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PLÜCKER FORMS AND THE THETA MAP

By SONIA BRIVIO and ALESSANDRO VERRA

Abstract. Let $SU_X(r,0)$ be the moduli space of semistable vector bundles of rank $r$ and trivial determinant over a smooth, irreducible, complex projective curve $X$. The theta map $\theta_r : SU_X(r,0) \to \mathbb{P}^N$ is the rational map defined by the ample generator of Pic$SU_X(r,0)$. The main result of the paper is that $\theta_r$ is generically injective if $g \gg r$ and $X$ is general. This partially answers the following conjecture proposed by Beauville: $\theta_r$ is generically injective if $X$ is not hyperelliptic. The proof relies on the study of the injectivity of the determinant map $d_E : \wedge^r H^0(E) \to H^0(\det E)$, for a vector bundle $E$ on $X$, and on the reconstruction of the Grassmannian $G(r,rm)$ from a natural multilinear form associated to it, defined in the paper as the Plücker form. The method applies to other moduli spaces of vector bundles on a projective variety $X$.

1. Introduction. In this paper we introduce the elementary notion of Plücker form of a pair $(E,S)$, where $E$ is a vector bundle of rank $r$ on a smooth, irreducible, complex projective variety $X$ and $S \subset H^0(E)$ is a subspace of dimension $rm$. Then we apply this notion to the study of the moduli space $SU_X(r,0)$ of semistable vector bundles of rank $r$ and trivial determinant on a curve $X$. Let

$$\theta_r : SU_X(r,0) \longrightarrow \mathbb{P}(H^0(L)^*)$$

be the so called theta map, defined by the ample generator $L$ of Pic$SU_X(r,0)$, [DN]. Assume $X$ has genus $g$, we prove the following main result:

**Main Theorem.** $\theta_r$ is generically injective if $X$ is general and $g \gg r$.

The theorem gives a partial answer to the following conjecture, or optimistic speculation, proposed by Beauville in [B3] 6.1:

**Speculation.** $\theta_r$ is generically injective if $X$ is not hyperelliptic.

To put in perspective our result we briefly recall some open problems on $\theta_r$ and some known results, see [B3]. A serious difficulty in the study of $\theta_r$ is represented by its indeterminacy locus, which is quite unknown. Raynaud bundles and few more constructions provide examples of points in this locus when $r \gg 0$, cf. [CGT, R]. In particular, there exists an integer $r(X) > 0$ such that $\theta_r$ is not a morphism as soon as $r > r(X)$. As a matter of fact related to this situation, some basic questions are still unsolved. For instance:

- is $\theta_r$ generically finite onto its image for any curve $X$?
• is \( \theta_r \) an embedding if \( r \) is very low and \( X \) is general?
• compute \( r(g) := \min\{r(X), X \text{ curve of genus } g\} \).

On the side of known results only the case \( r = 2 \) is well understood: \( \theta_2 \) is an embedding unless \( X \) is hyperelliptic of genus \( g \geq 3 \), see [B1, BV1, vGI]. Otherwise \( \theta_2 \) is a finite 2:1 cover of its image, [DR]. For \( r = 3 \) it is conjectured that \( \theta_3 \) is a morphism and this is proved for \( g \leq 3 \), see [B3, 6.2] and [B2]. To complete the picture of known results we have to mention the case of genus two. In this case \( \theta_r \) is generically finite, see [B2, BV2].

To prove our main theorem we apply a more general method, working in principle for more moduli spaces of vector bundles over a variety \( X \) of arbitrary dimension. Let us briefly describe it.

Assume \( X \) is embedded in \( \mathbb{P}^n \) and consider a pair \((E,S)\) such that: (i) \( E \) is a vector bundle of rank \( r \) on \( X \), (ii) \( S \) is a subspace of dimension \( rm \) of \( H^0(E) \), (iii) \( \det E \cong \mathcal{O}_X(1) \). Under suitable stability conditions there exists a coarse moduli space \( \mathcal{S} \) for \((E,S)\), see for instance [L] for an account of this theory. Let \( p_i : X^m \rightarrow X \) be the \( i \)th projection and let

\[
e_{S,E} : S \otimes \mathcal{O}_{X^m} \rightarrow \bigoplus_{i=1,\ldots,m} p_i^*E
\]

be the natural map induced by evaluating global sections. We will assume that \( e_{E,S} \) is generically an isomorphism for general pairs \((E,S)\). For such a pair the degeneracy scheme \( \mathbb{D}_{E,S} \) of \( e_{E,S} \) is a divisor in \( X^m \), moreover

\[
\mathbb{D}_{E,S} \in \mathcal{O}_{X^m}(1,\ldots,1),
\]

where \( \mathcal{O}_{X^m}(1,\ldots,1) := p_1^*\mathcal{O}_X(1) \otimes \cdots \otimes p_m^*\mathcal{O}_X(1) \). In this paper \( \mathbb{D}_{E,S} \) is defined as the Plücker form of \((E,S)\). The construction of the Plücker form of \((E,S)\) defines a rational map

\[
\theta_{r,m} : \mathcal{S} \rightarrow \mathcal{O}_{X^m}(1,\ldots,1),
\]

sending the moduli point of \((E,S)\) to \( \mathbb{D}_{E,S} \). Assume \( X = \mathbb{G} \), where \( \mathbb{G} \) is the Plücker embedding of the Grassmannian \( G(r,rm) \). Then consider the pair \((\mathcal{U}^*,H)\), where \( \mathcal{U} \) is the universal bundle of \( \mathbb{G} \) and \( H = H^0(\mathcal{U}^*) \). In this case the Plücker form of \((\mathcal{U}^*,H)\) is the zero locus

\[
\mathbb{D}_\mathbb{G} \in \mathcal{O}_{\mathbb{G}^m}(1,\ldots,1)
\]

of a natural multilinear form related to \( \mathbb{G} \). More precisely \( \mathbb{G} \) is embedded in \( \mathbb{P}(\wedge^r V) \), where \( V = H^* \), and \( \mathbb{D}_\mathbb{G} \) is the zero locus of the map

\[
d_{r,m} : (\wedge^r V)^m \rightarrow \wedge^rmV \cong \mathbb{C},
\]

induced by the wedge product. In the first part of the paper we prove that \( \mathbb{G} \) is uniquely reconstructed from \( \mathbb{D}_\mathbb{G} \) as soon as \( m \geq 3 \). We prove the following:
Theorem. Let $m \geq 3$ and let $x \in \mathbb{P}(\wedge^r V)$, then $x \in G$ iff the following conditions hold true:

1. $(x, \ldots, x) \in (\mathbb{P}(\wedge^r V))^m$ is a point of multiplicity $m - 1$ for $D_G$,
2. $\text{Sing}_{m-1}(D_G)$ has tangent space of maximal dimension at $(x, \ldots, x)$.

It follows essentially from this result that the previous map $\theta_{r,m}$ is generically injective, provided some suitable conditions are satisfied.

Indeed let $(E, S)$ be a pair as above and let $g_{E,S} : X \to G_{E,S}$ be the classifying map in the Grassmannian $G_{E,S}$ of $r$ dimensional subspaces of $S^*$. In Section 4 we use the previous theorem to prove the following:

Theorem. $\theta_{r,m}$ is generically injective under the following assumptions:

1. $\text{Aut}(X)$ is trivial and $m \geq 3$,
2. $g_{E,S}$ is a morphism birational onto its image,
3. the determinant map $d_{E,S} : \wedge^r S \to H^0(\mathcal{O}_X(1))$ is injective.

However the main emphasis of this paper is on the case where $X \subset \mathbb{P}^n$ is a general curve of genus $g$ and $\mathcal{O}_X(1)$ has degree $r(m + g - 1)$. Assuming this, we consider the moduli space $\mathcal{S}_r$ of pairs $(E, H^0(E))$, where $E$ is a stable vector bundle of determinant $\mathcal{O}_X(1)$ and $h^1(E) = 0$. Let $t$ be an $r$-root of $\mathcal{O}_X(1)$, then $\mathcal{S}_r$ is birational to $\text{SU}_X(r, 0)$ via the map

$$\alpha : \mathcal{S}_r \longrightarrow \text{SU}_X(r, 0),$$

sending the moduli point of $(E, H^0(E))$ to the moduli point of $E(-t)$. In the second half of the paper we prove that

$$\theta_{r,m} \circ \alpha^{-1} = \beta \circ \theta_r,$$

where $\theta_r$ is the theta map of $\text{SU}_X(r, 0)$ and $\beta$ is a rational map. Moreover we prove that the assumptions of the latter theorem are satisfied if $X$ is general of genus $g \gg r$. Then it follows that $\theta_r$ is generically injective as soon as $X$ is general of genus $g \gg r$.

This completes the description of the proof of the main theorem of this paper. It seems interesting to use Plücker forms for further applications.

2. Plücker forms. Let $V$ be a complex vector space of positive dimension $rm$ and let $\wedge^r V$ be the $r$-exterior power of $V$. On $\wedge^r V$ we consider the multilinear form

$$d_{r,m} : (\wedge^r V)^m \longrightarrow \wedge^{rm} V \simeq \mathbb{C},$$

such that

$$d_{r,m}(w_1, \ldots, w_m) := w_1 \wedge \cdots \wedge w_m.$$
Notice that $d_{r,m}$ is symmetric if $r$ is even and skew symmetric if $r$ is odd. We fix $m$ copies $V_1, \ldots, V_m$ of $V$ and the spaces $\mathbb{P}_s := \mathbb{P}^r V_s$, $s = 1, \ldots, m$, of dimension $N := \binom{rm}{r} - 1$. Then we consider the Segre embedding

$$\mathbb{P}_1 \times \cdots \times \mathbb{P}_m \hookrightarrow \mathbb{P}^{(N+1)m-1}$$

and its projections $\pi_s : \mathbb{P}_1 \times \cdots \times \mathbb{P}_m \to \mathbb{P}_s$, $s = 1, \ldots, m$. The form $d_{r,m}$ defines the following hyperplane section of $\mathbb{P}_1 \times \cdots \times \mathbb{P}_m$:

$$(2) \quad \mathbb{D}_{r,m} := \{(w_1, \ldots, w_m) \in \mathbb{P}_1 \times \cdots \times \mathbb{P}_m \mid d_{r,m}(w_1, \ldots, w_m) = 0\}.$$ 

**Definition 2.1.** $\mathbb{D}_{r,m}$ is the Plücker form of $\mathbb{P}^r V^m$.

$\mathbb{D}_{r,m}$ is an element of the linear system $|O_{\mathbb{P}_1 \times \cdots \times \mathbb{P}_m}(1, \ldots, 1)|$, where

$$O_{\mathbb{P}_1 \times \cdots \times \mathbb{P}_m}(1, \ldots, 1) = \pi_1^* O_{\mathbb{P}_1}(1) \otimes \cdots \otimes \pi_m^* O_{\mathbb{P}_m}(1).$$

Let $e_1, \ldots, e_{rm}$ be a basis of $V$ and let $\mathcal{I}$ be the set of all naturally ordered sets $I := i_1 < \cdots < i_r$ of integers in $[1, rm]$. We fix in $\mathbb{P}^r V$ the basis

$$e_I^{(s)} := e_{i_1} \wedge \cdots \wedge e_{i_r}, \quad I = i_1 < \cdots < i_r \in \mathcal{I}.$$ 

Then any vector of $\mathbb{P}^r V$ is of the form $\sum p_I I^{(s)} I_{I_1} \cdots p_{I_m} I^{(s)} I_{I_m} \wedge I_{I_m}$, where the coefficients $p_I I^{(s)}$ are the standard Plücker coordinates on $\mathbb{P}_s$. This implies that

$$d_{r,m}(w_1, \ldots, w_m) = \sum_{I_1 \cup \cdots \cup I_m = \{1, \ldots, rm\}} p_{I_1}^{(1)} \cdots p_{I_m}^{(m)} e_{I_1} \wedge \cdots \wedge e_{I_m}$$

for each $(w_1, \ldots, w_m) \in (\mathbb{P}^r V)^m$. Note that, to give a decomposition

$$I_1 \cup \cdots \cup I_m = \{1, \ldots, rm\}$$

as above, is equivalent to give a permutation $\sigma : \{1, \ldots, rm\} \to \{1, \ldots, rm\}$ which is strictly increasing on each of the intervals

$$\mathbb{U}_1 := [1, r], \ \mathbb{U}_2 := [r + 1, 2r], \ \ldots, \ \mathbb{U}_m := [(m - 1)r + 1, mr].$$

Let $\mathcal{P}$ be the set of these permutations, then we conclude that

$$d_{r,m}(w_1, \ldots, w_m) = \sum_{\sigma \in \mathcal{P}} \text{sgn}(\sigma) p_{\sigma(\mathbb{U}_1)}^{(1)} \cdots p_{\sigma(\mathbb{U}_m)}^{(m)} e_{1} \wedge \cdots \wedge e_{rm}.$$ 

Assume that $w := (w_1, \ldots, w_m) \in (\mathbb{P}^r V)^m$ is a vector defining the point $o \in \mathbb{P}_1 \times \cdots \times \mathbb{P}_m$, we want to compute the Taylor series of $\mathbb{D}_{r,m}$ at $o$. Let $t := (t_1, \ldots, t_m) \in (\mathbb{P}^r V)^m$, then we have the identity

$$d_{r,m}(w + \epsilon t_1, \ldots, w_m + \epsilon t_m) = \sum_{k=0,\ldots,m} \partial_{w}^{m-k} d_{r,m}(t) \epsilon^k.$$
We will say that the function
\[ \partial_w^{m-k} d_{r,m} : (\wedge^r V)^m \longrightarrow \mathbf{C}, \]
sending \( t \) to the coefficient \( \partial_w^{m-k} d_{r,m}(t) \) of \( \epsilon^k \), is the \( k \)th polar of \( d_{r,m} \) at \( w \), cf. [D]. Let \( S := s_1 < \cdots < s_k \) be a strictly increasing sequence of \( k \) elements of \( M := \{1, \ldots, m\} \). We will put \( k := |S| \). Moreover, for \( w = (w_1, \ldots, w_m) \in (\wedge^r V)^m \), we define \( w_S := w_{s_1} \wedge \cdots \wedge w_{s_k} \). Note that \( \partial_w^0(t) = d(w_1, \ldots, w_m) \) for each \( t \). If \( m - k \geq 1 \) it turns out that
\[
\partial_w^{m-k} d_{r,m}(t) = \sum_{|S|=k} \text{sgn}(\sigma_S) w_{M-S} \wedge t_S,
\]
where \( \sigma_S : M \rightarrow M \) is the permutation \( (1, \ldots, m) \mapsto (j_1, \ldots, j_{m-k}, s_1, \ldots, s_k) \) such that \( S = s_1 < \cdots < s_k \) and \( j_1 < \cdots < j_{m-k} \).

**Definition 2.2.** Let \( W := \wedge^r V \) then
\[ q : \mathbb{P}(W^m) \longrightarrow \mathbb{P}_1 \times \cdots \times \mathbb{P}_m \]
is the rational map sending the point defined by the vector \( w = (w_1, \ldots, w_m) \) of \( W^m \) to the \( m \)-tuple of points defined by the vectors \( w_1, \ldots, w_m \).

Note that the pull-back of \( d_{r,m} \) by \( q \) is a homogeneous polynomial
\[ q^* d_{r,m} \in \text{Sym}^n W^* = H^0(\mathcal{O}_{\mathbb{P}(W)}(m)). \]

We mention, without its non difficult proof, the following result

**Proposition 2.3.** \( \partial_w^{m-k} (d_{r,m}) \) is the \( k \)th polar form at \( w \) of \( q^* d_{r,m} \).

Let \( \delta \in \mathbb{P}(W^m) \) be the point defined by \( w = (w_1, \ldots, w_m) \) and let \( o = q(\delta) \). For the tangent spaces to \( \mathbb{P}(W^m) \) at \( \delta \) and to \( \mathbb{P}_1 \times \cdots \times \mathbb{P}_m \) at \( o \) one has
\[ T_{\mathbb{P}(W^m), \delta} = W^m/\langle w \rangle \]
\[ T_{\mathbb{P}_1 \times \cdots \times \mathbb{P}_m, o} = W/\langle w_1 \rangle \oplus \cdots \oplus W/\langle w_m \rangle. \]
Moreover the tangent map
\[ dq_\delta : W^m/\langle w \rangle \longrightarrow W/\langle w_1 \rangle \oplus \cdots \oplus W/\langle w_m \rangle \]
is exactly the map sending
\[ (t_1, \ldots, t_m) \mod \langle w \rangle \longrightarrow (t_1 \mod \langle w_1 \rangle, \ldots, t_m \mod \langle w_m \rangle). \]
In particular we have
\[ \ker dq_\delta = \{(c_1 w_1, \ldots, c_m w_m), (c_1, \ldots, c_m) \in \mathbb{C}^m \}/\langle w \rangle. \]
We can now use \( dq_\delta \) to study some properties of \( \text{Sing}(\mathbb{D}_{r,m}) \). We consider the \( k \)-osculating tangent cone \( C^k_o \subset T_{\mathbb{P}_1 \times \cdots \times \mathbb{P}_m, o} \) to \( \mathbb{D}_{r,m} \) at \( o \).
**Lemma 2.4.** Keeping the above notations one has:

1. \( \text{Sing}_k(D_{r,m}) = \{ o \in D_{r,m} | \partial_w^{m-i}(d_{r,m}) = 0, \ i \leq k-1 \} \).

2. \( C_k = dq_\partial(\{ t \in W^m \text{ mod}(w) | \partial_w^{m-i}(d_{r,m})(t) = 0, \ i \leq k \} ) \).

**Proof.** By the previous description of \( dq_\partial \) any one dimensional subspace \( l \) of \( T_{P_1 \times \cdots \times P_m,o} \) is the isomorphic image by \( dq_\partial \) of the tangent space at \( \partial \) to an affine line

\[
L_t := \{ w + \epsilon t | \epsilon \in \mathbb{C} \} \subset \mathbb{P}(W^m),
\]

for some \( t = (t_1, \ldots, t_m) \in W^m \). On the other hand the pull-back of the Taylor series of \( d_{r,m} \) to \( L_t \) is

\[
d_{r,m}(w + \epsilon t) = \sum_{i=0, \ldots, m} \partial_w^{m-i}(d_{r,m})(t)\epsilon^i,
\]

this implies (1) and (2). \qed

Let \( o \in P_1 \times \cdots \times P_m \) be the point defined by the vector \( (w_1, \ldots, w_m) \) and let \( v \in T_{P_1 \times \cdots \times P_m,o} \) be a tangent vector to an arc of curve

\[
\{ w_1 + \epsilon t_1, \ldots, w_m + \epsilon t_m, \epsilon \in \mathbb{C} \}.
\]

Applying the lemma and the equality (3), it follows:

**Theorem 2.5.**

(i) \( o \in \text{Sing}_k(D_{r,m}) \iff w_S = 0, \forall S \in \mathcal{I}, |S| = m - k + 1 \).

(ii) \( v \) is tangent to \( \text{Sing}_k(D_{r,m}) \) at \( o \) iff

\[
\sum_{s \in S} \text{sgn}(\sigma_s)w_{S-\{s\}} \wedge t_s = 0, \forall S \in \mathcal{I}, |S| = m - k + 1,
\]

where \( \sigma_s \) is the permutation of \( S \) shifting \( s \) to the bottom and keeping the natural order in \( S - s \).

**Proof.** (i) By Lemma 2.4(1), \( o \in \text{Sing}_k(D_{r,m}) \) iff the \( i \)th polar \( \partial_w^i(d_{r,m}) \) is zero for \( i \leq k - 1 \). This is equivalent to \( w_S = 0 \) for \( |S| = m - k + 1 \). (ii) As above, consider a tangent vector \( v \) at \( o \) to the arc of curve \( \{ w_1 + \epsilon t_1, \ldots, w_m + \epsilon t_m, \epsilon \in \mathbb{C} \} \). By Lemma 2.4(2), \( v \) is tangent to \( \text{Sing}_k(D_{r,m}) \) at \( o \) iff the coefficient of \( \epsilon \) in \( (w + \epsilon t)_S \) is zero, \( \forall |S| = m - k + 1 \). This is equivalent to the condition

\[
\sum_{s \in S} \text{sgn}(\sigma_s)w_{S-\{s\}} \wedge t_s = 0, \forall S \in \mathcal{I}, |S| = m - k + 1.
\]

\( \square \)

**Corollary 2.6.** The Plücker form \( D_{r,m} \) has no point of multiplicity \( \geq m \).

**Proof.** Assume \( D_{r,m} \) has multiplicity \( \geq m \) at \( o \). Then \( w_S = 0, \forall S \) with \( |S| = 1 \). This means \( w_1 = \cdots = w_m = 0 \), which is impossible. \( \square \)
We are especially interested to the behavior of $D_{r,m}$ along its intersection with the diagonal

\[(4) \quad \Delta \subset \mathbb{P}_1 \times \cdots \times \mathbb{P}_m \subset \mathbb{P}^{(N+1)m-1}.
\]

We recall that $\Delta$ spans the projectivized space of the symmetric tensors of $(\wedge^r V)^\otimes m$. Moreover, $\Delta$ is the $m$-Veronese embedding of $\mathbb{P}(\wedge^r V)$. If $r$ is odd $d_{r,m}$ is skew symmetric and $D_{r,m}$ contains $\Delta$. If $r$ is even then

$$D_{r,m} \cdot \Delta$$

is an interesting hypersurface of degree $m$ in the projective space $\Delta$.

Applying Theorem 2.5 to a point $o$ in the diagonal, we have:

**Corollary 2.7.** Let $o \in \Delta$. Then:

(i) $o \in \text{Sing}_k(D_{r,m}) \iff w^{\wedge m-k+1} = 0$;

(ii) $v \in T_{\text{Sing}_k(D_{r,m}),o}$ if and only if

$$\sum_{j \in S} sgn(\sigma_s) w^{\wedge (m-k)} \wedge t_j = 0, \quad \forall S \in \mathcal{I}, |S| = m - k + 1.$$

**Remark 2.8.** Let $o \in \Delta$ be as above, it follows from the corollary that:

$$o \in \Delta \cap \text{Sing}_{m-1}(D_{r,m}) \iff w \wedge w = 0.$$  

It is easy to see that $\Delta \subset \text{Sing}_{m-1}(D_{r,m})$ if $r$ is odd. Let $r$ be even then

$$G \subset \Delta \cap \text{Sing}_{m-1}(D_{r,m}),$$

where $G$ is the Plücker embedding in $\Delta = \mathbb{P}(\wedge^r V)$ of the Grassmannian $G(r, V)$. However it is not true that the equality holds in the latter case. In fact the equation $w \wedge w = 0$ defines $G$ if and only if $r = 2$, see [Ha2].

### 3. Plücker forms and Grassmannians

In this section we will keep the notation $G$ for the Plücker embedding of $G(r, V)$. Our purpose is now to show that $G$ is uniquely reconstructed from $D_{r,m}$ and the diagonal $\Delta$. More precisely we will show the following:

**Theorem 3.1.** Let $m \geq 3$, then

$$G = \{ o \in \Delta \cap \text{Sing}_{m-1}(D_{r,m}) \mid \dim T_{\text{Sing}_{m-1}(D_{r,m}), o} \text{ is maximal} \}.$$  

For the proof we need some preparation. The following result of linear algebra will be useful: let $E$ be a vector space of dimension $d$ and let $w \in \wedge^r E$ be a non zero vector. Consider the linear map

$$\mu_w^s : \wedge^s E \longrightarrow \wedge^{r+s} E$$

sending $t$ to $w \wedge t$. We have:
PROPOSITION 3.2. Let $d - 2r \geq s$, then $\mu^s_w$ has rank $\geq \binom{d-r}{s}$ and the equality holds if and only if the vector $w$ is decomposable.

Proof. We fix, with the previous notations, a basis $\{e_1, \ldots, e_d\}$ of $E$ and the corresponding basis $\{e_I, I = i_1 < \cdots < i_r\}$ of $\wedge^r E$. Let $e_{I_0} := e_1 \wedge \cdots \wedge e_r$ so that $I_0 = 1 < 2 < \cdots < r$. Since $w$ is non zero we can assume that $w = e_{I_0} + \sum_{I \neq I_0} a_I e_I$.

Let $W^-, W^+$ be the subspaces of $E$ respectively generated by $\{e_1, \ldots, e_r\}$ and $\{e_{r+1}, \ldots, e_d\}$. Then we have the direct sum decomposition

$$\wedge^{r+s} E = E^+ \oplus E^-,$$

where $E^+$ and $E^-$ are defined as follows:

$$E^+ = \{e_{I_0} \wedge u, u \in \wedge^s W^+\} \quad \text{and} \quad E^- = \left\{ \sum_{i=1, \ldots, r} e_i \wedge v_i, v_i \in \wedge^{r+s-1} E \right\}.$$

Let $p^+ : \wedge^{r+s} E \to E^+$ be the projection map. Since $w = e_{I_0} + \sum_{I \neq I_0} a_I e_I$, the map

$$(p^+ \circ \mu^s_w)|_{\wedge^s W^+} : \wedge^s W^+ \to E^+$$

is just the map $u \to e_{I_0} \wedge u$, in particular it is an isomorphism. This implies that

$$\text{rank } \mu^s_w \geq \text{rank } (p^+_r \circ \mu^s_w) = \dim \wedge^s W^+ = \binom{d-r}{s}.$$

Let $w$ be decomposable, then there is no restriction to assume $w = e_{I_0}$ and it follows $\dim \Im \mu^s_w = \binom{d-r}{s}$. Now let us assume that $w$ is not decomposable. To complete the proof it suffices to show that, in this case,

$$\text{(5)} \quad \dim \Im \mu^s_w \geq \binom{d-s}{r}.$$

By the above remarks $\mu^s_w$ is injective on $\wedge^s W^+$. Hence the inequality (5) holds iff

$$\text{(6)} \quad \mu^s_w(\wedge^s W^+) \neq \Im \mu^s_w.$$

On the other hand $p^+ \circ \mu^s_w : \wedge^r W^+ \to E^+$ is an isomorphism and $\dim \wedge^s W^+ = \binom{d-r}{s}$. Therefore inequality (6) is satisfied iff there exists a vector $\tau \in \wedge^{r+s} E$ such that

$$\text{(7)} \quad 0 \neq \tau \in \Im \mu^s_w \cap \text{Ker } p^+.$$

So, to complete the proof, it remains to show the following:

CLAIM. Let $d - 2r \geq s$ and $w$ be not decomposable. Then there exists a vector $\tau$ as above.
Proof. By induction on \( s \). If \( s = 1 \) we have \( \dim \text{Im} \mu_1^w \geq d - r \). It is proved in [G] Prop. 6.27, that the strict inequality holds iff \( w \) is not decomposable. Hence we have \( \dim \text{Im} \mu_1^w > d - r \) and there exists a non zero \( \tau \in \text{Im} \mu_1^w \cap \text{Ker} p^+ \).

Now assume that \( \tau \in \text{Im} \mu_1^{s-1} \) is a non zero vector satisfying the induction hypothesis. Let \( N = \{ v \in E \mid \tau \wedge v = 0 \} \). Then \( N \) is the Kernel of the map \( \mu_1^s : E \to \wedge^{r+s-1} E \) and, by the first part of the proof, \( \dim N \leq r + s - 1 \). Since we are assuming \( s + r \leq d - r \), it follows that we can find a vector \( e_k \in \{ e_1, \ldots, e_d \} \) such that

\[
e_k \wedge e_1 \wedge \cdots \wedge e_r \neq 0 \quad \text{and} \quad e_k \notin N.
\]

Then for such a vector we have

\[
0 \neq e_k \wedge \tau = \sum b_J e_k \wedge e_J, \quad |J| = r + s - 1, \quad I_0 \notin \{ J \cup k \}
\]

and, moreover, \( e_k \wedge \tau \in \text{Im} \mu_1^s w \). Hence the claim follows. \( \square \)

From now on we will assume \( m \geq 3 \). Moreover we identify \( \wedge^r V \) to its image via the diagonal embedding

\[
\delta : \wedge^r V \longrightarrow (\wedge^r V)_m,
\]

sending \( w \) to \( \delta(w) := (w, \ldots, w) \). Let \( o \in \Delta \) be the point defined by \( w = (w, \ldots, w) \). From Corollary 2.7(i), we have that

\[
\Delta \cap \text{Sing}_{m-1}(D_{r,m}) = \{ o \in \Delta \mid w \wedge w = 0 \}.
\]

Moreover let \( (t_1, \ldots, t_m) \in (\wedge^r V)_m \), and let \( v \) be a tangent vector at \( o \) to

\[
\{(w + \epsilon t_1, \ldots, w + \epsilon t_m), \epsilon \in C \} \subset P_1 \times \cdots \times P_m,
\]

it follows from Corollary 2.7 that \( v \) is tangent to \( \text{Sing}_{m-1}(D_{r,m}) \) at \( o \) iff

\[
w \wedge t_j + t_i \wedge w = 0, \quad 1 \leq i < j \leq m,
\]

in the vector space \( \wedge^{2r} V \). Let

\[
\vartheta : (\wedge^r V)_m \longrightarrow (\wedge^r V/\langle w \rangle)_m
\]

be the natural quotient map, where \( (\wedge^r V/\langle w \rangle)_m = T_{P_1 \times \cdots \times P_m, o} \). Consider

\[
T_o := \{ (t_1, \ldots, t_m) \in (\wedge^r V)_m \mid w \wedge t_j + t_i \wedge w = 0, \quad 1 \leq i < j \leq m \}
\]

and note that, by the latter remark, one has

\[
\vartheta^{-1}(T_{\text{Sing}_{m-1}(D_{r,m}), o}) = T_o.
\]
For any point \( o \in \Delta \cap \text{Sing}_{m-1}(\mathbb{D}_{r,m}) \) we define

\[
(9) \quad c_o = \text{codimension of } T_{\text{Sing}_{m-1}(\mathbb{D}_{r,m}),o} \text{ in } T_{\mathbb{P}_1 \times \cdots \times \mathbb{P}_m,o},
\]

Since \( \vartheta \) is surjective, it is clear that \( c_o \) is the codimension of \( T_o \) in \((\wedge^r V)^m\).

**Lemma 3.3.** Let \( c_o \) be as above and let

(i) \( c_o \geq mB \) if \( r \) is even and \( m \geq 3 \),

(ii) \( c_o \geq (m-1)B \) if \( r \) is odd and \( m \geq 3 \),

(iii) \( c_o = m-1 \) if \( m \leq 2 \).

Moreover the equality holds in (i) and (ii) iff \( w \) is a decomposable vector.

**Proof.** Let \( w^\perp \subset \wedge^r V \) be the orthogonal space of \( w = (w, \ldots, w) \) with respect to the bilinear form

\[
\wedge : \wedge^r V \times \wedge^r V \longrightarrow \wedge^{2r} V.
\]

Moreover let \( N \subset (\wedge^r V)^m \) be the subspace defined by the equations

\[
(-1)^r t_i + t_j = 0, \quad 1 \leq i < j \leq m.
\]

It is easy to check that

\[
T_o = N + (w^\perp)^m.
\]

Let \( m \geq 3 \) then \( N \) is the diagonal subspace if \( r \) is odd and \( N = (0) \) if \( r \) is even. By Proposition 3.2, we have that \( \text{codim } w^\perp \geq B \) and moreover the equality holds iff \( w \) is a decomposable vector. This implies (i), (ii) and the latter statement. Let \( m \leq 2 \) then \( N \) is either the diagonal subspace or the space of pairs \( (t, -t), t \in \wedge^r V \). Arguing as above it follows that \( c_o = (m-1)B \), i.e. \( c_o = m-1 \). This completes the proof. \( \square \)

**Proof of Theorem 3.1.** The proof is now immediate: let \( o \in \Delta \cap \text{Sing}_{m-1}(\mathbb{D}_{r,m}) \). It is obvious that the codimension \( c_o \) is minimal iff \( \dim T_{\text{Sing}_{m-1}(\mathbb{D}_{r,m}),o} \) is maximal. Assume \( m \geq 3 \), by Lemma 3.3 \( c_o \) is minimal iff \( o \in G \). \( \square \)

Keeping our usual notations we have

\[
G^m \subset \mathbb{P}_1 \times \cdots \times \mathbb{P}_m \subset \mathbb{P}((N+1)^m-1),
\]

where the latter inclusion is the Segre embedding and \( G \) is the previous Plücker embedding. The restriction of \( \mathbb{D}_{r,m} \) to \( G^m \) has a geometric interpretation given in the next lemma.

Let \( o = (w_1, \ldots, w_m) \in G^m \). Then we have \( w_s := v^{(s)}_1 \wedge \cdots \wedge v^{(s)}_r \), where \( v^{(s)}_1, \ldots, v^{(s)}_r \in V_s \) and \( s = 1, \ldots, m \). In particular \( w_s \) is a decomposable vector, so it defines a point \( l_s \) in \( G \). The vector space corresponding to \( l_s \) is generated by the basis \( v^{s}_1, \ldots, v^{s}_r \). We will denote its projectivization by \( L_s \).
LEMMA 3.4. The following conditions are equivalent:
(i) $o \in \mathbb{D}_{r,m}$,
(ii) $w_1 \wedge \cdots \wedge w_m = 0$,
(iii) $\{v_i^j\}, 1 \leq i \leq r, 1 \leq j \leq m$, is not a basis of $V$,
(iv) there exists a hyperplane in $\mathbb{P}(V)$ containing $L_1 \cup \cdots \cup L_m$.

Proof. Immediate. \qed

LEMMA 3.5. $\mathbb{D}_{r,m}$ cuts on $G^m$ an integral hyperplane section.

Proof. Consider the correspondence
$$I = \{ (l_1, \ldots, l_m, H) \in G^m \times \mathbb{P}(V^*) \mid L_1 \cup \cdots \cup L_m \subset H \} ,$$
and its projections $p_1 : I \to G^m$ and $p_2 : I \to \mathbb{P}(V^*)$. Note that the fibre of $p_2$ at any $H$ is the product of Grassmannians of $r - 1$ spaces in $H$, which is irreducible. Hence $I$ is irreducible. On the other hand we have $p_1(I) = \mathbb{D}_{r,m} \cap G^m$ by Lemma 3.4(iv). Hence the latter intersection is irreducible. Since $O_{G^m}(1)$ is not divisible in $\text{Pic}(G^m)$, it follows that $\mathbb{D}_{r,m} \cdot G^m$ is integral. \qed

On $G$ we consider the universal bundle $U_r$. We recall that $U_r$ is uniquely defined by its Chern classes, unless $m = 2$. Let $l \in G$ and let $L \subset \mathbb{P}(V)$ be the space corresponding to $l$. Then the fibre of $U_r^*$ at $l$ is $H^0(O_L(1))$, moreover $H^0(U_r^*) = V^* = H^0(O_{\mathbb{P}(V)}(1))$. Let $\pi_s : G^m \to G$ be the projection onto the $s$th factor. On $G^m$ we consider the vector bundle of rank $rm$
$$\mathcal{F} : = \bigoplus_{s=1,\ldots,m} \pi_s^*U_r^* .$$
For any point $o = (l_1, \ldots, l_m) \in G^m$, we have
$$\mathcal{F}_o = (U_r^*)_{l_1} \oplus \cdots \oplus (U_r^*)_{l_m} = H^0(O_{L_1}(1)) \oplus \cdots \oplus H^0(O_{L_m}(1)) .$$
In particular the natural evaluation map
$$e^{m}_v : V^* \otimes O_{G^m} \longrightarrow \mathcal{F},$$
is a morphism of vector bundles of the same rank $rm$.

Definition 3.6. $\mathbb{D}_G$ is the degeneracy locus of $e^{m}_v$.

Theorem 3.7. $\mathbb{D}_G = \mathbb{D}_{r,m} \cdot G^m$.

Proof. Let $o = (l_1, \ldots, l_m) \in G^m$, then $e^{m}_o$ is the natural restriction map
$$H^0(O_{\mathbb{P}(V)}(1)) \longrightarrow H^0(O_{L_1}(1)) \oplus \cdots \oplus H^0(O_{L_m}(1)) .$$
Note that \( ev^m_o \) is an isomorphism iff \( L_1 \cup \cdots \cup L_m \) is not in a hyperplane of \( \mathbb{P}(V) \). This implies that \( \mathbb{D}_G \) is a divisor. Moreover \( \mathbb{D}_G = \mathbb{D}_{r,m} \cap G^m \) by Lemma 3.4 and \( \mathbb{D}_G \in |O_{G^m}(1, \ldots, 1)|. \) Hence \( \mathbb{D}_G = \mathbb{D}_{r,m} \cdot G^m. \) \( \square \)

4. **Plücker forms and moduli of vector bundles.** In this section we consider any integral, smooth projective variety \( X \subset \mathbb{P}^n \) of dimension \( d \geq 1. \) We assume that \( X \) is linearly normal and not degenerate.

**Definition 4.1.** \((E, S)\) is a good pair on \( X \) if
(i) \( E \) is a vector bundle of rank \( r \) on \( X \),
(ii) \( \det E \cong O_X(1) \),
(iii) \( S \subset H^0(E) \) is a subspace of dimension \( rm \),
(iv) \( E \) is globally generated by \( S \),
(v) the classifying map of \((E, S)\) is a morphism birational onto its image.

Given \((E, S)\) we have the dual space \( V := S^* \) and its Plücker form
\[
\mathbb{D}_{r,m} \subset \mathbb{P}(\wedge^r V)^m.
\]

We want to use it. Let us fix preliminarily some further notations:

**Definition 4.2.**
(i) \( G_{E, S} \) is the Plücker embedding of the Grassmannian \( G(r, V) \),
(ii) \( U_{E, S} \) is the universal bundle of \( G_{E, S} \),
(iii) \( d_{E, S} : \wedge^r S \to H^0(O_X(1)) \) is the standard determinant map,
(iv) \( \lambda_{E, S} : \mathbb{P}^n \to \mathbb{P}(\wedge^r V) \) is the projectivized dual of \( d_{E, S} \),
(v) \( g_{E, S} : X \to G_{E, S} \) is the classifying map defined by \( S \).

We recall that \( g_{E, S} \) associates to \( x \in X \) the parameter point of the space \( \text{Im} ev^*_x \), where \( ev : S \otimes O_X \to E \) is the evaluation map. It is well known that \( g_{E, S} \) is defined by the subspace \( \text{Im} d_{E, S} \) of \( H^0(O_X(1)) \), in particular
\[
g_{E, S} = \lambda_{E, S}|_X.
\]

Since \( E \) is globally generated by \( S \) and \( g_{E, S} \) is a birational morphism, the next three lemmas describe standard properties.

**Lemma 4.3.** One has \( E \cong \lambda^*_E U^*_E, S \) and \( S = \lambda^*_E H^0(U^*_E, S) \) for any good pair \((E, S)\).

We say that the good pairs \((E_1, S_1), (E_2, S_2)\) are isomorphic if there exists an isomorphism \( u : E_1 \to E_2 \) such that \( u^* S_1 = S_2 \).

**Lemma 4.4.** Let \((E_1, S_1)\) and \((E_2, S_2)\) be good pairs. Then the following conditions are equivalent:
(i) \( d_{E_1, S_1} = d_{E_2, S_2} \circ (\wedge^r \alpha) \) for some isomorphism \( \alpha : S_1 \to S_2 \).
(ii) \( f^*E_1 \cong E_2 \) and \( f^*S_1 = S_2 \) for some automorphism \( f \in \text{Aut}(X) \).

**Proof.** (i)⇒(ii). The projectivized dual of \( \wedge^r \alpha \) induces an isomorphism \( a : G_{E_2,S_2} \rightarrow G_{E_1,S_1} \) such that \( g_{E_1,S_1} = a \circ g_{E_2,S_2} \). On the other hand, \( g_{E_i,S_i} : X \rightarrow G_{E_i,S_i} \) is a morphism birational onto its image for \( i = 1,2 \). Hence \( a \) lifts to an automorphism \( f : X \rightarrow X \) with the required properties. (ii)⇒(i). It suffices to put \( \alpha = f^* \). □

Let \( \rho_i : X^m \rightarrow X \) be the projection onto the \( i \)th factor of \( X^m \). Then \( \text{ev}_{E,S} : S \otimes O_X^m \rightarrow \bigoplus_{i=1,\ldots,m} \rho_i^*E := E \) is the morphism defined as follows. Let \( U \subset X^m \) be open, we observe that \( E(U) = E(U)^m \). Then we define the map \( \text{ev}_{E,S}(U) : S \rightarrow E(U)^m \) as the natural restriction map. Since \( \text{ev}_{E,S} \) is a morphism of vector bundles of the same rank, its degeneracy locus is either \( X^m \) or a divisor \( D_{E,S} \in |O_X^m(1,\ldots,1)| \).

**Definition 4.5.** We will say that the divisor \( D_{E,S} \) is the determinant divisor, or the Plücker form, of the pair \((E,S)\).

If the previous locus is \( X^m \) we will say that \((E,S)\) has no Plücker form.

**Lemma 4.6.** Let \((E_1,S_1)\) and \((E_2,S_2)\) be isomorphic good pairs. Then \( D_{E_2,S_2} = D_{E_1,S_1} \).

**Proof.** Let \( u : E_1 \rightarrow E_2 \) be an isomorphism such that \( u^*S_2 = S_1 \). Then, by taking the pull back of \( u \) to \( \text{ev}_{E_1,S_1} : S_1 \otimes O_X \rightarrow E_1 \), we obtain \( \text{ev}_{E_2,S_2} \). This implies that \( D_{E_1,S_1} = D_{E_2,S_2} \). □

**Remark 4.7.** Note that \( D_{E,S} \) contains the multidiagonal \( \Delta_m \), i.e. the set of all the points \((x_1,\ldots,x_m) \in X^m \) such that \( x_i = x_j \) for some distinct \( i,j \in \{1,\ldots,m\} \). Moreover, \( \Delta_m \) is a divisor in \( X^m \) iff \( \text{dim } X = 1 \). In this case \( D_{E,S} \) is reducible:

**Proposition 4.8.** Assume that \( X \) is a curve, then

\[ D_{E,S} = (r + \epsilon) \Delta_m + D_{E,S}^*. \]

where \( \epsilon \geq 0 \) and the support of the divisor \( D_{E,S}^* \) is the Zariski closure of the set

\[ \{ (x_1,\ldots,x_m) \in X^m - \Delta_m \mid \exists s \in S, s(x_i) = 0, i = 1,\ldots,m \}. \]

**Proof.** Let \( x = (x_1,\ldots,x_m) \in \Delta_m \). Then \( \text{ev}_{E,S} \) has rank \( \leq rm - r \) at \( x \). This implies that \( x \) is a point of multiplicity \( \geq r \) of the determinant divisor \( D_{E,S} \). Hence \( \Delta_m \) is a component of \( D_{E,S} \) of multiplicity \( \geq r \). This implies the statement. □
Actually, \( \epsilon = 0 \) if \( E \) is a general semistable vector bundle on the curve \( X \). It is enough to verify this property in the case \( E = L \oplus r \) and \( S = H^0(E) \), where \( L \) is a general line bundle on \( X \) of degree \( m + g - 1 \). In this case the Plücker form of \((E, S)\) is indeed \( r \) times the Plücker form of \((L, H^0(L))\).

It is also non difficult to compute that \( D_{E,S} - r \Delta_m \) is numerically equivalent to \( a^* r \Theta \), where \( a : X^m \to \text{Pic}^m(X) \) is the natural Abel map and \( \Theta \subset \text{Pic}^m(X) \) is a theta divisor. Finally we consider the commutative diagram

\[
\begin{array}{ccc}
X^m & \xrightarrow{g^m_{E,S}} & \mathbb{G}^m_{E,S} \\
\downarrow & & \downarrow \\
(P^n)^m & \xrightarrow{\lambda^m_{E,S}} & (P^N)^m
\end{array}
\]

where the vertical arrows are the inclusion maps.

**Lemma 4.9.** Let \( D_{E,S} \) be the Plücker form of a good pair \((E, S)\), then

\[ D_{E,S} = (\lambda^m_{E,S})^* D_{r,m}. \]

**Proof.** Lifting by \( g^m_{E,S} \) the map \( ev^m : V \otimes \mathcal{O}_{E,S} \to \bigoplus_{i=1,\ldots,m} \pi_i^* U^m_{E,S} \), one obtains the map \( ev_{E,S} : S \otimes O_{X^m} \to \bigoplus_{i=1,\ldots,m} \rho_i^* E \). From the commutativity of the above diagram it follows that \( D_{E,S} = (\lambda^m_{E,S})^* D_{r,m} = (g^m_{E,S})^* D_{G_{E,S}} \). \( \square \)

To a good pair \((E, S)\) we have associated its Plücker form \( D_{E,S} \). Now we want to prove that, under suitable assumptions, a good pair \((E, S)\) is uniquely reconstructed from \( D_{E,S} \). To this purpose we define the following projective variety in the ambient space \( \mathbb{P}^n \) of \( X \).

**Definition 4.10.** \( \Gamma_{E,S} \) is the closure of the set of points \( x \in \mathbb{P}^n \) such that:

(i) \( D_{E,S} \) has multiplicity \( m - 1 \) at the point \( o = (x, \ldots, x) \in (\mathbb{P}^n)^m \),

(ii) the tangent space to \( \text{Sing}(D_{E,S}) \) at \( o \) has maximal dimension.

**Theorem 4.11.** Assume that \( d_{E,S} \) is injective and \( m \geq 3 \). Then:

(i) \( \Gamma_{E,S} \) is a cone in \( \mathbb{P}^n \) with directrix the Grassmannian \( \mathbb{G}_{E,S} \).

(ii) the vertex of the cone \( \Gamma_{E,S} \) is the center of the projection \( \lambda_{E,S} \).

**Proof.** Since \( \lambda_{E,S} \) is the projective dual of \( d_{E,S} \), the tensor product map

\[ d_{E,S}^{\otimes m} : (\wedge^r S)^\otimes m \to H^0(\mathcal{O}_{X}(1))^\otimes m \]

is precisely the pull-back map

\[ (\lambda^m_{E,S})^* : H^0(\mathcal{O}_{(\mathbb{P}(\wedge^r V))^m}(1, \ldots, 1)) \to H^0(\mathcal{O}_{\mathbb{P}^n}(1, \ldots, 1)). \]

Moreover it is injective. Let \( F \in H^0(\mathcal{O}_{(\mathbb{P}(\wedge^r V))^m}(1, \ldots, 1)) \) be the polynomial of multidegree \((1, \ldots, 1)\) defining \( D_{r,m} \). Then we can choose coordinates on
(\mathbb{P}(\wedge^r V))^m \text{ and } (\mathbb{P}^n)^m \text{ so that } d_{E,S}^m(F) = F. \text{ Assume that } \lambda_{E,S}^m \text{ is a morphism at the point } o \in (\mathbb{P}^n)^m \text{, then it follows that:}

(a) \lambda_{E,S}^m(o) \in \text{Sing}_{m-1}(\mathbb{D}_{r,m}) \text{ iff } o \in \text{Sing}_{m-1}(\mathbb{D}_{E,S}).

(b) the codimension is equal for the tangent spaces to \text{Sing}_{m-1}(\mathbb{D}_{r,m}) \text{ at } \lambda_{E,S}^m(o) \text{ and to } \text{Sing}_{m-1}(\mathbb{D}_{E,S}) \text{ at } o.

Assume that \( o = (x, \ldots, x) \) is a diagonal point in \((\mathbb{P}^n)^m \). Then \( x \in \Gamma_{E,S} \text{ iff } o \text{ satisfies (i) and (ii) in Definition 4.10. By (a) and (b), conditions (i) and (ii) hold true for } o \text{ iff they hold true for } \lambda_{E,S}^m(o) \text{ as a point of } \mathbb{D}_{r,m}. \text{ Finally, by Theorem 3.1, } \lambda_{E,S}(o) \text{ satisfies (i) and (ii) iff } x \text{ belongs to the Grassmannian } G_{E,S}. \text{ Hence } \Gamma_{E,S} \text{ is a cone over } G_{E,S} \text{ with vertex the center of } \lambda_{E,S}. \square

We are now able to show the main result of the current section.

**Theorem 4.12.** Let \((E_1, S_1)\) and \((E_2, S_2)\) be good pairs defining the same Plücker form \( \mathbb{D} \subset (\mathbb{P}^n)^m \). Assume that \( m \geq 3 \) and \( d_{E_i,S_i} \) is injective for any \( i = 1, 2 \), then there exists \( f \in \text{Aut}(X) \) such that \( f^*E_2 \cong E_1 \) and \( f^*S_2 = S_1 \).

**Proof.** Let \( \Gamma \) be the closure of the set of diagonal points \( o = (x, \ldots, x) \in \mathbb{D} \) of multiplicity \( m - 1 \) and tangent space \( T_{\text{Sing}_{m-1}(\mathbb{D}),o} \) of maximal dimension. By Theorem 4.11, \( \Gamma \) is a cone in \( \mathbb{P}^n \): its directrix is the Grassmannian \( G_{E_i,S_i} \) and its vertex is the center of the projection \( \lambda_{E_i,S_i} \), both for \( i = 1 \) and \( i = 2 \). Since the projection maps \( \lambda_{E_i,S_i} \) have the same center, there exist an isomorphism \( \sigma: G_{E_2,S_2} \to G_{E_1,S_1} \) such that \( \lambda_{E_1,S_1} = \sigma \circ \lambda_{E_2,S_2} \). Since \( m \geq 3 \), then \( \sigma = \wedge^r \alpha^* \) for an isomorphism \( \alpha: S_1 \to S_2 \), see [Ha2, p. 122]. Then, applying Lemma 4.4, it follows \( f^*E_1 \cong E_2 \) and \( f^*S_1 = S_2 \) for some \( f \in \text{Aut}(X) \). \square

To conclude this section we briefly summarize, in a general statement, how to deduce from the previous results the generic injectivity of some natural maps, defined on a moduli space of good pairs as above. Therefore we assume that a coarse moduli space \( S \) exists for the family of good pairs \((E, S)\) under consideration. This is, for instance the case when \( E \) is stable with respect to the polarization \( \mathcal{O}_X(1) \) and \( S = H^0(E) \). Then there exists a natural map

\[
\theta_{r,m}: S \longrightarrow |\mathcal{O}_X^m(1, \ldots, 1)|
\]

sending the moduli point of \((E, S)\) to its determinant divisor \( \mathbb{D}_{E,S} \). Let \((E_1, S_1)\) and \((E_2, S_2)\) be good pairs as above defining two general points of \( S \). Assume that \( \mathbb{D}_{E_1,S_1} = \mathbb{D}_{E_2,S_2} \). Then we know from Theorem 4.12 that then \((E_1, S_1)\) and \((E_2, S_2)\) are isomorphic if \( m \geq 3 \), \( \text{Aut}(X) = 1 \) and

\[
d_{E_i,S_i}: \wedge^r S_i \longrightarrow H^0(\mathcal{O}_X(1)).
\]

is injective. This implies the next statement:
THEOREM 4.13. Let $m \geq 3$ and $\text{Aut}(X) = 1$. Assume $d_{E,S} : N^r S \rightarrow H^0(\mathcal{O}_X(1))$ is injective for good pairs $(E,S)$ with moduli in a dense open subset of $S$. Then $\theta_{r,m}$ is generically injective.

5. Plücker forms and the theta map of $SU_X(r,0)$. Now we apply the preceding arguments to study the theta map of the moduli space $SU_X(r,0)$ of semistable vector bundles of rank $r$ and trivial determinant over a curve $X$ of genus $g \geq 2$. By definition the theta map

$$\theta_r : SU_X(r,0) \longrightarrow \mathbb{P}(H^0(L)^*)$$

is just the rational map defined by the ample generator $L$ of $SU_X(r,0)$. We prove our main result:

THEOREM 5.1. Let $X$ be general and $g \gg r$, then $\theta_r$ is generically injective.

To prove the theorem we need some preparation. At first we replace the space $SU_X(r,0)$ by a suitable translate of it, namely the moduli space

$$S_r$$

of semistable vector bundles $E$ on $X$ having rank $r$ and fixed determinant $\mathcal{O}_X(1)$ of degree $r(m+g-1)$. We assume that $X$ has general moduli and that $\mathcal{O}_X(1)$ is general in $\text{Pic}^{r(m+g-1)}(X)$, with $m \geq 3$ and $r \geq 2$. In particular $\mathcal{O}_X(1)$ is very ample: we also assume that $X$ is embedded in $\mathbb{P}^n$ by $\mathcal{O}_X(1)$.

We recall that $S_r$ is biregular to $SU_X(r,0)$, the biregular map being induced by tensor product with an $r$th root of $\mathcal{O}_X(-1)$.

PROPOSITION 5.2. Let $E$ be a semistable vector bundle on $X$ with general moduli in $S_r$. Then:

(i) $h^0(E) = rm$ and $(E,H^0(E))$ is a good pair,

(ii) the Plücker form of $(E,H^0(E))$ exists.

Proof. (i) It suffices to produce one semistable vector bundle $E$ on $X$, of degree $r(m+g-1)$ and rank $r$, such that $h^0(E) = rm$ and $(E,H^0(E))$ is a good pair in the sense of Definition 4.1. Then the statement follows because the conditions defining a good pair are open. Let $L \in \text{Pic}^{m+g-1}(X)$ be general, then $h^0(L) = m$ and $L$ is globally generated. Since $m \geq 3$, $L$ defines a morphism birational onto its image

$$f : X \longrightarrow \mathbb{P}(H^0(L)^*).$$

Putting $E := L^\oplus r$ we have a globally generated, semistable vector bundle such that $h^0(E) = rm$. Hence, to prove that $(E,H^0(E))$ is a good pair, it remains to show
that its classifying map
\[ g_E : X \rightarrow \mathbb{G}_E : = G(r, H^0(E)^*) \]
is birational onto its image. We observe that \( H^0(E) = H_1 \oplus \cdots \oplus H_r \), where \( H_i \) is just a copy of \( H^0(L) \), \( i = 1, \ldots, r \). Let \( f_i : X \rightarrow \mathbb{P}(H_i^*) \) be the corresponding copy of \( f \), for any \( i = 1, \ldots, r \). Then \( g_E : X \rightarrow \mathbb{G}_E \) can be described as follows: let \( \mathbb{P}(E^*_x) \subset \mathbb{P}(H^0(E)^*) \) be the linear embedding induced by the evaluation map, it turns out that \( \mathbb{P}(E^*_x) \) is the linear span of \( f_1(x), \ldots, f_r(x) \). This implies that \( g_E = u \circ (f_1 \times \cdots \times f_r) \), where
\[ u : \mathbb{P}(H_1^*) \times \cdots \times \mathbb{P}(H_r^*) \rightarrow \mathbb{G}_E \]
is the rational map sending \( (y_1, \ldots, y_r) \) to the linear span of the points \( y_i \in \mathbb{P}(H_i^*) \subset \mathbb{P}(H^0(E)^*) \), \( i = 1, \ldots, r \). Since \( f \) is birational onto its image, the same is true for the map \( f_1 \times \cdots \times f_r \). Moreover \( u \) is clearly birational onto its image. Hence \( g_E \) is birational onto its image. Finally \( g_E \) is a morphism, since \( L^{\oplus r} \) is globally generated. This completes the proof of (i).

(ii) Again it suffices to produce one good pair \((E, H^0(E))\) with the required property. It is easy to see that this is the case if \( E = L^{\oplus r} \) as in (i). \( \square \)

Now we consider the rational map
\[ \theta_{r,m} : S_r \rightarrow |O_X(1, \ldots, 1)| \]
sending the moduli point \([E] \in S_r\) of a general \( E \) to the Plücker form
\[ \mathbb{D}_E \subset |O_X(1, \ldots, 1)| \]
of the pair \((E, H^0(E))\). Let \( t \in \text{Pic}^{m+g-1}(X) \) be an \( r \)-root of \( O_X(1) \), then we have a map
\[ a_t : X^m \rightarrow \text{Pic}^{g-1}(X) \]
sending \((x_1, \ldots, x_m)\) to \( O_X(t - x_1 - \cdots - x_m) \). It is just the natural Abel map \( a : X^m \rightarrow \text{Pic}^m(X) \), multiplied by \(-1\) and composed with the tensor product by \( t \). Fixing a Poincaré bundle \( \mathcal{P} \) on \( X \times \text{Pic}^{g-1}(X) \) we have the sheaf
\[ R^1q_{2*}(q_1^*E(-t) \otimes \mathcal{P}), \]
where \( q_1, q_2 \) are the natural projection maps of \( X \times \text{Pic}^{g-1}(X) \). It is well known the support of this sheaf is either \( \text{Pic}^{g-1}(X) \) or a Cartier divisor \( \Theta_E \), see [BNR]. Moreover, due to the choice of \( t \), one has
\[ \Theta_E \in |r\Theta|, \]
where $\Theta := \{ N \in \text{Pic}^{g-1}(X) \mid h^0(N) \geq 1 \}$ is the natural theta divisor of $\text{Pic}^{g-1}(X)$. In particular, one has $h^0(E \otimes N(-t)) = h^1(E \otimes N(-t))$ so that

$$\text{Supp} \Theta_E = \{ N \in \text{Pic}^{g-1}(X) \mid h^0(E \otimes N(-t)) \geq 1 \}.$$  

Finally, it is well known that there exists a suitable identification $| r \Theta | = \mathbb{P}(H^0(\mathcal{L})^*)$ such that $\theta_r([E]) = \Theta_E$, [BNR]. Computing Chern classes it follows

$$a^*_t \Theta_E + r \Delta_m \in | \mathcal{O}_{X_m}(1, \ldots, 1)|,$$

where $\Delta_m \subset X^m$ is the multidiagonal divisor. On the other hand, $r \Delta_m$ is a component of $D_E$ by Proposition 4.8. Moreover, it follows from the definition of determinant divisor that $D_E$ contains $a^{-1}_t(\Theta_E)$. Therefore we have

$$a^*_t \Theta_E + r \Delta_m = D_E. \quad (11)$$

Let $\alpha : | r \Theta | \to | \mathcal{O}_{X_m}(1, \ldots, 1)|$ be the linear map sending $D \in | r \Theta |$ to $a^*_t D + r \Delta_m$. We conclude the following from the latter equality:

**Proposition 5.3.** $\theta_{r,m}$ factors through the theta map $\theta_r$, that is $\theta_{r,m} = \alpha \circ \theta_r$.

**Proof of Theorem 5.1.** Let $\theta_{r,m} : S_r \to | \mathcal{O}_{X_m}(1, \ldots, 1)|$ be as above. We have $\text{Aut}(X) = 1$ and $m \geq 3$. We know that $(E, H^0(E))$ is a good pair if $[E] \in S_r$ is general and that $\theta_{r,m}$ factors through the theta map $\theta_r$. Theorem 4.13 says that $\theta_{r,m}$ is generically injective if $(E, H^0(E))$ is a good pair and the determinant map

$$d_E : \wedge^r H^0(E) \longrightarrow H^0(\mathcal{O}_X(1))$$

is injective for a general $[E]$. This is proved in the next section. \hfill $\Box$

6. **The injectivity of the determinant map.** Let $(X, E)$ be a pair such that $X$ is a smooth irreducible curve of genus $g$ and $E$ is a semistable vector bundle of rank $r$ on $X$ and degree $r(g - 1 + m)$, with $m \geq 3$. If $E$ is a general semistable vector bundle on $X$, it follows that:

(i) $(E, H^0(E))$ is a good pair,

(ii) its Plücker form exists.

(see Definition 4.1 and Proposition 5.2). It is therefore clear that the previous conditions are satisfied on a dense open set $U$ of the moduli space of $(X, E)$.

**Assumption.** From now on we will assume that $(X, E)$ defines a point of $U$, so that $X$ is a general curve of genus $g$ and $E$ is semistable and satisfies (i) and (ii).

In this section we prove the following result:
THEOREM 6.1. Let $X$ and $E$ be sufficiently general and $g \gg r$, then:

(i) the determinant map $d_E : \bigwedge^r H^0(E) \to H^0(\det E)$ is injective,

(ii) the classifying map $g_E : X \to G_E$ is an embedding.

Since $m \geq 3$, $\det E = \mathcal{O}_X(1)$ is very ample. So we will assume as usual that the curve $X$ is embedded in $\mathbb{P}^n = \mathbb{P}(H^0(\mathcal{O}_X(1))^*)$. Let us also recall that $G_E \subset \mathbb{P}^{(rm)} - 1$ denotes the Plücker embedding of the Grassmannian $G(r, H^0(E)^*)$. Let

$$\lambda_E : \mathbb{P}^n \to \mathbb{P}^{(rm)} - 1$$

be the projectivized dual of $d_E$. We have already remarked in Section 4 that $g_E$ is just the restriction $\lambda_E|_X$. This immediately implies that:

**Lemma 6.2.** $d_E$ is injective $\iff$ $\lambda_E$ is surjective $\iff$ the curve $g_E(X)$ spans the Plücker space $\mathbb{P}^{(rm)} - 1$.

Since $(E, H^0(E))$ is a good pair, $g_E : X \to g_E(X)$ is a birational morphism. Let

$$\langle g_E(X) \rangle \subset \mathbb{P}^{(rm)} - 1$$

be the linear span of $g_E(X)$. Then the previous Theorem 6.1 is an immediate consequence of the following one:

**Theorem 6.3.** For a general pair $(X, E)$ as above $g_E$ is an embedding and

$$\dim \langle g_E(X) \rangle \geq r(m - 1) + g.$$

In other words, the statement says that $g_E$ is an embedding and that $d_E$ has rank $> r(m - 1) + g$. This theorem and the previous lemma imply the following:

**Corollary 6.4.** For a general $(X, E)$, $d_E$ is injective if $g \geq \binom{rm}{r} - r(m - 1) - 1$.

Hence the proof of Theorem 6.1 also follows.

**Proof of Theorem 6.3.** To prove the theorem, hence Theorem 6.1, we observe that the moduli space of all pairs $(X, E)$ is an integral, quasi-projective variety defined over the moduli space $M_g$ of $X$. On the other hand, the conditions in the statement of the theorem are open. Therefore, it suffices to construct one pair $(X, E)$ such that $E$ is semistable, $h^0(E) = rm$ and these conditions are satisfied. We will construct such a pair by induction on the genus

$$g \geq 0$$

of $X$. For $g = 0$ we have $X = \mathbb{P}^1$ and $E = \mathcal{O}_{\mathbb{P}^1}(m - 1)^r$. 
LEMMA 6.5. Let $X = \mathbb{P}^1$ and $E = \mathcal{O}_{\mathbb{P}^1}(m-1)^r$, with $m \geq 2$. Then $d_E$ is surjective and $g_E$ is an embedding.

Proof. The proof of the surjectivity of $d_E$ is standard. It also follows from the results in [T]. In order to deduce that $g_E$ is an embedding recall that $g_E$ is defined by $\text{Im} d_E$, hence by the complete linear system $|\mathcal{O}_{\mathbb{P}^1}(r(m-1))|$. □

Now we assume by induction that the statement is true for $g$ and prove it for $g+1$.

Let $(X, E)$ be a general pair such that $X$ has genus $g$. We recall that then $X$ is a general curve of genus $g$ and $(E, H^0(E))$ is a good pair admitting a Plücker form.

By induction $g_E$ is an embedding and $\dim \langle g_E(X) \rangle \geq r(m-1)+g$. We need to prove various lemmas.

LEMMA 6.6. The evaluation map $ev_{x,y} : H^0(E) \to E_x \oplus E_y$ is surjective for general $x, y \in X$.

Proof. If not we would have $h^0(E(-x-y)) > h^0(E) - 2r = r(m-2)$, for any pair $(x, y) \in X^2$. This implies that $h^0(E(-x-y-z_1-\cdots-z_{m-2})) \geq 1, \forall (x, y, z_1, \ldots, z_{m-2}) \in X^m$ and hence that $(E, H^0(E))$ has no Plücker form. But then, by Proposition 5.2 (ii), $(X, E)$ is not general: a contradiction. □

From now on we put

$$ C := g_E(X). $$

Choosing $x, y$ so that $ev_{x,y}$ is surjective, we have a linear embedding

$$ E_x^* \oplus E_y^* \subset H^0(E)^* $$

induced by the dual map $ev_{x,y}^*$. This induces an inclusion of Plücker spaces

$$ \mathbb{P}^{(r)} := \mathbb{P}(\wedge^r (E_x^* \oplus E_y^*)) \subset \mathbb{P}^{(m)} := \mathbb{P}(\wedge^r H^0(E)^*) $$

and of their corresponding Grassmannians

$$ G_{x,y} := G(r, (E_x^* \oplus E_y^*)) \subset G_E. $$

LEMMA 6.7. Assume $\langle C \rangle$ is a proper subspace of the Plücker space of $G_E$. Let $x, y$ be general points of $X$. Then $\langle G_{x,y} \rangle$ is not in $\langle C \rangle$.

Proof. For a general $x \in X$ consider the linear map $\pi : H^0(E)^* \to H^0(E(-x))^*$ dual to the inclusion $H^0(E(-x)) \subset H^0(E)$. It induces a surjective linear projection

$$ \wedge^r \pi : \mathbb{P}(\wedge^r H^0(E)^*) \rightarrow \mathbb{P}(\wedge^r H^0(E(-x))^*), $$
with the linear span $\langle \sigma \rangle$ of $\sigma := \{L \in \mathbb{G}_E \mid \dim(L \cap E^*_x) \geq 1\}$. In particular $\wedge^r \pi$ restricts to a rational map between Grassmannians

$$f : \mathbb{G}_E \longrightarrow \mathbb{G}_{E(-x)},$$

where $\mathbb{G}_{E(-x)} := G(r, H^0(E(-x))^*) \simeq G(r, (m - 1)r)$. Let $l \in \mathbb{G}_E$ be the parameter point of the space $L$, then $f(l)$ is the parameter point of $\pi(L)$. Clearly $f$ is defined at $l$ iff $L \cap E^*_x = 0$. Moreover, the closure of the fibre of $f$ at $f(l)$ is the Grassmannian $G(r, L \oplus E^*_x)$. In particular, the closure of the fibre at $f(y)$ is $\mathbb{G}_{x,y}$, for a general $y \in X$. We distinguish two cases:

1. $f(C)$ spans the Plücker space of $\mathbb{G}_{E(-x)}$. Since $f = \wedge^r \pi_{|_{\mathbb{G}_E}}$ and $\wedge^r \pi$ is linear, it follows that $\bigcup_{y \in C} \langle \mathbb{G}_{x,y} \rangle$ spans the Plücker space of $\mathbb{G}_E$. Since $\langle C \rangle$ is proper in it, we conclude that $\langle \mathbb{G}_{x,y} \rangle$ is not in $\langle C \rangle$ for some $y$, hence for general points $x, y \in X$.

2. $f(C)$ does not span the Plücker space of $\mathbb{G}_{E(-x)}$. Since the Plücker form of $(E, H^0(E))$ exists and $m \geq 3$, we can fix $x, y, z_1, \ldots, z_{m-2} \in X$ so that $h^0(E(-x - y - z)) = 0$, where $z := z_1 + \cdots + z_{m-2}$. Then we have $H^0(E(-x)) \cap H^0(E(-y - z)) = 0$ in $H^0(E)$. Putting $E^*_x := E^*_{z_1} \oplus \cdots \oplus E^*_{z_{m-2}}$, it follows that

$$\pi_{|(E^*_x \oplus E^*_y)} : E^*_y \oplus E^*_x \longrightarrow H^0(E(-x))^*$$

is an isomorphism, that is, $\wedge^r \pi$ induces the following isomorphism of projective spaces:

$$i_{y,z} : \mathbb{P}(\wedge^r (E^*_y \oplus E^*_x)) \longrightarrow \mathbb{P}(\wedge^r H^0(E(-x))^*).$$

On the other hand, $\mathbb{P}(\wedge^r (E^*_y \oplus E^*_x))$ is spanned by the union of its natural linear subspaces $\langle \mathbb{G}_{y,z_i} \rangle = \mathbb{P}(\wedge^r (E^*_y \oplus E^*_y)), i = 1, \ldots, m - 2$. Since $\langle f(C) \rangle$ is a proper subspace of $\mathbb{P}(\wedge^r H^0(E(-x))^*)$, it follows that $\langle \mathbb{G}_{y,z_i} \rangle$ is not in $\langle C \rangle$, for some $i = 1, \ldots, m - 2$.

Now we assume that $\langle C \rangle$ is a proper subspace of the Plücker space of $\mathbb{G}_E$ and fix general points $x, y \in X$ so that the conditions of the previous lemma are satisfied. Keeping the previous notations let $P \subset \mathbb{P}^{rm-1}$ be the tautological image of $\mathbb{P}(E^*)$ and let $P_z := \mathbb{P}(E^*_z), z \in X$. We observe that the Grassmannian $\mathbb{G}_{x,y}$ is ruled by smooth rational normal curves of degree $r$ passing through $x$ and $y$. More precisely, let

$$\mathbb{P}^{2r-1} := \mathbb{P}(E^*_x \oplus E^*_y)$$

and for $t \in \mathbb{G}_{x,y}$ let

$$P_t \subset \mathbb{P}^{2r-1} \subset \mathbb{P}^{rm-1}$$

be the projectivized space corresponding to $t$. We have:
LEMMA 6.8. For a general \( t \in \mathbb{G}_{x,y} \) there exists a unique Segre product \( S := \mathbb{P}^1 \times \mathbb{P}^{r-1} \) such that \( P_x \cup P_y \cup P_t \subset S \subset \mathbb{P}^{2r-1} \). Moreover:

(i) the ruling of \( S \) is parametrized by a degree \( r \) rational normal curve

\[
R \subset \mathbb{G}_{x,y} \subset \mathbb{G}_E \subset \mathbb{P}^{(rm)} - 1,
\]

(ii) the universal bundle \( U_r \) of \( \mathbb{G}_E \) restricts to \( \mathcal{O}_{\mathbb{P}^1(-1) \oplus r} \) on \( R \),

(iii) the restriction map \( H^0(U^*) \to H^0(\mathcal{O}_{\mathbb{P}^1(1)} \oplus r) \) is surjective.

Proof. Since \( x,y \) are general in \( X \), Lemma 6.6 implies that \( P_x \cap P_y = \emptyset \). Since \( t \) is general in \( \mathbb{G}_{x,y} \), we have \( P_t \cap P_x = P_t \cap P_y = \emptyset \). It is a standard fact that the union of all lines in \( \mathbb{P}^{2r-1} \) meeting \( P_x, P_y \) and \( P_t \) is the Segre embedding \( S \subset \mathbb{P}^{2r-1} \) of the product \( \mathbb{P}^1 \times \mathbb{P}^{r-1} \), which is actually the unique Segre variety containing the above linear spaces, see [Ha2, p. 26, 2.12]. It is also well known that \( S \) is the tautological image of the projective bundle associated to \( \mathcal{O}_{\mathbb{P}^1(1) \oplus r} \), see [Ha1]. Therefore, the map assigning to each point \( p \in \mathbb{P}^1 \) the fiber of \( S \) over \( p \) is the classifying map of \( \mathcal{O}_{\mathbb{P}^1(1) \oplus r} \). So it defines an embedding of \( \mathbb{P}^1 \) into the Grassmannian \( \mathbb{G}_{x,y} \), whose image is a rational normal curve \( R \). This implies (ii) and (iii). \qed

Let \( t \in \mathbb{G}_{x,y} \) be a sufficiently general point, where \( x,y \) are general in \( X \). Then, by Lemma 6.7, \( t \) is not in the linear space \( \langle C \rangle \). Since \( \mathbb{G}_{x,y} \) is ruled by the family of curves \( R \), we can also assume that \( C \cup R \) is a nodal curve with exactly two nodes in \( x \) and \( y \). So far we have constructed a nodal curve

\[ \Gamma := C \cup R \]

such that

(i) \( \Gamma \) has arithmetic genus \( g + 1 \) and degree \( r(m + g) \),

(ii) \( \dim \langle \Gamma \rangle \geq \dim \langle C \rangle + 1 = r(m - 1) + g + 1 \).

LEMMA 6.9.

(i) The curve \( \Gamma \) is smoothable in \( \mathbb{G}_E \),

(ii) \( h^1(\mathcal{O}_\Gamma(1)) = 0 \) and \( h^0(\mathcal{O}_\Gamma(1)) = r(m + g) - g \).

Let \( U_r \) be the universal bundle on \( \mathbb{G}_E \), we have also the vector bundle on \( \Gamma \):

\[ F := U_r^* \otimes \mathcal{O}_\Gamma. \]

LEMMA 6.10.

(i) The restriction map \( H^0(U_r^*) \to H^0(F) \) is an isomorphism,

(ii) \( h^1(F) = 0 \) and \( h^0(F) = rm \).

LEMMA 6.11. Let \( x_1, \ldots, x_m \) be general points on \( C \). Then \( h^0(F(-x_1 - \cdots - x_m)) = 0 \).
Proof. Let us recall that $C = g_E(X)$ and that $E \cong \mathcal{U}_r^* \otimes \mathcal{O}_C$. Under the assumptions made at the beginning of this section, $X$ is a general curve of genus $g$, $(E, H^0(E))$ is a good pair admitting a Plücker form. This implies that $h^0(E(-x_1 - \cdots - x_m)) = 0$, where $x_1, \ldots, x_m$ are general points on $X$. Notice also that $F \otimes \mathcal{O}_C \cong E$ and that, by the previous lemma, the restriction map $H^0(F) \to H^0(E)$ is an isomorphism.

Let $d := x_1 + \cdots + x_m$ and let $s \in H^0(F(-d))$. Then $s$ is zero on $X$ because $h^0(E(-d)) = 0$. In particular $s$ is zero on $\{x, y\} = C \cap R$. Hence its restriction on $R$ is a global section $s|_R$ of $\mathcal{O}_R(-x - y)$. But $F \otimes \mathcal{O}_R(-x - y)$ is $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus r}$ so that $s|_R = 0$. Hence $s$ is zero on $Γ$ and $h^0(F(-d)) = 0$. □

We are now able to complete the proof of Theorem 6.3, postponing the proofs of Lemmas 6.9 and 6.10.

Completion of the proof of Theorem 6.3. We start from a curve $Γ = C \cup R$ as above. Therefore the component $C = g_E(X)$ is the embedding in $\mathbb{G}_E$ of a curve $X$ with general moduli and, by the previous lemma, there exists $(x_1, \ldots, x_m) \in C^m$ such that $h^0(F(-x_1 - \cdots - x_m)) = 0$. Now recall that, by Lemma 6.9, the curve $Γ$ is smoothable in $\mathbb{G}_E$. This means that there exists a flat family

$$\{X_t, t \in T\}$$

of curves $X_t \subset \mathbb{G}_E$ such that: (1) $T$ is integral and smooth, (2) for a given $o \in T$ one has $X_o = Γ$, (3) $X_t$ is smooth for $t \neq o$.

Let

$$E_t := \mathcal{U}_r^* \otimes \mathcal{O}_{X_t}.$$  

For $t$ general we have $h^1(E_t) = h^1(F) = 0$, by semicontinuity, and hence $h^0(E_t) = rm$. For the same reason, the determinant map $d_t : \wedge^r H^0(E_t) \to H^0(\mathcal{O}_{X_t}(1))$ has rank bigger or equal to the rank of $d_o : \wedge^r H^0(F) \to H^0(\mathcal{O}_Γ(1))$. This is equivalent to say that

$$\dim\langle X_t \rangle \geq \dim\langle Γ \rangle \geq r(m - 1) + g + 1.$$  

Then, for $t$ general, the pair $(X_t, E_t)$ satisfies the statement of Theorem 6.3.

To complete the proof of the theorem, it remains to show that $E_t$ is semistable for a general $t$. It is well known that $E_t$ is semistable if it admits theta divisor, see [B3]. This is equivalent to say that

$$\Theta_t := \{N \in \text{Pic}^m(X_t) \mid h^0(E_t \otimes N^{-1}) \geq 1\} \neq \text{Pic}^m(X_t),$$

therefore $E_t$ is semistable if

$$D_t := \{(z_1, \ldots, z_m) \in X_t^m \mid h^0(E_t(-z_1 - \cdots - z_m)) \geq 1\} \neq X_t^m.$$
To prove that $D_t \neq X_t^m$ for a general $t$, we fix in $\mathbb{G}_E^m \times T$ the family
\[ A := \{ (z_1, \ldots, z_m; t) \in \mathbb{G}_E^m \times T \mid z_1, \ldots, z_m \in X_t - \operatorname{Sing}(X_t) \}, \]
which is integral and smooth over $T$. Then we consider its closed subset
\[ D := \{ (z_1, \ldots, z_m; t) \in A \mid h^0(U_t^\ast \otimes O_{X_t}(-z_1 - \cdots - z_m)) \geq 1 \}. \]

It suffices to show that $D$ is proper, so that $D_t \neq X_t^m$ for a general $t$. Since $E_o = F$, Lemma 6.11 implies that $D \cap X_o^m$ is proper. Indeed there exists a point $(x_1, \ldots, x) \in C_m \subset X_o^m$ so that $h^0(F(-x_1 - \cdots - x_m)) = 0$. Hence $D$ is proper. \[ \square \]

Proof of Lemma 6.9. (i) We will put $\mathbb{G} := \mathbb{G}_E$. We recall that $\Gamma$ is smoothable in $\mathbb{G}$ if there exists an integral variety $\mathcal{X} \subset \mathbb{G} \times T$ such that:

(a) the projection $p : \mathcal{X} \to T$ is flat,
(b) for some $o \in T$ the fibre $\mathcal{X}_o$ is $\Gamma$,
(c) if $t \in T - \{o\}$, the fibre $\mathcal{X}_t$ is smooth of genus $g + 1$.

To prove that $\Gamma$ is smoothable we use a well known argument, see [S], [HH]. Consider the natural map $\phi : T_{\mathbb{G}|\Gamma} \to N_{\Gamma|\mathbb{G}}$, where $N_{\Gamma|\mathbb{G}}$ is the normal bundle of $\Gamma$ in $\mathbb{G}$. The Cokernel of $\phi$ is a sheaf $T^1_S$, supported on $S := \operatorname{Sing}(\Gamma)$. It is known as the $T^1$-sheaf of Lichtenbaum-Schlessinger. Finally, $\phi$ fits into the following exact sequence induced by the inclusion $\Gamma \subset \mathbb{G}$:
\[ 0 \to T_{\Gamma} \to T_{\mathbb{G}|\Gamma} \xrightarrow{\phi} N_{\Gamma|\mathbb{G}} \to T^1_S \to 0. \]

Let $\mathcal{N}'$ be the image of $\phi$ in $N_{\Gamma|\mathbb{G}}$. The condition $h^1(\mathcal{N}') = 0$ implies that $\Gamma$ is smoothable in $\mathbb{G}$, [S] prop. 1.6. To show that $h^1(N') = 0$ it is enough to show that $h^1(T_{\mathbb{G}|\Gamma}) = 0$, this is a standard argument following from the exact sequence
\[ 0 \to T_{\Gamma} \to T_{\mathbb{G}|\Gamma} \to N' \to 0. \]

To prove that $h^1(T_{\mathbb{G}|\Gamma}) = 0$ we use the Mayer-Vietoris exact sequence
\[ 0 \to T_{\mathbb{G}|\Gamma} \to T_{\mathbb{G}|C} \oplus T_{\mathbb{G}|R} \to T_{\mathbb{G}|S} \to 0. \]

The associated long exact yields the restriction map
\[ \rho : H^0(T_{\mathbb{G}|C}) \oplus H^0(T_{\mathbb{G}|R}) \to H^0(T_{\mathbb{G}|S}). \]

At first we show its surjectivity: it suffices to show that
\[ \rho : 0 \oplus H^0(T_{\mathbb{G}|R}) \to H^0(T_{\mathbb{G}|S}) \]
is surjective. Recall that $S$ consists of two points $x, y$ and that $T^1_S = O_S$. Then, tensoring by $T_{\mathbb{G}|R}$ the exact sequence
\[ 0 \to O_R(-x - y) \to O_R \to O_S \to 0, \]
the surjectivity of $\rho$ follows if $h^1(\mathcal{T}_G|_R(-x - y)) = 0$. To prove this consider the standard Euler sequence defining the tangent bundle to $G$:

$$0 \longrightarrow \mathcal{U}_r \otimes \mathcal{U}_r^* \longrightarrow O_G \oplus \mathcal{O}_r \longrightarrow \mathcal{T}_G \longrightarrow 0.$$ 

Then restrict it to $R$ and tensor by $O_R(-x - y)$. The term in the middle of such a sequence is $M := \mathcal{O}_{\mathbb{P}_1} \otimes \mathcal{O}_{\mathbb{P}_1}(-1)^{\oplus r}$. This just follows because $\mathcal{U}_r^* \otimes \mathcal{O}_R \cong \mathcal{O}_{\mathbb{P}_1}(1)^{\oplus r}$. Since $h^1(M) = 0$, it follows that $h^1(\mathcal{T}_G|_R(-x - y)) = 0$. Hence $\rho$ is surjective. The surjectivity of $\rho$ and the vanishing of $h^1(\mathcal{T}_G|_R)$ and $h^1(\mathcal{T}_G|_C)$ clearly imply that $h^1(\mathcal{T}_G|_R) = 0$. Hence we are left to show that $h^1(\mathcal{T}_G|_R) = h^1(\mathcal{T}_G|_C) = 0$. Since $\mathcal{T}_G|_R \cong \mathcal{O}_{\mathbb{P}_1} \otimes \mathcal{O}_{\mathbb{P}_1}(1)^{\oplus r}$, the former vanishing is immediate. To prove that $h^1(\mathcal{T}_G|_C) = 0$ the argument is similar. Restricting the above Euler sequence to $C$ we obtain the exact sequence

$$0 \longrightarrow E^* \otimes E \longrightarrow E^{\oplus rm} \longrightarrow \mathcal{T}_G|_C \longrightarrow 0,$$

since $\mathcal{U}_r^*|_C \cong E$. Then $h^1(E) = 0$ implies $h^1(\mathcal{T}_G|_C) = 0$.

(ii) To prove $h^1(\mathcal{O}_T(1)) = 0$ it suffices to consider the long exact sequence associated to the Mayer-Vietoris exact sequence

$$0 \longrightarrow \mathcal{O}_T(1) \longrightarrow \mathcal{O}_C(1) \oplus \mathcal{O}_R(1) \longrightarrow \mathcal{O}_{x,y}(1) \longrightarrow 0.$$

For degree reasons we have $h^1(\mathcal{O}_C(1)) = h^1(\mathcal{O}_R(1)) = 0$. Hence it suffices to show that the restriction $H^0(\mathcal{O}_C(1)) \oplus H^0(\mathcal{O}_R(1)) \to \mathcal{O}_{x,y}$ is surjective. This follows from the surjectivity of the restriction $H^0(\mathcal{O}_R(1)) \to \mathcal{O}_{x,y}$. \hfill $\square$

**Proof of Lemma 6.10.** Tensoring by $F$ the standard Mayer-Vietoris exact sequence

$$0 \longrightarrow \mathcal{O}_T \longrightarrow \mathcal{O}_C \oplus \mathcal{O}_R \longrightarrow \mathcal{O}_{x,y} \longrightarrow 0$$

we have the exact sequence

$$0 \longrightarrow F \longrightarrow E \oplus \mathcal{O}_{\mathbb{P}_1}(1)^{\oplus r} \longrightarrow F \otimes \mathcal{O}_{x,y} \longrightarrow 0.$$

Passing to the associated long exact sequence we obtain

$$0 \longrightarrow H^0(F) \xrightarrow{u} H^0(E) \oplus H^0(\mathcal{O}_{\mathbb{P}_1}(1)^{\oplus r}) \xrightarrow{\rho} H^0(F \otimes \mathcal{O}_{x,y}) \longrightarrow H^1(F) \cdots.$$

Restricting $\rho$ to $H^0(E) \oplus 0$ or $0 \oplus H^0(\mathcal{O}_{\mathbb{P}_1}(1)^{\oplus r})$ we have the following maps

$$\rho_C : H^0(E) \longrightarrow E_x \oplus E_y,$$

and

$$\rho_R : H^0(\mathcal{O}_{\mathbb{P}_1}(1)^{\oplus r}) \longrightarrow \mathcal{O}_{\mathbb{P}_1,x}(1)^{\oplus r} \oplus \mathcal{O}_{\mathbb{P}_1,y}(1)^{\oplus r}. $$
These are the usual evaluation maps and we know they are surjective. It follows from the surjectivity of $\rho$ and the above long exact sequence that $h^0(F) = rm = h^0(U^*_r)$ and $h^1(F) = 0$. Thus, to complete the proof, it suffices to show that $H^0(U^*_r) \to H^0(F)$ is injective. This is clear because the composition of maps $H^0(U^*_r) \to H^0(F) \to H^0(E)$ is injective. \hfill $\square$

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