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Conjugacy and centralizers in groups of piecewise projective homeomorphisms

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Abstract. In 2013, Monod introduced a family of Thompson-like groups which provides natural counterexamples to the von Neumann–Day conjecture. We construct a characterization of conjugacy and an invariant and use them to compute centralizers in one group of this family.

1. Introduction

The von Neumann conjecture states that a group is non-amenable if and only if it contains non-abelian free subgroups. It was formulated in 1957 by Mahlon Marsh Day and disproved in 1980 by Alexander Ol'shanskii in [15] through a non-amenable Tarski monster group without any non-abelian free subgroup. The historically first potential counterexample to such conjecture is Thompson's group F of piecewise-linear homeomorphisms of the real line. The group F does not contain any non-abelian free subgroup, but is still not known to be amenable.

Nicolas Monod introduced in [14] a class of groups $H(A)$ depending on a subring A of $\mathbb{R}$ providing another family of counterexamples of the von Neumann–Day conjecture [14]. Monod's groups are very natural and "Thompson-like" as they are described by piecewise projective homeomorphisms of the real line. Later on, Yash Lodha and Justin Moore [11] found that $H(\mathbb{Z}[1/\sqrt{2}])$ contains a finitely presented subgroup, thus providing the first torsion-free finitely presented counterexample.

Thompson-like groups have been extensively studied from the point of view of decision problems. Decision problems play an important role in group theory, giving a measure of the complexity of groups. A finitely presented group G has *solvable conjugacy problem* if there exists an algorithm which, given that $y, z \in G$, determines whether or not there is an element $g \in G$ such that $g^{-1} y g = z$. This problem has been studied for many classes of groups and is generally unsolvable. The conjugacy problem has been studied for several Thompson-like groups [1, 2, 5–8, 10, 13, 16, 17]. Monod's groups share commonalities used in approaches used to study the conjugacy problem, such as being a topological full group. In this paper, we exploit such commonalities to understand conjugacy in Monod's group $H := H(\mathbb{R})$ and find a criterion (see Corollary 3.17) to establish conjugacy within the group.

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Matthew Brin and Craig Squier construct in [3] a conjugacy invariant in the infinitely generated group $\mathrm{PL}_+(\mathbb{R})$ of all piecewise-linear homeomorphisms of the real line with finitely many breakpoints and use it to compute element centralizers by adapting techniques developed in [12]. This invariant has been revisited later in [6, 13] and we adapt it in Theorem 4.2 to produce our own version of this invariant and compute centralizers by the following theorem.

Theorem A. *Given that $z \in H$, then*

$$C_H(z) \cong (\mathbb{Z}, +)^n \times (\mathbb{R}, +)^m \times H^k,$$

for suitable $k, m, n \in \mathbb{Z}_{\geq 0}$.

Several of our results adapt to the general Monod groups $H(A)$ for a subring A of $\mathbb{R}$, but there are some for which the proofs given for H do not immediately apply to the groups $H(A)$. More precisely, the results of Section 3 can be easily rephrased and proved for $H(A)$, while those from Sections 4 and 5 may extend too, but our proofs do not apply to $H(A)$.

The work is organized as follows: in Section 2, we define Monod groups and present some basic properties, some of which shared with Thompson's group F. In Section 3, we discuss a characterization of conjugacy, which is an adaptation of the *Stair algorithm*, developed by Kassabov and the first author in [10]. In Section 4, we define a conjugacy invariant (the *Mather invariant*) for a class of elements by adapting techniques developed in [12], and we show the relation between the Stair algorithm and the Mather invariant. In Section 5, we compute the centralizer subgroups of elements from H as applications of the preceding tools.

2. Monod's groups

In this section, we will discuss groups of piecewise projective orientation-preserving homeomorphisms of $\mathbb{R}\mathrm{P}^1$ which stabilize infinity and discuss some of their properties. These groups are called *Monod's groups* and they were introduced by Nicolas Monod in [14].

We now introduce the notation that will be used in the paper. If A is a subring of $\mathbb{R}$ with unit, the group of Möbius transformations $\mathrm{PSL}_2(A)$, under composition of functions, is the group of transformations of the real projective line $\mathbb{R}\mathrm{P}^1 = \mathbb{R} \cup \{\infty\}$ of the form $f : t \mapsto \frac{at+b}{ct+d}$ for $a, b, c, d \in A$, where the determinant of the associated matrix $M_f = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is equal to 1. We say that f is *hyperbolic* if $|\operatorname{tr}(M_f)| > 2$. We consider the group $\mathrm{PPSL}_2(A)$ of piecewise projective homeomorphisms of $\mathbb{R}\mathrm{P}^1$ with multiplication given by composition of functions. We say that $f \in \mathrm{PPSL}_2(A)$ if there are finitely many points $t_0, t_1, \ldots, t_{n+1} \in \mathbb{R}\mathrm{P}^1$ so that on each interval $(-\infty, t_0]$, $[t_i, t_{i+1}]$, $i = 0, 1, \ldots, n-1$, and $[t_n, \infty)$ the map is a Möbius transformation

$$f : t \mapsto \frac{a_i t + b_i}{c_i t + d_i},$$

where $a_i d_i - c_i b_i = 1$, for suitable $a_i, b_i, c_i, d_i \in A$. *Monod's group* $H(A)$ is the subgroup of $\mathrm{PPSL}_2(A)$, where $f(\infty) = \infty$ and the points $t_0, \ldots, t_n$ lie in the set $\mathcal{P}_A$ of fixed points of hyperbolic Möbius transformations in $\mathrm{PSL}_2(A)$. In the case $A = \mathbb{R}$, we simply write $H(\mathbb{R}) = H$. We say that a point $t_0 \in \mathcal{P}_A$ is a *breakpoint* of $f \in \mathrm{PPSL}(A)$ if there exists a $\varepsilon > 0$ such that there do not exist $a, b, c, d \in A$, where $ad - cb = 1$ and $f(t) = \frac{at+b}{ct+d}$ on $(t_0 - \varepsilon, t_0 + \varepsilon)$.

One of the requirements to adapt the Stair algorithm to this setting is to be able to simultaneously send a tuple of intervals to another such tuple, which means having a form of transitivity. We need H to act order k-transitively on $\mathbb{RP}^1$ and this is a property shared with Thompson's group F. The proof of the following result is analogous to the one for $\mathrm{PL}_+(\mathbb{R})$.

Lemma 2.1. *Let $t_1 < t_2 < \cdots < t_k$ and $s_1 < s_2 < \cdots < s_k$ be elements from $\mathbb{RP}^1 \setminus \{\infty\}$. Then there exists $f \in H$ such that $f(t_i) = s_i$, for all $i = 1, 2, \ldots, k$.*

Proof. For all $i \in \{1, 2, \ldots, k-1\}$, let us consider the intervals $[t_i, t_{i+1}]$ and $[s_i, s_{i+1}]$. Since $\mathrm{PSL}_2(\mathbb{R})$ is 2-transitive on $\mathbb{RP}^1$ (see [9, Theorem 5.2.1 (ii)]), there exists an element $f_i \in \mathrm{PSL}_2(\mathbb{R})$ such that

$$f_i(t_i) = s_i \quad \text{and} \quad f_i(t_{i+1}) = s_{i+1}.$$

Thus it is enough to glue together these maps with two functions $f_0, f_k \in \mathrm{PSL}_2(\mathbb{R})$ defined on $(-\infty, t_1]$ and $[t_k, +\infty)$, respectively, as

$$f_0(t) = \frac{a_0 t + b_0}{d_0} \quad \text{and} \quad f_{k+1}(t) = \frac{a_k t + b_k}{d_k},$$

where $a_0 d_0 = a_k d_k = 1$ and $a_0, b_0, d_0, a_k, b_k, d_k$ are chosen in such a way that $f_0(t_1) = s_1$ and $f_k(t_k) = s_k$. To finish, we construct the following element from H:

$$f(t) := \begin{cases} f_0(t), & \text{if } t \in (-\infty, t_1], \\ f_i(t), & \text{if } t \in [t_i, t_{i+1}], \\ f_k(t), & \text{if } t \in [t_k, +\infty), \end{cases}$$

for $i \in \{1, 2, \ldots, k-1\}$, so that $f(t_i) = s_i$, for all $i \in \{1, 2, \ldots, k\}$. ∎

Remark 2.2. The proof that Lemma 2.1 is true for H does not immediately carry over to $H(A)$, for a subring A of $\mathbb{R}$, as we are not aware of a transitivity result for fixed points of hyperbolic Möbius transformations. In this paper, we sometimes make use of Lemma 2.1 and, in these instances, our proofs do not immediately carry over to $H(A)$, although it is not clear that they cannot be achieved through a different route. Several of the results of this paper carry over to $H(A)$, while for others we cannot immediately say that they do.

If $f \in H(A)$, there are finitely many points $t_1, t_2, \ldots, t_n \in \mathcal{P}_A$ such that on each interval $(-\infty, t_1]$, $[t_i, t_{i+1}]$ for $i = 1, \ldots, n-1$, and $[t_n, +\infty)$ we have

$$f : t \mapsto \frac{a_i t + b_i}{c_i t + d_i},$$

where $a_i d_i - c_i b_i = 1$, for suitable $a_i, b_i, c_i, d_i \in A$. Since $f(\pm\infty) = \pm\infty$, we must have $c_1 = c_n = 0$ and so $f : t \mapsto (a_0 t + b_0)/d_0$ and $f : t \mapsto (a_n t + b_n)/d_n$ on $(-\infty, t_1]$ and $[t_n, +\infty)$, respectively, where $a_0 d_0 = 1 = a_n d_n$, for $a_0, a_n, b_0, b_n \in A$. Then we can say that elements in $H(A)$ have *affine germs* at $\pm\infty$. In other words, when $t \in (-\infty, t_1]$, we rewrite f in this interval as $f(t) = a_0^2 t + a_0 b_0$, for all $t \in (-\infty, t_1]$, since $a_0 d_0 = 1$. Similarly, we can rewrite f as $f(t) = a_n^2 t + a_n b_n$, for all $t \in [t_n, +\infty)$, since $a_n d_n = 1$.

Remark 2.3 ([4]). Notice that, for all elements in $H(A)$, the germs at infinity satisfy that the slopes a_0^2 and a_n^2 are units of the ring A. Thus, if the only units of A are ± 1, the first and last parts of maps in $H(A)$ are translations. For instance, if $A = \mathbb{Z}$, the only possibility is that $a_0^2 = a_n^2 = 1$.

A property that is inherently used while studying the conjugacy problem in the works [10, 13] which we will adapt to work for Monod's group H is that the Thompson–Stein groups $\mathrm{PL}_{A,G}(I)$, defined for a subring A of $\mathbb{R}$ and a subgroup G of the positive units of A, are full groups.

Definition 2.4. Let G be a group of homeomorphisms of some topological space X.

(a) A homeomorphism h of X *locally agrees* with G if for every point $p \in X$, there exists a neighborhood U of p and an element $g \in G$ such that

$$h\big|_U = g\big|_U.$$

We denote the set of all homeomorphisms of X which locally agree with G by $[G]$.

(b) The group G is *full* if every homeomorphism of X that locally agrees with G belongs to G. In other words, G is a full group if $G = [G]$.

Lemma 2.5. *Monod's group $H(A)$ is a full group for any subring A of $\mathbb{R}$.*

Proof. Given a subring A of $\mathbb{R}$ and $h \in [H(A)]$, compactness of $\mathbb{R}\mathrm{P}^1$ implies that h has only finitely many breakpoints, as it locally agrees with maps from $H(A)$. Moreover, h must have affine germs around $\pm\infty$, since it coincides with some element from $H(A)$ and so $h \in H(A)$. Therefore, $[H(A)] \subseteq H(A)$ and so $H(A)$ is a full group. ∎

We finally recall another property of Monod's group which is shared with Thompson's group F (see [14]).

Lemma 2.6. *Monod's group $H(A)$ is torsion-free for any subring A of $\mathbb{R}$.*

For more properties of Monod's groups, we encourage the interested reader to consult the references [4, 14].

3. The Stair algorithm

In this section, we adapt the *Stair algorithm* developed in [5, 10]. If there exists a conjugator between two elements, this algorithm allows us to construct such conjugator from an "initial germ". The algorithm constructs the conjugator by looking at necessary conditions it should satisfy and building it piece by piece until we reach the so-called "final box" and ending of the construction. We show that, if a conjugator exists, it has to coincide with the homeomorphism we construct. In the following, if $y, z \in H$ and there is a $g \in H$ such that $g^{-1}yg = z$, we will write $y^g = z$.

3.1. Notations

Let us fix some notation. Given an $h \in H$, we define the *support* of h to be $\mathrm{supp}(f) = \{t \in \mathbb{R} \mid f(t) \neq t\}$.

Definition 3.1. Let G be any subset of H. We define $G^>$ as the subset of G of all maps that lie above the diagonal, that is,

$$G^> := \{g \in G \mid g(t) > t,\ t \in \mathbb{R}\}.$$

Similarly, we define $G^<$. A homeomorphism $g \in G^> \cup G^<$ is called a *one-bump function*. Moreover, for every $-\infty \leq p < q \leq +\infty$, we define $G(p,q)$ as the set of elements of G with support contained inside (p,q), that is,

$$G(p,q) := \{g \in G \mid g(t) = t,\ t \notin (p,q)\}.$$

We also define the subset

$$G^>(p,q) := \{g \in G \mid g(t) = t,\ \forall t \notin (p,q) \text{ and } g(t) > t,\ \forall t \in (p,q)\}.$$

Analogously, we define $G^<(p,q)$. If $g \in G^>(p,q) \cup G^<(p,q)$, we say that g is a *one-bump function on* (p,q).

Remark 3.2. If G is a subgroup, then $g \in G^>$ if, and only if, $g^{-1} \in G^<$.

Since elements $f \in H$ are defined for all real numbers, we will define suitable "boxes" for real numbers around $\pm\infty$. In order to work with numbers sufficiently close to $\pm\infty$, we give the next definition.

Definition 3.3. A property $\mathcal{P}$ holds *for t negative sufficiently large* (respectively, *for t positive sufficiently large*) to mean that there exists a real number $L < 0$ such that $\mathcal{P}$ holds for every $t \leq L$ (respectively, there is a positive real number R so that $\mathcal{P}$ holds for every $t \geq R$).

3.2. Necessary conditions

In [10], Kassabov and the first author worked with the initial and final slopes of elements from $\mathrm{PL}_+([0,1])$. If two elements from $\mathrm{PL}_+([0,1])$ are conjugate, they coincide on suitable "boxes" around 0 and 1. Let us define similar concepts for elements from H.

Given that $y \in H$, let us denote the slope of y for t negative sufficiently large as

$$y'(-\infty) := \lim_{t \to -\infty} y'(t).$$

Similarly, we denote the slope of y for t positive sufficiently large by $y'(+\infty)$. However, if two elements from H have the same slopes for t negative sufficiently large, they do not necessarily coincide around $-\infty$. Thus, in order to ensure that two elements coincide for t negative sufficiently large, we give the following definition.

Definition 3.4. We define the *germ of* $y \in H$ *at* $-\infty$ as the pair

$$y_{-\infty} := \big(y'(-\infty), y(L) - y'(-\infty)L\big),$$

where L is the largest real number for which y is the affine map with slope $y'(-\infty)$ on the interval $(-\infty, L]$. If y is affine on $\mathbb{R}$, then L can be taken to be any real number. We call $y_{-\infty}$ the *initial germ*. Analogously, we define the *final germ* $y_{+\infty}$.

We remark that, for an element $y \in H$, the initial germ $y_{-\infty}$ and the final germ $y_{+\infty}$ are elements of the *affine group* $\mathrm{Aff}(\mathbb{R})$, which is defined as the semidirect product $\mathrm{Aff}(\mathbb{R}) := \mathbb{R}_{>0} \ltimes \mathbb{R}$, where $\mathbb{R}_{>0}$ denotes the multiplicative group $(\mathbb{R}_{>0}, \cdot)$ and $\mathbb{R}$ denotes the additive group $(\mathbb{R}, +)$. The operation of this group is $(a, b)(c, d) := (ac, b + ad)$. The identity element is $(1, 0)$ and inverses are given by $(a, b)^{-1} = (a^{-1}, -a^{-1}b)$.

The following observation on slopes is the first necessary condition we test for conjugacy. Its proof is a straightforward calculation.

Lemma 3.5. *Let $y, z \in H$ be such that $y^g = z$, for some $g \in H$. Then for t negative (respectively, positive) sufficiently large one has that $y'(-\infty) = z'(-\infty)$ (respectively, $(y'(+\infty) = z'(+\infty))$).*

The next necessary condition we observe is that if the conjugacy classes of the germs of $y, z \in H$ at $-\infty$, or at $+\infty$, are different, then y and z cannot be conjugate.

Lemma 3.6. *For any $y, z \in H$ such that $y^g = z$ for some $g \in H$, the conjugacy classes $y_{-\infty}^{\mathrm{Aff}(\mathbb{R})}$ and $z_{-\infty}^{\mathrm{Aff}(\mathbb{R})}$ of $y_{-\infty}$ and $z_{-\infty}$ inside* $\mathrm{Aff}(\mathbb{R})$ *coincide. Similarly,*

$$y_{+\infty}^{\mathrm{Aff}(\mathbb{R})} = z_{+\infty}^{\mathrm{Aff}(\mathbb{R})}.$$

Proof. Assume that $g_{-\infty} = (a^2, ab)$ and $y_{-\infty} = (a_0^2, a_0 b_0)$. Since $y^g = z$, it is straightforward to see that, for t negative sufficiently large, we have that

$$\begin{aligned} z_{-\infty} = (y^g)_{-\infty} = y_{-\infty}^{g_{-\infty}} &= (a^{-2}, -a^{-1}b) \cdot (a_0^2, a_0 b_0) \cdot (a^2, ab) \\ &= \big(a_0^2, a_0 b_0 a^{-2} + (a_0^2 - 1)a^{-1}b\big). \end{aligned}$$

Thus $y_{-\infty}^{\mathrm{Aff}(\mathbb{R})} = z_{-\infty}^{\mathrm{Aff}(\mathbb{R})}$. Similarly, we see that $y_{+\infty}^{\mathrm{Aff}(\mathbb{R})} = z_{+\infty}^{\mathrm{Aff}(\mathbb{R})}$. ∎

From now on, if $y_{-\infty}$ and $z_{-\infty}$ are conjugate in $\mathrm{Aff}(\mathbb{R})$, we will denote it by $y_{-\infty} \sim_{\mathrm{Aff}(\mathbb{R})} z_{-\infty}$.

3.3. Initial and final boxes

In this subsection, we see that a possible conjugator between two given elements is determined by its germs inside suitable boxes.

Lemma 3.7 (Initial and final boxes). *Let $y, z \in H^{>}(-\infty, p)$ for some $-\infty < p \le +\infty$ and let $g \in H$ be such that $y^g = z$. Then there exists a constant $L \in \mathbb{R}$ (depending on y and z) such that g is affine on the initial box $(-\infty, L]^2$. An analogous result holds, for $y, z \in H^{>}(p, +\infty)$ for some $-\infty \le p < +\infty$ and a final box $[R, +\infty)^2$.*

Proof. By Lemma 3.5, there exists an $L < \min\{0, p\}$ such that $y'(t) = z'(t)$ for $t \le L$. Up to replacing L by a suitable $L_1 < L$, we can assume that $y'(t) = z'(t)$ for every $t \le L$. Assume that $g_{-\infty} = (a^2, ab)$ and $y_{-\infty} = (a_0^2, a_0 b_0)$. Then, following the same calculations of Lemma 3.6, we have that

$$y(t) = a_0^2 t + a_0 b_0 \quad \text{and} \quad z(t) = a_0^2 t + a_0 b_0 a^{-2} + a^{-1} b (a_0^2 - 1)$$

for all $t \le L$ and for suitable $a, b \in \mathbb{R}$.

We can rewrite our goal as follows: if we define

$$\widetilde{L} := \sup\{r \mid g \text{ is affine on } (-\infty, r]\},$$

then $\widetilde{L} \ge \min\{L, g^{-1}(L)\}$. Let us assume the opposite, that is, $\widetilde{L} < \min\{L, g^{-1}(L)\}$ and

$$g(t) = \begin{cases} a^2 t + ab, & \text{if } t \in (-\infty, \widetilde{L}], \\ \frac{\bar{a}t + \bar{b}}{\bar{c}t + \bar{d}}, & \text{if } t \in [\widetilde{L}, L_2), \end{cases}$$

for suitable $\bar{a}, \bar{b}, \bar{c}, \bar{d} \in \mathbb{R}$ and $\widetilde{L} < L_2 \le L$ so that g has a breakpoint at $\widetilde{L}$. Without loss of generality, we can assume that $L_2 = L$.

Since $\widetilde{L} < L < 0$ and $z \in H^{>}(-\infty, p)$, we have $L < z(L)$ and so there exists a real number $\sigma > 1$ such that $\sigma\widetilde{L} < \widetilde{L} < L$ and $\widetilde{L} < z(\sigma\widetilde{L}) < L$. On the other hand, $\widetilde{L} < g^{-1}(L)$ and so $\sigma\widetilde{L} < g^{-1}(L)$. Thus we have $g(\sigma\widetilde{L}) < L$, which means that y is affine around $g(\sigma\widetilde{L})$ and

$$\begin{aligned} y(g(\sigma\widetilde{L})) = y(a^2\sigma\widetilde{L} + ab) &= a^2(a_0^2\sigma\widetilde{L}) + a_0^2 ab + a_0 b_0 \\ &= a^2(a_0^2\sigma\widetilde{L} + a^{-2}a_0 b_0 + a^{-1} b a_0^2 - a^{-1} b + a^{-1} b) \\ &= a^2 z(\sigma\widetilde{L}) + ab. \end{aligned} \tag{3.1}$$

Since $gz(t) = yg(t)$ for every real number t, then (3.1) returns

$$g\big(z(\sigma\widetilde{L})\big) = a^2\big(z(\sigma\widetilde{L})\big) + ab, \tag{3.2}$$

for any real number $\sigma > 1$. By definition of g we also have that

$$g\big(z(\sigma\widetilde{L})\big) = \frac{\bar{a}\big(z(\sigma\widetilde{L})\big) + \bar{b}}{\bar{c}\big(z(\sigma\widetilde{L})\big) + \bar{d}}. \tag{3.3}$$

Then equating (3.2) and (3.3) we see that

$$a^2\big(z(\sigma\widetilde{L})\big) + ab = \frac{\bar{a}\big(z(\sigma\widetilde{L})\big) + \bar{b}}{\bar{c}\big(z(\sigma\widetilde{L})\big) + \bar{d}}. \tag{3.4}$$

By rewriting (3.4), we get that

$$a^2\bar{c}\big(z(\sigma\widetilde{L})\big)^2 + (ab\bar{c} + a^2\bar{d})z(\sigma\widetilde{L}) + ab\bar{d} = \bar{a}z(\sigma\widetilde{L}) + \bar{b}. \tag{3.5}$$

Equation (3.5) is a polynomial equation that holds for all $\sigma > 1$ such that $\sigma\widetilde{L} < \widetilde{L} < L$ and so, since there is an interval worth of such σ's, either $a^2 = 0$ or $\bar{c} = 0$. If $a^2 = 0$, then g would not be a homeomorphism for $t < L$, which is impossible. If $\bar{c} = 0$, then (3.5), coupled with the fact that $\bar{a}\bar{d} = 1$, implies that

$$a^2 z(\sigma\widetilde{L}) + ab = \bar{a}^2 z(\sigma\widetilde{L}) + \bar{a}\bar{b},$$

and so $g(t) = a^2 t + ab$ for $t \in (-\infty, M]$ for some $M > \widetilde{L}$, a contradiction to the definition of the breakpoint $\widetilde{L}$. Hence, in all cases we have a contradiction to the assumption that $\widetilde{L} < \min\{L, g^{-1}(L)\}$ and so we have that $\widetilde{L} \geq \min\{L, g^{-1}(L)\}$. The proof for the final box is similar. ∎

Remark 3.8. We notice that Lemma 3.7 also holds for $y, z \in H^{<}(-\infty, p)$, by just applying its statement to y^{-1} and z^{-1}.

3.4. Building a candidate conjugator

In this subsection, we prove several lemmas which show how to build a conjugator if it exists. If this is the case, then we prove that the conjugator must be unique. Given two elements $y, z \in H$, the set of their conjugators is a coset of the centralizer of either y or z. Thus it is important to begin by obtaining properties of centralizers, which we will do next. After that, we will identify y and z inside a box close to the initial box using a suitable conjugator, as mentioned before. Then we repeat this process and build more pieces of this potential conjugator until we reach the final affinity box. We omit the proof of some of the lemmas, since they follow word-by-word from the original ones in [10] with a slight adaptation in which we use the initial germs. The proofs of the following two results are the same as those of [10, Lemma 4.4] and [10, Corollary 4.5].

Lemma 3.9. *Let $z \in H$ and suppose that there exist real numbers λ and μ satisfying $-\infty < \lambda \leq \mu < +\infty$, $z(t) \leq \lambda$, for all $t \in (-\infty, \mu]$ and that there is $g \in H$ so that $g(t) = t$ for all $t \in (-\infty, \lambda]$ and $g^{-1}zg(t) = z(t)$ for each $t \in (-\infty, \mu]$. Then g is the identity map up to μ.*

In case of $z \in H^{<}$, the previous lemma yields the following consequence.

Corollary 3.10. *Let $z \in H^{<}$ and $g \in H$ such that $g_{-\infty} = (1, 0)$ and $g^{-1}zg = z$. Then g is the identity map.*

The preceding two results allow us to construct a group monomorphism between the group of centralizers of elements from H and the group $\mathrm{Aff}(\mathbb{R})$ as well as showing uniqueness of conjugators

Lemma 3.11. *Given that* $z \in H^<$*, the map*

$$\varphi_z : C_H(z) \to \mathrm{Aff}(\mathbb{R}), \quad g \mapsto g_{-\infty}$$

is a group monomorphism.

Proof. First of all, for each $g_1, g_2 \in C_H(z)$, with $(g_1)_{-\infty} = (a_0^2, a_0 b_0)$ and $(g_2)_{-\infty} = (\bar{a}_0^2, \bar{a}_0 \bar{b}_0)$, there exists $L \in \mathbb{R}$ so that $g_1 g_2(t) = a_0^2 \bar{a}_0^2 t + a_0^2 \bar{a}_0 \bar{b}_0 + a_0 b_0$ on $(-\infty, L]$. Then

$$(g_1 g_2)_{-\infty} = (a_0^2 \bar{a}_0^2, a_0 b_0 + a_0^2 \bar{a}_0 \bar{b}_0) = (a_0^2, a_0 b_0) \cdot (\bar{a}_0^2, \bar{a}_0 \bar{b}_0) = (g_1)_{-\infty} (g_2)_{-\infty},$$

so that φ_z is a well-defined group homomorphism. To show injectivity, suppose that $\varphi(g_1) = \varphi(g_2)$ for $g_1, g_2 \in C_H(z)$. Then $(a_0^2, a_0 b_0) = (\bar{a}_0^2, \bar{a}_0 \bar{b}_0)$. Thus there exists a number $L \in \mathbb{R}$ so that $g_1(t) = g_2(t)$ for all $t \in (-\infty, L]$. Let us define $g := g_1 g_2^{-1}$. We have that $g(t) = t$ for each $t \in (-\infty, L]$. Moreover, we have that $g^{-1} z g = z$. It follows from Corollary 3.10 that $g(t) = t$ for all $t \in \mathbb{R}$, which implies that $g_1(t) = g_2(t)$ for each $t \in \mathbb{R}$. Therefore, φ_z is a monomorphism. ∎

Proposition 3.12 (Uniqueness). *Let* $y, z \in H^<$ *and* $g \in H$ *be maps so that* $y^g = z$*. Then the conjugator* g *is uniquely determined by its initial germ* $g_{-\infty}$*.*

Proof. Let us assume that there are $g_1, g_2 \in H$ so that $g_1^{-1} y g_1 = z$ and $g_2^{-1} y g_2 = z$ and with the same initial germ. Then $(g_1 g_2^{-1})^{-1} y (g_1 g_2^{-1}) = y$. Defining $g := g_1 g_2^{-1}$, we get that $g(t) = t$ for all $t \in (-\infty, L]$, which implies that the initial germ of g is $g_{-\infty} = (1, 0)$. By Corollary 3.10, the unique centralizer of y with initial germ $(1, 0)$ is the identity map. Then $g(t) = t$ for all $t \in \mathbb{R}$. Therefore, $g_1 = g_2$, which proves the uniqueness of a conjugator with a given initial germ if it exists. ∎

The next lemma gives a tool to identify the graphs of y and z inside suitable boxes via some candidate conjugator.

Lemma 3.13 (Identification lemma). *Let* $y, z \in H^<$ *and* $L \in \mathbb{R}$ *be such that* $y(t) = z(t)$ *for all* $t \in (-\infty, L]$*. Then there exists* $g \in H$ *so that* $z(t) = g^{-1} y g(t)$ *for every* $t \in (-\infty, z^{-1}(L)]$ *and* $g(t) = t$ *in* $(-\infty, L]$*. Moreover, this element* g *is uniquely determined on* $(L, z^{-1}(L)]$*.*

Proof. We start showing that, if such a $g \in H$ exists, then it is uniquely determined on $(L, z^{-1}(L)]$. In fact, if such a $g \in H$ exists then, for each $t \in (L, z^{-1}(L)]$, we have that $y(g(t)) = g(z(t)) = z(t)$, since $z(t) \le L$. Therefore, $g(t) = y^{-1} z(t)$, for every $t \in (L, z^{-1}(L)]$.

To show existence, we just define

$$g(t) := \begin{cases} t, & \text{if } t \in (-\infty, L], \\ y^{-1} z(t), & \text{if } t \in \left[L, z^{-1}(L)\right] \end{cases}$$

and we extend it to the real line from the point $(z^{-1}(L), y^{-1}(L))$, by gluing some order-preserving affine map defined on $[z^{-1}(L), +\infty)$. We also define $g(\pm\infty) = \pm\infty$. ∎

We repeatedly apply Lemma 3.13 so that, if we iterate it N times, we can build g on $(-\infty, z^{-N}(L)]$ and this will be the key step for the Stair algorithm in the next subsection. We conclude this subsection with a result whose proof can be obtained word for word from [10, Lemma 4.13].

Lemma 3.14. *Let $y, z \in H^<$. Assume that $g \in H$ and $n \in \mathbb{Z}_{>0}$. Then $y^g = z$ if, and only if, $(y^n)^g = z^n$.*

3.5. The Stair algorithm for H

We now adapt to H the Stair algorithm from [10] which constructs the unique candidate conjugator between two elements $y, z \in H$ with a given initial germ $(a^2, ab) \in \mathrm{Aff}(\mathbb{R})$, that is, an element $g \in H$ such that, if there exists an $h \in H$ so that $h_{-\infty} = (a^2, ab)$ and $y^g = z$, then $h = g$.

Theorem 3.15 (Stair algorithm). *Let $y, z \in H^<$ and let $(-\infty, L]^2$ be the initial box given by y and z. Assume that $(a^2, ab) \in \mathrm{Aff}(\mathbb{R})$ so that $a^2L + ab \leq L$. Then there exists $N \in \mathbb{Z}_{>0}$ such that the unique candidate conjugator $g \in H$ between y and z with initial germ $g_{-\infty} = (a^2, ab)$ is given by*

$$g(t) = y^{-N} g_0 z^N(t), \quad \text{for } t \in \big(-\infty, z^{-N}(L)\big],$$

and affine otherwise, where $g_0 \in H$ is an arbitrary homeomorphism which is affine on $(-\infty, L]^2$ and so that $(g_0)_{-\infty} = (a^2, ab)$.

Remark 3.16. We observe that the hypothesis on (a^2, ab) is a mild one. It ensures that $g_0(L) \leq L$ and so, up to replacing g_0 by g_0^{-1} and switching the role of y and z, we can always assume that $a^2L + ab \leq L$.

Before giving the proof of Theorem 3.15, we observe the following corollary and make a comment about completely characterizing conjugacy in Monod's group H.

Corollary 3.17. *Let $y, z \in H^<$ and let $(-\infty, L]^2$ and $[R, +\infty)^2$ be, respectively, the initial and the final box given by y and z. There is a $g \in H$ such that $y^g = z$ if and only if there is some $(a^2, ab) \in \mathrm{Aff}(\mathbb{R})$ so that $a^2L + ab \leq L$ and*

$$\lim_{N \to \infty} y^{-N} g_0 z^N(t)$$

is affine inside $[R, +\infty)^2$ and where $g_0 \in H$ is an arbitrary homeomorphism which is affine on $(-\infty, L]^2$ and so that $(g_0)_{-\infty} = (a^2, ab)$.

Remark 3.18. Corollary 3.17 gives a characterization of conjugacy inside Monod's group H. However, as is, such characterization cannot be used to construct a finite set of candidate conjugators and thus use them to solve the conjugacy problem in Monod's group $H(A)$ for a suitable subring $A \subseteq \mathbb{R}$ in a manner analogous to what was done for the Thompson–Stein groups $\mathrm{PL}_{A,G}([0, 1])$ in [10] for a suitable subring $A \subseteq \mathbb{R}$ and subgroup $G \subseteq U(A)_{>0}$ of the group of the positive units of A. We now explain better why not.

Lemma 5.4 in [10] shows that there are only finitely conjugators between y and z whose initial and final slopes lie within a bounded interval. After having used a suitable isomorphism (see Lemma 5.7) and thus considering a version of H over the unit interval $[0,1]$, we prove in Lemma 5.9 a similar result for centralizers with first and second derivatives lying within bounded intervals. Even if Lemma 5.9 can indeed be generalized to study conjugators with bounded first and second derivatives (to get a result analogous to [10, Lemma 5.4]), we cannot use the same trick of [10, Lemmas 7.1 and 7.2] to bound both derivatives. By replacing a conjugator g with $y^n g$, we can only bound the first derivative (so that it lives in a bounded interval), but have no available bound on the second derivative appearing in Lemma 5.9: in other words, we can bound the a appearing in $g_{-\infty} = (a^2, ab)$, but not the b and so we would have to test continuum many b's (or equivalently, continuum many initial germs) to find candidate conjugators.

Proof of Theorem 3.15. First of all, we notice that we will consider $y, z \in H^<$ such that their initial germs are in the same conjugacy class in $\mathrm{Aff}(\mathbb{R})$, otherwise y and z cannot be conjugate to each other by Lemma 3.6. Moreover, we further assume that $(a^2, ab) \in \mathrm{Aff}(\mathbb{R})$ conjugates $y_{-\infty}$ to $z_{-\infty}$ in $\mathrm{Aff}(\mathbb{R})$, otherwise there cannot be a conjugator g for y and z with initial germ $g_{-\infty} = (a^2, ab)$, again by Lemma 3.6. Now, let $[R, +\infty)^2$ be the final box and let $N \in \mathbb{Z}_{>0}$ be sufficiently large so that $\min\{z^{-N}(L), y^{-N}(a^2 L + ab)\} > R$.

We will now build a candidate conjugator g between y^N and z^N as the product of two functions g_0 and g_1 and then use Lemma 3.14. Since $y, z \in H^<$, a direct calculation shows that y^N and z^N are affine in the initial and final boxes of y and z, so we can take them as the initial and final boxes of y^N and z^N. We define g_0 as $g_0(t) := a^2 t + ab$, on $(-\infty, L]^2$ and extend it to the real line so that $g_0 \in H$. Our assumption on (a^2, ab) ensures that $g_0(L) \le L$. Now we define $y_1 := g_0^{-1} y g_0$ and we look for a conjugator g_1 between y_1^N and z^N. We remark that y_1^N and z^N coincide on $(-\infty, L]$, since $y_1^N = g_0^{-1} y^N g_0 = z^N$ and $y^N, z^N \in H^<$ are affine on $(-\infty, L]$.

Making use of the Identification Lemma 3.13, we define a $g_1 \in H$ so that

$$g_1(t) := \begin{cases} t, & \text{if } t \in (-\infty, L], \\ y_1^{-N} z^N(t), & \text{if } t \in \big[L, z^{-N}(L)\big]. \end{cases}$$

By construction we have that $g_1^{-1} y_1^N g_1 = z^N$ on $(-\infty, z^{-N}(L)]$. We now construct a function g on $(-\infty, z^{-N}(L)]$ by defining $g(t) := g_0 g_1(t)$, for $t \in (-\infty, z^{-N}(L)]$. We observe that the last part of g is defined inside in the final box since $t = z^{-N}(L) > R$ and

$$g\big(z^{-N}(L)\big) = g_0 g_1\big(z^{-N}(L)\big) > R.$$

Moreover, by construction, g is a conjugator for y^N and z^N on $(-\infty, z^{-N}(L)]$, that is, $g = y^{-N} g z^N$ on $(-\infty, z^{-N}(L)]$. Therefore,

$$g(t) = y^{-N} g z^N(t) = y^{-N} g_0 g_1 z^N(t) = y^{-N} g_0 z^N(t),$$

since $g_1 z^N(t) = z^N(t)$ for every $t \in (-\infty, z^{-N}(L)]$.

If g is not an affine Möbius function on $[R, z^{-N}(L)]$, then g cannot be extended to a conjugator of y^N and z^N and the uniqueness of the shape of g (Proposition 3.12) says that continuing the Stair algorithm will build a function that cannot be a conjugator and, therefore, a conjugator with initial germ (a^2, ab) cannot exist or it would coincide with g on $(-\infty, z^{-N}(L)]$. In the case that g is an affine Möbius function on $[R, z^{-N}(L)]$, we extend g to the whole real line by extending its affine piece on $[R, z^{-N}(L)]$. The map that we construct (which we still call g) lies in H.

By Lemma 3.7 and Proposition 3.12, if there exists a conjugator between y^N and z^N, with initial germ (a^2, ab), it must be equal to g. Then we just check if g conjugates y^N to z^N. If g conjugates y^N to z^N then, by Lemma 3.14, g is a conjugator between y and z, as desired. ■

Remark 3.19. Let us suppose that $y, z \in H^< \cup H^>$. In order to be conjugate, Lemma 3.6 says that their initial germ must be in the same conjugacy class in $\mathrm{Aff}(\mathbb{R})$. Similarly, their final germs must be in the same conjugacy class in $\mathrm{Aff}(\mathbb{R})$. In other words, either both y and z are in $H^<$ or both are in $H^>$. Furthermore, since $g^{-1}yg = z$ if and only if $g^{-1}y^{-1}g = z^{-1}$, we can reduce the study to the case where they are both in $H^<$.

Remark 3.20. The Stair algorithm for $H^<$ can be reversed. This means that we can apply it in order to build a candidate for a conjugator between $y, z \in H^>$. Thus, given an element $(a^2, ab) \in \mathrm{Aff}(\mathbb{R})$, we can determine whether or not there is a conjugator g with final germ $g_{+\infty} = (a^2, ab)$. The proof is similar. We just begin to construct g from the final box.

We observe that the proof of Stair algorithm does not depend on the choice of g_0, the only requirement on it is that it be affine on the initial box and $g_{0_{-\infty}} = (a^2, ab)$. Moreover, it gives a way to find candidate conjugators, if they exist, and we have chosen an initial germ.

The following are two examples of construction of candidate conjugators via the Stair algorithm. In the first example, the candidate is indeed a conjugator, while in the second it is not.

Example 3.21. Consider the maps $y(t) = t - 1$ and

$$z(t) = \begin{cases} \frac{2t-2}{\frac{-3}{2}t+2}, & \text{if } t \in [0,1], \\ \frac{-2t+2}{\frac{-3}{2}t+1}, & \text{if } t \in [1,2], \\ t-1, & \text{otherwise.} \end{cases}$$

Notice that $y, z \in H^<$ and that their initial and final germs are equal. Moreover, we have that $L = 0$ and $R = 2$. Now we take $(1, -1) \in \mathrm{Aff}(\mathbb{R})$ and construct a candidate conjugator between y^4 and z^4. We follow the procedure of the proof of the Stair algorithm and define the maps $g_0(t) := t - 1$ and

$$g_1(t) := \begin{cases} \frac{\frac{1}{2}t}{-\frac{3}{2}t+2}, & \text{if } t \in [0,1], \\ t, & \text{otherwise.} \end{cases}$$

We then define $g := g_0 g_1$ and see that

$$g(t) := \begin{cases} \frac{2t-2}{\frac{-3}{2}t+2}, & \text{if } t \in [0,1], \\ t-1, & \text{otherwise.} \end{cases} \quad \text{and} \quad g^{-1}(t) = \begin{cases} \frac{2t+2}{\frac{3}{2}t+2}, & \text{if } t \in [-1,0], \\ t+1, & \text{otherwise.} \end{cases}$$

We notice that $g \in H$. A direct calculation shows that g conjugates y^4 to z^4. By Lemma 3.14, the element g is a conjugator between y and z.

Example 3.22. Consider the maps $y(t) = t - 1$ and

$$z(t) = \begin{cases} \frac{-2t+2}{\frac{-3}{2}t+1}, & \text{if } t \in [1,2], \\ t-1, & \text{otherwise.} \end{cases}$$

Notice that $y, z \in H^{<}$ and that their initial and final germs are equal. We observe that $L = 1$ and $R = 2$. Now we take $(1, 0) \in \mathrm{Aff}(\mathbb{R})$ and construct a candidate conjugator between y^3 and z^3. We follow the procedure of the proof of the Stair algorithm and define the maps $g_0(t) = t$ and

$$g_1(t) = \begin{cases} \frac{-\frac{7}{2}t+3}{-\frac{3}{2}t+1}, & \text{if } t \in [1,2], \\ \frac{-5t+9}{\frac{-3}{2}t+\frac{5}{2}}, & \text{if } t \in [2,3], \\ t, & \text{otherwise.} \end{cases}$$

We then define $g := g_0 g_1$ and see that

$$g(t) = \begin{cases} \frac{-\frac{7}{2}t+3}{-\frac{3}{2}t+1}, & \text{if } t \in [1,2], \\ \frac{-5t+9}{\frac{-3}{2}t+\frac{5}{2}}, & \text{if } t \in [2,3], \\ t, & \text{otherwise.} \end{cases}$$

We notice that g is not a linear Möbius function on $[2, 3]$. Thus, by Theorem 3.15, the element g cannot be a conjugator between y^3 and z^3. By Lemma 3.14, g cannot be a conjugator between y and z as well.

Remark 3.23. Although this section is stated for H for the sake of consistency of the paper, there are proofs that all the results of this section hold for $H(A)$ too, with the following provisions.

(1) The elements L and R defined for the affinity boxes in Lemma 3.7 must live in A. This can always be achieved since, given any initial box $(-\infty, L]^2$, we can then take an $L' \leq L$ in $L \in A$ and consider the box $(-\infty, L']^2$. Similarly, we can do that for the final one.

(2) Lemma 3.6 needs to be stated by saying that $y_{-\infty}^{\mathrm{Aff}(A)} = z_{-\infty}^{\mathrm{Aff}(A)}$, where the affine group of A is the subgroup of $\mathrm{Aff}(\mathbb{R})$ defined by $\mathrm{Aff}(A) = (U(A)_{>0}, \cdot) \ltimes (A, +)$, where $U(A)_{>0}$ is the group of the positive units of A. Similarly, we must have that $y_{+\infty}^{\mathrm{Aff}(A)} = z_{+\infty}^{\mathrm{Aff}(A)}$.

4. The Mather invariant

We now construct a conjugacy invariant for a class of functions, called *Mather invariant*, by adapting ideas from [12,13]. While in the previous section we worked with $y, z \in H^{<}$, in this section we will work with $y, z \in H^{>}$ as it helps with the arguments and we can do so without loss of generality because of Remark 3.19. We construct such an invariant to deal with the case $y'(\pm\infty) = z'(\pm\infty) = 1$, where the point of view of the Stair algorithm cannot be used to cover all cases when computing element centralizers of elements which will be studied in Section 5.

In the remainder of this section, we assume that $y, z \in H^{>}$ such that $y(t) = z(t) = t + b_0$ if $t \in (-\infty, L]$ and $z(t) = y(t) = t + b_1$ if $t \in [R, +\infty)$, for some suitable $b_0, b_1 > 0$, where L and R are, respectively, sufficiently large negative and positive real numbers.

Let $N \in \mathbb{Z}_{>0}$ be large enough so that

$$y^N\left(\left(y^{-1}(L), L\right)\right) \cup z^N\left(\left(z^{-1}(L), L\right)\right) \subset (R, +\infty).$$

We intend to find a map $s \in H$ such that $s(y^k(L)) = k$, for every $k \in \mathbb{Z}$. We thus define the map s as

$$\begin{aligned} s\colon \mathbb{R} &\to \mathbb{R} \\ t &\mapsto s(t) := \begin{cases} s_{-1}(t), & \text{if } t \in (-\infty, L], \\ s_j(t), & \text{if } t \in \left[y^j(L), y^{j+1}(L)\right], \\ s_{N-1}(t), & \text{if } t \in \left[y^{N-1}(L), +\infty\right), \end{cases} \end{aligned}$$

where

$$\begin{aligned} s_{-1}\colon \left[y^{-1}(L), L\right] &\to [-1, 0] \\ t &\mapsto \frac{t - L}{b_0}, \\ s_{N-1}\colon \left[y^{N-1}(L), y^N(L)\right] &\to [N-1, N] \\ t &\mapsto \frac{t - y^{N-1}(L)}{b_1} + N - 1, \\ s_j\colon \left[y^j(L), y^{j+1}(L)\right] &\to [j, j+1] \\ t &\mapsto \frac{t - y^j(L)}{y^{j+1}(L) - y^j(L)} + j, \quad \forall j = 0, 1, \ldots, N-2. \end{aligned}$$

Since L is a fixed point of some hyperbolic element from $\mathrm{PSL}_2(\mathbb{R})$, so is $y^j(L)$, for every $j = 0, 1, \ldots, N-2, N-1$. Also, since all of the s_i's are affine with strictly positive slopes, they can all be written as $s_i(t) = a_i^2 t + a_i b_i$ for suitable $a_i, b_i \in \mathbb{R}$ and so $s \in H$. Moreover, it is clear that $s(y^k(L)) = k$, for all $k \in \mathbb{Z}$. If we define $\bar{y} := sys^{-1}$ and $\bar{z} := szs^{-1}$, we get that both functions are well defined and lie in H. Now, we observe that

(i) if $t \in (-\infty, 0] \cup [N-1, +\infty)$, then $\bar{y}(t) = \bar{z}(t) = t + 1$;

(ii) $\bar{y}^N, \bar{z}^N \in H$.

We define the circles

$$C_0 = (-\infty, 0]/\{t \sim t+1\} \quad \text{and} \quad C_1 = [N-1, +\infty)/\{t \sim t+1\}.$$

Let us consider the natural projections $p_0\colon (-\infty, 0] \to C_0$ and $p_1\colon [N-1, +\infty) \to C_1$. Then we restrict $\bar{y}^N$ to the interval $[-1, 0]$ such that p_0 surjects it onto C_0. Since N is sufficiently large so that $\bar{y}^N((\bar{y}^{-1}(L), L)) \subset [R, +\infty)$, it follows that $\bar{y}^N$ maps $[-1, 0]$ to $[R, +\infty)$. Passing to quotients, we define $\bar{y}^\infty\colon C_0 \to C_1$ such as $\bar{y}^\infty([t]) = [\bar{y}^N(t)]$ making the following diagram commutative:

$$\begin{array}{ccc} [-1,0] & \xrightarrow{\bar{y}^N} & [N-1,+\infty) \\ {\scriptstyle p_0}\downarrow & \circlearrowleft & \downarrow{\scriptstyle p_1} \\ C_0 & \dashrightarrow_{\bar{y}^\infty} & C_1 \end{array}$$

We emphasize that the map $\bar{y}^\infty$ does not depend on the specific chosen value of N, since if $m \geq N$, then $\bar{y}^m(t) = \bar{y}^{m-N}(\bar{y}^N(t))$, where $\bar{y}^N(t) \in (R, +\infty)$ and

$$\bar{y}^N(t) \sim \bar{y}^N(t) + 1 = \bar{y}\big(\bar{y}^N(t)\big) \sim \cdots \sim \bar{y}^{m-N}\big(\bar{y}^N(t)\big) = \bar{y}^m(t).$$

Similarly, we define the map $\bar{z}^\infty$. We remark that both these maps are piecewise-Möbius homeomorphisms from the circle C_0 to the circle C_1. They are called the *Mather invariants* of $\bar{y}$ and $\bar{z}$.

Assume now that there exists a $g \in H$ such that $gz = yg$. By conjugating by s, we get the equation $\overline{gz} = \bar{y}\bar{g}$, where $\bar{g} \in H$. Since $\bar{y}$ and $\bar{z}$ are equal to the translation $t \mapsto t + 1$ around $\pm\infty$, then the equation $\overline{gz} = \bar{y}\bar{g}$ implies that $\bar{g} \in H$ is periodic for real numbers that are sufficiently large positive and negative and so, around $-\infty$, where $\bar{g}(t) = a^2 t + ab$ is affine, we must have that $a^2 = 1$ so $\bar{g}$ is a translation, otherwise $\bar{g}$ would not be periodic. Similarly, $\bar{g}$ is a translation around $+\infty$. Thus g itself is a translation around $\pm\infty$. We record this observation for independent later use.

Remark 4.1. If an affine map $g(t) = a^2 t + ab$ commutes with a translation $z(t) = t + k$, then $a^2 = 1$.

Now, going back to the argument above, we see that it induces the equation $\overline{gz}^N = \bar{y}^N \bar{g}$, where $\bar{g}$ is periodic on $(-\infty, 0) \cup (N-1, +\infty)$ since it commutes with $\bar{y}$ and $\bar{z}$ on such intervals, and so $\bar{g}$ passes to quotients and becomes

$$v_{1,m}\bar{z}^\infty = \bar{y}^\infty v_{0,\ell}, \tag{4.1}$$

as done in [5, 13], since $\bar{g}, \bar{y}, \bar{z} \in H$ and $v_{0,\ell} := p_0\bar{g}$ and $v_{1,m} := p_1\bar{g}$ are rotations of the circles C_0 and C_1, respectively, where ℓ, m are the translation terms of g on $(-\infty, 0)$ and $(N - 1 + \infty)$, respectively.

The proof of the next result shows the relation between the Stair algorithm and the Mather invariant.

Theorem 4.2. *Let $y, z \in H^{>}$ be such that $y(t) = z(t) = t + b_0$ for $t \in (-\infty, L]$ and $y(t) = z(t) = t + b_1$ for $t \in [R, +\infty)$ and let $\bar{y}^{\infty}, \bar{z}^{\infty}\colon C_0 \to C_1$ be the corresponding Mather invariants. Then y and z are conjugate in H if and only if $\bar{y}^{\infty}$ and $\bar{z}^{\infty}$ differ by rotations $v_{0,\ell}$ and $v_{1,m}$ of the domain and range circles, for some $\ell, m \in \mathbb{R}$:*

$$\begin{array}{ccc} C_0 & \xrightarrow{\bar{z}^{\infty}} & C_1 \\ {\scriptstyle v_{0,\ell}}\downarrow & \circlearrowleft & \downarrow{\scriptstyle v_{1,m}} \\ C_0 & \xrightarrow[\bar{y}^{\infty}]{} & C_1 \end{array}$$

Proof. The calculations above yield that, if $g \in H$ conjugates y and z, then (4.1) is satisfied, which is equivalent to say that $\bar{y}^{\infty}$ and $\bar{z}^{\infty}$ differ by rotations $v_{0,\ell}$ and $v_{1,m}$ of the domain and range circles, for some $\ell, m \in \mathbb{R}$.

Conversely, let us assume that there are $\ell, m \in \mathbb{R}$ such that equation (4.1) is satisfied. Then we choose $g_0 \in H$ which is affine in the initial box with a initial germ $(g_0)_{-\infty} = (1, \ell)$. Then we define a map g as the following pointwise limit:

$$g(t) := \lim_{n \to +\infty} y^n g_0 z^{-n}(t).$$

By the Stair algorithm in Theorem 3.15, we have that $gz = yg$, where $g \in \mathrm{Homeo}_+(\mathbb{R})$ and it is such that, on any bounded interval, it coincides with the restriction of some function $\mathrm{PPSL}(\mathbb{R})$ to such an interval. We need to show that $g \in H$. By construction, g has finitely many breakpoints in $(-\infty, N-1]$. Conjugating both sides of the equation $gz = yg$ by s, we get

$$\overline{gz} = \overline{yg}.$$

For all $t \in (-\infty, 0] \cup [N-1, +\infty)$, we have that $\bar{y}(t) = \bar{z}(t) = t + 1$. Thus $\bar{g}(t+1) = \bar{g}(t) + 1$, for each $t \in (-\infty, 0] \cup [N-1, +\infty)$ and we can pass to quotients. Moreover, as argued above during the definition of the Mather invariants, we have that $\bar{g}$ is a translation $\bar{g}(t) = t + \ell$ on $(-\infty, 0]$, while we still need to show that $\bar{g}$ is a translation on $[N-1, +\infty)$. Up to switch the role of $\bar{y}$ and $\bar{z}$, we can assume that $\ell \geq 0$. Passing the equation $\bar{g}\bar{z}^N = \bar{y}^N \bar{g}$ to quotients, we obtain that

$$\bar{g}_{ind}\bar{z}^{\infty}\big([t]\big) = \bar{y}^{\infty} v_{0,\ell}\big([t]\big),$$

for a suitable well-defined $\bar{g}_{ind}$. By (4.1), we have that

$$\bar{g}_{ind}\bar{z}^{\infty}\big([t]\big) = v_{1,m}\bar{z}^{\infty}\big([t]\big),$$

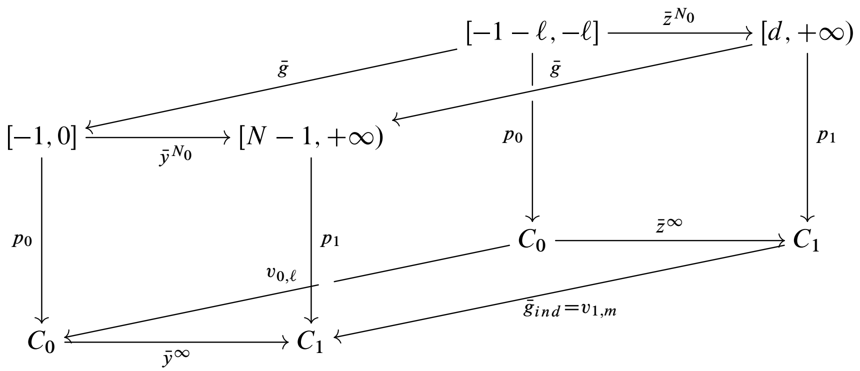
and so, by the cancelation law, we have that

$$\bar{g}_{ind} = v_{1,m}$$

so that $\bar{g}_{ind}$ is a rotation by m of the circle C_1. We now choose $N_0 \geq N$ large enough so that $d := \bar{z}^{N_0}(-1-\ell) \geq N-1$ and

$$\bar{z}^{N_0}\big([-1-\ell,-\ell]\big) = \bar{z}^{N_0}\big(\big[-1-\ell, z(-1-\ell)\big]\big) = \big[d, z(d)\big] = [d, d+1],$$
$$\bar{g}(d) = \bar{y}^{N_0}\big(\bar{g}(-1-\ell)\big) = \bar{y}^{N_0}(-1) \geq N-1.$$

Hence the following commutative diagram holds:



To finish the proof, we need to see that $\bar{g}$ is an affine Möbius map on $[N-1,+\infty)$, which will mean that $\bar{g} \in H$. From the previous commutative diagram we get $v_{1,m}p_1 = p_1\bar{g}$, which implies that $[\bar{g}(t)] = [t+m]$ for $[t] \in C_1$. By definition of the equivalence relation and the fact that $\bar{g}$ is a periodic continuous function, we have that there exists some $r \in \mathbb{Z}$ such that, $\bar{g}(t) := t+m+r$, for all $t \in [N-1,+\infty)$. Therefore, $\bar{g} \in H$. ∎

Remark 4.3. The results of this section rely on Lemma 2.1 and so, in order to generalize them to $H(A)$, one needs to prove a generalized version of Lemma 2.1 for $H(A)$. In particular, the construction of the homeomorphism s at the beginning of this section requires a version of Lemma 2.1 for $H(A)$ to construct the maps s_j for $j = 0, \dots, N = 2$. This is thus true for Section 5 too.

5. Centralizers

In this section, we use the conjugacy tools we have just constructed to calculate the centralizers of elements from H. We start by performing some calculations for centralizers of elements in Aff($\mathbb{R}$) and use this information to classify the centralizers of elements from H.

5.1. Centralizers in Aff($\mathbb{R}$)

Since Lemma 3.11 gives a monomorphism $\varphi_z\colon C_H(z) \to \mathrm{Aff}(\mathbb{R})$, it makes sense to investigate centralizers in Aff($\mathbb{R}$).

If $(a, b) \in \mathrm{Aff}(\mathbb{R})$ and $a \neq 1$, then $(c, d) \in C_{\mathrm{Aff}(\mathbb{R})}(a, b)$ if and only if

$$(c, d) = (a^{-1}, -a^{-1}b)(c, d)(a, b) = (c, -a^{-1}b + a^{-1}d + a^{-1}bc)$$

which is equivalent to $d = \frac{b(c-1)}{a-1}$, and so

$$C_{\mathrm{Aff}(\mathbb{R})}(a, b) = \left\{\left(c, \frac{b(c-1)}{a-1}\right) \in \mathrm{Aff}(\mathbb{R}) \mid c \in \mathbb{R}_{>0}\right\} \cong (\mathbb{R}, +).$$

If $(1, b) \in \mathrm{Aff}(\mathbb{R})$ and $(c, d) \in C_{\mathrm{Aff}(\mathbb{R})}(a, b)$, we get

$$(c, d) = (1, -b)(c, d)(1, b) = \big(c, b(c-1) + d\big)$$

which implies that

$$d = b(c-1) + d \Rightarrow b(c-1) = 0 \Rightarrow b = 0 \quad \text{or} \quad c = 1.$$

If $b = 0$, then

$$C_{\mathrm{Aff}(\mathbb{R})}(1, 0) = \mathrm{Aff}(\mathbb{R}).$$

If $b \neq 0$, then

$$C_{\mathrm{Aff}(\mathbb{R})}(1, b) = \big\{(1, d) \mid d \in (\mathbb{R}, +)\big\} \cong (\mathbb{R}, +).$$

We collect our calculations in the following result.

Lemma 5.1. *If* $(1, 0) \neq (a, b) \in \mathrm{Aff}(\mathbb{R})$*, then* $C_{\mathrm{Aff}(\mathbb{R})}(a, b) \cong (\mathbb{R}, +)$.

5.2. Centralizers in H

We start by noticing that, since H is torsion-free by Lemma 2.6, the subgroup $C_H(z)$ is infinite for any non-trivial $z \in H$. Next, we will divide the study of centralizers of the elements from H into several cases.

First, we consider $z \in H$ without breakpoints, that is, the case where z is an affine map. Let us consider that $z(t) = a^2 t + ab$ for all $t \in \mathbb{R}$. If $a \neq \pm 1$, then we have the following result.

Proposition 5.2. *Let* $z \in H$ *be so that* $z(t) = a^2 t + ab$*, for all* $t \in \mathbb{R}$*, with* $a \neq \pm 1$*. Then* $C_H(z) \cong (\mathbb{R}, +)$.

Proof. First, notice that, in this case, $z_{-\infty} = z_{+\infty} = (a^2, ab)$. A direct calculation shows that

$$T := \left\{f \in H \mid f(t) = ct + \frac{ab(c-1)}{a^2-1},\ \forall t \in \mathbb{R},\ c > 0\right\}$$

is a subgroup of $C_H(z)$. Using the map φ_z from Lemma 3.11, we have that

$$\varphi_z(T) = \left\{\left(c, \frac{ab(c-1)}{a^2-1}\right) \mid c \in (\mathbb{R}_{>0}, \cdot)\right\} = C_{\mathrm{Aff}(\mathbb{R})}(a^2, ab).$$

From $\varphi_z(T) \leq \varphi_z(C_H(z)) \leq C_{\mathrm{Aff}(\mathbb{R})}(a^2, ab)$, we get $\varphi_z(C_H(z)) = C_{\mathrm{Aff}(\mathbb{R})}(a^2, ab)$. Since φ_z is a group monomorphism, we have that $C_H(z) \cong C_{\mathrm{Aff}(\mathbb{R})}(a^2, ab)$. Therefore, $C_H(z) \cong (\mathbb{R}, +)$ by Lemma 5.1. ∎

From the previous result, we have the following.

Corollary 5.3. *Let $y \in H$ be an element such that $y = g^{-1}zg$, where $z, g \in H$ and z is an affine map $z(t) = a^2t + ab$, with $a^2 \neq 1$. Then $C_H(y) \cong (\mathbb{R}, +)$.*

Now we consider the case $z(t) = a^2t + ab$ for all $t \in \mathbb{R}$ and with $a = \pm 1$, that is, z is a translation.

Proposition 5.4. *If $z \in H^<$ is a translation, then $C_H(z) \cong (\mathbb{R}, +)$.*

Proof. Let $g \in C_H(z)$. Since $g \in H$, we have that $g(t) = a_0^2 t + a_0 b_0$, for some $a_0, b_0 \in \mathbb{R}$, $t \in (-\infty, L]$ and for suitable $L \in \mathbb{R}$. The map g commutes with the translation $z(t) = t + k$, for some $k < 0$, so g is periodic of period $|k|$ and so, by Remark 4.1, we have that $a_0^2 = 1$. Hence g is a translation around $-\infty$ and it is periodic, so we must have that $g(t) = t + b_0$ for every $t \in \mathbb{R}$. Therefore, if φ_z is the map of Lemma 3.11, we have that

$$C_H(z) \cong \varphi_z\big(C_H(z)\big) \cong \big\{(1, b_0) \mid b_0 \in \mathbb{R}\big\} \cong (\mathbb{R}, +). \qquad \blacksquare$$

The previous proposition implies the following result.

Corollary 5.5. *Let $y \in H$ be an element such that $y = g^{-1}zg$, where $z, g \in H$ and z is a translation. Then $C_H(y) \cong (\mathbb{R}, +)$.*

Let us now consider $z \in H^<$ such that $z'(-\infty) \neq z'(+\infty)$ and z has breakpoints. We start with the following result.

Lemma 5.6. *Let $z \in H^<$ such that its initial and final affinity boxes with respect to z and itself are $(-\infty, L]^2$ and $[R, +\infty)^2$, respectively, and so that $z'(-\infty) \neq z'(+\infty)$. Let $s \in \mathbb{Z}_{>0}$ be such that $z^s(R) < L$. Then either z^{-s} is not affine on $[z^s(R), L]$ or z^{-2s} is not affine on $[z^{2s}(R), L]$.*

Proof. First of all, since $z \in H^<$, then $z^{-1} \in H^>$. Let us suppose that $z(t) = a_0^2 t + a_0 b_0$ on $(-\infty, L]$ and $z(t) = a_n^2 t + a_n b_n$ on $[R, +\infty)$. Then, by hypothesis, $a_0^2 = z'(-\infty) \neq z'(+\infty) = a_n^2$. Moreover, $z^{-1}(t) = a_0^{-2}t - a_0^{-1}b_0$ on $(-\infty, z(L)]$ and $z^{-1}(t) = a_n^{-2}t - a_n^{-1}b_n$ on $[z(R), +\infty)$. Then, since $z^{-1} \in H^>$, we have that z^{-s} is affine on $(-\infty, z^s(L)]$, with initial germ

$$(z^{-s})_{-\infty} = \left(a_0^{-2s}, -\sum_{j=1}^{s} a_0^{-2j+1} b_0\right).$$

Moreover, z^{-s} is affine on $[z^s(R), +\infty)$, which contains $[R, +\infty)$ with final germ

$$(z^{-s})_{+\infty} = \left(a_n^{-2s}, -\sum_{j=1}^{s} a_n^{-2j+1} b_n\right).$$

Let us assume, by contradiction, that both z^{-s} and z^{-2s} are affine on $[z^s(R), L]$ and $[z^{2s}(R), L]$, respectively, and that their germs on these two intervals are (a, b) and (c, d), respectively. Since $z^{-2s} = z^{-s} \circ z^{-s}$, we get that z^{-2s} is affine on $[z^s(R), L]$, because z^{-s}

is affine on $[z^s(R), L]$ by our assumption and z^{-s} is affine on $[R, z^{-s}(L)] \subset [R, +\infty)$, with germ

$$\left(a_n^{-2s}, -\sum_{j=1}^{s} a_n^{-2j+1} b_n\right)(a, b).$$

Moreover, z^{-2s} is also affine on $[z^{2s}(R), z^s(L)]$, since z^{-s} is affine on $(-\infty, z^s(L)]$ and on $[z^s(R), L]$ by our assumption, with germ

$$(a, b)\left(a_0^{-2s}, -\sum_{j=1}^{s} a_0^{-2j+1} b_0\right).$$

By comparing the germ (c, d) of z^{-2s} on $[z^{2s}(R), L]$ with the germs of the same map z^{-2s} on the two subintervals $[z^s(R), L]$ and $[z^{2s}(R), z^s(L)]$ of the interval $[z^{2s}(R), L]$, we get

$$\left(a_n^{-2s}, -\sum_{j=1}^{s} a_n^{-2j+1} b_n\right)(a, b) = (c, d) = (a, b)\left(a_0^{-2s}, -\sum_{j=1}^{s} a_0^{-2j+1} b_0\right).$$

From this, we must have that

$$a_n^{-2s} a = a a_0^{-2s}.$$

Since the group $(\mathbb{R}_{>0}, \cdot)$ is abelian, we have that $a_0^{-2s} = a_n^{-2s}$. However, we are considering $z \in H^{<}$ such that the $a_0^2 \neq a_n^2$, so that $a_0^{-2s} \neq a_n^{-2s}$ and we have a contradiction. Therefore, either z^{-s} is not affine on $[z^s(R), L]$ or z^{-2s} is not affine on $[z^{2s}(R), L]$. ■

Lemma 5.7. *The group H is isomorphic to the group K of all piecewise Möbius transformations of* $[0, 1]$ *to itself with finitely many breakpoints.*

Proof. We explicitly construct an isomorphism $\nabla : H \to K$.

We use Lemma 2.1 to construct an element $h \in H$ such that $h(-3) = \frac{1}{3}$ and $h(-1) = \frac{1}{2}$. Now consider the map

$$f(t) = \begin{cases} \frac{-1}{t}, & t \in (-\infty, -3], \\ h(t), & t \in [-3, -1], \\ \frac{t+2}{t+3}, & t \in [-1, +\infty). \end{cases}$$

We now define

$$\nabla(g)(t) = \begin{cases} fgf^{-1}(t), & t \in (0, 1), \\ t, & t \in \{0, 1\} \end{cases}$$

and notice that a direct calculation shows that $\mathrm{im}(\nabla) \subseteq K$. Thus the map $\nabla : H \to K$ is well defined and it is clearly a group isomorphism with an obvious inverse. ■

Remark 5.8. The isomorphism ∇ of the proof of Lemma 5.7 switches $-\infty$ with 0 and $+\infty$ with 1 and allows us to study maps in Monod's group from a bounded point of view which will be useful in the proof of Lemma 5.9. Moreover, a straightforward calculation shows that, if $y, z \in H$ are such that $y_{-\infty} = z_{-\infty}$ and $y_{+\infty} = z_{+\infty}$, then the initial and final affinity boxes of y and z correspond to initial and final *Möbius boxes* of $\nabla(y)$ and $\nabla(z)$, where the images coincide and are Möbius and a conjugator has to be Möbius.

In the next result, we will freely use the isomorphism $\nabla : H \to K$ of Lemma 5.7.

Lemma 5.9. *Let $z \in H^<$ be such that $z(t) = a^2 t + ab$ at $-\infty$ with $a^2 > 1$. Then there exists $\varepsilon > 0$ such that the only $g \in C_H(z)$ with $1-\varepsilon < \tilde{g}'(0) < 1+\varepsilon$ and $-\varepsilon < \tilde{g}''(0) < \varepsilon$, where $\tilde{g} = \nabla(g)$, is $g = \mathrm{id}$.*

Proof. Let us consider $\tilde{z}$ to be a conjugate version of z from the proof of Lemma 5.7; that is, $\tilde{z} = \nabla(z)$. Let $[0, \alpha]$ and $[\beta, 1]$ be, respectively, the initial and final Möbius boxes of $\tilde{z}$ (see Remark 5.8) for suitable $0 < \alpha < \beta < 1$. By Lemma 5.6, there exists an $N_1 \in \mathbb{Z}_{>0}$ such that $\tilde{z}^{-N_1}$ has a breakpoint μ_1 on $[\tilde{z}^{N_1}(\beta), \alpha]$. We now consider a real number α' such that $0 < \alpha' < \mu_1 < \alpha$ and we take a new initial (and smaller) Möbius box $[0, \alpha']$ for z. We use Lemma 5.6 again and find that there exists $N_2 \in \mathbb{Z}_{>0}$ such that $\tilde{z}^{-N_2}$ has a breakpoint μ_2 on $[\tilde{z}^{N_2}(\beta), \alpha']$. Without loss of generality, assume that $\tilde{z}^{N_2}(\beta) \le \tilde{z}^{N_1}(\beta)$. Then there exists $\varepsilon > 0$ such that $\{\mu_2 < \mu_1\} \subseteq I_\varepsilon := [\tilde{z}^{N_2}(\frac{\beta+\varepsilon}{1+\varepsilon}), (1-\varepsilon)\alpha]$.

Fact 5.10. *Let $0 < \varepsilon < \frac{1}{3}$ and $g \in C_H(z)$ such that*

$$1-\varepsilon < \tilde{g}'(0) < 1+\varepsilon \quad \textit{and} \quad -\varepsilon < \tilde{g}''(0) < \varepsilon.$$

Then $|\tilde{g}(t) - \mathrm{id}(t)| < 3\varepsilon + 2\varepsilon^2$, for all $t \in [0, \alpha]$, and so the family of functions $\tilde{g}$ can be seen as uniformly converging to the identity function id *on the interval* $[0, \alpha]$.

Proof of Fact 5.10. Let us consider $\tilde{g} = \nabla(g)$ so that $\tilde{g}(t) = \frac{at+b}{ct+d}$ on $[0, \alpha]$, where $ad - bc = 1$. Then $\tilde{g}(0) = 0$ and, consequently, $b = 0$ and $ad = 1$. Let us define $\tilde{g}'(0) := \lambda$ and $\tilde{g}''(0) = \rho$. Since

$$\tilde{g}'(t) = \frac{1}{(ct+d)^2} \quad \text{and} \quad \tilde{g}''(t) = -\frac{2c}{(ct+d)^3},$$

we have that $\lambda = \frac{1}{d^2}$ and $\rho = -\frac{2c}{d^3}$. Therefore, $d^2 = \frac{1}{\lambda}$ and $c = \frac{-\rho d^3}{2}$. Observe that

$$\tilde{g}(t) = \frac{at}{ct+d} = \frac{t}{cdt+d^2} = \frac{t}{\frac{-\rho t}{2\lambda^2} + \frac{1}{\lambda}} = \frac{2\lambda^2 t}{-\rho t + 2\lambda}$$

and so

$$\begin{aligned}
\left|\tilde{g}(t) - \mathrm{id}(t)\right| &= \left|\frac{2\lambda^2 t}{-\rho t + 2\lambda} - t\right| \\
&= \left|\frac{2\lambda^2 t - 2\lambda t + \rho t^2}{-\rho t + 2\lambda}\right| \\
&\le |2\lambda^2 t - 2\lambda t + \rho t^2| \\
&\le 2|\lambda| \cdot |\lambda - 1| \cdot |t| + |\rho| \cdot |t| \\
&\le 2(1+\varepsilon)\varepsilon + \varepsilon \\
&\le 3\varepsilon + 2\varepsilon^2,
\end{aligned}$$

where at the various steps we have observed that $|t| \leq 1$, $|\lambda| \leq 1+\varepsilon$, $|\lambda - 1| \leq \varepsilon$ and, since $|\rho| < \varepsilon < \frac{1}{3}$, we have that

$$|-\rho t + 2\lambda| \geq \big|2\lambda - |\rho t|\big| \geq |2\lambda - \varepsilon| \geq \big|2(1-\varepsilon) - \varepsilon\big| = |2 - 3\varepsilon| \geq 1. \qquad \blacksquare$$

Fact 5.11. *Let $t_0 \in (0, \alpha)$. Then, for any $1 - \varepsilon < \tilde{g}'(0) = \lambda < 1 + \varepsilon$, there is at most one $g \in C_H(z)$ such that $-\varepsilon < \tilde{g}''(0) = \rho < \varepsilon$ and such that $\tilde{g}^{-1}(t_0) = t_0$.*

Proof of Fact 5.11. We write $\tilde{g}$ on the open interval $(0, \alpha)$ using the expression that was computed in the proof of Fact 5.10. Assume that $\tilde{g}(t_0) = t_0$, then

$$t_0 = \frac{2\lambda^2 t_0}{-\rho t_0 + 2\lambda}$$

and so

$$1 = \frac{2\lambda^2}{-\rho t_0 + 2\lambda}$$

and so

$$-\rho t_0 + 2\lambda = 2\lambda^2$$

and so

$$\rho = \frac{2\lambda - 2\lambda^2}{t_0}.$$

If we assume that $\lambda = 1 + \tau$ for $-\varepsilon \leq \tau \leq \varepsilon$, then

$$\rho = \frac{2(1+\tau) - 2(1+\tau)^2}{t_0} = \frac{-2\tau - 2\tau^2}{t_0}.$$

For any $-\varepsilon \leq \tau \leq \varepsilon$, the expression above returns a unique ρ. In case such an expression returns $|\rho| \geq \varepsilon$, then g cannot exist. On the other hand, if such an expression returns $|\rho| < \varepsilon$, then the pair (τ, ρ) satisfies the required conditions. Therefore, for each λ we obtain at most one g satisfying the requirements. $\blacksquare$

End of the proof of Lemma 5.9. Since we know that

(i) μ_i is a breakpoint for $\tilde{z}^{-N_i}$,

(ii) $\tilde{z}^{-N_i}(\mu_i) \in [0, \alpha]$, and

(iii) $\tilde{g}$ is a Möbius transformation on $[0, \alpha]$,

it follows that μ_i is a breakpoint for $\tilde{g}\tilde{z}^{-N_i}$. On the other hand, when we consider $\tilde{z}^{-N_i}\tilde{g}$, the map $\tilde{g}$ pushes the breakpoint μ_i of $\tilde{z}^{-N_i}$ to $\tilde{g}^{-1}(\mu_i)$, then $\tilde{g}^{-1}(\mu_i)$ is a breakpoint for $\tilde{z}^{-N_i}\tilde{g}$.

By construction, the set of breakpoints of $\tilde{g}\tilde{z}^{N_i}$ on I_ε is $B := \{\delta_1 < \cdots < \delta_k\} \supseteq \{\mu_1 < \mu_2\}$ and the set of breakpoints of $\tilde{z}^{N_i}\tilde{g}$ on $\tilde{g}^{-1}(I_\varepsilon)$ is $\tilde{g}^{-1}(B) = \{\tilde{g}^{-1}(\delta_1) < \cdots < \tilde{g}^{-1}(\delta_k)\}\{\tilde{g}^{-1}(\mu_1) < \tilde{g}^{-1}(\mu_2)\}$. However, since $g \in C_H(z)$, then $\tilde{g}\tilde{z}^{N_i}(t) = \tilde{z}^{N_i}\tilde{g}(t)$, for every $t \in I_\varepsilon$ and so $\tilde{g}^{-1}(\delta_i) = \delta_i$ for $i = 1, \ldots, k$ and in particular $\tilde{g}^{-1}(\mu_i) = \mu_i$ for $i = 1, 2$.

By Fact 5.11, there can exist at most one $\mathrm{id} \neq g \in C_H(z)$ fixing μ_1 and, since $\tilde{g}$ fixes 0 too, it cannot also fix μ_2, otherwise g would be the identity map by [9, Corollary 2.5.3]. Similarly, there can exist at most one $\mathrm{id} \neq g \in C_H(z)$ fixing μ_2 and such a map cannot fix μ_1 too. Then the only way to avoid a contradiction and have a $g \in C_H(z)$ such that $\tilde{g}'(0)$ and $\tilde{g}''(0)$ satisfy the given conditions with respect to the chosen $\varepsilon > 0$ is that $g = \mathrm{id}$. ∎

We now show that in many cases centralizers are infinite cyclic.

Proposition 5.12. *Let $z \in H^<$ be such that $z(t) = a^2 t + ab$ around $-\infty$ and $a^2 > 1$. Then $C_H(z)$ is a discrete subgroup of $(\mathbb{R}, +)$ and so it is isomorphic to $(\mathbb{Z}, +)$.*

Proof. By Lemma 5.9, the subgroup $C_H(z)$ is a discrete set. Since $C_H(z) \cong \varphi_z(C_H(z)) \leq C_{\mathrm{Aff}(\mathbb{R})}(z) \cong (\mathbb{R}, +)$ and the subgroups of $(\mathbb{R}, +)$ are either discrete (then isomorphic to $(\mathbb{Z}, +)$) or dense, we get $C_H(z) \cong (\mathbb{Z}, +)$. ∎

5.2.1. Mather invariant and centralizers. As done in Section 4, we consider $z \in H^>$ that is a translation around $\pm\infty$ and we use the Mather invariant of z in order to understand centralizers.

Proposition 5.13. *Consider that $z \in H^>$ such that $z(t) = t + b_0$ for $t \in (-\infty, L]$ and $z(t) = t + b_1$ for $t \in [R, +\infty)$. Then either $C_H(z) \cong (\mathbb{Z}, +)$ or $C_H(z) \cong (\mathbb{R}, +)$.*

Proof. We follow notations from Section 4. Let $N \in \mathbb{Z}_{>0}$ be large enough so that

$$z^N\big((z^{-1}(L), L)\big) \subset (R, +\infty).$$

Up to conjugating z with s, we will work with $z(t) = t + 1$. We define the relation $t \sim t+1$ and construct the circles $C_0 := (-\infty, 0]/\sim$ and $C_1 := [N-1, +\infty)/\sim$. By Theorem 4.2, a $g \in H$ is a centralizer of z if and only if the following equation is satisfied:

$$z^\infty v_{0,\ell} = v_{1,m} z^\infty. \tag{5.1}$$

We now consider the map $V_0 \colon \mathbb{R} \to \mathbb{R}$ defined by $V_0(t) = t + \ell$, which is a lift of $v_{0,\ell}$, that is, it makes the following diagram commute:

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{V_0} & \mathbb{R} \\ {\scriptstyle p_0}\downarrow & \circlearrowleft & \downarrow{\scriptstyle p_0} \\ C_0 & \xrightarrow[v_{0,\ell}]{} & C_0 \end{array}$$

Similarly, $V_1(t) = t + m$ makes the following diagram commute:

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{V_1} & \mathbb{R} \\ {\scriptstyle p_1}\downarrow & \circlearrowleft & \downarrow{\scriptstyle p_1} \\ C_1 & \xrightarrow[v_{1,m}]{} & C_1 \end{array}$$

Let $Z : \mathbb{R} \to \mathbb{R}$ be a lift of z^∞. The previous two commutative diagrams and (5.1) form three faces of a commutative cube analogous to that appearing in the proof of Theorem 4.2 and so they imply that $ZV_0 = V_1 Z$. In other words, for $t \in \mathbb{R}$, we have that

$$Z(t+\ell) = ZV_0(t) = V_1 Z(t) = Z(t) + m, \tag{5.2}$$

which means that the graph of Z is shifted back to itself. If the lift of z^∞ does not have breakpoints, the graph of Z is affine. Thus there are infinitely many pairs $\ell, m \in \mathbb{R}$ for which the graph can be shifted back to itself and so, for each $\ell \in \mathbb{R}$, there exists an $m \in \mathbb{R}$ so that (5.2) holds. Consequently, the image of the map φ_z from Lemma 3.11 is so that $\varphi_z(C_H(z)) \cong (\mathbb{R}, +)$. Otherwise, the lift of z^∞ has breakpoints and the set of candidates for ℓ forms a discrete subgroup of $(\mathbb{R}, +)$. Then $\varphi_z(C_H(z)) \cong (\mathbb{Z}, +)$. Therefore, we have either $C_H(z) \cong (\mathbb{Z}, +)$ or $C_H(z) \cong (\mathbb{R}, +)$. ■

We see two examples: in the first one, the subgroup of centralizers is isomorphic to $(\mathbb{R}, +)$, while in the second it is isomorphic to $(\mathbb{Z}, +)$.

Example 5.14. If we conjugate $y(t) = t + 1$ by

$$g(t) = \begin{cases} \frac{t-2}{\frac{3}{2}t-2}, & \text{if } t \in [0,1], \\ t+1, & \text{otherwise,} \end{cases}$$

then we get that

$$z(t) = \begin{cases} \frac{2t+2}{\frac{3}{2}t+2}, & \text{if } t \in [-1,0], \\ \frac{t-2}{\frac{3}{2}t-2}, & \text{if } t \in [0,1], \\ t+1, & \text{otherwise,} \end{cases}$$

see Figure 1. Then $C_H(z) \cong (\mathbb{R}, +)$ by Corollary 5.5.

Example 5.15. Let us consider

$$z(t) = \begin{cases} \frac{t-2}{\frac{3}{2}t-2}, & \text{if } t \in [0,1], \\ t+1, & \text{otherwise,} \end{cases}$$

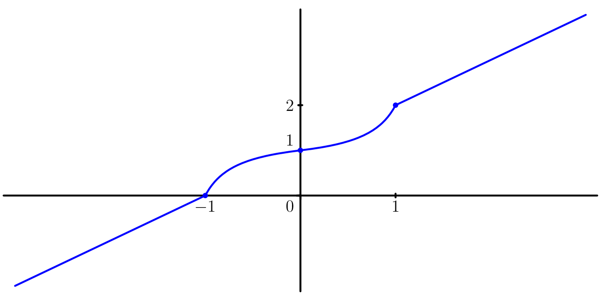


Figure 1. Graph of z, from Example 5.14.

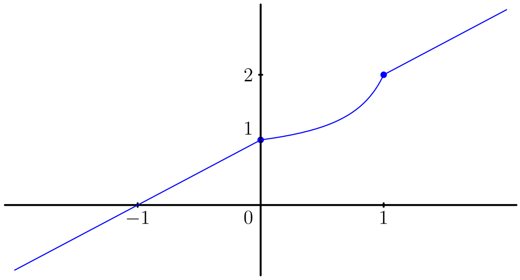


Figure 2. Graph of z, from Example 5.15.

see Figure 2. Note that $z \in H^{>}$ and that $L = 0$ and $R = 1$. Its inverse is given by

$$z^{-1}(t) = \begin{cases} \frac{2t-2}{\frac{3}{2}t-1}, & \text{if } t \in [1,2], \\ t-1, & \text{otherwise.} \end{cases}$$

If $N = 2$, then we have that

$$z^2\big((z^{-1}(0), 0)\big) \subset [1, +\infty).$$

Notice that we do not need to conjugate by the map s, since it is a translation by one around $\pm\infty$; that is $\bar{z} := z$. Moreover,

$$z^2(t) = \begin{cases} \frac{t-1}{\frac{3}{2}t-\frac{1}{2}}, & \text{if } t \in [-1,0], \\ \frac{\frac{5}{2}t-4}{\frac{3}{2}t-2}, & \text{if } t \in [0,1], \\ t+2, & \text{otherwise.} \end{cases}$$

Considering the relation $t \sim t+1$, we define $C_0 := (-\infty, 0]/t \sim t+1$ and $C_1 := [1, +\infty)/t \sim t+1$. Then we get the Mather invariant

$$\begin{aligned} z^{\infty}\colon C_0 &\to C_1 \\ [t] &\mapsto z^{\infty}\big([t]\big) = \big[z^2(t)\big]. \end{aligned}$$

The lift of this map making the following diagram commute

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{Z} & \mathbb{R} \\ {\scriptstyle p_0}\downarrow & \circlearrowright & \downarrow{\scriptstyle p_1} \\ C_0 & \xrightarrow[z^{\infty}]{} & C_1 \end{array}$$

is given by the periodic extension of the restriction of z^2 (see Figure 3) defined on $[-1, 0]$ by

$$Z(t) = z^2(t-x) + x,$$

if $x - 1 \le t \le x$, where $x \in \mathbb{Z}$ (see Figure 4). Then the centralizer of Z is $(\mathbb{Z}, +)$. Moreover, notice that $Z \notin H$.

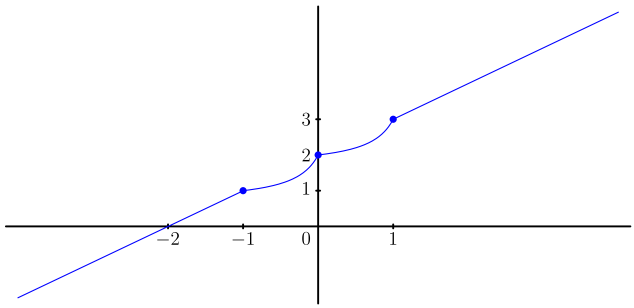


Figure 3. Graph of z^2.

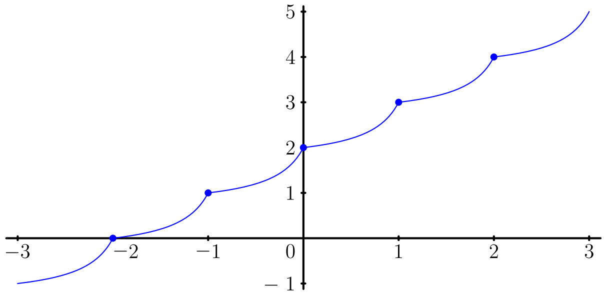


Figure 4. Graph of the lift Z.

5.2.2. Main result about centralizers. We can now give a structure result for centralizers in H (Theorem A in Section 1).

Theorem 5.16. *Given that $z \in H$, then*

$$C_H(z) \cong (\mathbb{Z}, +)^n \times (\mathbb{R}, +)^m \times H^k,$$

for suitable $k, m, n \in \mathbb{Z}_{\geq 0}$.

Proof. The element z has finitely many (possibly unbounded) intervals of fixed points, so its boundary $\partial \operatorname{Fix}(z) = \{t_0 < t_1 < \cdots < t_n\}$ has only finitely points. If $g \in C_H(z)$, then g fixes $\partial \operatorname{Fix}(z)$ setwise. Moreover, since g is order-preserving, it must fix t_i for each $i = 1, \ldots, n$. As a consequence, we can restrict to study centralizers in each of the subgroups

$$H\big([t_i, t_{i+1}]\big) = \big\{h \in H \mid h(t) = t, \ \forall t \notin [t_i, t_{i+1}]\big\} \cong H,$$

where $i = 0, 1, \ldots, n-1$. If $z(t) = t$ on $[t_i, t_{i+1}]$, then it is easy to see that

$$C_{H([t_i, t_{i+1}])}(z) = H\big([t_i, t_{i+1}]\big) \cong H.$$

Otherwise, Corollaries 5.3 and 5.5 and Propositions 5.12 and 5.13 cover the remaining cases (when z is conjugate to an affine map or entirely above or below the diagonal) showing that either $C_{H([t_i, t_{i+1}])}(z) \cong (\mathbb{R}, +)$ or $C_{H([t_i, t_{i+1}])}(z) \cong (\mathbb{Z}, +)$. ■

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