TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 366, Number 7, July 2014, Pages 3729–3745
S 0002-9947(2014)05992-8
Article electronically published on March 14, 2014

ON THE ORDER OF ARC-STABILISERS IN ARC-TRANSITIVE GRAPHS WITH PRESCRIBED LOCAL GROUP

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ABSTRACT. Let Γ be a connected G-arc-transitive graph, let uv be an arc of Γ and let L be the permutation group induced by the action of the vertex-stabiliser G_v on the neighbourhood $\Gamma(v)$. We study the problem of bounding $|G_{uv}|$ in terms of L and the order of Γ.

1. INTRODUCTION

All graphs considered in this paper are finite, simple and connected. A graph Γ is said to be *G-vertex-transitive* if G is a subgroup of $\mathrm{Aut}(\Gamma)$ acting transitively on the vertex-set $\mathrm{V}(\Gamma)$ of Γ. Similarly, Γ is said to be *G-arc-transitive* if G acts transitively on the arc-set $\mathrm{A}(\Gamma)$ of Γ (an *arc* is an ordered pair of adjacent vertices).

For a vertex v of Γ and for $G \leq \mathrm{Aut}(\Gamma)$, let $\Gamma(v)$ be the neighbourhood of v in Γ and let $G_v^{\Gamma(v)}$ be the permutation group induced by the action of the stabiliser G_v on $\Gamma(v)$. We shall often refer to the group $G_v^{\Gamma(v)}$ as the *local group* of the pair (Γ, G). Of course, if G acts transitively on the arcs of Γ, then the local group $G_v^{\Gamma(v)}$ is transitive and (up to permutation isomorphism) independent of the choice of v. If Γ is a G-arc-transitive graph and L is a permutation group which is permutation isomorphic to $G_v^{\Gamma(v)}$, then we say that the pair (Γ, G) is *locally-L*.

In [11], the following notion was introduced: a transitive permutation group L is called *graph-restrictive* if there exists a constant $c(L)$ such that, for every locally-L pair (Γ, G) and for every arc (u, v) of Γ, the inequality $|G_{uv}| \leq c(L)$ holds. Proving that certain permutation groups are graph-restrictive is a classical topic in algebraic graph theory; for example, the famous result of Tutte on cubic arc-transitive graphs [9, 10] states that the symmetric group of degree 3 is graph-restrictive and the still open conjectures of Weiss [13] and Praeger [7] claim that every primitive as well as every quasiprimitive permutation group is graph-restrictive. The problem of determining which transitive permutation groups are graph-restrictive was proposed in [11]. A survey of the state of this problem can be found in [5].

There are several reasons why one might like to control the order of the arc-stabiliser G_{uv} in a locally-L pair (Γ, G) even when the local group L is not graph-restrictive. While $|G_{uv}|$ can be arbitrarily large in this case, it would often suffice to obtain a good bound on $|G_{uv}|$ in terms of $|\mathrm{V}(\Gamma)|$; for example, if $|G_{uv}|$ can be bounded by a reasonably tame function of $|\mathrm{V}(\Gamma)|$, then the method described in [1]

Received by the editors July 24, 2012 and, in revised form, October 10, 2012.

2010 *Mathematics Subject Classification.* Primary 20B25; Secondary 05E18.

Key words and phrases. Arc-transitive graphs, arc-stabiliser, graph-restrictive group, local group.

can be applied to obtain a complete list of all locally-L pairs on a small number of vertices.

Bounding $|G_{uv}|$ in terms of $|\mathrm{V}(\Gamma)|$ and the local group $G_v^{\Gamma(v)}$ is precisely the goal that we pursue in this paper. In fact, it is not hard to see that there is always an exponential upper bound on $|\mathrm{Aut}(\Gamma)_{uv}|$ in terms of $|\mathrm{V}(\Gamma)|$ (see Theorem 5). It is thus very natural to ask for which local groups a subexponential upper bound exists. This question motivates the following definition.

Definition 1. Let $f\colon \mathbb{N} \to \mathbb{R}$ be a function and let L be a transitive permutation group. If, for every locally-L pair (Γ, G) and every arc (u, v) of Γ, the inequality $|G_{uv}| \leq f(|V(\Gamma)|)$ holds, then L is called *f-graph-restrictive*. On the other hand, if, for every integer n, there exists a locally-L pair (Γ, G) with $|\mathrm{V}(\Gamma)| \geq n$ and $|G_{uv}| \geq f(|\mathrm{V}(\Gamma)|)$, then L is called *f-graph-unrestrictive*.

Note that, for each transitive permutation group L, there exists a function f such that L is both f-graph-restrictive and f-graph-unrestrictive (for example, let $f(n)$ be the largest possible order of G_{uv} in a locally-L pair (Γ, G) with Γ having order n). On the other hand, finding such a function explicitly is quite difficult in general. We therefore define graph-restrictiveness and graph-unrestrictiveness over a class of functions.

Definition 2. Let $\mathcal{C}$ be a class of functions. If L is f-graph-restrictive (respectively, f-graph-unrestrictive) for some function $f \in \mathcal{C}$, then we say that L is *$\mathcal{C}$-graph-restrictive* (respectively, *$\mathcal{C}$-graph-unrestrictive*). If L is both $\mathcal{C}$-graph-restrictive and $\mathcal{C}$-graph-unrestrictive, then we say that L has *graph-type $\mathcal{C}$*.

Given a transitive group L, we would like to find a "natural" class of functions $\mathcal{C}$ such that L has graph-type $\mathcal{C}$. The classes of functions that will particularly interest us are: the class Cons of constant functions, the class Poly of functions of the form $f(n) = n^\alpha$ for some $\alpha > 0$ and the class Exp of functions of the form $f(n) = \alpha^n$ for some $\alpha > 1$.

We also define the intermediate classes: the class SubPoly of functions of the form $f(n)$ such that $f(n)$ is unbounded and $\frac{\log(f(n))}{\log(n)} \to 0$ as $n \to \infty$ and the class SubExp of functions of the form $f(n)$ such that $\frac{\log(f(n))}{\log(n)}$ is unbounded and $\frac{\log(f(n))}{n} \to 0$ as $n \to \infty$. Every transitive permutation group is Exp-graph-restrictive (see Theorem 5). It is then an elementary exercise in analysis to show that a transitive permutation group has graph-type exactly one of Cons, SubPoly, Poly, SubExp or Exp.

Problem 3. Given a transitive permutation group L, find $\mathcal{C}$ in $\{\mathrm{Cons}, \mathrm{SubPoly}, \mathrm{Poly}, \mathrm{SubExp}, \mathrm{Exp}\}$ such that L has graph-type $\mathcal{C}$.

We now give a brief summary of the results which are proved in the rest of the paper. In Section 2, we show that every transitive permutation group is Exp-graph-restrictive. In Section 3, we show that the imprimitive wreath product of two non-trivial transitive permutation groups is Exp-graph-unrestrictive and hence has graph-type Exp.

In Section 4, we consider a permutation group L that is transitive and admits a system of imprimitivity consisting of two blocks A and B and show that L is Poly-graph-unrestrictive unless L is regular. Moreover, we show that if the pointwise stabiliser of A in L is non-trivial, then L is actually Exp-graph-unrestrictive and thus has graph-type Exp.

In Section 5, we consider the imprimitive permutation groups of degree 6 that do not admit a system of imprimitivity consisting of two blocks of size 3. There exist five such groups up to permutation isomorphism. Two of them are imprimitive wreath products and hence have graph-type Exp. We show that the remaining three groups are SubExp-graph-unrestrictive. However, for none of these three groups were we able to decide whether it has graph-type SubExp or Exp.

Finally, in Section 6, we apply these results to solve Problem 3 for permutation groups of degree at most 7, except for the three undecided cases mentioned in the previous paragraph. It turns out that there is a unique transitive permutation group of degree at most 7 with graph-type Poly (namely, the dihedral group of order 12 in its natural action on six points) while there are many with graph-types Cons and Exp. On the other hand, we know of no transitive permutation group with graph-type SubPoly or SubExp.

Question 4. Does there exist a transitive permutation group with graph-type SubPoly or SubExp?

2. General exponential upper bound

In this section, we show that every transitive permutation group is Exp-graph-restrictive.

Theorem 5. *Let L be a transitive permutation group and let L_ω be a point-stabiliser in L. Then L is f-graph-restrictive where $f(n) = |L_\omega|^{\frac{n-2}{2}}$. In particular, L is* Exp-*graph-restrictive.*

Proof. Let (Γ, G) be a locally-L pair, let $n = |\mathrm{V}(\Gamma)|$ and let (u, v) be an arc of Γ. Recall that a group A is called a *section* of a group B provided that A is isomorphic to a quotient of some subgroup of B.

We shall now recursively construct an increasing sequence of subsets S_i of $\mathrm{V}(\Gamma)$ and a decreasing subnormal sequence of subgroups G_i of G,

$$S_0 \subseteq S_1 \subseteq \cdots \subseteq S_m \quad \text{and} \quad G_0 \trianglerighteq G_1 \trianglerighteq \cdots \trianglerighteq G_m, \tag{1}$$

such that conditions (i)–(v) are fulfilled. (We use the notation $\Gamma[S]$ to denote the subgraph of a graph Γ induced by a set of vertices $S \subseteq \mathrm{V}(\Gamma)$.)

(i) $S_0 = \{u, v\}$ and $|S_i| \geq |S_{i-1}| + 2$ for every $i \in \{1, \dots, m\}$,
(ii) $\Gamma[S_i]$ is connected for every $i \in \{0, \dots, m\}$,
(iii) $G_i = G_{(S_i)}$ for every $i \in \{0, \dots, m\}$,
(iv) G_{i-1}/G_i is a section of L_ω for every $i \in \{1, \dots, m\}$,
(v) $G_m = 1$.

Let $S_0 = \{u, v\}$ and let $G_0 = G_{(S_0)} = G_{uv}$. Clearly conditions (i)–(iv) are satisfied with $m = 1$. Suppose now that for some $k \geq 0$ the sets $S_0, \dots, S_k$ and the groups $G_0, \dots, G_k$ are defined and that they satisfy conditions (i)–(iv) with $m = k$.

Let S be the set of all vertices of Γ that are fixed by G_k and let S' be the vertex-set of the connected component of $\Gamma[S]$ that contains S_k. Then clearly $S_k \subseteq S' \subseteq S$ hence $G_k \leq G_{(S)} \leq G_{(S')} \leq G_{(S_k)} = G_k$ and $G_{(S')} = G_k$.

If $S' = \mathrm{V}(\Gamma)$, then we let $m = k$ and terminate the construction. Observe that in this case the group G_m is trivial, as required by condition (v).

If S' is a proper subset of $\mathrm{V}(\Gamma)$, then it follows from the definition of S' that there exists a vertex in $\mathrm{V}(\Gamma) \setminus S'$, say x, which is adjacent to some vertex, say w, in

S' and which is not fixed by G_k. Moreover, since $\Gamma[S']$ is connected and contains at least two vertices, there exists a neighbour of w, say z, which is contained in S'.

Let $X = x^{G_k}$, let $S_{k+1} = S' \cup X$ and let $G_{k+1} = G_{(S_{k+1})}$. We need to show that the extended sequences $(S_i)_i$ and $(G_i)_i$ still satisfy conditions (i)–(iv) with $m = k+1$. Indeed, since x is not fixed by G_k, we see that $|X| \geq 2$ and thus condition (i) holds. Similarly, conditions (ii) and (iii) hold by construction of S_{k+1} and G_{k+1}. To show (iv), observe that G_{k+1} is the kernel of the action of G_k on X. Hence G_k/G_{k+1} is isomorphic to the permutation group G_k^X induced by the action of G_k on X. However, G_k^X can also be viewed as the permutation group induced by the action of $G_k^{\Gamma(w)}$ on X, and is thus isomorphic to a quotient of $G_k^{\Gamma(w)}$. Since $G_k \leq G_{wz}$, it follows that G_k^X is a section of $G_{wz}^{\Gamma(w)}$. Since the latter group is isomorphic to L_ω, this shows that condition (iv) holds as well.

The result of the above construction is thus a pair of sequences (1) satisfying conditions (i)–(v). Now observe that condition (i) implies that $m \leq \frac{n-2}{2}$. On the other hand, condition (iv) implies that $|G_i/G_{i+1}| \leq |L_\omega|$ for every $i \in \{1, \ldots, m\}$. In view of condition (v), this implies that $|G_0| \leq |L_\omega|^m$. Since $G_0 = G_{uv}$, this completes the proof. □

Remark. The upper bound provided by the function f in Theorem 5 is rather crude and can be improved if some further information about the permutation group L is taken into consideration. For example, if p is the smallest prime dividing $|L_\omega|$, then the orbit X introduced in the proof of Theorem 5 is of length at least p and thus condition (i) can be replaced by $|S_i| \geq |S_{i+1}| + p$. The definition of f in Theorem 5 can then be replaced by $f(n) = |L_\omega|^{\frac{n-2}{p}}$.

3. Imprimitive wreath products

In this section, we show that the imprimitive wreath product of two non-trivial transitive permutation groups is Exp-graph-unrestrictive. First we need the following lemma.

Lemma 6. *Let T be a transitive permutation group and let T_ω be a point-stabiliser in T. Then there exists a sequence of locally-T pairs (Γ_i, H_i), with $|V(\Gamma_i)| \to \infty$ as $i \to \infty$, such that, for every $i \geq 1$, the stabiliser of an arc of Γ_i in H_i has order $|T_\omega|^2$.*

Proof. There are several ways to prove the existence of such a sequence. We construct it recursively using the theory of covering projections of graphs. We refer the reader to [3] for further information on this topic. In particular, we refer the reader to [3, Section 6] for the definition of a homological p-cover of a graph.

Let k be the degree of T, let $\Gamma_1 = \mathrm{K}_{k,k}$ be the complete bipartite graph with bipartition sets of size k, and let $H_1 = (T \times T) \rtimes \mathrm{S}_2$ acting arc-transitively on Γ_1 in the natural way. Note that (Γ_1, H_1) is locally-T and an arc-stabiliser has order $|T_\omega|^2$.

Suppose now that (Γ_i, H_i) has already been constructed for some $i \geq 1$. Let Γ_{i+1} be a homological 2-cover of Γ_i and let H_{i+1} be the lift of H_i along the covering projection $\Gamma_{i+1} \to \Gamma_i$ (note that by [3, Proposition 6.4] the group H_i indeed lifts along this covering projection). Note that the vertex-stabilisers in H_{i+i} and in H_i are isomorphic and, moreover, that they induce permutation isomorphic groups on the respective neighbourhoods. In particular, the pair (Γ_{i+1}, H_{i+1}) is locally-T and an arc-stabiliser has order $|T_\omega|^2$. □

Theorem 7. *Let R and T be non-trivial transitive permutation groups, let m be the degree of R and let T_ω be a point-stabiliser in T. Then the imprimitive wreath product $R \operatorname{wr} T$ is f-graph-unrestrictive where $f(n) = \frac{|T_\omega|^2|R|^{\frac{n}{m}}}{m^2}$. In particular, $R \operatorname{wr} T$ has graph-type* Exp.

Proof. Let Δ and Ω be the sets on which R and T act, respectively. Then $R \operatorname{wr} T$ is a permutation group on $\Delta \times \Omega$. Writing $\Omega = \{1, 2, \dots, k\}$ yields that $R \operatorname{wr} T$ is isomorphic to the semidirect product $R^k \rtimes T$ where the action of $(a_1, \dots, a_k) \in R^k$ and $b \in T$ on $\Delta \times \Omega$ is given by:

$$(\delta, \omega)^{(a_1,\dots,a_k)} = (\delta^{a_\omega}, \omega) \quad \text{and} \quad (\delta, \omega)^b = (\delta, \omega^b).$$

By Lemma 6, there exists a sequence of locally-T pairs (Γ_i, H_i), with $|V(\Gamma_i)| \to \infty$, such that, for every $i \geq 1$, the stabiliser of an arc of Γ_i in H_i has order $|T_\omega|^2$. Let Λ_i be the lexicographic product of Γ_i with the edgeless graph on the vertex-set Δ, that is, the graph with vertex-set $\Delta \times \mathrm{V}(\Gamma_i)$ and two vertices (δ_1, v_1) and (δ_2, v_2) adjacent in Λ_i whenever v_1 and v_2 are adjacent in Γ. Note that $|\mathrm{V}(\Lambda_i)| = m|\mathrm{V}(\Gamma_i)|$ and that $|\mathrm{A}(\Lambda_i)| = m^2|\mathrm{A}(\Gamma_i)|$.

Observe that the imprimitive wreath product $G_i = R \operatorname{wr} H_i$ acts on Λ_i as an arc-transitive group of automorphisms and that the local group $(G_i)_v^{\Lambda_i(v)}$ is permutation isomorphic to $R \operatorname{wr} T$.

Let (u, v) be an arc of Γ_i and let $(\tilde{u}, \tilde{v})$ be an arc of Λ_i. Since Γ_i is H_i-arc-transitive and Λ_i is G_i-arc-transitive, it follows that:

$$|(G_i)_{\tilde{u}\tilde{v}}| = \frac{|G_i|}{|\mathrm{A}(\Lambda_i)|} = \frac{|R|^{|\mathrm{V}(\Gamma_i)|}|H_i|}{m^2|\mathrm{A}(\Gamma_i)|} = \frac{|(H_i)_{uv}||R|^{\frac{|\mathrm{V}(\Lambda_i)|}{m}}}{m^2} = \frac{|T_\omega|^2|R|^{\frac{|\mathrm{V}(\Lambda_i)|}{m}}}{m^2}.$$

Since $|\mathrm{V}(\Lambda_i)| \to \infty$ as $i \to \infty$, the result follows. □

4. System of imprimitivity consisting of two blocks

In this section, we consider a permutation group L that is transitive and admits a system of imprimitivity consisting of two blocks A and B and show that L is Poly-graph-unrestrictive unless it is regular. Moreover, we show that if the pointwise stabiliser of A in L is non-trivial, then L is actually Exp-graph-unrestrictive.

To do this, we must first define the graphs $\mathrm{C}(k, r, s)$, which were first defined by Praeger and Xu [8]. Let r and s be positive integers with $r \geq 3$ and $1 \leq s \leq r - 1$. Let $\mathrm{C}(k, r, 1)$ be the lexicographic product $\mathrm{C}_r[k\mathrm{K}_1]$ of a cycle of length r and an edgeless graph on k vertices. In other words, $\mathrm{V}(\mathrm{C}(k, r, 1)) = \mathbb{Z}_k \times \mathbb{Z}_r$ with (u, i) being adjacent to (v, j) if and only if $i - j \in \{-1, 1\}$. A path in $\mathrm{C}(k, r, 1)$ is called *traversing* if it contains at most one vertex from $\mathbb{Z}_k \times \{y\}$ for each $y \in \mathbb{Z}_r$. For $s \geq 2$, let $\mathrm{C}(k, r, s)$ be the graph with vertices being the traversing paths in $\mathrm{C}(k, r, 1)$ of length $s - 1$ and with two such $(s-1)$-paths being adjacent in $\mathrm{C}(k, r, s)$ if and only if their union is a traversing path in $\mathrm{C}(k, r, 1)$ of length s.

Clearly, $\mathrm{C}(k, r, s)$ is a connected $2k$-valent graph with rk^s vertices. There is an obvious action of the wreath product $\mathrm{S}_k \operatorname{wr} \mathrm{D}_r$ as a group of automorphisms of $\mathrm{C}(k, r, 1)$. (We denote the symmetric group and the dihedral group in their natural action on n points by S_n and D_n, respectively.) Moreover, every automorphism of

$\mathrm{C}(k, r, 1)$ has a natural induced action as an automorphism of $\mathrm{C}(k, r, s)$. We use these graphs to prove the following result.

Theorem 8. *Let L be a transitive permutation group of degree $2k$ with a system of imprimitivity consisting of two blocks A and B. Let L_ω be a point-stabiliser and let $L_{(A)}$ be the pointwise stabiliser of A in L. Let $\ell \geq 1$ and $m \geq 2$ be integers, let $\Gamma = \mathrm{C}(k, 2\ell m, m-1)$ and let uv be an arc of Γ. Then there exists a group $G \leq \mathrm{Aut}(\Gamma)$ such that (Γ, G) is locally-L and $|G_{uv}| = |L_{(A)}|^{2m(\ell-1)}|L_\omega|^m$.*

Proof. Let K be the kernel of the action of L on $\{A, B\}$. Clearly, K is a normal subgroup of index 2 in L. Fix an element $h \in L \setminus K$ and observe that $A^h = B$ and $B^h = A$. We can label the points of A by $\mathbb{Z}_k \times \{0\}$ and the points of B by $\mathbb{Z}_k \times \{1\}$ in such a way that $(x, 0)^h = (x, 1)$ for every $x \in \mathbb{Z}_k$. With respect to this fixed labeling, we can view L as a subgroup of $\mathrm{S}_k \,\mathrm{wr}\, \mathrm{S}_2 = (\mathrm{S}_k \times \mathrm{S}_k) \rtimes \mathrm{S}_2$, with $K \leq \mathrm{S}_k \times \mathrm{S}_k$, where, for every $(a, b) \in K \leq \mathrm{S}_k \times \mathrm{S}_k$ and every $x \in \mathbb{Z}_k$, we have

$$(x, 0)^{(a,b)} = (x^a, 0) \text{ and } (x, 1)^{(a,b)} = (x^b, 1). \tag{2}$$

Let $r \in \mathrm{S}_k$ be such that $(x, 1)^h = (x^r, 0)$ for every $x \in \mathbb{Z}_k$. Then $(x, 0)^{h^2} = (x, 1)^h = (x^r, 0)$ and $(x, 1)^{h^2} = (x^r, 0)^h = (x^r, 1)$, implying that

$$h^2 = (r, r). \tag{3}$$

To summarise, the action of h on $A \cup B$ is given by:

$$(x, 0)^h = (x, 1) \quad \text{and} \quad (x, 1)^h = (x^r, 0). \tag{4}$$

If $(a, b) \in K$ and $x \in \mathbb{Z}_k$, then $(x, 1)^{(a,b)^h} = (x, 0)^{(a,b)h} = (x^a, 0)^h = (x^a, 1)$ and $(x, 0)^{(a,b)^h} = (x, 0)^{h(a,b)^{h^2}h^{-1}} = (x, 1)^{(a^r,b^r)h^{-1}} = (x^{b^r}, 1)^{h^{-1}} = (x^{b^r}, 0)$. This shows that

$$(a, b)^h = (b^r, a). \tag{5}$$

Let $\Lambda = \mathrm{C}(k, 2\ell m, 1)$. Recall that $\mathrm{V}(\Lambda) = \mathbb{Z}_k \times \mathbb{Z}_{2\ell m}$ and that $\mathrm{S}_k \,\mathrm{wr}\, \mathrm{D}_{2\ell m}$ acts naturally as a group of automorphisms of Λ. We now define some permutations of $\mathrm{V}(\Lambda)$. Let $(x, y) \in \mathbb{Z}_k \times \mathbb{Z}_{2\ell m}$ be a vertex of Λ, let $(x, y)^s = (x, y+1)$ and let $(x, y)^t = (x, -y)$. Moreover, for any $c \in \mathrm{S}_k$ and any $i \in \mathbb{Z}_{2\ell m}$, let

$$(x, y)^{[c]_i} = \begin{cases} (x^c, y) & \text{if } y = i, \\ (x, y) & \text{otherwise.} \end{cases}$$

Note that for every $c, d \in \mathrm{S}_k$ and every $i, j \in \mathbb{Z}_{2\ell m}$, we have

$$[c]_i[d]_i = [cd]_i \text{ and } [c]_i[d]_j = [d]_j[c]_i \text{ if } i \neq j. \tag{6}$$

Clearly, s, t and $[c]_i$ are automorphisms of Λ. Note that $t^2 = 1$ and $tst = s^{-1}$. Moreover, for every $i, j \in \mathbb{Z}_{2\ell m}$, we have

$$[c]_i^{s^j} = [c]_{i+j} \text{ and } [c]_i^t = [c]_{-i}. \tag{7}$$

A typical element of $L_{(A)}$ can be written in the form $(1, b)$ with $b \in \mathrm{S}_k$, and moreover, an element $(a, b) \in K$ belongs to $L_{(A)}$ if and only if $a = 1$. Now let

$$M_0 = \{[b]_0 \mid (1, b) \in K\} \tag{8}$$

and note that $M_0 \leq \mathrm{Aut}(\Lambda)$. For $a \in \mathrm{S}_k$ and $i \in \mathbb{Z}_{2\ell m}$, let $\chi(a, i)$ be the automorphism of Λ defined by

$$\chi(a, i) = [a]_i\,[a]_{i+2m}\,[a]_{i+4m}\,\cdots\,[a]_{i+(2\ell-2)m}. \tag{9}$$

Note that for every $a, b \in \mathrm{S}_k$ the following holds:

$$
(10)\qquad
\begin{array}{ll}
(10.1) & \chi(a,i) = \chi(a,j) \text{ whenever } i \equiv j \bmod 2m,\\
(10.2) & \chi(a,i)\chi(b,i) = \chi(ab,i),\\
(10.3) & \chi(a,i) \text{ and } \chi(b,j) \text{ commute whenever } i \not\equiv j \bmod 2m,\\
(10.4) & \chi(a,i)^s = \chi(a,i+1),\\
(10.5) & \chi(a,i)^t = \chi(a,-i).
\end{array}
$$

Let

$$N_0 = \{\chi(a,0)\chi(b,m) \mid (a,b) \in K\}.$$

Using (10.2) and (10.3), one can see that N_0 is a subgroup of $\mathrm{Aut}(\Lambda)$. Recall that r is the element of S_k such that $h^2 = (r,r)$ and let

$$\sigma = \chi(r,-1)s.$$

Using (10.3) and (10.4), it is easy to see that, for $i \in \{1,\ldots,2m\}$, we have

$$\sigma^i = \chi(r,-i)\ldots\chi(r,-2)\chi(r,-1)s^i. \tag{11}$$

For any integer i, let

$$M_i = (M_0)^{\sigma^i} \text{ and } N_i = (N_0)^{\sigma^i}. \tag{12}$$

We will now show that for every element $(a,b) \in K$ and for every $j \in \mathbb{Z}_{2\ell m}$, we have:

$$(M_0)^{[a]_j} = (M_0)^{[b]_j} = M_0. \tag{13}$$

Indeed, by definition, a typical element of M_0 is of the form $[\beta]_0$ such that $(1,\beta) \in K$. If $j \neq 0$, then by (6) $[b]_j$ commutes with $[\beta]_0$ and hence centralises M_0. Suppose now that $j = 0$. Then, by (6), we have $[\beta]_0^{[b]_0} = [\beta^b]_0$. Since $(1,\beta)$ and (a,b) are elements of K and since $(1,\beta)^{(a,b)} = (1,\beta^b)$, we see that $(1,\beta^b) \in K$, which, by definition of M_0, implies that $[\beta^b]_0$ (and thus $[\beta]_0^{[b]_0}$) is in M_0. This shows that $[b]_j$ normalises M_0 for every j. Now recall that, by (5), we have $(a,b)^h = (b^r,a)$. Since h normalises K, this shows that $(b^r,a) \in K$. By applying the above argument with (b^r,a) in place of (a,b), we see that also $[a]_j$ normalises M_0, thus proving (13).

By (9) and (13), for every element $(a,b) \in K$ and for every $j \in \mathbb{Z}_{2\ell m}$, we have:

$$(M_0)^{\chi(a,j)} = (M_0)^{\chi(b,j)} = M_0. \tag{14}$$

In particular, since $(r,r) \in K$, the group M_0 is normalised by $\chi(r,j)$. Together with (7), (11) and (12), the latter implies that:

$$M_i = (M_0)^{s^i} = \{[\beta]_i \mid (1,\beta) \in K\}. \tag{15}$$

Let us now show that the following holds for every $(a,b) \in K$ and $i,j \in \mathbb{Z}_{2\ell m}$:

$$(M_i)^{\chi(a,j)} = (M_i)^{\chi(b,j)} = (M_i)^{\chi(r,j)} = M_i. \tag{16}$$

Indeed:

$$
\begin{aligned}
(M_i)^{\chi(a,j)} &\overset{(15)}{=} (M_0)^{s^i\chi(a,j)}\\
&\overset{(10.4)}{=} (M_0)^{\chi(a,j-i)s^i}\\
&\overset{(14)}{=} (M_0)^{s^i}\\
&\overset{(15)}{=} M_i,
\end{aligned}
$$

and similarly, $(M_i)^{\chi(b,j)} = M_i$, thus proving (16).

Furthermore, the definition (12) of N_i, (10.3) and (11) imply that

$$N_i = \{\chi(a,i)\chi(b,i+m) \mid (a,b) \in K\} \text{ for every } i \in \{1,\dots,m-1\}. \tag{17}$$

We shall now show that $N_{m+i} = N_i$. Let $(a,b) \in K$ and observe that:

$$\begin{aligned}
\big(\chi(a,0)\chi(b,m)\big)^{\sigma^m} &\overset{(11)}{=} \big(\chi(a,0)\chi(b,m)\big)^{\chi(r,-m)\dots\chi(r,-2)\chi(r,-1)s^m}\\
&\overset{(10.3)}{=} \big(\chi(a,0)\chi(b,m)\big)^{\chi(r,-m)s^m}\\
&\overset{(10.1,\ 10.2)}{=} \big(\chi(a,0)\chi(b^r,m)\big)^{s^m}\\
&\overset{(10.3,\ 10.4)}{=} \chi(b^r,0)\chi(a,m).
\end{aligned}$$

Since $(a,b) \in K$, by (5) we have $(a,b)^h = (b^r,a) \in K$ and hence $\chi(b^r,0)\chi(a,m) \in N_0$. This shows that σ^m normalises N_0. However, $N_m = (N_0)^{\sigma^m}$ by definition, implying that $N_m = N_0$, and thus

$$N_{i+m} = N_i \text{ for every integer } i. \tag{18}$$

Let us consider the automorphism τ of Λ defined by

$$\tau = \chi(r,m+1)\chi(r,m+2)\dots\chi(r,2m-1)t. \tag{19}$$

Note that (7) and (15) imply

$$(M_i)^\tau = M_{-i} \text{ for every integer } i. \tag{20}$$

We shall now show that

$$(N_i)^\tau = N_{-i} \text{ for every integer } i. \tag{21}$$

It follows from (10.3) that τ centralises N_0 and hence (21) holds for $i = 0$. By (18), it thus suffices to show (21) for $i \in \{1,\dots,m-1\}$. By (17), a typical element of N_i is of the form $\chi(a,i)\chi(b,m+i)$ for some $(a,b) \in K$. Then

$$\begin{aligned}
\big(\chi(a,i)\chi(b,m+i)\big)^\tau &\overset{(19)}{=} \big(\chi(a,i)\chi(b,m+i)\big)^{\chi(r,m+1)\chi(r,m+2)\dots\chi(r,2m-1)t}\\
&\overset{(10.2,\ 10.3)}{=} \big(\chi(b^r,m+i)\chi(a,i)\big)^t\\
&\overset{(10.1,\,10.5)}{=} \chi(b^r,m-i)\chi(a,-i).
\end{aligned}$$

Since $(a,b) \in K$, by (5) we have $(b^r,a) \in K$ and hence $\chi(b^r,m-i)\chi(a,-i) \in N_{m-i}$. This shows that $(N_i)^\tau = N_{m-i}$, which, by (18), equals N_{-i}. This completes the proof of (21).

Let M (N, respectively) be the group generated by all M_i (N_i, respectively), $i \in \mathbb{Z}$. By (15) it follows that $M_i = M_{i+2\ell m}$. From this and from (17), we deduce that $M = \langle M_0,\dots,M_{2\ell m-1}\rangle$ and $N = \langle N_0,\dots,N_{m-1}\rangle$. Further, by (10.3), (17) and (18), it follows that

$$M = M_0 \times \dots \times M_{2\ell m-1} \text{ and } N = N_0 \times \dots \times N_{m-1}. \tag{22}$$

Let $G = \langle M,N,\sigma,\tau\rangle$. By (20) and (21), it follows that M and N are normalised by τ. By definition, they are also normalised by σ, implying that $\langle M,N\rangle$ is normal in G. Recall that $\mathrm{V}(\Lambda)$ admits a natural partition $\mathcal{P} = \{\mathbb{Z}_k \times \{y\} \mid y \in \mathbb{Z}_{2\ell m}\}$ and observe that $\mathcal{P}$ is in fact the set of orbits of $\langle M,N\rangle$ on $\mathrm{V}(\Lambda)$ and hence is G-invariant.

By the definitions of σ and τ, it follows that the permutation group induced by the action of G on $\mathcal{P}$ is isomorphic to the dihedral group $\mathrm{D}_{2\ell m}$ in its natural action of degree $2\ell m$.

We will now show that $\langle M, N\rangle$ is the kernel of the action of G on $\mathcal{P}$. Observe that it suffices to show that $G/\langle M, N\rangle$ acts faithfully on $\mathcal{P}$ or, equivalently, that $G/\langle M, N\rangle$ is isomorphic to $\mathrm{D}_{2\ell m}$. We will show this by proving that $\sigma^{2\ell m}$, τ^2 and $\sigma\sigma^\tau$ are all contained in $\langle M, N\rangle$. We begin by computing $\sigma\sigma^\tau$. Observe that by (10.3) and (10.4) we have $s^{\chi(a,i)} = \chi(a, i-1)\chi(a^{-1}, i)s$. By applying this repeatedly and using (10.1), (10.2) and (10.3) we deduce that

$$s^{\chi(r,m+1)\chi(r,m+2)\cdots\chi(r,2m-1)} = \chi(r, m)\chi(r^{-1}, -1)s,$$

and therefore

$$\begin{array}{rcl}\sigma^\tau & = & (\chi(r,-1)s)^{\chi(r,m+1)\chi(r,m+2)\cdots\chi(r,2m-1)t} \\ & \overset{(10.2,\ 10.3)}{=} & (\chi(r,-1)\chi(r,m)\chi(r^{-1},-1)s)^t \\ & \overset{(10.1,\ 10.5)}{=} & \chi(r,m)s^{-1}.\end{array}$$

Computing $\sigma\sigma^\tau$ is now easy:

$$\sigma\sigma^\tau \;=\; \chi(r,-1)s\chi(r,m)s^{-1} \;=\; \chi(r,-1)\chi(r,m-1). \tag{23}$$

By (3), we know that $h^2 = (r, r) \in K$ and hence, by (17), $\chi(r, i)\chi(r, m+i) \in N_i$ and, in particular, $\sigma\sigma^\tau \in N$. Now,

$$\begin{array}{rcl}\tau^2 & = & \big(\chi(r,m+1)\cdots\chi(r,2m-1)t\big)^2 \\ & \overset{(10.5)}{=} & \chi(r,m+1)\cdots\chi(r,2m-1)\chi(r,1)\cdots\chi(r,m-1) \\ & \overset{(10.3)}{=} & (\chi(r,1)\chi(r,m+1))(\chi(r,2)\chi(r,m+2))\cdots(\chi(r,m-1)\chi(r,2m-1)).\end{array}$$

Again, $\chi(r, i)\chi(r, m+i) \in N_i$ for all $i \in \mathbb{Z}_{2\ell m}$ and hence $\tau^2 \in N$. Finally, by (11), we have

$$\begin{array}{rcl}\sigma^{2m} & = & \chi(r,0)\chi(r,1)\cdots\chi(r,2m-1) \\ & = & (\chi(r,0)\chi(r,m))(\chi(r,1)\chi(r,m+1))\cdots(\chi(r,m-1)\chi(r,2m-1)) \in N.\end{array}$$

We have just shown that σ^{2m}, τ^2 and $\sigma\sigma^\tau$ are all contained in N, and therefore, that $\langle M, N\rangle$ is the kernel of the action of G on $\mathcal{P}$, as claimed.

We shall now determine the order of $\langle M, N\rangle$. Note that by (16), (17) and (22), the group N normalises M, implying that $\langle M, N\rangle = MN$. Let us first determine the order of $M \cap N$.

Let α be an arbitrary element of $M \cap N$. Since $\alpha \in N$, we can write

$$\alpha = \prod_{j=0}^{m-1} \chi(a_j, j)\chi(b_j, j+m)$$

for some permutations $a_0, a_1, \dots, a_{m-1}, b_0, b_1, \dots, b_{m-1}$ of $\mathbb{Z}_k$ such that $(a_j, b_j) \in K$ for all $j \in \{0, \dots, m-1\}$. If we write $a_{j+m} = b_j$, we obtain

$$\alpha = \prod_{j=0}^{2m-1} \chi(a_j, j) \tag{24}$$

with $(a_j, a_{j+m}) \in K$ for all $j \in \{0, \dots, m-1\}$. Now observe that for every $i \in \mathbb{Z}_{2\ell m}$, the set $\mathbb{Z}_k \times \{i\} \subseteq \mathrm{V}(\Lambda)$ is preserved by α. Furthermore, for every $j \in \{0, \dots, 2m-1\}$, the restriction of α to $\mathbb{Z}_k \times \{j\}$ (when viewed as a permutation on $\mathbb{Z}_k$) is in fact a_j. On the other hand, since $\alpha \in M$, such a restriction a_j has to satisfy the defining property $(1, a_j) \in K$.

Conversely, let $a_0, a_1, \ldots, a_{2m-1}$ be arbitrary permutations of $\mathbb{Z}_k$ such that $(1, a_j) \in K$ for all $j \in \{0, \ldots, 2m-1\}$ and let α be as in (24). We shall now show that α is an element of $M \cap N$.

The fact that α is in M follows directly from (15) and from the fact that $(1, a_j) \in K$ for each j. In order to prove that $\alpha \in N$, it suffices to show that $(a_j, a_{j+m}) \in K$ (and thus $\chi(a_j, j)\chi(a_{j+m}, j+m) \in N_j$) for every $j \in \{0, \ldots, m-1\}$. Fix such an integer j and let $a = a_j$ and $b = a_{j+m}$. Since $(1, a) \in K$ and since h normalises K, it follows that $(1, a)^{h^{-1}} \in K$. However, $(1, a)^{h^{-1}} = (a, 1)$, implying that $(a, b) = (a, 1)(1, b) \in K$, as required. We have thus shown that $\alpha \in M \cap N$.

To summarise, the intersection $M \cap N$ consists precisely of those α from (24) for which $(1, a_j) \in K$ for all $j \in \{0, \ldots, 2m-1\}$. In particular,

$$|M \cap N| = |\{b \mid (1, b) \in K\}|^{2m} = |L_{(A)}|^{2m},$$

which implies that

$$|\langle M, N\rangle| = |M||N|/|M \cap N| = |L_{(A)}|^{2\ell m}|K|^m/|L_{(A)}|^{2m} = |L_{(A)}|^{2m(\ell-1)}(|L|/2)^m.$$

Thus:

$$|G| = |\mathrm{D}_{2\ell m}||\langle M, N\rangle| = 4\ell m|L_{(A)}|^{2m(\ell-1)}(|L|/2)^m. \tag{25}$$

We shall now turn our attention to the graph Γ. Recall that $\Gamma = \mathrm{C}(k, 2\ell m, m-1)$ and that the vertices of Γ are the traversing paths of length $m-2$ in Λ; that is:

$$\mathrm{V}(\Gamma) = \{(u_1, i)(u_2, i+1)\ldots(u_{m-1}, i+m-2) \mid (u_1, \ldots, u_{m-1}) \in (\mathbb{Z}_k)^{m-1}, i \in \mathbb{Z}_{2\ell m}\}.$$

Further, recall that since every automorphism of Λ preserves the set of all such paths, there exists a natural faithful action of G as a group of automorphisms of Γ. Moreover, observe that for any fixed $i \in \mathbb{Z}_{2\ell m}$, the group N acts transitively on the set $\mathrm{V}_i = \{(u_1, i)(u_2, i+1)\ldots(u_{m-1}, i+m-2) \mid (u_1, \ldots, u_{m-1}) \in (\mathbb{Z}_k)^{m-1}\}$, and that the group $\langle\sigma\rangle$ cyclically permutes the family $\{\mathrm{V}_i \mid i \in \mathbb{Z}_{2\ell m}\}$. Since the latter family is a partition of $\mathrm{V}(\Gamma)$, this shows that $\langle N, \sigma\rangle$ (and thus G) is transitive on $\mathrm{V}(\Gamma)$.

Let $v = (0, 1)(0, 2)\ldots(0, m-1)$ be a traversing path in Λ of length $m-2$, interpreted as a vertex of Γ. Observe that the neighbourhood $\Gamma(v)$ decomposes into a disjoint union of subsets

$$\begin{aligned}\Gamma^-(v) &= \{(j, 0)(0, 1)\ldots(0, m-2) \mid j \in \mathbb{Z}_k\},\\ \Gamma^+(v) &= \{(0, 2)\ldots(0, m-1)(j, m) \mid j \in \mathbb{Z}_k\},\end{aligned}$$

with $\Gamma^-(v)$ and $\Gamma^+(v)$ being blocks of imprimitivity for G_v. Hence there is a natural identification of the sets $\Gamma^-(v)$ and $\Gamma^+(v)$ with the sets $\mathbb{Z}_k \times \{0\}$ and $\mathbb{Z}_k \times \{m\}$, respectively. This induces an action of G_v on $\mathbb{Z}_k \times \{0, m\}$.

Recall that the sets A and B were identified with the sets $\mathbb{Z}_k \times \{0\}$ and $\mathbb{Z}_k \times \{1\}$, which gives rise to a further identification of $\Gamma^-(v)$ with A and $\Gamma^+(v)$ with B (where $(x, 1) \in B$ is identified with $(x, m) \in \Gamma^+(v)$). In particular, G_v can be viewed as acting on $A \cup B$. Since $\Gamma^-(v)$ and $\Gamma^+(v)$ are blocks of imprimitivity for G_v, so are A and B.

Observe that an element of G_v, when viewed as acting on $A \cup B$, preserves each of A and B setwise if and only if, when viewed as acting on $\mathrm{V}(\Lambda)$, it preserves each of the sets $\{i\} \times \mathbb{Z}_{2\ell m} \in \mathcal{P}$. Now recall that the kernel of the action of G on $\mathcal{P}$ is

$\langle M, N\rangle$ and hence the kernel of the action of G_v on the partition $\{A, B\}$ of $A \cup B$ is $\langle M, N\rangle_v$.

We shall now prove that $G_v^{\Gamma(v)}$ is permutation isomorphic to L. We begin by proving that $M_v^{\Gamma(v)} \leq N_v^{\Gamma(v)}$. Note that by (22), it suffices to prove that $(M_i)_v^{\Gamma(v)} \leq N_v^{\Gamma(v)}$ for all $i \in \{0, \ldots, 2\ell m - 1\}$. Clearly, if $i \notin \{0, m\}$, then $(M_i)_v$ acts trivially on $\mathbb{Z}_k \times \{0, m\}$ and hence on $\Gamma(v)$. Further, observe that both M_0 as well as M_m fix the vertex v. Recall that a typical element of M_0 is of the form $[b]_0$ for $b \in \mathrm{S}_k$ such that $(1, b) \in K$. Since K is normal in L, it follows that $(1, b)^{h^{-1}} \in K$. However, $(1, b)^{h^{-1}} = (b, 1)$. It follows that the element $\chi(b, 0)\chi(1, m)$ is in N_0. However, the latter element clearly fixes v and induces the same permutation on $\Gamma(v)$ as the element $[b]_0$ of M_0 does. In particular, $(M_0)^{\Gamma(v)} \leq N_v^{\Gamma(v)}$. A similar computation shows that $(M_m)^{\Gamma(v)} \leq N_v^{\Gamma(v)}$, and therefore, that $M_v^{\Gamma(v)} \leq N_v^{\Gamma(v)}$.

We shall now show that $N_v^{\Gamma(v)}$ is permutation isomorphic to K. Recall that $N = N_0 \times \cdots \times N_{m-1}$ and note that, for $i \neq 0$, the subgroup $(N_i)_v$ acts trivially on $\mathbb{Z}_k \times \{0, m\}$, and hence on $\Gamma(v)$. Moreover, $(N_0)_v = N_0$, implying that $N_v^{\Gamma(v)} = (N_0)^{\Gamma(v)}$. Since N_0 consists of the elements $\chi(a, 0)\chi(b, m)$ for $(a, b) \in K$, the identification of $\Gamma(v)$ with $\mathbb{Z}_k \times \{0, m\}$ and also with $A \cup B$ clearly implies that $(N_0)^{\Gamma(v)}$ is permutation isomorphic to K and hence $N_v^{\Gamma(v)}$ is permutation isomorphic to K, as claimed.

Since $M_v^{\Gamma(v)} \leq N_v^{\Gamma(v)}$, this implies that $\langle M, N\rangle_v^{\Gamma(v)}$ is permutation isomorphic to K. Now recall that $\langle M, N\rangle_v^{\Gamma(v)}$ is in fact the kernel of the action of $G_v^{\Gamma(v)}$ on the partition of $\{A, B\}$ of $A \cup B$ (after the usual identification of $\Gamma(v)$ with $A \cup B$). To conclude the proof that $G_v^{\Gamma(v)}$ is permutation isomorphic to L, it thus suffices to exhibit an element of G_v which acts on $A \cup B$ as the permutation h.

By (11), we have $\sigma^m = \chi(r, -m) \ldots \chi(r, -1)s^m$. By (19), it follows that

$$\tau^{-1}\sigma^m = \chi(r, m)ts^m. \tag{26}$$

Clearly, both $\chi(r, m)$ and ts^m fix v and hence so does $\tau^{-1}\sigma^m$. Let us now compute the permutation induced by $\tau^{-1}\sigma^m$ on $A \cup B$. Let $(j, 1)$ be an arbitrary element of B and recall that this element is represented by $(0, 2) \ldots (0, m-1)(j, m) \in \Gamma^+(v)$. Now, note that

$$\begin{aligned} \big((0,2)\ldots(0,m-1)(j,m)\big)^{\tau^{-1}\sigma^m} &\overset{(26)}{=} \big((0,2)\ldots(0,m-1)(j,m)\big)^{\chi(r,m)ts^m} \\ &\overset{(9)}{=} \big((0,2)\ldots(0,m-1)(j^r,m)\big)^{ts^m} \\ &= \big((j^r,0)(0,1)\ldots(0,m-2)\big). \end{aligned}$$

This shows that $(j, 1)^{\tau^{-1}\sigma^m} = (j^r, 0) \in A$.

Similarly, let $(j, 0)$ be an arbitrary element of A. This element is represented by $(j, 0)(0, 1) \ldots (0, m-2) \in \Gamma^-(v)$ and an analogous computation yields:

$$\big((j,0)(0,1)\ldots(0,m-2)\big)^{\tau^{-1}\sigma^m} = (0,2)\ldots(0,m-1)(j,m).$$

This shows that $(j, 0)^{\tau^{-1}\sigma^m} = (j, 1) \in B$. By (4), this shows that $\tau^{-1}\sigma^m$, viewed as a permutation on $A \cup B$, is equal to h. This concludes the proof of the fact that $G_v^{\Gamma(v)}$ is permutation isomorphic to L.

Since Γ is G-vertex-transitive and L is transitive, it follows that Γ is G-arc-transitive. It remains to determine the order of the arc-stabiliser G_{uv} of an arc

uv of Γ. Since $|\mathrm{V}(\Gamma)| = 2\ell m k^{m-1}$ and the valence of Γ is $2k$, it follows that $|\mathrm{A}(\Gamma)| = 4\ell m k^m$. Further, since L is a transitive permutation group of degree $2k$, it follows that $|L|/2k = |L_\omega|$, where L_ω is a point-stabiliser in L. Using (25) for the order of G, we thus get the following:

$$|G_{uv}| = \frac{4\ell m |L_{(A)}|^{2m(\ell-1)}(|L|/2)^m}{4\ell m k^m} = |L_{(A)}|^{2m(\ell-1)}|L_\omega|^m.$$

□

Corollary 9. *Let L be a transitive permutation group of degree $2k$ with a system of imprimitivity consisting of two blocks A and B, let L_ω be a point-stabiliser in L and let $L_{(A)}$ be the pointwise stabiliser of A in L. Then L is f-graph-unrestrictive where $f(n) = |L_{(A)}|^{\frac{n}{k}-4}|L_\omega|^2$. In particular, if $L_{(A)} \neq 1$, then L has graph-type* Exp.

Moreover, if $L_\omega \neq 1$, then L is Poly*-graph-unrestrictive; in fact, L is n^α-graph-unrestrictive for every α with $0 < \alpha < \frac{\log |L_\omega|}{\log k}$.*

Proof. For $\ell \geq 1$, let $\Gamma_\ell = \mathrm{C}(k, 4\ell, 1)$. By Theorem 8 (applied with $m = 2$), there exists $G_\ell \leq \mathrm{Aut}(\Gamma_\ell)$ such that (Γ_ℓ, G_ℓ) is locally-L and $|(G_\ell)_{uv}| = |L_{(A)}|^{4(\ell-1)}|L_\omega|^2$. Since $|\mathrm{V}(\Gamma_\ell)| = 4k\ell$, the latter is equal to $|L_{(A)}|^{\frac{|\mathrm{V}(\Gamma_\ell)|}{k}-4}|L_\omega|^2$. This shows that L is f-graph-unrestrictive where $f(n) = |L_{(A)}|^{\frac{n}{k}-4}|L_\omega|^2$, which is a function in the class Exp provided that $L_{(A)} \neq 1$.

Now suppose that $L_\omega \neq 1$. For $m \geq 2$, let $\Gamma_m = \mathrm{C}(k, 2m, m-1)$. By Theorem 8 (applied with $\ell = 1$), there exists $G_m \leq \mathrm{Aut}(\Gamma_m)$ such that (Γ_m, G_m) is locally-L and $|(G_m)_{uv}| = |L_\omega|^m$.

Let $n = |\mathrm{V}(\Gamma_m)|$ and observe that $n = 2mk^{m-1}$. Let $c = |L_\omega|$ and fix α such that $0 < \alpha < \frac{\log c}{\log k}$. Then there exists $\epsilon > 0$ such that $\alpha = \frac{\log c}{\log(k+\epsilon)}$. Furthermore, since $\frac{n}{(k+\epsilon)^m} = \frac{2mk^{m-1}}{(k+\epsilon)^m} \to 0$ as $m \to \infty$, there exists m_0 such that for every $m > m_0$, we have $n < (k+\epsilon)^m$, and thus $\frac{\log n}{\log(k+\epsilon)} < m$. Hence

$$n^\alpha = n^{\frac{\log c}{\log(k+\epsilon)}} = c^{\frac{\log n}{\log(k+\epsilon)}} < c^m = |G_{uv}|.$$

This proves that L is f-graph-unrestrictive for $f(n) = n^\alpha$, and, in particular, that L is Poly-graph-unrestrictive. □

5. Imprimitive groups of degree 6 that do not admit a system of imprimitivity consisting of two blocks

The results proved so far in this paper together with previously known results are enough to settle Problem 3 for transitive permutation groups of degree at most 7 with the exception of three groups of degree 6 that are imprimitive but do not admit a system of imprimitivity consisting of two blocks of size 3 (see Section 6 for details). In this section, we will show that these three groups are SubExp-graph-unrestrictive.

By [2], there are five transitive groups of degree 6 that are imprimitive but do not admit a system of imprimitivity consisting of two blocks. Using the taxonomy of [2], they are $A_4(6)$, $2A_4(6)$, $S_4(6d)$, $S_4(6c)$ and $2S_4(6)$. Note that $2S_4(6)$ is in fact isomorphic to the wreath product $\mathbb{Z}_2 \operatorname{wr} S_3$ in its imprimitive action on 6 points, while $A_4(6)$ is isomorphic to $\mathbb{Z}_2^2 \rtimes C_3$ viewed as a subgroup of index 2 in the wreath product $\mathbb{Z}_2 \operatorname{wr} C_3$. The group $A_4(6)$ is thus a normal subgroup of index 4 in $2S_4(6)$

and $2S_4(6)/A_4(6) \cong \mathbb{Z}_2^2$. The remaining three groups $2A_4(6)$, $S_4(6d)$ and $S_4(6c)$ are precisely the three groups L with the property that $A_4(6) < L < 2S_4(6)$. The main result of this section is the following:

Theorem 10. *Let L be a transitive permutation group of degree 6 that is imprimitive but does not admit a system of imprimitivity consisting of two blocks. Then L is f-graph-unrestrictive where $f(n) = 4^{\sqrt{\frac{n}{3}}-1}$.*

It will be convenient for us to describe the five groups L satisfying $A_4(6) \leq L \leq 2S_4(6)$ as subgroups of the symmetric group on the set $\Omega = \{0, 1, \ldots, 5\}$ in terms of generators. We shall follow the notation introduced in [2] as closely as possible. Let a, b, e, f be the following permutations of Ω:

$$a = (0\ 2\ 4)(1\ 3\ 5), \quad b = (1\ 5)(2\ 4), \quad e = (1\ 4)(2\ 5), \quad f = (0\ 3). \tag{27}$$

Then by [2, Appendix A], the groups $A_4(6)$ and $2S_4(6)$ can be expressed as:

$$A_4(6) = \langle a, e\rangle \quad \text{and} \quad 2S_4(6) = \langle f, a, b\rangle. \tag{28}$$

The rest of the section is devoted to the proof of Theorem 10. In this endeavour, the following lemma will prove very useful. (For a graph Γ, a vertex $v \in \mathrm{V}(\Gamma)$ and a group $G \leq \mathrm{Aut}(\Gamma)$, let $G_v^{[1]}$ denote the kernel of the action of G_v on $\Gamma(v)$.)

Lemma 11. *Let Λ be a graph, let v be a vertex of Λ and let A and B be vertex-transitive groups of automorphisms of Λ such that $B \trianglelefteq A$. Then, for every permutation group L such that $B_v^{\Lambda(v)} \leq L \leq A_v^{\Lambda(v)}$, there exists C such that $B \leq C \leq A$, $C_v^{\Lambda(v)} = L$ and $C_v^{[1]} = A_v^{[1]}$.*

Proof. Let $\pi\colon A_v \to A_v^{\Lambda(v)}$ be the epimorphism that maps each $g \in A_v$ to the permutation induced by g on $\Lambda(v)$. Observe that $B_v \leq \pi^{-1}(L) \leq A_v$ and $(\pi^{-1}(L))^{\Lambda(v)} = \pi(\pi^{-1}(L)) = L$. Finally, let $C = B\pi^{-1}(L)$. Clearly, $B \leq C \leq A$ and $C_v = B\pi^{-1}(L) \cap A_v$. By the modular law, the latter equals $(B \cap A_v)\pi^{-1}(L)$ and hence $C_v = \pi^{-1}(L)$. In particular, the group $A_v^{[1]}$ (being the kernel of π) is contained in C_v and thus is equal to $C_v^{[1]}$. Moreover, $C_v^{\Lambda(v)} = \pi(C_v) = \pi(\pi^{-1}(L)) = L$. ☐

The proof of Theorem 10 makes use of certain cubic arc-transitive graphs having a large nullity over the field $\mathbb{F}_2$ of order 2, the existence of which was proved in [4]. Let us explain this in more detail.

Let Γ be a graph with vertex-set V, let $\mathbb{F}$ be a field and let $\mathbb{F}^V$ be the set of all functions from V to $\mathbb{F}$, viewed as an $\mathbb{F}$-vector space (with addition and multiplication by scalars defined pointwise). The $\mathbb{F}$-*nullspace* of Γ is then the set of all elements $x \in \mathbb{F}^V$ such that for every $v \in V$ we have

$$\sum_{u \in \Gamma(v)} x(u) = 0.$$

The $\mathbb{F}$-nullspace is clearly a subspace of the vector space $\mathbb{F}^V$ and its dimension over $\mathbb{F}$ is called the $\mathbb{F}$-*nullity* of Γ. We can now state and prove the crucial step in the proof of Theorem 10.

Theorem 12. *Let L be one of the five imprimitive permutation groups of degree 6 admitting no blocks of imprimitivity of size 3. Let Γ be a connected cubic graph, let M be the $\mathbb{F}_2$-nullspace of Γ and suppose that M is non-trivial. Suppose further that $\mathrm{Aut}(\Gamma)$ contains two subgroups H and G with H acting regularly on the set of arcs of*

Γ, *with* G *acting regularly on the set of* 2*-arcs of* Γ *and with* $H \leq G$. *Let* $\Lambda = \Gamma[2K_1]$ *be the lexicographic product of* Γ *with the edgeless graph* $2K_1$ *on two vertices. Then there exists an arc-transitive subgroup* $C \leq \mathrm{Aut}(\Lambda)$, *such that the pair* (Λ, C) *is locally-L and the stabiliser of an arc of* Λ *in* C *has order* $|M||L|/48 \geq |M|/4$.

Proof. Let V denote the vertex-set of Γ. We will think of the vertex-set of Λ as $\mathbb{F}_2 \times V$. Observe that there is a natural embedding of G as well as of the additive group of $\mathbb{F}_2^V$ into $\mathrm{Aut}(\Lambda)$ given by

$$(a, v)^g = (a, v^g) \quad \text{and} \quad (a, v)^x = (a + x(v), v)$$

for every $(a, v) \in \mathbb{F}_2 \times V$, $g \in G$ and $x \in \mathbb{F}_2^V$. In this sense, we may view G and $\mathbb{F}_2^V$ as subgroups of $\mathrm{Aut}(\Lambda)$.

Let $\mathbf{1} \in \mathbb{F}_2^V$ be the constant function mapping each vertex to 1 and let $N = \langle \mathbf{1} \rangle \oplus M \leq \mathbb{F}_2^V \leq \mathrm{Aut}(\Lambda)$. Finally, let A and B be the subgroups of $\mathrm{Aut}(\Lambda)$ defined by

$$A = \langle N, G \rangle \qquad \text{and} \qquad B = \langle M, H \rangle.$$

Since M is the $\mathbb{F}_2$-nullspace of Γ and $G \leq \mathrm{Aut}(\Gamma)$, it follows that M is normalised by G and hence by H. Moreover, the automorphism $\mathbf{1}$ of Λ is clearly centralised by both M and G and hence N is also normalised by G. In fact, since M and N act trivially on the partition $\{\mathbb{F}_2 \times \{v\} : v \in V\}$ of $\mathrm{V}(\Lambda)$, while G and H act faithfully, we see that $N \cap G = M \cap H = 1$, and thus

$$A = N \rtimes G \text{ and } B = M \rtimes H. \tag{29}$$

Let us now show that $B \trianglelefteq A$. It is clear from the definition that $B \leq A$. We have already noted that G normalises M. Moreover, H has index 2 in G and hence is normal in G. It follows that G normalises $B = M \rtimes H$. Note that A is generated by B, G and $\mathbf{1}$. Since $\mathbf{1}$ centralises B, this implies that $B \trianglelefteq A$.

Let v be a vertex of Γ and let $\tilde{v} = (0, v)$ be the corresponding vertex of Λ. Let us now show that

$$A_{\tilde{v}}^{\Lambda(\tilde{v})} = N_{\tilde{v}}^{\Lambda(\tilde{v})} G_{\tilde{v}}^{\Lambda(\tilde{v})} \quad \text{and} \quad B_{\tilde{v}}^{\Lambda(\tilde{v})} = M_{\tilde{v}}^{\Lambda(\tilde{v})} H_{\tilde{v}}^{\Lambda(\tilde{v})}. \tag{30}$$

Let $a \in A$. By (29), $a = xg$ for some $x \in N$ and $g \in G$. Then $a \in A_{\tilde{v}}$ if and only if $(0, v) = (0, v)^a = (x(v), v^g)$ if and only if $g \in G_v$ and $x \in N_{\tilde{v}}$. Note that an element $g \in G$ fixes $v \in \mathrm{V}(\Gamma)$ if and only if (when viewed as an automorphism of Λ) it fixes $\tilde{v}$. In this sense we can write $G_v = G_{\tilde{v}}$. We have thus shown that $A_{\tilde{v}} = N_{\tilde{v}} G_{\tilde{v}}$.

On the other hand, $B_{\tilde{v}} = A_{\tilde{v}} \cap B = N_{\tilde{v}} G_{\tilde{v}} \cap MH = M_{\tilde{v}} H_{\tilde{v}}$. To conclude the proof of (30), apply the epimorphism that maps each $a \in A_{\tilde{v}}$ to the permutation induced by a on $\Lambda(\tilde{v})$ to the equalities $A_{\tilde{v}} = N_{\tilde{v}} G_{\tilde{v}}$ and $B_{\tilde{v}} = M_{\tilde{v}} H_{\tilde{v}}$.

We shall now prove that the permutation groups $B_{\tilde{v}}^{\Lambda(\tilde{v})}$ and $A_{\tilde{v}}^{\Lambda(\tilde{v})}$ are permutation isomorphic to the groups $A_4(6)$ and $2S_4(6)$, respectively. Let $\Gamma(v) = \{s, t, u\}$ and label the six neighbours $(0, s), (1, s), (0, t), (1, t), (0, u), (1, u)$ of $\tilde{v}$ by $0, 3, 2, 5, 4, 1$, respectively. Since H acts regularly on the set of the arcs of Γ, it follows that $H_v^{\Gamma(v)} = \langle (s\ t\ u) \rangle \cong C_3$. Similarly, since G acts regularly on the set of the 2-arcs of Γ, it follows that $G_v^{\Gamma(v)} = \langle (s\ t\ u), (t\ u) \rangle \cong \mathrm{S}_3$. In particular, when H_v and G_v are viewed as acting on $\Lambda(\tilde{v})$, we see that

$$H_{\tilde{v}}^{\Lambda(\tilde{v})} = \langle a \rangle, \ G_{\tilde{v}}^{\Lambda(\tilde{v})} = \langle a, b \rangle, \ \text{where } a = (0\ 2\ 4)(1\ 3\ 5), \ b = (1\ 5)(2\ 4). \tag{31}$$

Note that the permutations a and b above are the same as the permutations defined in (27). Let us define the following permutations of $\Lambda(\tilde{v})$:

$$e = (1\ 4)(2\ 5),\ f = (0\ 3)\ \text{ and }\ k = (0\ 3)(1\ 4).$$

Note that e and f are as in (27) and that $k = e^a$. Moreover, note that $\langle e, k\rangle \cong \mathbb{Z}_2^2$ and that it is in fact the group consisting of all even permutations of $\langle f, f^a, f^{a^2}\rangle$. We shall now show that

$$M_{\tilde{v}}^{\Lambda(\tilde{v})} = \langle e, k\rangle = \langle e, e^a\rangle \ \text{ and }\ N_{\tilde{v}}^{\Lambda(\tilde{v})} = \langle f, f^a, f^{a^2}\rangle. \tag{32}$$

Since $M \neq 0$ and Γ is connected and arc-transitive, there exists $x \in M$ such that $x(v) = 0$ and $x(s) = 1$ (in particular, $x \in M_{\tilde{v}}$). Since x belongs to the $\mathbb{F}_2$-nullspace of Γ, it follows that $x(t) + x(u) = 1$. Let γ be the permutation of $\Lambda(\tilde{v})$ induced by the action of x on $\Lambda(\tilde{v})$. Since $M_{\tilde{v}}$ is normalised by $G_{\tilde{v}}$, it follows that $M_{\tilde{v}}^{\Lambda(\tilde{v})}$ is normalised by $G_{\tilde{v}}^{\Lambda(\tilde{v})}$ and hence $\gamma, \gamma^a, \gamma^{a^2} \in M_{\tilde{v}}$. Observe that $\{\gamma, \gamma^a, \gamma^{a^2}\} = \{e, k, ek\}$, and thus $\langle e, k\rangle \leq M_{\tilde{v}}^{\Lambda(\tilde{v})}$. On the other hand, since M is the $\mathbb{F}_2$-nullity of Γ, it follows that $|M_{\tilde{v}}^{\Lambda(\tilde{v})}| \leq 4$, and thus $M_{\tilde{v}}^{\Lambda(\tilde{v})} = \langle e, k\rangle$, as claimed.

Since Γ is G-arc-transitive, there exists $g \in G$ such that $(s, v)^g = (v, s)$. Let $y = \mathbf{1} + x + x^g$. Then $y(v) = 1 + x(v) + x(s) = 1 + 0 + 1 = 0$ and hence $y \in N_{\tilde{v}}$. Moreover, since $x, x^g \in M$ it follows that $x(s) + x(t) + x(u) = 0 = x^g(s) + x^g(t) + x^g(u)$ and therefore $y(s) + y(t) + y(u) = 1$. It follows that $y^{\Lambda(\tilde{v})} \in N_{\tilde{v}}^{\Lambda(\tilde{v})} \setminus M_{\tilde{v}}^{\Lambda(\tilde{v})}$ and hence $N_{\tilde{v}}^{\Lambda(\tilde{v})}$ has order at least 8. On the other hand, $N_{\tilde{v}}^{\Lambda(\tilde{v})}$ acts trivially on the partition $\{\{0,3\},\{1,4\},\{2,5\}\}$ of $\Lambda(\tilde{v})$, and is therefore a subgroup of $\langle f, f^a, f^{a^2}\rangle \cong \mathbb{Z}_2^3$. This proves that $N_{\tilde{v}}^{\Lambda(\tilde{v})} = \langle f, f^a, f^{a^2}\rangle$, as claimed.

Finally, if we combine (30), (31) and (32) we obtain $A_{\tilde{v}}^{\Lambda(\tilde{v})} = N_{\tilde{v}}^{\Lambda(\tilde{v})} G_{\tilde{v}}^{\Lambda(\tilde{v})} = \langle f, f^a, f^{a^2}, a, b\rangle = \langle f, a, b\rangle$ and $B_{\tilde{v}}^{\Lambda(\tilde{v})} = M_{\tilde{v}}^{\Lambda(\tilde{v})} H_{\tilde{v}}^{\Lambda(\tilde{v})} = \langle e, e^a, a\rangle = \langle a, e\rangle$. Hence, by (28), we see that

$$A_{\tilde{v}}^{\Lambda(\tilde{v})} = 2S_4(6) \quad \text{and} \quad B_{\tilde{v}}^{\Lambda(\tilde{v})} = A_4(6). \tag{33}$$

In particular, this shows that $B_{\tilde{v}}^{\Lambda(\tilde{v})} \leq L \leq A_{\tilde{v}}^{\Lambda(\tilde{v})}$. Since B is normal in A, Lemma 11 implies that there exists $C \leq \mathrm{Aut}(\Lambda)$ with $B \leq C \leq A$, $A_{\tilde{v}}^{[1]} = C_{\tilde{v}}^{[1]}$ and such that the pair (Λ, C) is locally-L.

Since A and C are both transitive on $\mathrm{V}(\Lambda)$ and since $A_{\tilde{v}}^{[1]} = C_{\tilde{v}}^{[1]}$, it follows that $|A : C| = |A_v : C_v| = |A_{\tilde{v}}^{\Lambda(\tilde{v})} : C_{\tilde{v}}^{\Lambda(\tilde{v})}| = |2S_4(6) : L| = 48/|L|$. By (29) we see that $|A| = |N||G| = 2|M||G|$. Moreover, since G is transitive on $\mathrm{V}(\Gamma)$ and $|G_v| = 6$, it follows that $|A| = 12|M||\mathrm{V}(\Gamma)| = 6|M||\mathrm{V}(\Lambda)|$. Hence $|C| = |A|/|A : C| = |M||V(\Lambda)||L|/8$. Since C is transitive on the $6|\mathrm{V}(\Lambda)|$ arcs of Λ, this implies that the stabiliser of an arc of Λ in C has order $|M||L|/48$. Finally, since $|L| \geq 12$, we have $|M||L|/48 \geq |M|/4$. $\square$

In order to apply Theorem 12 to prove Theorem 10, we need to establish the existence of appropriate cubic arc-transitive graphs with large $\mathbb{F}_2$-nullity. The existence of such graphs follows immediately from [4, Construction 8, Proposition 9], where the following was proved:

Theorem 13. *For every positive integer m there exists a connected cubic graph of order $6 \cdot 2^{2m}$ and $\mathbb{F}_2$-nullity at least $4 \cdot 2^m$ which admits an arc-regular group of automorphisms H and a 2-arc-regular group of automorphisms G such that $H \leq G$.*

We can now combine Theorem 12 and Theorem 13 to prove Theorem 10.

Proof of Theorem 10. Let L be one of the five imprimitive permutation groups of degree 6 admitting no system of imprimitivity with blocks of size 3. Let m be a positive integer. By Theorem 13, there exists a connected cubic graph Γ_m of order $6 \cdot 2^{2m}$ and $\mathbb{F}_2$-nullity at least $4 \cdot 2^m$ which admits an arc-regular group of automorphisms H_m and a 2-arc-regular group of automorphisms G_m such that $H_m \leq G_m$. Let $\Lambda_m = \Gamma_m[2K_1]$ and let $n = |\mathrm{V}(\Lambda_m)| = 12 \cdot 2^{2m}$. By Theorem 12, there exists an arc-transitive subgroup $C_m \leq \mathrm{Aut}(\Lambda_m)$, such that the pair (Λ_m, C_m) is locally-L and the stabiliser of an arc of Λ_m in C_m has order at least $\frac{|M|}{4} = 4^{2\cdot 2^m - 1} = 4^{2\cdot\sqrt{\frac{n}{12}}-1} = 4^{\sqrt{\frac{n}{3}}-1}$. Thus L is f-graph-unrestrictive where $f(n) = 4^{\sqrt{\frac{n}{3}}-1}$. □

6. Transitive permutation groups of degree at most 7

In this section, we determine the status of Problem 3 for L a transitive permutation group of degree at most 7. The results are summarised in Table 1, where we use the notation from [2].

Since L has degree at most 7, it follows that L is graph-restrictive if and only if it is regular or primitive (see [5, Proposition 14]). In this case, L has graph-type Cons. We may thus assume that L is neither regular nor primitive. Since permutation groups of prime degree are primitive, it thus suffices to consider the cases when L has degree 4 or 6.

The only transitive permutation group of degree 4 which is neither regular nor primitive is D_4. This group is permutation isomorphic to the imprimitive wreath product $\mathbb{Z}_2 \,\mathrm{wr}\, \mathbb{Z}_2$ and, by Theorem 7, has graph-type Exp.

We now consider the case when L has degree 6. Suppose that L admits a system of imprimitivity consisting of two blocks of size 3 and let A be one of these blocks. If $L_{(A)} \neq 1$, then by Corollary 9 L has graph-type Exp. If $L_{(A)} = 1$, then it is easily checked that L must be permutation isomorphic to D_6. It follows by Corollary 9 that L is Poly-graph-unrestrictive. On the other hand, it follows from [12, Theorem A] that D_6 is Poly-graph-restrictive and hence D_6 has graph-type Poly.

Table 1. Graph-types of transitive permutation groups of degree at most 7

	No. of grps.	Graph-type
Regular or primitive	26	Cons
D_6	1	Poly
D_4, $\mathbb{Z}_3 \,\mathrm{wr}\, \mathbb{Z}_2$, $\mathbb{Z}_2 \,\mathrm{wr}\, \mathbb{Z}_3$, $\mathrm{S}_3 \,\mathrm{wr}\, \mathbb{Z}_2$, $\mathbb{Z}_2 \,\mathrm{wr}\, \mathrm{S}_3$, $F_{18}(6)\colon 2$, $F_{36}(6)$	7	Exp
$A_4(6)$, $S_4(6c)$, $S_4(6d)$	3	SubExp or Exp

It thus remains to deal with transitive permutation groups of degree 6 which are imprimitive but do not admit a system of imprimitivity consisting of two blocks of size 3. As we saw in Section 5, there are five such groups, two of them are wreath products and hence have graph-type Exp. As for the remaining three, it follows from Theorem 10 that they are SubExp-graph-unrestrictive. Unfortunately, for none of these three groups were we able to decide whether it has graph-type SubExp or Exp.

References

[1] Marston Conder and Peter Dobcsányi, *Trivalent symmetric graphs on up to 768 vertices*, J. Combin. Math. Combin. Comput. **40** (2002), 41–63. MR1887966 (2002m:05105)

[2] John H. Conway, Alexander Hulpke, and John McKay, *On transitive permutation groups*, LMS J. Comput. Math. **1** (1998), 1–8 (electronic). MR1635715 (99g:20011)

[3] Aleksander Malnič, Dragan Marušič, and Primož Potočnik, *Elementary abelian covers of graphs*, J. Algebraic Combin. **20** (2004), no. 1, 71–97, DOI 10.1023/B:JACO.0000047294.42633.25. MR2104822 (2005g:05070)

[4] Primož Potočnik, Pablo Spiga, and Gabriel Verret, *On the nullspace of arc-transitive graphs over finite fields*, J. Algebraic Combin. **36** (2012), no. 3, 389–401, DOI 10.1007/s10801-011-0340-2. MR2969068

[5] Primož Potočnik, Pablo Spiga, and Gabriel Verret, *On graph-restrictive permutation groups*, J. Combin. Theory Ser. B **102** (2012), no. 3, 820–831, DOI 10.1016/j.jctb.2011.11.006. MR2900820

[6] Cheryl E. Praeger, *Inflations of graphs with large automorphism groups. I*, Ars Combin. **26** (1988), no. A, 171–192. Second International Conference on Combinatorial Mathematics and Computing (Canberra, 1987). MR992685 (90d:05116)

[7] Cheryl E. Praeger, *Finite quasiprimitive group actions on graphs and designs*, Groups—Korea '98 (Pusan), de Gruyter, Berlin, 2000, pp. 319–331. MR1751103 (2001f:20004)

[8] Cheryl E. Praeger and Ming Yao Xu, *A characterization of a class of symmetric graphs of twice prime valency*, European J. Combin. **10** (1989), no. 1, 91–102. MR977184 (89m:05059)

[9] W. T. Tutte, *A family of cubical graphs*, Proc. Cambridge Philos. Soc. **43** (1947), 459–474. MR0021678 (9,97g)

[10] W. T. Tutte, *On the symmetry of cubic graphs*, Canad. J. Math. **11** (1959), 621–624. MR0109794 (22 #679)

[11] Gabriel Verret, *On the order of arc-stabilizers in arc-transitive graphs*, Bull. Aust. Math. Soc. **80** (2009), no. 3, 498–505, DOI 10.1017/S0004972709000550. MR2569924 (2010k:20006)

[12] Gabriel Verret, *On the order of arc-stabilisers in arc-transitive graphs*, II, Bull. Aust. Math. Soc. **87** (2013), 441–447. MR3063588

[13] R. Weiss, *s-transitive graphs*, Algebraic methods in graph theory, Vol. I, II (Szeged, 1978), Colloq. Math. Soc. János Bolyai, vol. 25, North-Holland, Amsterdam, 1981, pp. 827–847. MR642075 (83b:05071)

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