x) = c_G \int_{-\tau/2}^{\tau/2} |\mathbf{c}(s)|^{-2} \, \mathrm{d}s \int_\Omega P(x,\omega)^{1/2-is} \, \widehat{f}(\omega, s) \, \mathrm{d}\omega \qquad \forall x \in \mathscr{T},$$

and

$$\sum_{x \in \mathscr{T}} \big| f(x) \big|^2 = c_G \int_{-\tau/2}^{\tau/2} \int_\Omega \big| \widehat{f}(\omega, s) \big| \, |\mathbf{c}(s)|^{-2} \, \mathrm{d}s \, \mathrm{d}\omega.$$

In particular if f is radial, then the formulae above simplify to those in Theorem 1.8.

1.8 Imaginary powers

Recall that $H^1(\mathscr{T})$ is the atomic Hardy space defined in the Introduction. We prove that if u is in $\mathbb{R} \setminus \{0\}$, then $\mathscr{L}^{iu}$ is unbounded from $H^1(\mathscr{T})$ to $L^1(\mathscr{T})$, and that the same is true of $\mathscr{R}$. The proof is similar, but easier, than the proof of the corresponding statement on symmetric spaces of the noncompact type [MMV4, Theorems 5.1 and 5.3].

Proposition 1.10. *The Riesz transform $\mathscr{R}$ and, for each complex z with* $\operatorname{Re} z \geq 0$ *and $z \neq 0$, the operator $\mathscr{L}^z$, are unbounded from $H^1(\mathscr{T})$ to $L^1(\mathscr{T})$.*

Proof. By Proposition 1.4, $\mathscr{T}$ possesses the Cheeger isoperimetric property, for $b_2 > 0$, as observed above. By the Federer–Fleming inequality (see Theorem 1.5)

$$\big\| |\nabla f| \big\|_1 \geq c_{FF} \, \big\| f \big\|_1 .$$

Denote by $x \cdot o$ a neighbour of the reference point o, and consider the function $f := \delta_{x \cdot o} - \delta_o$. We shall prove that the function $\mathscr{L}^{-1/2} f$ is not in $L^1(\mathscr{T})$.

We observe that this implies that the Riesz transform $\mathscr{R}$ does not map $H^1(\mathscr{T})$ into $L^1(\mathscr{T})$. Indeed, by Cheeger's inequality,

$$\big\| |\mathscr{R} f| \big\|_1 \geq c_{FF} \, \big\| \mathscr{L}^{-1/2} f \big\|_1 ,$$

and the right hand side is infinite.

Thus, in order to prove that $\mathscr{R}$ is unbounded from $H^1(\mathscr{T})$ to $L^1(\mathscr{T})$, it remains to prove that $\mathscr{L}^{-1/2}f$ is not in $L^1(\mathscr{T})$. Denote by $k_{\mathscr{L}^{-1/2}}$ the convolution kernel of $\mathscr{L}^{-1/2}$. Then

$$\begin{aligned}\mathscr{L}^{-1/2}f(p\cdot o) &= f * k_{\mathscr{L}^{-1/2}}(p\cdot o)\\ &= \int_G f(py\cdot o)\, k_{\mathscr{L}^{-1/2}}(y^{-1}\cdot o)\,\mathrm{d}y\\ &= \int_G \big[\delta_{x\cdot o}(py\cdot o) - \delta_o(py\cdot o)\big]\ k_{\mathscr{L}^{-1/2}}(y\cdot o)\,\mathrm{d}y\\ &= k_{\mathscr{L}^{-1/2}}(p^{-1}x\cdot o) - k_{\mathscr{L}^{-1/2}}(p^{-1}\cdot o).\end{aligned} \tag{1.18}$$

We have used the fact that $k_{\mathscr{L}^{-1/2}}(p^{-1}\cdot o) = k_{\mathscr{L}^{-1/2}}(p\cdot o)$ in the third equality above. Now recall that $k_{\mathscr{L}^{-1/2}}$ is radial and observe that $p^{-1}x\cdot o$ and $p^{-1}\cdot o$ are neighbours. Indeed, markboth $d(p^{-1}x\cdot o, p^{-1}\cdot o) = d(x\cdot o, \cdot o) = 1$ by the left invariance of d with respect to the group G of automorphisms of $\mathscr{T}$, and the assumption that $x\cdot o$ be a neighbour of o.

We recall the following asymptotics of $k_{\mathscr{L}^{-1/2}}$, proved in [CMS3, Proposition 3.2]:

$$k_{\mathscr{L}^{-1/2}}(x\cdot o) \sim c\,\frac{q^{-|x\cdot o|}}{|x\cdot o|^{1/2}}$$

as $|x|$ tends to infinity. In fact, this result is not explicitly stated in Proposition 3.2 of [CMS3]. However, it is straightforward to check that the proof of that proposition extends almost *verbatim* to cover the case of interest to us. By integrating in polar co-ordinates around o, it is straighforward to check that $k_{\mathscr{L}^{-1/2}}$ is nonintegrable on $\mathscr{T}$. Moreover, there exists a constant C such that if $x\cdot o$, $x'\cdot o$ are neighbours, $|x'\cdot o| = |x\cdot o| + 1$, and $|x\cdot o| \geq R$, then

$$\big|k_{\mathscr{L}^{-1/2}}(x\cdot o) - k_{\mathscr{L}^{-1/2}}(x'\cdot o)\big| \geq C\,\frac{q^{-|x\cdot o|}}{|x\cdot o|^{1/2}}.$$

By combining this inequality and (1.18), we see that

$$\begin{aligned}\big\|\mathscr{L}^{-1/2}f\big\|_1 &\geq \int_{|p\cdot o|\geq R}\big|\mathscr{L}^{-1/2}f(p\cdot o)\big|\,\mathrm{d}(p\cdot o)\\ &\geq C\int_{|x\cdot o|\geq R}\frac{q^{-|x\cdot o|}}{|x\cdot o|^{1/2}}\,\mathrm{d}(x\cdot o)\\ &= C\sum_{p:d(p,o)\geq R}\frac{q^{-|p|}}{|p|^{1/2}},\end{aligned}$$

which is infinite, as required. This completes the proof of the unboundedness of $\mathscr{R}$ from $H^1(\mathscr{T})$ to $L^1(\mathscr{T})$.

Next we prove that if u is in $\mathbb{R}\setminus\{0\}$, then $\mathscr{L}^{iu}$ is unbounded from $H^1(\mathscr{T})$ to $L^1(\mathscr{T})$. The idea of the proof is similar to that above. Denote by $k_{\mathscr{L}^{iu}}$ the kernel of $\mathscr{L}^{iu}$, and let f be as above. Observe that

$$\mathscr{L}^{iu}f(p\cdot o) = k_{\mathscr{L}^{iu}}\big(p^{-1}x\cdot o\big) - k_{\mathscr{L}^{iu}}\big(p^{-1}\cdot o\big)\qquad \forall p\in G.$$

The kernel $k_{\mathscr{L}^{iu}}$ has the following asymptotic expansion

$$k_{\mathscr{L}^{iu}}\big(x\cdot o\big)\sim\frac{c_{iu}}{\Gamma(-iu)}\,\frac{q^{-|x\cdot o|}}{|x\cdot o|^{1-iu}}$$

as $|x|$ tends to infinity [CMS3]. Therefore there exists a constant C such that if $x\cdot o$, $x'\cdot o$ are neighbours, $|x'\cdot o| = |x\cdot o|+1$, and $|x\cdot o|\geq R$, then

$$\big|k_{\mathscr{L}^{iu}}(x\cdot o) - k_{\mathscr{L}^{iu}}(x'\cdot o)\big|\geq C\,\frac{q^{-|x\cdot o|}}{|x\cdot o|}.$$

Thus,

$$\begin{aligned}\big\|\mathscr{L}^{iu}f\big\|_1 &\geq \int_{|p\cdot o|\geq R}\big|\mathscr{L}^{iu}f(p\cdot o)\big|\,\mathrm{d}(p\cdot o)\\ &\geq \int_{|p\cdot o|\geq R}\big|k_{\mathscr{L}^{iu}}(p^{-1}x\cdot o) - k_{\mathscr{L}^{iu}}(p^{-1}\cdot o)\big|\,\mathrm{d}(p\cdot o)\\ &\geq C\int_{|x\cdot o|\geq R}\frac{q^{-|x\cdot o|}}{|x\cdot o|}\,\mathrm{d}(x\cdot o).\end{aligned}$$

Now, the right hand side is equal to

$$
\begin{aligned}
C \sum_{p: d(p,o) \geq R} \frac{q^{-|p|}}{|p|} &= C \, \frac{q+1}{q} \sum_{j \geq R} j^{-1} \\
&= \infty,
\end{aligned}
$$

as required. This completes the proof of the unboundedness of $\mathscr{L}^{iu}$ from $H^1(\mathscr{T})$ to $L^1(\mathscr{T})$. $\square$

Chapter 2

A one-parameter family of Hardy-type spaces

2.1 The spaces $\mathfrak{X}^\gamma(\mathscr{G})$

For θ in $[0,\pi]$, we denote by S_θ the half-line $(0,\infty)$ if $\theta=0$, and the sector

$$\big\{z\in\mathbb{C} : z\neq 0, \text{ and } \big|\arg z\big| < \theta\big\}$$

if $\theta>0$. We recall that, given a number θ in $[0,\pi)$ and an operator A on a Banach space $\mathscr{B}$, we say that A is *sectorial* of angle θ if

(i) the spectrum of A is contained in the closed sector $\overline{S}_\theta$;

(ii) the following *resolvent estimate* holds:

$$\sup_{\lambda\in\mathbb{C}\setminus\overline{S}_{\theta'}} \big\|\!\big|\lambda\,(\lambda-\mathscr{L})^{-1}\big|\!\big\|_{\mathscr{B}} < \infty \qquad \forall\theta'\in(\theta,\pi).$$

Observe that condition (ii) above may be reformulated as follows

$$\sup_{\lambda\in S_{\theta'}} \big\|\!\big|\lambda\,(\lambda+\mathscr{L})^{-1}\big|\!\big\|_{\mathscr{B}} < \infty \qquad \forall\theta'\in[0,\theta). \tag{2.1}$$

Good references for the theory of sectorial operators are [Ha] and [MC].

Theorem 2.1. *Let $\mathscr{G}$ be a locally finite connected graph. Then $\mathscr{L}$ is a sectorial operator of angle $\pi/2$ on $L^1(\mathscr{G})$ and on $L^\infty(\mathscr{G})$.*

Proof. We give full details for $L^1(\mathscr{G})$. The proof carries over almost *verbatim* to the case of $L^\infty(\mathscr{G})$.

We already observed that the $L^1(\mathscr{G})$ spectrum $\sigma_1(\mathscr{L})$ of the Laplacian is contained in the closure of the right half plane (see the remark at the end of Proposition 1.2). Moreover, by the easy part of the Hille–Yosida theorem,

$$\left\vert\kern-0.25ex\left\vert\kern-0.25ex\left\vert (\lambda+\mathscr{L})^{-1} \right\vert\kern-0.25ex\right\vert\kern-0.25ex\right\vert_{L^1(\mathscr{G})} \leq \frac{A_0}{\operatorname{Re}\lambda} \qquad \forall \lambda : \operatorname{Re}\lambda > 0, \tag{2.2}$$

where $A_0 = \sup_{t>0} |||\mathscr{H}_t|||_{L^1(\mathscr{G})}$. By Proposition 1.2, $A_0 = 1$. Now choose $\theta < \pi/2$ and observe that (2.2) implies that

$$\left\vert\kern-0.25ex\left\vert\kern-0.25ex\left\vert te^{i\theta}(te^{i\theta}+\mathscr{L})^{-1} \right\vert\kern-0.25ex\right\vert\kern-0.25ex\right\vert_{L^1(\mathscr{G})} \leq \frac{1}{\cos\theta} \qquad \forall t > 0.$$

Thus, $\mathscr{L}$ satisfies (2.1) for all $\theta' < \pi/2$, whence $\mathscr{L}$ is sectorial of angle $\pi/2$ on $L^1(\mathscr{G})$, as required. □

Corollary 2.2. *Suppose that $\mathscr{G}$ is a locally finite connected graph. For every complex number γ with $\operatorname{Re}\gamma > 0$ the operator $\mathscr{L}^\gamma$ is bounded on $L^1(\mathscr{G})$ and on $L^\infty(\mathscr{G})$.*

Proof. We give details for $L^1(\mathscr{G})$. The proof in the case of $L^\infty(\mathscr{G})$ is almost identical, and is omitted.

By Proposition 1.1, the operator $\mathscr{L}$ is bounded on $L^1(\mathscr{G})$. Consequently, $\mathscr{L}^k$ is bounded on $L^1(\mathscr{G})$ for all positive integers k.

If γ is a complex number such that $0 < \operatorname{Re}\gamma < 1$, recall [MC, Definition 5.1] that $\mathscr{L}^\gamma$ is given by the following Balakhrishnan formula

$$\mathscr{L}^\gamma f = \frac{\sin(\gamma\pi)}{\pi} \int_0^\infty \lambda^{\gamma-1} \left(\lambda+\mathscr{L}\right)^{-1} \mathscr{L} f \,\mathrm{d}\lambda \qquad \forall f \in L^1(\mathscr{G}).$$

Observe that the integral is convergent as a Bochner integral in $L^1(\mathscr{G})$. Indeed,

$$\int_0^\infty \left\| \lambda^{\gamma-1} \left(\lambda+\mathscr{L}\right)^{-1} \mathscr{L} f \right\|_1 \,\mathrm{d}\lambda$$

may be written as the sum of the integrals over the intervals $(0,1)$ and $[1,\infty)$. The integral over $(0,1)$ may be estimated by

$$\left\|f\right\|_1 \int_0^1 \lambda^{\mathrm{Re}\,\gamma-1} \left|\!\left|\!\left|\left(\lambda+\mathscr{L}\right)^{-1}\mathscr{L}\right|\!\right|\!\right|_1 \mathrm{d}\lambda.$$

Notice that the operator norm inside the integral is uniformly bounded with respect to λ in $(0,\infty)$, for we may write

$$\left(\lambda+\mathscr{L}\right)^{-1}\mathscr{L} = -\lambda\left(\lambda+\mathscr{L}\right)^{-1} + \mathscr{I},$$

and the first summand on the right hand side is uniformly bounded with respect to λ in $(0,\infty)$, because $\mathscr{L}$ is sectorial of angle $\pi/2$ in $L^1(\mathscr{G})$. The integral over $[1,\infty)$ may be estimated by

$$\left\|\mathscr{L}f\right\|_1 \int_1^\infty \lambda^{\mathrm{Re}\,\gamma-1} \left|\!\left|\!\left|\left(\lambda+\mathscr{L}\right)^{-1}\right|\!\right|\!\right|_1 \mathrm{d}\lambda \leq C \left|\!\left|\!\left|\mathscr{L}\right|\!\right|\!\right|_1 \left\|f\right\|_1 \int_1^\infty \lambda^{\mathrm{Re}\,\gamma-2}\,\mathrm{d}\lambda$$

where we have used the sectoriality of $\mathscr{L}$ on $L^1(\mathscr{G})$ and (2.1). Thus, $\mathscr{L}^\gamma$ is bounded on $L^1(\mathscr{G})$ [Ha, Proposition 3.1.1].

Now suppose that $\mathrm{Re}\,\gamma > 1$ and γ is not an integer. Then

$$\mathscr{L}^\gamma = \mathscr{L}^{\gamma-[\mathrm{Re}\,\gamma]}\mathscr{L}^{[\mathrm{Re}\,\gamma]}$$

so that $\mathscr{L}^\gamma$ is bounded, being the composition of two bounded operators. □

Remark 2.3. The operator $\mathscr{L}^\gamma$ is also injective on $L^1(\mathscr{G})$. Indeed by abstract nonsense [Ha, Proposition 3.1.1 (d)], if $\mathscr{L}$ is injective, then so is $\mathscr{L}^\gamma$. It remains to prove that $\mathscr{L}$ is injective on $L^1(\mathscr{G})$. Suppose that $\mathscr{L}u = 0$ for some u in $L^1(\mathscr{T})$. In our discrete setting we have the continuous inclusion $L^1(\mathscr{G}) \subset L^2(\mathscr{G})$, and by Proposition 1.4 the Laplacian is invertible on $L^2(\mathscr{T})$. Thus $\mathscr{L}u = 0$ implies $u = 0$, as required.

Corollary 2.2 allows us to define a family of Hardy-type spaces as follows.

Definition 2.4. Suppose that $\gamma > 0$. We denote by $\mathfrak{X}^\gamma(\mathscr{G})$ the space $\mathscr{L}^\gamma\big[L^1(\mathscr{G})\big]$, endowed with the norm that makes $\mathscr{L}^\gamma$ an isometry, i.e., for every f in the range of $\mathscr{L}^\gamma$, we set

$$\left\|f\right\|_{\mathfrak{X}^\gamma(\mathscr{G})} := \left\|\mathscr{L}^{-\gamma}f\right\|_{L^1(\mathscr{G})}.$$

Remark 2.5. Observe that the spaces $\mathfrak{X}^\gamma(\mathscr{G})$ form a descending family of spaces. Indeed, suppose that $0 < \gamma_1 < \gamma_2$ and that f is in $\mathfrak{X}^{\gamma_2}(\mathscr{G})$. Then there exists a function g in $L^1(\mathscr{G})$ such that $f = \mathscr{L}^{\gamma_2} g$. We may write $f = \mathscr{L}^{\gamma_1}\big[\mathscr{L}^{\gamma_2-\gamma_1} g\big]$. The function $\mathscr{L}^{\gamma_2-\gamma_1} g$ is in $L^1(\mathscr{G})$, because $\mathscr{L}^{\gamma_2-\gamma_1}$ is bounded on $L^1(\mathscr{G})$. Hence f is the image of a function in $L^1(\mathscr{G})$ via the operator $\mathscr{L}^{\gamma_1}$, whence it is in $\mathfrak{X}^{\gamma_1}(\mathscr{G})$, as required.

Theorem 2.1 can be modified in order to prove an analogous result for the spaces $\mathfrak{X}^\gamma(\mathscr{G})$. We shall need to prove the contractivity of the heat semigroup on $\mathfrak{X}^\gamma(\mathscr{G})$.

Theorem 2.6. *Suppose that $\mathscr{G}$ is a locally finite connected graph. The following hold:*

(i) the heat semigroup is contractive on $\mathfrak{X}^\gamma(\mathscr{G})$;

(ii) $\mathscr{L}$ is a sectorial operator of angle $\pi/2$ on $\mathfrak{X}^\gamma(\mathscr{G})$.

Proof. First we prove (i). Suppose that f is in $\mathfrak{X}^\gamma(\mathscr{G})$. Then $\mathscr{L}^{-\gamma} f$ is in $L^1(\mathscr{G})$ and $\|\mathscr{L}^{-\gamma} f\|_1 = \|f\|_{\mathfrak{X}^\gamma(\mathscr{G})}$, for $\mathscr{L}^\gamma$ is an isometric isomorphism between $L^1(\mathscr{G})$ and $\mathfrak{X}^\gamma(\mathscr{G})$. Therefore

$$\begin{aligned}
\big\|\mathrm{e}^{-t\mathscr{L}} f\big\|_{\mathfrak{X}^\gamma(\mathscr{G})} &= \big\|\mathscr{L}^{-\gamma}\mathrm{e}^{-t\mathscr{L}}\big[\mathscr{L}^\gamma(\mathscr{L}^{-\gamma} f)\big]\big\|_{L^1(\mathscr{G})} \\
&= \big\|\big[\mathscr{L}^{-\gamma}\mathrm{e}^{-t\mathscr{L}}\mathscr{L}^\gamma\big](\mathscr{L}^{-\gamma} f)\big\|_{L^1(\mathscr{G})} \\
&= \big\|\mathrm{e}^{-t\mathscr{L}}(\mathscr{L}^{-\gamma} f)\big\|_{L^1(\mathscr{G})}.
\end{aligned}$$

In the second equality above we have used the fact that $\mathscr{L}^\gamma$ and $\mathrm{e}^{-t\mathscr{L}}$ commute, because they are functions of $\mathscr{L}$ and bounded operators on $L^1(\mathscr{G})$. Therefore $\mathscr{L}^{-\gamma}\mathrm{e}^{-t\mathscr{L}}\mathscr{L}^\gamma = \mathrm{e}^{-t\mathscr{L}}$, and

$$\begin{aligned}
\big\|\mathrm{e}^{-t\mathscr{L}} f\big\|_{\mathfrak{X}^\gamma(\mathscr{G})} &= \big\|\mathrm{e}^{-t\mathscr{L}}(\mathscr{L}^{-\gamma} f)\big\|_{L^1(\mathscr{G})} \\
&\le \big\|\mathscr{L}^{-\gamma} f\big\|_{L^1(\mathscr{G})} \\
&= \big\|f\big\|_{\mathfrak{X}^\gamma(\mathscr{G})},
\end{aligned}$$

as required.

To prove (ii), observe that the easy part of the Hille–Yosida theorem implies that

$$|||(\lambda + \mathscr{L})^{-1}|||_{\mathfrak{X}^\gamma(\mathscr{G})} \leq \frac{A_\gamma}{\operatorname{Re}\lambda} \qquad \forall \lambda : \operatorname{Re}\lambda > 0, \tag{2.3}$$

where $A_\gamma = \sup_{t>0} |||\mathrm{e}^{-t\mathscr{L}}|||_{\mathfrak{X}^\gamma(\mathscr{G})}$, so $A_\gamma = 1$ because of (i). Therefore the spectrum of $\mathscr{L}$ in $\mathfrak{X}^\gamma(\mathscr{G})$ is contained in the closure of the right half plane. Now, choose $\theta < \pi/2$, and observe that if $t > 0$, then (2.3) implies that

$$\sup_{t>0} |||t\mathrm{e}^{i\theta}\,(t\mathrm{e}^{i\theta} + \mathscr{L})^{-1}|||_{\mathfrak{X}^\gamma(\mathscr{G})} \leq \frac{1}{\cos\theta}.$$

Thus, $\mathscr{L}$ satisfies (2.1) for all $\theta' < \pi/2$, whence $\mathscr{L}$ is sectorial of angle $\pi/2$ on $L^1(\mathscr{G})$, as required. □

A straightforward but interesting consequence of the theory above, and of Corollary 2.2 in particular, is the following result.

Corollary 2.7. *Suppose that $\mathscr{G}$ is a locally finite connected graph, that u is a real number, and that $0 < \gamma < \gamma'$. The following hold:*

(i) $\mathscr{L}^{iu}$ is bounded from $\mathfrak{X}^\gamma(\mathscr{G})$ to $L^1(\mathscr{G})$.

(ii) $\mathscr{L}^{iu}$ is bounded from $\mathfrak{X}^{\gamma'}(\mathscr{G})$ to $\mathfrak{X}^\gamma(\mathscr{G})$.

Proof. By definition of $\mathfrak{X}^\gamma(\mathscr{G})$, $\mathscr{L}^{iu}$ is bounded from $\mathfrak{X}^\gamma(\mathscr{G})$ to $L^1(\mathscr{G})$ if and only if $\mathscr{L}^{iu}\mathscr{L}^\gamma$ is bounded on $L^1(\mathscr{G})$. Observe that $\mathscr{L}^{iu}\mathscr{L}^\gamma = \mathscr{L}^{iu+\gamma}$, which, in fact, is bounded on $L^1(\mathscr{G})$ by Corollary 2.2. This proves (i).

To prove (ii), observe that $\mathscr{L}^{\gamma'-\gamma}$ is an isometric isomorphism between $\mathfrak{X}^\gamma(\mathscr{G})$ and $\mathfrak{X}^{\gamma'}(\mathscr{G})$. Thus, $\mathscr{L}^{iu}$ is bounded from $\mathfrak{X}^{\gamma'}(\mathscr{G})$ to $\mathfrak{X}^\gamma(\mathscr{G})$ if and only if $\mathscr{L}^{iu}\mathscr{L}^{\gamma'-\gamma}$ is bounded on $L^1(\mathscr{G})$. This is true by Corollary 2.2. □

It may be worth observing that $\mathscr{L}^{iu}$ is unbounded on $L^1(\mathscr{G})$, but is bounded on $L^p(\mathscr{G})$ for all p in $(1, \infty)$. Corollary 2.7 (i) contains a useful endpoint estimate for $\mathscr{L}^{iu}$ when $p = 1$. This result may be thought of as an analogue of the classical result that $(-\Delta)^{iu}$ maps $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$. We would like to emphasise the fact that the result above holds in great generality, with minimal assumptions on the geometry of the graph.

2.2 Interpolation

In this section we prove that if p is in $(1,2)$, then $L^p(\mathscr{G})$ is the complex interpolation space with index $2/p'$ between $\mathfrak{X}^\gamma(\mathscr{G})$ and $L^2(\mathscr{G})$.

Suppose that (X^0, X^1) is an interpolation pair of Banach spaces, i.e., X^0 and X^1 are Banach spaces both continuously included in a topological vector space V. For every θ in $(0,1)$ consider the interpolation space $(X^0, X^1)_{[\theta]}$, which we denote simply by X_θ, obtained via Calderón's complex interpolation method [Ca]. The notation we adopt is consistent with that of [BL, Chapter 4].

Proposition 2.8. *Suppose that (X^0, X^1) and (Y^0, Y^1) are interpolation pairs of Banach spaces. Suppose further that T is a bounded linear map from $X^0 + X^1$ to Y^0+Y^1, such that the restrictions $T : X^0 \to Y^0$ and $T : X^1 \to Y^1$ are isomorphisms. Then for every θ in $(0,1)$ the restriction $T : X_\theta \to Y_\theta$ is an isomorphism.*

Proof. For every θ in $[0,1]$ denote by T_θ the restriction of T to X_θ.
Define $\mathscr{S} : Y_0 + Y_1 \to X_0 + X_1$ by setting

$$\mathscr{S}(y_0 + y_1) = T_0^{-1} y_0 + T_1^{-1} y_1.$$

It is straightforward to check that the operator $\mathscr{S}$ is well defined, bounded and linear. Moreover $\mathscr{S}T$ is the identity on $X_0 + X_1$ and $T\mathscr{S}$ is the identity on $Y_0 + Y_1$. Thus $\mathscr{S} = T^{-1}$. Hence $\mathscr{S}_\theta = T_\theta^{-1}$. Finally, $\mathscr{S}_\theta : Y_\theta \to X_\theta$ is bounded by interpolation. This concludes the proof of the proposition. □

Corollary 2.9. *Suppose that θ is in $(0,1)$. If p_θ is $2/(2-\theta)$, then*

$$\big(\mathfrak{X}^\gamma(\mathscr{T}), L^2(\mathscr{T})\big)_{[\theta]} = L^{p_\theta}(\mathscr{T}).$$

Proof. The required equality is a consequence of Proposition 2.8, with $\mathscr{L}^\gamma$ in place of T. Indeed, we first observe that $\mathscr{L}^\gamma$ is a bounded operator from $L^1(\mathscr{T})+L^2(\mathscr{T})$ to $\mathfrak{X}^\gamma(\mathscr{T}) + L^2(\mathscr{T})$. Furthermore, its restriction to $L^1(\mathscr{T})$ is an isomorphism between $L^1(\mathscr{T})$ and $\mathfrak{X}^\gamma(\mathscr{T})$ (by definition of $\mathfrak{X}^\gamma(\mathscr{T})$), and its restriction to $L^2(\mathscr{T})$ is an isomorphism of $L^2(\mathscr{T})$ for 0 does not belong to the $L^2(\mathscr{T})$ spectrum of $\mathscr{L}$, hence of $\mathscr{L}^\gamma$ (see [Ha, Proposition 3.1.1 (d)]). Then we may apply Proposition 2.8 with $\mathscr{L}^\gamma$ in place of T, $X^0 = L^1(\mathscr{T})$, $Y^0 = \mathfrak{X}^\gamma(\mathscr{T})$, $X^1 = L^2(\mathscr{T}) = Y^1$. A well known interpolation theorem states that

$$\big(L^1(\mathscr{T}), L^2(\mathscr{T})\big)_{[\theta]} = L^p(\mathscr{T}).$$

By Proposition 2.8, the restriction of $\mathscr{L}^\gamma$ to $L^p(\mathscr{T})$ is an isomorphism between $L^p(\mathscr{T})$ and $\big(\mathfrak{X}^\gamma(\mathscr{T}), L^2(\mathscr{T})\big)_{[\theta]}$. But the restriction of $\mathscr{L}^\gamma$ to $L^p(\mathscr{T})$ is just $\mathscr{L}^\gamma$, which is an isomorphism of $L^p(\mathscr{T})$. Hence $\big(\mathfrak{X}^\gamma(\mathscr{T}), L^2(\mathscr{T})\big)_{[\theta]}$ and $L^p(\mathscr{T})$ are isomorphic Banach spaces, as required. □

2.3 The annihilator of all bounded harmonic functions

Denote by $\mathscr{H}^\infty(\mathscr{G})$ the space of all bounded harmonic functions on $\mathscr{G}$. For reasons which will become apparent after Section 2.6, where the atomic theory for the spaces $\mathfrak{X}^k(\mathscr{G})$ is developed, it is natural to consider the annihilator of $\mathscr{H}^\infty(\mathscr{G})$ in $L^1(\mathscr{G})$, defined by

$$\mathscr{H}^\infty(\mathscr{G})^\perp := \big\{f \in L^1(\mathscr{G}) : \langle f, H\rangle = 0 \text{ for all } H \text{ in } \mathscr{H}^\infty(\mathscr{G})\big\},$$

where $\langle\cdot,\cdot\rangle$ denotes the duality between $L^1(\mathscr{G})$ and $L^\infty(\mathscr{G})$. In view of Remark 2.21 below it is natural to speculate whether $\mathscr{H}^\infty(\mathscr{G})^\perp$ and $\mathfrak{X}^1(\mathscr{G})$ agree.

Proposition 2.10. *Suppose that $\mathscr{G}$ is a locally finite connected graph. Then $\mathfrak{X}^\gamma(\mathscr{G})$ is properly contained in $\mathscr{H}^\infty(\mathscr{G})^\perp$ for every $\gamma > 0$.*

Proof. First we show that $\mathfrak{X}^\gamma(\mathscr{G})$ is contained in $\mathscr{H}^\infty(\mathscr{G})^\perp$ for all γ in $(0,1)$. Suppose that f is in $\mathfrak{X}^\gamma(\mathscr{G})$, and denote by g the unique function in $L^1(\mathscr{G})$ such that $\mathscr{L}^\gamma g = f$.

Then for every H in $\mathscr{H}^\infty$

$$\begin{aligned}\langle f, H\rangle &= \langle \mathscr{L}^\gamma g, H\rangle \\ &= \Big\langle \sum_{x\in\mathscr{G}} g(x)\,\mathscr{L}^\gamma \delta_x, H\Big\rangle \\ &= \sum_{x\in\mathscr{G}} g(x)\,\langle \mathscr{L}^\gamma \delta_x, H\rangle.\end{aligned}$$

We observe that $\langle \mathscr{L}^\gamma \delta_x, H\rangle = \langle \delta_x, \mathscr{L}^\gamma H\rangle$. Now, recall that $\mathscr{L}^\gamma$ may be defined in terms of the Balakrishnan integral, so that, at least formally,

$$\begin{aligned}\langle \mathscr{L}^\gamma \delta_x, H\rangle &= \frac{\sin(\gamma\pi)}{\pi}\Big\langle \int_0^\infty \lambda^{\gamma-1}\big(\lambda+\mathscr{L}\big)^{-1}\mathscr{L}\delta_x\,\mathrm{d}\lambda, H\Big\rangle \\ &= \frac{\sin(\gamma\pi)}{\pi}\int_0^\infty \lambda^{\gamma-1}\big\langle\big(\lambda+\mathscr{L}\big)^{-1}\mathscr{L}\delta_x, H\big\rangle\,\mathrm{d}\lambda.\end{aligned} \tag{2.4}$$

It is not hard to justify the formal steps above, working backwards. Now observe that $\big(\lambda+\mathscr{L}\big)^{-1}$ and $\mathscr{L}$ commute as bounded operators on $L^1(\mathscr{G})$, so that $\big(\lambda+\mathscr{L}\big)^{-1}\mathscr{L}\delta_x = \mathscr{L}\big(\lambda+\mathscr{L}\big)^{-1}\delta_x$. Furthermore, a straighforward application of Fubini's theorem shows that for every function φ in $L^1(\mathscr{G})$ the following formula holds

$$\big\langle \mathscr{L}\varphi, H\big\rangle = \big\langle \varphi, \mathscr{L}H\big\rangle.$$

We use this formula with $\big(\lambda+\mathscr{L}\big)^{-1}\delta_x$ in place of φ, and conclude that the right hand side of (2.4) vanishes, for H is harmonic. Hence $\langle f, H\rangle = 0$, so that f is in $\mathscr{H}^\infty(\mathscr{G})^\perp$, as required.

Since the spaces $\mathfrak{X}^\gamma(\mathscr{G})$ form a descending family as γ increases, $\mathfrak{X}^\gamma(\mathscr{G})$ is contained in $\mathscr{H}^\infty(\mathscr{G})^\perp$ also for every γ in $[1,\infty)$.

Next we show that $\mathfrak{X}^\gamma(\mathscr{G})$ is strictly contained in $\mathscr{H}^\infty(\mathscr{G})^\perp$ for every γ in $(0,1)$. Denote by $(\mathscr{L}^\gamma)^\star$ be the adjoint of $\mathscr{L}^\gamma$, thought of as an operator acting on $L^1(\mathscr{G})$. Then, $(\mathscr{L}^\gamma)^\star$ is simply the operator $\mathscr{L}^\gamma$ acting on $L^\infty(\mathscr{G})$. By [Br, Corollary 2.18],

$$\mathscr{H}^\infty(\mathscr{G})^\perp = \big(\mathrm{Ker}(\mathscr{L}^\star)\big)^\perp = \overline{\mathrm{Ran}(\mathscr{L})},$$

where the closure of $\mathrm{Ran}(\mathscr{L})$ is in the $L^1(\mathscr{G})$ norm. Notice that

$$\mathrm{Ker}(\mathscr{L}^\star) = \mathrm{Ker}\big((\mathscr{L}^\gamma)^\star\big)$$

(see [Ha, Proposition 3.1.1 (d)]), so that $\mathscr{H}^\infty(\mathscr{G})^\perp = \overline{\mathrm{Ran}(\mathscr{L}^\gamma)}$. So all we need to prove is that $\mathfrak{X}^\gamma(\mathscr{G}) = \mathrm{Ran}(\mathscr{L}^\gamma)$ is not closed in $L^1(\mathscr{G})$. We already pointed out that the $L^1(\mathscr{G})$ spectrum $\sigma_1(\mathscr{L})$ of the Laplacian is contained in the closure of the right half plane (see the remark at the end of Proposition 1.2). Observe that 0 belongs to the $L^1(\mathscr{G})$ spectrum of the Laplacian. Indeed, if 0 were in the resolvent set, then $\mathscr{L}$ would be invertible in $L^1(\mathscr{G})$ with bounded inverse. In particular, $\mathscr{L}$ would be surjective. But this is impossible for $\mathrm{Ran}(\mathscr{L})$ is contained in the proper subspace of $L^1(\mathscr{G})$ of all functions with vanishing integral. By the spectral mapping theorem for powers of sectorial operators [Ha, Proposition 3.1.1 (j)],

$$\sigma_1(\mathscr{L}^\gamma) = \{z^\gamma : z \in \sigma_1(\mathscr{L})\}.$$

In particular, $\sigma_1(\mathscr{L}^\gamma)$ is contained in the closure of the right half-plane, and contains 0. A well known result [Ha, Corollary A.3.5] states that the boundary of the spectrum of an operator is contained in its approximate point spectrum. Thus, in our case, $\partial\sigma_1(\mathscr{L}^\gamma) \subset A\sigma_1(\mathscr{L}^\gamma)$, where $\partial\sigma_1(\mathscr{L}^\gamma)$ is the boundary of $\sigma_1(\mathscr{L}^\gamma)$, and $A\sigma_1(\mathscr{L}^\gamma)$ is the approximate point spectrum (see [Ha, Appendix A]), defined as

$$A\sigma_1(\mathscr{L}^\gamma) = \{\lambda \in \mathbb{C} : \mathrm{Ker}(\lambda - \mathscr{L}^\gamma) \neq \{0\} \text{ or } \mathrm{Ran}(\lambda - \mathscr{L}^\gamma) \text{ is not closed}\}.$$

Thus, in particular, 0 belongs to the approximate point spectrum $A\sigma_1(\mathscr{L}^\gamma)$. Since $\mathrm{Ker}(\mathscr{L}^\gamma) = \mathrm{Ker}(\mathscr{L}) = \{0\}$ (as operators acting on $L^1(\mathscr{G})$) [Ha, Proposition 3.1.1 (d)], $\mathrm{Ran}(\mathscr{L}^\gamma)$ is not closed, whence $\mathrm{Ran}(\mathscr{L}^\gamma)$ is properly contained in $\mathscr{H}^\infty(\mathscr{G})^\perp$, as required. □

We already know (see Corollary 2.7) that $\mathscr{L}^{iu}$ is bounded from $\mathfrak{X}^\gamma(\mathscr{G})$ to $L^1(\mathscr{G})$ for each positive number γ. In view of the proper containments

$$\mathfrak{X}^\gamma(\mathscr{G}) \subset \mathscr{H}^\infty(\mathscr{G})^\perp \subset L^1(\mathscr{G}),$$

it is natural to speculate whether $\mathscr{L}^{iu}$ is bounded from $\mathscr{H}^\infty(\mathscr{G})^\perp$, thought of as a closed subspace of $L^1(\mathscr{G})$, to $L^1(\mathscr{G})$. We shall see that this fails. Therefore $\mathscr{H}^\infty(\mathscr{G})^\perp$ cannot serve as an analogue of the classical Hardy space $H^1(\mathbb{R}^n)$ in the setting of (possibly nonhomogeneous) trees.

Proposition 2.11. *Suppose that $\mathscr{G}$ is a homogeneous tree. For each u in $\mathbb{R} \setminus \{0\}$ the operator $\mathscr{L}^{iu}$ is unbounded from $\mathscr{H}^\infty(\mathscr{G})^\perp$, endowed with the $L^1(\mathscr{G})$ norm, to $L^1(\mathscr{G})$.*

Proof. Recall that $\mathscr{H}^\infty(\mathscr{G})^\perp$ is the closure of $\mathrm{Ran}(\mathscr{L})$ in $L^1(\mathscr{G})$ (see, for instance, [Br, Corollary 2.18]). Thus, it suffices to show that there exists a sequence $\{f_n\}$ of functions in $L^1(\mathscr{G})$ such that $\lim_{n\to\infty} \left\|\mathscr{L}^{iu}(\mathscr{L} f_n)\right\|_{L^1(\mathscr{G})} = \infty$, but $\left\|\mathscr{L} f_n\right\|_{L^1(\mathscr{G})}$ is bounded.

We shall use the fact that $\mathscr{L}^{-1}\delta_o(x) = q^{1-|x|}/(q-1)$ (to prove this formula just compute the Laplacian of both sides). Set

$$f_n := \mathbf{1}_{B_n(o)}\, \mathscr{L}^{-1}\delta_o.$$

Obviously f_n is in $L^1(\mathscr{G})$ (its support is finite). The function f_n is harmonic in $B_{n+1}(o)^c$, for it vanishes in $B_n(o)^c$, and in $B_{n-1}(o) \setminus \{o\}$, for it agrees with $\mathscr{L}^{-1}\delta_o$ therein. Thus, the support of $\mathscr{L} f_n$ is contained in $\{o\} \cup S_n(o) \cup S_{n+1}(o)$. Observe that $\mathscr{L} f_n(o) = \mathscr{L}\big(\mathscr{L}^{-1}\delta_o\big)(o) = 1$ (at least when $n \geq 3$), that

$$\begin{aligned}\mathscr{L} f_n(x) &= \frac{q}{q-1}\, q^{-n} - \frac{1}{q+1}\, \frac{q}{q-1}\, q^{1-n} \\ &= \frac{q^{1-n}}{q^2-1} \qquad \forall x \in S_n(o),\end{aligned}$$

and that

$$\begin{aligned}\mathscr{L} f_n(x) &= -\frac{1}{q+1}\, \frac{q}{q-1}\, q^{1-n} \\ &= -\frac{q^{1-n}}{q^2-1} \qquad \forall x \in S_{n+1}(o).\end{aligned}$$

Therefore

$$\begin{aligned}\left\|\mathscr{L} f_n\right\|_{L^1(\mathscr{G})} &= 1 + \frac{q^{1-n}}{q^2-1}\, \mu\big(S_n(o)\big) + \frac{q^{1-n}}{q^2-1}\, \mu\big(S_{n+1}(o)\big) \\ &= \frac{2q}{q-1}.\end{aligned}$$

Now, notice that for each p in $(1,\infty)$

$$\begin{aligned}\big\|f_n-\mathscr{L}^{-1}\delta_o\big\|^p_{L^p(\mathscr{G})} &= \sum_{|y|\geq n+1}\big|\mathscr{L}^{-1}\delta_o(y)\big|^p \\ &= \frac{q^p}{(q-1)^p}\sum_{|y|\geq n+1} q^{-p|y|} \\ &= \frac{q^p}{(q-1)^p}\,\frac{q+1}{q}\sum_{j\geq n+1} q^{(1-p)j},\end{aligned}$$

which tends to 0 as n tends to infinity. Since $\mathscr{L}^{1+iu}$ is bounded on $L^p(\mathscr{G})$, $\mathscr{L}^{iu}\big(\mathscr{L}f_n\big)$ tends to $\mathscr{L}^{iu}\delta_o$ in $L^p(\mathscr{G})$ as n tends to infinity. This implies that $\mathscr{L}^{iu}\big(\mathscr{L}f_n\big)$ is pointwise convergent to $\mathscr{L}^{iu}\delta_o$, which agrees with the convolution kernel $k_{\mathscr{L}^{iu}}$ of the operator $\mathscr{L}^{iu}$. It is well known [CMS3, Proposition 3.2] that $k_{\mathscr{L}^{iu}}$ is not in $L^1(\mathscr{G})$. By Fatou's theorem

$$\begin{aligned}\liminf_{n\to\infty}\sum_{x\in\mathscr{T}}\big|\mathscr{L}^{iu}\big(\mathscr{L}f_n\big)(x)\big| &\geq \sum_{x\in\mathscr{T}}\liminf_{n\to\infty}\big|\mathscr{L}^{iu}\big(\mathscr{L}f_n\big)(x)\big| \\ &= \sum_{x\in\mathscr{T}}\big|k_{\mathscr{L}^{iu}}(x)\big| \\ &= \infty.\end{aligned}$$

Thus, $\big\|\mathscr{L}\big(\mathscr{L}^{iu}f_n\big)\big\|_{L^1(\mathscr{G})}$ tends to infinity, whereas $\big\|\mathscr{L}f_n\big\|_{L^1(\mathscr{G})}=\dfrac{2q}{q-1}$. Consequently $\mathscr{L}^{iu}$ is unbounded from $\mathscr{H}^\infty(\mathscr{G})^\perp$ to $L^1(\mathscr{G})$, as required. □

2.4 The space $\mathfrak{X}^{1/2}(\mathscr{G})$

Now we focus on the space $\mathfrak{X}^{1/2}(\mathscr{G})$. The main result of this section is the following.

Theorem 2.12. *Suppose that $\mathscr{G}$ is a locally finite connected graph that possesses Cheeger's isoperimetric property. Assume that f is in $L^1(\mathscr{G})$. Then the following are equivalent:*

(i) f belongs to $\mathfrak{X}^{1/2}(\mathscr{G})$;

(ii) the Riesz transform $|\mathscr{R}f|$ is in $L^1(\mathscr{G})$.

Furthermore,

$$\begin{aligned} c_{FF}\ \|f\|_{\mathfrak{X}^{1/2}(\mathscr{G})} &\le \|f\|_1 + \||\mathscr{R}f|\|_1 \\ &\le \big[\|\!|\mathscr{L}^{1/2}|\!\|_1 + \sqrt{2}\big]\ \|f\|_{\mathfrak{X}^{1/2}(\mathscr{G})} \qquad \forall f \in L^1(\mathscr{G}), \end{aligned} \tag{2.5}$$

where c_{FF} denotes the constant in the Federer–Fleming inequality.

Proof. First we prove that (i) implies (ii), and that the right hand inequality in (2.5) holds. Suppose that f is in $\mathfrak{X}^{1/2}(\mathscr{G})$. Then there exists an integrable function g such that $f = \mathscr{L}^{1/2}g$, whence

$$\begin{aligned} \||\mathscr{R}f|\|_1 &= \||\nabla\mathscr{L}^{-1/2}\mathscr{L}^{1/2}g|\|_1 \\ &= \||\nabla g|\|_1 . \end{aligned}$$

Observe that

$$\begin{aligned} \||\nabla g|\|_1 &= \sum_{x\in\mathscr{G}} \Big[\frac{1}{2}\sum_{y\sim x}\big|g(x)-g(y)\big|^2\Big]^{1/2} \\ &\le \frac{1}{\sqrt{2}}\sum_{x\in\mathscr{G}}\sum_{y\sim x}\big|g(x)-g(y)\big| \\ &\le \frac{1}{\sqrt{2}}\sum_{x\in\mathscr{G}}\Big[\nu(x)\,\big|g(x)\big| + \sum_{y\sim x}\big|g(y)\big|\Big] \\ &\le \frac{1}{\sqrt{2}}\Big[\ \|g\|_1 + \sum_{x\in\mathscr{G}}\sum_{y\sim x}\big|g(y)\big|\Big]. \end{aligned}$$

Now we use Fubini's theorem to conclude that

$$\begin{aligned}\sum_{x\in\mathscr{G}}\sum_{y\sim x}|g(y)| &= \sum_{y\in\mathscr{G}}\sum_{x\sim y}|g(y)| \\ &= \sum_{y\in\mathscr{G}}\nu(y)|g(y)| \\ &= \|g\|_1 .\end{aligned}$$

By combining the last two formulae above, we see that $\||\nabla g|\|_1 \leq \sqrt{2}\ \|g\|_1$. Therefore

$$\begin{aligned}\||\mathscr{R} f|\|_1 &\leq \sqrt{2}\ \|g\|_1 \\ &= \sqrt{2}\ \|f\|_{\mathfrak{X}^{1/2}(\mathscr{G})} .\end{aligned}$$

To conclude the proof of this part, we need to produce an upper bound of $\|f\|_1$ in terms of $\|f\|_{\mathfrak{X}^{1/2}(\mathscr{G})}$. Observe that $f = \mathscr{L}^{1/2}\mathscr{L}^{-1/2}f$, so that

$$\begin{aligned}\|f\|_1 &= \|\mathscr{L}^{1/2}\mathscr{L}^{-1/2}f\|_1 \\ &\leq |||\mathscr{L}^{1/2}|||_1\ \|\mathscr{L}^{-1/2}f\|_1 \\ &= |||\mathscr{L}^{1/2}|||_1\ \|f\|_{\mathfrak{X}^{1/2}(\mathscr{G})} .\end{aligned}$$

By combining these estimates, we obtain that

$$\|f\|_1 + \||\mathscr{R} f|\|_1 \leq \big[|||\mathscr{L}^{1/2}|||_1 + \sqrt{2}\big]\ \|f\|_{\mathfrak{X}^{1/2}(\mathscr{G})} ,$$

as required.

Next we prove that (ii) implies (i), and that the left hand inequality in (2.5) holds. Suppose that f and $|\mathscr{R} f|$ are in $L^1(\mathscr{G})$. We apply the Federer–Fleming inequality (see Theorem (1.5)) to $\mathscr{L}^{-1/2}f$, and obtain that

$$\||\nabla\mathscr{L}^{-1/2}f|\|_1 \geq c_{FF}\ \|\mathscr{L}^{-1/2}f\|_1 .$$

Thus, $\mathscr{L}^{-1/2}f$ is in $L^1(\mathscr{G})$. Set $g := \mathscr{L}^{-1/2}f$. Then $f = \mathscr{L}^{1/2}g$, i.e, f belongs to $\mathfrak{X}^{1/2}(\mathscr{G})$, and the corresponding norm estimate holds, as required. $\square$

It may be worth recording the following straightforward consequence of Theorem 2.12.

Corollary 2.13. *If $\gamma < 1/2$, then the Riesz transform is unbounded from $\mathfrak{X}^\gamma(\mathscr{T})$ to $L^1(\mathscr{T})$.*

Proof. This follows from Theorem 2.12 and the fact that if $\gamma < 1/2$, then $\mathfrak{X}^\gamma(\mathscr{T})$ contains properly $\mathfrak{X}^{1/2}(\mathscr{T})$. □

2.5 Some properties of harmonic functions

In this section we prove a few properties of k-harmonic functions on graphs that will be important in the sequel, especially in Section 2.6.

Definition 2.14. We say that a function f on $\mathscr{G}$ is *k-harmonic* if $\mathscr{L}^k f$ vanishes identically on $\mathscr{G}$. If B is a ball in $\mathscr{G}$, we say that f is *k-harmonic* in B if $\mathscr{L}^k f = 0$ on B.

Notice that if x is a point in $\mathscr{G}$, then $\mathscr{L}^{-k}\delta_x$ is k-harmonic in $\mathscr{G} \setminus \{x\}$.

Proposition 2.15. *Suppose that $\mathscr{G}$ is a locally finite connected graph, which possesses the Cheeger isoperimetric property. Assume that f is a function with support contained in a ball B of radius r, that is orthogonal to all functions that are k-harmonic in B. The following hold:*

(i) the support of $\mathscr{L}^{-k}f$ is contained in B;

(ii) if $\mathscr{G}$ is a tree, and $r \leq k-1$, then f vanishes identically;

(iii) if $\mathscr{G}$ is a tree, and $r = k$, then there exists a constant c such that $f = c\,\mathscr{L}^k\delta_{c_B}$.

Recall that if $\mathscr{G}$ is a tree and possesses Cheeger's isoperimetric inequality, then every vertex has must have at least three neighbours (see Proposition 1.4 and Corollary 1.7).

Proof. Observe that f is in $L^2(\mathscr{G})$. Since $\mathscr{L}^{-1}$ is bounded on $L^2(\mathscr{G})$, so is $\mathscr{L}^{-k}$, and we may consider $\mathscr{L}^{-k}f$, which belongs to $L^2(\mathscr{G})$. Observe that for each y in B^c,

$$(\mathscr{L}^{-k}f, \delta_y) = (f, \mathscr{L}^{-k}\delta_y)$$

by the self adjointness of $\mathscr{L}$. Clearly $\mathscr{L}^{-k}\delta_y$ is a k-harmonic function on B; hence the last inner product vanishes because f is, by assumption, orthogonal to all k-harmonic functions on B. Therefore $\mathscr{L}^{-k}f(y) = 0$, for every y in B^c, and (i) is proved.

Next we prove (ii). Set $g = \mathscr{L}^{-k}f$, whence $f = \mathscr{L}^k g$, and denote by c_B the centre of B. By (i), the support of g is contained in B. Suppose that z is a point in B such that $d(c_B, z) = r$. We shall prove that $g(z) = 0$. Then the support of g is, in fact, contained in the ball with centre c_B and radius $r-1$. By arguing recursively, it follows that g vanishes identically, whence so does f.

Suppose now that $d(c_B, z) = r$. The idea is to show that there exists a point w_k such that $d(c_B, w_k) = r+k$ and $f(w_k)$ is a constant multiple of $g(z)$. Since w_k does not belong to B, $f(w_k) = 0$, whence $g(z) = 0$. To construct w_k, we argue as follows. Denote by w_1 a neighbour of z such that $d(c_B, w_1) = d(c_B, z) + 1$. Then

$$\mathscr{L}g(w_1) = -\frac{1}{\nu(w_1)}\, g(z).$$

Here we used the fact that $\mathscr{G}$ is a tree (each point has at least two neighbours). Now, if w_2 is a neighbour of w_1 such that $d(c_B, w_2) = d(c_B, w_1) + 1$, then

$$\mathscr{L}^2 g(w_2) = \mathscr{L}\big(\mathscr{L}g\big)(w_2) = -\frac{1}{\nu(w_2)}\, \mathscr{L}g(w_1) = \frac{1}{\nu(w_2)\,\nu(w_1)}\, g(z).$$

By arguing recursively, we see that if $d(c_B, w_k) = d(c_B, z) + k$, and w_k is a neighbour of w_{k-1}, then

$$\begin{aligned}\mathscr{L}^k g(w_k) &= \mathscr{L}\big(\mathscr{L}^{k-1}g\big)(w_k) = -\frac{1}{\nu(w_k)}\, \mathscr{L}^{k-1}g(w_{k-1}) = \cdots \\ &= \frac{(-1)^k}{\nu(w_k)\cdots\nu(w_1)}\, g(z).\end{aligned}$$

Since the support of f is contained in B, and $d(c_B, w_k) = r+k > r$, $f(w_k) = 0$. Thus $\mathscr{L}^k g(w_k) = f(w_k) = 0$, whence $g(z) = 0$.

Finally we prove (iii). By arguing as in the proof of (i), we see that the support of g is reduced to the point c_B. Then g is of the form $c\,\delta_{c_B}$, for some constant c, so that $f = c\,\mathscr{L}^k\delta_{c_B}$, as required. □

It is interesting to speculate whether properties (ii)-(iii) in the proposition above hold on every locally finite connected graph $\mathscr{G}$.

In the case where $\mathscr{G}$ is a (possibly nonhomogeneous) tree in which every vertex has at least three neighbours, functions which are harmonic on a ball B have a bounded harmonic extension to all of $\mathscr{G}$. We prove this in the next theorem. Notice that we do not assume that $\mathscr{G}$ has bounded geometry.

Proposition 2.16. *Assume that $\mathscr{G}$ is a locally finite (possibly nonhomogeneous) tree such that*

$$\min_{x\in\mathscr{G}} \nu(x) \geq 3.$$

Suppose that R is a positive integer and that f is a function defined on the ball $B_R(x_0)$ and harmonic on $B_{R-1}(x_0)$. Then there exists a bounded harmonic extension F of f to the whole graph $\mathscr{G}$.

Proof. We shall construct F explicitly. First we construct this extension in the case where $\mathscr{G}$ is a homogeneous tree, where the details of the construction are slightly simpler. Since $\mathscr{L}$ is invariant under the group of isometries of $\mathscr{G}$, we may assume that $x_0 = o$. Of course, $F = f$ on $B_R(o)$.

Suppose that n is a positive integer and that $d(z, o) = R+n$. Then $d\big(z, B_R(o)\big) = n$; denote by y the unique point in $B_R(o)$ such that $d(z, y) = n$ and by x the unique point in $B_{R-1}(o)$ such that $d(z, x) = n + 1$. Set

$$F(z) := f(y) + \frac{q^n - 1}{q^n(q-1)} \big(f(y) - f(x)\big). \tag{2.6}$$

Clearly the coefficient of $f(y) - f(x)$ is smaller than 1, as q is at least 2. So

$$|F(z)| \leq 3 \max_{w\in B(o,R)} |f(w)|,$$

hence F is bounded. We postpone for a moment the verification that the function F thus defined is harmonic on $\mathscr{G}$, and explain the idea behind the definition above. Suppose that $|y| = R$; we want to define F at the neighbours of y of length $n + 1$, so that $\mathscr{L}F(y) = 0$. This can be done in many ways. What we need is that

$\frac{1}{q+1}\sum_{z\sim y,|z|=R+1} F(z) = F(y) - \frac{1}{q+1}F(x)$. Amongst all these possibilities, definition (2.6) corresponds to the choice that F be constant on all the points $z \sim y$ with $|z| = n+1$. We remark that this choice minimises the quantity

$$\max\{|F(z)| : |z| = R+1,\ z \sim y,\ \mathscr{L}F(y) = 0\},$$

hence the supremum norm of F. If $d(y,o) > R$, then we define $F(y)$ recursively, and obtain formula (2.6).

It remains to prove that $\mathscr{L}F = 0$. If $d(y,o) = R$, then

$$\begin{aligned}\mathscr{L}F(y) &= F(y) - \frac{1}{q+1}\sum_{w\sim y} F(w)\\ &= F(y) - \frac{1}{q+1}F(x) - \frac{q}{q+1}F(z).\end{aligned}$$

Now we use the definition (2.6) of $F(z)$ and see that

$$\begin{aligned}\mathscr{L}F(y) &= F(y) - \frac{1}{q+1}F(x) - \frac{q}{q+1}\Big[F(y) + \frac{q-1}{q(q-1)}(F(x) - F(y))\Big]\\ &= 0,\end{aligned}$$

whence F is harmonic on $B(o,R)$. If z is a point such that $d(z, B(o,R)) = n > 0$, then, by definition of F (see (2.6)),

$$\begin{aligned}\mathscr{L}F(z) &= F(z) - \frac{1}{q+1}\sum_{w\sim z} F(w)\\ &= F(y) + \frac{q^n-1}{q^n(q-1)}(F(y) - F(x))\\ &\quad - \frac{1}{q+1}\Big[F(y) + \frac{q^{n-1}-1}{q^{n-1}(q-1)}(F(y) - F(x))\Big]\\ &\quad - \frac{q}{q+1}\Big[F(y) + \frac{q^{n+1}-1}{q^{n+1}(q-1)}(F(y) - F(x))\Big]\\ &= (F(x) - F(y))\Big[\frac{q^n-1}{q^n(q-1)} - \frac{1}{q+1}\frac{q^{n-1}-1}{q^{n-1}(q-1)} - \frac{q}{q+1}\frac{q^{n+1}-1}{q^{n+1}(q-1)}\Big]\\ &= \frac{F(x)-F(y)}{q^{n-1}(q-1)}\ \frac{(q+1)(q^n-1) - q(q^{n-1}-1) - (q^{n+1}-1)}{q(q+1)}\\ &= 0.\end{aligned}$$

Hence F is harmonic on $\mathscr{G}$ and extends f.

We now show how to modify this construction for graphs satisfying the assumptions of the theorem. Suppose that z is in $\mathscr{G}$ and that $d\big(z, B_R(x_0)\big) = n$, let y and x be as before and let $y = z_0, z_1, \ldots, z_{n-1}, z_n = z$ be the geodesic path joining y to z. Set

$$F(z) := F(y) + [F(y) - F(x)] \sum_{i=0}^{n-1} \frac{1}{\prod_{j=0}^{i}(\nu(z_j) - 1)}.$$

Since $\nu(x) \geq 3$ for every x in $\mathscr{G}$, $\prod_{j=0}^{i}(\nu(z_j) - 1) \geq 2^i$, whence

$$\sum_{i=0}^{n-1} \frac{1}{\prod_{j=0}^{i}(\nu(z_j) - 1)} \leq \sum_{i=0}^{n-1} \frac{1}{2^i} \leq 1$$

and $|F(z)| \leq 3 \max_{w \in B_R(x_0)} \big|f(w)\big|$. Thus, F is bounded. It remains to prove that F is harmonic. Clearly, F is harmonic in $B_{R-1}(x_0)$, for there it agrees with f. Let y be a point of $B(x_0, R) \backslash B(x_0, R-1)$. Then

$$\begin{aligned}
\mathscr{L}F(y) &= F(y) - \frac{1}{\nu(y)} \sum_{w \sim y} F(w) \\
&= F(y) - \frac{1}{\nu(y)} F(x) - \frac{\nu(y) - 1}{\nu(y)} F(z) \\
&= F(y) - \frac{1}{\nu(y)} F(x) - \frac{\nu(y) - 1}{\nu(y)} \Big[F(y) + \frac{1}{\nu(y) - 1}(F(x) - F(y))\Big] \\
&= 0,
\end{aligned}$$

so F is harmonic on $B(x_0, R)$. Now we compute the $\mathscr{L}F(z_n)$ where z_n is a point at distance n from $B(x_0, R)$. In order to obtain cleaner formulae, it is convenient to set

$$S_{n-1} := \sum_{i=0}^{n-1} \frac{1}{\prod_{j=0}^{i}(\nu(z_j) - 1)}.$$

Observe that

$$\begin{aligned}\mathscr{L}F(z_n) &= F(z_n) - \frac{1}{\nu(z_n)}\sum_{w\sim z_n} F(w)\\ &= F(y) + S_{n-1}(F(y)-F(x)) - \frac{1}{\nu(z_n)}\Big[F(y) + S_{n-2}(F(y)-F(x))\Big]\\ &\quad - \frac{\nu(z_n)-1}{\nu(z_n)}\Big[F(y) + S_n(F(y)-F(x))\Big]\\ &= (F(x)-F(y))\Big[S_{n-1} - \frac{1}{\nu(z_n)}S_{n-2} - \frac{\nu(z_n)-1}{\nu(z_n)}S_n\Big].\end{aligned}$$

Now, we write $S_{n-2} + \dfrac{1}{\prod_{j=0}^{n-1}(\nu(z_j)-1)}$ instead of S_{n-1} and $S_{n-2} + \dfrac{1}{\prod_{j=0}^{n-1}(\nu(z_j)-1)} + \dfrac{1}{\prod_{j=0}^{n}(\nu(z_j)-1)}$ instead of S_n in the last line, observe that all the terms containing S_{n-2} cancel out, and obtain that

$$\begin{aligned}\mathscr{L}F(z_n) = (F(x)-F(y))\Big[&\frac{1}{\prod_{j=0}^{n-1}(\nu(z_j)-1)}\\ &- \frac{\nu(z_n)-1}{\nu(z_n)}\Big(\frac{1}{\prod_{j=0}^{n-1}(\nu(z_j)-1)} + \frac{1}{\prod_{j=0}^{n}(\nu(z_j)-1)}\Big)\Big]\\ = 0.&\end{aligned}$$

Thus F is harmonic on $\mathscr{G}$, as required. □

Remark 2.17. Proposition 2.16 above concerning the extension of an harmonic function defined on a ball does not hold on a generic graph possessing Cheeger's isoperimetric property. There is a topological obstruction. To prove this, consider a graph that supports Cheeger's isoperimetric inequality which admits a subgraph consisting of a triangle with vertices a, b, c, connected to the rest of the graph only via the edge $[c, c']$. By Proposition 2.18, this graph possess Cheeger's inequality. Consider now a point o in $\mathscr{G}$ such that $d(o, c') = n-1$, $d(o, c) = n$ (whence $d(o, a) = d(o, b) = n+1$) and a function f defined on $B_{n+1}(o)$ and harmonic on $B_n(o)$ such that $f(d) = 0$ and $f(c) = 1$ (these values are clearly arbitrary, we only need that $f(c) \neq f(d)$). Since

f is harmonic in c,

$$\begin{aligned} 0 &= \mathscr{L} f(c) \\ &= f(c) - \frac{f(a) + f(b) + f(d)}{3}, \end{aligned}$$

whence $f(a) + f(b) = 3$. Now we prove that if this relation is true, f cannot be harmonic in a and b. If f is harmonic in a,

$$\begin{aligned} 0 &= \mathscr{L} f(a) \\ &= f(a) - \frac{f(b) + f(c)}{2}, \end{aligned}$$

i.e. $2f(a) = f(b) + 1$. Similarly, if f is harmonic in b, then $2f(b) = f(a) + 1$. These two relations are compatible if and only of $f(a) = f(b) = 1$. However, this contradicts the harmonicity of f in c, which would imply $f(a) + f(b) = 3$.

We now state and prove a proposition, which has already been used in Remark 2.17 above. Suppose that $\mathscr{G}$ is a connected graph, which possesses Cheeger isoperimetric property with Cheeger constant β, and consider a finite connected graph $\mathscr{G}'$. The graph $\widetilde{\mathscr{G}}$ is defined as follows. Its vertices are those of $\mathscr{G} \cup \mathscr{G}'$. The edges of $\mathscr{G}$ are the edges of $\mathscr{G}$, those of $\mathscr{G}'$ and another edge $[x, y]$, which joins a vertex x in $\mathscr{G}$ and a vertex y' in $\mathscr{G}'$.

Proposition 2.18. *The graph $\widetilde{\mathscr{G}}$ constructed above possesses Cheeger's isoperimetric property.*

Proof. Suppose that $\mathscr{G}_0$ is a finite subgraph of $\widetilde{\mathscr{G}}$. We consider the following three cases separately:

(i) $\mathscr{G}_0 \subset \mathscr{G}'$;

(ii) $\mathscr{G}_0 \subset \mathscr{G}$;

(iii) $\mathscr{G}_0 \cap \mathscr{G} \neq \emptyset$ and $\mathscr{G}_0 \cap \mathscr{G}' \neq \emptyset$.

In the first case, $L(\partial \mathscr{G}_0) \geq 1$. Hence

$$\frac{L\big(\partial \mathscr{G}_0\big)}{\mu(\mathscr{G}_0)} \geq \frac{1}{\mu(\mathscr{G}')} > 0.$$

In the second case, we observe that if z is a point in $\mathscr{G}$ different from x, then its measure $\mu_{\mathscr{G}}$ as an element of $\mathscr{G}$ is the same as its measure $\mu_{\widetilde{\mathscr{G}}}$ as an element of $\widetilde{\mathscr{G}}$. Moreover, $\mu_{\widetilde{\mathscr{G}}}(x) = \mu_{\mathscr{G}}(x) + 1$. Hence $\mu_{\widetilde{\mathscr{G}}}(\mathscr{G}_0) \leq \mu_{\mathscr{G}}(\mathscr{G}_0) + 1$. Furthermore $L\big(\partial_{\widetilde{\mathscr{G}}}\mathscr{G}_0\big) \geq L\big(\partial_{\mathscr{G}}\mathscr{G}_0\big)$. Recall that $L\big(\partial_{\mathscr{G}}\mathscr{G}_0\big)$ is defined to be the cardinality of the boundary of the complement of $\mathscr{G}_0$ in $\mathscr{G}$, and similarly $L\big(\partial_{\widetilde{\mathscr{G}}}\mathscr{G}_0\big)$ is the cardinality of the boundary of the complement of $\mathscr{G}_0$ in $\widetilde{\mathscr{G}}$. If x is not in $\mathscr{G}_0$, the boundaries $\partial_{\widetilde{\mathscr{G}}}\mathscr{G}_0^c$ and $\partial_{\mathscr{G}}\mathscr{G}_0^c$ coincide. If, instead, x belongs to $\mathscr{G}_0$, then y is in $\partial_{\widetilde{\mathscr{G}}}\mathscr{G}_0^c$ (for $\mathscr{G}_0$ is contained in $\mathscr{G}$), but not in $\partial_{\mathscr{G}}\mathscr{G}_0^c$. Therefore

$$\begin{aligned}
\frac{L\big(\partial_{\widetilde{\mathscr{G}}}\mathscr{G}_0\big)}{\mu_{\widetilde{\mathscr{G}}}\big(\mathscr{G}_0\big)} &\geq \frac{L\big(\partial_{\mathscr{G}}\mathscr{G}_0\big)}{\mu_{\mathscr{G}}\big(\mathscr{G}_0\big)+1} \\
\text{(because } \mu_{\mathscr{G}}\big(\mathscr{G}_0\big) \geq 1) \qquad &\geq \frac{L\big(\partial_{\mathscr{G}}\mathscr{G}_0\big)}{2\,\mu_{\mathscr{G}}\big(\mathscr{G}_0\big)} \\
&\geq \frac{\beta}{2}.
\end{aligned}$$

In the third case, we consider the set $\mathscr{G}_1 := \mathscr{G}_0 \cup \mathscr{G}'$. Clearly, $\mu_{\widetilde{\mathscr{G}}}(\mathscr{G}_1) \geq \mu_{\widetilde{\mathscr{G}}}(\mathscr{G}_0)$, and $\mu_{\widetilde{\mathscr{G}}}(\mathscr{G}_1) = \mu_{\widetilde{\mathscr{G}}}(\mathscr{G}_0 \cap \mathscr{G}) + \mu_{\widetilde{\mathscr{G}}}(\mathscr{G}')$. Observe that $\mathscr{G}_0 \cap \mathscr{G}$ is a finite graph of the kind considered in the second case above. Therefore $\mu_{\widetilde{\mathscr{G}}}(\mathscr{G}_0 \cap \mathscr{G}) \leq \mu_{\mathscr{G}}(\mathscr{G}_0 \cap \mathscr{G}) + 1$, whence

$$\begin{aligned}
\mu_{\widetilde{\mathscr{G}}}(\mathscr{G}_0) &\leq \mu_{\widetilde{\mathscr{G}}}(\mathscr{G}_1) \\
&\leq \mu_{\mathscr{G}}(\mathscr{G}_0 \cap \mathscr{G}) + 1 + \mu_{\widetilde{\mathscr{G}}}(\mathscr{G}').
\end{aligned} \tag{2.7}$$

Now, we estimate the measure of the boundary. Notice that $\partial_{\mathscr{G}}(\mathscr{G}_0 \cap \mathscr{G})^c$ is contained in $\partial_{\widetilde{\mathscr{G}}}\mathscr{G}_0^c$. Indeed, suppose that z is in $\partial_{\mathscr{G}}(\mathscr{G}_0 \cap \mathscr{G})^c$. Then there exists a neighbour w of z in $\mathscr{G}_0 \cap \mathscr{G}$, which, in turn, is obviously contained in $\mathscr{G}_0$. Hence

$$L\big(\partial_{\mathscr{G}}(\mathscr{G}_0 \cap \mathscr{G})\big) \leq L\big(\partial_{\widetilde{\mathscr{G}}}(\mathscr{G}_0)\big)$$

This, and (2.7), imply that

$$\begin{aligned}
\frac{L\big(\partial_{\widetilde{\mathscr{G}}}\mathscr{G}_0\big)}{\mu_{\widetilde{\mathscr{G}}}\big(\mathscr{G}_0\big)} &\geq \frac{L\big(\partial_{\mathscr{G}}\big(\mathscr{G} \cap \mathscr{G}_0\big)\big)}{\mu_{\mathscr{G}}\big(\mathscr{G} \cap \mathscr{G}_0\big) + 1 + \mu_{\widetilde{\mathscr{G}}}(\mathscr{G}')} \\
&= \frac{L\big(\partial_{\mathscr{G}}\big(\mathscr{G} \cap \mathscr{G}_0\big)\big)}{\mu_{\mathscr{G}}(\mathscr{G}_0 \cap \mathscr{G})} \frac{\mu_{\mathscr{G}}(\mathscr{G}_0 \cap \mathscr{G})}{\mu_{\mathscr{G}}\big(\mathscr{G} \cap \mathscr{G}_0\big) + 1 + \mu_{\widetilde{\mathscr{G}}}(\mathscr{G}')} \\
&\geq \frac{\beta}{2 + \mu(\mathscr{G}')},
\end{aligned}$$

This complees the proof of (iii), and of the proposition. □

2.6 Atomic decomposition for $\mathfrak{X}^\gamma(\mathscr{G})$

In this section we address the question whether $\mathfrak{X}^\gamma(\mathscr{G})$ admits an atomic characterisation. In the case where γ is a positive integer, the answer is in the affirmative. Quite surprisingly, the answer is negative whenever γ is not an integer. In fact, it turns out that if k is an integer ≥ 1, γ lies in the interval $(k-1, k)$, and f is a function in $\mathfrak{X}^\gamma(\mathscr{G})$ with compact support, then f is in $\mathfrak{X}^k(\mathscr{G})$.

Definition 2.19. Suppose that k is a nonnegative integer. An $\mathfrak{X}^k(\mathscr{G})$-*atom* is a function A with support in a ball B that is orthogonal to all functions, which are k-harmonic on B, and satisfies the following *size condition*

$$\big\|A\big\|_2 \leq \mu(B)^{-1/2}.$$

Remark 2.20. Suppose that $\mathscr{G}$ is a locally finite graph, which possesses the Cheeger isoperimetric property. Then every $\mathfrak{X}^k(\mathscr{G})$-atom A belongs to $\mathfrak{X}^k(\mathscr{G})$, for $\mathscr{L}^{-k}A$ has finite support by Proposition 2.15 (i), hence it belongs to $L^1(\mathscr{G})$. Consequently, finite linear combinations of $\mathfrak{X}^k(\mathscr{G})$-atoms belong to $\mathfrak{X}^k(\mathscr{G})$.

Remark 2.21. Suppose that $\mathscr{G}$ is a locally finite (possibly nonhomogeneous) tree, i.e. a connected locally finite graph with no loops.

By Proposition 2.15 (ii)-(iii), if B is a ball of radius r, and $r \leq k-1$, then there are no nontrivial atoms with support in B; if $r = k$, then all $\mathfrak{X}^k(\mathscr{G})$ atoms are constant multiples of $\mathscr{L}\delta_{c_B}$. The number of $\mathfrak{X}^k(\mathscr{G})$-atoms with support in balls of radius r increases with r.

Furthermore, by Proposition 2.16, when $\mathscr{G}$ is a tree in which every vertex has at least three neighbours, the condition of orthogonality in the definition of $\mathfrak{X}^k(\mathscr{G})$-atoms may be replaced by the condition of orthogonality to *all bounded harmonic functions* on $\mathscr{G}$, i.e., the space $\mathscr{H}^\infty(\mathscr{G})$ considered in Section 2.3.

Definition 2.22. Fix an integer k and a number $s \geq k$. We define $\mathfrak{X}_s^k(\mathscr{G})$ to be the space of all functions f, which may be written as

$$\sum_j c_j A_j, \tag{2.8}$$

where A_j is a $\mathfrak{X}^k(\mathscr{G})$-atom with support contained in a ball of radius at most s, and $\sum_j |c_j| < \infty$. We endow $\mathfrak{X}_s^k(\mathscr{G})$ with the natural atomic norm:

$$\|f\|_{\mathfrak{X}_s^k(\mathscr{G})} := \inf \Big\{ \sum_j |c_j| : f = \sum_j c_j A_j \Big\},$$

where the infimum is taken over all decompositions of f of the form (2.8).

It is natural to speculate whether $\mathfrak{X}_s^k(\mathscr{G})$ is independent of s, as long as $s \geq k$, and whether $\mathfrak{X}_s^k(\mathscr{G})$ agrees with $\mathfrak{X}^k(\mathscr{G})$. The answer is in the affirmative, at least if $\mathscr{G}$ has bounded geometry, as shown in the next proposition. The proof will require the following estimate, which holds for graphs with bounded geometry: for every positive integer k

$$\mu_k := \sup\{\mu(B) : B \in \mathscr{B}, r_B \leq k\} < \infty. \tag{2.9}$$

This validity of this estimate on graphs with bounded geometry depends on the fact that each point in $\mathscr{G}$ has at most $N := \sup_{x \in \mathscr{G}} \nu(x)$ neighbours, so that a ball of radius k has at most

$$1 + N + (N-1)^2 + \cdots + (N-1)^k$$

points and that the measure of each point is at most N (recall that the measure of a point is, by definition, the number of its neighbours, see the beginning of the introduction).

Proposition 2.23. *Suppose that $\mathscr{G}$ is a connected graph which possesses the Cheeger isoperimetric property and has bounded geometry. The following hold:*

(i) if s_1 and s_2 are real numbers $\geq k$, then $\mathfrak{X}^k_{s_1}(\mathscr{G}) = \mathfrak{X}^k_{s_2}(\mathscr{G})$;

(ii) if s is a real number $\geq k$, then $\mathfrak{X}^k(\mathscr{G}) = \mathfrak{X}^k_s(\mathscr{G})$;

(iii) if f is in $\mathfrak{X}^k(\mathscr{G})$, then there exists a unique summable sequence $\{c_x\}_{x\in\mathscr{G}}$ such that

$$f = \sum_{x\in\mathscr{G}} c_x \, \mathscr{L}^k \delta_x.$$

In particular, $c_x = g(x)$ for every x in $\mathscr{G}$, where g is the unique function in $L^1(\mathscr{G})$ such that $f = \mathscr{L}^k g$.

Proof. First we prove (i). Suppose that $s_1 < s_2$. Every $\mathfrak{X}^k(\mathscr{G})$-atom with support in a ball of radius at most s_1 is clearly a $\mathfrak{X}^k(\mathscr{G})$-atom with support in a ball of radius at most s_2, so that $\mathfrak{X}^k_{s_1}(\mathscr{G}) \subseteq \mathfrak{X}^k_{s_2}(\mathscr{G})$ trivially, and

$$\|f\|_{\mathfrak{X}^k_{s_2}(\mathscr{G})} \leq \|f\|_{\mathfrak{X}^k_{s_1}(\mathscr{G})} \,.$$

It remains to prove that $\mathfrak{X}^k_{s_2}(\mathscr{G}) \subseteq \mathfrak{X}^k_{s_1}(\mathscr{G})$. Suppose preliminarly that A is a $\mathfrak{X}^k(\mathscr{G})$-atom with support in a ball B of radius r with $s_1 < r \leq s_2$. By Proposition 2.15 (i), $\mathscr{L}^{-k}A$ is supported in B, so that

$$\mathscr{L}^{-k}A = \sum_{y\in B} c(y)\, \delta_y,$$

for suitable constants $c(y)$. Hence

$$A = \sum_{y\in B} c(y)\ \big\|\mathscr{L}^k\delta_y\big\|_2\ \mu\big(B_k(y)\big)^{1/2}\, \frac{\mathscr{L}^k\delta_y}{\big\|\mathscr{L}^k\delta_y\big\|_2\ \mu\big(B_k(y)\big)^{1/2}}.$$

Thus, we have written A as a linear combination of the $\mathfrak{X}^k(\mathscr{G})$-atoms

$$\frac{\mathscr{L}^k\delta_y}{\big\|\mathscr{L}^k\delta_y\big\|_2\ \mu\big(B_k(y)\big)^{1/2}}$$

with support in the balls $B_k(y)$ of radius k. Consequently

$$\begin{aligned}\|A\|_{\mathfrak{X}^k_{s_1}(\mathscr{G})} &\leq \sum_{y\in B} |c(y)| \, \big\|\mathscr{L}^k\delta_y\big\|_2 \, \mu\big(B_k(y)\big)^{1/2} \\ &\leq \mu_k^{1/2} \, |||\mathscr{L}^k|||_2 \sum_{y\in B} |c(y)| \, \nu(y)^{1/2} \\ &\leq \mu_k^{1/2} \, |||\mathscr{L}^k|||_2 \, \#(B)^{1/2} \Big[\sum_{y\in B} |c(y)|^2 \, \nu(y)\Big]^{1/2};\end{aligned} \tag{2.10}$$

we have used the Cauchy–Schwarz inequality in the last inequality above. Now observe that

$$\begin{aligned}\Big[\sum_{y\in B} |c(y)|^2 \, \nu(y)\Big]^{1/2} &= \big\|\mathscr{L}^{-k}A\big\|_2 \\ &\leq |||\mathscr{L}^{-k}|||_2 \, \big\|A\big\|_2 \\ &\leq |||\mathscr{L}^{-k}|||_2 \, \mu(B)^{-1/2},\end{aligned} \tag{2.11}$$

where we have used the boundedness of $\mathscr{L}^{-k}$ on $L^2(\mathscr{G})$ and the size estimate of A. By combining these estimates, we obtain that

$$\begin{aligned}\big\|A\big\|_{\mathfrak{X}^k_{s_1}(\mathscr{G})} &\leq \mu_k^{1/2} \, |||\mathscr{L}^k|||_2 \, \#(B)^{1/2} \, |||\mathscr{L}^{-k}|||_2 \, \mu(B)^{-1/2} \\ &\leq \mu_k^{1/2} \, |||\mathscr{L}^k|||_2 \, |||\mathscr{L}^{-k}|||_2.\end{aligned} \tag{2.12}$$

Suppose now that f is a generic function in $\mathfrak{X}^k_{s_2}(\mathscr{G})$. Then for every $\varepsilon > 0$, there exist atoms $\{A_j\}$ with support contained in balls of radius at most s_2, and complex numbers $\{c_j\}$ such that $f = \sum_j c_j \, A_j$, and $\sum_j |c_j| \leq \big\|f\big\|_{\mathfrak{X}^k_{s_2}(\mathscr{G})} + \varepsilon$. Then (2.12) implies that

$$\begin{aligned}\big\|f\big\|_{\mathfrak{X}^k_{s_1}(\mathscr{G})} &\leq \sum_j |c_j| \, \big\|A_j\big\|_{\mathfrak{X}^k_{s_1}(\mathscr{G})} \\ &\leq \mu_k^{1/2} \, |||\mathscr{L}^k|||_2 \, |||\mathscr{L}^{-k}|||_2 \sum_j |c_j| \\ &\leq \mu_k^{1/2} \, |||\mathscr{L}^k|||_2 \, |||\mathscr{L}^{-k}|||_2 \, \big(\big\|f\big\|_{\mathfrak{X}^k_{s_2}(\mathscr{G})} + \varepsilon\big).\end{aligned}$$

By taking the infimum of both sides with respecto to ε, we obtain the estimate

$$\big\|f\big\|_{\mathfrak{X}^k_{s_1}(\mathscr{G})} \leq \mu_k^{1/2} \, |||\mathscr{L}^k|||_2 \, |||\mathscr{L}^{-k}|||_2 \, \big\|f\big\|_{\mathfrak{X}^k_{s_2}(\mathscr{G})} \, ,$$

as required to complete the proof of (i).

Next we prove (ii). By (i), it suffices to prove that $\mathfrak{X}^k(\mathscr{G}) = \mathfrak{X}^k_k(\mathscr{G})$.

First we prove that $\mathfrak{X}^k(\mathscr{G}) \subseteq \mathfrak{X}^k_k(\mathscr{G})$. If f is in $\mathfrak{X}^k(\mathscr{G})$, then there exists a function g in $L^1(\mathscr{G})$ such that $\mathscr{L}^k g = f$. We may write $g = \sum_{x\in\mathscr{G}} g(x)\,\delta_x$, with $\sum_{x\in\mathscr{G}} \big|g(x)\big|\nu(x) < \infty$. Then

$$f = \mathscr{L}^k g = \sum_{x\in\mathscr{G}} g(x)\,\mathscr{L}^k \delta_x;$$

here we have used the boundedness of $\mathscr{L}$, hence of $\mathscr{L}^k$, on $L^1(\mathscr{G})$ to interchange $\mathscr{L}^k$ with the sum. Furthermore,

$$\begin{aligned}
\big\|f\big\|_{\mathfrak{X}^k_k(\mathscr{G})} &\leq \sum_{x\in\mathscr{G}} \big|g(x)\big| \;\big\|\mathscr{L}^k\delta_x\big\|_2 \;\mu\big(B_k(x)\big)^{1/2} \\
&\leq \mu_k^{1/2}\, \big|\!\big|\!\big|\mathscr{L}^k\big|\!\big|\!\big|_2 \sum_{x\in\mathscr{G}} \big|g(x)\big|\,\nu(x)^{1/2} \\
&\leq \mu_k^{1/2}\, \big|\!\big|\!\big|\mathscr{L}^k\big|\!\big|\!\big|_2 \sum_{x\in\mathscr{G}} \big|g(x)\big|\,\nu(x) \\
&= \mu_k^{1/2}\, \big|\!\big|\!\big|\mathscr{L}^k\big|\!\big|\!\big|_2 \;\big\|f\big\|_{\mathfrak{X}^k(\mathscr{G})}
\end{aligned}$$

(we have used the fact that $\nu(x) \geq 1$ in the last inequality above), so that f may be written as a linear combination of $\mathfrak{X}^k(\mathscr{G})$-atoms at scale k with summable coefficients. Thus, f is in $\mathfrak{X}^k_k(\mathscr{G})$.

Next we prove that $\mathfrak{X}^k_k(\mathscr{G}) \subseteq \mathfrak{X}^k(\mathscr{G})$. Suppose that f is in $\mathfrak{X}^k_k(\mathscr{G})$. Then for every $\varepsilon > 0$ there exist a sequence $\{A_j\}$ of $\mathfrak{X}^k(\mathscr{G})$-atoms at scale k and a sequence of ‘complex numbers $\{c_j\}$ such that $f = \sum_j c_j\,A_j$, and

$$\sum_j \big|c_j\big| \leq \big\|f\big\|_{\mathfrak{X}^k_k(\mathscr{G})} + \varepsilon.$$

Notice that the function $\mathscr{L}^{-k} f = \sum_j c_j\,\mathscr{L}^{-k} A_j$ is in $L^1(\mathscr{G})$. Indeed, by the triangle inequality and Schwarz's inequality

$$\begin{aligned}\left\|\mathscr{L}^{-k}f\right\|_1 &\leq \sum_j \left|c_j\right| \, \left\|\mathscr{L}^{-k}A_j\right\|_1 \\ &\leq \sum_j \left|c_j\right| \, \left\|\mathscr{L}^{-k}A_j\right\|_2 \, \mu(B_j)^{1/2},\end{aligned}$$

where B_j is the support of A_j. We have used the fact that, by Proposition 2.15, the support of $\mathscr{L}^{-k}A_j$ is contained in the support of A_j, for A_j is orthogonal to all k-harmonic functions. Observe that

$$\begin{aligned}\left\|\mathscr{L}^{-k}A_j\right\|_2 &\leq \left|\!\left|\!\left|\mathscr{L}^{-k}\right|\!\right|\!\right|_2 \, \left\|A_j\right\|_2 \\ &\leq \left|\!\left|\!\left|\mathscr{L}^{-k}\right|\!\right|\!\right|_2 \, \mu(B_j)^{-1/2}\end{aligned}$$

by the boundedness of $\mathscr{L}^{-k}$ on $L^2(\mathscr{G})$ and the size estimate of A_j. By combining the last two estimates, we obtain that

$$\begin{aligned}\left\|f\right\|_{\mathfrak{X}^k(\mathscr{G})} &= \left\|\mathscr{L}^{-k}f\right\|_1 \\ &\leq \left|\!\left|\!\left|\mathscr{L}^{-k}\right|\!\right|\!\right|_2 \sum_j \left|c_j\right| \\ &\leq \left|\!\left|\!\left|\mathscr{L}^{-k}\right|\!\right|\!\right|_2 \left(\left\|f\right\|_{\mathfrak{X}^k_k(\mathscr{G})} + \varepsilon\right).\end{aligned}$$

By taking the infimum of both sides with respect to $\varepsilon > 0$, we find that f is in $\mathfrak{X}^k(\mathscr{G})$, and

$$\left\|f\right\|_{\mathfrak{X}^k(\mathscr{G})} \leq \left|\!\left|\!\left|\mathscr{L}^{-k}\right|\!\right|\!\right|_2 \, \left\|f\right\|_{\mathfrak{X}^k_k(\mathscr{G})},$$

as required to conclude the proof of (ii), and of the proposition.

Finally, we prove (iii). Since f is in $\mathfrak{X}^k(\mathscr{G})$ and $\mathscr{L}^k$ is injective on $L^2(\mathscr{G})$ (hence on $L^1(\mathscr{G})$), there exists a unique function g in $L^1(\mathscr{G})$ such that $f = \mathscr{L}^k g$.

On the one hand, $g = \sum_{x\in\mathscr{G}} g(x)\, \delta_x$, so that

$$f = \sum_{x\in\mathscr{G}} g(x)\, \mathscr{L}^k \delta_x,$$

with $\displaystyle\sum_{x\in\mathscr{G}} |g(x)|\, \nu(x) < \infty$, and at least one representation of f of the required form exists. On the other hand, if $f = \sum_{x\in\mathscr{G}} c_x\, \mathscr{L}^k \delta_x$ and we have that $\displaystyle\sum_{x\in\mathscr{G}} \left|c_x\right| \nu(x) < \infty$, then the function

$$g_1 := \sum_{x \in \mathscr{G}} c_x \, \delta_x$$

is in $L^1(\mathscr{G})$, and $\mathscr{L}^k g_1 = f$. Therefore $\mathscr{L}^k(g - g_1) = 0$, hence $g = g_1$ by the injectivity of $\mathscr{L}^k$ on $L^1(\mathscr{G})$. Thus, $c_x = g(x)$, as required. $\square$

Remark 2.24. Notice that we have not used the assumption that $\mathscr{G}$ has bounded geometry in the proof of the containment $\mathfrak{X}^k_k(\mathscr{G}) \subseteq \mathfrak{X}^k(\mathscr{G})$. Therefore, this containment holds on every connected locally finite graph which possesses the Cheeger isoperimetric property.

The proof of Proposition 2.23 (i) suggests that if $\mathscr{G}$ has not bounded geometry, then $\mathfrak{X}^k_k(\mathscr{G})$ may be strictly contained in $\mathfrak{X}^k(\mathscr{G})$. This is indeed the case.

Proposition 2.25. *Suppose that k is a positive integer and that $\mathscr{G}$ is a tree such that there exists a sequence of points $\{x_j\}$ such that $\nu(x_j)/\mu\big(B_k(x_j)\big)$ tends to 0 as j tends to infinity. Then $\mathfrak{X}^k_k(\mathscr{G})$ is strictly contained in $\mathfrak{X}^k(\mathscr{G})$.*

Proof. We argue by *reductio ad absurdum*. Suppose that $\mathfrak{X}^k(\mathscr{G}) \subseteq \mathfrak{X}^k_k(\mathscr{G})$. Then there exists a constant C such that

$$\big\|f\big\|_{\mathfrak{X}^k_k(\mathscr{G})} \leq C \, \big\|f\big\|_{\mathfrak{X}^k(\mathscr{G})} \qquad \forall f \in \mathfrak{X}^k(\mathscr{G}). \tag{2.13}$$

Consider the sequence $\{x_j\}$ of points in the statement of the proposition, and the functions $f_j := \mathscr{L}^k \delta_{x_j}$. Clearly,

$$\big\|f_j\big\|_{\mathfrak{X}^k(\mathscr{G})} = \big\|\mathscr{L}^{-k} f_j\big\|_1 = \big\|\delta_{x_j}\big\|_1 = \nu(x_j). \tag{2.14}$$

Observe that f_j belongs to $\mathfrak{X}^k_k(\mathscr{G})$, because f_j is a multiple of a $\mathfrak{X}^k_k$-atom. Therefore there exists a representation $f_j = \sum_\ell c_j^\ell \, A_\ell$ of f_j as a (possibly infinite) linear combination of $\mathfrak{X}^k_k$-atoms A_ℓ. The support of A_ℓ is precisely a ball of radius k, $B_k(z_\ell)$ say, and

$$\big\|f\big\|_{\mathfrak{X}^k_k(\mathscr{G})} \asymp \sum_\ell |c_j^\ell|.$$

By Proposition 2.15 (iii), there is only one $\mathfrak{X}^k_k$-atom with support in $B_k(z_\ell)$, which is of the form $d_\ell \mathscr{L}^k \delta_{z_\ell}$. Thus,

$$\delta_{x_j} = \mathscr{L}^{-k} f_j = \sum_\ell c_j^\ell \, \mathscr{L}^{-k} A_\ell = \sum_\ell c_j^\ell \, d_\ell \, \delta_{z_\ell}.$$

Therefore there exists ℓ_0 such that $x_j = z_{\ell_0}$, and that $c_j^\ell = 0$ whenever $\ell \neq \ell_0$. Furthermore and $c_j^{\ell_0} = 1/d_{\ell_0}$. Observe that

$$\big|d_{\ell_0}\big| \leq \frac{1}{\big\|\mathscr{L}^k \delta_{x_j}\big\|_2 \; \mu\big(B_k(x_j)\big)^{1/2}}.$$

Thus,

$$\big\|f_j\big\|_{\mathfrak{X}^k_k(\mathscr{G})} \asymp \big|c_j^{\ell_0}\big| = \big|d_{\ell_0}^{-1}\big| \geq \big\|\mathscr{L}^k \delta_{x_j}\big\|_2 \; \mu\big(B_k(x_j)\big)^{1/2}.$$

We then deduce from (2.13) and (2.14) that there exists a constant C such that

$$\big\|\mathscr{L}^k \delta_{x_j}\big\|_2 \; \mu\big(B_k(x_j)\big)^{1/2} \leq C \, \nu(x_j), \tag{2.15}$$

for all positive integers j. Notice that $\delta_{x_j} = \mathscr{L}^{-k} \mathscr{L}^k \delta_{x_j}$, whence

$$\big\|\delta_{x_j}\big\|_2 \leq \big|\!\big|\!\big|\mathscr{L}^{-k}\big|\!\big|\!\big|_2 \; \big\|\mathscr{L}^k \delta_{x_j}\big\|_2 \,.$$

This and (2.15) imply that

$$\big\|\delta_{x_j}\big\|_2 \; \mu\big(B_k(x_j)\big)^{1/2} \leq C \, \big|\!\big|\!\big|\mathscr{L}^{-k}\big|\!\big|\!\big|_2 \, \nu(x_j).$$

Since $\big\|\delta_{x_j}\big\|_2 = \nu(x)^{1/2}$, the inequality above implies that

$$\mu\big(B_k(x_j)\big)^{1/2} \leq C \, \big|\!\big|\!\big|\mathscr{L}^{-k}\big|\!\big|\!\big|_2 \, \nu(x_j)^{1/2}.$$

which cannot possibly hold for j large. $\square$

Now we restrict our analysis to homogeneous trees $\mathscr{T}$, with degree $q \geq 2$, and show that if γ is not a positive integer, then $\mathfrak{X}^\gamma(\mathscr{T})$ does not admit an atomic decomposition.

Theorem 2.26. *Suppose that $\mathscr{T}$ is a homogeneous tree, that k is a positive integer, and that γ is in $(k-1,k)$. If f is a function in $\mathfrak{X}^\gamma(\mathscr{T})$ with compact support, then f belongs to $\mathfrak{X}^k(\mathscr{T})$. Moreover, $\mathfrak{X}^k(\mathscr{T})$ is not dense in $\mathfrak{X}^\gamma(\mathscr{T})$.*

Proof. We prove the result in the case where $k=1$. The proof for $k \geq 2$ is similar and is omitted.

Step I. We show that if γ is in $(0,1)$, and f is a function with finite support in $\mathfrak{X}^\gamma(\mathscr{T})$, then f belongs, in fact, to $\mathfrak{X}^1(\mathscr{T})$.

Since f is in $\mathfrak{X}^\gamma(\mathscr{T})$, there exist a function g in $L^1(\mathscr{T})$ such that $f = \mathscr{L}^\gamma g$. This implies that

$$\begin{aligned} \widehat{f}(s,\omega) &= \big(\mathscr{L}^\gamma g\big)\widehat{\ }(s,\omega) \\ &= (1-\gamma(s))^\gamma\, \widehat{g}(s,\omega), \end{aligned} \tag{2.16}$$

where $\widehat{f}$ and $\widehat{g}$ denote the Helgason–Fourier transforms of f and g, respectively. Since f has compact support in the ball $B_N(o)$, say, by (the easy part of) the Paley–Wiener type theorem [CS, Theorem 1], $\widehat{f}$ extends to a τ-periodic entire function of exponential type N uniformly in ω, i.e. there exists a constant C such that

$$\big|\widehat{f}(z,\omega)\big| \leq C\, q^{|\operatorname{Im} z| N} \qquad \forall z \in \mathbb{C} \quad \forall \omega \in \Omega.$$

Furthermore, $\widehat{f}$ is continuous on $\mathbb{T}\times\Omega$ and satisfies the following symmetry condition:

$$\int_\Omega P^{1/2-is}(x,\omega)\, \widehat{f}(s,\omega)\, \mathrm{d}\nu(\omega) = \int_\Omega P^{1/2+is}(x,\omega)\, \widehat{f}(-s,\omega)\, \mathrm{d}\nu(\omega). \tag{2.17}$$

Thus the function $\big(1-\gamma(z)\big)^\gamma\, \widehat{g}(z,\omega)$ is entire for every ω in Ω. Recall that $1-\gamma(z)$ vanishes at the points of the set $\pm i/2 + \tau\mathbb{Z}$. Since g is in $L^1(\mathscr{T})$, its Helgason–Fourier transform $\widehat{g}(z,\omega)$ is a continuous τ-periodic function on the strip $\overline{\mathbf{S}}_{1/2}$. Thus, $\widehat{f}(z,\omega) = 0$ for every z in $\{\tau\mathbb{Z} \pm i/2\}$. Since $\widehat{f}$ is entire, its zeros have at least order 1. Moreover $\big(1-\gamma\big)^{-1}$ is a meromorphic τ-periodic function in $\mathbb{C}$ with simple poles in $\{\tau\mathbb{Z} \pm i/2\}$. Therefore $\big(1-\gamma\big)^{-1}\, \widehat{f}(\cdot,\omega)$ is an entire function for every ω in Ω. Since $\gamma(-z) = \gamma(z)$ for all z in $\mathbb{C}$ and $\widehat{f}$ satisfies the symmetry condition (2.17),

$$\int_\Omega P^{1/2-is}(x,\omega)\, \frac{\widehat{f}(s,\omega)}{1-\gamma(s)}\, \mathrm{d}\nu(\omega) = \int_\Omega P^{1/2+is}(x,\omega)\, \frac{\widehat{f}(-s,\omega)}{1-\gamma(-s)}\, \mathrm{d}\nu(\omega).$$

Finally, $\big(1-\gamma\big)^{-1}\ \widehat{f}(\cdot,\omega)$ is clearly continuous on $\mathbb{T}\times\Omega$. By the Paley–Wiener theorem for the Fourier–Helgason transform [CS, Theorem 1], there exists a compactly supported function h such that

$$\big(1-\gamma(z)\big)\ \widehat{h}(z,\omega)=\widehat{f}(z,\omega),$$

i.e. $f=\mathscr{L}h$. Hence f belongs to $\mathfrak{X}^1(\mathscr{T})$, for h is obviously in $L^1(\mathscr{T})$.

Step II. Suppose that f is in $\mathfrak{X}^\gamma(\mathscr{T})$, and that there exists a sequence $\{\varphi_n\}$ of functions in $\mathfrak{X}^1(\mathscr{T})$ that is convergent to f in the $\mathfrak{X}^\gamma(\mathscr{T})$ norm. Observe that

$$\begin{aligned}\big\|f-\varphi_n\big\|_{\mathfrak{X}^\gamma(\mathscr{T})} &= \big\|\mathscr{L}^{-\gamma}f-\mathscr{L}^{-\gamma}\varphi_n\big\|_{L^1(\mathscr{T})}\\ &= \big\|\mathscr{L}^{-\gamma}f-\mathscr{L}^{1-\gamma}\big(\mathscr{L}^{-1}\varphi_n\big)\big\|_{L^1(\mathscr{T})}.\end{aligned}$$

Since φ_n is in $\mathfrak{X}^1(\mathscr{T})$, $\mathscr{L}^{-1}\varphi_n$ is in $L^1(\mathscr{T})$, so that $\mathscr{L}^{1-\gamma}\big(\mathscr{L}^{-1}\varphi_n\big)$ is in $\mathfrak{X}^{1-\gamma}(\mathscr{T})$. Therefore $\mathscr{L}^{-\gamma}f$ is approximated in the $L^1(\mathscr{T})$ norm by a sequence of functions in $\mathfrak{X}^{1-\gamma}(\mathscr{T})$. By Proposition 2.10, $\mathfrak{X}^{1-\gamma}(\mathscr{T})$ is contained in $\mathscr{H}^\infty(\mathscr{T})^\perp$, the annihilator in $L^1(\mathscr{T})$ of the space of all bounded harmonic functions. Clearly, this is a proper closed subspace of $L^1(\mathscr{T})$. Thus, $\mathscr{L}^{-\gamma}f$ cannot by approximated by the sequence $\mathscr{L}^{1-\gamma}\big(\mathscr{L}^{-1}\varphi_n\big)$ whenever $\mathscr{L}^{-\gamma}f$ does not belong to $\mathscr{H}^\infty(\mathscr{T})^\perp$. □

2.7 The heat semigroup is not uniformly bounded on $H^1(\mathscr{T})$

The theory of the Hardy-type spaces $\mathfrak{X}^\gamma(\mathscr{G})$ we developed in the previous sections of this chapter hinges on two basic facts:

(i) if $\gamma>0$, then $\mathscr{L}^\gamma$ is bounded on $L^1(\mathscr{G})$;

(ii) the bottom of the $L^2(\mathscr{G})$ spectrum of $\mathscr{L}$ is positive.

Property (i), in turn, was established as a consequence of the fact that the heat operator $\mathscr{H}_t$ is contractive on $L^1(\mathscr{G})$ (in fact, the uniform boundedness of the heat semigroup would have sufficed).

In this section we work on a homogeneous trees $\mathscr{T}$, and we prove that the heat semigroup is bounded, but not uniformly bounded on the Hardy-type space $H^1(\mathscr{T})$, defined in the introduction. In fact, we show that the operator norm of $\mathscr{H}_t$ on $H^1(\mathscr{T})$ grows linearly with t, as t tends to infinity. This is the reason for which it is far easier to work with the spaces $\mathscr{L}^k\big(L^1(\mathscr{T})\big)$ rather than with $\mathscr{L}^k\big(H^1(\mathscr{T})\big)$. The main result of this section is the following.

Theorem 2.27. *Suppose that $\mathscr{T}$ is a homogeneous tree of degree ≥ 3. Then there exist two positive constants c, C such that*

$$c\,(1+t) \leq \big\|\!\big|\mathscr{H}_t\big\|\!\big|_{H^1(\mathscr{T})} \leq C\,(1+t) \qquad \forall t>0. \tag{2.18}$$

In order to prove this theorem, we need a characterisation of convolution operators on $H^1(\mathscr{T})$ with positive kernels (see Lemma 2.32) and some pointwise estimates of the heat kernel h_t (see Lemma 4.2), together with some definitions and a few preliminary results.

A. Carbonaro, G. Mauceri and S. Meda [CMM] defined and studied certain spaces $H^1(M)$ and $BMO(M)$ in the case where M is a measured metric space satisfying Cheeger's isoperimetric inequality together with some other mild requirements. In particular, the theory developed in [CMM] applies to homogeneous trees. Their structure, however, is so simple that in many cases most of the technicalities necessary to prove results in the general setting of [CMM] are unnecessary in the case of homogeneous trees. This is the reason for which we shall give new proofs of some of the results in [CMM].

For the sake of precision, we must also say that, strictly speaking, the theory developed in [CMM] requires that atoms in $H^1(\mathscr{T})$ be associated to balls of radius at least 4. This is irrelevant in the case of homogeneous trees, and we may, and shall, assume that atoms are supported in balls of radius 1.

We denote by $\mathscr{B}_1$ the collection of all balls in $\mathscr{T}$ of radius 1. Consider the space $BMO(\mathscr{T})$ of all functions with bounded mean oscillation, defined by

$$BMO(\mathscr{T}) := \big\{f : \|f\|_{BMO(\mathscr{T})} < \infty\big\},$$

where $\|f\|_{BMO(\mathscr{T})}$ denotes the seminorm

$$\|f\|_{BMO(\mathscr{T})} = \sup_{B\in\mathscr{B}_1} \Big[\frac{1}{\mu(B)} \sum_{x\in B} \big|f(x) - f_B\big|^2\Big]^{1/2};$$

here f_B denotes the average of f over the ball B, i.e. $f_B = \frac{1}{\mu(B)} \sum_{x\in B} f(x)$. By abuse of notation we still denote by $BMO(\mathscr{T})$ the quotient space $BMO(\mathscr{T})/\mathbb{C}$, endowed with the (semi-)norm $\|\cdot\|_{BMO(\mathscr{T})}$. We shall also need an equivalent norm on $BMO(\mathscr{T})$ that we introduce in the next proposition.

Proposition 2.28. *For every f in $BMO(\mathscr{T})$*

$$\|f\|_{BMO(\mathscr{T})} = \frac{1}{\sqrt{2}} \sup_{B\in\mathscr{B}_1} \frac{1}{\mu(B)} \Big[\sum_{x\in B}\sum_{y\in B} \big|f(x) - f(y)\big|^2\Big]^{1/2}.$$

Proof. Choose a ball B in $\mathscr{B}_1$, and two points x and y in B. Then

$$\begin{aligned}
&\big|f(x) - f_B\big|^2 \\
&= \big|f(x) - f(y) + f(y) - f_B\big|^2 \\
&= \big|f(x) - f(y)\big|^2 + \big|f(y) - f_B\big|^2 - 2\ \mathrm{Re}\,\big[\big(f(x) - f(y)\big)\,\big(\overline{f(y) - f_B}\big)\big].
\end{aligned}$$

We write $f = u + iv$, with u and v real, take the average of both sides on the ball B, and observe that

$$\begin{aligned}
&\frac{1}{\mu(B)} \sum_{x\in B} \mathrm{Re}\,\big[\big(f(x) - f(y)\big)\,\big(\overline{f(y) - f_B}\big)\big] \\
&= \frac{u(y) - u_B}{\mu(B)} \sum_{x\in B} \big(u(x) - u(y)\big) + \frac{v(y) - v_B}{\mu(B)} \sum_{x\in B} \big(v(x) - v(y)\big) \\
&= -\big|f_B - f(y)\big|^2.
\end{aligned}$$

Therefore

$$\frac{1}{\mu(B)} \sum_{x\in B} |f(x) - f_B|^2 = \frac{1}{\mu(B)} \sum_{x\in B} \big|f(x) - f(y)\big|^2 - |f(y) - f_B|^2.$$

By taking the average (with respect to the variable y) of both sides on B, we see that

$$\frac{2}{\mu(B)}\sum_{x\in B}\big|f(x)-f_B\big|^2=\frac{1}{\mu(B)^2}\sum_{x\in B}\sum_{y\in B}\big|f(x)-f(y)\big|^2.$$

The required formula follows from this by taking the supremum of both sides over all balls B in $\mathscr{B}_1$. □

Definition 2.29. A function $f:\mathscr{T}\to\mathbb{R}$ is in the *Lipschitz class* $\Lambda_1(\mathscr{T})$ if

$$\sup_{x\in\mathscr{T}}\sup_{y\sim x}\big|f(x)-f(y)\big|<\infty. \tag{2.19}$$

We endow $\Lambda_1(\mathscr{T})$ with the seminorm

$$\big\|f\big\|_{\Lambda_1(\mathscr{T})}:=\sup_{x\in\mathscr{T}}\sup_{y\sim x}|f(x)-f(y)|.$$

Usually a Lipschitz function on a metric space (X,d) is a function f which satisfies the following: there exists a constant L such that

$$\big|f(x)-f(y)\big|\le L\,d(x,y)\qquad\forall x,y\in X. \tag{2.20}$$

Observe that this is equivalent to Definition 2.29 when $X=\mathscr{T}$.

Indeed, on the one hand if f satisfies (2.20), then clearly it satisfies (2.19).

On the other hand, if f satisfies (2.19), and x and y are two points in $\mathscr{T}$ at distance n, then there exists a unique segment $[x,y]$ of exactly $n+1$ points $x=z_0,z_1,\ldots,z_n=y$, which joins x and y. Then

$$\begin{aligned}\big|f(x)-f(y)\big|&\le\sum_{j=1}^{n}\big|f(z_j)-f(z_{j-1})\big|\\&\le n\,L\\&=L\,d(x,y),\end{aligned}$$

and f satisfies (2.20).

Proposition 2.30. *For a function f on $\mathscr{T}$ the following are equivalent:*

(i) f *is in* $BMO(\mathscr{T})$;

(ii) $\left|\nabla f\right|$ *is in* $L^\infty(\mathscr{T})$;

(iii) f *is in* $\Lambda_1(\mathscr{T})$.

Moreover the three seminorms $\left\|\cdot\right\|_{BMO(\mathscr{T})}$, $\left\||\nabla\cdot|\right\|_{L^\infty(\mathscr{T})}$ *and* $\left\|\cdot\right\|_{\Lambda_1(\mathscr{T})}$ *are equivalent.*

Proof. Clearly it suffices to prove the equivalence of the three norms. Recall that

$$\left\||\nabla f|\right\|_{L^\infty(\mathscr{T})} = \sup_{x\in\mathscr{T}} \left[\frac{1}{2}\sum_{y\sim x}\left|f(x)-f(y)\right|^2\right]^{1/2}.$$

We claim that

$$(q+2)\left\|f\right\|_{BMO} \geq \left\||\nabla f|\right\|_{L^\infty(\mathscr{T})} \geq \frac{1}{\sqrt{2}}\left\|f\right\|_{\Lambda_1(\mathscr{T})}. \tag{2.21}$$

Indeed, observe that $\mu(B) = q+2$ for each ball in $\mathscr{B}_1$. Therefore

$$\begin{aligned}(q+2)\left\|f\right\|_{BMO} &= \frac{1}{\sqrt{2}}\sup_{B\in\mathscr{B}_1}\left[\sum_{z\in B}\sum_{y\in B}\left|f(z)-f(y)\right|^2\right]^{1/2} \\ &\geq \frac{1}{\sqrt{2}}\sup_{x\in\mathscr{T}}\left[\sum_{y\in B}\left|f(x)-f(y)\right|^2\right]^{1/2} \\ &= \left\||\nabla f|\right\|_{L^\infty(\mathscr{T})},\end{aligned}$$

and the left hand inequality in (2.21) is proved. To prove the right hand inequality in (2.21), observe that

$$\begin{aligned}\left\||\nabla f|\right\|_{L^\infty(\mathscr{T})} &\geq \frac{1}{\sqrt{2}}\sup_{x\in\mathscr{T}}\left[\sup_{y\sim x}\left|f(x)-f(y)\right|^2\right]^{1/2} \\ &= \frac{1}{\sqrt{2}}\sup_{x\in\mathscr{T}}\sup_{y\sim x}\left|f(x)-f(y)\right| \\ &= \frac{1}{\sqrt{2}}\left\|f\right\|_{\Lambda_1(\mathscr{T})},\end{aligned}$$

as required.

On the other hand, if f is Lipschitz and z and y are points in a ball B in $\mathscr{B}_1$, then $d(z,y) \leq 2$ and

$$\big|f(z) - f(y)\big| \leq 2\, \big\|f\big\|_{\Lambda_1(\mathscr{T})}.$$

Therefore

$$\begin{aligned}\big\|f\big\|_{BMO} &= \frac{1}{\mu(B)\sqrt{2}} \sup_{B\in\mathscr{B}_1} \Big[\sum_{z\in B}\sum_{y\in B} \big|f(z)-f(y)\big|^2\Big]^{1/2} \\ &\leq \frac{1}{\mu(B)\sqrt{2}} \sup_{B\in\mathscr{B}_1} 2\, \big\|f\big\|_{\Lambda_1(\mathscr{T})}\, \mu(B) \\ &= \sqrt{2}\, \big\|f\big\|_{\Lambda_1(\mathscr{T})}.\end{aligned}$$

The required equivalence of the three norms $\big\|\cdot\big\|_{BMO(\mathscr{T})}$, $\big\||\nabla\cdot|\big\|_{L^\infty(\mathscr{T})}$ and $\big\|\cdot\big\|_{\Lambda_1(\mathscr{T})}$ is proved. □

We now observe that the dual of $H^1(\mathscr{T})$ may be identified with $BMO(\mathscr{T})$.

Theorem 2.31. *The following hold*

(i) for every f in $BMO(\mathscr{T})$ the functional ℓ, initially defined on finite linear combinations of $H^1(\mathscr{T})$-atoms by the rule

$$\ell(g) = \sum_{x\in\mathscr{T}} f(x)\, g(x),$$

extends to a bounded functional on $H^1(\mathscr{T})$. Furthermore,

$$|||\ell|||_{H^1(\mathscr{T})} \leq \|f\|_{BMO(\mathscr{T})};$$

(ii) there exists a constant C such that for every continuous linear functional ℓ on $H^1(\mathscr{T})$ there exists a function f^ℓ in $BMO(\mathscr{T})$ such that

$$\|f^\ell\|_{BMO(\mathscr{T})} \leq C\, |||\ell|||_{H^1(\mathscr{T})}$$

and

$$\ell(g) = \sum_{x\in\mathscr{T}} f^\ell(x)\, g(x)$$

for all finite linear combination of $H^1(\mathscr{T})$-atoms.

The proof of this result may be obtained along the lines of the proof of [CMM, Theorem 5.1]. In fact, the proof therein may be simplified considerably, due to the simplicity of the structure of $\mathscr{T}$. However, since this result is slightly out of the main line of this thesis, we omit its proof.

Lemma 2.32. *Assume that T is a left invariant operator on $L^1(\mathscr{T})$ with nonnegative (radial) kernel k. The following are equivalent:*

(i) the restriction of T to $H^1(\mathscr{T})$ is a bounded operator on $H^1(\mathscr{T})$;

(ii) the kernel k satisfies

$$\sum_{x\in\mathscr{T}} |x|\, k(x) < \infty. \tag{2.22}$$

Furthermore,

$$\frac{q-1}{q+1}\,\frac{1}{\sqrt{2(q+2)}} \sum_{x\in\mathscr{T}} |x|\, k(x) \leq |||T|||_{H^1(\mathscr{T})} \leq \|k\|_1 + \sum_{x\in\mathscr{T}} |x|\, k(x). \tag{2.23}$$

Proof. In this proof, $\langle\cdot,\cdot\rangle$ denotes the duality between $H^1(\mathscr{T})$ and $BMO(\mathscr{T})$.

Suppose that (i) holds. Then for every atom a in $H^1(\mathscr{T})$ and for every b in $BMO(\mathscr{T})$

$$\left|\langle Ta, b\rangle\right| \leq |||T|||_{H^1(\mathscr{T})} \left\|a\right\|_{H^1(\mathscr{T})} \left\|b\right\|_{BMO(\mathscr{T})}. \tag{2.24}$$

By Proposition 2.30, an element b in $BMO(\mathscr{T})$ may be represented by a Lipschitz function in $\Lambda_1(\mathscr{T})$, which, with abuse of notation, we still denote by b, such that $b(o) = 0$.

Fix a neighbour p of o and consider the function $a' := \delta_o - \delta_p$. It is straightforward to check that $a'/\sqrt{2(q+2)}$ is an $H^1(\mathscr{T})$-atom. Denote by $\mathscr{T}_p$ the subset of $\mathscr{T}$ consisting of all points x such that the (unique) geodesic joining x and o does not contain p. Notice that x belongs to $\mathscr{T}_p$ if and only if $d(x,p) > d(x,o)$. Set

$$b(x) := |x|\ \mathbf{1}_{\mathscr{T}_p}(x).$$

Clearly b is a Lipschitz function with Lipschitz constant 1. For each positive integer n, denote by $\mathscr{T}_{p,n}$ the subtree of $\mathscr{T}$, defined by

$$\mathscr{T}_{p,n} := \mathscr{T}_p \cap B_n(o),$$

and by b_n the "truncated version" of b, defined by

$$b_n(x) = |x| \; \mathbf{1}_{\mathscr{T}_{p,n}}(x) + n \; \mathbf{1}_{\mathscr{T}_p \setminus \mathscr{T}_{p,n}}.$$

It is straightforward to check that b_n is in $\Lambda_1(\mathscr{T})$ and that

$$\big\|b_n\big\|_{\Lambda_1(\mathscr{T})} = \big\|b\big\|_{\Lambda_1(\mathscr{T})} = 1 \qquad \forall n \in \mathbb{N} \setminus \{0\}.$$

We *claim* that

$$\frac{q-1}{q+1} \sum_{x \in \mathscr{T}} |x| \, k(x) \leq \sup_n \big|\langle Ta', b_n \rangle\big| \tag{2.25}$$

By combining (2.24) and (2.25), we conclude that

$$\frac{q-1}{q+1} \sum_{x \in \mathscr{T}} |x| \, k(x) \leq \sqrt{2(q+2)} \; \big\|\!\big|T\big|\!\big\|_{H^1(\mathscr{T})},$$

as required to show that (i) implies (ii) and that the left hand inequality in (2.23) holds.

Thus, it remains to prove the claim (2.25). Denote by k_0 the profile of k, defined by

$$k_0\big(|x|\big) = k(x) \qquad \forall x \in \mathscr{T}.$$

Observe that $Ta'(x) = k_0\big(d(x,o)\big) - k_0\big(d(p,x)\big)$. Since b_n is bounded and, by assumption, Ta' is in $L^1(\mathscr{T})$, the pairing between Ta' and b_n is given by

$$\langle Ta', b_n \rangle = \sum_{x \in \mathscr{T}} \big[k_0\big(d(x,o)\big) - k_0\big(d(p,x)\big)\big] \, b_n(x). \tag{2.26}$$

We integrate in polar co-ordinates around o, use the fact that for each positive integer j

$$\#\big[S_j(o) \cap \mathscr{T}_p\big] = q^j, \tag{2.27}$$

and see that the right hand side of (2.26) may be written as the sum of the following four terms

$$\begin{aligned}
S_1 &= \sum_{x\in\mathscr{T}_{p,n}} k_0\big(d(x,o)\big)\,|x| = \sum_{j=0}^{n} k_0(j)\,j\,q^j \\
S_2 &= -\sum_{x\in\mathscr{T}_{p,n}} k_0\big(d(p,x)\big)\,|x| = -\sum_{j=0}^{n} k_0(j+1)\,j\,q^j \\
S_3 &= n \sum_{x\in\mathscr{T}_{p}\setminus\mathscr{T}_{p,n}} k_0\big(d(x,o)\big) = n\sum_{j=n+1}^{\infty} k_0(j)\,q^j \\
S_4 &= -n \sum_{x\in\mathscr{T}_{p}\setminus\mathscr{T}_{p,n}} k_0\big(d(p,x)\big) = -n\sum_{j=n+1}^{\infty} k_0(j+1)\,q^j.
\end{aligned} \tag{2.28}$$

We change variables in S_2 and S_4 ($j+1=\ell$), group similar terms together, and find that

$$\begin{aligned}
S_1+S_2+S_3+S_4 = k_0(1)\,q + \sum_{j=2}^{n} k_0(j)\,\big[jq^j-(j-1)q^{j-1}\big] \\
+ n\,k_0(n+1)\,\big(q^{n+1}-q^n\big) + n\sum_{j=n+2}^{\infty} k_0(j)\,\big[q^j-q^{j-1}\big].
\end{aligned}$$

The positivity of k_0 implies that all the summands on the right hand side are non-negative. This, and the fact that

$$j\,q^j-(j-1)\,q^{j-1} = \frac{q-1}{q}\,j\,q^j + q^{j-1} \geq \frac{q-1}{q}\,j\,q^j$$

for all positive integer j imply that

$$\begin{aligned}
S_1+S_2+S_3+S_4 &\geq \frac{q-1}{q}\sum_{j=1}^{n} k_0(j)\,j\,q^j \\
&= \frac{q-1}{q+1}\sum_{1\leq|x|\leq n} k(x)\,|x|.
\end{aligned}$$

Thus,

$$\sup_n\,\langle Ta',b_n\rangle \geq \frac{q-1}{q+1}\sum_{x\in\mathscr{T}} k(x)\,|x|.$$

and the claim (2.25) is proved.

Next we prove that (ii) implies (i), and that

$$\left|\!\left|\!\left|T\right|\!\right|\!\right|_{H^1(\mathscr{T})} \leq \left\|k\right\|_1 + \sum_{x\in\mathscr{T}} k(x)\,|x|. \tag{2.29}$$

It suffices to prove that for every function f in $H^1(\mathscr{T})$, and for every b in $\Lambda_1(\mathscr{T})$, with norm ≤ 1, the following estimate holds

$$\left|\langle Tf,b\rangle\right| \leq \left\|f\right\|_{H^1(\mathscr{T})} \Big[\left\|k\right\|_1 + \sum_{x\in\mathscr{T}} k(x)\,|x|\Big].$$

Observe that this conclusion follows from the seemingly weaker estimate

$$\left|\langle Ta,b\rangle\right| \leq \left\|k\right\|_1 + \sum_{x\in\mathscr{T}} k(x)\,|x| \tag{2.30}$$

for all $H^1(\mathscr{T})$-atoms a (C independent of a). Indeed, suppose that (2.30) holds, and consider a function f in $H^1(\mathscr{T})$. For every $\varepsilon > 0$, we may write $f = \sum_j c_j\, a_j$, where the a_j's are atoms, and $\sum_j |c_j| \leq \left\|f\right\|_{H^1(\mathscr{T})} + \varepsilon$. Since T is bounded on $L^1(\mathscr{T})$, $Tf = \sum_j c_j\, Ta_j$, so that

$$\begin{aligned}
\left\|Tf\right\|_{H^1(\mathscr{T})} &\leq \sum_j |c_j|\ \left\|Ta_j\right\|_{H^1(\mathscr{T})} \\
&\leq \Big[\left\|k\right\|_1 + \sum_{x\in\mathscr{T}} k(x)\,|x|\Big] \sum_j |c_j| \\
&\leq \Big[\left\|k\right\|_1 + \sum_{x\in\mathscr{T}} k(x)\,|x|\Big] \big(\left\|f\right\|_{H^1(\mathscr{T})} + \varepsilon\big).
\end{aligned}$$

The required estimate (2.29) follows from this by taking the infimum of both sides with respect to ε, and then the supremum over all f in $H^1(\mathscr{T})$ with $\left\|f\right\|_{H^1(\mathscr{T})} \leq 1$.

Thus, it remains to prove (2.30). Suppose that a is an $H^1(\mathscr{T})$-atom with support in $B_1(z)$ for some z in $\mathscr{T}$. Since $\sum_{x\in\mathscr{T}} a(x) = 0$ and T is bounded on $L^1(\mathscr{T})$, $\sum_{x\in\mathscr{T}} Ta(x) = 0$. Clearly,

$$a = \sum_{w\in B_1(z)} c_w\,\delta_w,$$

for suitable constants c_w, so that

$$\begin{aligned} Ta(x) &= \sum_{w\in B_1(z)} c_w\, T\delta_w(x) \\ &= \sum_{w\in B_1(z)} c_w\, k_0\big(d(w,x)\big). \end{aligned}$$

Observe that

$$\begin{aligned} \sum_{x\in\mathscr{T}} \big|Ta(x)\big|\,\big|b(x)\big| &\leq \sum_{w\in B_1(z)} \big|c_w\big| \sum_{x\in\mathscr{T}} k_0\big(d(w,x)\big)\,\big[\big|b(x)-b(z)\big|+\big|b(z)\big|\big] \\ &\leq \sum_{w\in B_1(z)} \big|c_w\big| \sum_{x\in\mathscr{T}} k_0\big(d(w,x)\big)\,\big[d(x,z)+\big|b(z)\big|\big] \\ &\leq \sum_{w\in B_1(z)} \big|c_w\big| \sum_{x\in\mathscr{T}} k_0\big(d(w,x)\big)\,\big[d(x,w)+1+\big|b(z)\big|\big]. \end{aligned}$$

Now, the inner sum is equal to $\sum_{x\in\mathscr{T}}\big|k(x)\big|\,\big[|x|+1+\big|b(z)\big|\big]$, which is clearly finite, and independent of w in $B_1(z)$, and $\sum_{w\in B_1(z)}\big|c_w\big|\leq 1$.

Thus, the duality between $H^1(\mathscr{T})$ and $BMO(\mathscr{T})$ is given by integration, and we may write

$$\begin{aligned} \big|\langle Ta,b\rangle\big| &= \Big|\sum_{x\in\mathscr{T}} Ta(x)\,b(x)\Big| \\ &= \Big|\sum_{x\in\mathscr{T}} Ta(x)\,\big[b(x)-b(z)\big]\Big| \\ &\leq \sum_{x\in\mathscr{T}} \big|Ta(x)\big|\,d(x,z). \end{aligned}$$

By arguing as above, we see that the right hand side is dominated by

$$\big\|k\big\|_1 + \sum_{x\in\mathscr{T}} \big|k(x)\big|\,|x|,$$

as required to conclude the proof of (2.30) and of (ii). □

We shall prove the following (see Lemma 4.2 below): for each positive integer N there exists a constant C, independent of t, such that

$$h_t(x) \leq C\, q^{-|x|}\,\big(|x|-\beta t\big)^{-N}\, t^{-1/2} \qquad \forall x : |x| \geq (\beta+1)\,t \quad \forall t\geq 1. \tag{2.31}$$

We postpone the proof of (2.31) just because we prefer to group together all the estimate of h_t we need in this thesis. Some of them are related to the heat maximal operator and will be needed only much later.

Lemma 2.33. *There exist two positive contants c and C such that*

$$c\,t \leq \sum_{x\in\mathscr{T}} h_t(x)\,|x| \leq C\,t \qquad \forall t \geq 1.$$

Proof. We recall the following concentration phenomenon, proved by G. Medolla and A.G. Setti [MS2], concerning heat diffusion on trees. Denote by R a function on $[0,\infty)$ such that $\sqrt{t}/R(t)$ tends to 0 as t tends to ∞, and set

$$\alpha(t) := \frac{q-1}{q+1}\,t - R(t), \qquad \beta(t) := \frac{q-1}{q+1}\,t + R(t),$$

and

$$A(t) := \big\{x \in \mathscr{T} : \alpha(t) \leq |x| \leq \beta(t)\big\}.$$

Then

$$\lim_{t\to\infty} \sum_{x\in A(t)} h_t(x) = 1.$$

Now, observe that

$$\begin{aligned}\sum_{x\in\mathscr{T}} h_t(x)\,|x| &\leq 4\beta t \sum_{x:|x|\leq 4\beta t} h_t(x) + \sum_{x:|x|\geq 4\beta t} h_t(x)\,|x| \\ &\leq 4\beta t\, + C_N \sum_{x:|x|\geq 4\beta t} q^{-|x|}\,|x|^{1-N} \\ &\leq C\,t,\end{aligned}$$

as required to conclude the proof of the upper bound. We have used (2.31) in the second inequality above.

As for the lower bound, note that

$$\begin{aligned}\sum_{x\in\mathscr{T}} h_t(x)\,|x| &\geq \alpha(t) \sum_{x:|x|\geq\alpha(t)} h_t(x) \\ &\geq C\,t,\end{aligned}$$

by the definition of $\alpha(t)$ and the aforementioned result of G. Medolla and A.G. Setti. This concludes the proof of the lemma. $\square$

We are now ready to prove Theorem 2.27.

Proof. First we show that $\mathscr{H}_t$ is bounded on $H^1(\mathscr{T})$, and that the function $t \mapsto |||\mathscr{H}_t|||_{H^1(\mathscr{T})}$ is bounded on compact subintervals of $[0,\infty)$. Recall that

$$\mathscr{H}_t = \mathscr{I} + \sum_{k=1}^{\infty} \frac{(-t\mathscr{L})^k}{k!} \qquad \forall t \in \mathbb{R}.$$

Notice that the operator $\mathscr{L}$ is bounded on $H^1(\mathscr{T})$. Indeed, if f is in $H^1(\mathscr{T})$, then $\mathscr{L}f = \sum_{x\in\mathscr{T}} f(x)\,\mathscr{L}\delta_x$. It is straightforward to check that $\mathscr{L}\delta_x$ is a multiple of an $H^1(\mathscr{T})$-atom, so that

$$\begin{aligned} \|f\|_{H^1(\mathscr{T})} &\le \sum_{x\in\mathscr{T}} |f(x)|\ \|\mathscr{L}\delta_x\|_{H^1(\mathscr{T})} \\ &\le C \sum_{x\in\mathscr{T}} |f(x)| \\ &\le C\ \|f\|_{H^1(\mathscr{T})}\,; \end{aligned}$$

the last inequality follows from the continuous containment $H^1(\mathscr{T}) \subseteq L^1(\mathscr{T})$. Therefore

$$\begin{aligned} \|\mathscr{H}_t f\|_{H^1(\mathscr{T})} &\le \Big[1 + \sum_{k=1}^{\infty} \frac{(C|t|)^k}{k!}\Big]\ \|f\|_{H^1(\mathscr{T})} \\ &\le \mathrm{e}^{C|t|}\ \|f\|_{H^1(\mathscr{T})} \qquad \forall t \in \mathbb{R}, \end{aligned}$$

as required.

Next, we prove that $t \mapsto |||\mathscr{H}_t|||_{H^1(\mathscr{T})}$ stays away from 0 on compact intervals of $[0,\infty)$. Observe that

$$\|f\|_{H^1(\mathscr{T})} = \|\mathrm{e}^{t\mathscr{L}}\mathrm{e}^{-t\mathscr{L}} f\|_{H^1(\mathscr{T})} \le \mathrm{e}^{Ct}\ \|\mathrm{e}^{-t\mathscr{L}} f\|_{H^1(\mathscr{T})} \qquad \forall t > 0.$$

This implies that

$$|||\mathscr{H}_t|||_{H^1(\mathscr{T})} \ge \mathrm{e}^{-Ct} \qquad \forall t > 0,$$

as required.

To conclude the proof it suffices to show that there exist positive constants c and C such that

$$c\,t \leq \left\| \mathscr{H}_t \right\|_{H^1(\mathscr{T})} \leq C\,t \qquad \forall t \geq 1.$$

This follows directly from Lemmata 2.32 and 2.33.

The proof of Theorem 2.27 is complete. □

We end this chapter by proving the following criterion of independent interest.

Proposition 2.34. *Suppose that f is a function on $\mathscr{T}$ such that $|\cdot|\, f(\cdot)$ belongs to $L^1(\mathscr{T})$, and that $\sum_{x\in\mathscr{T}} f(x) = 0$. Then f belongs to $H^1(\mathscr{T})$.*

Proof. First we prove that

$$\sum_{x\in\mathscr{T}} |f(x)|\, |b(x)| < \infty$$

for every function b in $\Lambda_1(\mathscr{T})$. Indeed, $|b(x)| \leq |b(x) - b(o)| + |b(o)|$, so that

$$\begin{aligned}\sum_{x\in\mathscr{T}} |f(x)|\, |b(x)| &= \sum_{x\in\mathscr{T}} |f(x)|\, |b(x) - b(o)| + |b(o)| \sum_{x\in\mathscr{T}} |f(x)| \\ &\leq \left\|b\right\|_{\Lambda_1(\mathscr{T})} \sum_{x\in\mathscr{T}} |f(x)|\, |x| + |b(o)|\; \left\|f\right\|_{L^1(\mathscr{T})}\,,\end{aligned}$$

which is finite by assumption.

Thus, the pairing $\sum_{x\in\mathscr{T}} f(x)\, b(x)$ is well defined for each b in $\Lambda_1(\mathscr{T})$, i.e. for every b in $BMO(\mathscr{T})$ (see Proposition 2.30). By assumption, the integral of f vanishes, so that

$$\begin{aligned}\Big|\sum_{x\in\mathscr{T}} f(x)\, b(x)\Big| &= \Big|\sum_{x\in\mathscr{T}\setminus\{o\}} f(x)\,\big(b(x) - b(o)\big)\Big| \\ &\leq \left\|b\right\|_{\Lambda_1(\mathscr{T})} \sum_{x\in\mathscr{T}} |f(x)|\, |x|.\end{aligned}$$

Thus, f is in $H^1(\mathscr{T})$, with norm bounded from above by $\sum_{x\in\mathscr{T}} |f(x)|\, |x|$, as required. □

Chapter 3

Duality

In this chapter we consider a (possibly nonhomogeneous) tree $\mathscr{G}$ which possesses Cheeger's isoperimetric inequality and has bounded geometry. Recall that this is implied by the requirement that there exists a constant ν_0 such that

$$3 \leq \nu(x) \leq \nu_0 \qquad \forall x \in \mathscr{G}.$$

We shall determine the Banach dual of $\mathfrak{X}^\gamma(\mathscr{G})$, for each positive γ. First we consider the case where γ is a positive integer, also denoted by k in the sequel.

3.1 The space $\mathfrak{X}^k_{\mathrm{fin}}(\mathscr{G})$

Recall that, under our standing assumptions on $\mathscr{G}$, the spaces $\mathfrak{X}^k(\mathscr{G})$ and $\mathfrak{X}^k_k(\mathscr{G})$ agree, and the corresponding norms are equivalent (see Proposition 2.23 (ii)).

Definition 3.1. We denote by $\mathfrak{X}^k_{\mathrm{fin}}(\mathscr{G})$ the subspace of all functions in $\mathfrak{X}^k(\mathscr{G})$ with finite support.

A function f belongs to $\mathfrak{X}^k_{\mathrm{fin}}(\mathscr{G})$ if and only if there exist a finite set $K \subset \mathscr{G}$ and constants $\{c_x\}_{x\in K}$ such that

$$f = \sum_{x\in K} c_x \, \mathscr{L}^k \delta_x. \tag{3.1}$$

This decomposition in terms of $\mathfrak{X}^k_k(\mathscr{G})$-atoms is unique, for, by Proposition 2.23 (iii) every function in $\mathfrak{X}^k_k(\mathscr{G})$ admits a unique decomposition as a linear combination of $\mathfrak{X}^k(\mathscr{G})$-atoms. We endow $\mathfrak{X}^k_{\mathrm{fin}}(\mathscr{G})$ with the $\mathfrak{X}^k(\mathscr{G})$-norm

$$\|f\|_{\mathfrak{X}^k_{\mathrm{fin}}(\mathscr{G})} = \sum_{x \in K} |c_x|.$$

Of course, this norm is equivalent to the $\mathfrak{X}^k_k(\mathscr{G})$-norm. Since finitely supported functions are norm-dense in $L^1(\mathscr{G})$, this implies that $\mathfrak{X}^k_{\mathrm{fin}}(\mathscr{G})$ is norm-dense in $\mathfrak{X}^k(\mathscr{G})$.

Recall that the analogue for Riemannian manifolds of the space $\mathfrak{X}^k_{\mathrm{fin}}(\mathscr{G})$ has been considered in [MMV3], where it is shown to play a key role in the study of the Banach dual of the Hardy type spaces defined therein. In the case of Riemannian manifolds it is quite hard to show that the norms $\|\cdot\|_{\mathfrak{X}^k_{\mathrm{fin}}(\mathscr{G})}$ and $\|\cdot\|_{\mathfrak{X}^k(\mathscr{G})}$ are equivalent on $\mathfrak{X}^k_{\mathrm{fin}}(\mathscr{G})$.

3.2 The space $\mathfrak{Y}^k(\mathscr{G})$

We recall our standing assumption that $\mathscr{G}$ is a (possibly nonhomogeneous) tree which possesses Cheeger's isoperimetric inequality and has bounded geometry. First we define the space $\mathfrak{Y}^k(\mathscr{G})$, which we shall prove to be isometrically isomorphic to the dual of $\mathfrak{X}^k(\mathscr{G})$ (see Theorem 3.6 below).

Definition 3.2. Denote by $\mathscr{Y}^k(\mathscr{G})$ the vector space of all complex valued functions G on $\mathscr{G}$ such that $\mathscr{L}^k G$ is a bounded function on $\mathscr{G}$. We endow $\mathscr{Y}^k(\mathscr{G})$ with the seminorm

$$\|G\|_{\mathscr{Y}^k} := \|\mathscr{L}^k G\|_\infty .$$

Observe that $\|\mathscr{L}^k G\|_\infty = 0$ if and only if G is k-harmonic on $\mathscr{G}$. Hence $\|\cdot\|_\infty$ is a genuine norm on the quotient space $\mathscr{Y}^k(\mathscr{G})/\mathscr{H}^k(\mathscr{G})$, where $\mathscr{H}^k(\mathscr{G})$ is the space of k-harmonic functions on $\mathscr{G}$. We denote by $\mathfrak{Y}^k(\mathscr{G})$ the quotient space above, endowed with the quotient norm. If G is in $\mathscr{Y}^k(\mathscr{G})$, we denote by $\mathbb{G}$ the coset $G + \mathscr{H}^k(\mathscr{G})$ in $\mathfrak{Y}^k(\mathscr{G})$. Thus,

$$\|\mathbb{G}\|_{\mathfrak{Y}^k(\mathscr{G})} := \|\mathscr{L}^k G\|_\infty . \tag{3.2}$$

For later purposes, we record that $\mathscr{L}\big(\mathscr{Y}^1(\mathscr{G})\big)$ agrees with $L^\infty(\mathscr{G})$ if and only if every function in $L^\infty(\mathscr{G})$ is the Laplacian of some other function. This is indeed true under the assumption made throughout this chapter. It would be interesting to know whether this property continues to hold on all connected graphs with bounded geometry.

One of the key steps in [MMV3] in the determination of the dual of the analogue on Riemannian manifolds of the Hardy-type space $\mathfrak{X}^k(\mathscr{G})$ considered above, was a careful investigation of the solvability of the generalised Poisson equation $\mathscr{L}^k u = g$ for every datum g locally in L^2. The counterpart of that analysis in our setting is contained in the following proposition, which states that for each function g on $\mathscr{G}$ the equation $\mathscr{L}^k u = g$ is solvable.

Proposition 3.3. *Suppose that g is a function on a locally finite tree $\mathscr{G}$. The following hold:*

(i) there exists a function u on $\mathscr{G}$ such that $\mathscr{L}u = g$;

(ii) if every vertex of $\mathscr{G}$ has at least three neighbours, and if g is bounded, then we may select a function u in $\Lambda_1(\mathscr{G})$ such that $\mathscr{L}u = g$, and

$$\|u\|_{\Lambda_1(\mathscr{G})} \leq 3\, \|g\|_\infty .$$

In particular, if u is harmonic (i.e., if $g = 0$), then u is constant.

Proof. First we prove (i), by constructing u recursively.
Set $u(o) = 0$ and $u(x) = -g(o)$ for every $x \sim o$, so that

$$\mathscr{L}u(o) = 0 - \frac{1}{\nu(o)} \sum_{x\sim o} \big(- g(o)\big) = g(o). \tag{3.3}$$

Suppose that u has already been defined on the ball $B_n(o)$, for some $n \geq 1$, in such a way that $\mathscr{L}u = g$ on the ball $B_{n-1}(o)$. Pick a point z such that $d(o,z) = n+1$ and denote by y be the unique point in $\mathscr{G}$ such that $d(o,y) = n$ and $y \sim z$. Moreover denote by x be the unique point in $\mathscr{G}$ such that $d(o,x) = n-1$ and $x \sim y$. We set

$$u(z) := \frac{\nu(y)}{\nu(y)-1} \big[u(y) - g(y)\big] - \frac{u(x)}{\nu(y)-1}. \tag{3.4}$$

We remark that u has the same value at all points z such that $d(o,z)=n+1$ and $z\sim y$. Observe that

$$\mathscr{L}u(y)=u(y)-\frac{\nu(y)-1}{\nu(y)}\,u(z)-\frac{1}{\nu(y)}\,u(x).$$

We use formula (3.4), and substitute $\dfrac{\nu(y)}{\nu(y)-1}\big[u(y)-g(y)\big]-\dfrac{u(x)}{\nu(y)-1}$ in place of $u(z)$ in the formula above. Rearranging terms, we find that

$$\mathscr{L}u(y)=g(y).$$

Now u is defined on $B_{n+1}(o)$, and $\mathscr{L}u=g$ on $B_n(o)$, thereby concluding the proof of (i).

Next we prove (ii). We shall prove that the function u, constructed in the proof of (i), belongs to the Lipschitz class $\Lambda_1(\mathscr{G})$. Suppose that y and z are neighbours. We need to estimate $\big|u(z)-u(y)\big|$. Recall that $u(o)=0$ and that $u(x)=-g(o)$ for every $x\sim o$, so that $\big|u(x)-u(o)\big|\leq\big\|g\big\|_\infty$. Thus, without loss of generality, we may assume that $d(z,o)=n$, with $n\geq 2$, and that $d(y,o)=d(z,o)-1$. The geodesic segment $[o,z]$ is of the form $[o,x_1,\ldots,x_{n-2},y,z]$. For notational convenience we shall write x_0 instead of o, x_{n-1} instead of y and x_n instead of z. By (3.4),

$$u(x_n)=\frac{\nu(y)}{\nu(y)-1}\big[u(x_{n-1})-g(x_{n-1})\big]-\frac{u(x_{n-2})}{\nu(y)-1}\qquad\forall n\geq 2.$$

Thus,

$$u(x_n)-u(x_{n-1})=\frac{1}{\nu(x_{n-1})-1}\big[u(x_{n-1})-u(x_{n-2})\big]-\frac{\nu(x_{n-1})}{\nu(x_{n-1})-1}\,g(x_{n-1}).$$

We *claim* that

$$u(x_n)-u(x_{n-1})=-\sum_{k=1}^{n-1}\frac{\nu(x_k)}{\prod_{i=k}^{n-1}(\nu(x_i)-1)}\,g(x_k)-\frac{g(o)}{\prod_{k=1}^{n-1}(\nu(x_k)-1)}.\qquad(3.5)$$

We prove the claim by induction on n. If $n=1$, then by construction of u we have $u(x_1)=-g(o)$ and $u(o)=0$, so equation (3.5) holds. Now we suppose that (3.5) holds for all n up to N, and prove it for $N+1$. Indeed,

$$u(x_{N+1})-u(x_N)=\frac{1}{\nu(x_N)-1}\big[u(x_N)-u(x_{N-1})\big]-\frac{\nu(x_N)}{\nu(x_N)-1}\,g(x_N).$$

We use the induction hypothesis, and substitute the right hand side of (3.5) (with N in place of n) in place of $u(x_N) - u(x_{N-1})$. Then $u(x_{N+1}) - u(x_N)$ will be written as the sum of

$$\begin{aligned} &- \frac{1}{\nu(x_N) - 1} \sum_{k=1}^{N-1} \frac{\nu(x_k)}{\prod_{i=k}^{N-1} (\nu(x_i) - 1)} \, g(x_k) - \frac{\nu(x_N)}{\nu(x_N) - 1} \, g(x_N) \\ &= - \sum_{k=1}^{N} \frac{\nu(x_k)}{\prod_{i=k}^{N} (\nu(x_i) - 1)} \, g(x_k), \end{aligned}$$

and

$$- \frac{1}{\nu(x_N) - 1} \frac{1}{\prod_{k=1}^{N-1} (\nu(x_k) - 1)} \, g(o) = - \frac{1}{\prod_{k=1}^{N} (\nu(x_k) - 1)} \, g(o),$$

which proves the claim (3.5).

Now (3.5), the triangle inequality, and the boundedness of g imply that

$$\begin{aligned} \left| u(z) - u(y) \right| &\leq \sum_{k=1}^{n-1} \frac{\nu(x_k)}{\prod_{i=k}^{n-1} (\nu(x_i) - 1)} \left| g(x_k) \right| + \frac{1}{\prod_{k=1}^{n-1} (\nu(x_k) - 1)} \left| g(o) \right| \\ &\leq \left\| g \right\|_\infty \Big[\sum_{k=1}^{n-1} \frac{\nu(x_k)}{\prod_{i=k}^{n-1} (\nu(x_i) - 1)} + \frac{1}{\prod_{k=1}^{n-1} (\nu(x_k) - 1)} \Big]. \end{aligned}$$

Observe that $\nu(x_k)/(\nu(x_k) - 1) \leq 3/2$, because $\nu(x) \geq 3$, and that $\nu(x_j) - 1 \geq 2$ for all $j = 1, \ldots, n-1$. Clearly the expression within square brackets above is dominated by $\dfrac{3}{2} \displaystyle\sum_{k=1}^{n-1} 2^{k+1-n} + 2^{1-n}$, which is less than 3. Hence

$$|u(z) - u(y)| \leq 3 \, \left\| g \right\|_\infty ,$$

as required to conclude the proof of (ii), and of the proposition. □

Observe that we do not assume $\mathscr{G}$ to have bounded geometry in Proposition 3.3 above.

Corollary 3.4. *Suppose that g is a function on $\mathscr{G}$ and that k is a positive integer. Then there exists a function u on $\mathscr{G}$ such that $\mathscr{L}^k u = f$.*

Proof. This follows directly from the proposition above. □

Observe that there is a chain of continuous inclusions

$$L^\infty(\mathscr{G}) \subset \mathfrak{Y}^1(\mathscr{G}) \subset \mathfrak{Y}^2(\mathscr{G}) \subset \cdots$$

Specifically, to a function G in $L^\infty(\mathscr{G})$ we associate the coset $G+\mathscr{H}^j$ in $\mathfrak{Y}^j(\mathscr{G})$, and

$$\left\|G+\mathscr{H}^j\right\|_{\mathfrak{Y}^j(\mathscr{G})} = \left\|\mathscr{L}^j G\right\|_\infty \le 2^j \ \left\|G\right\|_\infty ;$$

see Proposition 1.1. Similarly, we associate to the coset $G+\mathscr{H}^k$ in $\mathfrak{Y}^k(\mathscr{G})$, the coset $G+\mathscr{H}^{k+j}$ in $\mathfrak{Y}^{k+j}(G)$. Clearly

$$\left\|G+\mathscr{H}^{k+j}\right\|_{\mathfrak{Y}^{k+j}(\mathscr{G})} = \left\|\mathscr{L}^{k+j}G\right\|_\infty \le 2^j \ \left\|\mathscr{L}^k G\right\|_\infty = 2^j \ \left\|G+\mathscr{H}^k\right\|_{\mathfrak{Y}^k(\mathscr{G})} .$$

Also note that $\mathscr{L}^j$ acts on $\mathfrak{Y}^k(\mathscr{G})$ as follows. For each coset $G+\mathscr{H}^k$ in $\mathfrak{Y}^k(\mathscr{G})$, we set

$$\mathscr{L}^j\big(G+\mathscr{H}^k\big) = \begin{cases} \mathscr{L}^j G+\mathscr{H}^{k-j} & \text{if } j<k \\ \mathscr{L}^j G & \text{if } j\ge k. \end{cases}$$

In particular, if $j\ge k$, then $\mathscr{L}^j G$ is a bounded function.

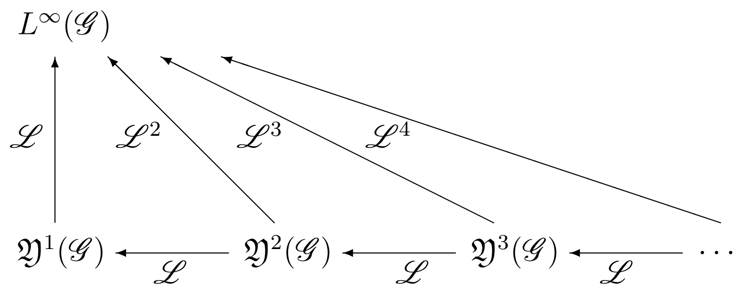
Furthermore $\mathscr{L}^k$ is an isometric isomorphism between $\mathfrak{Y}^k(\mathscr{G})$ and $L^\infty(\mathscr{G})$, for it is surjective by Corollary 3.4, and

$$\left\|\mathscr{L}^k(G+\mathscr{H}^k)\right\|_\infty = \left\|\mathscr{L}^k G\right\|_\infty = \left\|G+\mathscr{H}^k\right\|_{\mathfrak{Y}^k(\mathscr{G})} .$$

Consequently, $\mathscr{L}^{-k}$ is an isomorphism between $L^\infty(\mathscr{G})$ and $\mathfrak{Y}^k(\mathscr{G})$. By Corollary 3.4, for each G in $L^\infty(\mathscr{G})$ there exists a function $\widetilde{G}$ on $\mathscr{G}$ such that $\mathscr{L}^k\widetilde{G}=G$. Then

$$\mathscr{L}^{-k}G = \widetilde{G}+\mathscr{H}^k.$$

The situation described above is pictorially illustrated by the following commutative diagram, in which each arrow is an isometric isomorphism of Banach spaces.



Remark 3.5. We observe that for every function G with bounded k-Laplacian, for every k-harmonic function H and for every f in $\mathfrak{X}^k_{\text{fin}}(\mathscr{G})$

$$\begin{aligned}\sum_{x\in\mathscr{G}} f(x)\,\big[G(x)+H(x)\big] &= \sum_{x\in K} \mathscr{L}^{-k}f(x)\,\mathscr{L}^k G(x)\\ &= \langle \mathscr{L}^{-k}f, \mathscr{L}^k G\rangle,\end{aligned} \tag{3.6}$$

where the pairing in the last line is the standard duality beetween $L^1(\mathscr{G})$ and $L^\infty(\mathscr{G})$.

Indeed,

$$\sum_{x\in\mathscr{G}} f(x)\,\big[G(x)+H(x)\,\big] = \sum_{x\in\mathscr{G}} f(x)\,G(x)$$

for f is orthogonal to all harmonic functions on $\mathscr{G}$. Now $f = \mathscr{L}^k\mathscr{L}^{-k}f = 0$, and recall that $\mathscr{L}^{-k}f$ has finite support, because it may be written as a finite sum of $\mathfrak{X}^k_k(\mathscr{G})$-atoms. Now, the required conclusion follows from the trivial fact that

$$\sum_{x\in\mathscr{G}} \mathscr{L}^k\varphi(x)\,G(x) = \sum_{x\in\mathscr{G}} \varphi(x)\,\mathscr{L}^k G(x)$$

for every function φ with finite support.

In order to state the main result of this section, we need more notation, and a few preliminary observations.

Consider the linear map i, that associates to an element $\mathbb{G}$ of $\mathfrak{Y}^k(\mathscr{G})$ the functional $\Lambda_{\mathbb{G}}$ on $\mathfrak{X}^k_{\text{fin}}(\mathscr{G})$, defined by

$$\Lambda_{\mathbb{G}}(f) := \langle \mathscr{L}^{-k}f, \mathscr{L}^k G\rangle \qquad \forall G\in\mathbb{G}, \tag{3.7}$$

where the pairing above denotes the standard duality beetween $L^1(\mathscr{G})$ and $L^\infty(\mathscr{G})$. The map i is well defined by (3.7). Indeed, the right hand side of (3.7) makes sense, because the support of $\mathscr{L}^{-k}f$ is contained in the support of f, which is finite, for f belongs to $\mathfrak{X}^k_{\mathrm{fin}}(\mathscr{G})$. Furthermore, the right hand side above does not depend on the representative G of $\mathbb{G}$, because any other such representative is of the form $G+H$, where H is k-harmonic, whence $\mathscr{L}^k(G+H) = \mathscr{L}^k G$.

Furthermore

$$\begin{aligned} \big|\Lambda_{\mathbb{G}} f\big| &= \big|\langle \mathscr{L}^{-k} f, \mathscr{L}^k G\rangle\big| \\ &\leq \big\|\mathscr{L}^{-k} f\big\|_1 \ \big\|\mathscr{L}^k G\big\|_\infty \\ &= \big\|f\big\|_{\mathfrak{X}^k(\mathscr{G})} \ \big\|\mathbb{G}\big\|_{\mathfrak{Y}^k(\mathscr{G})} \qquad \forall f \in \mathfrak{X}^k_{\mathrm{fin}}(\mathscr{G}). \end{aligned} \tag{3.8}$$

Since $\mathfrak{X}^k_{\mathrm{fin}}(\mathscr{G})$ is norm-dense in $\mathfrak{X}^k(\mathscr{G})$, the functional $\Lambda_{\mathbb{G}}$ extends to a unique continuous linear functional (also denoted by $\Lambda_{\mathbb{G}}$) on $\mathfrak{X}^k(\mathscr{G})$. Therefore $\Lambda_{\mathbb{G}}$ belongs to $[\mathfrak{X}^k(\mathscr{G})]^*$ and

$$\big\|\Lambda_{\mathbb{G}}\big\|_{[\mathfrak{X}^k(\mathscr{G})]^*} \leq \big\|\mathbb{G}\big\|_{\mathfrak{Y}^k(\mathscr{G})} . \tag{3.9}$$

Thus, the map $i : \mathbb{G} \mapsto \Lambda_{\mathbb{G}}$ is a linear contractive map from $\mathfrak{Y}^k(\mathscr{G})$ to $[\mathfrak{X}^k(\mathscr{G})]^*$.

Theorem 3.6. *The space $\mathfrak{Y}^k(\mathscr{G})$ is isometrically isomorphic to the Banach dual of $\mathfrak{X}^k(\mathscr{G})$ via the map i defined above.*

Proof. First we prove that i is *injective.* Suppose that $\mathbb{G}$ is an element of $\mathfrak{Y}^k(\mathscr{G})$ such that $\Lambda_{\mathbb{G}}(f) = 0$ for all f in $\mathfrak{X}^k_{\mathrm{fin}}(\mathscr{G})$. Then, in particular, $\Lambda_{\mathbb{G}}(\mathscr{L}^k\delta_x) = 0$ for every $x \in \mathscr{G}$. Therefore

$$0 = \Lambda_{\mathbb{G}}(\mathscr{L}^k\delta_x) = \langle \delta_x, \mathscr{L}^k G\rangle = \mathscr{L}^k G(x), \tag{3.10}$$

whence G is k-harmonic and $\mathbb{G}$ is the null element in $\mathfrak{Y}^k(\mathscr{G})$; thus i is injective, as required.

Next we prove that i is *surjective.* Suppose that Λ is a continuous linear functional on $\mathfrak{X}^k(\mathscr{G})$. We must show that there exists $\mathbb{G}$ in $\mathfrak{Y}^k(\mathscr{G})$ such that $i(\mathbb{G}) = \Lambda$. Since $\mathscr{L}^k$ is an isometric isomorphism between $L^1(\mathscr{G})$ and $\mathfrak{X}^k(\mathscr{G})$, $\Lambda \circ \mathscr{L}^k$ is a continuous linear functional on $L^1(\mathscr{G})$. Therefore there exists a bounded function g

such that

$$\Lambda(\mathscr{L}^k F) = \langle F, g \rangle \qquad \forall F \in L^1(\mathscr{G}).$$

Furthermore $\left\|\left|\Lambda \circ \mathscr{L}^k\right|\right\|_{L^1(\mathscr{G})} = \|g\|_\infty$. Since $\mathscr{L}^k$ is an isometric isomorphism between $L^1(\mathscr{G})$ and $\mathfrak{X}^k(\mathscr{G})$,

$$\left\|\left|\Lambda\right|\right\|_{[\mathfrak{X}^k(\mathscr{G})]^*} = \left\|g\right\|_\infty . \tag{3.11}$$

By Corollary 3.4, the equation $\mathscr{L}^k u = g$, with datum g, has a solution, G say. Consider the coset $\mathbb{G} := G + \mathscr{H}^k(\mathscr{G})$ in $\mathfrak{Y}^k(\mathscr{G})$.
Since $\mathscr{L}^k G = g$, $\left\|G + \mathscr{H}^k(\mathscr{G})\right\|_{\mathfrak{Y}^k(\mathscr{G})} = \left\|g\right\|_\infty$. By combining this and (3.11), we may conclude that

$$\left\|\left|\Lambda\right|\right\|_{[\mathfrak{X}^k(\mathscr{G})]^*} = \left\|G + \mathscr{H}^k(\mathscr{G})\right\|_{\mathfrak{Y}^k(\mathscr{G})} . \tag{3.12}$$

Furthermore, for every finitely supported function F

$$\begin{aligned} \Lambda(\mathscr{L}^k F) &= \langle F, \mathscr{L}^k G \rangle \\ &= \langle \mathscr{L}^{-k}\big(\mathscr{L}^k F\big), \mathscr{L}^k G \rangle \\ &= \Lambda_{\mathbb{G}}(\mathscr{L}^k F). \end{aligned} \tag{3.13}$$

Since $\mathfrak{X}^k_{\text{fin}}(\mathscr{G})$ is dense in $\mathfrak{X}^k(\mathscr{G})$, we may conclude that $\Lambda = \Lambda_{\mathbb{G}}$. Thus the map $i : \mathbb{G} \mapsto \Lambda_{\mathbb{G}}$ is surjective from $\mathfrak{Y}^k(\mathscr{G})$ to $\big[\mathfrak{X}^k(\mathscr{G})\big]^*$, and it is an isometry.

Therefore i is an isometric isomorphism, as required. □

3.3 Sectoriality of $\mathscr{L}$ and the spaces $\mathfrak{Y}^\gamma(\mathscr{G})$

Suppose that k is a positive integer. We may define the action of $\mathscr{L}$ on $\mathfrak{Y}^k(\mathscr{G})$ by the following

$$\mathscr{L}\big(G + \mathscr{H}^k(\mathscr{G})\big) := \mathscr{L}G + \mathscr{H}^k(\mathscr{G}). \tag{3.14}$$

Observe that $\mathscr{L}^k G$ is bounded, whence so it is $\mathscr{L}^k(\mathscr{L}G)$, and the right hand side is an element of $\mathfrak{Y}^k(\mathscr{G})$. Furthermore, the map $\mathscr{L}$ thus defined is a bounded linear operator on $\mathfrak{Y}^k(\mathscr{G})$. Indeed,

$$\begin{aligned}\left\|\mathscr{L}\big(G+\mathscr{H}^k(\mathscr{G})\big)\right\|_{\mathfrak{Y}^k(\mathscr{G})} &= \left\|\mathscr{L}^{k+1}G\right\|_\infty \\ &\leq 2\left\|\mathscr{L}^k G\right\|_\infty \\ &= 2\left\|G+\mathscr{H}^k(\mathscr{G})\right\|_{\mathfrak{Y}^k(\mathscr{G})}.\end{aligned}$$

We define the operator $\mathrm{e}^{-t\mathscr{L}}$ on $\mathfrak{Y}^k(\mathscr{G})$ by

$$\mathrm{e}^{-t\mathscr{L}}\big(G+\mathscr{H}^k(\mathscr{G})\big) := G+\sum_{j=1}^{\infty}\frac{(-t)^j}{j!}\,\mathscr{L}^j G+\mathscr{H}^k(\mathscr{G}). \tag{3.15}$$

It is straightforward to check that the right hand side does not depend on the representative G chosen in $\mathbb{G}$.

Proposition 3.7. *The following hold:*

(i) for each nonnegative real number t the operator $\mathrm{e}^{-t\mathscr{L}}$ defined above is contractive on $\mathfrak{Y}^k(\mathscr{G})$;

(ii) the operator $\mathscr{L}$ is sectorial on $\mathfrak{Y}^k(\mathscr{G})$.

Proof. First we prove (i). Suppose that $\mathbb{G}$ is in $\mathfrak{Y}^k(\mathscr{G})$, and that G is a representative of $\mathbb{G}$. Then

$$\begin{aligned}\left\|\mathrm{e}^{-t\mathscr{L}}\mathbb{G}\right\|_{\mathfrak{Y}^k(\mathscr{G})} &= \left\|\mathscr{L}^k G+\mathscr{L}^k\sum_{j=1}^{\infty}\frac{(-t)^j}{j!}\,\mathscr{L}^j G\right\|_\infty \\ &= \left\|\mathscr{L}^k G+\sum_{j=1}^{\infty}\frac{(-t)^j}{j!}\,\mathscr{L}^j\mathscr{L}^k G\right\|_\infty.\end{aligned}$$

With a slight abuse of notation, observe that the argument of the norm above is $\mathrm{e}^{-t\mathscr{L}}\mathscr{L}^k G$, where $\mathrm{e}^{-t\mathscr{L}}$ denotes here the Markovian semigroup generated by $\mathscr{L}$ and acting on $L^\infty(\mathscr{G})$. This semigroup is contractive (see Proposition 1.2 above), so that

$$\begin{aligned}\left\|\mathrm{e}^{-t\mathscr{L}}\mathbb{G}\right\|_{\mathfrak{Y}^k(\mathscr{G})} &\leq \left\|\mathscr{L}^k G\right\|_\infty \\ &= \left\|\mathbb{G}\right\|_{\mathfrak{Y}^k(\mathscr{G})}.\end{aligned}$$

This proves (i).

Next we prove (ii). Notice that the infinitesimal generator of the semigroup $\{e^{-t\mathscr{L}} : t \geq 0\}$, acting on $\mathfrak{Y}^k(\mathscr{G})$, is the operator $\mathscr{L}$. Then (ii) follows from (i), much in the same wat as Theorem 2.6 (ii) follows from Theorem 2.6 (i). We leave the details to the interested reader. □

The sectoriality of $\mathscr{L}$ on $\mathfrak{Y}^k(\mathscr{G})$, established in the previous lemma, has a number of interesting consequence. We group some of them in the next proposition.

Proposition 3.8. *Suppose that σ is a complex number with $0 < \operatorname{Re}\sigma < 1$. The following hold:*

(i) $\mathscr{L}^\sigma$ is defined via the Balakrishnan integral, given, for each $\mathbb{G}$ in $\mathfrak{Y}^1(\mathscr{G})$, by

$$\mathscr{L}^\sigma \mathbb{G} = \frac{\sin(\sigma\pi)}{\pi} \int_0^\infty \lambda^{\sigma-1} \big(\lambda + \mathscr{L}\big)^{-1} \mathscr{L}\mathbb{G} \, \mathrm{d}\lambda.$$

The integral above is convergent as a Bochner integral in $\mathfrak{Y}^1(\mathscr{G})$, and $\mathscr{L}^\sigma$ is a bounded operator on $\mathfrak{Y}^1(\mathscr{G})$;

(ii) the kernel of $\mathscr{L}^\sigma$ is trivial, i.e., $\mathscr{L}^\sigma$ is injective on $\mathfrak{Y}^1(\mathscr{G})$;

(iii) $\big(\mathscr{L}^{-1}\big)^\sigma = \big(\mathscr{L}^\sigma\big)^{-1}$;

(iv) $\mathscr{L}^\sigma$ is a bounded operator from $\mathfrak{Y}^1(\mathscr{G})$ to $L^\infty(\mathscr{G})$.

Proof. Properties (i)-(iii) follow by abstract nonsense from the sectoriality of $\mathscr{L}$. A proof of them for general sectorial operators may be found in [Ha, Proposition 3.1.1].

To prove (iv), observe that the integral in (i) above defining $\mathscr{L}^\sigma\mathbb{G}$ is convergent as a Bochner integral in $L^\infty(\mathscr{G})$; in particular, there exists a constant C such that

$$\int_0^\infty \lambda^{\operatorname{Re}\sigma-1} \, \big\|\big(\lambda + \mathscr{L}\big)^{-1} \mathscr{L}G\big\|_\infty \, \mathrm{d}\lambda \leq \frac{C}{\operatorname{Re}\sigma\,\big(1-\operatorname{Re}\sigma\big)} \, \big\|\mathscr{L}G\big\|_\infty \, . \tag{3.16}$$

Indeed, the integral above may be written as the sum of the integrals over $(0,1)$ and $(1,\infty)$. The integral over $(0,1)$ may be estimated by

$$\begin{aligned}\int_0^1 \lambda^{\operatorname{Re}\sigma-1} \, \big|\!\big|\!\big|\big(\lambda + \mathscr{L}\big)^{-1}\big|\!\big|\!\big|_{L^\infty(\mathscr{G})} \, \big\|\mathscr{L}G\big\|_\infty \, \mathrm{d}\lambda &\leq \big\|\mathscr{L}G\big\|_\infty \int_0^1 \lambda^{\operatorname{Re}\sigma-1} \, \mathrm{d}\lambda \\ &= \frac{\big\|\mathscr{L}G\big\|_\infty}{\operatorname{Re}\sigma};\end{aligned}$$

we have used the fact that the operator norm of the resolvent is bounded by 1, because $\mathscr{L}$ generates a contraction semigroup on $L^\infty(\mathscr{G})$. The integral over $(1,\infty)$ may be estimated by

$$\begin{aligned}\int_1^\infty \lambda^{\operatorname{Re}\sigma-2} \left|\!\left|\!\left|\lambda\left(\lambda+\mathscr{L}\right)^{-1}\right|\!\right|\!\right|_{L^\infty(\mathscr{G})} \left\|\mathscr{L}G\right\|_\infty \,\mathrm{d}\lambda &\leq C\, \left\|\mathscr{L}G\right\|_\infty \int_1^\infty \lambda^{\operatorname{Re}\sigma-2}\,\mathrm{d}\lambda \\ &= \left\|\mathscr{L}G\right\|_\infty \frac{C}{1-\operatorname{Re}\sigma};\end{aligned}$$

we have used the sectoriality of $\mathscr{L}$ on $L^\infty(\mathscr{G})$, which is another consequence of the fact that $\mathscr{L}$ generates a strongly contraction semigroup on $L^\infty(\mathscr{G})$. □

Definition 3.9. Suppose that $0<\gamma<1$. We define $\mathfrak{Y}^\gamma(\mathscr{G})$ to be the vector subspace $\mathscr{L}^{1-\gamma}\big(\mathfrak{Y}^1(\mathscr{G})\big)$ of $\mathfrak{Y}^1(\mathscr{G})$, endowed with the norm

$$\left\|\mathbb{G}\right\|_{\mathfrak{Y}^\gamma(\mathscr{G})} := \left\|\mathscr{L}^{\gamma-1}\mathbb{G}\right\|_{\mathfrak{Y}^1(\mathscr{G})}.$$

By Proposition 3.8 (ii), $\mathscr{L}^{1-\gamma}$ is injective, so that $\mathscr{L}^{1-\gamma}$ is an isometric isomorphism between $\mathfrak{Y}^\gamma(\mathscr{G})$ and $\mathfrak{Y}^1(\mathscr{G})$.

Note that the elements of $\mathfrak{Y}^\gamma(\mathscr{G})$ are cosets in $\mathfrak{Y}^1(\mathscr{G})$.

The composition rule for fractional powers of sectorial operators, sometimes referred to as the *first law of exponents*, gives, for each γ in $(0,1)$,

$$\mathscr{L} = \mathscr{L}^\gamma\mathscr{L}^{1-\gamma}$$

(see [Ha, Proposition 3.3.1 (c)] for the general statement). Observe that

$$L^\infty(\mathscr{G}) = \mathscr{L}\big(\mathfrak{Y}^1(\mathscr{G})\big) = \mathscr{L}^\gamma\big[\mathscr{L}^{1-\gamma}\big(\mathfrak{Y}^1(\mathscr{G})\big)\big] = \mathscr{L}^\gamma\big[\mathfrak{Y}^\gamma(\mathscr{G})\big].$$

Therefore $\mathscr{L}^\gamma$ is a surjective operator from $\mathfrak{Y}^\gamma(\mathscr{G})$ to $L^\infty(\mathscr{G})$. Also, by Proposition 3.8 (ii), $\mathscr{L}^\gamma$ is injective on $\mathfrak{Y}^1(\mathscr{G})$, hence, *a fortiori*, injective on $\mathfrak{Y}^\gamma(\mathscr{G})$. Therefore $\mathscr{L}^\gamma$ is a bijective operator between $\mathfrak{Y}^\gamma(\mathscr{G})$ and $L^\infty(\mathscr{G})$. Furthermore, for an element $\mathbb{G}$ in $\mathfrak{Y}^\gamma(\mathscr{G})$, we have that

$$\left\|\mathscr{L}^\gamma\mathbb{G}\right\|_\infty = \left\|\mathbb{G}\right\|_{\mathfrak{Y}^\gamma(\mathscr{G})}.$$

The situation is described by the following commutative diagram, in which each arrow is an isometric isomorphism of Banach spaces.

$$\begin{array}{ccc} \mathfrak{Y}^1(\mathscr{G}) & \xrightarrow{\mathscr{L}^{1-\gamma}} & \mathfrak{Y}^\gamma(\mathscr{G}) \\ & {\scriptstyle \mathscr{L}}\searrow & \downarrow {\scriptstyle \mathscr{L}^\gamma} \\ & & L^\infty(\mathscr{G}) \end{array}$$

3.4 Duality between $\mathfrak{X}^\gamma(\mathscr{G})$ and $\mathfrak{Y}^\gamma(\mathscr{G})$

So far, we have proved that for each positive integer k the Banach dual of $\mathfrak{X}^k(\mathscr{G})$ may be identified with $\mathfrak{Y}^k(\mathscr{G})$. It remains the problem to determine the Banach dual of $\mathfrak{X}^\gamma(\mathscr{G})$ for all nonintegral positive numbers.

Theorem 3.10. *Suppose that* $0 < \gamma < 1$. *The space* $\mathfrak{Y}^\gamma(\mathscr{G})$ *is isometrically isomorphic to the Banach dual of* $\mathfrak{X}^\gamma(\mathscr{G})$ *via the map* i, *that associates to an element* $\mathbb{G}$ *of* $\mathfrak{Y}^\gamma(\mathscr{G})$ *the functional* $\Lambda_{\mathbb{G}}$ *on* $\mathfrak{X}^\gamma(\mathscr{G})$, *defined by*

$$\Lambda_{\mathbb{G}}(f) := \langle \mathscr{L}^{-\gamma} f, \mathscr{L}^\gamma G \rangle \qquad \forall G \in \mathbb{G} \quad \forall f \in \mathfrak{X}^\gamma(\mathscr{G}), \tag{3.17}$$

where the pairing above is the standard duality beetween $L^1(\mathscr{G})$ *and* $L^\infty(\mathscr{G})$.

Proof. First we prove that the linear functional $\Lambda_{\mathbb{G}}$, defined in (3.17), is bounded on $\mathfrak{X}^\gamma(\mathscr{G})$. Observe that

$$\begin{aligned} \left|\Lambda_{\mathbb{G}}(f)\right| &\leq \left\|\mathscr{L}^{-\gamma} f\right\|_1 \left\|\mathscr{L}^\gamma G\right\|_\infty \\ &= \left\|f\right\|_{\mathfrak{X}^\gamma(\mathscr{G})} \left\|\mathbb{G}\right\|_{\mathfrak{Y}^\gamma(\mathscr{G})}, \end{aligned}$$

so that

$$\left|\!\left|\!\left|\Lambda_{\mathbb{G}}\right|\!\right|\!\right|_{\mathfrak{X}^\gamma(\mathscr{G})^*} \leq \left\|\mathbb{G}\right\|_{\mathfrak{Y}^\gamma(\mathscr{G})}.$$

Conversely, suppose that Λ is a bounded linear functional on $\mathfrak{X}^\gamma(\mathscr{G})$. Then $\Lambda \circ \mathscr{L}^\gamma$ is a bounded linear functional on $L^1(\mathscr{G})$. Therefore there exists a bounded function

g such that

$$\big(\Lambda \circ \mathscr{L}^\gamma\big)h = \langle h, g\rangle \qquad \forall h \in L^1(\mathscr{G}), \tag{3.18}$$

where the pairing denotes the duality between $L^1(\mathscr{G})$ and $L^\infty(\mathscr{G})$. Furthermore,

$$\big\|\!\big|\Lambda\big|\!\big\|_{\mathfrak{X}^\gamma(\mathscr{G})^*} = \big\|\!\big|\Lambda \circ \mathscr{L}^\gamma\big|\!\big\|_{L^1(\mathscr{G})^*} = \big\|g\big\|_\infty ;$$

the first equality follows from the fact that $\mathscr{L}^\gamma$ is an isometric isomorphism between $L^1(\mathscr{G})$ and $\mathfrak{X}^\gamma(\mathscr{G})$, and the second from the fact that $L^\infty(\mathscr{G})$ is isometrically isomorphic to $L^\infty(\mathscr{G})$.

Now, there exists a unique class $\mathbb{G}$ in $\mathfrak{Y}^\gamma(\mathscr{G})$ such that $\mathscr{L}^\gamma\mathbb{G} = g$, and for each f in $\mathfrak{X}^\gamma(\mathscr{G})$, the function $\mathscr{L}^{-\gamma}f$ is in $L^1(\mathscr{G})$. Thus we may rewrite (3.18) as

$$\begin{aligned}\Lambda(f) &= \big(\Lambda \circ \mathscr{L}^\gamma\big)\mathscr{L}^{-\gamma}f = \langle \mathscr{L}^{-\gamma}f, g\rangle = \langle \mathscr{L}^{-\gamma}f, \mathscr{L}^\gamma\mathbb{G}\rangle \\ &= \Lambda_{\mathbb{G}}(f) \qquad \forall f \in \mathfrak{X}^\gamma(\mathscr{G}).\end{aligned}$$

Hence $\Lambda = \Lambda_{\mathbb{G}}$, as required.

The proof of the theorem is complete. □

We conclude this section with a couple of remarks.

Lemma 3.11. *The Laplacian $\mathscr{L}$ is a contractive operator from $\Lambda_1(\mathscr{G})$ to $L^\infty(\mathscr{G})$.*

Proof. Observe that

$$\begin{aligned}\mathscr{L}f(x) &= f(x) - \frac{1}{\nu(x)}\sum_{y\sim x} f(y) \\ &= \frac{1}{\nu(x)}\sum_{y\sim x}\big[f(x) - f(y)\big].\end{aligned}$$

Therefore

$$\begin{aligned}\big|\mathscr{L}f(x)\big| &\leq \frac{1}{\nu(x)}\sum_{y\sim x}\big|f(x) - f(y)\big| \\ &\leq \|f\|_{\Lambda_1(\mathscr{G})}.\end{aligned}$$

The required statement follows by taking the supremum of both sides over all x in $\mathscr{G}$. □

Proposition 3.12. *The Banach spaces $\mathfrak{Y}^1(\mathscr{G})$ and $BMO(\mathscr{G})$ are isomorphic.*

Proof. Recall (see Proposition 2.30) that $BMO(\mathscr{G})$ may be identified with $\Lambda_1(\mathscr{G})/\mathbb{C}$, endowed with the norm

$$\left\|u+\mathbb{C}\right\|_{BMO(\mathscr{G})} := \left\|u\right\|_{\Lambda_1(\mathscr{G})}.$$

Moreover, an element of $\mathfrak{Y}^1(\mathscr{G})$ is a coset $G+\mathscr{H}^1(\mathscr{G})$, where G is a function on $\mathscr{G}$ such that $\mathscr{L}G$ is bounded, endowed with the norm

$$\left\|G+\mathscr{H}^1(\mathscr{G})\right\|_{\mathfrak{Y}^1(\mathscr{G})} := \left\|\mathscr{L}G\right\|_{\infty}.$$

By Proposition 3.3 (ii), given G such that $\mathscr{L}g$ is bounded, there exists a Lipschitz function u such that $\mathscr{L}u=\mathscr{L}G$, with $\left\|u\right\|_{\Lambda_1(\mathscr{G})} \leq 3\left\|\mathscr{L}G\right\|_{\infty}$.
Observe that $u+\mathscr{H}^1(\mathscr{G}) = G+\mathscr{H}^1(\mathscr{G})$, for $\mathscr{L}(u-G)=0$. In other words, every coset in $\mathfrak{Y}^1(\mathscr{G})$ has a representative in $\Lambda_1(\mathscr{G})$. By Lemma 3.11, $\left\|\mathscr{L}u\right\|_{\infty} \leq \left\|u\right\|_{\Lambda_1(\mathscr{G})}$. Therefore we have

$$\left\|\mathscr{L}G\right\|_{\infty} = \left\|\mathscr{L}u\right\|_{\infty} \leq \left\|u\right\|_{\Lambda_1(\mathscr{G})} \leq 3\left\|\mathscr{L}G\right\|_{\infty}.$$

In other words,

$$\left\|u+\mathscr{H}^1(\mathscr{G})\right\|_{\mathfrak{Y}^1(\mathscr{G})} \leq \left\|u+\mathbb{C}\right\|_{BMO(\mathscr{G})} \leq 3\left\|u+\mathscr{H}^1(\mathscr{G})\right\|_{\mathfrak{Y}^1(\mathscr{G})}. \tag{3.19}$$

Now, define a map $\mathscr{J} : BMO(\mathscr{G}) \to \mathfrak{Y}^1(\mathscr{G})$ as follows. Given an element $u+\mathbb{C}$ in $BMO(\mathscr{G})$ (here u is in $\Lambda_1(\mathscr{G})$), set

$$\mathscr{J}(u+\mathbb{C}) = u+\mathscr{H}^1(\mathscr{G}).$$

Notice that $\mathscr{J}$ is well defined, because any other representative of $u+\mathbb{C}$ is of the form $u+\text{const}$, and $u+\text{const}+\mathscr{H}^1(\mathscr{G}) = u+\mathscr{H}^1(\mathscr{G})$.

Observe that $\mathscr{J}$ is *injective*. Indeed, if $\mathscr{J}\big(u+BMO(\mathscr{G})\big)=0$, then $u+\mathscr{H}^1(\mathscr{G}) = \mathscr{H}^1(\mathscr{G})$, i.e., u is harmonic. By Proposition 3.3 (ii), $\left\|u\right\|_{\Lambda_1(\mathscr{G})}=0$, i.e., u is constant, and $\left\|u+\mathbb{C}\right\|_{BMO(fG)}=0$, as required.

Now we prove that $\mathscr{J}$ is *surjective*. Suppose that $G+\mathscr{H}^1(\mathscr{G})$ is in $\mathfrak{Y}^1(\mathscr{G})$. Then there exists a Lipschitz function u such that $u+\mathscr{H}^1(\mathscr{G}) = G+\mathscr{H}^1(\mathscr{G})$. Consider the element $u+\mathbb{C}$ in $BMO(\mathscr{G})$. Clearly, $\mathscr{J}(u+\mathbb{C}) = u+\mathscr{H}^1(\mathscr{G})$, as required.

Finally, (3.19) implies that the map $\mathscr{J}$ is bicontinuous, so that $\mathscr{J}$ is an isomorphism, as required.

The proof of the proposition is complete. $\square$

Chapter 4

Maximal operators

4.1 The heat maximal operator

In this section we prove various estimates of the heat kernel h_t, and of $\mathscr{L} h_t$. First, we consider the *heat maximal operator*, which acts on a function f on $\mathscr{T}$ as

$$\mathscr{H}_* f = \sup_{t>0} |\mathscr{H}_t f|.$$

We prove that $\mathscr{H}_*\big(\mathscr{L}\delta_o\big)$ is not integrable on $\mathscr{T}$. As a consequence, the heat maximal operator is unbounded from $\mathfrak{X}^1(\mathscr{T})$ to $L^1(\mathscr{T})$, so that the Hardy-type space

$$H^1_{\mathscr{H}}(\mathscr{T}) := \{f \in L^1(\mathscr{T}) : \mathscr{H}_* f \in L^1(\mathscr{T})\}$$

does not coincide with $\mathfrak{X}^1(\mathscr{T})$. It is an interesting question, which we have not been able to answer, whether $H^1_{\mathscr{H}}(\mathscr{T})$ is included in $\mathfrak{X}^1(\mathscr{T})$ or not.

Theorem 4.1. *The heat maximal operator $\mathscr{H}_*$ is unbounded from $\mathfrak{X}^1(\mathscr{T})$ to $L^1(\mathscr{T})$. In particular, there exists a constant $c > 0$ such that*

$$\mathscr{H}_*\big(\mathscr{L}\delta_o\big)(x) \geq c\,\frac{q^{-|x|}}{1+|x|} \qquad \forall x \in \mathscr{T},$$

so that $\mathscr{H}_*\big(\mathscr{L}\delta_o\big)$ is not in $L^1(\mathscr{T})$.

Proof. The heat semigroup $\mathscr{H}_t$ commutes with the Laplacian, so that

$$\mathscr{H}_t\big(\mathscr{L}\delta_o\big) = \mathscr{L}h_t \qquad \forall t > 0,$$

and its spherical Fourier transform is given by

$$\big[\mathscr{H}_t\big(\mathscr{L}\delta_o\big)\big]^{\sim}(s) = \big(1-\gamma(s)\big)\, \mathrm{e}^{-t(1-\gamma(s))} \qquad \forall t > 0 \quad \forall s \in \mathbb{T}.$$

By the inversion formula (1.14) for the spherical Fourier transform,

$$\mathscr{L}h_t(x) = 2c_G\, q^{-|x|/2} \int_{-\tau/2}^{\tau/2} \frac{1-\gamma(s)}{\mathbf{c}(-s)}\, \mathrm{e}^{-t(1-\gamma(s))}\, q^{is|x|}\, \mathrm{d}s. \tag{4.1}$$

Observe that the integrand in (4.1) is holomorphic on the closure of the rectangle with vertices $\pm\tau/2$, $\pm\tau/2 + i/2$. Then we may integrate on the boundary of this rectangle, observe that, by periodicity, the contributions of the integrals over the vertical sides cancel out, use Cauchy's theorem, and conclude that

$$\begin{aligned} \mathscr{L}h_t(x) &= 2c_G\, q^{-|x|/2} \int_{-\tau/2}^{\tau/2} \frac{1-\gamma(s)}{\mathbf{c}(-s)}\, \mathrm{e}^{-t(1-\gamma(s))}\, q^{is|x|}\, \mathrm{d}s \\ &= 2c_G\, q^{-|x|/2} \int_{-\tau/2}^{\tau/2} \frac{1-\gamma(s+i/2)}{\mathbf{c}(-s-i/2)}\, \mathrm{e}^{-t(1-\gamma(s+i/2))}\, q^{i(s+i/2)|x|}\, \mathrm{d}s \qquad (4.2)\\ &= 2c_G\, q^{-|x|} \int_{-\tau/2}^{\tau/2} \frac{1-\gamma(s+i/2)}{\mathbf{c}(-s-i/2)}\, \mathrm{e}^{-t(1-\gamma(s+i/2))}\, q^{is|x|}\, \mathrm{d}s. \end{aligned}$$

Notice that

$$\begin{aligned} 1-\gamma(s+i/2) &= 1 - \frac{1}{q+1}\big(q^{is} + q^{-is+1}\big) \\ &= \frac{q+1-q^{is}-q^{-is+1}}{q+1} \qquad (4.3)\\ &= \frac{(1-q^{-is})(q-q^{is})}{q+1}. \end{aligned}$$

and that

$$\begin{aligned} \frac{1}{\mathbf{c}(-s-1/2)} &= \frac{q+1}{q^{1/2}} \frac{q^{-is+1/2}-q^{is-1/2}}{q^{-is+1}-q^{is-1}} \\ &= (q+1) \frac{q^{-is}-q^{is-1}}{q^{-is+1}-q^{is-1}} \qquad (4.4)\\ &= (q+1) \frac{q-q^{2is}}{q^2-q^{2is}}. \end{aligned}$$

Thus,

$$\frac{1-\gamma(s+i/2)}{\mathbf{c}(-s-1/2)} = (1-q^{-is})(q-q^{is})\frac{q-q^{2is}}{q^2-q^{2is}} = \frac{(q-q^{2is})(1-q^{-is})}{q+q^{is}}.$$

We insert this in the last integral in (4.2), change variables, and obtain that

$$\mathscr{L}h_t(x) = \frac{q^{1-|x|}}{2\pi(q+1)}\, I(|x|,t), \tag{4.5}$$

where we have set

$$I(n,t) = \int_{-\pi}^{\pi} \frac{(q-\mathrm{e}^{2iu})\,(1-\mathrm{e}^{-iu})}{q+\mathrm{e}^{iu}}\; \mathrm{e}^{\Phi(u;n,t)}\,\mathrm{d}u. \tag{4.6}$$

Here the phase Φ is given by

$$\begin{aligned} \Phi(u;n,t) &= -t\,\frac{(1-\mathrm{e}^{-iu})(q-\mathrm{e}^{iu})}{q+1} + inu \\ &= -t\frac{(1-\mathrm{e}^{-iu})(q-1+1-\mathrm{e}^{iu})}{q+1} - inu \\ &= -t\,\big(1-\cos u + i\beta\sin u\big) - inu \\ &= -t\,(1-\cos u) + i(nu - \beta t\sin u), \end{aligned} \tag{4.7}$$

and $\beta = \dfrac{q-1}{q+1}$. Since $\mathscr{L}h_t$ is real, the imaginary part of $I(n,t)$ must vanish. We compute the real part of the integrand. Denote by η the function on $[-\pi,\pi]$, defined by

$$\eta(u) := \frac{q-\mathrm{e}^{2iu}}{q+\mathrm{e}^{iu}}.$$

A straightforward computation shows that

$$\begin{aligned} \operatorname{Re}\eta(u) &= \frac{q^2+(q-1)\cos u - q\cos(2u)}{\left|q+\mathrm{e}^{iu}\right|^2} \\ \operatorname{Im}\eta(u) &= \frac{(q+1)\sin u + q\sin(2u)}{\left|q+\mathrm{e}^{iu}\right|^2}. \end{aligned}$$

Therefore

$$I(n,t) = \int_{-\pi}^{\pi} \big(1-\cos u + i\sin u\big)\,\big(\operatorname{Re}\eta + i\operatorname{Im}\eta\big)\,\big(\cos\operatorname{Im}\Phi + i\sin\operatorname{Im}\Phi\big)\,\mathrm{e}^{\operatorname{Re}\Phi}\,\mathrm{d}u.$$

We denote by A the product of the first three factors in the integral above. Then

$$I(n,t) = \int_{-\pi}^{\pi} \operatorname{Re} A(u)\,\mathrm{e}^{\operatorname{Re}\Phi}\,\mathrm{d}u.$$

The only part of the integral which matters is a small neighbourhood of the origin. Indeed, denote by ψ a smooth cutoff function, which is supported in the interval $[-10^{-2}, 10^{-2}]$, it is equal to 1 in $[-10^{-3}, 10^{-3}]$ and such that $0 \le \psi \le 1$. Then

$$I(n,t) = I^{\psi}(n,t) + I^{1-\psi}(n,t),$$

where

$$I^{\psi}(n,t) = \int_{-\pi}^{\pi} \psi(u)\ \operatorname{Re} A(u)\,\mathrm{e}^{\operatorname{Re}\Phi}\,\mathrm{d}u$$

and

$$I^{1-\psi}(n,t) = \int_{-\pi}^{\pi} \big(1-\psi(u)\big)\ \operatorname{Re} A(u)\,\mathrm{e}^{\operatorname{Re}\Phi}\,\mathrm{d}u.$$

We shall estimate I^{ψ} and $I^{1-\psi}$ separately.

First we consider $I^{1-\psi}$. Observe that

$$\big|I^{1-\psi}(n,t)\big| \le \big\|\operatorname{Re} A\big\|_{\infty} \int_{[-\pi,-10^{-3}]\cup[10^{-3},\pi]} \mathrm{e}^{\operatorname{Re}\Phi}\,\mathrm{d}u.$$

Notice that if $10^{-3} \le |u| \le \pi$, then

$$\operatorname{Re}\Phi(u) = -t\,\big(1-\cos u\big) \le -t\,\big(1-\cos 10^{-3}\big), \tag{4.8}$$

whence

$$\big|I^{1-\psi}(n,t)\big| \le \big\|\operatorname{Re} A\big\|_{\infty}\,\mathrm{e}^{-ct}, \qquad \forall n \in \mathbb{N}, \tag{4.9}$$

where $c = 1 - \cos 10^{-3}$.

Next we estimate $I^{\psi}(n,t)$. Observe that

$$\begin{aligned}\operatorname{Re} A(u) = \big(1-\cos u\big)\ \operatorname{Re}\eta\ \cos(\operatorname{Im}\Phi)\\ -\sin u\ \ \operatorname{Im}\eta\ \ \cos(\operatorname{Im}\Phi)\\ -\big(1-\cos u\big)\ \ \operatorname{Im}\eta\ \ \sin(\operatorname{Im}\Phi)\\ -\sin u\ \ \operatorname{Re}\eta\ \ \sin(\operatorname{Im}\Phi).\end{aligned} \tag{4.10}$$

Correspondingly, we write

$$I^{\psi} = I_1^{\psi} - I_2^{\psi} - I_3^{\psi} - I_4^{\psi},$$

where

$$I_1^{\psi}(n,t) = \int_{-\pi}^{\pi} \psi(u)\,\big(1-\cos u\big)\ \operatorname{Re}\eta\ \cos(\operatorname{Im}\Phi)\,\mathrm{e}^{\operatorname{Re}\Phi}\,\mathrm{d}u,$$

$$I_2^{\psi}(n,t) = \int_{-\pi}^{\pi} \psi(u)\ \sin u\ \ \operatorname{Im}\eta\ \ \cos(\operatorname{Im}\Phi)\,\mathrm{e}^{\operatorname{Re}\Phi}\,\mathrm{d}u,$$

$$I_3^{\psi}(n,t) = \int_{-\pi}^{\pi} \psi(u)\,\big(1-\cos u\big)\ \ \operatorname{Im}\eta\ \ \sin(\operatorname{Im}\Phi)\,\mathrm{e}^{\operatorname{Re}\Phi}\,\mathrm{d}u$$

and

$$I_4^{\psi}(n,t) = \int_{-\pi}^{\pi} \psi(u)\ \sin u\ \ \operatorname{Re}\eta\ \ \sin(\operatorname{Im}\Phi)\,\mathrm{e}^{\operatorname{Re}\Phi}\,\mathrm{d}u.$$

Observe that

$$1-\cos u \asymp u^2 \qquad \sin u \asymp u \qquad \operatorname{Im}\eta(u) \asymp u \qquad \forall u \in \operatorname{supp}(\psi),$$

and that $\operatorname{Re}\eta$ is bounded (and bounded away from 0) in $[-\pi,\pi]$. Therefore

$$\begin{aligned}|I_1^{\psi}(n,t)| &\le C\ \big\|\operatorname{Re}\eta\big\|_{\infty} \int_{-10^{-2}}^{10^{-2}} u^2\,\mathrm{e}^{-c'tu^2}\,\mathrm{d}u\\ &= C\ \big\|\operatorname{Re}\eta\big\|_{\infty} \int_{-10^{-2}\sqrt{t}}^{10^{-2}\sqrt{t}} \frac{v^2}{t}\,\mathrm{e}^{-c'v^2}\,\frac{\mathrm{d}v}{\sqrt{t}}\\ &\le C\ \big\|\operatorname{Re}\eta\big\|_{\infty}\ t^{-3/2} \qquad \forall t\in[1,\infty) \quad \forall n\in\mathbb{N}.\end{aligned}$$

We have made the change of variables $v = u\sqrt{t}$ in the first integral above.

By arguing similarly, we may show that

$$|I_2^\psi| \leq C \, \big\|\mathrm{Im}\,\eta\big\|_\infty \, t^{-3/2} \qquad |I_3^\psi| \leq C \, \big\|\mathrm{Im}\,\eta\big\|_\infty \, t^{-2} \qquad \forall t \in [1,\infty).$$

It remains to estimate $I_4^\psi(n,t)$. We notice that $\mathrm{Re}\,\eta(0) = \beta \neq 0$, and write

$$\mathrm{Re}\,\eta = \big[\,\mathrm{Re}\,\eta - \mathrm{Re}\,\eta(0)\big] + \mathrm{Re}\,\eta(0).$$

Correspondingly, we write

$$\begin{aligned} I_4^\psi(n,t) = &\int_{-\pi}^{\pi} \psi(u)\, \sin u \,\big[\,\mathrm{Re}\,\eta - \mathrm{Re}\,\eta(0)\big]\, \sin(\mathrm{Im}\,\Phi)\, \mathrm{e}^{\mathrm{Re}\,\Phi}\, \mathrm{d}u \\ &+ \mathrm{Re}\,\eta(0) \int_{-\pi}^{\pi} \psi(u)\, \sin u\, \sin(\mathrm{Im}\,\Phi)\, \mathrm{e}^{\mathrm{Re}\,\Phi}\, \mathrm{d}u, \end{aligned} \tag{4.11}$$

and estimate the two integrals on the right hand side separately.

Since $\mathrm{Re}\,\eta$ is even and smooth, $\mathrm{Re}\,\eta - \mathrm{Re}\,\eta(0)$ vanishes at the origin at least of order 2, whence the absolute value of the first integral may be majorised by

$$C \, \big\|\mathrm{Re}\,\eta''\big\|_\infty \int_0^{10^{-2}} u^3 e^{-c'tu^2}\, \mathrm{d}u \leq C \, \big\|\mathrm{Re}\,\eta''\big\|_\infty \, t^{-2} \qquad \forall t \in [1,\infty) \quad \forall n \in \mathbb{N}.$$

It remains to estimate the second integral in (4.11). Observe that

$$\begin{aligned} \mathrm{Im}\,\Phi(u) &= nu - t\beta \sin u \\ &= u(n - t\beta) + t\beta(u - \sin u), \end{aligned}$$

and that

$$\begin{aligned} &\sin \mathrm{Im}\,\Phi(u) \\ &= \sin\big(u(n-t\beta)\big)\, \cos\big(t\beta(u - \sin u)\big) + \cos\big(u(n-t\beta)\big)\, \sin\big(t\beta(u-\sin u)\big). \end{aligned}$$

Accordingly, we write

$$\int_{-\pi}^{\pi} \psi(u)\, \sin u\, \sin(\mathrm{Im}\,\Phi)\, \mathrm{e}^{\mathrm{Re}\,\Phi}\, \mathrm{d}u = J_1(n,t) + J_2(n,t),$$

where

$$J_1(n,t) = \int_{-\pi}^{\pi} \psi(u)\, \sin u\, \sin\big(u(n-t\beta)\big)\, \cos\big(t\beta(u-\sin u)\big)\, \mathrm{e}^{\mathrm{Re}\,\Phi}\, \mathrm{d}u \tag{4.12}$$

and

$$J_2(n,t) = \int_{-\pi}^{\pi} \psi(u)\, \sin u\, \cos\big(u(n - t\beta)\big)\, \sin\big(t\beta(u - \sin u)\big)\, \mathrm{e}^{\operatorname{Re}\Phi}\, \mathrm{d}u \tag{4.13}$$

The main part is $J_1(n,t)$. We first estimate $J_2(n,t)$ from above. Clearly

$$0 \le t\beta(u - \sin u) \le t\beta \frac{u^3}{6} \tag{4.14}$$

and

$$t\beta \frac{u^3}{6} \le 1 \qquad \text{iff} \qquad u \le \Big(\frac{6}{\beta}\Big)^{1/3} t^{-1/3}.$$

This suggests to write the integral in (4.13) as the sum of the integrals over the interval $[-ct^{-1/3}, ct^{-1/3}]$ and over the set $\{ct^{-1/3} \le |u| \le \pi\}$, where c is short for $(6/\beta)^{1/3}$. We observe that the integrand in J_2 is even. Furthermore, we use the trivial estimate $\psi(u)\,\big|\cos\big(u(n - t\beta)\big)\big| \le 1$ in both integrals, the estimate (4.14) in the first, and obtain that

$$\begin{aligned}
\big|J_2(n,t)\big| &\le 2\int_0^{ct^{-1/3}} u\, t\beta\, \frac{u^3}{6}\, \mathrm{e}^{-c'tu^2}\, \mathrm{d}u + 2\int_{ct^{-1/3}}^{10^{-2}} \mathrm{e}^{-c'tu^2}\, \mathrm{d}u \\
&\le \frac{\beta}{3}\, t^{1-5/2} \int_0^{ct^{1/6}} v^4 \mathrm{e}^{-c'v^2}\, \mathrm{d}v + \frac{2}{\sqrt{t}} \int_{ct^{1/6}}^{10^{-2}t^{1/6}} \mathrm{e}^{-c'v^2}\, \mathrm{d}v \\
&\le C\, t^{-3/2} + C\, t^{-3/2}\, \mathrm{e}^{-c' 10^{-1} t^{1/3}} \\
&\le C\, t^{-3/2} \qquad \forall t \in [1,\infty) \quad \forall n \in \mathbb{N};
\end{aligned} \tag{4.15}$$

we have made the change of variables $v = u\sqrt{t}$ in the second inequality above.

Finally, we estimate $J_1(n,t)$. We *claim* that there exist positive constants c and C, and, for each nonnegative integer n, a positive number $t(n)$ such that

$$c\,(1+n)^{-1} \le J_1\big(n, t(n)\big) \le C\,(1+n)^{-1} \qquad \forall n \in \mathbb{N}. \tag{4.16}$$

Taking this for granted, we have that

$$\begin{aligned}
\sup_{t>0} \big|\mathscr{L} h_t(x)\big| &= \sup_{t>0} \frac{q^{1-|x|}}{2\pi(q+1)}\, I(|x|, t) \\
&\ge c'\, q^{-|x|}\, J_1\big(|x|, t(|x|)\big) \\
&\ge c' \frac{q^{-|x|}}{1+|x|} \qquad \forall x \in \mathscr{T},
\end{aligned}$$

as required. This pointwise estimate, in turn, implies that

$$\begin{aligned}
\Big\|\sup_{t>0}\big|\mathscr{L}h_t\big|\Big\|_1 &= \sum_{x\in\mathscr{T}}\sup_{t>0}\big|\mathscr{L}h_t(x)\big| \\
&\geq c'\sum_{x\in\mathscr{T}}\frac{q^{-|x|}}{1+|x|} \\
&= c\sum_{n\in\mathbb{N}}(1+n)^{-1} \\
&= \infty,
\end{aligned}$$

as required to complete the proof of the theorem.

It remains to prove (4.16). We choose $t(n)$ so that $\big|n-\beta\, t(n)\big| = \sqrt{t(n)}$. A straightforward calculation shows that there are two possible choices of $t(n)$, namely

$$\beta\, t(n) = n + \sqrt{\frac{n}{\beta}} + \frac{1}{2\beta} + o(1)$$

and

$$\beta\, t(n) = n - \sqrt{\frac{n}{\beta}} + \frac{1}{2\beta} + o(1).$$

Here $o(1)$ denotes a function which tends to 0 as n tends to infinity. For notational convenience, hereafter in this proof we shall omit the dependence of t on n, and, for instance, we shall simply write $\sqrt{t}$ in place of $\big|n-\beta\, t\big|$ and $J(n,t)$ instead of $J_1\big(n,t(n)\big)$. We refer to formula (4.12).
We write $J_1(n,t)$, change variables ($v = u\sqrt{t}$), and obtain that

$$\begin{aligned}
&\big|J_1(n,t)\big| \\
&= \Big|\int_{-\pi}^{\pi}\psi(u)\,\sin u\,\sin(\sqrt{t}u)\,\cos(t\beta(u-\sin u))\,\mathrm{e}^{-t(1-\cos u)}\,\mathrm{d}u\Big| \\
&= \Big|\int_{-\pi\sqrt{t}}^{\pi\sqrt{t}}\psi\Big(\frac{v}{\sqrt{t}}\Big)\,\sin\frac{v}{\sqrt{t}}\,\sin v\,\cos\Big[t\beta\Big(\frac{v}{\sqrt{t}}-\sin\frac{v}{\sqrt{t}}\Big)\Big]\,\mathrm{e}^{-t(1-\cos(v/\sqrt{t}))}\,\frac{\mathrm{d}v}{\sqrt{t}}\Big| \\
&= \frac{1}{t}\Big|\int_{-\pi\sqrt{t}}^{\pi\sqrt{t}}\psi\Big(\frac{v}{\sqrt{t}}\Big)\,\frac{\sin(v/\sqrt{t})}{v/\sqrt{t}}\,v\,\sin v\,\cos\Big[t\beta\Big(\frac{v}{\sqrt{t}}-\sin\frac{v}{\sqrt{t}}\Big)\Big]\,\mathrm{e}^{-t(1-\cos(v/\sqrt{t}))}\,\mathrm{d}v\Big|.
\end{aligned}$$

It is straightforward to check that the integrand in the last integral is pointwise convergent to $v\,\sin v\mathrm{e}^{-v^2/2}$ as t tends to infinity.

We *claim* that the function $|v|e^{-\cos 10^{-2}v^2/2}$ is an integrable majorant of the integrand that does not depend on t.

Indeed, the absolute value of the integrand is clearly dominated by

$$\mathbf{1}_{[-10^{-2}\sqrt{t},10^{-2}\sqrt{t}]}(v)\,|v|\,\mathrm{e}^{-t(1-\cos(v/\sqrt{t}))}.$$

Set $\omega_t(v) := t(\cos(v/\sqrt{t})-1)$, and observe that both ω_t and its first derivative vanish at 0, and that

$$\omega_t''(v) = -\cos(v/\sqrt{t}).$$

We expand ω_t using McLaurin's formula with Lagrange form of the remainder, and obtain

$$\omega_t(v) = -\frac{v^2}{2}\cos\left(\frac{v\theta}{\sqrt{t}}\right)$$

for suitable $\theta \in (0, v)$. Since v is in $[-10^{-2}\sqrt{t}, 10^{-2}\sqrt{t}]$,

$$\begin{aligned}\omega_t(v) &\leq -\frac{v^2}{2}\inf_{|v|\leq 10^{-2}\sqrt{t}}\cos\left(\frac{v\theta}{\sqrt{t}}\right)\\ &\leq -\frac{v^2}{2}\cos 10^{-2},\end{aligned} \tag{4.17}$$

thereby proving the claim. Consequently,

$$\big|J_1(n,t)\big| \sim \frac{1}{t}\left|\int_{-\infty}^{+\infty} v\sin v\,\mathrm{e}^{-v^2/2}\,\mathrm{d}v\right|.$$

To conclude the proof of (4.16), it remains to show that the integral above does not vanish. Indeed, observe that

$$\begin{aligned}\int_{-\infty}^{\infty} v\ \sin v\,\mathrm{e}^{-v^2/2}\,\mathrm{d}v &= -\int_{-\infty}^{+\infty}\sin v\frac{\mathrm{d}}{\mathrm{d}v}(\,\mathrm{e}^{-v^2/2})\,\mathrm{d}v\\ &= \int_{-\infty}^{\infty}\cos v\,\mathrm{e}^{-v^2/2}\,\mathrm{d}v\\ &= \mathscr{F}[\,\mathrm{e}^{-(\cdot)^2/2}](1)\\ &\neq 0,\end{aligned} \tag{4.18}$$

as required. □

Next we prove a comparatively easy, but useful, estimate of $h_t(x)$ when t is small compared to $|x|$.

Lemma 4.2. *For each positive integer N there exists a constant C, independent of t, such that*

$$h_t(x) \leq C\, q^{-|x|} \big(|x| - \beta t\big)^{-N}\, t^{-1/2} \qquad \forall x : |x| \geq (\beta+1)\, t \quad \forall t \geq 1.$$

Proof. We shall prove the required estimate via spherical Fourier inversion formula. By arguing as in the proof of Theorem 4.1, it is straightforward to establish the following formula

$$h_t(x) = \frac{q^{1-|x|}}{2\pi} \int_{-\pi}^{\pi} \frac{q - \mathrm{e}^{2iu}}{q^2 - \mathrm{e}^{2iu}}\, \mathrm{e}^{\Phi(u;|x|,t)}\, \mathrm{d}u \qquad \forall x \in \mathscr{T} \quad \forall t > 0,$$

where

$$\Phi(u, |x|, t) = -t(1 - \cos u) + i(|x|u - \beta t \sin u).$$

Observe that

$$\partial_u \Phi(u, |x|, t) = -t \sin u + i(|x| - \beta t \cos u).$$

Denote by D the differential operator, defined by

$$D = \frac{1}{\partial_u \Phi(u, |x|, t)}\, \partial_u,$$

and by D^* its formal adjoint with respect to the Lebesgue measure. For notational convenience, we write $\psi(u)$ in place of $1/\partial_u \Phi(u, |x|, t)$, and we denote by M_ψ the operator of multiplication by ψ, i.e.,

$$M_\psi \varphi = \psi\, \varphi$$

for any reasonable function φ. Therefore

$$D^* \varphi = -(\partial_u M_\psi) \varphi.$$

In particular, in the rest of this proof, we shall work with

$$\varphi(u) = \frac{q - \mathrm{e}^{2iu}}{q^2 - \mathrm{e}^{2iu}}.$$

It is straightforward to check that φ and all its derivatives are bounded in the uniform norm on $[-\pi, \pi]$. For every positive integer k we may write

$$\begin{aligned} h_t(x) &= \frac{q^{1-|x|}}{2\pi} \int_{-\pi}^{\pi} \varphi(u)\, D^k \mathrm{e}^{\Phi(u;|x|,t)}\, \mathrm{d}u \\ &= (-1)^k \, \frac{q^{1-|x|}}{2\pi} \int_{-\pi}^{\pi} \big(\partial_u M_\psi\big)^k \varphi(u)\; \mathrm{e}^{\Phi(u;|x|,t)}\, \mathrm{d}u. \end{aligned}$$

Here we have used the fact that, by periodicity, the boundary terms arising from integration by parts cancel out. Observe that

$$\big|\psi(u)\big| = \frac{1}{\big|-t\sin u + i(|x| - \beta t\cos u)\big|} \le \frac{1}{\big||x| - \beta t\big|}. \tag{4.19}$$

We *claim* that for each positive integer j there exist smooth bounded functions $\varphi_1, \ldots, \varphi_j$, with bounded derivatives such that

$$\partial_u^j \psi(u) = \sum_{h=1}^{j} t^h\, \varphi_h(u)\, \psi(u)^{h+1}. \tag{4.20}$$

In fact, each function φ_h is a constant multiple of a finite product of linear combinations of sines and cosines.

We argue by induction. Since

$$\partial_u \psi(u) = t\,\big[\cos u - i\beta \sin u\big]\, \psi(u)^2,$$

the required property holds for $j = 1$. Suppose that (4.20) holds for all positive integers $\le j-1$, and consider $\partial_u^j \psi$. By Leibnitz's rule

$$\begin{aligned} \partial_u^j \psi(u) &= \partial_u\big(\partial_u^{j-1}\psi\big)(u) \\ &= \sum_{h=1}^{j-1} t^h\, \big[\partial_u \varphi_h(u)\, \psi(u)^{h+1} + \varphi_h(u)\,(h+1)\, \psi(u)^h\, \partial_u \psi(u)\big] \\ &= \sum_{h=1}^{j-1} \big[t^h\, \partial_u \varphi_h(u)\, \psi(u)^{h+1} + t^{h+1}\,(h+1)\,\varphi_h(u)\,(\cos u - i\beta\sin u)\,\psi(u)^{h+2}\big], \end{aligned}$$

which, after relabeling, has the required form. This proves the claim.

A straightforward consequence of the claim and of (4.19) is that for each positive integer j there exists a constant C such that

$$\left|\partial_u^j \psi(u)\right| \leq C \sum_{h=1}^{j} \frac{t^h}{\big(|x|-\beta t\big)^{h+1}}.$$

Observe also that for each positive integer h

$$\frac{t^h}{\big(|x|-\beta t\big)^{h+1}} \leq \frac{1}{|x|-\beta t} \qquad \forall x : |x| \geq (\beta+1)\, t. \tag{4.21}$$

Another induction argument shows that $\big(\partial_u M_\psi\big)^k \varphi$ may be written as a linear combination of terms of the form

$$\psi^{\beta_0} \left(\partial_u \psi\right)^{\beta_1} \cdots \left(\partial_u^k \psi\right)^{\beta_k},$$

with coefficients given by linear combination (with constant coefficients) of derivatives of φ. Here $\beta_0, \beta_1, \ldots, \beta_k$ are nonnegative integers and

$$\beta_0 + \beta_1 + \ldots + \beta_k = k.$$

We combine this and (4.21), and obtain that $\left|\big(\partial_u M_\psi\big)^k \varphi(u)\right|$ may be estimated by

$$C \left(|x|-\beta t\right)^{-\beta_0-\beta_1-\ldots-\beta_k} \qquad \forall x : |x| \geq (\beta+1)\, t.$$

Therefore

$$\left|\big(DM_\psi\big)^k \varphi(u)\right| \leq C \left(|x|-\beta t\right)^{-\beta_0-\beta_1-\ldots-\beta_k} \qquad \forall x : |x| \geq (\beta+1)\, t. \tag{4.22}$$

As a consequence,

$$\begin{aligned} h_t(x) &\leq C\, q^{-|x|} \left(|x|-\beta t\right)^{-k} \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{Re}\,\Phi(u;|x|,t)}\, \mathrm{d}u \\ &\leq C\, q^{-|x|} \left(|x|-\beta t\right)^{-k} t^{-1/2} \qquad \forall x : |x| \geq (\beta+1)\, t; \end{aligned}$$

the last inequality follows from the fact that

$$\begin{aligned} \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{Re}\,\Phi(u;|x|,t)}\, \mathrm{d}u &\leq \int_{-\pi}^{\pi} \mathrm{e}^{-t(1-\cos u)}\, \mathrm{d}u \\ &\leq C\, t^{-1/2} \qquad \forall t \geq 1. \end{aligned}$$

This proves the required estimate. □

For any real number c we define the heat maximal operator with parameter c, which acts on a function f on $\mathscr{T}$ as

$$\mathscr{H}_*^c f = \sup_{t\geq 1} t^c\, |\mathscr{H}_t f|.$$

The corresponding Hardy-type space is

$$H^1_{\mathscr{H},c}(\mathscr{T}) = \{f \in L^1(\mathscr{T})\colon \mathscr{H}_*^c f \in L^1(\mathscr{T})\},$$

which is endowed with the norm

$$\big\|f\big\|_{H^1_{\mathscr{H},c}(\mathscr{T})} = \big\|f\big\|_1 + \big\|\mathscr{H}_*^c f\big\|_1 .$$

In the final part of this section we investigate the link between $H^1_{\mathscr{H},c}(\mathscr{T})$ and $\mathfrak{X}^1(\mathscr{T})$, with respect to the parameter c.

Theorem 4.3. *If c is negative, then $\mathscr{H}_*^c\big(\mathscr{L}\delta_o\big)$ is in $L^1(\mathscr{T})$. Consequently, the heat maximal operator $\mathscr{H}_*^c$ is bounded from $\mathfrak{X}^1(\mathscr{T})$ to $L^1(\mathscr{T})$, and $\mathfrak{X}^1(\mathscr{T})$ is included in $H^1_{\mathscr{H},c}(\mathscr{T})$.*

Proof. First, we observe that $\mathscr{H}_*^c\big(\mathscr{L}\delta_o\big)(o)$ is finite, for it is equal to

$$2c_G \sup_{t\geq 1}\; t^c \left|\int_{-\pi}^{\pi} \big(1-\gamma(s)\big)\, \mathrm{e}^{-t(1-\gamma(s))}\, \mathbf{c}(-s)^{-1}\, \mathrm{d}s\right|,$$

which is dominated by

$$4c_G \int_{-\pi}^{\pi} \big|\mathbf{c}(-s)^{-1}\big|\, \mathrm{d}s,$$

because c is negative and $1-\gamma(s) = 1-\dfrac{2q^{1/2}}{q+1}\,\cos(s\log q)$ is nonnegative and bounded above by 2. The integral above is convergent, and the required estimate follows.

Thus, we only need to prove that $\mathscr{H}_*^c\big(\mathscr{L}\delta_o\big)$ is in $L^1(\mathscr{T}\setminus\{o\})$. We retain the notation of the proof of Theorem 4.1. By (4.5),

$$t^c\, \mathscr{L} h_t(x) = \frac{q^{1-|x|}}{2\pi(q+1)}\, t^c\, I(|x|,t),$$

where $I(n,t)$ is defined in (4.6). A careful examination of the proof of Theorem 4.1 shows that

$$I(n,t) = J_1(n,t) + O(t^{-3/2}),$$

where

$$J_1(n,t) = \int_{-\pi}^{\pi} \psi(u)\, \sin u\, \sin\big(u(n-\beta t)\big)\, \cos\big(\beta t(u-\sin u)\big)\, \mathrm{e}^{\mathrm{Re}\,\Phi}\, \mathrm{d}u.$$

Trivially,

$$\big|J_1(n,t)\big| \le \int_{-\pi}^{\pi} \psi(u)\, |u|\, \mathrm{e}^{-tu^2}\, \mathrm{d}u.$$

Changing variables ($u\sqrt{t} = v$), the last integral transforms to

$$t^{-1} \int_{\mathbb{R}} \psi(v/\sqrt{t})\, |v|\, \mathrm{e}^{-v^2}\, \mathrm{d}v \sim t^{-1}.$$

Thus, $J_1(n,t) = O(t^{-1})$, whence $I(n,t) = O(t^{-1})$. We may conclude that there exists a constant C, independent of t and x such that

$$\begin{aligned} \sup_{t\ge |x|^{1-\varepsilon}} t\, \big|\mathscr{H}_t(\mathscr{L}\delta_o)(x)\big| &\le C\, q^{-|x|} \sup_{t\ge |x|^{1-\varepsilon}} t^{c-1} \\ &\le C\, q^{-|x|}\, |x|^{-(1-\varepsilon)(1-c)}. \end{aligned}$$

We choose $\varepsilon < -c/(1-c)$, so that $(1-\varepsilon)(1-c) > 1$. Then the right hand side above is in $L^1(\mathscr{T})$.

It remains to show that the maximal function $\sup_{1\le t<|x|^{1-\varepsilon}} t^c\, \big|\mathscr{H}_t(\mathscr{L}\delta_o)(x)\big|$ is in $L^1(\mathscr{T})$. In this case the imaginary part of the phase function Φ (see formula (4.7)) does not vanish, and we may integrate by parts as many times as needed. Specifically, notice that

$$\partial_u(\mathrm{Im}\,\Phi) = n - \beta t \cos u \ge n - \beta n^{1-\varepsilon} \ge (1-\beta)\, n, \tag{4.23}$$

which does not vanish if $n \ge 1$ (recall that $\beta = \dfrac{q-1}{q+1} < 1$). We follow the lines of the proof of Lemma 4.2. We consider the differential operator D, defined by

$$D = \frac{1}{i\partial_u\big(\,\mathrm{Im}\,\Phi(u,|x|,t)\big)}\, \partial_u,$$

and denote by D^* its formal adjoint with respect to the Lebesgue measure. For notational convenience, we write $\zeta(u)$ in place of $1/i\partial_u\big(\operatorname{Im}\Phi(u,|x|,t)\big)$, and denote by M_ζ the operator of multiplication by ζ, i.e.,

$$M_\zeta\varphi = \zeta\,\varphi$$

for any reasonable function φ. Therefore

$$D^*\varphi = -\partial_u(M_\zeta\varphi).$$

In particular, in the rest of this proof, we shall work with

$$\varphi(u) = \frac{q-\mathrm{e}^{2iu}}{q+\mathrm{e}^{iu}}\,(1-\mathrm{e}^{-iu})\,\mathrm{e}^{\operatorname{Re}\Phi}.$$

It is straightforward to check that for each positive integer j there exists a constant C such that

$$\big|\partial_u^j\varphi(u)\big| \le C\,t^j \qquad \forall u\in[-\pi,\pi] \quad \forall t\in[1,\infty).$$

We may integrate by parts k times and obtain

$$\begin{aligned}\mathscr{L}h_t(x) &= \frac{q^{1-|x|}}{2\pi}\int_{-\pi}^{\pi}\varphi(u)\,D^k\mathrm{e}^{i\operatorname{Im}\Phi(u;|x|,t)}\,\mathrm{d}u\\ &= (-1)^k\,\frac{q^{1-|x|}}{2\pi}\int_{-\pi}^{\pi}\big(\partial_u M_\zeta\big)^k\varphi(u)\;\mathrm{e}^{i\operatorname{Im}\Phi(u;|x|,t)}\,\mathrm{d}u.\end{aligned}$$

Here we have used the fact that, by periodicity, the boundary terms arising from integration by parts cancel out.

We *claim* that for each positive integer j there exist smooth bounded functions $\varphi_1,\ldots,\varphi_j$, with bounded derivatives such that

$$\partial_u^j\zeta(u) = \sum_{h=1}^{j} t^h\,\varphi_h(u)\,\zeta(u)^{h+1}. \tag{4.24}$$

In fact, each function φ_h is a constant multiple of a finite product of linear combinations of sines and cosines.

We argue by induction. Since

$$\partial_u \zeta(u) = -i\beta t \, \sin u \, \zeta(u)^2,$$

the required property holds for $j = 1$. Suppose that (4.24) holds for all positive integers up to $j-1$, and consider $\partial_u^j \zeta$. By Leibnitz's rule

$$\begin{aligned}
\partial_u^j \zeta(u) &= \partial_u \big(\partial_u^{j-1} \zeta\big)(u) \\
&= \sum_{h=1}^{j-1} t^h \left[\partial_u \varphi_h(u) \, \zeta(u)^{h+1} + \varphi_h(u) \, (h+1) \, \zeta(u)^h \, \partial_u \zeta(u)\right] \\
&= \sum_{h=1}^{j-1} \left[t^h \, \partial_u \varphi_h(u) \, \zeta(u)^{h+1} - i\beta t^{h+1} \, (h+1) \, \varphi_h(u) \, (\sin u) \, \zeta(u)^{h+2}\right],
\end{aligned}$$

which, after relabeling, has the required form. This proves the claim.

A straightforward consequence of the claim and of (4.23) is that for each positive integer j there exists a constant C such that

$$\big|\partial_u^j \zeta(u)\big| \leq C \sum_{h=1}^{j} \frac{t^h}{|x|^{h+1}}.$$

Another tedious, albeit straightforward, induction argument shows that

$$\begin{aligned}
\Big|\big(DM_\zeta\big)^k \varphi(u)\Big| &\leq C \, \mathrm{e}^{\mathrm{Re}\, \Phi} \, t^k \, |\zeta|^k \\
&\leq C \, \mathrm{e}^{\mathrm{Re}\, \Phi} \left(\frac{t}{|x|}\right)^k \qquad \forall u \in [-\pi, \pi].
\end{aligned} \tag{4.25}$$

We have used (4.23) in the last inequality. As a consequence,

$$\begin{aligned}
\big|\mathscr{L} h_t(x)\big| &\leq C \, q^{-|x|} \left(\frac{t}{|x|}\right)^k \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{Re}\, \Phi(u;|x|,t)} \, \mathrm{d}u \\
&\leq C \, q^{-|x|} \left(\frac{t}{|x|}\right)^k \min\big(1, t^{-1/2}\big) \qquad \forall x : |x|^{1-\varepsilon} \geq t;
\end{aligned}$$

the last inequality follows from the fact that

$$\begin{aligned}
\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{Re}\, \Phi(u;|x|,t)} \, \mathrm{d}u &\leq \int_{-\pi}^{\pi} \mathrm{e}^{-t(1-\cos u)} \, \mathrm{d}u \\
&\leq C \, t^{-1/2} \qquad \forall t \geq 1,
\end{aligned}$$

and that $\mathrm{e}^{\mathrm{Re}\,\Phi(u;|x|,t)} \leq 1$. Thus,

$$\sup_{1\leq t<|x|^{1-\varepsilon}} t^c \left|\mathscr{H}_t(\mathscr{L}\delta_o)(x)\right| \leq C\, q^{-|x|}\, |x|^{-\varepsilon k}.$$

By choosing k so that $\varepsilon k > 1$, the right hand side of the inequality above is in $L^1(\mathscr{T} \setminus \{o\})$.

This completes the proof of the theorem. $\square$

We conclude this section with the following result.

Theorem 4.4. *Suppose that f is in $H^1_{\mathscr{H},c}(\mathscr{T})$ for some c in $(1,\infty)$. Then f is in $\mathfrak{X}^1(\mathscr{T})$. Furthermore there exists a constant C, independent of f, such that*

$$\left\|f\right\|_{\mathfrak{X}^1(\mathscr{T})} \leq C\ \left\|f\right\|_{H^1_{\mathscr{H},c}(\mathscr{T})}.$$

Proof. The proof hinges on the following reproducing formula

$$\delta_o = 2\int_0^\infty \mathscr{L} h_t * h_t \,\mathrm{d}t,$$

and its consequence

$$g = 2\int_0^\infty g * \mathscr{L} h_t * h_t \,\mathrm{d}t, \tag{4.26}$$

which may be established via spherical Fourier analysis along the lines of the proof of similar formulae for the Poisson semigroup (see Lemma 4.10 below).

We need to prove that if $\mathscr{H}^c_* f$ is in $L^1(\mathscr{T})$, then $\mathscr{L}^{-1} f$ is in $L^1(\mathscr{T})$, with a corresponding control of the norm. Set $g := \mathscr{L}^{-1} f$. By (4.26),

$$g = 2\int_0^\infty \mathscr{H}_t f * h_t \,\mathrm{d}t = 2\int_0^\infty \langle t\rangle^c \mathscr{H}_t f * h_t \frac{\mathrm{d}t}{\langle t\rangle^c},$$

where $\langle t\rangle^c = \max(1, t^c)$. Therefore

$$\begin{aligned}\left\|g\right\|_1 &\leq C \int_0^\infty \left\|\mathscr{H}^c_* f * h_t\right\|_1 \frac{\mathrm{d}t}{\langle t\rangle^c} \\ &\leq C\ \left\|\mathscr{H}^c_* f\right\|_1 \int_0^\infty \frac{\mathrm{d}t}{\langle t\rangle^c}.\end{aligned}$$

The integral on the right hand side is convergent, because $c > 1$, and the required conclusion follows. $\square$

Corollary 4.5. *Suppose that $c_1 < 0$ and that $c_2 > 1$. Then*

$$H^1_{\mathscr{H},c_2}(\mathscr{T}) \subset \mathfrak{X}^1(\mathscr{T}) \subset H^1_{\mathscr{H},c_1}(\mathscr{T}).$$

It is natural to speculate whether the inclusion on the left holds for all $c_2 > 0$. We leave this for further investigations.

We conjecture that the following estimate holds: there exists a positive constant C such that

$$\big|\mathscr{H}_t\big(\mathscr{L}\delta_o\big)(x)\big| \leq \begin{cases} C\, q^{-|x|}\, \dfrac{\big||x|-\beta\, t\big|}{t^{3/2}} & \text{if } \big||x|-\beta t\big| \geq \sqrt{t} \\ \\ C\, q^{-|x|}\, \dfrac{1}{\big||x|-\beta\, t\big|\,\sqrt{t}} & \text{if } \big||x|-\beta t\big| \leq \sqrt{t}. \end{cases}$$

4.2 The Poisson maximal operator

Recall that $\{\mathscr{P}_t\}$ denotes the Poisson semigroup. For any real number c in we consider the *Poisson maximal operator* $\mathscr{P}^c_*$ *with parameter* c, which acts on a function f on $\mathscr{T}$ by

$$\mathscr{P}^c_* f = \sup_{t\geq 1} t^c \,\big|\mathscr{P}_t f\big|.$$

We shall write $\mathscr{P}_* f$, instead of $\mathscr{P}^0_*$. We then define $H^1_{\mathscr{P},c}(\mathscr{T})$ by

$$H^1_{\mathscr{P},c}(\mathscr{T}) = \{f \in L^1(\mathscr{T}) : \mathscr{P}^c_* f \in L^1(\mathscr{T})\}.$$

We endow $H^1_{\mathscr{P},c}(\mathscr{T})$ with the norm

$$\big\|f\big\|_{H^1_{\mathscr{P},c}(\mathscr{T})} = \big\|f\big\|_1 + \big\|\mathscr{P}^c_* f\big\|_1\,.$$

The analogue of $\mathscr{P}^c_*$ on symmetric spaces of the noncompact type was considered by J.-Ph. Anker [An], who proved that for every c in $[0,1)$ there exists a constant C such that

$$\big\|\mathscr{P}^c_* f\big\|_1 \leq C\,\big(\,\big\|f\big\|_1 + \big\||\mathscr{R}f|\big\|_1\,\big), \tag{4.27}$$

where $\mathscr{R}$ denotes the Riemannian Riesz transform. It is reasonable to conjecture that a similar result holds on trees. This is indeed true, and is proved in the next theorem. We need the following simple lemma.

Lemma 4.6. *The maximal functions* $\sup_{0<t\leq 1} |h_t|$ *and* $\sup_{0<t\leq 1} |p_t|$ *are in* $L^1(\mathscr{T})$.

Proof. Notice that $h_t = \mathrm{e}^{-t} \sum_{n=0}^{\infty} \frac{(t\varsigma)^n}{n!}$, where ς denotes the measure $\frac{1}{\nu(x)} \sum_{y\sim o} \delta_y$. Therefore

$$\sup_{0<t\leq 1} |h_t| \leq \sum_{n=0}^{\infty} \frac{\varsigma^n}{n!},$$

whence

$$\Big\| \sup_{0<t\leq 1} |h_t| \Big\|_1 \leq \sum_{n=0}^{\infty} \frac{\|\varsigma\|_1^n}{n!} \leq \mathrm{e}^{\|\varsigma\|_1}$$

Similarly, $p_t = \sum_{n=0}^{\infty} \frac{t^n\, k_{\mathscr{L}^{1/2}}^{(*n)}}{n!}$, where $k_{\mathscr{L}^{1/2}}$ denotes the convolution kernel of $\mathscr{L}^{1/2}$, which is known to belong to $L^1(\mathscr{T})$. Therefore

$$\sup_{0<t\leq 1} |p_t| \leq \sum_{n=0}^{\infty} \frac{\big|k_{\mathscr{L}^{1/2}}^{(*n)}\big|}{n!},$$

whence

$$\Big\| \sup_{0<t\leq 1} |p_t| \Big\|_1 \leq \sum_{n=0}^{\infty} \frac{\|k_{\mathscr{L}^{1/2}}\|_1^n}{n!} \leq \mathrm{e}^{\|k_{\mathscr{L}^{1/2}}\|_1},$$

as required. □

Theorem 4.7. *The following hold:*

(i) $\mathscr{L}^{1/2}\delta_o$ *is in* $H^1_{\mathscr{P},c}(\mathscr{T})$ *for every* c *in* $[0,1)$*;*

(ii) $\mathfrak{X}^{1/2}(\mathscr{T})$ *is contained in* $H^1_{\mathscr{P},c}(\mathscr{T})$ *for every* c *in* $[0,1)$*. Furthermore, there exists a constant* C*, independent of* f*, such that*

$$\|f\|_{H^1_{\mathscr{P},c}(\mathscr{T})} \leq C\, \|f\|_{\mathfrak{X}^{1/2}(\mathscr{T})} \qquad \forall f \in \mathfrak{X}^{1/2}(\mathscr{T});$$

(iii) *if f is in $H^1_{\mathscr{P},c}(\mathscr{T})$ for some c in $(1,\infty)$, then f is in $\mathfrak{X}^{1/2}(\mathscr{T})$. Furthermore, there exists a constant C, independent of f, such that*

$$\|f\|_{\mathfrak{X}^{1/2}(\mathscr{T})} \leq C\,\|f\|_{H^1_{\mathscr{P},c}(\mathscr{T})}\,;$$

(iv) *the maximal operator $f \mapsto \mathscr{P}^1_* f$ is unbounded from $\mathfrak{X}^{1/2}(\mathscr{T})$ to $L^1(\mathscr{T})$. In particular, $\mathscr{L}^{1/2}\delta_o$ is not in $H^1_{\mathscr{P},1}(\mathscr{T})$.*

Remark 4.8. Theorem 4.7 (iv) says that

$$\mathfrak{X}^{1/2}(\mathscr{T}) \not\subset H^1_{\mathscr{P},1}(\mathscr{T}).$$

Recall that we already know that $H^1_{\mathscr{R}}(\mathscr{T}) = \mathfrak{X}^{1/2}(\mathscr{T})$, so that

$$H^1_{\mathscr{R}}(\mathscr{T}) \neq H^1_{\mathscr{P},1}(\mathscr{T}),$$

a phenomenon that does not have a counterpart in the Euclidean setting.

First we need more notation and a technical lemma. The kernel p_t of the Poisson semigroup $\mathscr{P}_t$ is given by the following well known subordination formula (see, for instance, [Y, formula (2), p. 260] or [St1, formula (*), p. 47])

$$p_t = t\int_0^\infty (4\pi s)^{-1/2}\,\mathrm{e}^{-t^2/(4s)}\,h_s\,\frac{\mathrm{d}s}{s}. \tag{4.28}$$

We need some information concerning the behaviour of $p_t(o)$, which is analysed in the next lemma. Observe that there exists a constant C such that

$$p_t(o) \leq C\,t^{-3/4}\,\mathrm{e}^{-b_2 t} \tag{4.29}$$

for all t in $[1,\infty)$ (recall that b_2 is the bottom of the L^2 spectrum of $\mathscr{L}$). Indeed, $p_t(o) \leq \|p_t\|_2$, which, by [Se1, Lemma 3 (i)], is dominated by $C\,t^{-3/4}\,\mathrm{e}^{-b_2 t}$, as required.

We shall also require estimates of the time derivative of p_t, which can be readily computed from formula (4.28). We see that

$$\partial_t p_t = \int_0^\infty \Big[1-\frac{t^2}{2s}\Big](4\pi s)^{-1/2}\,\mathrm{e}^{-t^2/(4s)}\,h_s\,\frac{\mathrm{d}s}{s}. \tag{4.30}$$

Some estimates involving $\partial_t p_t$ are given in the next lemma.

Lemma 4.9. *The following hold:*

(i) for each c in $[0,1)$ the maximal function

$$\sup_{t\geq 1} t^c \left|\int_1^\infty \left[1-\frac{t^2}{2s}\right](4\pi s)^{-1/2}\,\mathrm{e}^{-t^2/(4s)}\,h_s\,\frac{\mathrm{d}s}{s}\right|$$

is in $L^1(\mathscr{T})$;

(ii) for every $a>0$ the maximal function

$$\sup_{t\geq 1} t^a \left|\int_0^1 \left[1-\frac{t^2}{2s}\right](4\pi s)^{-1/2}\,\mathrm{e}^{-t^2/(4s)}\,h_s\,\frac{\mathrm{d}s}{s}\right|$$

is in $L^1(\mathscr{T})$.

Proof. To prove (i), write $t^c = s^{c/2}\,(t^c/s^{c/2})$. Therefore

$$\begin{aligned}&\sup_{t\geq 1} t^c \left|\int_1^\infty \left[1-\frac{t^2}{2s}\right](4\pi s)^{-1/2}\,\mathrm{e}^{-t^2/(4s)}\,h_s\,\frac{\mathrm{d}s}{s}\right| \\ &\leq (4\pi)^{-1/2}\int_1^\infty s^{(c-3)/2}\,h_s\,\sup_{t\geq 1}\frac{t^c}{s^{c/2}}\left[1+\frac{t^2}{2s}\right]\mathrm{e}^{-t^2/(4s)}\,\mathrm{d}s.\end{aligned}$$

The supremum inside the integral is finite, and independent of s, for it agrees with the supremum over the positive reals of the function $v\mapsto v^c\,(1+v^2)\,\mathrm{e}^{-v/4}$. Thus,

$$\sup_{t>0} t^c \left|\int_1^\infty \left[1-\frac{t^2}{2s}\right](4\pi s)^{-1/2}\,\mathrm{e}^{-t^2/(4s)}\,h_s\,\frac{\mathrm{d}s}{s}\right| \leq C\int_1^\infty s^{(c-3)/2}\,h_s\,\mathrm{d}s.$$

To complete the proof of (i), it suffices to observe that

$$\left\|\int_1^\infty s^{(c-3)/2}\,h_s\,\mathrm{d}s\right\|_1 \leq \int_1^\infty s^{(c-3)/2}\,\left\|h_s\right\|_1\,\mathrm{d}s = \int_1^\infty s^{(c-3)/2}\,\mathrm{d}s,$$

which is convergent, because, by assumption, $c<1$.

Next we prove (ii). Observe that if $t\geq 1$, then

$$t^2 = t^2-\frac{1}{2}+\frac{1}{2}\geq\frac{t^2}{2}+\frac{1}{2},$$

so that

$$\mathrm{e}^{-t^2/(8s)} \leq \mathrm{e}^{-t^2/(16s)}\,\mathrm{e}^{-1/(16s)} \leq \mathrm{e}^{-t^2/(16)}\,\mathrm{e}^{-1/(16s)} \qquad \forall s \in (0,1].$$

Moreover

$$\left|1-\frac{t^2}{2s}\right|\mathrm{e}^{-t^2/(4s)} \leq C\,\mathrm{e}^{-t^2/(8s)},$$

where $C = \sup_{v>0}\left|1-v\right|\mathrm{e}^{-v/4}$. By combining these estimates, we see that

$$\begin{aligned}
&\sup_{t\geq 1} t^a \left|\int_0^1 \left[1-\frac{t^2}{2s}\right](4\pi s)^{-1/2}\,\mathrm{e}^{-t^2/(4s)}\,h_s\,\frac{\mathrm{d}s}{s}\right| \\
&\leq C \sup_{t\geq 1} t^a\,\mathrm{e}^{-t^2/(16)} \int_0^1 s^{-3/2}\,\mathrm{e}^{-1/(16s)}\,h_s\,\mathrm{d}s \\
&\leq C \int_0^1 s^{-3/2}\,\mathrm{e}^{-1/(16s)}\,h_s\,\mathrm{d}s.
\end{aligned}$$

To conclude the proof of (ii), it suffices to recall that $\left\|h_s\right\|_1 = 1$, so that the $L^1(\mathscr{T})$ norm of the last integral is dominated by $\int_0^1 s^{-3/2}\,\mathrm{e}^{-1/(16s)}\,\mathrm{d}s$, which is convergent, as required. □

Another important formula, which will be the key to prove Theorem 4.7 (iii), is given in the next lemma.

Lemma 4.10. *The following reproducing formula of Calderón type holds*

$$\delta_o = 4\int_0^\infty \mathscr{L}^{1/2}p_t * \mathscr{L}^{1/2}p_t\,t\,\mathrm{d}t.$$

Furthermore, if g is in $L^q(\mathscr{G})$ for all q in $(1,2]$, then

$$g = 4\int_0^\infty g * \mathscr{L}^{1/2}p_t * \mathscr{L}^{1/2}p_t\,t\,\mathrm{d}t$$

in $L^q(\mathscr{G})$ (hence pointwise) for all q in $(1,2]$.

Proof. In order to prove the first formula observe that both sides are radial functions. Then it suffices to show that their spherical Fourier transforms agree. Indeed, for every z in $\mathbb{T}$,

$$\widetilde{\delta_o}(z) = \langle \varphi_z, \delta_o\rangle = \varphi_z(o) = 1,$$

and, at least formally,

$$
\begin{aligned}
\Big[\int_0^\infty \mathscr{L}^{1/2}p_t * \mathscr{L}^{1/2}p_t \,\mathrm{d}t\Big]^\sim(z) &= \int_0^\infty \big[\mathscr{L}^{1/2}p_t * \mathscr{L}^{1/2}p_t\big]^\sim(z)\, t\,\mathrm{d}t \\
&= \int_0^\infty \big(1-\gamma(z)\big)\,\mathrm{e}^{-2t(1-\gamma(z))^{1/2}}\, t\,\mathrm{d}t.
\end{aligned}
\tag{4.31}
$$

We now change variables, and the right hand side becomes $\int_0^\infty v\,\mathrm{e}^{-2v}\,\mathrm{d}v$, which is equal to $1/4$, and the required formula follows. It remains to justify the first equality in the chain of equalities above. It is useful to observe that $\int_0^\infty \mathscr{L}^{1/2}p_t * \mathscr{L}^{1/2}p_t\, t\,\mathrm{d}t$ is convergent as a Bochner integral in $L^q(\mathscr{G})$ for every q in $(1,2]$. Indeed,

$$
\begin{aligned}
\int_0^\infty \big\|\mathscr{L}^{1/2}p_t * \mathscr{L}^{1/2}p_t\big\|_q \,\mathrm{d}t &\le \int_0^\infty \big\|\mathscr{L}^{1/2}p_t\big\|_1 \,\big\|\mathscr{L}^{1/2}p_t\big\|_q \, t\,\mathrm{d}t \\
&\le \big\|k_{\mathscr{L}^{1/2}}\big\|_1^2 \int_0^\infty \big\|p_t\big\|_q \, t\,\mathrm{d}t,
\end{aligned}
$$

and the last integral is convergent because of [Se1, Lemma 3 (i)].

Now note that for every z in $\mathbb{T}$ the spherical function φ_z is in $L^{q'}(\mathscr{G})$ for all q in $(1,2)$. Thus,

$$
\begin{aligned}
&\sum_{x\in\mathscr{G}} \int_0^\infty \big|\varphi_z(x)\big|\,\big|\mathscr{L}^{1/2}p_t * \mathscr{L}^{1/2}p_t(x)\big|\, t\,\mathrm{d}t \\
&\le \big\|\varphi_z\big\|_{q'} \int_0^\infty \Big\|\mathscr{L}^{1/2}p_t * \mathscr{L}^{1/2}p_t\Big\|_q \, t\,\mathrm{d}t \\
&\le C\,\big\|k_{\mathscr{L}^{1/2}}\big\|_1^2,
\end{aligned}
$$

where C depends on q, but not on z in $\mathbb{T}$. We have used the previous estimate in the last inequality above. Thus, the first equality in (4.31) follows from Fubini's theorem.

To prove the second formula in the statement of the lemma, we first prove that it holds pointwise. Notice that, at least formally,

$$
\begin{aligned}
g(x) &= g * \delta_o(x) \\
&= 4\, g * \Big[\int_0^\infty \mathscr{L}^{1/2}p_t * \mathscr{L}^{1/2}p_t\, t\,\mathrm{d}t\Big](x) \\
&= 4 \int_0^\infty g * \mathscr{L}^{1/2}p_t * \mathscr{L}^{1/2}p_t(x)\, t\,\mathrm{d}t,
\end{aligned}
$$

as required. The last equality is readily justified by Fubini's theorem, for

$$\begin{aligned}
&\sum_{y\in\mathscr{G}} \int_0^\infty \big|g(y)\big| \, \big|\mathscr{L}^{1/2}p_t * \mathscr{L}^{1/2}p_t(y^{-1}x)\big| \, t \, \mathrm{d}t \\
&\leq \big\|g\big\|_q \int_0^\infty \Big\|\mathscr{L}^{1/2}p_t * \mathscr{L}^{1/2}p_t\Big\|_{q'} \, t \, \mathrm{d}t \\
&\leq C \, \big\|k_{\mathscr{L}^{1/2}}\big\|_1^2 .
\end{aligned}$$

Finally, $\int_0^\infty g * \mathscr{L}^{1/2}p_t * \mathscr{L}^{1/2}p_t \, t \, \mathrm{d}t$ is convergent as a Bochner integral in $L^q(\mathscr{G})$ for every q in $(1,2]$. Indeed, fix q in $(1,2]$, and choose p in $(1,q)$. By Minkowski's generalised inequality and the Kunze–Stein property

$$\begin{aligned}
\Big\|\int_0^\infty g * \mathscr{L}^{1/2}p_t * \mathscr{L}^{1/2}p_t \, \mathrm{d}t\Big\|_q &\leq \int_0^\infty \big\|g * \mathscr{L}^{1/2}p_t * \mathscr{L}^{1/2}p_t\big\|_q \, t \, \mathrm{d}t \\
&\leq \big\|g\big\|_p \int_0^\infty \big\|\mathscr{L}^{1/2}p_t * \mathscr{L}^{1/2}p_t\big\|_q \, t \, \mathrm{d}t.
\end{aligned}$$

By Young's inequality the last integral is bounded above by

$$\begin{aligned}
\int_0^\infty \big\|\mathscr{L}^{1/2}p_t\big\|_1 \, \big\|\mathscr{L}^{1/2}p_t\big\|_q \, t \, \mathrm{d}t &\leq \big\|k_{\mathscr{L}^{1/2}}\big\|_1^2 \int_0^\infty \big\|p_t\big\|_q \, t \, \mathrm{d}t \\
&\leq C \, \big\|k_{\mathscr{L}^{1/2}}\big\|_1^2 ;
\end{aligned}$$

we have used the estimates of $\big\|p_t\big\|_q$ in [Se1, Lemma 3 (i)] to prove that the last integral is convergent.

This completes the proof of the lemma. □

We are now ready to prove Theorem 4.7.

Proof. (of Theorem 4.7) First we prove (i). We need to prove that $\mathscr{P}_*^c\big(\mathscr{L}^{1/2}\delta_o\big)$ is in $L^1(\mathscr{T})$ for every c in $[0,1)$; Observe that

$$\mathscr{P}_t\big(\mathscr{L}^{1/2}\delta_o\big) = -\partial_t \mathscr{P}_t\delta_o = -\partial_t p_t.$$

Therefore

$$\mathscr{P}_*^c\big(\mathscr{L}^{1/2}\delta_o\big) = \sup_{t\geq 1} t^c \, \big|\partial_t p_t\big|,$$

and the required conclusion follows from Lemma 4.9 (i) and (ii).

Next we prove (ii). Suppose that f is in $\mathfrak{X}^{1/2}(\mathscr{T})$. Then there exists g in $L^1(\mathscr{T})$ such that $f = \mathscr{L}^{1/2} g$, so that

$$\mathscr{P}_t f = f * p_t = \mathscr{L}^{1/2} g * p_t.$$

We may write $g = \sum_{x\in\mathscr{T}} g(x)\,\delta_x$, whence

$$\mathscr{P}_t f = \sum_{x\in\mathscr{T}} g(x)\,\mathscr{L}^{1/2}\delta_x * p_t,$$

and

$$\big\|\mathscr{P}_*^c f\big\|_1 \leq \sum_{x\in\mathscr{T}} \big|g(x)\big| \;\big\|\mathscr{P}_*^c\big(\mathscr{L}^{1/2}\delta_x\big)\big\|_1 \,.$$

Recall that both $\mathscr{P}_t$ and $\mathscr{L}^{1/2}$ are left invariant operators. Therefore $\big\|\mathscr{P}_*^c\big(\mathscr{L}^{1/2}\delta_x\big)\big\|_1$ is independent of x. By (i),

$$\big\|\mathscr{P}_*^c\big(\mathscr{L}^{1/2}\delta_o\big)\big\|_1 < \infty,$$

whence

$$\begin{aligned}\big\|\mathscr{P}_*^c f\big\|_1 &\leq \big\|g\big\|_1 \;\big\|\mathscr{P}_*^c\big(\mathscr{L}^{1/2}\delta_o\big)\big\|_1 \\ &= \big\|f\big\|_{\mathfrak{X}^{1/2}(\mathscr{T})} \;\big\|\mathscr{P}_*^c\big(\mathscr{L}^{1/2}\delta_o\big)\big\|_1 \,,\end{aligned}$$

thereby concluding the proof of (ii).

Next we prove (iii). Observe that

$$\mathscr{P}_t f = \mathscr{P}_t \mathscr{L}^{1/2}\mathscr{L}^{-1/2} f = -\partial_t \mathscr{P}_t \mathscr{L}^{-1/2} f.$$

The last equality has the following meaning. For a generic f in $L^1(\mathscr{T})$, the function $\mathscr{L}^{-1/2} f$ is not necessarily in $L^1(\mathscr{T})$ (for $\mathscr{L}^{-1/2}$ is unbounded on $L^1(\mathscr{T})$), but it belongs to $L^p(\mathscr{T})$ for every p in $(1,\infty)$, because f is in $L^p(\mathscr{T})$, and $\mathscr{L}^{-1/2}$ is bounded on $L^p(\mathscr{T})$ for each p in $(1,\infty)$. The restriction of $\mathscr{L}^{1/2}$ to $L^p(\mathscr{T})$ is the infinitesimal generator of $\{\mathscr{P}_t\}$, thought of as a semigroup acting on $L^p(\mathscr{T})$.

We need to prove that f is in $\mathfrak{X}^{1/2}(\mathscr{T})$, i.e., that $\mathscr{L}^{-1/2}f$ is in $L^1(\mathscr{T})$, with a corresponding control of the norm. We have already observed that $\mathscr{L}^{-1/2}f$ is in $L^q(\mathscr{G})$ for all q in $(1,\infty]$. Therefore we may apply Lemma 4.10, and write

$$\begin{aligned}\mathscr{L}^{-1/2}f &= 4\int_0^\infty \mathscr{L}^{-1/2}f * \mathscr{L}^{1/2}p_t * \mathscr{L}^{1/2}p_t\, t\, \mathrm{d}t \\ &= 4\int_0^\infty \mathscr{P}_t f * \mathscr{L}^{1/2}p_t\, t\, \mathrm{d}t \qquad \forall x\in\mathscr{T}.\end{aligned}$$

Then

$$\begin{aligned}\big|\mathscr{L}^{-1/2}f\big| &\leq 4\int_0^\infty \big|\mathscr{P}_t f\big| * \big|\mathscr{L}^{1/2}p_t\big|\, t\, \mathrm{d}t \\ &= 4\int_0^\infty \big[\max(1,t^c)\big|P_t f\big|\big] * \big[\min(t,t^{1-c})\,\big|\mathscr{L}^{1/2}p_t\big|\big]\, \mathrm{d}t.\end{aligned}$$

Clearly,

$$\begin{aligned}\max(1,t^c)\,\big|P_t f\big| &\leq \sup_{0<t\leq 1}\big|P_t f\big| + \sup_{t\geq 1} t^c\,\big|P_t f\big| \\ &\leq \big|f\big| * \sup_{0<t\leq 1}\big|p_t\big| + \mathscr{P}_*^c f\end{aligned}$$

Thus,

$$\big|\mathscr{L}^{-1/2}f\big| \leq 4\left[\left(\big|f\big| * \sup_{0<t\leq 1}\big|p_t\big| + \mathscr{P}_*^c f\right) * \int_0^\infty \min(1,t^{1-c})\,\big|\mathscr{L}^{1/2}p_t\big|\,\mathrm{d}t\right]$$

Observe that, by Lemma 4.6,

$$\Big\|\big|f\big| * \sup_{0<t\leq 1}\big|p_t\big| + \mathscr{P}_*^c f\Big\|_1 \leq C\,\big\|f\big\|_1 + \big\|\mathscr{P}_*^c f\big\|_1$$

Consequently, by standard convolution inequalities,

$$\big\|\mathscr{L}^{-1/2}f\big\|_1 \leq C\left(\,\big\|f\big\|_1 + \big\|\mathscr{P}_*^c f\big\|_1\,\right)\,\Big\|\int_0^\infty \min(1,t^{1-c})\,\big|\mathscr{L}^{1/2}p_t\big|\,\mathrm{d}t\Big\|_1 .$$

The integral on the right hand side is convergent. Indeed, choose c' in $[0,1)$ such that $c'>2-c$. This is possible, because $c>1$ by assumption. Note that

$$\sup_{0<t\leq 1}\big|\mathscr{L}^{1/2}p_t\big| \leq \big|k_{\mathscr{L}^{1/2}}\big| * \sup_{0<t\leq 1}\big|p_t\big|$$

By this and (i) there exists an integrable function h such that

$$\left|\mathscr{L}^{1/2}p_t\right| \leq \min\left(1, t^{-c'}\right) h.$$

Hence

$$\left\|\int_0^\infty t^{1-c}\left|\mathscr{L}^{1/2}p_t\right| \mathrm{d}t\right\|_1 \leq \left\|h\right\|_1 \int_0^\infty \min(1, t^{1-c}) \min\left(1, t^{-c'}\right) \mathrm{d}t.$$

The last integral is convergent, because $c + c' > 2$. By combining these estimate we may conclude that

$$\left\|\mathscr{L}^{-1/2}f\right\|_1 \leq C\left(\left\|f\right\|_1 + \left\|\mathscr{P}_*^c f\right\|_1 \right)$$

as required to conclude the proof of (iii).

Finally we prove (iv).

We shall follow the idea of the proof of Theorem 4.1.

The Poisson semigroup $\mathscr{P}_t$ commutes with the powers of the Laplacian, so that

$$\mathscr{P}_t\left(\mathscr{L}^{1/2}\delta_o\right) = \mathscr{L}^{1/2}p_t \qquad \forall t > 0,$$

and its spherical Fourier transform is given by

$$\left[\mathscr{P}_t\left(\mathscr{L}^{1/2}\delta_o\right)\right]^{\sim}(s) = \left(1 - \gamma(s)\right)^{1/2} \mathrm{e}^{-t(1-\gamma(s))^{1/2}} \qquad \forall t > 0 \quad \forall s \in \mathbb{T}.$$

By the inversion formula (1.14) for the spherical Fourier transform,

$$\mathscr{L}^{1/2}p_t(x) = 2c_G\, q^{-|x|/2} \int_{-\tau/2}^{\tau/2} \left(1 - \gamma(s)\right)^{1/2} \mathrm{e}^{-t(1-\gamma(s))^{1/2}}\, q^{is|x|}\, \mathbf{c}(-s)^{-1}\, \mathrm{d}s. \qquad (4.32)$$

Observe that the integrand in (4.32) is holomorphic on the closure of the rectangle with vertices $\pm\tau/2$, $\pm\tau/2 + i/2$. Then we may integrate on the boundary of this rectangle, observe that, by periodicity, the contributions of the integrals over the vertical sides cancel out, use Cauchy's theorem, and conclude that

$$\begin{aligned}\mathscr{L}^{1/2}p_t(x) &= 2c_G\, q^{-|x|/2}\int_{-\tau/2}^{\tau/2}\frac{\big(1-\gamma(s)\big)^{1/2}}{\mathbf{c}(-s)}\,\mathrm{e}^{-t(1-\gamma(s))^{1/2}}\,q^{is|x|}\,\mathrm{d}s\\ &= 2c_G\, q^{-|x|/2}\int_{-\tau/2}^{\tau/2}\frac{\big(1-\gamma(s+i/2)\big)^{1/2}}{\mathbf{c}(-s-i/2)}\,\mathrm{e}^{-t(1-\gamma(s+i/2))^{1/2}}\,q^{i(s+i/2)|x|}\,\mathrm{d}s\\ &= 2c_G\, q^{-|x|}\int_{-\tau/2}^{\tau/2}\frac{\big(1-\gamma(s+i/2)\big)^{1/2}}{\mathbf{c}(-s-i/2)}\,\mathrm{e}^{-t(1-\gamma(s+i/2))^{1/2}}\,q^{is|x|}\,\mathrm{d}s.\end{aligned} \tag{4.33}$$

We recall here for the readers' convenience equation (4.3)

$$1-\gamma(s+i/2)=\frac{(1-q^{-is})(q-q^{is})}{q+1}$$

and equation (4.4)

$$\frac{1}{\mathbf{c}(-s-1/2)}=(q+1)\frac{q-q^{2is}}{q^2-q^{2is}}.$$

We insert this in the last integral in (4.33), change variables, and obtain that

$$\mathscr{L}^{1/2}p_t(x)=\frac{q^{1-|x|}}{2\pi}\,I(|x|,t), \tag{4.34}$$

where we have set

$$I(n,t)=\int_{-\pi}^{\pi}\Big[\frac{(1-e^{-is})(q-e^{is})}{q+1}\Big]^{1/2}\,\frac{q-e^{2is}}{q^2-e^{2is}}\,\mathrm{e}^{\Phi(u;n,t)}\,\mathrm{d}u. \tag{4.35}$$

Here the phase Φ is given by

$$\Phi(u;n,t)=-t\,\Big[\frac{(1-\mathrm{e}^{-iu})(q-\mathrm{e}^{iu})}{q+1}\Big]^{1/2}+inu. \tag{4.36}$$

Now we need to give an explicit expression of the above square root. The computations in (4.7) imply

$$\frac{(1-\mathrm{e}^{-iu})(q-\mathrm{e}^{iu})}{q+1}=(1-\cos u+i\beta\sin u),$$

where $\beta = \dfrac{q-1}{q+1}$. To extract the root we pass to polar cohordinates (r, ϑ). Thus

$$\begin{aligned} r^2(u) &= (1-\cos u)^2 + \beta^2 \sin^2 u \\ &= 1 - 2\cos u + \cos^2 u + \beta^2 \sin^2 u \\ &= 1 + \beta^2 - 2\cos u + (1-\beta^2)\cos^2 u. \end{aligned} \tag{4.37}$$

For further reference we remark that the above equation implies that r is an even function, and it is asymptotic to $\beta\,|u|$ as u tends to zero.
The angle ϑ has the following expression

$$\vartheta(u) = \arctan \frac{\beta \sin u}{1-\cos u}.$$

If $u > 0$ this can also be written as

$$\vartheta(u) = \frac{\pi}{2} - \arctan \frac{1-\cos u}{\beta \sin u},$$

thus

$$\frac{\vartheta(u)}{2} = \frac{\pi}{4} - \frac{1}{2} \arctan \frac{1-\cos u}{\beta \sin u}. \tag{4.38}$$

If $u < 0$ instead we have

$$\frac{\vartheta(u)}{2} = -\frac{\pi}{4} - \frac{1}{2} \arctan \frac{1-\cos u}{\beta \sin u}.$$

Note that $u \mapsto \dfrac{1-\cos u}{\beta \sin u}$ is an odd function, so the above equations implies that also $\vartheta/2$ is odd.

With the above expressions for r and ϑ, we have

$$(1-\cos u + \beta \sin u)^{1/2} = r^{1/2}\left(\cos\frac{\vartheta}{2} + i \sin\frac{\vartheta}{2}\right). \tag{4.39}$$

In particular we have the following asymptotic as u tends to $0^{\pm}$

$$(1-\cos u + \beta \sin u)^{1/2} \sim (1 \pm i)\sqrt{\frac{\beta}{2}\,|u|}. \tag{4.40}$$

We also observe that, since $\vartheta/2$ is odd and r is even, the real part of $(1-\cos u + \beta \sin u)^{1/2}$ is an even function of u, while its imaginary part is odd.

We shall need a more careful analysis of $(1-\cos u+\beta\,\sin u)^{1/2}$ near the origin. Equation (4.37) and the McLaurin's formula for the cosine yield

$$\begin{aligned} r^{1/2}(u) &= \Big[1+\beta^2-2+u^2-\frac{u^4}{12}+(1-\beta^2)\Big(1-\frac{u^2}{2}+\frac{u^4}{4!}\Big)^2+o(u^4)\Big]^{1/4} \\ &= \Big[\beta^2u^2+\Big(\frac{1-\beta^2}{3}-\frac{1}{12}\Big)u^4+o(u^4)\Big]^{1/4} \\ &= \sqrt{\beta|u|}+\frac{1}{4}\Big(\frac{1-\beta^2}{3}-\frac{1}{12}\Big)u^{5/2}+o(u^{5/2}). \end{aligned}$$

In a similar way equation (4.38) leads to

$$\sin\Big(\frac{\vartheta(u)}{2}\Big)=\frac{\sqrt{2}}{2}-\frac{u\sqrt{2}}{8\beta}+o(u).$$

We conclude that, for u positive we have

$$\operatorname{Im}(1-\cos u+i\beta\sin u)^{1/2}=u^{1/2}\sqrt{\frac{\beta}{2}}-u^{3/2}\,\frac{\sqrt{2}}{8\sqrt{\beta}}+o(u^{3/2}). \tag{4.41}$$

An analogous computation implies

$$\operatorname{Re}(1-\cos u+i\beta\sin u)^{1/2}=u^{1/2}\sqrt{\frac{\beta}{2}}+u^{3/2}\,\frac{\sqrt{2}}{8\sqrt{\beta}}+o(u^{3/2}). \tag{4.42}$$

In order to simplify the notation, we denote as $\varphi(u)$ the square root we just studied.

Now we go back to the analysis of the integral $I(n,t)$, defined in (4.35). Since $\mathscr{L}^{1/2}p_t$ is real, the imaginary part of $I(n,t)$ must vanish. We compute the real part of the integrand. Denote by η the function on $[-\pi,\pi]$, defined by

$$\eta(u):=\frac{q-\mathrm{e}^{2iu}}{q^2-\mathrm{e}^{iu}}.$$

A straightforward computation shows that

$$\begin{aligned} \operatorname{Re}\eta(u) &= \frac{q^3-q^2\cos(2u)-q\cos(2u)+1}{\big|q^2-\mathrm{e}^{2iu}\big|^2} \\ \operatorname{Im}\eta(u) &= -\sin(2u)\,\frac{q^2-q}{\big|q^2-\mathrm{e}^{2iu}\big|^2}. \end{aligned} \tag{4.43}$$

Therefore

$$I(n,t)=\int_{-\pi}^{\pi}\big(\operatorname{Re}\varphi+i\operatorname{Im}\varphi\big)\big(\operatorname{Re}\eta+i\operatorname{Im}\eta\big)\big(\cos\operatorname{Im}\Phi+i\sin\operatorname{Im}\Phi\big)\,\mathrm{e}^{\operatorname{Re}\Phi}\,\mathrm{d}u.$$

We denote by A the product of the first three factors in the integral above. Then

$$I(n,t)=\int_{-\pi}^{\pi}\operatorname{Re}A(u)\,\mathrm{e}^{\operatorname{Re}\Phi}\,\mathrm{d}u.$$

The only part of the integral which matters is a small neighbourhood of the origin. Indeed, denote by ψ an even and smooth cutoff function, which is supported in the interval $[-\varepsilon,\varepsilon]$ (see the lines immediately above equation (4.49), where we impose conditions on ε), it is equal to 1 in $[-\varepsilon/2,\varepsilon/2]$ and such that $0\le\psi\le 1$. Then

$$I(n,t)=I^{\psi}(n,t)+I^{1-\psi}(n,t),$$

where

$$I^{\psi}(n,t)=\int_{-\pi}^{\pi}\psi(u)\;\operatorname{Re}A(u)\,\mathrm{e}^{\operatorname{Re}\Phi}\,\mathrm{d}u$$

and

$$I^{1-\psi}(n,t)=\int_{-\pi}^{\pi}\big(1-\psi(u)\big)\;\operatorname{Re}A(u)\,\mathrm{e}^{\operatorname{Re}\Phi}\,\mathrm{d}u.$$

We shall estimate I^{ψ} and $I^{1-\psi}$ separately.

First we consider $I^{1-\psi}$. Observe that

$$\big|I^{1-\psi}(n,t)\big|\le\big\|\operatorname{Re}A\big\|_{\infty}\int_{[-\pi,-\varepsilon/2]\cup[\varepsilon/2,\pi]}\mathrm{e}^{\operatorname{Re}\Phi}\,\mathrm{d}u.$$

Notice that if $\varepsilon/2\le|u|\le\pi$, then (4.40) implies that there exists a positive constant c such that

$$\operatorname{Re}\Phi(u)=-t\;\operatorname{Re}\varphi\le -c\,\varepsilon^{1/2}\,t, \tag{4.44}$$

whence

$$\big|I^{1-\psi}(n,t)\big|\le 2\pi\;\big\|\operatorname{Re}A\big\|_{\infty}\,\mathrm{e}^{-c't},\qquad\forall n\in\mathbb{N}. \tag{4.45}$$

Next we estimate $I^{\psi}(n,t)$. Observe that

$$\begin{aligned}\operatorname{Re}A(u)=\operatorname{Re}\varphi\;\operatorname{Re}\eta\;\cos(\operatorname{Im}\Phi)&\\ -\operatorname{Im}\varphi\;\operatorname{Im}\eta\;\cos(\operatorname{Im}\Phi)&\\ -\operatorname{Re}\varphi\;\operatorname{Im}\eta\;\sin(\operatorname{Im}\Phi)&\\ -\operatorname{Im}\varphi\;\operatorname{Re}\eta\;\sin(\operatorname{Im}\Phi).&\end{aligned} \tag{4.46}$$

Correspondingly, we write

$$I^\psi = I_1^\psi - I_2^\psi - I_3^\psi - I_4^\psi$$

where

$$I_1^\psi(n,t) = \int_{-\pi}^{\pi} \psi(u)\ \mathrm{Re}\,\varphi\ \mathrm{Re}\,\eta\ \cos(\mathrm{Im}\,\Phi)\,\mathrm{e}^{\mathrm{Re}\,\Phi}\,\mathrm{d}u,$$

$$I_2^\psi(n,t) = \int_{-\pi}^{\pi} \psi(u)\ \mathrm{Im}\,\varphi\ \ \mathrm{Im}\,\eta\ \ \cos(\mathrm{Im}\,\Phi)\,\mathrm{e}^{\mathrm{Re}\,\Phi}\,\mathrm{d}u,$$

$$I_3^\psi(n,t) = \int_{-\pi}^{\pi} \psi(u)\ \mathrm{Re}\,\varphi\ \ \mathrm{Im}\,\eta\ \ \sin(\mathrm{Im}\,\Phi)\,\mathrm{e}^{\mathrm{Re}\,\Phi}\,\mathrm{d}u,$$

and

$$I_4^\psi(n,t) = \int_{-\pi}^{\pi} \psi(u)\ \mathrm{Im}\,\varphi\ \ \mathrm{Re}\,\eta\ \ \sin(\mathrm{Im}\,\Phi)\,\mathrm{e}^{\mathrm{Re}\,\Phi}\,\mathrm{d}u.$$

Observe that all the integrands above are even, so we can substitute the integration from $-\pi$ to π with twice the integral from 0 to π. This fact allows us to use the expressions of r and ϑ for nonnegative u. Recall that (4.40) implies

$$\mathrm{Re}\,\varphi \asymp u^{1/2}, \quad \mathrm{Im}\,\varphi \asymp u^{1/2} \qquad \forall u \in \mathrm{supp}(\psi).$$

Also observe that

$$\mathrm{Im}\,\eta(u) \asymp u \qquad\qquad \forall u \in \mathrm{supp}(\psi).$$

Therefore

$$\begin{aligned} |I_2^\psi(n,t)| &\le C \int_0^\varepsilon u^{3/2}\,\mathrm{e}^{-ct\sqrt{u}}\ \mathrm{d}u \\ &= C \int_0^{\varepsilon^{1/2}t} \frac{v^6}{t^3}\,\mathrm{e}^{-cv}\,\frac{2v\ \mathrm{d}v}{t^2} \\ &\le C\,t^{-5} \int_0^\infty v^7\,e^{-cv}\ \mathrm{d}v \\ &\le C\,t^{-5} \qquad \forall t \in [1,\infty) \quad \forall n \in \mathbb{N}. \end{aligned}$$

We have made the change of variables $v = t\sqrt{u}$ in the first integral above.

By arguing similarly, we may show that

$$|I_3^\psi| \le C\,t^{-5} \qquad \forall t \in [1,\infty).$$

It remains to estimate $I_1^\psi(n,t)$ and $I_4^\psi(n,t)$. We notice that $\operatorname{Re}\eta(0) \neq 0$, and write

$$\operatorname{Re}\eta = \big[\operatorname{Re}\eta - \operatorname{Re}\eta(0)\big] + \operatorname{Re}\eta(0).$$

Correspondingly, we write

$$\begin{aligned} I_1^\psi(n,t) = 2 \int_0^\pi \psi(u)\, \operatorname{Re}\varphi \big[\operatorname{Re}\eta - \operatorname{Re}\eta(0)\big]\, \cos(\operatorname{Im}\Phi)\, \mathrm{e}^{\operatorname{Re}\Phi}\, \mathrm{d}u \\ + 2\operatorname{Re}\eta(0) \int_0^\pi \psi(u)\, \operatorname{Re}\varphi\, \cos(\operatorname{Im}\Phi)\, \mathrm{e}^{\operatorname{Re}\Phi}\, \mathrm{d}u, \end{aligned} \tag{4.47}$$

and estimate the two integrals on the right hand side separately.

Since $\operatorname{Re}\eta$ is even and smooth, $\operatorname{Re}\eta - \operatorname{Re}\eta(0)$ vanishes at the origin at least of order 2, whence the absolute value of the first integral may be majorised by

$$C\, \big\|\operatorname{Re}\eta''\big\|_\infty \int_0^\varepsilon u^{5/2}\, e^{-ct\sqrt{u}}\, \mathrm{d}u \leq C\, t^{-7} \qquad \forall t \in [1,\infty) \quad \forall n \in \mathbb{N}.$$

The very same reasoning can be applied to I_4^ψ, leaving us with the following integral

$$J(n,t) = 2\operatorname{Re}\eta(0) \int_0^\pi \psi(u)\, \big[\operatorname{Re}\varphi\, \cos(\operatorname{Im}\Phi) - \operatorname{Im}\varphi\, \sin(\operatorname{Im}\Phi)\big]\, e^{\operatorname{Re}\Phi}\, \mathrm{d}u, \tag{4.48}$$

which includes the remainder terms of both I_1^ψ and I_4^ψ.

Write the term in the square brackets in the above integral as

$$[\operatorname{Re}\varphi - \operatorname{Im}\varphi]\cos(\operatorname{Im}\Phi) + \operatorname{Im}\varphi[\cos(\operatorname{Im}\Phi) - \sin(\operatorname{Im}\Phi)],$$

and write accordingly $J = J_1 + J_2$. Consider first

$$J_1(n,t) = 2\operatorname{Re}\eta(0) \int_0^\pi \psi(u)\, \big[(\operatorname{Re}\varphi - \operatorname{Im}\varphi)\cos(\operatorname{Im}\Phi)\big]\, e^{\operatorname{Re}\Phi}\, \mathrm{d}u.$$

The asymptotic expressions (4.41) and (4.42) imply that

$$|\operatorname{Re}\varphi - \operatorname{Im}\varphi| \leq C\, u^{3/2} \qquad \forall u \in \operatorname{supp}(\psi).$$

Thus

$$\begin{aligned} |J_1(n,t)| &\leq C \int_0^\varepsilon u^{3/2} e^{-ct\sqrt{u}}\, \mathrm{d}u \\ &\leq C\, t^{-5} \qquad \forall t \in [1,\infty) \quad \forall n \in \mathbb{N}. \end{aligned}$$

In the remaining part we write

$$\operatorname{Im}\varphi = \Big(\operatorname{Im}\varphi - u^{1/2}\sqrt{\frac{\beta}{2}}\Big) + u^{1/2}\sqrt{\frac{\beta}{2}},$$

and we split J_2 into $J_3 + J_4$ accordingly. Using the asymptotic expression (4.41) it is possible to prove that J_3, i.e. the integral corresponding to $\big(\operatorname{Im}\varphi - u^{1/2}\sqrt{\beta/2}\big)$ is controlled by a constant multiple of t^{-5}.

We are left with the last integral, i.e.

$$J_4(n,t) = 2\operatorname{Re}\eta(0)\sqrt{\frac{\beta}{2}}\int_0^\pi \psi(u)\,u^{1/2}\,[\cos(\operatorname{Im}\Phi) - \sin(\operatorname{Im}\Phi)]\,e^{\operatorname{Re}\Phi}\,du.$$

We *claim* that there exist positive constants c and λ, such that

$$J_4\big(n,t(n)\big) \geq c\,(1+n)^{-3/2},$$

where $t(n) = \lambda\sqrt{n}$ for each nonnegative integer n.
Taking this for granted, we have that

$$\begin{aligned}\sup_{t>0} t\,\big|\mathscr{L}^{1/2}p_t(x)\big| &= \sup_{t>0} t\,\frac{q^{1-|x|}}{2\pi(q+1)}\,I(|x|,t)\\ &\geq c'\,q^{-|x|}\,t(|x|)\,J_4\big(|x|,t(|x|)\big)\\ &\geq c'\frac{q^{-|x|}}{1+|x|} \qquad \forall x\in\mathscr{T}.\end{aligned}$$

This pointwise estimate, in turn, implies that

$$\begin{aligned}\Big\|\sup_{t>0} t\,\big|\mathscr{L}^{1/2}p_t\big|\Big\|_1 &= \sum_{x\in\mathscr{T}}\sup_{t>0} t\,\big|\mathscr{L}^{1/2}p_t(x)\big|\\ &\geq c'\sum_{x\in\mathscr{T}}\frac{q^{-|x|}}{1+|x|}\\ &= c\sum_{n\in\mathbb{N}}(1+n)^{-1}\\ &= \infty,\end{aligned}$$

as required to complete the proof of the theorem.

In order to estimate J_4, we need a preliminary observation. Write

$$\operatorname{Im}\varphi = u^{1/2}\sqrt{\frac{\beta}{2}}\Big(\frac{\operatorname{Im}\varphi}{u^{1/2}\sqrt{\beta/2}}\Big).$$

The asymptotic expansion (4.41) yields

$$\frac{\operatorname{Im}\varphi}{u^{1/2}\sqrt{\beta/2}} = 1 - \frac{1}{4\beta}s + o(s).$$

This equation implies the existence of the limit for $u \to 0^+$ of the derivative

$$\frac{\mathrm{d}}{\mathrm{d}u}\frac{\operatorname{Im}\varphi}{u^{1/2}\sqrt{\beta/2}},$$

and this limit is equal to $-(4\beta)^{-1}$.
This fact allows us to choose ε small enough, so that $\frac{\mathrm{d}}{\mathrm{d}u}\frac{\operatorname{Im}\varphi}{u^{1/2}\sqrt{\beta/2}} < 0$ on $[0,\varepsilon]$, and then to use McLaurin's formula with Lagrange form of the remainder on $[0,\varepsilon]$. Thus

$$\operatorname{Im}\varphi = u^{1/2}\sqrt{\frac{\beta}{2}} + R_1 u^{3/2}, \tag{4.49}$$

where R_1 is a negative constant.
In the same way we also have

$$\operatorname{Re}\varphi = u^{1/2}\sqrt{\frac{\beta}{2}} + R_2 u^{3/2}, \tag{4.50}$$

and this time R_2 is positive.

Using these equations in the expression of Φ leads to

$$\operatorname{Re}\Phi = -t\left(u^{1/2}\sqrt{\frac{\beta}{2}} + R_2\, u^{3/2}\right) \tag{4.51}$$

and

$$\operatorname{Im}\Phi = -t\left(u^{1/2}\sqrt{\frac{\beta}{2}} + R_1\, u^{3/2}\right) + nu. \tag{4.52}$$

Now we go back to the estimate of J_4. Let λ be a positive real number, and choose $t = \lambda\sqrt{\beta/2}\,\sqrt{n}$. We denote as $F(n,\lambda)$ the integral $J_4(n,t)$ for this particular choice of t. Equations (4.51) and (4.52) yield

$$F(n,\lambda) = C\int_0^\pi \psi(u)\,u^{1/2}\,\big[\cos(-\lambda\sqrt{nu} + \lambda c_1\sqrt{n}u^{3/2} + nu) \\ - \sin(-\lambda\sqrt{nu} + \lambda c_1\sqrt{n}u^{3/2} + nu)\big]\,\exp(-\lambda\sqrt{nu} - \lambda c_2\sqrt{n}u^{3/2})\,\mathrm{d}u,$$

where c_1 and c_2 are two positive constants incorporating $\sqrt{\beta/2}$ and R_1 or R_2 respectively. We change variable $v = \sqrt{nu}$, so that we have

$$F(n,\lambda) = \frac{C}{n^{3/2}}\int_0^{\sqrt{n\pi}} \psi\Big(\frac{v^2}{n}\Big)\,v^2\,\Big[\cos\Big(\lambda c_1\frac{v^3}{n} - \lambda v + v^2\Big) \\ - \sin\Big(\lambda c_1\frac{v^3}{n} - \lambda v + v^2\Big)\Big]\,\exp\Big(-\lambda v - \lambda c_2\frac{v^3}{n}\Big)\,\mathrm{d}v.$$

The function $g(v) = 2v^2\,e^{-\lambda v}$ is an integrable majorant of the integrand, which, in turn, converges pointwise to the function

$$v \mapsto v^2\,e^{-\lambda v}\,[\cos(v^2 - \lambda v) - \sin(v^2 - \lambda v)] \qquad \forall v \in [0,\infty).$$

Consequently,

$$F(n,\lambda) \sim \frac{C}{n^{3/2}}\,G(\lambda),$$

where

$$G(\lambda) = \int_0^\infty v^2\,e^{-\lambda v}\,[\cos(v^2 - \lambda v) - \sin(v^2 - \lambda v)]\,\mathrm{d}v \qquad \forall\lambda \in (0,\infty).$$

It remains to show that G is not identically zero.

First of all we write

$$\begin{aligned}\frac{\sqrt{2}}{2}\,G(\lambda) &= \int_0^\infty v^2\,e^{-\lambda v}\,\cos\Big(v^2 - \lambda v + \frac{\pi}{4}\Big)\,\mathrm{d}v \\ &= \mathrm{Re}\int_0^\infty v^2\,e^{-\lambda v}\,\exp\Big[i\Big(v^2 - \lambda v + \frac{\pi}{4}\Big)\Big]\,\mathrm{d}v \qquad (4.53) \\ &= \mathrm{Re}\Big[e^{i\pi/4}\int_0^\infty v^2\,e^{iv^2}\,e^{-\lambda(1+i)v}\,\mathrm{d}v\Big]\end{aligned}$$

Now consider the auxiliary holomorphic function

$$f(z) = z^2\, e^{iz^2}\, e^{-\lambda(1+i)z} \qquad \forall z \in \mathbb{C},$$

and the circuit γ_R consisting in the real segment $[0, R]$, the circular arc $\{R\, e^{i\theta} \colon \theta \in [0, \pi/4]\}$ and the segment $\{s\, e^{i\pi/4} \colon s \in [0, R]\}$. Cauchy integral theorem then implies

$$\begin{aligned} 0 &= \int_{\gamma_R} f(z)\, \mathrm{d}z \\ &= \int_0^R f(s)\ \mathrm{d}s + \int_0^{\pi/4} f(Re^{i\theta})\, Re^{i\theta}\, i\ \mathrm{d}\theta - \int_0^R f(se^{i\pi/4})\, e^{i\pi/4}\, \mathrm{d}s. \end{aligned} \tag{4.54}$$

We claim that the integral over the arc tends to zero as R tends to infinity. Indeed

$$\begin{aligned} f(Re^{i\theta}) &= R^2\, e^{2i\theta}\ \exp\big[iR^2\big(\cos(2\theta) + i\sin(2\theta)\big)\big] \\ &\qquad\qquad \times \exp\big[-\lambda R(1+i)(\cos\theta + i\sin\theta)\big] \\ &= R^2\ \exp\big[i\big(2\theta + R^2\cos(2\theta) - \lambda R(\cos\theta + \sin\theta)\big)\big] \\ &\qquad\qquad \times \exp\big[-R^2\sin(2\theta) - \lambda R(\cos\theta - \sin\theta)\big]. \end{aligned}$$

So

$$|f(Re^{i\theta})\, Re^{i\theta}| \le \begin{cases} R^3\ \exp\big(-\lambda\, R\,(\sqrt{3} - 1)/2\big) & \text{if } \theta \in [0, \pi/6] \\ R^3\exp(-R^2/2) & \text{if } \theta \in (\pi/6, \pi/4]. \end{cases}$$

This inequality implies that the second integral in the last line of equation (4.54) tends to 0 as R tends to infinity. This fact implies that

$$\begin{aligned} \int_0^\infty f(s)\ \mathrm{d}s &= \int_0^\infty f(se^{i\pi/4})\, e^{i\pi/4}\ \mathrm{d}s \\ &= i\, e^{i\pi/4} \int_0^\infty s^2\, e^{-s^2}\, e^{-i\sqrt{2}\lambda s}\ \mathrm{d}s \\ &= i\, e^{i\pi/4} \Big(\int_0^\infty s^2\, e^{-s^2}\ \cos\big(\sqrt{2}\,\lambda s\big)\ \mathrm{d}s \\ &\qquad\qquad - i \int_0^\infty s^2\, e^{-s^2}\ \sin\big(\sqrt{2}\,\lambda s\big)\ \mathrm{d}s\Big). \end{aligned} \tag{4.55}$$

We note that the function

$$s \mapsto s^2\, e^{-s^2} \qquad \forall s \in \mathbb{R}$$

is even, so

$$\begin{aligned}\mathscr{F}\big((\cdot)^2\, e^{-(\cdot)^2}\big)(\xi) &= \int_{\mathbb{R}} s^2\, e^{-s^2}\, e^{-i\xi s}\, \mathrm{d}s \\ &= \int_{\mathbb{R}} s^2\, e^{-s^2}\, \cos(\xi s)\, \mathrm{d}s \\ &= 2\int_0^\infty s^2\, e^{-s^2}\, \cos(\xi s)\, \mathrm{d}s \qquad \forall \xi \in \mathbb{R}\end{aligned} \tag{4.56}$$

is, in turn, an even function in ξ. Since the Fourier transform is injective on $L^2(\mathbb{R})$, we conclude that there exists $\lambda > 0$ such that

$$\mathscr{F}\big((\cdot)^2\, e^{-(\cdot)^2}\big)(\sqrt{2}\,\lambda) \neq 0.$$

Equations (4.53), (4.55) and (4.56) then imply

$$\begin{aligned}G(\lambda) &= \sqrt{2}\ \mathrm{Re}\Big[-\Big(\frac{1}{2}\,\mathscr{F}\big((\cdot)^2\, e^{-(\cdot)^2}\big)(\sqrt{2}\,\lambda) + i\int_0^\infty s^2\, e^{-s^2}\, \sin(\sqrt{2}\,\lambda s)\, \mathrm{d}s\Big)\Big] \\ &= -\frac{\sqrt{2}}{2}\,\mathscr{F}\big((\cdot)^2\, e^{-(\cdot)^2}\big)(\sqrt{2}\,\lambda),\end{aligned}$$

so G is not identically zero. This concludes the proof of (iv) and of the theorem. □

We conclude this chapter with the following corollary of Theorems 4.1 and 4.7 (ii).

Corollary 4.11. *The heat maximal space $H^1_{\mathscr{H}}(\mathscr{T})$ is properly contained in the Poisson maximal space $H^1_{\mathscr{P}}(\mathscr{T})$.*

Proof. Observe that the containment is a straightforward consequence of the subordination formula (4.28). Indeed, changing variables, we see that

$$p_t = \frac{1}{\sqrt{\pi}}\int_0^\infty v^{-1/2}\, \mathrm{e}^{-v}\, h_{t^2/4v}\, \mathrm{d}v.$$

Therefore, for every f

$$\mathscr{P}_* f \leq \frac{1}{\sqrt{\pi}}\Big(\int_0^\infty v^{-1/2}\, \mathrm{e}^{-v}\, \mathrm{d}v\Big)\, \mathscr{H}_* f = \mathscr{H}_* f.$$

Hence $H^1_{\mathscr{H}}(\mathscr{T}) \subseteq H^1_{\mathscr{P}}(\mathscr{T})$.

If $c \in [0,1)$, Theorem 4.7 (ii) implies

$$H^1_{\mathscr{P}}(\mathscr{T}) \supset H^1_{\mathscr{P},c}(\mathscr{T}) \supset \mathfrak{X}^{1/2}(\mathscr{T}) \supset \mathfrak{X}^1(\mathscr{T}).$$

However, $\mathfrak{X}^1(\mathscr{T})$ is not contained in $H^1_{\mathscr{H}}(\mathscr{T})$ by Theorem 4.4, so that $H^1_{\mathscr{H}}(\mathscr{T})$ cannot possibly coincide with $H^1_{\mathscr{P}}(\mathscr{T})$. $\square$

Chapter 5

Spherical multipliers

5.1 More on the group of isometries of a tree

Recall that $\{\gamma_j : j \in \mathbb{Z}\}$ is a fixed two-sided geodesic such that $\gamma_0 = o$. We denote by σ an isometry of $\mathscr{T}$ that maps γ_i in γ_{i+1} for every i. Then, for j in $\mathbb{Z}$, σ^j is an isometry of $\mathscr{T}$ that maps γ_i to γ_{i+j}. The group G admits an Iwasawa-type decomposition $G = NAG_o$, investigated in [FTN, Ve]. Denote by A the subgroup of G generated by the one-step translation σ and by N the subgroup of G of all the elements that stabilises ω^+ and at least an element of $\mathscr{T}$. It is known that N can be characterised as the subgroup of G consisting in the elements that fix all the horocycles with respect to ω^+ [Ve, Lemma 3.1]. Moreover, the orbit of an element x of $\mathscr{T}$ under the action of N is the horocycle which contains x [Ve, Corollary 3.2]. It is well known that the group N is unimodular; we normalise its Haar measure μ so that $\mu(N \cap G_o) = 1$, as in [Ve, Lemma 3.3].

The analogy between G and semisimple Lie groups of rank one is apparent in the following theorem [Ve, Theorem 3.5].

Theorem 5.1. *Let G, N, G_o and σ be as above. Then for every g in G there exist n in N, j in $\mathbb{Z}$ and g_o in G_o such that $g = n\sigma^j g_o$. Furthermore, if f is a continuous compactly supported function on G, then*

$$\int_G f(g)\,\mathrm{d}g = \int_N \sum_{j\in\mathbb{Z}} q^{-j} \int_{G_o} f(n\sigma^j g_o)\,\mathrm{d}g_o\,\mathrm{d}\mu(n).$$

We remark that, contrary to what happens in the case of noncompact symmetric spaces, there is a lack of uniqueness in this Iwasawa-type decomposition. Indeed, if $g = n\sigma^j g_o = v\sigma^\ell h_o$, then $j = \ell$ and there exists n_o in $N \cap G_o$ such that $v = \sigma^j n_o \sigma^{-j}$ and $h_o = n_o^{-1} g_o$ (see [Ve, Remark 3.6]).

Going back to the tree, a vertex x is of the form $n\sigma^j \cdot o$, with n in N and j in $\mathbb{Z}$. It is straightforward to prove that the height of x (with respect to ω^+) is simply j. The next lemma establishes a relation between the height of a point and its distance from the origin, and may be seen as an analogue of [Io, Lemma 3].

Lemma 5.2. *For every n in N and for every j in $\mathbb{Z}$ such that $j \leq d(n\cdot o, o)$*

$$d(n\sigma^j \cdot o, o) = d(n \cdot o, o) - j.$$

In particular, this formula holds for every n in N and every nonpositive j in $\mathbb{Z}$.

Proof. Write x instead of $n\sigma^j \cdot o$, and denote by γ_ℓ the confluence point of $[x, \omega^+)$ in ω, i.e. $[\gamma_\ell, \omega_+) = [x, \omega_+) \cap \omega$ (see also [CMS1, pag. 6]). Note that by definition γ_ℓ lies on $[x, \omega^+)$, so $\ell \geq j$. We observe that such γ_ℓ exists, because, by the definition of N, every element of this group fixes a geodesic ray equivalent to $[\gamma_j, \omega_+)$.

On a tree the union of two geodesic segments with one extreme in common (but no other point) is again a geodesic segment, so

$$d(x, o) = d(x, \gamma_\ell) + d(\gamma_\ell, o). \tag{5.1}$$

Note that $d(\gamma_\ell, o)$ is the absolute value $|\ell|$, as o lies on the geodesic γ. Moreover, $d(x, \gamma_\ell)$ is always equal to $\ell - j$, as we already noted that $\ell \geq j$.

Now we consider the cases where $\ell \leq 0$ or $\ell > 0$ separately. If $\ell \leq 0$, then n fixes the origin and (5.1) reads

$$d(x, o) = (\ell - j) - \ell = -j = d(n \cdot o, o) - j.$$

Otherwise $\ell > 0$, and we have $d(n \cdot o, o) = 2\ell$, because $n \cdot o$ belongs to the same horocycle as o. Hence (5.1) becomes

$$d(x, o) = (\ell - j) + \ell = 2\ell - j = d(n \cdot o, o) - j.$$

We conclude that the required formula is true for every j in $\mathbb{Z}$. $\square$

This will be used in Section 5.3.

For p in $[1, \infty)$, we denote by $Q_p : N \to \mathbb{R}$ the function defined by

$$Q_p(n) = q^{-|n \cdot o|/p}. \tag{5.2}$$

Lemma 5.3. *Suppose that p is in $[1, 2)$. Then the function $n \mapsto |n \cdot o|^{\ell}\, Q_p(n)$ belongs to $L^1(N)$ for each nonnegative integer ℓ.*

Proof. We shall prove the result in the case where $\ell = 0$. The proof in the case $\ell \geq 1$ is similar, and is omitted.

For any nonnegative integer r, we set $T_r := \{v \in N : v \cdot o \in S_r(o)\}$. By [Ve, Lemma 3.11], $\mu(T_r)$ vanishes if r is odd, is equal to 1 if $r = 0$, and is equal to $q^{r/2}$ if r is even and nonzero. Then

$$\begin{aligned}
\int_N q^{-|n \cdot o|/p}\, \mathrm{d}\mu(n) &= \sum_{r \geq 1} q^{-r/p} \mu(T_r) + 1 \\
&= \sum_{j \geq 1} q^{-2j/p} q^{2j/2} + 1 \\
&= \sum_{j \geq 1} q^{j(1-2/p)} + 1,
\end{aligned}$$

which is convergent, because $1 \leq p < 2$, as required. $\square$

Finally we observe that this Iwasawa-type decomposition is consistent with the convolution on $\mathscr{T}$.

Let N and A be the subgroups of G defined above, and consider the semi-direct product NA, where A acts on N by conjugation. By [Ve, Lemma 3.8] the modular function Δ_{NA} of NA is given by

$$\Delta_{NA}(n\sigma^j) = q^{-j} \qquad \forall n \in N \quad \forall j \in \mathbb{Z}. \tag{5.3}$$

By [Ve, Theorem 3.5], we may also identify the convolution between a G_o–right-invariant and G_o–bi-invariant functions on G, with the convolution of the corresponding functions on the group NA. Explicitly, suppose that f is a G_o–right invariant function and that k is a G_o–bi-invariant function on G. Then

$$\begin{aligned} f *_G k(v\sigma^j g_0) &= \int_N \sum_{\ell\in\mathbb{Z}} q^{-\ell} \int_{G_o} f(n\sigma^\ell h_o)\, k(h_o^{-1}\sigma^{-\ell} n^{-1} n\sigma^j g_o)\, \mathrm{d}h_o\, \mathrm{d}\mu(n) \\ &= \int_N \sum_{\ell\in\mathbb{Z}} q^{-\ell} f(n\sigma^\ell)\, k(\sigma^{-\ell} n^{-1} v\sigma^j)\, \mathrm{d}\mu(n) \\ &= \int_N \sum_{\ell\in\mathbb{Z}} \Delta_{NA}(n\sigma^\ell)\, f(n\sigma^\ell)\, k(\sigma^{-\ell} n^{-1} v\sigma^j)\, \mathrm{d}\mu(n) \\ &= f *_{NA} k(v\sigma^j), \end{aligned}$$

where we have used the G_o–invariance of f the G_o–bi-invariance of k, and the fact that G_o has total mass 1. By [Ve, Theorem 3.5], the norms of k in $Cv_p(G)$ and in $Cv_p(NA)$ coincide.

5.2 A general transference principle

In this section we assume that the locally compact group Γ is the semi-direct product of two groups $\mathscr{M}$ and $\mathscr{H}$, where $\mathscr{M}$ is normal in Γ and $\mathscr{H}$ acts on $\mathscr{M}$ by conjugation. Right Haar measures on $\mathscr{M}$ and $\mathscr{H}$ will be denoted by $\mathrm{d}n$ and $\mathrm{d}h$, respectively. Then $\mathrm{d}g = \mathrm{d}n\,\mathrm{d}h$ is a right Haar measure on Γ. We denote by $\Delta_{\mathscr{M}}$ and $\Delta_{\mathscr{H}}$ the modular functions of $\mathscr{M}$ and $\mathscr{H}$, respectively, so that $\mathrm{d}\lambda(n) = \Delta_{\mathscr{M}}(n)\,\mathrm{d}n$ and

$\mathrm{d}\lambda(h) = \Delta_{\mathscr{H}}(h)\,\mathrm{d}h$ are left Haar measures on $\mathscr{M}$ and $\mathscr{H}$ respectively. Note that there is a slight abuse of notation here, for λ denotes a both a left invariant measure on $\mathscr{H}$ and a left Haar measure on $\mathscr{N}$.

For h in $\mathscr{H}$ and n in $\mathscr{M}$, denote by n^h the conjugate hnh^{-1}. Denote by $\mathscr{D}(h)^{-1}$ the Radon–Nykodim derivative $\mathrm{d}(n^h)/\,\mathrm{d}n$. It is not hard to check that $\mathscr{D}$ is an homomorphism of $\mathscr{H}$, i.e. $\mathscr{D}(hh_1) = \mathscr{D}(h)\,\mathscr{D}(h_1)$ for every h and h_1 in $\mathscr{H}$.

Remark 5.4. Observe that

$$\mathscr{D}(h)^{-1} = \mathrm{d}\lambda(n^h)/\,\mathrm{d}\lambda(n).$$

Indeed, note that the conjugation by h commutes with the inversion on $\mathscr{M}$, i.e. $(n^h)^{-1} = (n^{-1})^h$. Hence

$$\int_{\mathscr{M}} f(n^h)\,\mathrm{d}\lambda(n) = \int_{\mathscr{M}} f\big(\big((n^{-1})^h\big)^{-1}\big)\,\mathrm{d}\lambda(n).$$

Now note that the inversion in $\mathscr{M}$ transforms the left Haar measure to the right Haar measures, and conversely. Thus,

$$\begin{aligned}\int_{\mathscr{M}} f\big(\big(n^{-1})^h\big)^{-1}\big)\,\mathrm{d}\lambda(n) &= \int_{\mathscr{M}} f\big((v^h)^{-1}\big)\,\mathrm{d}v \\ &= \mathscr{D}(h)\int_{\mathscr{M}} f(n^{-1})\,\mathrm{d}n \\ &= \mathscr{D}(h)\int_{\mathscr{M}} f(v)\,\mathrm{d}\lambda(v).\end{aligned}$$

This fact will be used repeatedly in the proof of Theorem 5.6.

Remark 5.5. Observe that $\mathscr{D}$ may be extended to a homomorphism on the whole group Γ, by setting $\mathscr{D}(nh) := \mathscr{D}(h)$ for all n in $\mathscr{M}$ and h in $\mathscr{H}$. Recall that $(nh)(n_1h_1) = (nn_1)^h hh_1$. Thus,

$$\begin{aligned}\mathscr{D}\big((nh)(n_1h_1)\big) &= \mathscr{D}\big((nn_1)^h hh_1\big) = \mathscr{D}(hh_1) = \mathscr{D}(h)\,\mathscr{D}(h_1) \\ &= \mathscr{D}(nh)\,\mathscr{D}(n_1h_1).\end{aligned}$$

This observation applies to any homomorphism of $\mathscr{H}$.

It is well known that $\mathscr{D}(h)\,\Delta_{\mathscr{M}}(n)\,\Delta_{\mathscr{H}}(h)\,\mathrm{d}n\,\mathrm{d}h$ is a left Haar measure on Γ (see [HR, p. 211]), and that the following integral formulae hold

$$\begin{aligned}\int_\Gamma f(g)\,\mathrm{d}\lambda(g) &= \int_{\mathscr{M}}\int_{\mathscr{H}} f(nh)\,\mathscr{D}(h)\,\Delta_{\mathscr{M}}(n)\,\Delta_{\mathscr{H}}(h)\,\mathrm{d}n\,\mathrm{d}h\\ &= \int_{\mathscr{H}}\int_{\mathscr{M}} f(hn)\,\mathrm{d}n\,\mathrm{d}h.\end{aligned}$$

The space $L^1(\mathscr{M};Cv_p(\mathscr{H}))$ is the set of all distributions k on Γ such that for (almost) every n in $\mathscr{M}$ the distribution $k(n\cdot)$ induces a bounded convolution operator on $L^p(\mathscr{H})$, and the function $n \mapsto \|k(n\cdot)\|_{Cv_p(\mathscr{H})}$ is in $L^1(\mathscr{M})$. The space $L^1(\mathscr{M};Cv_p(\mathscr{H}))$ is endowed with the norm

$$\|k\|_{L^1(\mathscr{M};Cv_p(\mathscr{H}))} := \int_{\mathscr{M}} \|k(n\cdot)\|_{Cv_p(\mathscr{H})}\,\mathrm{d}\lambda(n). \tag{5.4}$$

Theorem 5.6. *Suppose that p is in $(1,\infty)$ and that $\Delta_{\mathscr{M}}^{-1/p'}\,k$ belongs to $L^1(\mathscr{M};Cv_p(\mathscr{H}))$. Then the operator $f\mapsto f*(\mathscr{D}^{-1/p}\,k)$ is bounded on $L^p(\Gamma)$, and*

$$\|f*(\mathscr{D}^{-1/p}\,k)\|_{L^p(\Gamma)} \le \|f\|_{L^p(\Gamma)}\,\|\Delta_{\mathscr{M}}^{-1/p'}\,k\|_{L^1(\mathscr{M};Cv_p(\mathscr{H}))}\,.$$

Notice that

$$\|\Delta_{\mathscr{M}}^{-1/p'}\,k\|_{L^1(\mathscr{M};Cv_p(\mathscr{H}))} = \int_{\mathscr{M}} \|k(n\cdot)\|_{Cv_p(\mathscr{H})}\,\Delta_{\mathscr{M}}(n)^{-1/p'}\,\mathrm{d}\lambda(n).$$

Proof. Notice that $(nh)^{-1}n_1h_1 = h^{-1}n^{-1}n_1h_1 = (n^{-1}n_1)^{h^{-1}}h^{-1}h_1$. Thus,

$$\begin{aligned}&f*(\mathscr{D}^{-1/p}\,k)(n_1h_1)\\ &= \int_{\mathscr{M}}\int_{\mathscr{H}} f(nh)\,\mathscr{D}^{-1/p}(h^{-1}h)\,k\big((nh)^{-1}n_1h_1\big)\,\mathscr{D}(h)\,\mathrm{d}\lambda(n)\,\mathrm{d}\lambda(h)\\ &= \int_{\mathscr{M}}\int_{\mathscr{H}} f(nh)\,\mathscr{D}^{-1/p}(h^{-1}h)\,k\big((n^{-1}n_1)^{h^{-1}}h^{-1}h_1\big)\,\mathscr{D}(h)\,\mathrm{d}\lambda(n)\,\mathrm{d}\lambda(h).\end{aligned}$$

We change variables ($(n^{-1}n_1)^{h^{-1}} = m^{-1}$) in the integral over $\mathscr{M}$.
Then $m^{-1} = h^{-1}n^{-1}n_1h$, so that $m = h^{-1}n_1^{-1}nh = (n_1^{-1}n)^{h^{-1}}$, and

$$\frac{\mathrm{d}\lambda(m)}{\mathrm{d}\lambda(n)} = \frac{\mathrm{d}\lambda\big((n_1^{-1}n)^{h^{-1}}\big)}{\mathrm{d}\lambda(n_1^{-1}n)}\,\frac{\mathrm{d}\lambda(n_1^{-1}n)}{\mathrm{d}\lambda(n)} = \mathscr{D}(h).$$

The second equality follows from the fact that $\mathscr{D}$ is a homomorphism (whence $\mathscr{D}(h^{-1})^{-1} = \mathscr{D}(h)$), and from the left invariance of λ.
We conclude that $\mathrm{d}\lambda(n) = \mathscr{D}^{-1}(h)\,\mathrm{d}\lambda(m)$, whence

$$\begin{aligned} & f * (\mathscr{D}^{-1/p}\, k)(n_1 h_1) \\ &= \int_{\mathscr{M}} \mathrm{d}\lambda(m) \int_{\mathscr{H}} f(n_1 m^h h)\, \mathscr{D}^{-1/p}(h^{-1}h_1)\; k\big(m^{-1}h^{-1}h_1\big)\, \mathrm{d}\lambda(h). \end{aligned}$$

We set $U(n_1, m, h) := f(n_1 m^h h)$, and view the inner integral as the convolution on $\mathscr{H}$ between $U(n_1, m, \cdot)$ and $\mathscr{D}^{-1/p}(\cdot)\, k(m^{-1}\cdot)$, evaluated at the point h_1. Therefore

$$\begin{aligned} & \big\| f * (\mathscr{D}^{-1/p}\, k)\big\|_{L^p(\Gamma)} \\ &= \Big(\int_{\mathscr{M}} \int_{\mathscr{H}} |f * (\mathscr{D}^{-1/p}\, k)(n_1 h_1)|^p\, \mathscr{D}(h_1)\, \mathrm{d}\lambda(h_1)\, \mathrm{d}\lambda(n_1)\Big)^{1/p} \\ &= \Big\| \Big\| \int_{\mathscr{M}} [U(n_1, m, \cdot) *_{\mathscr{H}} (\mathscr{D}^{-1/p}\, k)(m^{-1}\cdot)](h_1)\, \mathscr{D}^{1/p}(h_1)\, \mathrm{d}\lambda(m) \Big\|_{L^p(\mathscr{H})} \Big\|_{L^p(\mathscr{M})} \end{aligned}$$

where the $L^p(\mathscr{M})$ norm is taken with respect to the left Haar measure of $\mathscr{M}$ and the variable n_1. Observe that the argument of the integral over $\mathscr{M}$ above may be written as

$$\int_{\mathscr{H}} U(n_1, m, h)\, \mathscr{D}^{-1/p}(h^{-1}h_1)\, k(m^{-1}h^{-1}h_1)\, \mathscr{D}^{1/p}(h_1)\, \mathrm{d}h.$$

Since $\mathscr{D}$ is an homomorphism, this simplifies to

$$\begin{aligned} & \int_{\mathscr{H}} U(n_1, m, h)\, \mathscr{D}^{1/p}(h)\, k(m^{-1}h^{-1}h_1)\, \mathrm{d}h \\ &= \big[\big(\mathscr{D}^{1/p}(\cdot)\, U(n_1, m, \cdot)\big) *_{\mathscr{H}} k(m^{-1}\cdot)\big](h_1). \end{aligned}$$

Therefore, by Minkowski's integral inequality,

$$\begin{aligned} & \big\| f * (\mathscr{D}^{-1/p}\, k)\big\|_{L^p(\Gamma)} \\ &\leq \int_{\mathscr{M}} \mathrm{d}\lambda(m)\; \Big\| \big\| \big(\mathscr{D}^{1/p}(\cdot)\, U(n_1, m, \cdot)\big) *_{\mathscr{H}} k(m^{-1}\cdot) \big\|_{L^p(\mathscr{H})} \Big\|_{L^p(\mathscr{M})}\,. \end{aligned} \tag{5.5}$$

By assumption, for every m in $\mathscr{M}$ the function $k(m^{-1}\cdot)$ is in $Cv_p(\mathscr{H})$, so that

$$\begin{aligned} & \big\| \big(\mathscr{D}^{1/p}(\cdot)\, U(n_1, m, \cdot)\big) *_{\mathscr{H}} k(m^{-1}\cdot) \big\|_{L^p(\mathscr{H})} \\ &\leq \big\| \mathscr{D}^{1/p}(\cdot)\, U(n_1, m, \cdot) \big\|_{L^p(\mathscr{H})}\; \big\| k(m^{-1}\cdot) \big\|_{Cv_p(\mathscr{H})}\,, \end{aligned}$$

and

$$\Big\| \big\| \big(\mathscr{D}^{1/p}(\cdot)\, U(n_1, m, \cdot)\big) *_{\mathscr{H}} k(m^{-1}\cdot) \big\|_{L^p(\mathscr{H})} \Big\|_{L^p(\mathscr{M})} \leq \Big\| \big\| \mathscr{D}^{1/p}(\cdot)\, U(n_1, m, \cdot) \big\|_{L^p(\mathscr{H})} \Big\|_{L^p(\mathscr{M})} \, \big\| k(m^{-1}\cdot) \big\|_{Cv_p(\mathscr{H})} .$$

Observe that

$$\Big\| \big\| \mathscr{D}^{1/p}(\cdot)\, U(n_1, m, \cdot) \big\|_{L^p(\mathscr{H})} \Big\|_{L^p(\mathscr{M})} = \Big[\int_{\mathscr{M}} \int_{\mathscr{H}} \big| f(n_1 m^h h) \big|^p \, \mathscr{D}(h) \, \mathrm{d}\lambda(n_1) \, \mathrm{d}\lambda(h) \Big]^{1/p}.$$

We change variables ($n_1 m^h = n$) in the inner integral, write $n_1 m^h = \big(n_1^{h^{-1}} m\big)^h$, and observe that

$$\begin{aligned} \frac{\mathrm{d}\lambda(n)}{\mathrm{d}\lambda(n_1)} &= \frac{\mathrm{d}\lambda\big(\big(n_1^{h^{-1}} m\big)^h\big)}{\mathrm{d}\lambda\big(n_1^{h^{-1}} m\big)} \, \frac{\mathrm{d}\lambda\big(n_1^{h^{-1}} m\big)}{\mathrm{d}\lambda\big(n_1^{h^{-1}}\big)} \, \frac{\mathrm{d}\lambda\big(n_1^{h^{-1}}\big)}{\mathrm{d}\lambda(n_1)} \\ &= \mathscr{D}(h)^{-1} \, \Delta_{\mathscr{M}}(m) \, \mathscr{D}(h^{-1})^{-1} \\ &= \Delta_{\mathscr{M}}(m), \end{aligned}$$

i.e., $\mathrm{d}\lambda(n_1) = \Delta_{\mathscr{M}}^{-1}(m) \, \mathrm{d}\lambda(n)$. Then

$$\Big\| \big\| \mathscr{D}^{1/p}(\cdot)\, U(n_1, m, \cdot) \big\|_{L^p(\mathscr{H})} \Big\|_{L^p(\mathscr{M})} = \Delta_{\mathscr{M}}^{-1/p}(m) \, \big\| f \big\|_{L^p(\Gamma)} .$$

By combining this and (5.5), we obtain that

$$\begin{aligned} \big\| f * k \big\|_{L^p(\Gamma)} &\leq \big\| f \big\|_{L^p(\Gamma)} \int_{\mathscr{M}} \big\| k(m^{-1}\cdot) \big\|_{Cv_p(\mathscr{H})} \, \Delta_{\mathscr{M}}^{-1/p}(m) \, \mathrm{d}\lambda(m) \\ &= \big\| f \big\|_{L^p(\Gamma)} \int_{\mathscr{M}} \big\| k(m\cdot) \big\|_{Cv_p(\mathscr{H})} \, \Delta_{\mathscr{M}}^{1/p}(m) \, \mathrm{d}m; \end{aligned} \tag{5.6}$$

the equality above is a consequence of the change of variables ($m^{-1} \mapsto m$), which transforms the left Haar measure into the right Haar measure. Finally,

$$\Delta_{\mathscr{M}}^{1/p}(m) \, \mathrm{d}m = \Delta_{\mathscr{M}}^{-1/p'}(m) \, \Delta_{\mathscr{M}}(m) \, \mathrm{d}m = \Delta_{\mathscr{M}}^{-1/p'}(m) \, \mathrm{d}\lambda(m),$$

which, together with (5.6), gives

$$\big\| f * k \big\|_{L^p(\Gamma)} \leq \big\| f \big\|_{L^p(\Gamma)} \, \big\| \Delta_{\mathscr{M}}^{-1/p'} \, k \big\|_{L^1(\mathscr{M}; Cv_p(\mathscr{H}))} ,$$

as required. □

We shall often apply Theorem 5.6 when either $\mathscr{M}$ is unimodular, or the action of $\mathscr{H}$ on $\mathscr{M}$ is trivial (i.e. when Γ is the direct product of $\mathscr{M}$ and $\mathscr{H}$). For the reader's convenience, we state the corresponding results in Corollaries (5.7) and (5.8) below.

Corollary 5.7. *Suppose that p is in $(1,\infty)$ and that $\mathscr{M}$ is unimodular. Assume that k belongs to $L^1(\mathscr{M};Cv_p(\mathscr{H}))$. Then the operator $f\mapsto f*(\mathscr{D}^{-1/p}\,k)$ is bounded on $L^p(\Gamma)$. Furthermore,*

$$\big\|f*(\mathscr{D}^{-1/p}\,k)\big\|_{L^p(\Gamma)}\leq \big\|f\big\|_{L^p(\Gamma)}\,\big\|k\big\|_{L^1(\mathscr{M};Cv_p(\mathscr{H}))}\,.$$

Corollary 5.8. *Suppose that p is in $(1,\infty)$ and that Γ is the direct product of two subgroups $\mathscr{M}$ and $\mathscr{H}$. Assume that $\Delta_{\mathscr{M}}^{-1/p'}\,k$ belongs to $L^1(\mathscr{M};Cv_p(\mathscr{H}))$. Then the operator $f\mapsto f*k$ is bounded on $L^p(\Gamma)$. Furthermore,*

$$\big\|f*k\big\|_{L^p(\Gamma)}\leq \big\|f\big\|_{L^p(\Gamma)}\,\big\|\Delta_{\mathscr{M}}^{-1/p'}\,k\big\|_{L^1(\mathscr{M};Cv_p(\mathscr{H}))}\,.$$

5.3 Spherical multipliers on a tree

We begin this section with some standard notations concerning harmonic analysis of $\mathbb{Z}$.

We denote by $\mathscr{F}$ the Fourier transformation on $\mathbb{Z}$, given by

$$\mathscr{F}F(s)=\sum_{d\in\mathbb{Z}}F(d)\,q^{-ids}\qquad\forall s\in\mathbb{T},\tag{5.7}$$

where $\mathbb{T}=\mathbb{R}/(\tau\mathbb{Z})$. We denote by $\mathscr{M}_p(\mathbb{T})$ the space of all (bounded) functions on $\mathbb{T}$ of the form $\mathscr{F}k$, where k is in $Cv_p(\mathbb{Z})$. The norm of a function $\mathscr{F}k$ in $\mathscr{M}_p(\mathbb{T})$ is then defined to be the norm of k in $Cv_p(\mathbb{Z})$.

The corresponding inversion formula is

$$F(d)=\frac{1}{\tau}\int_{\mathbb{T}}\mathscr{F}F(s)\,q^{ids}\;\mathrm{d}s\qquad\forall d\in\mathbb{Z}.$$

Clearly $\mathscr{F}F$ is τ-periodic on $\mathbb{R}$. Note that $\mathscr{M}_p(\mathbb{T})$ is contained in $L^\infty(\mathbb{T})$, because trivially $\mathscr{M}_p(\mathbb{T})$ is contained in $\mathscr{M}_2(\mathbb{T})$, and $\mathscr{M}_2(\mathbb{T})$ may be identified with $L^\infty(\mathbb{T})$.

Now we need a lemma on convolutors of $L^p(\mathbb{Z})$ whose Fourier transform extends to a holomorphic function in a strip. For each positive ε, we denote by Σ_ε the strip $\Sigma_\varepsilon := \{z \in \mathbb{C} : -\varepsilon < \operatorname{Im} z < 0\}$.

Theorem 5.9. *Suppose that p is in $[1,\infty)$, that φ is in $Cv_p(\mathbb{Z})$, and that $\mathscr{F}\varphi$ extends to a bounded holomorphic τ-periodic function in the strip Σ_ε for some positive ε. Then*

$$\left\|\varphi\,\mathbf{1}_{[J,\infty)}\right\|_{Cv_p(\mathbb{Z})} \leq \left\|\varphi\right\|_{Cv_p(\mathbb{Z})} + \Big(\frac{1}{q^\varepsilon - 1} + J\Big)\,\left\|\mathscr{F}\varphi\right\|_{H^\infty(\Sigma_\varepsilon)} \qquad \forall J \in \mathbb{N}.$$

Remark 5.10. The conclusion fails for a generic convolutor of $L^p(\mathbb{Z})$. For instance, it is well known that the function $\varphi(n) := n^{-1}\,\mathbf{1}_{\mathbb{Z}\setminus\{0\}}$ is in $Cv_p(\mathbb{Z})$ for all p in $(1,\infty)$. However, $\varphi\mathbf{1}_{[0,\infty)}$ is not a convolutor of $L^p(\mathbb{Z})$ for every p in $(1,\infty)$. Indeed, $\varphi\mathbf{1}_{[0,\infty)}$ is nonnegative. If $\varphi\mathbf{1}_{[0,\infty)}$ were a convolutor of $L^p(\mathbb{Z})$, then it would be a finite measure, because $\mathbb{Z}$ is amenable. This contradicts the fact that $\varphi\mathbf{1}_{[0,\infty)}$ is not integrable on $\mathbb{Z}$.

Proof. Observe that $\mathscr{F}\varphi$ is τ-periodic in the strip Σ_ε. A standard argument based on Cauchy's theorem allows us to move the path of integration from $[-\tau/2,\tau/2]$ to $[-\tau/2,\tau/2] - i\varepsilon$ (note that the integrals over the vertical sides of the rectangle $[-\tau/2,\tau/2] \times [-\varepsilon, 0]$ cancel out by periodicity), and obtain that

$$\varphi(j) = \frac{1}{\tau}\int_{\mathbb{T}} \mathscr{F}\varphi(s)\,q^{ijs}\,\mathrm{d}s = \frac{1}{\tau}\int_{\mathbb{T}} \mathscr{F}\varphi(s - i\varepsilon)\,q^{ij(s-i\varepsilon)}\,\mathrm{d}s.$$

Hence

$$\left|\varphi(j)\right| \leq \left\|\mathscr{F}\varphi\right\|_{H^\infty(\Sigma_\varepsilon)}\,q^{j\varepsilon} \qquad \forall j \in \mathbb{Z},$$

so that $\varphi\mathbf{1}_{(-\infty,-1]}$ is integrable on $\mathbb{Z}$, and

$$\left\|\varphi\mathbf{1}_{(-\infty,-1]}\right\|_{Cv_p(\mathbb{Z})} \leq \left\|\varphi\mathbf{1}_{(-\infty,-1]}\right\|_{L^1(\mathbb{Z})} \leq \frac{1}{q^\varepsilon - 1}\,\left\|\mathscr{F}\varphi\right\|_{H^\infty(\Sigma_\varepsilon)}.$$

Furthermore, trivially

$$\left|\varphi(j)\right| \leq \left\|\mathscr{F}\varphi\right\|_{L^\infty(\mathbb{T})} \qquad \forall j \in \mathbb{Z},$$

whence the function $\varphi\mathbf{1}_{[0,J-1]}$ satisfies the estimate

$$\left\|\varphi\mathbf{1}_{[0,J-1]}\right\|_{Cv_p(\mathbb{Z})} \leq \left\|\varphi\mathbf{1}_{[0,J-1]}\right\|_{L^1(\mathbb{Z})} \leq J \left\|\mathscr{F}\varphi\right\|_{L^\infty(\mathbb{T})} .$$

As a consequence

$$\begin{aligned}\left\|\varphi\mathbf{1}_{[J,\infty)}\right\|_{Cv_p(\mathbb{Z})} &\leq \left\|\varphi\right\|_{Cv_p(\mathbb{Z})} + \left\|\varphi\mathbf{1}_{(-\infty,-1]}\right\|_{Cv_p(\mathbb{Z})} + \left\|\varphi\mathbf{1}_{[0,J-1]}\right\|_{L^1(\mathbb{Z})} \\ &\leq \left\|\varphi\right\|_{Cv_p(\mathbb{Z})} + \frac{1}{q^\varepsilon-1} \left\|\mathscr{F}\varphi\right\|_{H^\infty(\Sigma_\varepsilon)} + J \left\|\mathscr{F}\varphi\right\|_{L^\infty(\mathbb{T})} \\ &\leq \left\|\varphi\right\|_{Cv_p(\mathbb{Z})} + \Big(\frac{1}{q^\varepsilon-1} + J\Big) \left\|\mathscr{F}\varphi\right\|_{H^\infty(\Sigma_\varepsilon)} ,\end{aligned}$$

as required. □

Recall that $\mathbf{S}_t$ is the strip $\{z \in \mathbb{C} : |\mathrm{Im}\, z| < t\}$. If f is a holomorphic function on $\mathbf{S}_t$, and v is in $(-t,t)$ then we denote by f_v the function on $\mathbb{R}$ defined by $f_v(u) = f(u+iv)$. We also denote by f_t and f_{-t} the boundary values of f, when they exist in the sense of distributions. Recall (see (1.12)) that the spherical Fourier transform $\widetilde{f}$ of a radial function f in $L^1(\mathscr{T})$ is

$$\widetilde{f}(z) = \sum_{x\in\mathscr{T}} f(x)\, \phi_z(x) \qquad \forall z \in \overline{\mathbf{S}}_{1/2}.$$

Since the map $z \mapsto \phi_z$ is even and τ-periodic in the strip $\mathbf{S}_{1/2}$, so is the function $\widetilde{f}$. We say that a holomorphic function in a strip $\mathbf{S}_{\delta(p)}$ is *Weyl–invariant* if it satisfies these conditions in $\mathbf{S}_{\delta(p)}$.

The main result of this section is the following.

Theorem 5.11. *Suppose that p is in $[1,\infty)\setminus\{2\}$, and that k is a radial function on $\mathscr{T}$. The following are equivalent:*

(i) k is in $Cv_p(\mathscr{T})$;

(ii) $\widetilde{k}$ is a holomorphic Weyl invariant function on $\mathbf{S}_{\delta(p)}$, and $\widetilde{k}_{\delta(p)}$ is in $\mathscr{M}_p(\mathbb{T})$.

Furthermore, there exists positive constants c and C, independent of k, such that

$$c \left\|\widetilde{k}_{\delta(p)}\right\|_{\mathscr{M}_p(\mathbb{T})} \leq \left\|k\right\|_{Cv_p(\mathscr{T})} \leq C \left\|\widetilde{k}_{\delta(p)}\right\|_{\mathscr{M}_p(\mathbb{T})} .$$

Proof. It is known that (i) implies (ii) and that the left hand inequality above holds (see [CMS2, Theorem 2.1]).

Thus, it remains to show that (ii) implies (i). Observe that it suffices to prove the result in the case where p is in $[1, 2)$. Indeed, if p is in $(2, \infty)$, and $\widetilde{k}_{\delta(p)}$ is in $\mathscr{M}_p(\mathbb{T})$, then $\widetilde{k}_{\delta(p)}$ is also in $\mathscr{M}_{p'}(\mathbb{T})$. Since p' is in $(1, 2)$, k is in $Cv_{p'}(\mathscr{T})$. A straightforward duality argument then shows that k is in $Cv_p(\mathscr{T})$, as required.

Now, assume that p is in $[1, 2)$. By the inversion formula (1.14),

$$k(x) = 2c_{_G}\, q^{-|x|/2} \int_{\mathbb{T}} \widetilde{k}(s)\, \mathbf{c}(-s)^{-1}\, q^{is|x|}\, \mathrm{d}s \qquad \forall x \in \mathscr{T}. \tag{5.8}$$

The integrand in (5.8) above is τ-periodic, and holomorphic in the rectangle $(-\tau/2, \tau/2) \times (-\delta(p), \delta(p))$. A standard argument based on Cauchy's theorem allows us to move the path of integration from $[-\tau/2, \tau/2]$ to $[-\tau/2, \tau/2] + i\delta(p)$ (the integrals over the vertical sides of the rectangle $[-\tau/2, \tau/2] \times [0, \delta(p)]$ cancel out by periodicity), and obtain that

$$k(x) = 2c_{_G}\, q^{-|x|/p} \int_{\mathbb{T}} \widetilde{k}\big(s + i\delta(p)\big)\, \mathbf{c}(-s - i\delta(p))^{-1}\, q^{is|x|}\, \mathrm{d}s.$$

We write $\big(\widetilde{k}\check{\mathbf{c}}^{-1}\big)_{\delta(p)}$ instead of $\widetilde{k}(\cdot + i\delta(p))\, \mathbf{c}(-\cdot - i\delta(p))^{-1}$. We introduce the function φ on $\mathbb{Z}$, defined by

$$\varphi(\ell) = 2c_{_G} \int_{\mathbb{T}} \big(\widetilde{k}\check{\mathbf{c}}^{-1}\big)_{\delta(p)}(s)\, q^{is\ell}\, \mathrm{d}s.$$

Then

$$k(x) = q^{-|x|/p}\, \varphi(|x|). \tag{5.9}$$

Suppose first that $p = 1$. We must prove that k belongs to $L^1(\mathscr{T})$. Since

$$\big\|k\big\|_{L^1(\mathscr{T})} = \sum_{x \in \mathscr{T}} q^{-|x|}\, \big|\varphi(|x|)\big| = \big|\varphi(0)\big| + \frac{q+1}{q} \sum_{d=1}^{\infty} \big|\varphi(d)\big|,$$

it suffices to prove that φ is in $L^1(\mathbb{Z})$. Obviously,

$$\varphi = 2c_{_G}\tau\, \mathscr{F}^{-1}\big(\widetilde{k}\check{\mathbf{c}}^{-1}\big)_{\delta(p)} = 2c_{_G}\tau\, \mathscr{F}^{-1}\widetilde{k}_{\delta(p)} *_{\mathbb{Z}} \mathscr{F}^{-1}(\check{\mathbf{c}}^{-1})_{\delta(p)}.$$

Since $(\check{\mathbf{c}}^{-1})_{\delta(p)}$ is smooth on $\mathbb{T}$, its inverse Fourier transform $\mathscr{F}^{-1}(\check{\mathbf{c}}^{-1})_{\delta(p)}$ is in $L^1(\mathbb{Z})$, by classical Fourier analysis. Furthermore $\mathscr{F}^{-1}\widetilde{k}_{\delta(p)}$ is in $L^1(\mathbb{Z})$ by assumption. Therefore φ is in $L^1(\mathbb{Z})$ and the proof in the case where $p=1$ is complete.

Now, assume that p is in $(1,2)$. In order to analyse k, it is convenient to view it as a function on the group NA. Denote by χ^+ and χ^- the characteristic functions of $\mathscr{T}$ defined by

$$\chi^+(v\sigma^j\cdot o)=\mathbf{1}_{[0,\infty)}(j)\qquad\text{and}\qquad\chi^-(v\sigma^j\cdot o)=\mathbf{1}_{(-\infty,-1]}(j).$$

Clearly

$$k=k\chi^-+k\chi^+.$$

In particular, we use formula (5.9), change variables (see Lemma 5.2)recall that $Q_p(v)=q^{-|v\cdot o|/p}$, and obtain that

$$\begin{aligned}\big(k\chi^-\big)(v\sigma^j\cdot o)&=q^{-(|v\cdot o|-j)/p}\,\varphi(|v\cdot o|-j)\,\mathbf{1}_{(-\infty,-1]}(j)\\&=q^{j/p}\,Q_p(n)\,\varphi(|v\cdot o|-j)\,\mathbf{1}_{(-\infty,-1]}(j).\end{aligned}\tag{5.10}$$

Step I: analysis of $k\chi^-$. Observe that

$$\begin{aligned}\big|\varphi(j)\big|&\leq 2c_{_G}\int_{\mathbb{T}}|\big(\widetilde{k}\check{\mathbf{c}}^{-1}\big)_{\delta(p)}(s)|\,\mathrm{d}s\\&\leq 2c_{_G}\tau\,\big\|\big(\widetilde{k}\check{\mathbf{c}}^{-1}\big)_{\delta(p)}\big\|_\infty\\&\leq 2c_{_G}\tau\,\big\|\widetilde{k}\check{\mathbf{c}}^{-1}\big\|_{L^\infty(\mathbf{S}_{\delta(p)})}\end{aligned}$$

for every integer j. As a consequence we obtain the following pointwise bound

$$\big|(k\chi^-)(v\sigma^j\cdot o)\big|\leq 2c_{_G}\tau\,\big\|\widetilde{k}\check{\mathbf{c}}^{-1}\big\|_{L^\infty(\mathbf{S}_{\delta(p)})}\;q^{j/p}\,\mathbf{1}_{(-\infty,-1]}(j)\,Q_p(v),\tag{5.11}$$

which we record for later use. Formula (5.10) and Corollary 5.7 (with $\Gamma=NA$ and $\mathscr{D}(v\sigma^j)=q^{-j}$) imply that

$$\begin{aligned}\big\|k\chi^-\big\|_{Cv_p(NA)}&\leq\big\|\mathscr{D}^{1/p}k\chi^-\big\|_{L^1(N;Cv_p(\mathbb{Z}))}\\&=\int_N\big\|\varphi(|v\cdot o|-\cdot)\,\mathbf{1}_{(-\infty,-1]}\big\|_{Cv_p(\mathbb{Z})}\;Q_p(v)\,\mathrm{d}v.\end{aligned}$$

Clearly the norm in $Cv_p(\mathbb{Z})$ is translation invariant; it is then straightforward to check that

$$\left\|\varphi(|v\cdot o|-\cdot)\mathbf{1}_{(-\infty,-1]}\right\|_{Cv_p(\mathbb{Z})}=\left\|\varphi\mathbf{1}_{[|v\cdot o|,\infty)}\right\|_{Cv_p(\mathbb{Z})}.$$

By Theorem 5.9 (with $2\delta(p)$ in place of ε, and $|v\cdot o|$ in place of J)

$$\left\|\varphi\mathbf{1}_{[|v\cdot o|,\infty)}\right\|_{Cv_p(\mathbb{Z})}\leq\left\|\varphi\right\|_{Cv_p(\mathbb{Z})}+\left(\frac{1}{q^{2\delta(p)}-1}+|v\cdot o|\right)\left\|\mathscr{F}\varphi\right\|_{H^\infty(\Sigma_{2\delta(p)})}.$$

By definition of φ and of the multiplier norm, $\left\|\varphi\right\|_{Cv_p(\mathbb{Z})}=2c_G\tau\left\|\left(\widetilde{k}\check{\mathbf{c}}^{-1}\right)_{\delta(p)}\right\|_{\mathscr{M}_p(\mathbb{T})}$. Notice that the function $\check{\mathbf{c}}_{\delta(p)}^{-1}$ is smooth on $\mathbb{R}$, and never vanishes. Therefore there exists a constant C such that

$$\left\|\left(\widetilde{k}\check{\mathbf{c}}^{-1}\right)_{\delta(p)}\right\|_{\mathscr{M}_p(\mathbb{T})}\leq C\left\|\widetilde{k}_{\delta(p)}\right\|_{\mathscr{M}_p(\mathbb{T})}.$$

Furthermore,

$$\begin{aligned}(2c_G\tau)^{-1}\left\|\mathscr{F}\varphi\right\|_{H^\infty(\Sigma_{2\delta(p)})}&=\left\|(\widetilde{k}\check{\mathbf{c}}^{-1})_{\delta(p)}\right\|_{H^\infty(\Sigma_{2\delta(p)})}\\&=\max\left[\left\|(\widetilde{k}\check{\mathbf{c}}^{-1})_{\delta(p)}\right\|_{L^\infty(\mathbb{T})},\left\|(\widetilde{k}\check{\mathbf{c}}^{-1})_{-\delta(p)}\right\|_{L^\infty(\mathbb{T})}\right]\\&\leq\left\|\check{\mathbf{c}}^{-1}\right\|_{H^\infty(\mathbf{S}_{\delta(p)})}\max\left[\left\|\widetilde{k}_{\delta(p)}\right\|_{L^\infty(\mathbb{T})},\left\|\widetilde{k}_{-\delta(p)}\right\|_{L^\infty(\mathbb{T})}\right]\\&=\left\|\check{\mathbf{c}}^{-1}\right\|_{H^\infty(\mathbf{S}_{\delta(p)})}\left\|\widetilde{k}_{\delta(p)}\right\|_{L^\infty(\mathbb{T})}\\&\leq\left\|\check{\mathbf{c}}^{-1}\right\|_{H^\infty(\mathbf{S}_{\delta(p)})}\left\|\widetilde{k}_{\delta(p)}\right\|_{\mathscr{M}_p(\mathbb{T})};\end{aligned}$$

we have used the Weyl-invariance of $\widetilde{k}$ in the last equality above. By combining the formulae above, we obtain that

$$\left\|k\chi^-\right\|_{Cv_p(NA)}\leq C\left\|\widetilde{k}_{\delta(p)}\right\|_{\mathscr{M}_p(\mathbb{T})}\int_N(1+|v\cdot o|)\,Q_p(v)\,\mathrm{d}v\leq C\left\|\widetilde{k}_{\delta(p)}\right\|_{\mathscr{M}_p(\mathbb{T})};$$

the last inequality follows from Lemma 5.3.

Step II: analysis of $k\chi^+$. Recall that the modular function on NA is $\Delta_{NA}(v\sigma^j)=q^{-j}$ (see (5.3)). Thus,

$$\begin{aligned}\left\|\Delta_{NA}^{-1/p'}k\chi^+\right\|_{L^1(NA)}&=\sum_{j\in\mathbb{Z}}q^{-j}\,q^{j/p'}\int_N|k\chi^+(v\sigma^j\cdot o)|\,\mathrm{d}\mu(v)\\&=\sum_{j\geq0}q^{-j/p}\int_N|k(v\sigma^j\cdot o)|\,\mathrm{d}\mu(v).\end{aligned}$$

Recall that the Abel transform (see [CMS2]) of $|k|$ is defined by

$$\mathscr{A}(|k|)(j) = q^{-j/2} \int_N \big|k(v\sigma^j \cdot o)\big| \, \mathrm{d}\mu(v).$$

By [CMS2, Theorem 2.5], the Abel transform of $|k|$ is an even function on $\mathbb{Z}$, equivalently

$$\int_N \big|k(v\sigma^j \cdot o)\big| \, \mathrm{d}\mu(v) = q^j \int_N \big|k(v\sigma^{-j} \cdot o)\big| \, \mathrm{d}\mu(v).$$

Altogether, we see that

$$\big\|\Delta_{NA}^{-1/p'} k\chi^+\big\|_{L^1(NA)} = \sum_{j\geq 0} q^{j/p'} \int_N |k(v\sigma^{-j} \cdot o)| \, \mathrm{d}\mu(v).$$

By the pointwise bound (5.11) the right hand side is dominated by

$$2c_{_G}\, \tau \, \big\|\widetilde{k}\check{\mathbf{c}}^{-1}\big\|_{L^\infty(\mathbf{S}_{\delta(p)})} \sum_{j\geq 0} q^{j/p'} \, q^{-j/p} \int_N Q_p(v) \, \mathrm{d}\mu(v).$$

Now, the integral over N is convergent, because $p > 1$ (see Lemma 5.3), and so is the series, because $p < 2 < p'$. Therefore

$$\big\|\Delta_{NA}^{-1/p'} k\chi^+\big\|_{L^1(NA)} \leq C \, \big\|\widetilde{k}\check{\mathbf{c}}^{-1}\big\|_{H^\infty(\mathbf{S}_{\delta(p)})} \leq C \, \big\|\widetilde{k}\big\|_{H^\infty(\mathbf{S}_{\delta(p)})} : \tag{5.12}$$

the last inequality follows from the fact that $\check{\mathbf{c}}^{-1}$ is bounded on $\mathbf{S}_{\delta(p)}$. Since $\widetilde{k}$ is bounded and Weyl-invariant on $\mathbf{S}_{\delta(p)}$,

$$\big\|\widetilde{k}\big\|_{H^\infty(\mathbf{S}_{\delta(p)})} \leq \big\|\widetilde{k}_{\delta(p)}\big\|_{L^\infty(\mathbb{T})} \leq \big\|\widetilde{k}_{\delta(p)}\big\|_{\mathscr{M}_p(\mathbb{T})} .$$

Step III: conclusion. By combining the estimates proved in *Step I* and *Step II*, we see that there exists a constant C, independent of k, such that

$$\begin{aligned} \big\|k\big\|_{Cv_p(NA)} &\leq \big\|k\chi^+\big\|_{Cv_p(NA)} + \big\|k\chi^-\big\|_{Cv_p(NA)} \\ &\leq C \, \big\|\widetilde{k}_{\delta(p)}\big\|_{\mathscr{M}_p(\mathbb{T})} . \end{aligned}$$

Since k is radial on $\mathscr{T}$, $\big\|k\big\|_{Cv_p(NA)} = \big\|k\big\|_{Cv_p(\mathscr{T})}$. Thus, k is in $Cv_p(\mathscr{T})$ and the required norm estimate holds.

This concludes the proof of the theorem. $\square$

It is worth to state independently the following version of Theorem 5.11, adapted to the Hardy-type spaces studied in this thesis.

Note that $\delta(1) = 1/2$ and recall that the spherical Fourier transform of the convolution kernel of the Laplacian is $1-\gamma$, where γ is defined in (1.6).

Corollary 5.12. *Suppose that α is in $(0,\infty)$, and that k is a radial function on $\mathscr{T}$. The following are equivalent:*

(i) k is the kernel of a bounded convolution operator T_k from $\mathfrak{X}^\alpha(\mathscr{T})$ to $L^1(\mathscr{T})$;

(ii) $\widetilde{k}$ is a holomorphic Weyl invariant function on $\mathbf{S}_{1/2}$, and $[(1-\gamma)^\alpha\,\widetilde{k}]_{1/2}$ is in $\mathscr{M}_1(\mathbb{T})$.

Furthermore, there exists positive constants c and C, independent of k, such that

$$c\,\left\|[(1-\gamma)^\alpha\,\widetilde{k}]_{1/2}\right\|_{\mathscr{M}_1(\mathbb{T})} \leq \left|\!\left|\!\left|T_k\right|\!\right|\!\right|_{\mathfrak{X}^\alpha(\mathscr{T});L^1(\mathscr{T})} \leq C\,\left\|[(1-\gamma)^\alpha\,\widetilde{k}]_{1/2}\right\|_{\mathscr{M}_1(\mathbb{T})}.$$

Proof. The thesis follows directly from the definition of $\mathfrak{X}^\alpha(\mathscr{T})$ and from Theorem 5.11. Indeed the convolution operator T_k is bounded from $\mathfrak{X}^\alpha(\mathscr{T})$ if and only if the operator $T_k\,\mathscr{L}^\alpha$ is bounded on $L^1(\mathscr{T})$. Now it is sufficient to note that the spherical Fourier transform of the convolution kernel of this operator is $(1-\gamma)^\alpha\,\widetilde{k}$ and to apply Theorem 5.11 to conclude the proof of the corollary. □

Remark 5.13. A multiplier satisfying condition (ii) of Corollary 5.12 for some $\alpha > 0$ may be unbounded on $\mathbf{S}_{1/2}$. In particular such a multiplier may have "singularities" at the points of the set $\pm i/2 + \tau\mathbb{Z}$, which are counterbalanced by the zeros of the function $(1-\gamma)^\alpha$. These multipliers, which are sometimes called *strongly singular*, are also considered in [MMV4] in the context of noncompact symmetric spaces.

5.4 Spherical multipliers on the product of trees

Suppose that $\mathscr{T}_1$ and $\mathscr{T}_2$ are two homogeneous trees, and consider the product $\mathscr{T}_1 \times \mathscr{T}_2$, hereafter denoted by $\mathscr{T}$. We assume that the degrees q_1 and q_2 of $\mathscr{T}_1$ and $\mathscr{T}_2$ are ≥ 2. Throughout this section we adopt the convention that variables or groups associated to $\mathscr{T}_j$, with $j = 1, 2$, will be denoted by the sub-script (or super-script) j. For instance, we denote by $G_1 = N_1 A_1 (G_1)_{o_1}$ and $G_2 = N_2 A_2 (G_2)_{o_2}$ the groups of isometries of $\mathscr{T}_1$ and $\mathscr{T}_2$ and their Iwasawa-type decompositions. Thus, we may define τ_1, τ_2, $\mathbb{T}_1$, $\mathbb{T}_2$, the spherical functions $\phi^1_{s_1}(x_1)$ and $\phi^2_{s_2}(x_2)$; we shall often write $\mathbb{T}$ instead of $\mathbb{T}_1 \times \mathbb{T}_2$. We shall frequently work with the product $\mathbf{S}_{\delta(p)} \times \mathbf{S}_{\delta(p)}$, which we denote by $\big(\mathbf{S}_{\delta(p)}\big)^2$, or simply by $\mathbf{S}^2_{\delta(p)}$. If f is a holomorphic function on $\mathbf{S}^2_p$, and v_1, v_2 are in the interval $\big(-\delta(p), \delta(p)\big)$, we denote by $f_{(v_1,v_2)}$ the function on $\mathbb{R}^2$ defined by $f_{(v_1,v_2)}(u_1, u_2) = f(u_1 + iv_1, u_2 + iv_2)$. We also denote by $f_{(\pm\delta(p),\pm\delta(p))}$ the boundary values of f, when they exist in the sense of distributions.

Recall that all the bounded spherical functions on $\mathscr{T}$ are of the form

$$\phi_{s_1,s_2}(x_1, x_2) := \phi^1_{s_1}(x_1)\, \phi^2_{s_2}(x_2),$$

where (s_1, s_2) is in $(\overline{\mathbf{S}}_{1/2})^2$ and (x_1, x_2) is in $\mathscr{T}$. We shall often write $\phi_s(x)$ instead of $\phi_{s_1,s_2}(x_1, x_2)$.

When dealing with products one of the troubles is notation, which soon becomes cumbersome. In this section we make the effort to keep the notational complication at a minimum. The price to pay is that formulae appear cleaner, but at the same time perhaps less transparent. We make a point to try to be as clear as possible. We adopt the point of view that whenever brevity does not affect the comprehension of a given formula, we always choose brevity.

In this section we are interested in continuous linear operators on $L^p(\mathscr{T})$, and in particular in those operators that are invariant by the action of $G = G_1 \times G_2$. Repeating the argument in Section 1.6, it is easy to see that these operators correspond to operators on $L^p(G/G_o)$ (where G_o is short for $(G_1)_{o_1} \times (G_2)_{o_2}$) given by convolution on the right by a G_o–bi-invariant function. We may identify G_o–bi-invariant functions on G with functions on $\mathscr{T}$ that are radial in each of their variables, and therefore we can define $Cv_p(\mathscr{T})$ in the same fashion as we did in the single tree setting.

Given a function $M : \mathbb{T} \to \mathbb{C}$, we say that M is *Weyl invariant* if it is even in each of the variables. Given a bounded Weyl invariant function M on $\mathbb{T}$, we may consider its inverse spherical Fourier transform k_M, given by

$$k_M(x) = 4c_{G_1} c_{G_2} \int_{\mathbb{T}} M(s_1, s_2)\, \phi_s(x)\, |\mathbf{c}_1(s_1)\, \mathbf{c}_2(s_2)|^{-2}\, \mathrm{d}s_1\, \mathrm{d}s_2 \qquad \forall x \in \mathscr{T}.$$

Observe that this function is radial in each of its varables, as the spherical functions are. Thus it is natural to speculate about what conditions on M imply that k_M bleongs to $Cv_p(\mathscr{T})$.

We also introduce the following notation: given two functions $f_i : X_i \to \mathbb{C}$, where $i = 1, 2$, we denote by $f_1 \otimes f_2$ the map with domain $X_1 \times X_2$ defined by $(f_1 \otimes f_2)(x_1, x_2) = f_1(x_1) f_2(x_2)$.

We shall often use the following simple result about convolutors of $L^p(\mathbb{Z}^2)$.

Lemma 5.14. *If φ belongs to $Cv_p(\mathbb{Z}^2)$ and j_1 is an integer, then $\varphi(j_1, \cdot)$ belongs to $Cv_p(\mathbb{Z})$. Moreover*

$$\sup_{j_1 \in \mathbb{Z}} \big\| \varphi(j_1, \cdot) \big\|_{Cv_p(\mathbb{Z})} \leq \big\| \varphi \big\|_{Cv_p(\mathbb{Z}^2)} .$$

Proof. We preliminarily observe that, given a function f on $\mathbb{Z}$, we have

$$\begin{aligned} \big(\varphi(j_1, \cdot) *_{\mathbb{Z}} f\big)(j_2) &= \sum_{\ell_2 \in \mathbb{Z}} \varphi(j_1, \ell_2)\, f(j_2 - \ell_2) \\ &= \sum_{(\ell_1, \ell_2) \in \mathbb{Z}^2} \varphi(\ell_1, \ell_2)\, \delta(j_1 - \ell_1)\, f(j_2 - \ell_2) \qquad (5.13) \\ &= \big(\varphi *_{\mathbb{Z}^2} (\delta_o \otimes f)\big)(j_1, j_2). \end{aligned}$$

Now suppose that f belongs to $L^p(\mathbb{Z})$, and that $\big\| f \big\|_{L^p(\mathbb{Z})} = 1$. Then we have

$$\begin{aligned} \big\| \varphi(j_1, \cdot) *_{\mathbb{Z}} f \big\|_{L^p(\mathscr{T}_2)} &= \Big[\sum_{\ell_2 \in \mathbb{Z}} \big| \big(\varphi(j_1, \cdot) *_{\mathbb{Z}} f\big)(\ell_2) \big|^p \Big]^{1/p} \\ &\leq \Big[\sum_{(\ell_1, \ell_2) \in \mathbb{Z}^2} \big| \big(\varphi(\ell_1, \cdot) *_{\mathbb{Z}} f\big)(\ell_2) \big|^p \Big]^{1/p}. \end{aligned}$$

We make use of (5.13) in the general term of the series above to get

$$\begin{aligned}\left\|\varphi(j_1,\cdot) *_{\mathbb{Z}} f\right\|_{L^p(\mathscr{T}_2)} &\leq \Big[\sum_{(\ell_1,\ell_2)\in\mathbb{Z}^2} \left|\big(\varphi *_{\mathbb{Z}^2} (\delta_o \otimes f)\big)(\ell_1,\ell_2)\right|^p\Big]^{1/p} \\ &= \left\|\varphi *_{\mathbb{Z}^2} (\delta_o \otimes f)\right\|_{L^p(\mathbb{Z}^2)} \\ &\leq \left\|\varphi\right\|_{Cv_p(\mathbb{Z}^2)} \left\|\delta_o \otimes f\right\|_{L^p(\mathbb{Z}^2)} .\end{aligned}$$

Note that $\left\|\delta_o \otimes f\right\|_{L^p(\mathbb{Z}^2)} = \left\|f\right\|_{L^p(\mathbb{Z})} = 1$, thus the above chains of inequalities imply that, for every integer j_1,

$$\left\|\varphi(j_1,\cdot)\right\|_{Cv_p(\mathbb{Z})} \leq \left\|\varphi\right\|_{Cv_p(\mathbb{Z}^2)} ,$$

that is the thesis. □

The next Lemma is a two variables version of Theorem 5.9. For each positive ε we set $\Sigma_\varepsilon^2 := \{(z_1,z_2)\in\mathbb{C}^2\colon -\varepsilon < \operatorname{Im} z_j < 0,\ j=1,2\}$.

Lemma 5.15. *Suppose that p is in $[1,\infty)$, that φ is in $Cv_p(\mathbb{Z}^2)$, and that $\mathscr{F}_{\mathbb{T}}\varphi$ extends to a bounded and holomorphic function in Σ_ε^2 for some positive ε. Also suppose that*

$$\left\|\mathscr{F}_{\mathbb{T}}\varphi\right\|_{H^\infty(\Sigma_\varepsilon^2)} = \left\|\mathscr{F}_{\mathbb{T}}\varphi\right\|_{L^\infty(\mathbb{T})} . \tag{5.14}$$

Then there exists a constant C, independent of φ, such that

$$\left\|\varphi\,\mathbf{1}_{[J_1,\infty)}\otimes\mathbf{1}_{[J_2,\infty)}\right\|_{Cv_p(\mathbb{Z}^2)} \leq C\,(1+J_1)\,(1+J_2)\ \left\|\varphi\right\|_{Cv_p(\mathbb{Z}^2)} \qquad \forall J_1,J_2\in\mathbb{N}.$$

Proof. Fix two positive integers J_1 and J_2, and consider the following (disjoint) subsets of $\mathbb{Z}^2$:

$$\begin{aligned}E_1 &= \{(j_1,j_2)\colon j_1\geq J_1,\ 0\leq j_2 < J_2\},\\ E_2 &= \{(j_1,j_2)\colon 0\leq j_1 < J_1,\ 0\leq j_2 < J_2\},\\ E_3 &= \{(j_1,j_2)\colon 0\leq j_1 < J_1,\ j_2\geq J_2\},\\ E_4 &= \{(j_1,j_2)\colon j_1 < 0,\ j_2 \geq 0\},\\ E_5 &= \{(j_1,j_2)\colon j_1 < 0,\ j_2 < 0\},\\ E_6 &= \{(j_1,j_2)\colon j_1 \geq 0,\ j_2 < 0\}.\end{aligned}$$

It is easily verified that the union of these subsets is $\mathbb{Z} \setminus \{(j_1, j_2)\colon j_1 \geq J_1,\ j_2 \geq J_2\}$. Since φ is in $Cv_p(\mathbb{Z}^2)$, it suffices to prove that $\varphi\,\mathbf{1}_{E_i}$ is in $Cv_p(\mathbb{Z}^2)$ for $i = 1, \ldots, 6$, with an appropriate bound on the convolution norm.

Step I: analysis of the restrictions to E_1 and E_3. We claim that there exists a constant C such that

$$\begin{aligned}\left\|\varphi\,\mathbf{1}_{E_1}\right\|_{Cv_p(\mathbb{Z}^2)} &\leq C\,(1+J_1)\,J_2\ \left\|\varphi\right\|_{Cv_p(\mathbb{Z}^2)}\,,\\ \left\|\varphi\,\mathbf{1}_{E_3}\right\|_{Cv_p(\mathbb{Z}^2)} &\leq C\,J_1\,(1+J_2)\ \left\|\varphi\right\|_{Cv_p(\mathbb{Z}^2)}\,.\end{aligned}$$

Note that by symmetry we can focus only on E_1.

Theorem 5.9 implies that, for every fixed integer j_2,

$$\left\|\varphi(\cdot, j_2)\mathbf{1}_{[J_1,\infty)}(\cdot)\right\|_{Cv_p(\mathbb{Z})} \leq C\,(1+J_1)\,\big(\,\left\|\varphi(\cdot, j_2)\right\|_{Cv_p(\mathbb{Z})} + \left\|\mathscr{F}_1\varphi(\cdot, j_2)\right\|_{H^\infty(\Sigma_\varepsilon)}\,\big). \tag{5.15}$$

For every fixed integer j_2,

$$\mathscr{F}_1[\varphi(\cdot, j_2)](s_1) = \int_{\mathbb{T}_2} \mathscr{F}\varphi(s_1, s_2)\,q_2^{is_2 j_2}\ \mathrm{d}s_2. \tag{5.16}$$

We may conclude that for every integer j_2 and for every s_1 in $\mathbb{T}_1$,

$$\begin{aligned}\left|\mathscr{F}_1[\varphi(\cdot, j_2)](s_1)\right| &\leq \tau_2\,\left\|\mathscr{F}\varphi\right\|_{H^\infty(\Sigma_\varepsilon^2)}\\ &\leq \tau_2\,\left\|\mathscr{F}\varphi\right\|_{L^\infty(\mathbb{T})}\\ &\leq \tau_2\,\left\|\mathscr{F}\varphi\right\|_{\mathscr{M}_p(\mathbb{T})}\,,\end{aligned}$$

where the first inequality follows from (5.16), simply passing the absolute value inside of the integral, while the second is true by hypothesis (5.14). Finally by definition $\left\|\mathscr{F}\varphi\right\|_{\mathscr{M}_p(\mathbb{T})} = \left\|\varphi\right\|_{Cv_p(\mathbb{Z}^2)}$, thus (5.15) reduces to

$$\left\|\varphi(\cdot, j_2)\mathbf{1}_{[J_1,\infty)}(\cdot)\right\|_{Cv_p(\mathbb{Z})} \leq C\,(1+J_1)\,\big(\,\left\|\varphi(\cdot, j_2)\right\|_{Cv_p(\mathbb{Z})} + \left\|\varphi\right\|_{Cv_p(\mathbb{Z}^2)}\,\big).$$

So

$$\begin{aligned}\left\|\varphi\,\mathbf{1}_{E_1}\right\|_{Cv_p(\mathbb{Z}^2)} &\leq \sum_{j_2\in\mathbb{Z}} \left\|\varphi(\cdot, j_2)\,\mathbf{1}_{E_1}(\cdot, j_2)\right\|_{Cv_p(\mathbb{Z})}\\ &\leq C\,(1+J_1)\,J_2\,\Big(\sup_{0\leq j_2<J_2}\ \left\|\varphi(\cdot, j_2)\right\|_{Cv_p(\mathbb{Z})} + \left\|\varphi\right\|_{Cv_p(\mathbb{Z}^2)}\Big).\end{aligned}$$

Now we can apply Lemma 5.14 in the last expression, to get

$$\left\|\varphi\,\mathbf{1}_{E_1}\right\|_{Cv_p(\mathbb{Z}^2)} \leq C\,(1+J_1)\,J_2\,\left\|\varphi\right\|_{Cv_p(\mathbb{Z}^2)},$$

which proves our claim.

Step II: analysis of the restriction to E_2. This is the simplest part. Indeed E_2 is finite and has $J_1\,J_2$ elements, so

$$\begin{aligned}\left\|\varphi\,\mathbf{1}_{E_2}\right\|_{Cv_p(\mathbb{Z}^2)} &\leq \left\|\varphi\,\mathbf{1}_{E_2}\right\|_{L^1(\mathbb{Z}^2)} \\ &\leq J_1\,J_2 \sup_{(j_1,j_2)\in E_2} |\varphi(j_1,j_2)|.\end{aligned}$$

It is easy to prove that

$$\begin{aligned}\sup_{(j_1,j_2)\in\mathbb{Z}^2} |\varphi(j_1,j_2)| &\leq \left\|\mathscr{F}_{\mathbb{T}}\varphi\right\|_{H^\infty(\Sigma^2_\varepsilon)} \\ &= \left\|\mathscr{F}_{\mathbb{T}}\varphi\right\|_{L^\infty(\mathbb{T})},\end{aligned}$$

where the equality above is true by hypothesis (5.14).
Clearly $\left\|\mathscr{F}_{\mathbb{T}}\varphi\right\|_{L^\infty(\mathbb{T})} \leq \left\|\mathscr{F}_{\mathbb{T}}\varphi\right\|_{\mathscr{M}_p(\mathbb{T})}$, and we conclude that

$$\left\|\varphi\,\mathbf{1}_{E_2}\right\|_{Cv_p(\mathbb{Z}^2)} \leq J_1\,J_2\,\left\|\mathscr{F}_{\mathbb{T}}\varphi\right\|_{\mathscr{M}_p(\mathbb{T})},$$

as required.

Step III: analysis of the restriction to E_5. We claim that there exists a constant C such that

$$\left\|\varphi\,\mathbf{1}_{E_5}\right\|_{Cv_p(\mathbb{Z}^2)} \leq C\,\left\|\varphi\right\|_{Cv_p(\mathbb{Z}^2)}.$$

By Fourier inversion, we have

$$\varphi(j_1,j_2) = \frac{1}{\tau_1\,\tau_2}\int_{\mathbb{T}_1}\int_{\mathbb{T}_2} \mathscr{F}_{\mathbb{T}}\varphi(s_1,s_2)\,q_1^{is_1j_1}\,q_2^{is_2j_2}\,\mathrm{d}s_1\,\mathrm{d}s_2$$

A standard argument based on the Cauchy's theorem allows us to move the path of integration from $[-\tau_1/2,\tau_1/2]$ to $[-\tau_1/2,\tau_1/2]-i\varepsilon$, and to do the same also in the second variable. Note that the integrals over the vertical sides of the rectangles cancel out by periodicity. Thus

$$\begin{aligned}\varphi(j_1, j_2) &= \frac{1}{\tau_1\,\tau_2} \int_{\mathbb{T}_1}\int_{\mathbb{T}_2} \mathscr{F}_{\mathbb{T}}\varphi(s_1 - i\varepsilon, s_2 - i\varepsilon)\, q_1^{ij_1(s_1-i\varepsilon)}\, q_2^{ij_2(s_2-i\varepsilon)}\ \mathrm{d}s_1\ \mathrm{d}s_2 \\ &= \frac{1}{\tau_1\,\tau_2}\, q_1^{j_1\varepsilon}\, q_2^{j_2\varepsilon} \int_{\mathbb{T}_1}\int_{\mathbb{T}_2} \mathscr{F}_{\mathbb{T}}\varphi(s_1 - i\varepsilon, s_2 - i\varepsilon)\, q_1^{is_1j_1}\, q_2^{is_2j_2}\ \mathrm{d}s_1\ \mathrm{d}s_2.\end{aligned}$$

These equalities yield

$$\begin{aligned}\left\|\varphi\,\mathbf{1}_{E_5}\right\|_{Cv_p(\mathbb{Z}^2)} &\leq \left\|\varphi\,\mathbf{1}_{E_5}\right\|_{L^1(\mathbb{Z}^2)} \\ &\leq \frac{1}{\tau_1\,\tau_2} \sum_{j_1,j_2<0} q_1^{j_1\varepsilon}\, q_2^{j_2\varepsilon}\ \left\|\mathscr{F}_{\mathbb{T}}\varphi(\cdot - i\varepsilon, \cdot - i\varepsilon)\right\|_{L^\infty(\mathbb{T})}.\end{aligned}$$

By (5.14)

$$\left\|\mathscr{F}_{\mathbb{T}}\varphi(\cdot - i\varepsilon, \cdot - i\varepsilon)\right\|_{L^\infty(\mathbb{T})} \leq \left\|\mathscr{F}_{\mathbb{T}}\varphi\right\|_{L^\infty(\mathbb{T})} \leq \left\|\mathscr{F}_{\mathbb{T}}\varphi\right\|_{\mathscr{M}_p(\mathbb{T})}.$$

We conclude that

$$\begin{aligned}\left\|\varphi\,\mathbf{1}_{E_5}\right\|_{Cv_p(\mathbb{Z}^2)} &\leq \frac{1}{\tau_1\,\tau_2}\,\frac{1}{q_1^{\varepsilon}-1}\,\frac{1}{q_2^{\varepsilon}-1}\ \left\|\mathscr{F}_{\mathbb{T}}\varphi\right\|_{\mathscr{M}_p(\mathbb{T})} \\ &= C\ \left\|\varphi\right\|_{Cv_p(\mathbb{Z}^2)}.\end{aligned}$$

This proves the claim for the restriction of φ to E_5.

Step IV: analysis for the restrictions to E_4 *and* E_6. We claim that there exists a constant C such that

$$\begin{aligned}\left\|\varphi\,(\mathbf{1}_{E_4} + \mathbf{1}_{E_5})\right\|_{Cv_p(\mathbb{Z}^2)} &\leq C\ \left\|\varphi\right\|_{Cv_p(\mathbb{Z}^2)}, \\ \left\|\varphi\,(\mathbf{1}_{E_6} + \mathbf{1}_{E_5})\right\|_{Cv_p(\mathbb{Z}^2)} &\leq C\ \left\|\varphi\right\|_{Cv_p(\mathbb{Z}^2)}.\end{aligned}$$

Observe that these inequalities, together with *Step III* imply

$$\begin{aligned}\left\|\varphi\,\mathbf{1}_{E_4}\right\|_{Cv_p(\mathbb{Z}^2)} &\leq C\ \left\|\varphi\right\|_{Cv_p(\mathbb{Z}^2)}, \\ \left\|\varphi\,\mathbf{1}_{E_6}\right\|_{Cv_p(\mathbb{Z}^2)} &\leq C\ \left\|\varphi\right\|_{Cv_p(\mathbb{Z}^2)}.\end{aligned}$$

By symmetry it suffices to prove the desired inequality for $\varphi\,(\mathbf{1}_{E_4}+\mathbf{1}_{E_5}) = \varphi\,\mathbf{1}_{\{j_1<0\}}$. The integration argument used in *Step III* (this time applied only in the first variable) leads to

$$\begin{aligned}\varphi(j_1,j_2) &= \frac{1}{\tau_1\,\tau_2}\int_{\mathbb{T}_1}\int_{\mathbb{T}_2}\mathscr{F}_{\mathbb{T}}\varphi(s_1,s_2)\,q_1^{is_1j_1}\,q_2^{is_2j_2}\,\mathrm{d}s_1\,\mathrm{d}s_2\\ &= \frac{1}{\tau_1\,\tau_2}\,q_1^{j_1\varepsilon}\int_{\mathbb{T}_1}\int_{\mathbb{T}_2}\mathscr{F}_{\mathbb{T}}\varphi(s_1-i\varepsilon,s_2)\,q_1^{is_1j_1}\,q_2^{is_2j_2}\,\mathrm{d}s_1\,\mathrm{d}s_2.\end{aligned}$$

Thus

$$\begin{aligned}\left\|\varphi\,\mathbf{1}_{\{j_1<0\}}\right\|_{Cv_p(\mathbb{Z}^2)} &\leq \frac{1}{\tau_1\,\tau_2}\sum_{j_1<0}q_1^{j_1\varepsilon}\,\left\|\mathscr{F}_{\mathbb{T}}\varphi(\cdot-i\varepsilon,\cdot)\right\|_{L^\infty(\mathbb{T})}\\ &\leq \frac{1}{\tau_1\,\tau_2}\,\frac{1}{q_1^\varepsilon-1}\,\left\|\mathscr{F}_{\mathbb{T}}\varphi\right\|_{\mathscr{M}_p(\mathbb{T})}\\ &\leq C\,\left\|\varphi\right\|_{Cv_p(\mathbb{Z}^2)},\end{aligned}$$

where the second inequality follows, as usual, from (5.14).

Step V: conclusion. We proved in the four steps above that there exists a constant C, independent of φ, such that

$$\left\|\varphi\,\mathbf{1}_{E_i}\right\|_{Cv_p(\mathbb{Z}^2)} \leq C\,(1+J_1)\,(1+J_2)\,\left\|\varphi\right\|_{Cv_p(\mathbb{Z}^2)}$$

for every $i=1,\dots,6$. Clearly

$$\left\|\varphi\,\mathbf{1}_{[J_1,\infty)}\otimes\mathbf{1}_{[J_2,\infty)}\right\|_{Cv_p(\mathbb{Z}^2)} \leq \left\|\varphi\right\|_{Cv_p(\mathbb{Z}^2)} + \sum_{i=1}^{6}\left\|\varphi\,\mathbf{1}_{E_i}\right\|_{Cv_p(\mathbb{Z}^2)},$$

so

$$\left\|\varphi\,\mathbf{1}_{[J_1,\infty)}\otimes\mathbf{1}_{[J_2,\infty)}\right\|_{Cv_p(\mathbb{Z}^2)} \leq C\,(1+J_1)\,(1+J_2)\,\left\|\varphi\right\|_{Cv_p(\mathbb{Z}^2)},$$

that is the thesis. □

The main result of this section is the following.

Theorem 5.16. *Suppose that p is in $[1,\infty)\setminus\{2\}$, that M is a Weyl invariant and holomorphic function on $\mathbf{S}^2_{\delta(p)}$, and that $M_{(\delta(p),\delta(p))}$ is in $\mathscr{M}_p(\mathbb{T})$. Then the inverse spherical Fourier transform k_M belongs to $Cv_p(\mathscr{T})$. Furthermore, there exists positive a constants C, independent of M, such that*

$$\left\|k_M\right\|_{Cv_p(\mathscr{T})} \leq C\,\left\|M_{(\delta(p),\delta(p))}\right\|_{\mathscr{M}_p(\mathbb{T})}.$$

Proof. Observe that by the same duality argument used in the beginning of the proof of Theorem 5.11 it suffices to prove the result in the case where p is in $[1,2)$.

Now, assume that p is in $[1,2)$. By the inversion formula (1.14),

$$\begin{aligned} k(x_1,x_2) = {} & 4c_{G_1}c_{G_2}\,q_1^{-|x_1|/2}\,q_2^{-|x_2|/2} \\ & \times \int_{\mathbb{T}_1}\int_{\mathbb{T}_2} M(s_1,s_2)\,\mathbf{c}(-s_1,-s_2)^{-1}\,q_1^{is_1|x_1|}\,q_2^{is_2|x_2|}\,\mathrm{d}s_1\,\mathrm{d}s_2, \end{aligned} \tag{5.17}$$

where we denoted by $\mathbf{c}$ the function $\mathbf{c}_1\otimes\mathbf{c}_2$. In what follows we shall write

$$\varphi(\ell_1,\ell_2) = c_{G_1}c_{G_2}\,\mathscr{F}\big(M\,\check{\mathbf{c}}^{-1}\big)_{(\delta(p),\delta(p))}(\ell_1,\ell_2) \qquad \forall(\ell_1,\ell_2)\in\mathbb{Z}^2, \tag{5.18}$$

where we denoted by $\mathscr{F}$ the Fourier transform on $\mathbb{T}$. Observe that the properties of M allow us to move that path of integration in both variables, from $[-\tau_j/2,\tau_j/2]$ to $[-\tau_j/2,\tau_j/2]+i\delta(p)$, so

$$k(x_1,x_2) = q_1^{-|x_1|/p}\,q_2^{-|x_2|/p}\,\varphi(|x_1|,|x_2|) \qquad \forall(x_1,x_2)\in\mathscr{T}_1\times\mathscr{T}_2. \tag{5.19}$$

Also note that

$$\mathscr{F}\big(M\,\check{\mathbf{c}}^{-1}\big)_{(\delta(p),\delta(p))} = \mathscr{F}M_{(\delta(p),\delta(p))} *_{\mathbb{Z}^2} \mathscr{F}\big(\check{\mathbf{c}}^{-1}\big)_{(\delta(p),\delta(p))},$$

and that $\mathscr{F}\big(\check{\mathbf{c}}^{-1}\big)_{(\delta(p),\delta(p))}$ is in $L^1(\mathbb{Z}^2)$. Thus $\mathscr{F}\big(M\,\check{\mathbf{c}}^{-1}\big)_{(\delta(p),\delta(p))}$ is a convolutor of $L^p(\mathbb{Z}^2)$ if and only if $\mathscr{F}M_{(\delta(p),\delta(p))}$ is, and

$$\left\|\mathscr{F}\big(M\,\check{\mathbf{c}}^{-1}\big)_{(\delta(p),\delta(p))}\right\|_{Cv_p(\mathbb{Z}^2)} \leq \left\|\mathscr{F}\big(\check{\mathbf{c}}^{-1}\big)_{(\delta(p),\delta(p))}\right\|_{L^1(\mathbb{Z}^2)}\,\left\|\mathscr{F}M_{(\delta(p),\delta(p))}\right\|_{Cv_p(\mathbb{Z}^2)}. \tag{5.20}$$

Suppose that $p=1$. Equation (5.19) in this case reads

$$k(x_1,x_2) = q_1^{-|x_1|}\,q_2^{-|x_2|}\,\varphi(|x_1|,|x_2|) \qquad \forall(x_1,x_2)\in\mathscr{T}_1\times\mathscr{T}_2.$$

We need to prove that k belongs to $L^1(\mathscr{T})$. It is easy to see that there exists a constant c_{q_1,q_2} such that

$$\big\|k\big\|_{L^1(\mathscr{T})} \leq c_{q_1,q_2} \sum_{d_1,d_2=0}^{\infty} |\varphi(d_1,d_2)|,$$

so it suffices to show that φ is in $L^1(\mathbb{Z}^2)$. This is a consequence of the hypothesis on M and of the preliminary observation (5.20).

Now suppose that p is in $(1,2)$. It is useful to view k as a function on the group $NA = N_1A_1 \times N_2A_2$. Denote by χ_1^+ and χ_1^- the characteristic functions of $(N_1 \times \mathbb{Z}^+) \times N_2A_2$ and $(N_1 \times \mathbb{Z}^-) \times N_2A_2$, respectively, and write

$$k = k\chi_1^- + k\chi_1^+.$$

We shall analyse $k\chi_1^+$ in *Step I* below. The term $k\chi_1^-$ will be further decomposed and we shall analyse it in *Steps II-III*.

Step I: analysis of $k\chi_1^+$. We apply Corollary 5.8 to the group $\Gamma = N_1A_1 \times N_2A_2$. Recall that the modular function of N_1A_1 is $\Delta_{N_1A_1}(v_1\sigma_1^{j_1}) = q_1^{-j_1}$, so

$$\begin{aligned} \big\|k\chi_1^+\big\|_{Cv_p(NA)} &\leq \big\|\Delta_{N_1A_1}^{-1/p'} k\chi_1^+\big\|_{L^1(N_1A_1;Cv_p(N_2A_2))} \\ &= \sum_{j_1\geq 0} q_1^{-j_1}\, q_1^{j/p'} \int_{N_1} \big\|k\chi_1^+(v_1\sigma_1^{j_1}\cdot o_1,\cdot)\big\|_{Cv_p(N_2A_2)}\, \mathrm{d}\mu(v_1). \end{aligned} \tag{5.21}$$

Note that k is radial in both variables and apply the Abel transform in the first variable to get

$$\begin{aligned} &\int_{N_1} \big\|k\chi_1^+(v_1\sigma_1^{j_1}\cdot o_1,\cdot)\big\|_{Cv_p(N_2A_2)}\, \mathrm{d}\mu(v_1) \\ &= q_1^{j_1} \int_{N_1} \big\|k\chi_1^+(v_1\sigma_1^{-j_1}\cdot o_1,\cdot)\big\|_{Cv_p(N_2A_2)}\, \mathrm{d}\mu(v_1) \end{aligned} \tag{5.22}$$

Before we continue, let us introduce the following auxiliary function

$$\varphi_{N_2A_2}(\ell_1, v_2\sigma_2^{j_2}) = 4c_{G_1}c_{G_2}\, q_2^{-|v_2\sigma_2^{j_2}\cdot o_2|/2}\, \mathscr{F}\big(M\check{\mathbf{c}}^{-1}\big)_{(\delta(p),0)}(\ell_1, |v_2\sigma_2^{j_2}\cdot o_2|) \qquad \forall (\ell_1, v_2\sigma_2^{j_2}) \in \mathbb{Z}\times N_2A_2.$$

Thus, for every nonnegative j_1, we have

$$k(v_1\sigma_1^{-j_1}\cdot o_1, v_2\sigma_2^{j_2}\cdot o_2) = q_1^{-(|v_1\cdot o_1|+j_1)/p}\,\varphi_{N_2A_2}(|v_1\cdot o_1|+j_1, v_2\sigma_2^{j_2}), \tag{5.23}$$

where we used Lemma 5.2 in the first variable. Combining equations (5.21), (5.22) and (5.23) we get

$$\begin{aligned}
&\left\|k\chi_1^+\right\|_{Cv_p(NA)}\\
&\leq \sum_{j\geq 0} q_1^{j_1/p'}\, q_1^{-j_1/p}\int_{N_1} Q_p(v_1)\,\left\|\varphi_{N_2A_2}(|v_1\cdot o_1|+j_1,\cdot)\right\|_{Cv_p(N_2A_2)}\,\mathrm{d}\mu(v_1)\\
&\leq \sum_{j\geq 0} q_1^{j_1/p'}\, q_1^{-j_1/p}\sup_{\ell_1\geq 0}\left\|\varphi_{N_2A_2}(\ell_1,\cdot)\right\|_{Cv_p(N_2A_2)}\int_{N_1} Q_p(v_1)\,\mathrm{d}\mu(v_1).
\end{aligned} \tag{5.24}$$

Observe that

$$\begin{aligned}
&\varphi_{N_2A_2}(\ell_1, v_2\sigma_2^{j_2}\cdot o_2)\\
&= 4c_{G_1}c_{G_2}\, q_2^{-|v_2\sigma_2^{j_2}\cdot o_2|/2}\int_{\mathbb{T}_2}\left[\int_{\mathbb{T}_1}\left(M\,\check{\mathbf{c}}^{-1}\right)_{(\delta(p),0)}(s_1,s_2)\, q_1^{is_1\ell_1}\,\mathrm{d}s_1\right] q_2^{is_2|v_2\sigma_2^{j_2}\cdot o_2|}\,\mathrm{d}s_2
\end{aligned}$$

and denote the inner integral in the above expression as $M_{\ell_1}(s_2)$. Then estimate (5.24), Lemma 5.3 and the single tree result (Theorem 5.11) imply

$$\begin{aligned}
\left\|k\chi_1^+\right\|_{Cv_p(NA)} &\leq C_p\sup_{\ell_1\geq 0}\left\|\varphi_{N_1A_1}(\ell_1\,\cdot)\right\|_{Cv_p(N_2A_2)}\\
&\leq C_p\sup_{\ell_1\geq 0}\left\|\left(M_{\ell_1}\right)_{\delta(p)}\right\|_{\mathscr{M}_p(\mathbb{T}_2)}\\
&\leq C_p\sup_{\ell_1\in\mathbb{Z}}\left\|\mathscr{F}_{\mathbb{T}_2}\left(M_{\ell_1}\right)_{\delta(p)}\right\|_{Cv_p(\mathbb{Z}_2)}.
\end{aligned} \tag{5.25}$$

A computation shows that

$$\begin{aligned}
\mathscr{F}_{\mathbb{T}_2}\left(M_{\ell_1}\right)_{\delta(p)}(\ell_2) &= \int_{\mathbb{T}_2}\int_{\mathbb{T}_1}\left(M\check{\mathbf{c}}^{-1}\right)_{(\delta(p),\delta(p))}(s_1,s_2)\, q_1^{is_1\ell_1}\, q_2^{is_2\ell_2}\,\mathrm{d}s_1\,\mathrm{d}s_2\\
&= (4c_{G_1}c_{G_2})^{-1}\,\varphi(\ell_1,\ell_2).
\end{aligned}$$

Finally (5.25) and Lemma 5.14 yield

$$\begin{aligned}
\left\|k\chi_1^+\right\|_{Cv_p(NA)} &\leq C_p\sup_{\ell_1\in\mathbb{Z}}\left\|\varphi(\ell_1,\cdot)\right\|_{Cv_p(\mathbb{Z}_2)}\\
&\leq C_p\,\left\|\varphi\right\|_{Cv_p(\mathbb{Z}^2)}\\
&\leq C_p\,\left\|M_{(\delta(p),\delta(p))}\right\|_{\mathscr{M}_p(\mathbb{T})}.
\end{aligned}$$

Step II: analysis of $k\chi_1^-$, part 1. To estimate the convolution norm of $k\chi_1^-$ we further decompose

$$k\chi_1^- = k\chi_1^-\chi_2^+ + k\chi_1^-\chi_2^-,$$

where χ_2^+ and χ_2^- are characteristic functions in the second variable, defined in the same fashion as χ_1^+ and χ_1^-.

First we consider $k\chi_1^-\chi_2^+$. We write $k\chi_1^-\chi_2^+ = k\chi_2^+ - k\chi_1^+ k\chi_2^+$, and we observe that we can obtain the estimate

$$\left\|k\chi_2^+\right\|_{Cv_p(NA)} \leq C_p \, \left\|M_{(\delta(p),\delta(p))}\right\|_{\mathscr{M}_p(\mathbb{T})}$$

following the procedure described in *Step I*, exchanging the roles of variables. Thus we need to consider $k\chi_1^+\chi_2^+$. Convolution inequality (1.5) implies

$$\begin{aligned}\left\|k\chi_1^+\chi_2^+\right\|_{Cv_p(NA)} &\leq \left\|\Delta_{N_1A_1}^{-1/p'}\,\Delta_{N_2A_2}^{-1/p'}\,k\chi_1^+\chi_2^+\right\|_{L^1(NA)} \\ &= \sum_{j_1,j_2\geq 0} q_1^{-j_1}\,q_2^{-j_2}\,q_1^{j_1/p'}\,q_2^{j_2/p'} \\ &\qquad \times \int_{N_1}\int_{N_2} |k(v_1\sigma_1^{j_1}\cdot o_1, v_2\sigma_2^{j_2}\cdot o_2)| \,\mathrm{d}\mu(s_1)\,\mathrm{d}\mu(s_2).\end{aligned} \tag{5.26}$$

The usual Abel transform argument, this time applied to both variables, implies the following equation

$$\begin{aligned}&\int_{N_1}\int_{N_2} |k(v_1\sigma_1^{j_1}\cdot o_1, v_2\sigma_2^{j_2}\cdot o_2)| \,\mathrm{d}\mu(s_1)\,\mathrm{d}\mu(s_2) \\ &= q_1^{j_1}\,q_2^{j_2}\int_{N_1}\int_{N_2} |k(v_1\sigma_1^{-j_1}\cdot o_1, v_2\sigma_2^{-j_2}\cdot o_2)| \,\mathrm{d}\mu(s_1)\,\mathrm{d}\mu(s_2).\end{aligned} \tag{5.27}$$

Since j_1 and j_2 are nonnegative, we may apply Lemma 5.2 in both variables, and we obtain

$$\begin{aligned}&\int_{N_1}\int_{N_2} |k(v_1\sigma_1^{-j_1}\cdot o_1, v_2\sigma_2^{-j_2}\cdot o_2)| \,\mathrm{d}\mu(s_1)\,\mathrm{d}\mu(s_2) \\ &= q_1^{-j_1/p}\,q_2^{-j_2/p}\int_{N_1}\int_{N_2} Q_p(v_1)\,Q_p(v_2)\,|\varphi(|v_1\cdot o_1|+j_1, |v_2\cdot o_2|+j_2)| \,\mathrm{d}\mu(s_1)\,\mathrm{d}\mu(s_2).\end{aligned} \tag{5.28}$$

Equations (5.26), (5.27) and (5.28), along with the trivial estimate $\|\varphi\|_{L^\infty(\mathbb{Z}^2)} \leq C\ \|M_{(\delta(p),\delta(p))}\|_{L^\infty(\mathbb{T})}$ imply that

$$\begin{aligned}\|k\chi_1^+\chi_2^+\|_{Cv_p(NA)} &\leq \sum_{j_1,j_2\geq 0} q_1^{j_1(1/p'-1/p)}\, q_2^{j_2(1/p'-1/p)}\ \|\varphi\|_{L^\infty(\mathbb{Z}^2)} \\ &\qquad\qquad \times \int_{N_1} Q_p(v_1)\ \mathrm{d}\mu(s_1) \int_{N_2} Q_p(v_2)\ \mathrm{d}\mu(s_2) \\ &\leq C_p\ \|M_{(\delta(p),\delta(p))}\|_{L^\infty(\mathbb{T})} \\ &\leq C_p\ \|M_{(\delta(p),\delta(p))}\|_{\mathscr{M}_p(\mathbb{T})}\,,\end{aligned}$$

where we also used Lemma 5.3 in the second inequality above.

Step III: analysis of $k\chi_1^-$, part 2. We are left with $k\chi_1^-\chi_2^-$. We apply the change of co-ordinates in Lemma 5.2 in both variables, and we obtain

$$\begin{aligned}&k(v_1\sigma_1^{j_1}\cdot o_1, v_2\sigma_2^{j_2}\cdot o_2)\,\chi^-(j_1)\,\chi^-(j_2) \\ &= q_1^{j_1/p}\, q_2^{j_2/p}\, Q_p(v_1)\, Q_p(v_2)\, \varphi(|v_1\cdot o_1| - j_1, |v_2\cdot o_2| - j_2)\, \mathbf{1}_{(-\infty,-1]^2}(j_1,j_2).\end{aligned}$$

Observe that $N_1A_1 \times N_2A_2$ is homomorphic to $(N_1\times N_2) \ltimes (A_1\times A_2)$, where the action of $A_1\times A_2$ on $N_1\times N_2$ is defined as the conjugation on the corresponding component. We may apply Corollary 5.7 (with $\Gamma = (N_1\times N_2)\ltimes(A_1\times A_2)$ and $\mathscr{D} = q_1^{-j_1}\, q_2^{-j_2}$) to obtain the following estimate

$$\begin{aligned}\|k\chi_1^-\chi_2^-\|_{Cv_p(NA)} \leq \int_{N_1}\int_{N_2} &\|\varphi(|v_1\cdot o_1| - \cdot, |v_2\cdot o_2| - \cdot)\, \mathbf{1}_{(-\infty,-1]^2}\|_{Cv_p(\mathbb{Z}^2)} \\ &\times Q_p(v_1)\, Q_p(v_2)\, \mathrm{d}\mu(v_1)\ \mathrm{d}\mu(v_2).\end{aligned} \tag{5.29}$$

It is straightforward to check that

$$\|\varphi(|v_1\cdot o_1| - \cdot, |v_2\cdot o_2| - \cdot)\, \mathbf{1}_{(-\infty,-1]^2}\|_{Cv_p(\mathbb{Z})} = \|\varphi \mathbf{1}_{[|v_1\cdot o_1|,\infty)} \otimes \mathbf{1}_{[|v_2\cdot o_2|,\infty)}\|_{Cv_p(\mathbb{Z})}\,.$$

We need to verify that φ satisfies the hypothesis of Lemma 5.15. Indeed the multiplier corresponding to φ is $4c_{G_1}c_{G_2}\left(M\check{\mathbf{c}}^{-1}\right)_{(\delta(p),\delta(p))}$, which extends to a bounded, holomorphic function on $\Sigma^2_{2\delta(p)}$. Moreover the Weyl invariance of M implies that $M\check{\mathbf{c}}^{-1}$ is even in both variables, thus

$$\|\left(M\check{\mathbf{c}}^{-1}\right)_{(\delta(p),\delta(p))}\|_{L^\infty(\mathbb{T})} = \|\left(M\check{\mathbf{c}}^{-1}\right)_{(\pm\delta(p),\pm\delta(p))}\|_{L^\infty(\mathbb{T})}\,.$$

Since $M\check{\mathbf{c}}^{-1}$ is holomorphic in $\mathbf{S}_{\delta(p)}$, the equation above implies that

$$\big\|\big(M\check{\mathbf{c}}^{-1}\big)_{(\delta(p),\delta(p))}\big\|_{H^\infty(\Sigma^2_{2\delta(p)})} = \big\|\big(M\check{\mathbf{c}}^{-1}\big)_{(\delta(p),\delta(p))}\big\|_{L^\infty(\mathbb{T})} .$$

Finally we can use Lemma 5.15 to get

$$\big\|\varphi(|v_1\cdot o_1|-\cdot,|v_2\cdot o_2|-\cdot)\,\mathbf{1}_{(-\infty,-1]^2}\big\|_{Cv_p(\mathbb{Z}^2)} \leq C\,\big(1+|v_1{\cdot}o_1|\big)\,\big(1+|v_2{\cdot}o_2|\big)\,\big\|\varphi\big\|_{Cv_p(\mathbb{Z}^2)} .$$

Recall that Q_p is integrable on N even if it is multiplied by a power of $|v\cdot o|$ (see Lemma 5.3), thus (5.29) yelds

$$\begin{aligned}\big\|k\chi_1^-\chi_2^-\big\|_{Cv_p(NA)} &\leq C\,\big\|\varphi\big\|_{Cv_p(\mathbb{Z}^2)}\int_{N_1}(1+|v_1\cdot o_1|)\,Q_p(v_1)\;\mathrm{d}\mu(v_1)\\ &\qquad\qquad\times\int_{N_2}(1+|v_2\cdot o_1|)\,Q_p(v_2)\;\mathrm{d}\mu(v_2)\\ &\leq C_p\,\big\|\varphi\big\|_{Cv_p(\mathbb{Z}^2)}\\ &\leq C_p\,\big\|M\big\|_{\mathscr{M}_p(\mathbb{T})} .\end{aligned}$$

Step IV: conclusion. By combining the estimates proved in *Steps I-II-III* we see that there exists a constant C, independent of M, such that

$$\begin{aligned}\big\|k\big\|_{Cv_p(NA)} &\leq \big\|k\chi_1^+\big\|_{Cv_p(NA)} + \big\|k\chi_1^-\chi_2^+\big\|_{Cv_p(NA)} + \big\|k\chi_1^-\chi_2^-\big\|_{Cv_p(NA)}\\ &\leq C\,\big\|M\big\|_{\mathscr{M}_p(\mathbb{T})} .\end{aligned}$$

Since k is radial in both of its variables, $\big\|k\big\|_{Cv_p(\mathscr{T})} = \big\|k\big\|_{Cv_p(NA)}$, so k belongs to $Cv_p(\mathscr{T})$, and the required estimate holds.

This concludes the proof of the theorem. □

Remark 5.17. Theorem 5.16 is the exact counterpart of the implication (ii) ⇒ (i) of Theorem 5.11. It is an interesting question whether it is true also the converse implication, that is a two-variables version of [CMS3, Theorem 2.1]. At least in the case of kernels k that factorises as $k_1\otimes k_2$ the answer to this question is in the affirmative.

Indeed consider a convolutor k on $L^p(\mathscr{T})$ such that

$$k(x,y)=k_1(x)\,k_2(y) \qquad\qquad \forall(x,y)\in\mathscr{T}_1\times\mathscr{T}_2,$$

where k_1 and k_2 are radial functions on $\mathscr{T}_1$ and $\mathscr{T}_2$ respectively. Since k is in $Cv_p(\mathscr{T})$, it is easy to prove that for every fixed x_0 in $\mathscr{T}_1$, $k(x_0, \cdot) = k_1(x_0)k_2(\cdot)$ is a convolutor of $L^p(\mathscr{T}_2)$. Thus [CMS3, Theorem 2.1] implies that $\widetilde{k}_2$ extends to a bounded, Weyl invariant, holomorphic function on $\mathbf{S}_{\delta(p)}$, whose boundary values belong to $\mathscr{M}_p(\mathbb{T}_2)$. Exchanging the role of the variables it is possible to prove that the same is true for $\widetilde{k}_1$. Now consider a function f in $L^p(\mathscr{T}_1)$ and a function g in $L^p(\mathscr{T}_2)$, so that $f \otimes g$ is in $L^p(\mathscr{T})$ and $\|f \otimes g\|_{L^p(\mathscr{T})} = \|f\|_{L^p(\mathscr{T}_1)} \|g\|_{L^p(\mathscr{T}_2)}$. Then $(f \otimes g) * (k_1 \otimes k_2) = (f * k_1) \otimes (g * k_2)$, and

$$\begin{aligned} \|k\|_{Cv_p(\mathscr{T})} &\geq \sup_{\|f\|_{L^p(\mathscr{T}_1)}=1} \sup_{\|g\|_{L^p(\mathscr{T}_2)}=1} \|(f \otimes g) * (k_1 \otimes k_2)\|_{L^p(\mathscr{T})} \\ &= \|k_1\|_{Cv_p(\mathscr{T}_1)} \|k_2\|_{Cv_p(\mathscr{T}_2)} . \end{aligned}$$

Thus the norm equivalence in Theorem 5.11, applied in both variables, implies that thers exists a positive constant C such that

$$\|k\|_{Cv_p(\mathscr{T})} \geq C \ \|(\widetilde{k}_1)_{\delta(p)}\|_{\mathscr{M}_p(\mathbb{T}_1)} \|(\widetilde{k}_2)_{\delta(p)}\|_{\mathscr{M}_p(\mathbb{T}_2)} .$$

Finally it suffices to show that

$$\|(\widetilde{k}_1)_{\delta(p)}\|_{\mathscr{M}_p(\mathbb{T}_1)} \|(\widetilde{k}_2)_{\delta(p)}\|_{\mathscr{M}_p(\mathbb{T}_2)} \geq \|\widetilde{k}_{(\delta(p),\delta(p))}\|_{\mathscr{M}_p(\mathbb{T})} . \tag{5.30}$$

Define φ as in (5.18), with $\widetilde{k}_1 \otimes \widetilde{k}_2$ in place of M, and note that also φ factorises as $\varphi_1 \otimes \varphi_2$ (where φ_1 and φ_2 are defined on $\mathbb{Z}$). By definition (5.30) is equivalent to

$$\|\varphi_1 \otimes \varphi_2\|_{Cv_p(\mathbb{Z}^2)} \leq \|\varphi_1\|_{Cv_p(\mathbb{Z}_1)} \|\varphi_2\|_{Cv_p(\mathbb{Z}_2)} .$$

Given a function F in $L^p(\mathbb{Z}^2)$, a straightforward computation implies that

$$F * (\varphi_1 \otimes \varphi_2) = [F * (\varphi_1 \otimes \delta_0)] * (\delta_0 \otimes \varphi_2),$$

so

$$\begin{aligned} \|F * \varphi\|_{L^p(\mathbb{Z}^2)} &= \|[F * (\varphi_1 \otimes \delta_0)] * (\delta_0 \otimes \varphi_2)\|_{L^p(\mathbb{Z}^2)} \\ &\leq \|\varphi_2\|_{Cv_p(\mathbb{Z}_2)} \|F * (\varphi_1 \otimes \delta_0)\|_{L^p(\mathbb{Z}^2)} \\ &\leq \|\varphi_1\|_{Cv_p(\mathbb{Z}_1)} \|\varphi_2\|_{Cv_p(\mathbb{Z}_2)} \|F\|_{L^p(\mathbb{Z}^2)} . \end{aligned}$$

We conclude that

$$\|\varphi\|_{Cv_p(\mathbb{Z}^2)} \leq \|\varphi_1\|_{Cv_p(\mathbb{Z}_1)} \|\varphi_2\|_{Cv_p(\mathbb{Z}_2)} ,$$

as required.

5.5 Hardy-type spaces on products

Denote by $\mathscr{L}_1$ and $\mathscr{L}_2$ the standard nearest neighbour Laplacians on $\mathscr{T}_1$ and $\mathscr{T}_2$. It is natural to speculate whether $L^1(\mathscr{T}_1 \times \mathscr{T}_2)$ admits subspaces that play for harmonic analysis on $\mathscr{T}_1 \times \mathscr{T}_2$ much the same role as the Hardy-type spaces $\mathfrak{X}^\alpha(\mathscr{G})$ plays for harmonic analysis on the graph $\mathscr{G}$.

It may be worth observing that the definition of the product Hardy space $H^1(\mathbb{R}^m \times \mathbb{R}^n)$ is not a straightforward generalisation of that of the classical Hardy space $H^1(\mathbb{R}^m)$.

As a preliminary observation, notice that $\mathscr{L}_1\mathscr{L}_2$ is injective on $L^2(\mathscr{T})$, hence on $L^1(\mathscr{T})$. This may be easily seen via spherical Fourier analysis. Indeed, it is straightforward to check that

$$\widetilde{k}_{\mathscr{L}_1\mathscr{L}_2}(s_1, s_2) = \big(1 - \gamma_1(s_1)\big)\,\big(1 - \gamma_2(s_2)\big) \qquad \forall (s_1, s_2) \in \mathbb{T}_1 \times \mathbb{T}_2.$$

Since γ_j is a continuous function on $\mathbb{T}_j$ and $1 - \gamma_j(s)$ does not vanish therein, the function

$$(s_1, s_2) \mapsto \frac{1}{\big(1 - \gamma_1(s_1)\big)\,\big(1 - \gamma_2(s_2)\big)}$$

is bounded on $\mathbb{T}_1 \times \mathbb{T}_2$, whence $\mathscr{L}_1\mathscr{L}_2$ is invertible on $L^2(\mathscr{T})$. Thus, $\mathscr{L}_1\mathscr{L}_2$ is injective: consequently so is $\big(\mathscr{L}_1\mathscr{L}_2\big)^\alpha$.

A slight generalisation of the argument above shows that $\mathscr{L}_1\mathscr{L}_2$ is, in fact, a Banach space isomorphism of $L^p(\mathscr{T}_1 \times \mathscr{T}_2)$ for every p in $(1, \infty)$. This simple fact has the following important consequence.

Definition 5.18. Suppose that α is a positive real number. Denote by $\mathfrak{X}^\alpha(\mathscr{T})$ the space $\big(\mathscr{L}_1\mathscr{L}_2\big)^\alpha\big(L^1(\mathscr{T})\big)$, endowed with the norm

$$\big\|f\big\|_{\mathfrak{X}^\alpha(\mathscr{T})} := \big\|\big(\mathscr{L}_1\mathscr{L}_2\big)^{-\alpha} f\big\|_{L^1(\mathscr{T})}\,.$$

Observe that the definition above makes sense because of the injectivity of $\big(\mathscr{L}_1\mathscr{L}_2\big)^\alpha$.

We adopt the same notation as in Section 2.2.

Corollary 5.19. *Suppose that θ is in $(0,1)$, and that $\alpha > 0$. If p_θ is $2/(2-\theta)$, then*

$$\big(\mathfrak{X}^\alpha(\mathscr{T}), L^2(\mathscr{T})\big)_{[\theta]} = L^{p_\theta}(\mathscr{T}).$$

Proof. The proof follows the lines of the proof of Corollary 2.9, with the role of $\mathscr{L}$ therein played here by the operator $\mathscr{L}_1\mathscr{L}_2$. We omit the details. □

Remark 5.20. Notice that Corollary 5.19 holds in the more general setting of product of two graphs with bounded geometry and spectral gap. We omit the details.

Next, we consider an application of the Hardy-type space $\mathfrak{X}^\alpha(\mathscr{T}_1 \times \mathscr{T}_2)$ to spherical multipliers. Notice that $\mathscr{T}_1 \times \mathscr{T}_2$ has a natural Laplacian $\mathscr{L}$, given by $\mathscr{L}_1 + \mathscr{L}_2$. It is well known that estimates for functions of $\mathscr{L}$, such as $\mathscr{L}^{iu}$, with u in $\mathbb{R}$, may be obtained as a consequence of Hörmander type multiplier results. However, such multiplier results do not apply to multipliers of the form

$$(s_1, s_2) \mapsto \big(1 - \gamma_1(s_1)\big)^{iu} \big(2 - \gamma_1(s_1) - \gamma_2(s_2)\big)^{iv},$$

which correspond to the operators $\mathscr{L}_1^{iu}(\mathscr{L}_1 + \mathscr{L}_2)^{iv}$.

The next result is a corollary of Theorem 5.16, and it is the counterpart of Corollary 5.12 in the present setting. We denote by $1 - \gamma = (1 - \gamma_1)(1 - \gamma_2)$ the spherical multiplier corresponding to $\mathscr{L}_1\mathscr{L}_2$.

Corollary 5.21. *Suppose that α is in $(0,\infty)$, that M is a holomorphic Weyl invariant function on $\mathbf{S}^2_{1/2}$, and that $[(1-\gamma)^\alpha\, M]_{(1/2,1/2)}$ is in $\mathscr{M}_1(\mathbb{T})$. Then the inverse spherical Fourier transform k_M is the kernel of a bounded convolution operator T_M from $\mathfrak{X}^\alpha(\mathscr{T})$ to $L^1(\mathscr{T})$. Furthermore, there exists a positive constant C, independent of M, such that*

$$\left|\!\left|\!\left| T_M \right|\!\right|\!\right|_{\mathfrak{X}^\alpha(\mathscr{T});L^1(\mathscr{T})} \leq C\ \left\| [(1-\gamma)^\alpha\, M]_{(1/2,1/2)} \right\|_{\mathscr{M}_1(\mathbb{T})}.$$

Proof. The thesis follows from Theorem 5.16 with $p = 1$. We only need to observe that the multiplier corresponding to $(\mathscr{L}_1\mathscr{L}_2)^\alpha$ is $(1-\gamma)^\alpha$. □

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