The tautological ring of $\mathcal{M}_{g,n}$ via Pandharipande–Pixton–Zvonkine r-spin relations

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Abstract

We use relations in the tautological ring of the moduli spaces $\overline{\mathcal{M}}_{g,n}$ derived by Pandharipande, Pixton, and Zvonkine from the Givental formula for the r-spin Witten class in order to obtain some restrictions on the dimensions of the tautological rings of the open moduli spaces $\mathcal{M}_{g,n}$. In particular, we give a new proof for the result of Looijenga (for $n = 1$) and Buryak et al. (for $n \geqslant 2$) that $\dim R^{g-1}(\mathcal{M}_{g,n}) \leqslant n$. We also give a new proof of the result of Looijenga (for $n = 1$) and Ionel (for arbitrary $n \geqslant 1$) that $R^i(\mathcal{M}_{g,n}) = 0$ for $i \geqslant g$ and give some estimates for the dimension of $R^i(\mathcal{M}_{g,n})$ for $i \leqslant g - 2$.

1. Introduction

The study of the tautological ring R^* of the moduli spaces of curves goes back to the classical papers of Mumford and Faber [Mum83, Fab99]; see also [Vak03, Pan02, Zvo12, Tav16]. The tautological ring of the moduli space of curves $\overline{\mathcal{M}}_{g,n}$ is additively generated by the so-called dual graphs decorated by ψ- and κ-classes. A dual graph determines a natural stratum in $\overline{\mathcal{M}}_{g,n}$, whose vertices correspond to irreducible components of a generic point in the stratum, whose leaves correspond to the marked points, and whose edges correspond to the nodes. We decorate each vertex with a non-negative integer equal to the geometric genus of the corresponding irreducible component. Each vertex is also equipped with a multi-index κ-class, and each half-edge, including the leaves, is equipped with a power of the ψ-class of the cotangent line bundle at the corresponding marked point or the corresponding branch of the node. There are many linear relations between these generators, called tautological relations.

We can restrict all these classes to the open moduli space $\mathcal{M}_{g,n}$. Then only the graphs with no edges can contribute non-trivially. These graphs just correspond to the classes $\prod_{i=1}^n \psi_i^{d_i} \kappa_{e_1,\ldots,e_k}$ for $d_i \geqslant 0$ and $e_i \geqslant 1$. There are still many relations among these classes that can be proved, in particular the relation $R^i(\mathcal{M}_{g,n}) = 0$ for $i \geqslant g$; see [Loo95, Ion02] and also a recent new proof in [CGJZ16]. In the case $i = g - 1$, one can prove that $\dim R^{g-1}(\mathcal{M}_{g,n}) \leqslant n$; see [Loo95, BSZ16] for the cases $n = 1$ and $n \geqslant 2$, respectively. In this paper, we give new proofs of all these results, as well as some restrictions on the dimensions of the tautological rings for $i \leqslant g - 2$. Note that, by the non-degeneracy of some matrix of intersection numbers, one can in fact show that

$\dim R^{g-1}(\mathcal{M}_{g,n}) = n$; we refer to [BSZ16] for that.

We use the tautological relations of Pandharipande–Pixton–Zvonkine [PPZ16]. Givental–Teleman theory [Giv01a, Tel12] provides a formula for a homogeneous semi-simple cohomological field theory as a sum over decorated dual graphs as above; see [Giv01b, DSS13, DOSS14, PPZ15]. These formulae can be explained as a result of a certain group action on non-homogeneous cohomological field theories applied to the rescaled Gromov–Witten theory of a finite number of points (also known as topological field theory or degree 0 cohomological field theory); see [FSZ10, Tel12, Sha09, PPZ15].

In some cases, we can obtain this way a graphical formula for a cohomological field theory whose properties we know independently. In particular, the graphical formula might contain classes (linear combinations of decorated dual graphs) that are of dimension higher than the homogeneity property allows for a cohomological field theory. Then these classes must be equal to zero and give us tautological relations. Alternatively, we might consider the graphical formula as a function of some parameter ϕ parametrizing a path on the underlying Frobenius manifold with $\phi = 0$ lying on the discriminant. If we know independently that the cohomological field theory is defined for any value of ϕ including $\phi = 0$, then all negative terms of the Laurent series expansion in ϕ near $\phi = 0$ also give tautological relations (see [Jan15a, Pan16] for more details). Once we have a relation for the decorated dual graphs in $\overline{\mathcal{M}}_{g,n+m}$, for $m \geqslant 0$, we can multiply it by an arbitrary tautological class, push it forward to $\overline{\mathcal{M}}_{g,n}$, and then restrict it to $\mathcal{M}_{g,n}$. This gives a relation among the classes $\prod_{i=1}^{n} \psi_i^{d_i} \kappa_{e_1,\dots,e_k}$, for $d_i \geqslant 0$ and $e_i \geqslant 1$, in $R^*(\mathcal{M}_{g,n})$.

In the case of the Witten r-spin class [Wit93, PV01], the graphical formula and its ingredients are discussed in detail in [Giv03, FSZ10, DNOPS15, PPZ16].

Both approaches mentioned above produce the same systems of tautological relations on $\overline{\mathcal{M}}_{g,n}$. Two particular paths on the underlying Frobenius manifold are worked out in detail in [PPZ16]; we are using one of them in this paper. Note that the results of Janda [Jan15a, Jan14, Jan15b] guarantee that these relations work in the Chow ring; see a discussion in [PPZ16].

1.1 Organization of the paper

In Section 2, we recall the relations of Pandharipande–Pixton–Zvonkine. In Section 3, we use them to give a new proof of the dimension of $R^{g-1}(\mathcal{M}_{g,n})$, up to one lemma whose proof takes up Section 4. In Section 5, we extend this proof scheme to show the vanishing of the tautological ring in all higher degrees. Finally, in Section 6, we give some bounds for the dimensions of the tautological rings in lower degrees.

2. Pandharipande–Pixton–Zvonkine relations

In this section, we recall the relations in the tautological ring of $\overline{\mathcal{M}}_{g,n}$ from [PPZ16] and put them in a convenient form for our further analysis.

2.1 Definition

Fix $r \geqslant 3$. Fix n primary fields $0 \leqslant a_1, \dots, a_n \leqslant r - 2$. All constructions below depend on an auxiliary variable ϕ; we fix its exponent $d < 0$. A tautological relation $T(g, n, r, a_1, \dots, a_n, d) = 0$ depends on these choices and is obtained as $T = r^{g-1} \sum_{k=0}^{\infty} \pi_*^{(k)} T_k / k!$, where T_k is the coefficient of ϕ^d in the expression in the decorated dual graphs of $\overline{\mathcal{M}}_{g,n+k}$ described below and $\pi^{(k)} \colon \overline{\mathcal{M}}_{g,n+k} \to \overline{\mathcal{M}}_{g,n}$ is the natural projection.

Consider the vector space of primary fields with basis $\{e_0, \ldots, e_{r-2}\}$. In the basis $\tilde{e}_i := \phi^{-i/(r-1)} e_i$, we define the scalar product $\eta_{ij} = \langle \tilde{e}_i, \tilde{e}_j \rangle := \phi^{-(r-2)/(r-1)} \delta_{i+j,r-2}$. Equip each vertex of genus h of valency v in a decorated dual graph with a tensor

$$\tilde{e}_{a_1} \otimes \cdots \otimes \tilde{e}_{a_v} \mapsto \phi^{(h-1)(r-2)/(r-1)} (r-1)^h \delta_{(r-1)|h-1-\sum_{i=1}^{v} a_i} \, .$$

Define matrices $(R_m^{-1})_a^b$ for $m \geqslant 0$ and $a, b = 0, \ldots, r-2$ in the basis $\tilde{e}_0, \ldots, \tilde{e}_{r-2}$. We set $(R_m^{-1})_a^b = 0$ if $b \not\equiv a - m \mod r - 1$. If $b \equiv a - m \mod r - 1$, then $(R_m^{-1})_a^b = (r(r-1)\phi^{r/(r-1)})^{-m} P_m(r, a)$, where $P_m(r, a)$, for $m \geqslant 0$, are the polynomials of degree $2m$ in r and a uniquely determined by the following conditions:

$$P_0(r, a) = 1 \, ,$$
$$P_m(r, a) - P_m(r, a-1) = \left((m - \tfrac{1}{2})r - a \right) P_{m-1}(r, a-1) \, ,$$
$$P_m(r, 0) = P_m(r, r-1) \, .$$

Equip the first n leaves with $\sum_{m=0}^{\infty} (R_m^{-1})_{a_i}^b \psi_i^m \tilde{e}_b$ for $i = 1, \ldots, n$. Equip the k extra leaves (the dilaton leaves) with $-\sum_{m=1}^{\infty} (R_m^{-1})_0^b \psi_{n+i}^{m+1} \tilde{e}_b$ for $i = 1, \ldots, k$. Equip each edge, where we denote by ψ' and ψ'' the ψ-classes on the two branches of the corresponding node, with

$$\frac{\eta^{i'i''} - \sum_{m',m''=0}^{\infty} (R_{m'}^{-1})_{j'}^{i'} \eta^{j'j''} (R_{m''}^{-1})_{j''}^{i''} (\psi')^{m'} (\psi'')^{m''}}{\psi' + \psi''} \tilde{e}_{i'} \otimes \tilde{e}_{i''} \, .$$

Then T_k is defined as the sum over all decorated dual graphs obtained by the contraction of all tensors assigned to their vertices, leaves, and edges, further divided by the order of the automorphism group of the graph.

2.2 Analysis of the relations

There are several observations about the formula introduced in the previous subsection.

(1) We obtain a decorated dual graph in $R^D(\overline{\mathcal{M}}_{g,n})$ if and only if the sum of the indices of the matrices R_m^{-1} used in its construction is equal to D.

(2) According to [PPZ16, Theorem 7], the relation $T(g, n, r, a_1, \ldots, a_n, d)$ is a sum of decorated dual graphs whose coefficients are polynomials in r.

(3) Let $A = \sum_{i=1}^{n} a_i$. Then $A \equiv g - 1 + D \mod r - 1$. We can assume $A = g - 1 + D + x(r-1)$ with $x \geqslant 0$ since D is bounded by $\dim \overline{\mathcal{M}}_{g,n} = 3g - 3 + n$, whereas the relations hold for r arbitrarily big. Collecting the powers of ϕ from the contributions above, we obtain $d(r-1) = A + (g-1)(r-2) - rD$. Substituting the expression for A, we have $d < 0$ if and only if $D \geqslant g + x$. The relevant cases in this paper are the cases $x = 0$ and $x = 1$.

These relations, valid for particular $r \geqslant 3$ and $0 \leqslant a_1, \ldots, a_n \leqslant r - 2$, are difficult to apply since we have almost no control over the κ-classes coming from the dilaton leaves. We solve this problem in the following way.

Let $x = 0$, and consider the degree $D = g$. We have relations with polynomial coefficients for all r much greater than g and $A = 2g - 1$. More precisely, for all integers $0 \leqslant a_1, \ldots, a_n \leqslant 2g - 1$ with $\sum_{i=1}^{n} a_i = 2g - 1$, we have a relation whose coefficients are polynomials of degree $2g$ in r. In other words, we have a polynomial in r whose coefficients are linear combinations of decorated dual graphs in degree g, and we can substitute any r sufficiently large. Possible integer values of r determine this polynomial completely, so its evaluation at any other complex value of r is again a relation.

Let $x = 1$, and consider the degree $D = g + 1$. We have relations with polynomial coefficients for all r much bigger than g and $A = 2g - 1 + r$. More precisely, for all integers $0 \leqslant a_1, \ldots, a_n \leqslant r - 2$ with $\sum_{i=1}^n a_i = 2g - 1 + r$, we have a relation whose coefficients are polynomials of degree $2g + 2$ in r.

Note that in both cases we do not, in general, have polynomiality in $a_1, \ldots, a_n$, but we have it for some special decorated dual graphs, under some extra conditions.

We argue below that a good choice of r in both cases is $r = \frac{1}{2}$ (note that we still have to explain what we mean in the case $x = 1$, since the sum A depends on r). In particular, this choice kills all dilaton leaves, and the only non-trivial term that contributes to the sum over k in the definition of $T(g, n, r, a_1, \ldots, a_n, d)$ in these cases is T_0.

2.3 P-polynomials at $r = \frac{1}{2}$

Recall the $P_m(r, a)$-polynomials of [PPZ16] introduced above, and define

$$Q_m(a) := \frac{(-1)^m}{2^m m!} \prod_{k=1}^{2m} \left(a + 1 - \frac{k}{2} \right).$$

LEMMA 2.1. We have $P_m\left(\frac{1}{2}, a\right) = Q_m(a)$.

Proof. We will use [PPZ16, Lemma 4.3]. It is clear that we have $Q_0(a) = 1$ and $Q_m(0) = Q_m\left(-\frac{1}{2}\right) = \delta_{m,0}$. Furthermore, we have

$$Q_m(a) - Q_m(a - 1) = \frac{(-1)^m}{2^m m!} \left(\prod_{k=1}^{2m} \left(a + 1 - \frac{k}{2} \right) - \prod_{k=1}^{2m} \left(a - \frac{k}{2} \right) \right)$$

$$= \frac{(-1)^m}{2^m m!} \left(\left(a + \frac{1}{2} \right) a - \left(a - m + \frac{1}{2} \right)(a - m) \right) \prod_{k=1}^{2m-2} \left(a - \frac{k}{2} \right)$$

$$= \frac{1}{2m} \left(-2am + m^2 - \frac{1}{2}m \right) Q_{m-1}(a - 1)$$

$$= \frac{1}{2} \left(m - \frac{1}{2} - 2a \right) Q_{m-1}(a - 1),$$

so the equations in the lemma are satisfied.

This does not allow us to conclude yet that our $Q_m(a)$ are equal to the $P_m\left(\frac{1}{2}, a\right)$, as the lemma only states uniqueness for the $P_m(r, a)$ as polynomials in a and r. However, we can prove equality by induction on m. The case $m = 0$ is given to be identically 1 in [PPZ16], agreeing with Q_0.

Now, assume $m > 0$ and $P_{m-1}\left(\frac{1}{2}, a\right) = Q_{m-1}(a)$. Then,

$$Q_m(a) - Q_m(a - 1) = \frac{1}{2} \left(m - \frac{1}{2} - 2a \right) Q_{m-1}(a - 1),$$

with the same relation for $P_m\left(\frac{1}{2}, a\right)$. Hence, $P_m\left(\frac{1}{2}, a\right) = Q_m(a) + c$. Using the same relation for $m + 1$, we get

$$\Delta_{m+1}(a) := P_{m+1}\left(\frac{1}{2}, a\right) - Q_{m+1}(a) = -\frac{c}{2} a^2 + \frac{2m - 1}{4} ac + d.$$

We then have

$$0 = \Delta_{m+1}\left(-\frac{1}{2}\right) - \Delta_{m+1}(0) = -\frac{c}{8} - \frac{2m - 1}{8} c = -\frac{m}{4} c.$$

Because $m > 0$ by assumption, this proves $c = 0$, so $P_m\left(\frac{1}{2}, a\right) = Q_m(a)$. $\square$

2.4 Simplified relations I

In this subsection, we discuss the relations that we can obtain from the substitution $r = \frac{1}{2}$ for the case of $x = 0$ in Subsection 2.2.

The polynomials $Q_m(a)$, for $m = 0, 1, 2, \ldots$, discussed in the previous subsection, have degree $2m$ and roots $-\frac{1}{2}, 0, \frac{1}{2}, 1, \ldots, m - \frac{3}{2}, m - 1$. Note that on the dilaton leaves in the relation of [PPZ16], we always have a coefficient $(R_m^{-1})_0^i$ for some $m \geqslant 1$. Since for $r = \frac{1}{2}$, we have $(R_m^{-1})_0^i = \left(-\frac{1}{4}\phi^{-1}\right)^{-m} Q_m(0) = 0$ for $m \geqslant 1$, the graphs with dilaton leaves do not contribute to the tautological relations.

In order to obtain a relation on $\mathcal{M}_{g,n}$ we first consider a relation on $\overline{\mathcal{M}}_{g,n+m}$ that we push forward to $\overline{\mathcal{M}}_{g,n}$ and then restrict to the open moduli space $\mathcal{M}_{g,n}$. Note that only graphs that correspond to a partial compactification of $\mathcal{M}_{g,n+m}$ can contribute non-trivially. Namely, it is a special case of the rational tails partial compactification, where we require in addition that at most one among the first n marked points can lie on each rational tail. We denote this compactification by $\mathcal{M}_{g,n+m}^{\mathrm{rt}[n]}$.

For instance, the dual graphs that can contribute non-trivially to a relation on $\mathcal{M}_{g,n+1}^{\mathrm{rt}[n]}$ are either the graph with one vertex and no edges or the graphs with two vertices of genus g and 0 and one edge connecting them, with leaves labeled by i and $n+1$ attached to the genus 0 vertex and all other leaves attached to the genus g vertex, for $i = 1, \ldots, n$. These graphs correspond to the divisors in $\mathcal{M}_{g,n+1}^{\mathrm{rt}[n]}$ that we denote by $D_{i,n+1}$.

More generally, we denote by D_I, for $I \subset \{1, \ldots, n+m\}$, the divisor in $\overline{\mathcal{M}}_{g,n+m}$ whose generic point is represented by a two-component curve with components of genus g and 0 connected through a node, such that all the points with labels in I lie on the component of genus 0 and all other points lie on the component of genus g. Then the divisors that belong to $\mathcal{M}_{g,n+m}^{\mathrm{rt}[n]}$ are those in which I contains at most one point with a label $1 \leqslant l \leqslant n$, and all dual graphs that we have to consider are the dual graphs of the generic points of the strata obtained by the intersection of these divisors.

We denote the relations on $\overline{\mathcal{M}}_{g,n}$ corresponding to the choice of the primary fields $a_1, \ldots, a_n$ by $\Omega_{g,n}^D(a_1, \ldots, a_n) = 0$, where D is the degree of the class. In this definition, we adjust the coefficient; namely, from now on, we ignore the pre-factor r^{g-1} in the definition of the relations, as well as the factor $\left(-\frac{1}{4}\phi^{-1}\right)^{-D}$ coming from the formula for the R-matrices in terms of the polynomials Q. Hence, $\Omega_{g,n}^D(\vec{a})$ is proportional to $T\left(g, n, \frac{1}{2}, \vec{a}, d(D)\right)$. We will also often write Ω for its restriction to various open parts of the moduli space, such as $\mathcal{M}_{g,n+m}^{\mathrm{rt}[n]}$.

Note that, as we discussed above, there is a condition on the possible degree of the class and the possible choices of the primary fields implied by the requirement that the degree of the auxiliary parameter ϕ must be negative.

We use the following relations in the rest of the paper: $\Omega_{g,n+m}^D(a_1, \ldots, a_{n+m})$, where $D \geqslant g$, $m \geqslant 0$, and $\sum_{i=1}^{n+m} a_i = g - 1 + D$ and all primary fields must be non-negative integers. We sometimes first multiply these relations by extra monomials of ψ-classes before we apply the pushforward to $\overline{\mathcal{M}}_{g,n}$ and/or restriction to $\mathcal{M}_{g,n}$.

2.5 Simplified relations II

In this subsection, we discuss the relations that we can obtain from the substitution $r = \frac{1}{2}$ for the case of $x = 1$ in Subsection 2.2.

Let us first list all the dual graphs representing the strata in $\mathcal{M}_{g,n+2}^{\mathrm{rt}[n]}$; see Figure 1. Note that
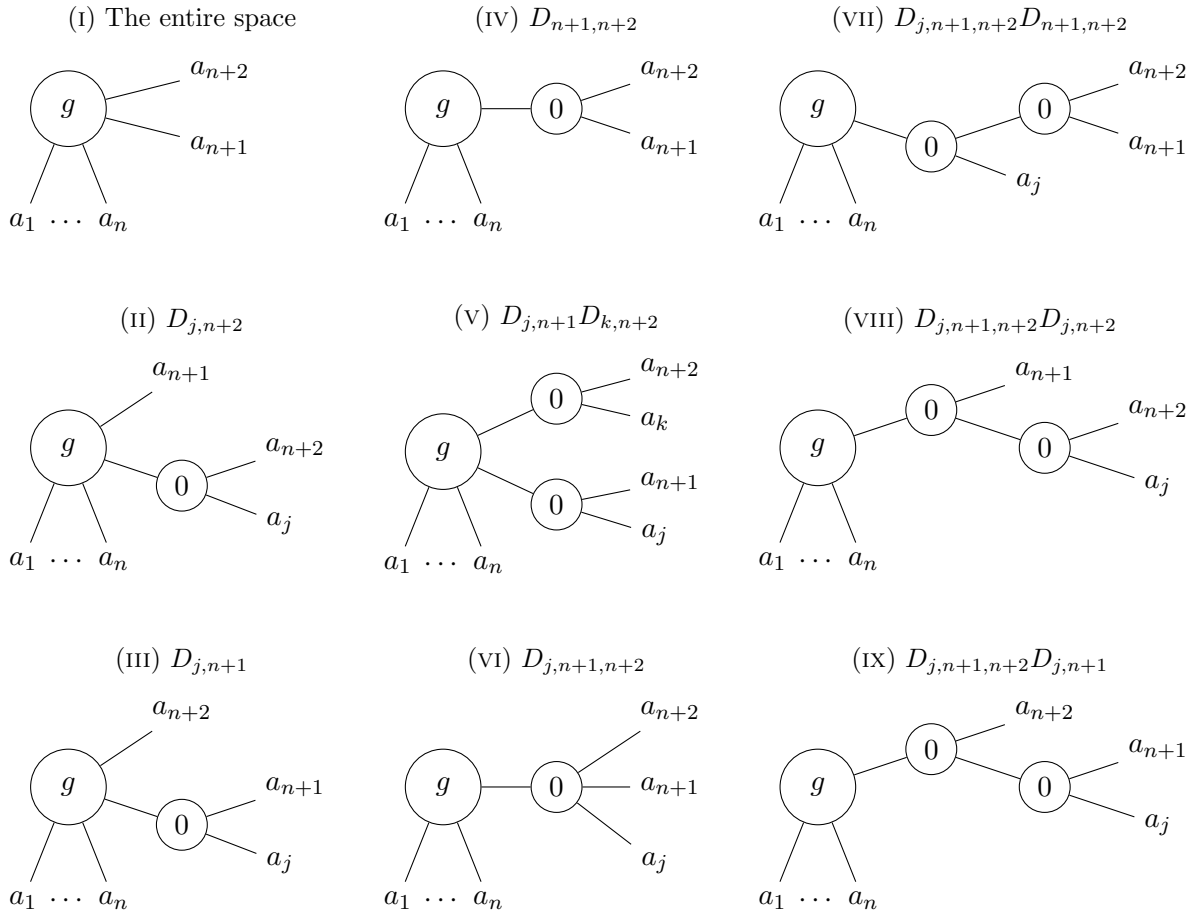
(I) The entire space   (IV) $D_{n+1,n+2}$ (VII) $D_{j,n+1,n+2}D_{n+1,n+2}$

(II) $D_{j,n+2}$ (V) $D_{j,n+1}D_{k,n+2}$ (VIII) $D_{j,n+1,n+2}D_{j,n+2}$

(III) $D_{j,n+1}$ (VI) $D_{j,n+1,n+2}$ (IX) $D_{j,n+1,n+2}D_{j,n+1}$

Figure 1: Strata in $\mathcal{M}_{g,n+2}^{\mathrm{rt}[2]}$.

under the extra condition $1 \leqslant a_i \leqslant r - 3 - a_{n+1} - a_{n+2}$ for any $1 \leqslant i \leqslant n$, the coefficients of all these graphs in $T(g, n+2, r, \vec{a}, -1)$, equipped in an arbitrary way with ψ- and κ-classes, are manifestly polynomial in $a_1, \ldots, a_{n+2}, r$. Indeed, this extra inequality guarantees that we can uniquely determine the primary fields on the edges in the Givental formula for all these nine graphs.

Thus, we have a sequence of tautological relations $T(g, n+2, r, \vec{a}, -1)$ in dimension $g+1$ defined for a big enough r and arbitrary non-negative integers $a_1, \ldots, a_{n+2}$ satisfying $a_1 + \cdots + a_{n+2} = 2g + r - 1$ and $1 \leqslant a_i \leqslant r - 3 - a_{n+1} - a_{n+2}$ for any $1 \leqslant i \leqslant n$. This gives us enough evaluations of the polynomial coefficients of the decorated dual graphs in $\mathcal{M}_{g,n+2}^{\mathrm{rt}[n]}$ to determine these polynomials completely. Thus, we can represent the values of these polynomial coefficients at an arbitrary point $(\tilde{a}_1, \ldots, \tilde{a}_{n+2}, \tilde{r}) \in \mathbb{C}^{n+3}$ as a linear combination of the Pandharipande–Pixton–Zvonkine relations. This representation is non-unique, since we have too many admissible points $(a_1, \ldots, a_{n+2}, r) \in \mathbb{Z}^{n+3}$ satisfying the conditions above. This non-uniqueness is not important for the coefficients of the decorated dual graphs in $\mathcal{M}_{g,n+2}^{\mathrm{rt}[n]}$, since we always get the values of their polynomial coefficients at the prescribed points, but the extensions of different linear combinations of the relations to the full compactification $\overline{\mathcal{M}}_{g,n+2}$ can be different. Indeed, the coefficients

of the graphs not listed in Figure 1 can be non-polynomial in $a_1, \ldots, a_{n+2}$ (but they are still polynomial in r).

We can choose one possible extension to the full compactification $\overline{\mathcal{M}}_{g,n+2}$ for each point $(\tilde{a}_1, \ldots, \tilde{a}_{n+2}, \tilde{r}) \in \mathbb{C}^{n+3}$. In particular, we always specialize to $r = \frac{1}{2}$, $a_{n+1} = \frac{3}{2}$, $a_{n+2} = -\frac{1}{2}$. The choice $r = \frac{1}{2}$ guarantees that we have no non-trivial dilaton leaves; that is, we have no κ-classes in the decorations of our graphs. We also divide the whole relation by the factor $\left(\frac{1}{2}\right)^{g-1} \left(-\frac{1}{4}\phi^{-1}\right)^{-1-g}$, as in the previous subsection.

Abusing the notation, we denote these relations by $\Omega_{g,n+2}^{g+1}(a_1, \ldots, a_n, \frac{3}{2}, -\frac{1}{2})$. These relations are defined for arbitrary complex numbers $a_1, \ldots, a_n$ satisfying $\sum_{i=1}^{n} a_i = 2g - \frac{3}{2}$. Of course, it is reasonable to use half-integer or integer primary fields $a_1, \ldots, a_n$ that would be the roots of the polynomials Q, since this gives us very good control over the possible degrees of the ψ-classes on the leaves and the edges of the dual graphs.

Let us stress once again that restriction of $\Omega_{g,n+2}^{g+1}(a_1, \ldots, a_n, \frac{3}{2}, -\frac{1}{2})$ to $\mathcal{M}_{g,n+2}^{\mathrm{rt}[n]}$ is well defined and can be obtained by the specialization of the polynomial coefficients of the dual graphs in Fig. 1 to the point $(a_1, \ldots, a_n, a_{n+1} = \frac{3}{2}, a_{n+2} = -\frac{1}{2}, r = \frac{1}{2})$. We analyze these polynomial coefficients in the next two sections. In the meantime, the extension of $\Omega_{g,n+2}^{g+1}(a_1, \ldots, a_n, \frac{3}{2}, -\frac{1}{2})$ from $\mathcal{M}_{g,n+2}^{\mathrm{rt}[n]}$ to $\overline{\mathcal{M}}_{g,n+2}$ is, in principle, not unique, and we only use that it exists.

3. The dimension of $R^{g-1}(\mathcal{M}_{g,n})$

In this section, we give a new proof of the result in [BSZ16] that $\dim R^{g-1}(\mathcal{M}_{g,n}) \leqslant n$.

3.1 Reduction to monomials in ψ-classes

In this subsection, we show that any monomial $\psi_1^{d_1} \cdots \psi_n^{d_n} \kappa_{e_1, \ldots, e_m}$ of degree $g-1$ can be expressed as a linear combination of monomials of degree $g-1$ which have only ψ-classes. We prove this fact by considering the relations $\Omega_{g,n+m}^{g-1+m}(a_1, \ldots, a_{n+m})$ for some appropriate choices of the primary fields.

PROPOSITION 3.1. *Let $g \geqslant 2$ and $n \geqslant 1$. The ring $R^{g-1}(\mathcal{M}_{g,n})$ is spanned by the monomials $\psi_1^{d_1} \cdots \psi_n^{d_n}$ for $d_1, \ldots, d_n \geqslant 0$ with $\sum_{i=1}^{n} d_i = g - 1$.*

Proof. The tautological ring of the open moduli space is generated by ψ- and κ-classes. Hence, a spanning set for the ring $R^{g-1}(\mathcal{M}_{g,n})$ is

$$\left\{ \psi_1^{d_1} \cdots \psi_n^{d_n} \kappa_{e_1, \ldots, e_m} \,\middle|\, m \geqslant 0, \, d_i \geqslant 0, \, e_j \geqslant 1, \, \sum_{i=1}^{n} d_i + \sum_{j=1}^{m} e_j = g - 1 \right\}.$$

Let $V \subset R^{g-1}(\mathcal{M}_{g,n})$ be the subspace spanned by the monomials

$$\left\{ \psi_1^{d_1} \cdots \psi_n^{d_n} \,\middle|\, \sum_{i=1}^{n} d_i = g - 1 \right\}.$$

We want to show that $R^{g-1}(\mathcal{M}_{g,n})/V = 0$. We do this by induction on the number m of indices of the κ-class.

Let us start with the case $m = 1$. Consider a relation $\Omega_{g,n+1}^{g}(a_1, \ldots, a_{n+1})$ for some admissible choice of the primary fields. In this case, we have contributions by the open stratum of smooth curves and by the divisors $D_{n+1,\ell}$ for $\ell = 1, \ldots, n$. The open stratum gives us the following

classes:

$$\sum_{\substack{d_1+\cdots+d_{n+1}=g \\ 0\leqslant d_i\leqslant a_i}} \prod_{i=1}^{n+1} Q_{d_i}(a_i) \prod_{i=1}^{n+1} \psi_i^{d_i}\,.$$

The condition $d_i \leqslant a_i$ follows from the identity $Q_d(a) = 0$ for $d > a$. The contribution of $-D_{n+1,\ell}$ is given by

$$\sum_{\substack{d_1+\cdots+d_n=g-1 \\ 0\leqslant d_i\leqslant a_i+\delta_{i\ell}(a_{n+1}-1)}} \prod_{i=1}^{n} Q_{d_i+\delta_{i\ell}}(a_i + \delta_{i\ell}a_{n+1}) \prod_{i\neq \ell,n+1} \psi_i^{d_i} D_{i,\ell}\pi^*\left(\psi_\ell^{d_\ell}\right).$$

Here $\pi\colon \mathcal{M}_{g,n+1}^{\mathrm{rt}[n]} \to \mathcal{M}_{g,n}$ is the natural projection. The sum of the pushforwards of these classes to $\mathcal{M}_{g,n}$ is equal to

$$0 = \sum_{\substack{d_1+\cdots+d_n+e=g-1 \\ d_i\geqslant 0,\, e\geqslant 1}} \prod_{i=1}^{n} Q_{d_i}(a_i)Q_{e+1}(a_{n+1}) \prod_{i=1}^{n} \psi_i^{d_i}\kappa_e \tag{3.1}$$

in $R^{g-1}(\mathcal{M}_{g,n})/V$. Thus we have equation (3.1) in $R^{g-1}(\mathcal{M}_{g,n})/V$ for each choice of $a_1,\ldots,a_{n+1}$ such that $\sum_{i=1}^{n+1} a_i = 2g - 1$.

If we choose the lexicographic order on the monomials $\psi_1^{d_1}\cdots\psi_n^{d_n}\kappa_e$, we can then choose the values of the a_i in such a way that the matrix of relations becomes lower triangular, in the following manner. For every monomial $\psi_1^{d_1}\cdots\psi_n^{d_n}\kappa_e$, we choose the relation with primary fields $a_1 = d_1 + g - 1$, $a_i = d_i$ for $i = 2,\ldots,n$, and $a_{n+1} = e + 1$. Equation (3.1) allows us to express this monomial in terms of similar monomials with the strictly larger exponent of ψ_1, so this set of relations does indeed give a lower-triangular matrix. This matrix is invertible; hence all monomials of the form $\psi_1^{d_1}\cdots\psi_n^{d_n}\kappa_e$ are equal to zero in $R^{g-1}(\mathcal{M}_{g,n})/V$.

Now, assume that all the monomials which have a κ-class with $m-1$ indices or fewer are equal to zero in $R^{g-1}(\mathcal{M}_{g,n})/V$. Consider a relation $\Omega_{g,n+1}^{g-1+m}(a_1,\ldots,a_n,b_1,\ldots,b_m)$. This relation, after the pushforward to $\mathcal{M}_{g,n}$, gives many terms with no κ-classes and also with κ-classes with at most $m-1$ indices, and also some terms with κ-classes with m indices. The latter terms are therefore equal to zero in $R^{g-1}(\mathcal{M}_{g,n})/V$; namely, we have

$$0 = \sum_{\substack{0\leqslant d_i\leqslant a_i \\ 1\leqslant e_j\leqslant b_j-1}} \left(\prod_{i=1}^{n} Q_{d_i}(a_i)\psi_i^{d_i}\right)\left(\prod_{j=1}^{m} Q_{e_j+1}(b_j)\right)\kappa_{e_1,\ldots,e_m} \tag{3.2}$$

for $\sum_{i=1}^{n} d_i + \sum_{j=1}^{m} e_j = g - 1$. Equation (3.2) is valid for each choice of the primary fields a_i and b_j such that $\sum_{i=1}^{n} a_i + \sum_{j=1}^{m} b_j = 2g - 2 + m$.

Choosing a monomial $\psi_1^{d_1}\cdots\psi_n^{d_n}\kappa_{e_1,\ldots,e_m}$, we can choose the primary fields to be $a_1 = d_1 + g - 1$, $a_i = d_i$ for $i = 2,\ldots,n$, and $b_j = e_j + 1$ for $j = 1,\ldots,m$. Again, this relation expresses our monomial as a linear combination of similar monomials with strictly higher exponent of ψ_1. By downward induction on this exponent, all monomials with m κ-indices vanish in $R^{g-1}(\mathcal{M}_{g,n})/V$ as well.

Thus $R^{g-1}(\mathcal{M}_{g,n})/V = 0$. In other words, any monomial which has a κ-class as a factor can be expressed as a linear combination of monomials in ψ-classes. $\qquad\square$

An immediate consequence of this proposition for $n = 1$ is the result of Looijenga.

COROLLARY 3.2 ([Loo95]). *For all $g \geqslant 2$, we have $R^{g-1}(\mathcal{M}_{g,1}) = \mathbb{Q}\psi_1^{g-1}$.*

3.2 Reduction to n generators

In this subsection, we prove the following proposition.

PROPOSITION 3.3. *For $n \geqslant 2$ and $g \geqslant 2$, every monomial of degree $g-1$ in ψ-classes and at most one κ_1-class can be expressed as a linear combination of the n classes*

$$\psi_1^{g-1},\ \psi_1^{g-2}\psi_2,\ldots,\ \psi_1^{g-2}\psi_n$$

with rational coefficients.

Together with the previous subsection this gives a new proof of the following statement.

THEOREM 3.4 ([BSZ16]). *For $n \geqslant 2$ and $g \geqslant 2$, we have $\dim_{\mathbb{Q}} R^{g-1}(\mathcal{M}_{g,n}) \leqslant n$.*

Remark 3.5. Note that the possible κ_1-class is added in Proposition 3.3 for a technical reason; it seems to be completely unnecessary in light of Proposition 3.1. In fact, when we include κ_1, we consider systems of generators approximately twice as large, but this allows us to obtain a much larger system of tautological relations. We do not know of any argument that would allow us to obtain the sufficient number of relations if we consider only monomials of ψ-classes as generators.

We reduce the number of generators by pushing forward enough relations via the map

$$\pi_*^{(2)}\colon R^{g+1}(\overline{\mathcal{M}}_{g,n+2}) \to R^{g-1}(\overline{\mathcal{M}}_{g,n}),$$

where $\pi^{(2)}$ is the forgetful morphism for the last two marked points (we abuse notation a little bit here, restricting the map $\pi^{(2)}$ to $\mathcal{M}_{g,n+2}^{\mathrm{rt}[n]} \to \mathcal{M}_{g,n}$). For $n \geqslant 2$, let us consider the following vector of primary fields:

$$\vec{a} := \left(a_1 = 2g - \tfrac{3}{2} - A, a_2, \ldots, a_n, a_{n+1} = \tfrac{3}{2}, a_{n+2} = -\tfrac{1}{2}\right), \tag{3.3}$$

where $a_i \in \mathbb{Z}_{\geqslant 0}$ for $i = 2, \ldots, n$ and $A = \sum_{i=2}^{n} a_i \leqslant g - 2$. We consider the following monomials in $R^{g-1}(\mathcal{M}_{g,n})$:

$$y := \psi_1^{g-2-A} \prod_{i=2}^{n} \psi_i^{a_i} \kappa_1 \quad \text{and} \quad x_\ell := \psi_1^{g-2-A} \prod_{i=2}^{n} \psi_i^{a_i + \delta_{i\ell}}, \quad \ell = 2, \ldots, n.$$

LEMMA 3.6. *The tautological relation $\pi_*^{(2)}\Omega_{g,n+2}^{g+1}(\vec{a})$, where $\vec{a}$ is defined in equation (3.3), has the following form:*

$$y \cdot \prod_{i=2}^{n} Q_{a_i}(a_i) Q_2\left(\tfrac{3}{2}\right)\left(Q_{g-1-A}\left(2g - \tfrac{3}{2} - A\right) - Q_{g-1-A}(2g - 2 - A)\right)$$

$$- \sum_{\ell=2}^{n} x_\ell \cdot \prod_{i=2}^{n} Q_{a_i + 2\delta_{i\ell}}\left(a_i + \tfrac{3}{2}\delta_{i\ell}\right)\left(Q_{g-1-A}\left(2g - \tfrac{3}{2} - A\right) - Q_{g-1-A}(2g - 2 - A)\right)$$

$$= \text{terms divisible by } \psi_1^{g-1-A}. \tag{3.4}$$

Proof. In order to prove this lemma, we have to analyze all strata in $\mathcal{M}_{g,n+2}^{\mathrm{rt}[n]}$. The list of strata is given in Figure 1. Each stratum should be decorated in all possible ways by the R-matrices with ψ-classes as described in Section 2.

There are several useful observations that simplify the computation. The leaf labeled by a_i, for $i = 2, \ldots, n$, is equipped with $\psi_i^{d_i} Q_{d_i}(a_i)$. This implies $d_i \leqslant a_i$. Since $Q_{>2}\left(\frac{3}{2}\right) = 0$ and $Q_{>0}\left(-\frac{1}{2}\right) = 0$, we conclude that the exponent of ψ_{n+1} is at most 2 and that of ψ_{n+2} is equal to 0. Note that we can obtain a monomial with κ_1-class in the pushforward only if we have ψ_{n+1}^2 in the original decorated graph.

Similar observations are also valid for the exponents of the ψ-classes at the nodes. Note that there are no ψ-classes on the genus 0 components in any strata except for the case of the dual graph VI, where we must have a ψ-class at one of the four points (three marked points and the node) of the genus 0 component; otherwise, the pushforward is equal to zero. So, for instance, we have ψ^d at the genus g branch of the node on the dual graph II with coefficient $-Q_{d+1}\left(a_j - \frac{1}{2}\right)$, so in this case we have $d \leqslant a_j - 1$. If we have ψ^d at the genus g branch of the node on the dual graph VIII, then the product of the coefficients that we have on the edges of this graph is equal to $Q_1\left(a_j - \frac{1}{2}\right) Q_{d+1}\left(a_j - \frac{1}{2} + \frac{3}{2} - 1\right)$, so in this case we have $d \leqslant a_j - 1$. And so on; one more example of a detailed analysis of the graphs VI–IX is given in Lemma 4.3 in the next section.

We see that we have severe restrictions on the possible powers of ψ-classes at all points but the one labeled by 1, where the exponent is bounded from below, also after the pushforward. Then it is easy to see by the analysis of the graph contributions as above that the exponent of ψ_1 is at least $g - 2 - A$. Let us list all the terms whose pushforwards to $\mathcal{M}_{g,n}$ contain the terms with ψ_1^{g-2-A}:

- One of the classes in $\mathcal{M}_{g,n+2}^{\mathrm{rt}[n]}$ corresponding to graph I is $\psi_1^{g-1-A} \prod_{i=2}^n \psi_i^{a_i} \psi_{n+1}^2$ with coefficient $\prod_{i=2}^n Q_{a_i}(a_i) Q_2\left(\frac{3}{2}\right) Q_{g-1-A}\left(2g - \frac{3}{2} - A\right)$. Its pushforward contains the monomial y and the terms divisible by ψ_1^{g-1-A}.

- Consider graph II for $j = 1$. Let $\pi_{n+2} \colon \overline{\mathcal{M}}_{g,n+2} \to \overline{\mathcal{M}}_{g,n+1}$ be the forgetful morphism for the $(n+2)$nd point. Up to irrelevant terms, one of the classes corresponding to this graph is $\prod_{i=2}^n \psi_i^{a_i} \psi_{n+1}^2 D_{1,n+2}(\pi_{n+2})^*\left(\psi_1^{g-2-A}\right)$ with coefficient $(-1)\prod_{i=2}^n Q_{a_i}(a_i) Q_2\left(\frac{3}{2}\right) \cdot Q_{g-1-A}(2g - 2 - A)$. Its pushforward is equal to the monomial y.

- Let $\pi_{n+1} \colon \overline{\mathcal{M}}_{g,n+2} \to \overline{\mathcal{M}}_{g,n+1}$ be the forgetful morphism for the $(n+1)$st point. One of the classes corresponding to graph III for $j = \ell$ is $\prod_{i\neq 1,\ell} \psi_i^{a_i} \psi_1^{g-1-A} D_{\ell,n+1}(\pi_{n+1})^*\left(\psi_\ell^{a_\ell+1}\right)$ with coefficient $(-1)\prod_{i\neq 1,\ell} Q_{a_i}(a_i) Q_{a_\ell+2}\left(a_\ell + \frac{3}{2}\right) Q_{g-1-A}\left(2g - \frac{3}{2} - A\right)$. The pushforward of this class contains the monomial x_ℓ and the terms divisible by ψ_1^{g-1-A}.

- Consider graph V for $j = \ell$ and $k = 1$. One of the classes corresponding to this graph is $\prod_{i\neq 1,\ell} \psi_i^{a_i} D_{\ell,n+1}(\pi_{n+1})^*\left(\psi_\ell^{a_\ell+1}\right) D_{1,n+2}(\pi_{n+2})^*\left(\psi_1^{g-2-A}\right)$ with coefficient $\prod_{i\neq 1,\ell} Q_{a_i}(a_i) \cdot Q_{a_\ell+2}\left(a_\ell + \frac{3}{2}\right) Q_{g-1-A}(2g - 2 - A)$. Its pushforward is equal to the monomial x_ℓ.

Collecting all these terms together, we obtain the left-hand side of equation (3.4). Then, it is easy to verify case by case that under the pushforward, all other graphs and all other possible decorations on these four graphs produce only monomials divisible by ψ_1^{g-1-A}. □

Let $a_j > 0$ for $j = 2, \ldots, n$. Consider a vector of primary fields $\vec{a}^{(j)}$ obtained from $\vec{a}$ by adding $\frac{1}{2}$ to a_1 and subtracting $\frac{1}{2}$ from a_j, that is,

$$\vec{a}^{(j)} := \left(2g - 1 - A, a_2, \ldots, a_{j-1}, a_j - \tfrac{1}{2}, a_{j+1}, \ldots, a_n, \tfrac{3}{2}, -\tfrac{1}{2}\right).$$

LEMMA 3.7. *The tautological relation* $\pi_*^{(2)}\Omega_{g,n+2}^{g+1}(\vec{a}^{(j)})$ *has the following form:*

$$y \cdot \prod_{i=2}^{n} Q_{a_i}\big(a_i - \tfrac{1}{2}\delta_{ij}\big)Q_2\big(\tfrac{3}{2}\big)\big(Q_{g-1-A}(2g-1-A) - Q_{g-1-A}\big(2g-\tfrac{3}{2}-A\big)\big)$$

$$- \sum_{\substack{\ell=2 \\ \ell \neq j}}^{n} x_\ell \cdot \prod_{i=2}^{n} Q_{a_i+2\delta_{i\ell}}\big(a_i + \tfrac{3}{2}\delta_{i\ell} - \tfrac{1}{2}\delta_{ij}\big)\big(Q_{g-1-A}(2g-1-A) - Q_{g-1-A}\big(2g-\tfrac{3}{2}-A\big)\big)$$

$$= \textit{terms divisible by } \psi_1^{g-1-A}.$$

Proof. The proof of this lemma repeats the proof of Lemma 3.6. It is only important to note that the terms that could produce the monomial x_j contribute trivially since they have a factor of $Q_{a_j+2}\big(a_j - \tfrac{1}{2} + \tfrac{3}{2}\big) = 0$ in their coefficients. $\qquad\square$

Remark 3.8. Note that we have the condition $a_j \geqslant 0$. Indeed, if $a_j = 0$, we can still try to use $\vec{a}^{(j)}$ as a possible vector of primary fields. But in this case, it can contain monomials with lower powers of ψ_1, and hence those relations cannot be used for our induction argument in increasing powers of ψ_1. To see this, consider graph II. The coefficient that we have in this case for the degree d of the ψ-class on the genus g branch of the node is equal to $Q_{d+1}\big(-\tfrac{1}{2} - \tfrac{1}{2}\big)$. Since -1 is not a zero of any polynomial $Q_{\geqslant 0}$, the degree d can be arbitrarily high, and therefore there is no restriction from below on the degree of ψ_1.

Let us distinguish now between zero and non-zero primary fields. Up to relabeling the marked points, we can assume

$$a_2 = a_3 = \cdots = a_s = 0 \quad \text{and} \quad a_i \geqslant 1, \quad i = s+1, \ldots, n.$$

Note that, by the definition of the Q-polynomials, the coefficient of y is not 0 in all relations in Lemmata 3.6 and 3.7. Dividing these relations by the coefficient of y, we obtain the $n - s + 1$ linearly independent relations

$$\mathrm{Rel}_0: \quad y - \sum_{l=2}^{n} \frac{Q_{a_l+2}(a_l + 3/2)}{Q_{a_l}(a_l)Q_2(3/2)} x_l = \text{terms divisible by } \psi_1^{g-1-A},$$

$$\mathrm{Rel}_j: \quad y - \sum_{l=2}^{n} \frac{Q_{a_l+2}(a_l + 3/2)}{Q_{a_l}(a_l)Q_2(3/2)}(1 - \delta_{j,l})x_l = \text{terms divisible by } \psi_1^{g-1-A}$$

for $j = s+1, \ldots, n$. Rescaling the generators by rational non-zero coefficients

$$\tilde{x}_l := -\frac{Q_{a_l+2}(a_l + 3/2)}{Q_{a_l}(a_l)Q_2(3/2)} x_l, \quad l = 2, \ldots, n,$$

we can represent the relations in the following matrix:

		y	$\tilde{x}_2$	$\cdots$	$\tilde{x}_s$	$\tilde{x}_{s+1}$	$\tilde{x}_{s+2}$	$\tilde{x}_{s+3}$	$\cdots$	$\tilde{x}_n$
	Rel_0	1	1	$\cdots$	1	1	1	1	$\cdots$	1
	Rel_{s+1}	1	1	$\cdots$	1	0	1	1	$\cdots$	1
$M :=$	Rel_{s+2}	1	1	$\cdots$	1	1	0	1	$\cdots$	1
	Rel_{s+3}	1	1	$\cdots$	1	1	1	0	$\cdots$	1
	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
	Rel_n	1	1	$\cdots$	1	1	1	1	$\cdots$	0

Let us take linear combinations of the above relations: $\widetilde{\text{Rel}}_j := \text{Rel}_0 - \text{Rel}_j$ for $j = s+1, \ldots, n$, and $\widetilde{\text{Rel}}_0 := \text{Rel}_0 - \sum_{j=s+1}^{n} \widetilde{\text{Rel}}_j$. We obtain

	y	$\tilde{x}_2$	$\cdots$	$\tilde{x}_s$	$\tilde{x}_{s+1}$	$\tilde{x}_{s+2}$	$\tilde{x}_{s+3}$	$\cdots$	$\tilde{x}_n$
$\widetilde{\text{Rel}}_0$	1	1	$\cdots$	1	0	0	0	$\cdots$	0
$\widetilde{\text{Rel}}_{s+1}$	0	0	$\cdots$	0	1	0	0	$\cdots$	0
$\widetilde{\text{Rel}}_{s+2}$	0	0	$\cdots$	0	0	1	0	$\cdots$	0
$\widetilde{\text{Rel}}_{s+3}$	0	0	$\cdots$	0	0	0	1	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\widetilde{\text{Rel}}_n$	0	0	$\cdots$	0	0	0	0	$\cdots$	1

The relation $\widetilde{\text{Rel}}_j$ expresses the monomial $\psi_1^{g-2-A} \prod_{i=2}^{n} \psi_i^{a_i+\delta_{ij}}$ as a linear combination of the generators with higher powers of ψ_1. The relation $\widetilde{\text{Rel}}_0$ expresses the monomial $\psi_1^{g-2-A} \prod_{i=2}^{n} \psi_i^{a_i} \kappa_1$ as a linear combination of the monomials $\psi_1^{g-2-A} \prod_{i=2}^{n} \psi_i^{a_i+\delta_{ij}}$, for $j = 2, \ldots, s$, and generators with higher powers of ψ_1. In case no primary field a_i is equal to 0 (that is, $s = 1$), any of the monomials $y, x_2, \ldots, x_n$ can be expressed in terms of the generators with strictly higher power of ψ_1.

3.2.1 *Reduction algorithm.* Consider a monomial $\psi_1^{g-1-\sum d_i} \psi_2^{d_2} \cdots \psi_n^{d_n}$. Let d_M be the maximal element in the list of the d_i with the lowest index. If $d_M \geqslant 2$, compute the relations $\widetilde{\text{Rel}}_j$ for the following vector of primary fields:

$$\left(2g - \tfrac{3}{2} - \sum_{i=2}^{n} d_i, d_2, \ldots, d_{M-1}, d_M - 1, d_{M+1}, \ldots, d_n, d_{n+1} = \tfrac{3}{2}, d_{n+2} = -\tfrac{1}{2}\right).$$

Since $d_M - 1 \geqslant 1$, we can use the relation $\widetilde{\text{Rel}}_M$ to express the monomial $\psi_1^{g-1-\sum d_i} \psi_2^{d_2} \cdots \psi_n^{d_n}$ as a linear combination of monomials with higher powers of ψ_1.

We are left to treat the vectors $\vec{d}$ with $d_i = 0$ or 1 for $i = 2, \ldots, n$. They correspond to the vertices of a unitary $(n-1)$-hypercube with non-negative coordinates. Let s be the number of d_i equal to 0, so the remaining $(n-1-s)$ d_i are equal to 1, for $s = 0, \ldots, n-1$. Let us distinguish between the different cases in s.

$s = n-1$: In this case, we have ψ_1^{g-1}, a generator.

$s = n-2$: In this case, we have the remaining $n-1$ generators $\psi_1^{g-2} \psi_i$ for $i = 2, \ldots, n$.

$1 \leqslant s \leqslant n-3$: This case can be treated as the case $s = 0$ for some smaller n discussed below. Let us argue by induction on n. For $n \leqslant 3$, the case $1 \leqslant s \leqslant n-3$ does not appear. Let us assume $n \geqslant 4$. We have at least one zero, so let us assume $d_j = 0$. Let $\pi_j^{(1)}$ be the morphism that forgets the jth marked point. If the monomial $\psi_1^{g-n+s} \psi_2^{d_2} \cdots \hat{\psi}_j \cdots \psi_n^{d_n}$ is expressed as a linear combination of generators in $R^{g-1}(\mathcal{M}_{g,n-1})$ (the space where the point with the label j is forgotten), then the pull-back of this relation via $\pi_j^{(1)}$ expresses $\psi_1^{g-n+s} \psi_2^{d_2} \cdots \hat{\psi}_j \cdots \psi_n^{d_n}$ as a linear combination of the pull-backs of the $n-1$ generators of $R^{g-1}(\mathcal{M}_{g,n-1})$, ψ_1^{g-1} and $\psi_1^{g-2} \psi_i$, where $i \neq 1, j$. To conclude, we observe that $\left(\pi_j^{(1)}\right)^* \psi_1^{g-1} = \psi_1^{g-1}$ and $\left(\pi_j^{(1)}\right)^* \psi_1^{g-2} \psi_i = \psi_1^{g-2} \psi_i$ for $i \neq 1, j$ on the open moduli spaces. Note that the same reasoning does not work in the case $s = n-2$ since the argument for $s = 0$ below uses the assumption $n \geqslant 3$.

3.2.2 *The case $s = 0$.* For $n \geqslant 3$, we show that the monomial $\psi_1^{g-n} \prod_{i=2}^{n} \psi_i^1$ can be expressed in terms of the generators ψ_1^{g-1} and $\psi_1^{g-2}\psi_i$ for $i = 2, \ldots, n$, concluding this way the proof of Proposition 3.3.

Now, let $\vec{v}_k$ be the vector of primary fields

$$\vec{v}_k := \left(a_1 = 2g - \frac{n+2+k}{2}, \underbrace{1, \ldots, 1}_{k}, \underbrace{\frac{1}{2}, \ldots, \frac{1}{2}}_{n-1-k}, a_{n+1} = \frac{3}{2}, a_{n+2} = -\frac{1}{2} \right).$$

Similarly as before, let

$$y := \psi_1^{g-n-1} \prod_{i=2}^{n} \psi_i^1 \kappa_1 \quad \text{and} \quad \tilde{x}_\ell := -\frac{Q_3(5/2)}{Q_1(1)Q_2(3/2)} \psi_1^{g-n-1} \psi_2^1 \cdots \psi_\ell^2 \cdots \psi_n^1, \quad \ell = 2, \ldots, n.$$

Consider the monomials

$$\psi_1^{g-n} \prod_{i=2}^{n} \psi_i \quad \text{and} \quad \psi_1^{g-n} \prod_{i=2}^{n} \psi_i^{1-\delta_{i\ell}} \kappa_1, \quad \ell = 2, \ldots, n.$$

The relations we used in the cases $s \geqslant 1$ imply that the difference of any two of these monomials is equal to a linear combination of the generators ψ_1^{g-1} and $\psi_1^{g-2}\psi_i$ for $i = 2, \ldots, n$. Let c_0 (respectively, c_1, c_2) be the sum of the coefficients of these monomials in the pushforwards of the relations $\Omega_{g,n+2}^{g+1}(\vec{v}_0)$ (respectively, $\Omega_{g,n+2}^{g+1}(\vec{v}_1)$, $\Omega_{g,n+2}^{g+1}(\vec{v}_2)$), and let $\hat{c}_i$ be the normalized coefficients that we get when we divide the relations by the coefficient of y.

Now we can expand, in this special case, the system of linear relations collected in the matrix M above. We have a new linear variable, $z := \psi_1^{g-n} \prod_{i=2}^{n} \psi_i$, which is equal to $\psi_1^{g-n} \prod_{i=2}^{n} \psi_i^{1-\delta_{i\ell}} \kappa_1$ up to generators, for $\ell = 2, \ldots, n$, and an extra linear relation Rel_* corresponding to the vector of primary fields $\vec{v}_2$. Since in this special case, in these relations all terms with the exponent of ψ_1 equal to $g - 1 - A$, where $A = n - 1$, are now identified with one another and collected in the variable z, these relations express $z, y, x_2, \ldots, x_n$ in terms of the monomials proportional to ψ_1^{g-A}. The matrix of this system of relations reads

	z	y	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{x}_4$	$\ldots$	$\tilde{x}_n$
Rel_0	$\hat{c}_0$	1	1	1	1	$\ldots$	1
Rel_2	$\hat{c}_1$	1	0	1	1	$\ldots$	1
Rel_3	$\hat{c}_1$	1	1	0	1	$\ldots$	1
Rel_4	$\hat{c}_1$	1	1	1	0	$\ldots$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
Rel_n	$\hat{c}_1$	1	1	1	1	$\ldots$	0
Rel_*	$\hat{c}_2$	1	0	0	1	$\ldots$	1

This matrix is non-degenerate if and only if $\hat{c}_2 - 2\hat{c}_1 + \hat{c}_0 \neq 0$. We prove this non-degeneracy in Proposition 4.1 in the next section. This completes the proof of Proposition 3.3 and, as a corollary, Theorem 3.4.

4. Non-degeneracy of the matrix

In this section, we compute the sum of the coefficients of the monomials $\psi_1^{g-n} \prod_{i=2}^{n} \psi_i$ and $\psi_1^{g-n} \left(\prod_{i=2}^{n} \psi_i^{1-\delta_{i\ell}} \right) \kappa_1$ for $\ell = 2, \ldots, n$ for the three particular sequences of the primary fields.

Let us recall the notation. We denote these sums of coefficients by

$$c_0 \quad \text{for the primary fields} \quad g - 1 - n, \tfrac{1}{2}, \ldots, \tfrac{1}{2};$$
$$c_1 \quad \text{for the primary fields} \quad g - \tfrac{3}{2} - n, 1, \tfrac{1}{2}, \ldots, \tfrac{1}{2};$$
$$c_2 \quad \text{for the primary fields} \quad g - 2 - n, 1, 1, \tfrac{1}{2}, \ldots, \tfrac{1}{2}.$$

We denote the sequence of the primary fields by $a_1, \ldots, a_n$. The primary fields at the two points that we forget are as usual $a_{n+1} = \tfrac{3}{2}$ and $a_{n+2} = -\tfrac{1}{2}$. For each c_i, for $i = 0, 1, 2$, we denote by $\hat{c}_i$ the normalized coefficient, namely,

$$\hat{c}_i := c_i \cdot \left(\left(Q_{g+1-n}(a_1) - Q_{g+1-n}\left(a_1 - \tfrac{1}{2}\right) \right) \prod_{i=2}^{n} Q_1(a_i) Q_2\left(\tfrac{3}{2}\right) \right)^{-1}, \tag{4.1}$$

where the sequence of primary fields is exactly the one used for the definition of the corresponding c_i.

The goal is to prove the following non-degeneracy statement.

PROPOSITION 4.1. *For any g and n satisfying $3 \leqslant n \leqslant g - 1$, we have $\hat{c}_0 - 2\hat{c}_1 + \hat{c}_2 \neq 0$.*

We prove this proposition below, in Subsection 4.3, after we compute the coefficients c_0, c_1, and c_2 explicitly.

4.1 A general formula

First, we prove a general formula for any set of primary fields $a_2, \ldots, a_n \in \left\{ \tfrac{1}{2}, 1 \right\}$.

LEMMA 4.2. *Let all a_i for $i = 2, \ldots, n$ be either $\tfrac{1}{2}$ or 1. We have $a_1 = 2g - \tfrac{3}{2} - \sum_{i=2}^{n} a_i$. A general formula for the sum of the coefficients of the classes $\psi_1^{g-n} \prod_{i=2}^{n} \psi_i$ and $\psi_1^{g-n} \prod_{i=2}^{n} \psi_i^{1-\delta_{i\ell}} \kappa_1$, for $\ell = 2, \ldots, n$, in the pushforward to $\mathcal{M}_{g,n}$ is given by*

$$\prod_{i=2}^{n} Q_1(a_i) \cdot \Bigg[(2g - 2 + n) Q_2\left(\tfrac{3}{2}\right) Q_{g-n}(a_1)$$

$$+ (2g - 2 + n) Q_1\left(\tfrac{3}{2}\right) \left(Q_{g+1-n}(a_1) - Q_{g+1-n}\left(a_1 - \tfrac{1}{2}\right) \right)$$

$$+ Q_{g+2-n}(a_1) - Q_{g+2-n}\left(a_1 - \tfrac{1}{2}\right)$$

$$+ Q_{g+2-n}(a_1 + 1) - Q_{g+2-n}\left(a_1 + \tfrac{3}{2}\right)$$

$$+ \left(Q_1\left(\tfrac{3}{2}\right) - Q_1(1) \right) Q_{g+1-n}(a_1)$$

$$+ \sum_{\ell=2}^{n} \frac{\left(Q_2\left(\tfrac{3}{2}\right) \left(Q_1(a_\ell) - Q_1\left(a_\ell - \tfrac{1}{2}\right) \right) \right) Q_{g-n}(a_1)}{Q_1(a_\ell)}$$

$$+ \sum_{\ell=2}^{n} \frac{\left(Q_3(a_\ell + 1) - Q_3\left(a_\ell + \tfrac{3}{2}\right) \right) Q_{g-n}(a_1)}{Q_1(a_\ell)}$$

$$+ \sum_{\ell=2}^{n} \frac{\left(Q_2\left(\tfrac{3}{2}\right) Q_0(a_\ell) - Q_2\left(a_\ell + \tfrac{3}{2}\right) \right) \left(Q_{g+1-n}(a_1) - Q_{g+1-n}\left(a_1 - \tfrac{1}{2}\right) \right)}{Q_1(a_\ell)} \Bigg]. \tag{4.2}$$

Proof. The proof of this lemma is based on the analysis of all possible strata in $\overline{\mathcal{M}}_{g,n+2}$ equipped with all possible monomials of ψ-classes that can potentially contribute non-trivially to

$\psi_1^{g-n} \prod_{i=2}^n \psi_i$ and $\psi_1^{g-n} \prod_{i=2}^n \psi_i^{1-\delta_{i\ell}} \kappa_1$, for $\ell = 2, \ldots, n$, under the pushforward. Note that we do not have to consider κ-classes on the strata in $\overline{\mathcal{M}}_{g,n+2}$ since the choice $r = \frac{1}{2}$ guarantees that there are no terms with κ-classes in the Pandharipande–Pixton–Zvonkine relations.

Recall that we denote by D_I, for $I \subset \{1, \ldots, n+2\}$, the divisor in $\overline{\mathcal{M}}_{g,n+2}$ whose generic point is represented by a two-component curve, with components of genus g and 0 connected through a node, such that all points with labels in I lie on the component of genus 0 and all other points lie on the component of genus g. In this case, we denote by ψ_0 the ψ-class corresponding to the node on the genus 0 component.

We denote by $\pi' \colon \overline{\mathcal{M}}_{g,n+2} \to \overline{\mathcal{M}}_{g,n+1}$ the map forgetting the marked point labeled by $n+2$, by $\pi'' \colon \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ the map forgetting the marked point labeled by $n+1$, and by π their composition $\pi = \pi'' \circ \pi'$. Note that $\pi'_* \big(\prod_{i=1}^{n+1} \psi_i^{d_i} \big) = \sum_{j \colon d_j > 0} \prod_{i=1}^{n+1} \psi_i^{d_i - \delta_{ij}}$, so, since in order to compute π_* we always first apply π'_*, below we typically mention the degree of which ψ-class is reduced. We do the same for π''_* in the relevant cases.

Let us now go through the full list of possible non-trivial contributions:

- The pushforward of the class $\psi_1^{g-n} \prod_{i=2}^n \psi_i^1 \psi_{n+1}^2$ contains $\psi_1^{g-n} \prod_{i=2}^n \psi_i^1$ with coefficient $(2g-2+n) \prod_{i=2}^n Q_1(a_i) Q_{g-n}(a_1) Q_2\big(\frac{3}{2}\big)$. This explains the first line of equation (4.2). It also contains the terms $\psi_1^{g-n} \prod_{i=2}^n \psi_i^{1-\delta_{i\ell}} \kappa_1$, for $\ell = 2, \ldots, n$, with coefficient $\prod_{i=2}^n Q_1(a_i) Q_{g-n}(a_1) \cdot Q_2\big(\frac{3}{2}\big)$.

- The pushforward of the class $\psi_1^{g-n} \prod_{i=2}^n \psi_i^{1-\delta_{i\ell}} D_{\ell,n+2} \psi_{n+1}^2$ also gives the term $\psi_1^{g-n} \prod_{i=2}^n \psi_i^{1-\delta_{i\ell}} \kappa_1$, with coefficient $-\prod_{i=2}^n Q_1\big(a_i - \delta_{i\ell}/2\big) Q_{g-n}(a_1) Q_2\big(\frac{3}{2}\big)$. The sum over ℓ of this and the previous coefficient is equal to the sixth line of equation (4.2).

- The pushforward of the class $\psi_1^{g+1-n} \prod_{i=2}^{n+1} \psi_i^1$, where at the first step the map π'_* decreases the degree of ψ_1, gives $\psi_1^{g-n} \prod_{i=1}^n \psi_i^1$ with coefficient $(2g-2+n) \prod_{i=2}^n Q_1(a_i) Q_{g+1-n}(a_1) Q_1\big(\frac{3}{2}\big)$.

- The pushforward of the class $(\pi')^*(\psi_1^{g-n}) \prod_{i=2}^{n+1} \psi_i^1 D_{1,n+2}$ gives $\psi_1^{g-n} \prod_{i=2}^n \psi_i^1$ with coefficient $-(2g-2+n) \prod_{i=2}^n Q_1(a_i) Q_{g+1-n}\big(a_1 - \frac{1}{2}\big) Q_1\big(\frac{3}{2}\big)$. The sum of this and the previous coefficient is equal to the second line of equation (4.2).

- The pushforward of the class $\psi_1^{g+2-n} \prod_{i=2}^n \psi_i^1$, where both π'_* and π''_* decrease the degree of ψ_1, gives $\psi_1^{g-n} \prod_{i=2}^n \psi_i^1$ with coefficient $\prod_{i=1}^n Q_1(a_i) Q_{g+2-n}(a_1)$.

- The pushforward of the class $(\pi')^*(\psi_1^{g+1-n}) \prod_{i=2}^n \psi_i^1 D_{1,n+2}$, where the map π''_* decreases the degree of ψ_1, gives $\psi_1^{g-n} \prod_{i=2}^n \psi_i^1$ with coefficient $-\prod_{i=2}^n Q_1(a_i) Q_{g+2-n}\big(a_1 - \frac{1}{2}\big)$. The sum of this and the previous coefficient is equal to the third line of equation (4.2).

- The pushforward of the class $(\pi'')^*(\psi_1^{g+1-n}) \prod_{i=2}^n \psi_i^1 D_{1,n+1}$, where at the first step the map π'_* decreases the degree of $(\pi'')^*\psi_1$, gives $\psi_1^{g-n} \prod_{i=2}^n \psi_i^1$ with coefficient $-\prod_{i=2}^n Q_1(a_i) \cdot Q_{g+2-n}\big(a_1 + \frac{3}{2}\big)$.

- Consider the following seven cases together: $\pi^*(\psi_1^{g-n}) \prod_{i=2}^n \psi_i^1 D_{1,n+1,n+2}(\psi_0 + \psi_1 + \psi_{n+1} + \psi_{n+2})$ and $\pi^*(\psi_1^{g-n}) \prod_{i=2}^n \psi_i^1 D_{1,n+1,n+2}(D_{1,n+1} + D_{1,n+2} + D_{n+1,n+2})$. By Lemma 4.3 below, the total sum of their pushforwards is equal to $\psi_1^{g-n} \prod_{i=2}^n \psi_i$ with coefficient $\prod_{i=2}^n Q_1(a_i) \cdot Q_{g+2-n}\big(a_1 - \frac{1}{2} + \frac{3}{2}\big)$. The sum of this and the previous coefficient is equal to the fourth line in equation (4.2).

- The pushforward of the class $\psi_1^{g-n} \prod_{i=2}^n \psi_i^{1-\delta_{i\ell}} D_{\ell,n+1}(\pi'')^*\psi_\ell^1$, where at the first step the map π'_* decreases the degree of $(\pi'')^*\psi_\ell$, gives $\psi_1^{g-n} \prod_{i=2}^n \psi_i^1$ with coefficient $-\prod_{i=2}^n Q_{1+2\delta_{i\ell}}\big(a_i + 3\delta_{i\ell}/2\big) Q_{g-n}(a_1)$.

- Consider the following seven cases together: $\psi_1^{g-n}\prod_{i=2}^n \psi_i^{1-\delta_{i\ell}}D_{\ell,n+1,n+2}\pi^*\psi_\ell^1(\psi_0+\psi_\ell+\psi_{n+1}+\psi_{n+2})$ and $\psi_1^{g-n}\prod_{i=2}^n \psi_i^{1-\delta_{i\ell}}D_{\ell,n+1,n+2}(D_{1,n+1}+D_{1,n+2}+D_{n+1,n+2})\pi^*\psi_\ell^1$. By Lemma 4.3 below, the total sum of their pushforwards is equal to $\psi_1^{g-n}\prod_{i=1}^n \psi_i$ with coefficient $\prod_{i=1}^n Q_{1+2\delta_{i\ell}}(a_i+\delta_{i\ell})Q_{g-n}(a_1)$. Note that this coefficient is always equal to 0, since $Q_3(2)=Q_3\left(\frac{3}{2}\right)=0$, but we included this term here and in equation (4.2) in any case in order to make the whole formula more transparent and homogeneous. The sum of this and the previous coefficient is equal to the seventh line in equation (4.2).

- The pushforward of the class $\psi_1^{g+1-n}\prod_{i=2}^n \psi_i^1\psi_{n+1}^1$, where, first, the map π'_* decreases the degree of ψ_{n+1}, so it becomes 0, and then the map π''_* decreases the degree of ψ_1, giving $\psi_1^{g-n}\prod_{i=2}^n \psi_i^1$ with coefficient $\prod_{i=2}^n Q_1(a_i)Q_{g+1-n}(a_1)Q_1\left(\frac{3}{2}\right)$.

- The pushforward of the class $\psi_1^{g+1-n}\prod_{i=2}^n \psi_i^1 D_{n+1,n+2}$, where the map π''_* decreases the degree of ψ_1, gives $\psi_1^{g-n}\prod_{i=2}^n \psi_i^1$ with coefficient $-\prod_{i=2}^n Q_1(a_i)\cdot Q_{g+1-n}(a_1)Q_1\left(\frac{3}{2}-\frac{1}{2}\right)$. The sum of this and the previous coefficient is equal to the fifth line of equation (4.2).

- The pushforward of the class $\psi_1^{g+1-n}\prod_{i=2}^n \psi_i^{1-\delta_{i\ell}}\psi_{n+1}^2$, where at the first step the map π'_* decreases the degree of ψ_1, gives $\psi_1^{g-n}\prod_{i=2}^n \psi_i^{1-\delta_{i\ell}}\kappa_1$ with coefficient $\prod_{i=2}^n Q_{1-\delta_{i\ell}}(a_i)\cdot Q_{g+1-n}(a_1)Q_2\left(\frac{3}{2}\right)$.

- The pushforward of the class $(\pi')^*(\psi_1^{g-n})\prod_{i=2}^n \psi_i^{1-\delta_{i\ell}}D_{1,n+2}\psi_{n+1}^2$ gives the term $\psi_1^{g-n}\prod_{i=2}^n \psi_i^{1-\delta_{i\ell}}\cdot\kappa_1$ with coefficient $-\prod_{i=2}^n Q_{1-\delta_{i\ell}}(a_i)Q_{g+1-n}\left(a_1-\frac{1}{2}\right)Q_2\left(\frac{3}{2}\right)$.

- The pushforward of the class $\psi_1^{g+1-n}\prod_{i=2}^n \psi_i^{1-\delta_{i\ell}}D_{\ell,n+1}(\pi'')^*\psi_\ell^1$, where at the first step the map π'_* decreases the degree of ψ_1, gives $\psi_1^{g-n}\prod_{i=2}^n \psi_i^1$ with coefficient $-\prod_{i=2}^n Q_{1-\delta_{i\ell}}(a_i)\cdot Q_{g+1-n}(a_1)Q_2\left(a_\ell+\frac{3}{2}\right)$.

- The pushforward of the class $(\pi')^*(\psi_1^{g-n})\prod_{i=2}^n \psi_i^{1-\delta_{i\ell}}D_{\ell,n+1}(\pi'')^*\psi_\ell^1 D_{1,n+2}$ gives $\psi_1^{g-n}\prod_{i=2}^n \psi_i^1$ with coefficient $\prod_{i=2}^n Q_{1-\delta_{i\ell}}(a_i)Q_{g+1-n}\left(a_1-\frac{1}{2}\right)Q_2\left(a_\ell+\frac{3}{2}\right)$. The sum over ℓ of this and the previous three coefficients is equal to the eighth line in equation (4.2).

Thus, we have explained how we obtain all terms in (4.2). Note that since $Q_{\geqslant 1}\left(-\frac{1}{2}\right)=0$, we can never have a non-trivial degree of ψ_{n+2} in our formulae. For the same reason, the degrees of $\psi_2,\ldots,\psi_n$ are bounded from above by 1 and the degree of ψ_{n+1} is bounded from above by 2. With this type of reasoning, it is easy to see by direct inspection that all other classes of degree $g+1$ do not contain any of the monomials $\psi_1^{g-n}\prod_{i=2}^n \psi_i$ and $\psi_1^{g-n}\prod_{i=2}^n \psi_i^{1-\delta_{i\ell}}\kappa_1$ for $\ell=2,\ldots,n$ with non-trivial coefficients in their pushforwards to $\mathcal{M}_{g,n}$. For instance, for an arbitrary a_ℓ, the class $(\pi'')^*(\psi_1^{g-n})\prod_{i=2}^n \psi_i^{1-\delta_{i\ell}}D_{\ell,n+2}(\pi')^*\psi_\ell^1 D_{1,n+1}$ gives as result $\psi_1^{g-n}\prod_{i=2}^n \psi_i^1$ with coefficient $\prod_{i=2}^n Q_{1-\delta_{i\ell}}(a_i)Q_{g+1-n}\left(a_1+\frac{3}{2}\right)Q_2\left(a_\ell-\frac{1}{2}\right)$. But since a_ℓ is either $\frac{1}{2}$ or 1 and $Q_2(0)=Q_2\left(\frac{1}{2}\right)=0$, this coefficient is equal to 0. $\qquad\square$

LEMMA 4.3. *Let the points 1, $n+1$, and $n+2$ have arbitrary primary fields α, β, and γ. Then the pushforward of the part of the class given by*

$$\prod_{i=2}^n \psi_i^{d_i}\left[D_{1,n+1,n+2}\pi^*\psi_1^{d_1}(\psi_0+\psi_1+\psi_{n+1}+\psi_{n+2})\right.$$

$$\left.+D_{1,n+1,n+2}(D_{1,n+1}+D_{1,n+2}+D_{n+1,n+2})\pi^*\psi_1^{d_1}\right]$$

is equal to $\prod_{i=1}^n \psi_i^{d_i}$ with coefficient $\prod_{i=2}^n Q_{d_i}(a_i)Q_{d_1+2}(\alpha+\beta+\gamma)$.

Proof. Indeed, the Givental formula for the deformed r-spin class (for a general r) in this case

implies that these seven summands have the following coefficients, up to a common factor:

$$\psi_0: \quad \left(R_{d_1+2}^{-1}\right)_{\alpha+\beta+\gamma}^{\alpha+\beta+\gamma-d_1-2} - \left(R_{d_1+1}^{-1}\right)_{\alpha+\beta+\gamma-1}^{\alpha+\beta+\gamma-d_1-2}\left(R_1^{-1}\right)_{r-1-(\alpha+\beta+\gamma)}^{r-2-(\alpha+\beta+\gamma)},$$

$$\psi_1: \quad -\left(R_{d_1+1}^{-1}\right)_{\alpha+\beta+\gamma-1}^{\alpha+\beta+\gamma-d_1-2}\left(R_1^{-1}\right)_{\alpha}^{\alpha-1},$$

$$\psi_{n+1}: \quad -\left(R_{d_1+1}^{-1}\right)_{\alpha+\beta+\gamma-1}^{\alpha+\beta+\gamma-d_1-2}\left(R_1^{-1}\right)_{\beta}^{\beta-1},$$

$$\psi_{n+2}: \quad -\left(R_{d_1+1}^{-1}\right)_{\alpha+\beta+\gamma-1}^{\alpha+\beta+\gamma-d_1-2}\left(R_1^{-1}\right)_{\gamma}^{\gamma-1},$$

$$D_{1,n+1}: \quad \left(R_{d_1+1}^{-1}\right)_{\alpha+\beta+\gamma-1}^{\alpha+\beta+\gamma-d_1-2}\left(R_1^{-1}\right)_{\alpha+\beta}^{\alpha+\beta-1},$$

$$D_{1,n+2}: \quad \left(R_{d_1+1}^{-1}\right)_{\alpha+\beta+\gamma-1}^{\alpha+\beta+\gamma-d_1-2}\left(R_1^{-1}\right)_{\alpha+\gamma}^{\alpha+\gamma-1},$$

$$D_{n+1,n+2}: \quad \left(R_{d_1+1}^{-1}\right)_{\alpha+\beta+\gamma-1}^{\alpha+\beta+\gamma-d_1-2}\left(R_1^{-1}\right)_{\gamma+\beta}^{\gamma+\beta-1}$$

(in addition to a common factor on the right-hand side we also omit the common factor $\pi^*\left(\psi_1^{d_1}\right)\prod_{i=2}^{n}\psi_i^{d_i}D_{1,n+1,n+2}$ on the left-hand side of this table).

After the substitution $r = \frac{1}{2}$, the first term above, $\left(R_{d_1+2}^{-1}\right)_{\alpha+\beta+\gamma}^{\alpha+\beta+\gamma-d_1-2}$, gives us the factor $Q_{d_1+2}(\alpha+\beta+\gamma)$, and times the common factor of $\prod_{i=2}^{n}Q_{d_i}(a_i)$, it is exactly the result we state in the lemma. We have to show that the other seven terms sum up to zero. Indeed, the other seven terms, after substitution $r = \frac{1}{2}$, are proportional to

$$Q_1\left(-\tfrac{1}{2}-\alpha-\beta-\gamma\right) + Q_1(\alpha) + Q_1(\beta) + Q_1(\gamma) - Q_1(\alpha+\beta) - Q_1(\alpha+\gamma) - Q_1(\gamma+\beta)\,.$$

Note that $Q_1\left(-\frac{1}{2}-x\right) = Q_1(x)$, so the expression above is proportional to

$$\left(\tfrac{1}{2}+\alpha+\beta+\gamma\right)(\alpha+\beta+\gamma) + \left(\tfrac{1}{2}+\alpha\right)(\alpha) + \left(\tfrac{1}{2}+\beta\right)(\beta) + \left(\tfrac{1}{2}+\gamma\right)(\gamma)$$
$$- \left(\tfrac{1}{2}+\alpha+\beta\right)(\alpha+\beta) - \left(\tfrac{1}{2}+\alpha+\gamma\right)(\alpha+\gamma) - \left(\tfrac{1}{2}+\beta+\gamma\right)(\beta+\gamma) = 0\,. \qquad \square$$

4.2 Special cases of the general formula

In this section, we use Lemma 4.2 in order to derive the formulae for c_0, c_1, and c_2. Since all our expressions are homogeneous (the sum of the indices of the polynomials Q is always equal to $g+1$), we define $\bar{Q}_m = (-2)^m Q_m$, for $m \geqslant 0$, to simplify notation.

We can substitute the values $\bar{Q}_1\left(\frac{3}{2}\right) = 3$, $\bar{Q}_1(1) = \frac{3}{2}$, $\bar{Q}_1\left(\frac{1}{2}\right) = \frac{1}{2}$, $\bar{Q}_1(0) = 0$, $\bar{Q}_2\left(\frac{5}{2}\right) = \frac{45}{4}$, $\bar{Q}_2(2) = \frac{15}{4}$, $\bar{Q}_2\left(\frac{3}{2}\right) = \frac{3}{4}$, $\bar{Q}_3\left(\frac{5}{2}\right) = \frac{15}{8}$, $\bar{Q}_3(2) = 0$ in equation (4.2). This gives use the following coefficients of $\bar{Q}_{g-n}(a_1)$, $\bar{Q}_{g+1-n}(a_1)$, and $\bar{Q}_{g+1-n}\left(a_1 - \frac{1}{2}\right)$, where we omit the global factor $\prod_{i=2}^{n}\bar{Q}_1(a_i)$:

in c_0: $\left(\frac{3}{2}g - \frac{9}{4} + \frac{3}{2}n\right)\bar{Q}_{g-n}(a_1) + \left(6g + \frac{3}{2} - 3n\right)\bar{Q}_{g+1-n}(a_1) + \left(-6g + 0 + 3n\right)\bar{Q}_{g+1-n}\left(a_1 - \frac{1}{2}\right),$

in c_1: $\left(\frac{3}{2}g - \frac{15}{4} + \frac{3}{2}n\right)\bar{Q}_{g-n}(a_1) + \left(6g + \frac{1}{2} - 3n\right)\bar{Q}_{g+1-n}(a_1) + \left(-6g + 1 + 3n\right)\bar{Q}_{g+1-n}\left(a_1 - \frac{1}{2}\right),$

in c_2: $\left(\frac{3}{2}g - \frac{21}{4} + \frac{3}{2}n\right)\bar{Q}_{g-n}(a_1) + \left(6g - \frac{1}{2} - 3n\right)\bar{Q}_{g+1-n}(a_1) + \left(-6g + 2 + 3n\right)\bar{Q}_{g+1-n}\left(a_1 - \frac{1}{2}\right).$

$$(4.3)$$

Note that the primary field a_1 has a different value in these three cases.

Furthermore, we are going to use that

$$\bar{Q}_{g+2-n}(a_1) - \bar{Q}_{g+2-n}\left(a_1 - \tfrac{1}{2}\right) = \frac{(a_1)\left(a_1 - \tfrac{1}{2}\right)\cdots(a_1 - g - 1 + n)}{(g+1-n)!}, \tag{4.4}$$

$$\bar{Q}_{g+2-n}(a_1+1) - \bar{Q}_{g+2-n}\left(a_1 + \tfrac{3}{2}\right) = \frac{-\left(a_1 + \tfrac{3}{2}\right)(a_1+1)\cdots\left(a_1 - g + \tfrac{1}{2} + n\right)}{(g+1-n)!}. \tag{4.5}$$

Let us combine these terms with the terms with $\bar{Q}_{g+1-n}$ computed above. In the case of c_0, the primary field a_1 is equal to $2g - 1 - n/2$. Then the sum of (4.3), (4.4), and (4.5) is equal to the following expression:

$$\frac{\left(2g - 1 - \tfrac{1}{2}n\right)\cdots\left(g - 2 + \tfrac{1}{2}n\right)}{(g+1-n)!} - \left(2g - 1 - \tfrac{1}{2}n\right)\bar{Q}_{g+1-n}\left(2g - \tfrac{3}{2} - \tfrac{1}{2}n\right)$$

$$- \left(4g + 1 - \tfrac{5}{2}n\right)\bar{Q}_{g+1-n}\left(2g - \tfrac{3}{2} - \tfrac{1}{2}n\right) + \left(4g + 1 - \tfrac{5}{2}n\right)\bar{Q}_{g+1-n}\left(2g - 1 - \tfrac{1}{2}n\right)$$

$$+ \left(2g + \tfrac{1}{2} - \tfrac{1}{2}n\right)\bar{Q}_{g+1-n}\left(2g - 1 - \tfrac{1}{2}n\right) - \frac{\left(2g + \tfrac{1}{2} - \tfrac{1}{2}n\right)\cdots\left(g - \tfrac{1}{2} + \tfrac{1}{2}n\right)}{(g+1-n)!}$$

$$= -\left(2g - 1 - \tfrac{1}{2}n\right)\frac{\left(2g - \tfrac{3}{2} - \tfrac{1}{2}n\right)\cdots\left(g - \tfrac{3}{2} + \tfrac{1}{2}n\right)}{(g-n)!}$$

$$+ \left(4g + 1 - \tfrac{5}{2}n\right)\frac{\left(2g - 1 - \tfrac{1}{2}n\right)\cdots\left(g - 1 + \tfrac{1}{2}n\right)}{(g-n)!}$$

$$- \left(2g + \tfrac{1}{2} - \tfrac{1}{2}n\right)\frac{\left(2g - \tfrac{1}{2} - \tfrac{1}{2}n\right)\cdots\left(g - \tfrac{1}{2} + \tfrac{1}{2}n\right)}{(g-n)!}$$

$$= \left(3g + \tfrac{5}{2} - 3n\right)\frac{\left(2g - 1 - \tfrac{1}{2}n\right)\cdots\left(g - 1 + \tfrac{1}{2}n\right)}{(g-n)!}$$

$$- \left(2g + \tfrac{1}{2} - \tfrac{1}{2}n\right)\frac{\left(2g - \tfrac{1}{2} - \tfrac{1}{2}n\right)\cdots\left(g - \tfrac{1}{2} + \tfrac{1}{2}n\right)}{(g-n)!}.$$

We can perform the same computation for c_1 and c_2. Recall in all three cases the term with $\bar{Q}_{g-n}$ and the overall coefficients $\prod_{i=2}^{n}\bar{Q}_1(a_i)$ in (4.2). We obtain the following expressions.

COROLLARY 4.4. *We have*

$$c_0 = \bar{Q}_1\left(\tfrac{1}{2}\right)^{n-1}\left[\left(\tfrac{3}{2}g - \tfrac{9}{4} + \tfrac{3}{2}n\right)\frac{\left(2g - \tfrac{1}{2} - \tfrac{1}{2}n\right)\cdots\left(g - 0 + \tfrac{1}{2}n\right)}{(g-n)!}\right.$$

$$- \left(2g + \tfrac{1}{2} - \tfrac{1}{2}n\right)\frac{\left(2g - \tfrac{1}{2} - \tfrac{1}{2}n\right)\cdots\left(g - \tfrac{1}{2} + \tfrac{1}{2}n\right)}{(g-n)!}$$

$$\left. + \left(3g + \tfrac{5}{2} - 3n\right)\frac{\left(2g - 1 - \tfrac{1}{2}n\right)\cdots\left(g - 1 + \tfrac{1}{2}n\right)}{(g-n)!}\right],$$

$$c_1 = \bar{Q}_1\left(\tfrac{1}{2}\right)^{n-2}\bar{Q}_1(1)\left[\left(\tfrac{3}{2}g - \tfrac{15}{4} + \tfrac{3}{2}n\right)\frac{\left(2g - 1 - \tfrac{1}{2}n\right)\cdots\left(g - \tfrac{1}{2} + \tfrac{1}{2}n\right)}{(g-n)!}\right.$$

$$- \left(2g + 0 - \tfrac{1}{2}n\right)\frac{\left(2g - 1 - \tfrac{1}{2}n\right)\cdots\left(g - 1 + \tfrac{1}{2}n\right)}{(g-n)!}$$

$$\left. + \left(3g + \tfrac{5}{2} - 3n\right)\frac{\left(2g - \tfrac{3}{2} - \tfrac{1}{2}n\right)\cdots\left(g - \tfrac{3}{2} + \tfrac{1}{2}n\right)}{(g-n)!}\right],$$

$$c_2 = \bar{Q}_1\left(\tfrac{1}{2}\right)^{n-3}\bar{Q}_1(1)^2\left[\left(\tfrac{3}{2}g - \tfrac{21}{4} + \tfrac{3}{2}n\right)\frac{\left(2g - \tfrac{3}{2} - \tfrac{1}{2}n\right)\cdots\left(g - 1 + \tfrac{1}{2}n\right)}{(g-n)!}\right.$$

$$- \left(2g - \tfrac{1}{2} - \tfrac{1}{2}n\right)\frac{\left(2g - \tfrac{3}{2} - \tfrac{1}{2}n\right)\cdots\left(g - \tfrac{3}{2} + \tfrac{1}{2}n\right)}{(g-n)!}$$

$$\left.+ \left(3g + \tfrac{5}{2} - 3n\right)\frac{\left(2g - 2 - \tfrac{1}{2}n\right)\cdots\left(g - 2 + \tfrac{1}{2}n\right)}{(g-n)!}\right].$$

4.3 Proof of the non-degeneracy

In this subsection, we prove Proposition 4.1. First, observe that $\bar{Q}_{g+1-n}(a_1) - \bar{Q}_{g+1-n}\left(a_1 - \tfrac{1}{2}\right)$ is equal to $(a_1)(a_1 - \tfrac{1}{2})\cdots(a_1 - g + n)/((g-n)!)$. We substitute $a_1 = 2g - 1 + \tfrac{1}{2}n$ for c_0 (respectively, $2g - \tfrac{3}{2} + \tfrac{1}{2}n$ for c_1 and $2g - 2 + \tfrac{1}{2}n$ for c_2) and combine the result of Corollary 4.4 and equation (4.1) in order to obtain the following formulae:

$$\tfrac{3}{4}\hat{c}_0 = \left(\tfrac{3}{2}g - \tfrac{9}{4} + \tfrac{3}{2}n\right)\frac{\left(2g - \tfrac{1}{2} - \tfrac{1}{2}n\right)}{\left(g - \tfrac{1}{2} + \tfrac{1}{2}n\right)\left(g - 1 + \tfrac{1}{2}n\right)} - \left(2g + \tfrac{1}{2} - \tfrac{1}{2}n\right)\frac{\left(2g - \tfrac{1}{2} - \tfrac{1}{2}n\right)}{\left(g - 1 + \tfrac{1}{2}n\right)} + \left(3g + \tfrac{5}{2} - 3n\right),$$

$$\tfrac{3}{4}\hat{c}_1 = \left(\tfrac{3}{2}g - \tfrac{15}{4} + \tfrac{3}{2}n\right)\frac{\left(2g - 1 - \tfrac{1}{2}n\right)}{\left(g - 1 + \tfrac{1}{2}n\right)\left(g - \tfrac{3}{2} + \tfrac{1}{2}n\right)} - \left(2g + 0 - \tfrac{1}{2}n\right)\frac{\left(2g - 1 - \tfrac{1}{2}n\right)}{\left(g - \tfrac{3}{2} + \tfrac{1}{2}n\right)} + \left(3g + \tfrac{5}{2} - 3n\right),$$

$$\tfrac{3}{4}\hat{c}_2 = \left(\tfrac{3}{2}g - \tfrac{21}{4} + \tfrac{3}{2}n\right)\frac{\left(2g - \tfrac{3}{2} - \tfrac{1}{2}n\right)}{\left(g - \tfrac{3}{2} + \tfrac{1}{2}n\right)\left(g - 2 + \tfrac{1}{2}n\right)} - \left(2g - \tfrac{1}{2} - \tfrac{1}{2}n\right)\frac{\left(2g - \tfrac{3}{2} - \tfrac{1}{2}n\right)}{\left(g - 2 + \tfrac{1}{2}n\right)} + \left(3g + \tfrac{5}{2} - 3n\right).$$

By an explicit computation, we obtain

$$\tfrac{3}{4}(\hat{c}_0 - 2\hat{c}_1 + \hat{c}_2) = \frac{S(g,n)}{\left(g - \tfrac{1}{2} + \tfrac{1}{2}n\right)\left(g - 1 + \tfrac{1}{2}n\right)\left(g - \tfrac{3}{2} + \tfrac{1}{2}n\right)\left(g - 2 + \tfrac{1}{2}n\right)},$$

where

$$S(g,n) = -g + \tfrac{11}{8}n - \tfrac{9}{4}g^2 + \tfrac{9}{8}gn - \tfrac{1}{2}g^3 + \tfrac{3}{4}g^2n - \tfrac{1}{4}n^3.$$

We want to prove that this polynomial is never equal to zero in the integer points (g,n) satisfying $3 \leqslant n \leqslant g-1$. We can make a change of variable $n = b+3$, $g = a+b+4$; then we want to prove that $S(a+b+4, b+3)$ never vanishes for any integer $a, b \geqslant 0$. This is indeed the case since all non-zero coefficients of the polynomial

$$S(a+b+4, b+3) = -\tfrac{201}{8} - \tfrac{173}{8}a - \tfrac{21}{2}b - 6a^2 - \tfrac{39}{8}ab - \tfrac{9}{8}b^2 - \tfrac{1}{2}a^3 - \tfrac{3}{4}a^2b$$

are negative, including the constant term. This completes the proof of Proposition 4.1.

5. The vanishing of $R^{\geqslant g}(\mathcal{M}_{g,n})$

In this section, we will give a new proof of the following theorem.

THEOREM 5.1 ([Loo95, Ion02]). *The tautological ring of $\mathcal{M}_{g,n}$ vanishes in degrees g and higher; that is, $R^{\geqslant g}(\mathcal{M}_{g,n}) = 0$.*

This theorem and Theorem 3.4 together constitute the generalized socle conjecture, as the bound $\dim R^{g-1}(\mathcal{M}_{g,n}) \geqslant n$ can be proved relatively simply; see, for example, [BSZ16]. This conjecture is a generalization of one of Faber's three conjectures on the tautological ring of $\mathcal{M}_g$; see [Fab99] for the original conjectures and [BSZ16] for the generalization.

The proof consists of three steps: in steps 1 and 2, we show that the pure ψ- and κ-classes

vanish, respectively, and in step 3 we reduce the mixed monomials to the pure cases. The first two steps will be proved in separate lemmata.

LEMMA 5.2. *Let $g \geqslant 0$ and $n \geqslant 1$. Any monomial in ψ-classes of degree at least $\max(g, 1)$ vanishes on $\mathcal{M}_{g,n}$.*

Remark 5.3. This lemma was originally conjectured by Getzler in [Get98].

Proof. For $g = 0$, this is well known; see, for example, [Zvo12, Proposition 2.13]. So let us assume $g \geqslant 1$.

We will prove that any monomial in ψ-classes of degree g vanishes. This clearly implies that any monomial of higher degree vanishes as well.

For this, look again at Ω^g, but now on $\overline{\mathcal{M}}_{g,n}$. When restricted to the open part $\mathcal{M}_{g,n}$, the only contributing graph is the one with one vertex of genus g, as the other graphs correspond to boundary divisors, by definition. Hence, the equation for the cohomological field theory reduces to

$$\Omega^g_{g,n}(a_1, \ldots, a_n)\big|_{\mathcal{M}_{g,n}} = \begin{cases} -\frac{1}{2} \prod_{i=1}^n \left(\sum_{m_i \geqslant 0} Q_{m_i}(a_i) \psi_i^{m_i} \right) & \text{if } \sum_{i=1}^n a_i = 2g - 1, \\ 0 & \text{else}. \end{cases}$$

We will prove the vanishing of all monomials using downward induction on the exponent d_1 of ψ_1, starting with the case $d_1 = g + 1$. This case trivially gives a zero, as this power cannot occur in a monomial of total degree g.

Now, assuming that all monomials with exponent of ψ_1 larger than d_1 vanish, consider the monomial $\psi_1^{d_1} \cdots \psi_n^{d_n}$ for any d_i summing up to g. For the relation, choose $a_i = d_i$ for all $i \neq 1$ and $a_1 = 2g - 1 - \sum_{i=2}^n a_i$. This means $Q_{m_i}(a_i) = 0$ unless $m_i \leqslant d_i$ or $i = 1$, so the only monomials with non-zero coefficients have exponent of ψ_i at most d_i for $i \neq 1$. Because the total degree is fixed, the only surviving monomial with exponent of ψ_1 equal to d_1 is the one we started with, and this relation expresses it in monomials with strictly larger exponent of ψ_1. By the induction hypothesis, this monomial must be zero. $\qquad\square$

Remark 5.4. Note that this argument breaks down for degrees lower than g, as the class does not vanish there. Therefore, to get relations in those degrees, one must push forward relations in higher degrees along forgetful maps *on the compactified moduli space*, which contain non-trivial contributions from boundary strata.

LEMMA 5.5. *Any multi-index κ-class of degree at least g vanishes on $\mathcal{M}_{g,n}$.*

Proof. Fix a degree $d \geqslant g$, and consider the pure (multi-index) κ-classes in this degree. Without loss of generality, we can assume the number of indices to be equal to d: this is certainly an upper bound, and adding an extra index zero only multiplies the class by a non-zero factor, using the dilaton equation on the definition of multi-index κ-classes.

We will consider $\Omega^g_{g,n+d}$. In order to get a relation in $R^d(\mathcal{M}_{g,n})$, we should multiply by a class σ of degree $2d - g$, push forward to $\overline{\mathcal{M}}_{g,n}$, and then restrict to $\mathcal{M}_{g,n}$. As we can now assume $d \geqslant g$, we have $2d - g \geqslant d$, and we can therefore choose $\sigma = \prod_{j=1}^d \psi_{n+j}^{f_j+1}$ with each $f_j \geqslant 0$. By choosing such a σ, we ensure that after pushforward and restriction to the open moduli space, none of the contributions from boundary divisors on $\overline{\mathcal{M}}_{g,n+d}$ survive and only the term with one vertex contributes.

We will use downward induction on the first index of the κ-class. The base case is a first index larger than d, and hence another index being negative, giving a trivial zero.

Now, assume that all κ-classes with first index larger than e_1 are zero. Fix a class $\kappa_{e_1,\ldots,e_d}$ of degree $d = \sum_{j=1}^d e_j$, choose a set of non-negative integers $\{a_j, f_j \,|\, 2 \leqslant j \leqslant d\}$ such that $a_j + f_j = e_j$, and set $a_1 = 2g - 1 - \sum_{j=2}^d a_j$ and $f_1 = 0$. We will consider

$$\pi_*^d\big(\sigma \cdot \Omega_{g,n+d}^g(0,\ldots,0,a_1,\ldots,a_d)\big)\big|_{\mathcal{M}_{g,n}} = \pi_*^d\left(\prod_{j=1}^d \psi_{n+j}^{f_j+1} \cdot -\frac{1}{2}\prod_{j=1}^d \sum_{m_j \geqslant 0} Q_{m_j}(a_j)\psi_{n+j}^{m_j}\right)\bigg|_{\mathcal{M}_{g,n}}$$

$$= -\frac{1}{2}\sum_{\substack{m_j \geqslant 0 \\ 1 \leqslant j \leqslant d}} \left(\prod_{j=1}^d Q_{m_j}(a_j)\right)\kappa_{f_1+m_1,\ldots,f_d+m_d}\,,$$

which vanishes. By our choice of a_j, for the product of Q-polynomials to be non-zero, we need $m_j \leqslant a_j$ for $j \neq 1$. Furthermore, by our choice of f_j, this shows that $f_j + m_j \leqslant e_j$ for $j \neq 1$. Because we look at a fixed degree d, this means $f_1 + m_1 \geqslant e_1$, with equality only occurring for $m_j = f_j$, with $j \neq 1$, and hence for the κ-class we started with. Hence this relation expresses our chosen class $\kappa_{e_1,\ldots,e_d}$ in terms of κ-classes with strictly higher first index, which we already know vanish. $\qquad\square$

Remark 5.6. Note that we cannot use the vanishing of the ψ-monomials in higher degrees and push these relations forward, as the κ-classes are defined by pushing forward ψ-classes on the compactified moduli space and then restricting to the open part, and not the other way around.

We are now ready to prove the theorem.

Proof of Theorem 5.1. For general monomial ψ-κ-classes, that is, classes of the form $\mu = \psi_i^{d_1}\cdots\psi_n^{d_n}\cdot\kappa_{e_1,\ldots,e_k}$, we will use induction on the total degree $d = \sum_{i=1}^n d_i + \sum_{j=1}^k e_j$. If all d_i are 0, we are in the case of Lemma 5.5, so we can assume that at least one of them is non-zero, that is, $\mu = \nu \cdot \psi_i$ for some i.

In degree $d = g$, we get that the degree of ν is $g-1$. By Proposition 3.1, we know that ν is a polynomial in ψ-classes. Therefore, so is $\mu = \nu \cdot \psi_i$. By Lemma 5.2, we know that μ vanishes.

For the induction step, we know by induction that ν is zero, hence μ is too. This finishes the proof of Theorem 5.1. $\qquad\square$

Because the proof of this theorem only uses the case $x = 0$ from Subsection 2.2, see also Subsection 2.4, and only fixes non-negative integer primary fields, all the relations are actually explicit on all of $\overline{\mathcal{M}}_{g,n}$. Hence, we get the following.

PROPOSITION 5.7. *The Pandharipande–Pixton–Zvonkine relations for $r = \frac{1}{2}$ give an algorithm for computing explicit tautological boundary formulae in the Chow ring for any tautological class on $\overline{\mathcal{M}}_{g,n}$ of codimension at least g. In particular, the intersection numbers of ψ-classes on $\overline{\mathcal{M}}_{g,n}$ can be computed with these relations for any $g \geqslant 0$ and $n \geqslant 1$ such that $2g - 2 + n > 0$.*

Remark 5.8. The first part of the statement is very similar to [CGJZ16, Theorem 5], which gave a reduction algorithm based on Pixton's double ramification cycle. It confirms an expectation on page 7 of [BJP15], that "(...)Pixton's relations are expected to uniquely determine the descendent theory, but the implication is not yet proven".

Note that the intersection numbers in ψ- and κ-classes can be expressed as intersection numbers of only ψ-classes by pulling back along forgetful maps; see [Zvo12, Corollary 3.23]. By Proposition 5.7, all these intersection numbers can then be computed using the PPZ relations.

Proof of Proposition 5.7. The first sentence follows by the comment above the proposition. For the second sentence, we will reduce polynomials in ψ-classes to smaller and smaller boundary strata using our explicit relation. This will be done in the form of an induction on $\dim \overline{\mathcal{M}}_{g,n} = 3g - 3 + n$, the 0-dimensional case $\overline{\mathcal{M}}_{0,3}$ being obvious.

For any $g_1 + g_2 = g$ and $I_1 \sqcup I_2 = \{1, \dots, n\}$ such that $2g_i + |I_i| - 1 > 0$, write $\rho_{I_1,I_2}^{g_1,g_2} : \overline{\mathcal{M}}_{g_1,|I_1|+1} \times \overline{\mathcal{M}}_{g_2,|I_2|+1} \to \overline{\mathcal{M}}_{g,n}$ for the attaching map, and $D_{I_1,I_2}^{g_1,g_2}$ for the divisor $(\rho_{I_1,I_2}^{g_1,g_2})_*(1)$. Similarly, write $\sigma \colon \overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n}$ for the gluing map, and δ_{irr} for $\sigma_*(1)$. Then these divisors together form the entire boundary of $\overline{\mathcal{M}}_{g,n}$, and $\rho^*(\psi_i) = \psi_i$ and $\sigma^*(\psi_i) = \psi_i$ for any choice of indices.

Now, let g and n be such that $3g - 3 + n > 0$, and choose a polynomial $p(\psi) \in R^{3g-3+n}(\overline{\mathcal{M}}_{g,n})$. Using stability, we see that $3g - 3 + n > g - 1$, so by Lemma 5.2, this class is zero on $\mathcal{M}_{g,n}$. Since the proof only uses relations without κ-classes, it can be given explicitly as a sum of the boundary divisors given above multiplied with other ψ-polynomials. By the projection formula, we have

$$\int_{\overline{\mathcal{M}}_{g,n}} \prod_{i=1}^n \psi_i^{d_i} D_{I_1,I_2}^{g_1,g_2} (\psi')^{d'} (\psi'')^{d''} = \int_{\overline{\mathcal{M}}_{g_1,|I_1|+1}} \left(\prod_{i \in I_1} \psi_i^{d_i} \right) \psi_{n+1}^{d'} \cdot \int_{\overline{\mathcal{M}}_{g_2,|I_2|+1}} \left(\prod_{i \in I_2} \psi_i^{d_i} \right) \psi_{n+2}^{d''},$$

$$\int_{\overline{\mathcal{M}}_{g,n}} \prod_{i=1}^n \psi_i^{d_i} \delta_{\mathrm{irr}} (\psi')^{d'} (\psi'')^{d''} = \int_{\overline{\mathcal{M}}_{g-1,n+2}} \left(\prod_{i=1}^n \psi_i^{d_i} \right) \psi_{n+1}^{d'} \psi_{n+2}^{d''},$$

where ψ' and ψ'' are the classes on the half-edges of the unique edge in the dual graphs of the divisors.

All spaces on the right-hand side have a strictly lower dimension, so by induction we can compute those numbers via the PPZ relations. $\qquad\square$

According to [CGJZ16, Section 3.5], Proposition 5.7 implies the following theorem.

COROLLARY 5.9 ([GV05, Theorem $\star$], *improved in* [FP05]). *Any codimension d tautological class can be expressed in terms of tautological classes supported on curves with at least $d - g + 1$ rational components.*

6. Dimensional bound for $R^{\leqslant g-2}(\mathcal{M}_{g,n})$

Similarly to [PPZ16, Theorem 6], our method also gives a bound for the dimension of the lower-degree tautological classes. For the statement of this proposition, recall that $p(n)$ denotes the number of partitions of n and $p(n,k)$ denotes the number of partitions of n of length at most k.

PROPOSITION 6.1. *We have*

$$\dim R^d(\mathcal{M}_{g,n}) \leqslant \sum_{k=0}^d \binom{n+k-1}{k} p(d-k, g-1-d).$$

Remark 6.2. If we use the natural interpretation of $\binom{k-1}{k}$ as $\delta_{k,0}$, this does indeed recover [PPZ16, Theorem 6] in the case $n = 0$.

Proof. We will exhibit an explicit spanning set of this cardinality, consisting of ψ-κ-classes, namely monomials in ψ-classes multiplied with a multi-index κ-class.

First, a less strict first bound can be obtained as follows: any ψ-κ-class is a product of factors ψ, of total degree k, and factors κ, of total degree $d - k$. There are $\binom{n+k-1}{k}$ different

monomials of degree k in n variables, and furthermore there are as many different multi-index κ-classes of degree $d - k$ as there are partitions of $d - k$, so $p(d - k)$. This gives the first bound

$$\dim R^d(\mathcal{M}_{g,n}) \leqslant \sum_{k=0}^{d} \binom{n + k - 1}{k} p(d - k) \,,$$

which is already close to the statement of the proposition.

To get the actual bound, we will show that any ψ-κ-class with at least $g - d$ κ-indices can be expressed in ψ-κ-classes with strictly fewer κ-indices. Following the logic of the previous paragraph, this proves the bound.

This reduction step is analogous to the proof of Lemma 5.5. Suppose that we have a class $\mu = \psi_1^{d_1} \cdots \psi_n^{d_n} \kappa_{e_1,\ldots,e_m}$ with $m \geqslant g - d$. Choose non-negative integers $\{f_i, a_i\}_{i=1}^{n+m}$ such that the following hold:

$$f_1 = 0 \,, \quad \sum_{i=1}^{n+m} f_i = d - g + m \,, \quad a_i + f_i = d_i \quad \text{for} \ 2 \leqslant i \leqslant n \,,$$

$$a_{n+j} + f_{n+j} = e_j + 1 \quad \text{for} \ 1 \leqslant j \leqslant m \,, \quad a_1 = 2g - 1 - \sum_{j=2}^{m} a_j \,.$$

Let $\sigma = \prod_{i=2}^{n+m} \psi_i^{f_i}$, and consider the class

$$\pi_*^m \big(\sigma \cdot \Omega_{g,n+m}^g (a_1, \ldots, a_{n+m})\big)\big|_{\mathcal{M}_{g,n}} \,.$$

By the second condition on our chosen numbers, which fixes the degree of σ, this expression gives a relation in $R^d(\mathcal{M}_{g,n})$.

There are no ψ-κ-classes with more than m κ-indices in this relation, and the coefficient of any ψ-κ-class with exactly m indices can only come from the open part of $\overline{\mathcal{M}}_{g,n+m}$, as each forgotten point must carry at least two ψ-classes, which would give too high degrees on any rational component. Therefore, the coefficient of $\psi_1^{p_1} \cdots \psi_n^{p_n} \kappa_{q_1,\ldots,q_m}$ must be $\prod_{i=1}^{n} Q_{p_i - f_i}(a_i) \cdot \prod_{j=1}^{m} Q_{q_j - f_{n+j}+1}(a_{n+j})$. This is only non-zero if $p_i \leqslant f_i + a_i = d_i$ for all $i \neq 1$ and $q_j \leqslant f_{n+j} + a_n + j - 1 = e_j$ for all j. This implies $p_1 \geqslant d_1$, with equality only if $p_i = d_i$ and $q_j = e_j$ for all i, j. Hence, this relation expresses the class μ as a linear combination of ψ-κ-classes with fewer than m κ-indices and ψ-κ-classes with strictly higher exponent of ψ_1. By induction on first the exponent of ψ_1 and then the number of κ-indices, all these classes can be reduced. $\qquad\square$

Remark 6.3. This argument breaks down for $m < g - d$, as the class σ would have to have a negative degree: our class only vanishes in degree at least g, and to get at most m-index κ-classes, we can only push forward m times, so the lowest degree relation would be in R^{g-m}.

The condition that partitions have length at most $g - 1 - d$ seems dual to Graber and Vakil's Theorem $\star$, Corollary 5.9; see [GV05, Theorem 1.1].

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