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On vector bundles over reducible curves with a node

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Abstract: Let C be a curve with two smooth components and a single node, and let $\mathcal{U}_C(w, r, \chi)$ be the moduli space of w-semistable classes of depth one sheaves on C having rank r on both components and Euler characteristic χ. In this paper, under suitable assumptions, we produce a projective bundle over the product of the moduli spaces of semistable vector bundles of rank r on each component and we show that it is birational to an irreducible component of $\mathcal{U}_C(w, r, \chi)$. Then we prove the rationality of the closed subset containing vector bundles with given fixed determinant.

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Introduction

Moduli spaces of vector bundles on curves have always been a central topic in Algebraic Geometry. The construction of moduli space of isomorphism classes of stable vector bundle of rank r and degree d on a smooth projective curve of genus $g \geq 2$ is due to Mumford; see [15]. Such a moduli space is a non-singular quasi-projective variety, whose compactification was obtained by Seshadri in [22], by introducing the S-equivalence relation between semistable vector bundles, and it is denoted by $\mathcal{U}_C(r, d)$. The compactification is a normal irreducible projective variety of dimension $r^2(g-1)+1$. When r and d are coprime, the notion of semistability coincides with that of stability, so $\mathcal{U}_C(r, d)$ parametrizes isomorphism classes of stable vector bundles. Moreover, in this case there exists a Poincaré bundle on $\mathcal{U}_C(r, d)$, see [20]. If $L \in \mathrm{Pic}^d(C)$ is a line bundle, the moduli space $\mathcal{S}U_C(r, L)$, parametrizing semistable vector bundles of rank r and fixed determinant L, is also of great interest. Indeed, up to a finite étale covering, the moduli space $\mathcal{U}_C(r, d)$ is isomorphic to the product of $\mathcal{S}U_C(r, L)$ and $\mathrm{Pic}^0(C)$. Hence, a lot of the geometry of $\mathcal{U}_C(r, d)$ is encoded in $\mathcal{S}U_C(r, L)$. Moreover, $\mathcal{S}U_C(r, L)$ is interesting on its own and it is a rational variety when r and d are coprime, see [14]. The geometry of these moduli spaces has been studied by many authors, in particular its relation with generalized theta functions; see [3] for a survey, and [9], [8], [7], [6], [5] and [11] for recent works by the authors.

Unfortunately, as soon as the base curve becomes singular, the above results do not apply anymore. For example, for a singular irreducible curve, in order to have a compact moduli space one possible approach consists in considering torsion-free sheaves instead of locally free, see [18] and [23]. This method was generalized for a reducible (but reduced) curve by Seshadri. The idea was to include in the moduli space also depth one sheaves and to introduce the notion of polarization w and of w-semistability. More precisely, we denote by $\mathcal{U}_C(w, r, \chi)$ the moduli space parametrizing w-semistable sheaves of depth one of rank r on each component and Euler characteristic χ.

In this paper we assume that C is a nodal reducible curve with two smooth irreducible components C_1 and C_2, of genera $g_i \geq 1$, with a single node p. We can obtain the curve by gluing C_1 and C_2 at the points q_1 and q_2. Under this hypothesis, the moduli space $\mathcal{U}_C(w, r, \chi)$ is a connected reducible projective variety, see

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[24] and [25]; each irreducible component has dimension $r^2(p_a(C)-1)+1$ and it corresponds to a possible pair of multidegree, see Section 2 for details. For problems about the stability of Kernel bundles on such curves the reader can see [10].

Under the above hypothesis, choose any $r \geq 2$ and fix a pair of integers (d_1, d_2) which are both coprime with r. The existence of Poincaré vector bundles on the moduli spaces $\mathcal{U}_{C_i}(r, d_i)$ allows us to produce a projective bundle $\pi : \mathbb{P}(\mathcal{F}) \to \mathcal{U}_{C_1}(r, d_1) \times \mathcal{U}_{C_2}(r, d_2)$, whose fiber at $([E_1], [E_2])$ is $\mathbb{P}(\mathrm{Hom}(E_{1,q_1}, E_{2,q_2}))$, see Lemma 3.1. Let $u \in \mathbb{P}(\mathcal{F})$, $u = (([E_1], [E_2]), [\sigma])$, where σ is a non-zero homomorphism $E_{1,q_1} \to E_{2,q_2}$. We can associate to u a depth one sheaf E_u on the curve C, which is obtained, roughly speaking, by gluing E_1 and E_2 along the fibers at q_1 and q_2 with σ. This is a vector bundle if and only if σ is an isomorphism. Our first concern is to study when E_u turns out to be w-semistable for some polarization w: we are able to give some necessary and sufficient conditions to ensure w-semistability (see Section 3). Then we turn our attention to the rational map

$$\varphi : \mathbb{P}(\mathcal{F}) \dashrightarrow \mathcal{U}_C(w, r, \chi)$$

sending u to E_u. Our first result (Theorem 4.1) can be summarized in the following statement:

Theorem A. *Let C be a reducible nodal curve as above. Let $r \geq 2$ and d_1 and d_2 be integers coprime with r. Set $\chi_i = d_i + r(1 - g_i)$ and $\chi = \chi_1 + \chi_2 - r$. For any pair (χ_1, χ_2) in a suitable non-empty subset of $\mathbb{Z}^2$ there exists a polarization w such that $\mathbb{P}(\mathcal{F})$ is birational to the irreducible component of the moduli space $\mathcal{U}_C(w, r, \chi)$ corresponding to the bidegree (d_1, d_2).*

The birational map of the statement is the map φ. We prove that it is an injective morphism on the open subset $\mathscr{U} \subset \mathbb{P}(\mathcal{F})$, given by points u where σ is an isomorphism. The image $\varphi(\mathscr{U})$ is a dense subset of the moduli space and its points are classes of vector bundles whose restriction to each component is stable (see Theorem 4.1). Moreover, when $g_i > r+1$, we can give some more information about the domain of φ as follows, see Theorem 4.3.

Theorem B. *Assume that the hypothesis of Theorem A holds. If $g_i > r + 1$, then for any pair (χ_1, χ_2) in a suitable non-empty subset of $\mathbb{Z}^2$ there exists a non-empty open subset $V_1 \times V_2$ of $\mathcal{U}_{C_1}(r, d_1) \times \mathcal{U}_{C_2}(r, d_2)$ and a polarization w such that $\varphi|_{\mathscr{U} \cup \mathscr{V}}$ is a morphism, where we set $\mathscr{V} = \pi^{-1}(V_1 \times V_2)$.*

Then, in analogy with the smooth case, for any $L \in \mathrm{Pic}(C)$ we define the variety $\mathcal{S}U_C(w, r, L)$ which is, roughly, the closure in $\mathcal{U}_C(w, r, \chi)_{d_1,d_2}$ of the locus parametrizing classes of vector bundles with fixed determinant L where $d_i = \deg(L|_{C_i})$. When r and d_i are coprime, as in the smooth case, we obtain the following result, see Theorem 5.2:

Theorem C. *Under the hypothesis of Theorem A, $\mathcal{S}U_C(w, r, L)$ is a rational variety.*

Recent results concerning rationality of these moduli spaces on reducible curves are obtained in [12] and [2] in the case of rank two, and in [4] for an integral irreducible nodal curve.

The paper is organized as follows. In Section 1 we fix notation about reducible nodal curves. In Section 2 we introduce the notion of depth one sheaves, of polarization and w-semistability and we recall general properties on their moduli spaces. In Section 3 we introduce the projective bundle $\mathbb{P}(\mathcal{F})$, we define the sheaf E_u associated to $u \in \mathbb{P}(\mathcal{F})$ and we study when it is w-semistable. In Section 4 we prove Theorems A and B. Finally, in Section 5 we deal with moduli spaces with fixed determinant and we prove Theorem C.

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1 Nodal reducible curves

In this paper we consider nodal reducible complex projective curves with two smooth irreducible components and one single node. Let C be such a curve; we consider a normalization map $\nu : C_1 \sqcup C_2 \to C$, where C_i is a smooth irreducible curve of genus $g_i \geq 1$. Hence $\nu^{-1}(x)$ is a single point except when x is the node p of C, in which case $\nu^{-1}(p) = \{q_1, q_2\}$ with $q_j \in C_j$. Since the restriction $\nu_{|C_i}$ is an isomorphism we identify C_1 and C_2 with the irreducible components of C.

Note that C can be embedded in a smooth surface X, on which C is an effective divisor $C = C_1 + C_2$ with $C_1 C_2 = 1$. Let $J_C = \mathcal{O}_X(-C)$ and $J_{C_i} = \mathcal{O}_X(-C_i)$ be the ideal sheaves of C and C_i respectively in X; then we have the inclusion $J_C \subset J_{C_i}$ and the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & \longrightarrow & \mathcal{O}_X(-C) & \longrightarrow & \mathcal{O}_X(-C_2) & \longrightarrow & \mathcal{O}_{C_1}(-C_2) \longrightarrow 0 \\
 & & & & \big\downarrow & & \big\downarrow & & \\
 & & & & \mathcal{O}_X & \xrightarrow{\simeq} & \mathcal{O}_X & & \\
 & & & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & J_{C_2}/J_C & \longrightarrow & \mathcal{O}_C & \longrightarrow & \mathcal{O}_{C_2} & \longrightarrow & 0
\end{array}$$

from which one deduces the isomorphism $J_{C_2}/J_C \simeq \mathcal{O}_{C_1}(-C_2)$. This gives the exact sequence

$$0 \to \mathcal{O}_{C_1}(-C_2) \to \mathcal{O}_C \to \mathcal{O}_{C_2} \to 0, \tag{1.1}$$

which is called the *decomposition sequence of* C. From it we can compute the Euler characteristic of $\mathcal{O}_C$:

$$\chi(\mathcal{O}_C) = \chi(\mathcal{O}_{C_1}(-C_2)) + \chi(\mathcal{O}_{C_2}).$$

Let $p_a(C) = 1 - \chi(\mathcal{O}_C)$ be the *arithmetic genus* of C. From the above relation we get that $p_a(C) = g_1 + g_2$.

Notation 1.1. We denote by $j_i \colon C_i \hookrightarrow C$ the natural inclusion of C_i in C, by $\mathcal{O}_{q_i}$ the stalk of $(j_i)_* \mathcal{O}_{C_i}$ in p and by $\mathcal{O}_p$ the stalk of $\mathcal{O}_C$ in p.

2 Moduli space of depth one sheaves

Let C be a smooth irreducible projective curve of genus $g \geq 1$. The moduli space of semistable vector bundles of rank r and degree d on C is denoted by $\mathcal{U}_C(r, d)$. Its points are S-equivalence classes of semistable vector bundles on the curve. We denote by $[E]$ the class of a vector bundle E. In [23] it is proved that $\mathcal{U}_C(r, d)$ is an irreducible and projective variety. Moreover, see [23] and [26], we have:

$$\dim \mathcal{U}_C(r, d) = \begin{cases} r^2(g-1) + 1 & g \geq 2 \\ \gcd(r, d) & g = 1. \end{cases} \tag{2.1}$$

In particular, when r and d are coprime, $\mathcal{U}_C(r, d)$ is a smooth variety, whose points parametrizes isomorphism classes of stable vector bundles. Moreover, for $g = 1$, we also have an isomorphism $\mathcal{U}_C(r, d) \simeq C$; see [1] and [26].

Let C be a nodal curve with a single node p and two smooth irreducible components C_1 and C_2. To construct compactifications of moduli spaces of vector bundles on C we introduce depth one sheaves, following the approach of Seshadri [23].

Definition 2.1. A coherent sheaf E on C is of *depth one* if every torsion section vanishes identically on some components of C.

A coherent sheaf E on C is of depth one if and only if the stalk at the node p is isomorphic to $\mathcal{O}_p^a \oplus \mathcal{O}_{q_1}^b \oplus \mathcal{O}_{q_2}^c$, see [23]. In particular, any vector bundle E on C is a sheaf of depth one. If E is a sheaf of depth one on C, then its restriction $E|_{C_i}$ is a torsion free sheaf on $C_i \setminus p$ (possibly identically zero). Moreover, any subsheaf of E is of depth one too.

Let E be a sheaf of depth one on C. We define the *relative rank* of E on the component C_i as the rank of the restriction $E_i = E_{|C_i}$ of E to C_i

$$r_i = \mathrm{Rk}(E_i) \tag{2.2}$$

and the *multirank* of E as the pair (r_1, r_2). We define the *relative degree* of E with respect to the component C_i as the degree of the restriction E_i

$$d_i = \deg(E_i) = \chi(E_i) - r_i\chi(\mathcal{O}_{C_i}), \tag{2.3}$$

where $\chi(E_i)$ is the Euler characteristic of E_i. The *multidegree* of E is the pair (d_1, d_2).

Definition 2.2. A *polarization* w of C is given by a pair of rational weights (w_1, w_2) such that $0 < w_i < 1$ and $w_1 + w_2 = 1$. For any sheaf E of depth one on C, of multirank (r_1, r_2) and $\chi(E) = \chi$, we define the *polarized slope* as

$$\mu_w(E) = \frac{\chi}{w_1 r_1 + w_2 r_2}.$$

Definition 2.3. Let E be a sheaf of depth one on C. E is called *w-semistable* if for any subsheaf $F \subseteq E$ we have $\mu_w(F) \leq \mu_w(E)$; E is called *w-stable* if $\mu_w(F) < \mu_w(E)$ for all proper subsheafs F of E.

For each w-semistable sheaf E of depth one on C there exists a finite filtration of sheaves of depth one on C:

$$0 = E^0 \subset E^1 \subset E^2 \subset \cdots \subset E^k = E$$

such that each quotient E^i/E^{i-1} is a w-stable sheaf of depth one on C with polarized slope $\mu_w(E^i/E^{i-1}) = \mu_w(E)$. This is called a *Jordan–Holder filtration* of E. The sheaf

$$Gr_w(E) = \bigoplus_{i=1}^{k} E^i/E^{i-1}$$

is called the *graduate sheaf associated to E* and it depends only on the isomorphism class of E. Let E and F be w-semistable sheaves of depth one on C. We say that E and F are S_w-equivalent if and only if $Gr_w(E) \simeq Gr_w(F)$. If E and F are w-stable sheaves then S_w-equivalence is just isomorphism, as in the smooth case.

There exists a moduli space $\mathcal{U}_C^s(w, (r_1, r_2), \chi)$ parametrizing isomorphism classes of w-stable sheaves of depth one on C of multirank (r_1, r_2) and given Euler characteristic χ, see [23]. It has a natural compactification $\mathcal{U}_C(w, (r_1, r_2), \chi)$, whose points correspond to S_w-equivalence classes of w-semistable sheaves of depth one on C of multirank (r_1, r_2) and given Euler characteristic χ. In particular, when $r_1 = r_2 = r$, we denote by $\mathcal{U}_C(w, r, \chi)$ the corresponding moduli space. In this case we have the following result (see [24] and [25]):

Theorem 2.1. *Let C be a nodal curve with a single node p and two smooth irreducible components C_i of genus $g_i \geq 1$, $i = 1, 2$. For a generic polarization w we have the following properties:*

(1) *any w-stable vector bundle $E \in \mathcal{U}_C(w, r, \chi)$ satisfies the following condition:*

$$w_i\chi(E) \leq \chi(E_i) \leq w_i\chi(E) + r, \tag{2.4}$$

where E_i is the restriction of E to C_i;

(2) *if a vector bundle E on C satisfies the above condition for $i = 1, 2$ and the restrictions E_1 and E_2 are semistable vector bundles, then E is w-semistable. Moreover, if at least one of the restrictions is stable, then E is w-stable;*

(3) *the moduli space $\mathcal{U}_C(w, r, \chi)$ is connected, each irreducible component has dimension $r^2(p_a(C) - 1) + 1$ and it corresponds to the choice of a multidegree (d_1, d_2) satisfying Conditions* 2.4.

Definition 2.4. We denote by $\mathcal{U}_C(w, r, \chi)_{d_1,d_2}$ the irreducible component of $\mathcal{U}_C(w, r, \chi)$ corresponding to the multidegree (d_1, d_2).

3 Construction of depth one sheaves

In this section we deal with the construction of depth one sheaves on a nodal curve C with two irreducible components and a single node. We begin with the following lemma:

Lemma 3.1. *Let C_1 and C_2 be smooth complex projective curves of genus $g_i \geq 1$, $i = 1, 2$, and $q_i \in C_i$. Fix $r \geq 2$ and $d_1, d_2 \in \mathbb{Z}$ such that r is coprime with both d_1 and d_2. Then there exists a projective bundle*

$$\pi : \mathbb{P}(\mathcal{F}) \to \mathcal{U}_{C_1}(r, d_1) \times \mathcal{U}_{C_2}(r, d_2)$$

such that the fiber over $([E_1], [E_2])$ is $\mathbb{P}(\mathrm{Hom}(E_{1,q_1}, E_{2,q_2}))$, where E_{i,q_i} is the fiber of E_i at the point q_i.

Proof. As r and d_i are coprime, there exists a Poincaré bundle $\mathcal{P}_i$ for the moduli space of semistable vector bundles on C_i of rank r and degree d_i, i.e. a vector bundle $\mathcal{P}_i$ on $\mathcal{U}_{C_i}(r, d_i) \times C_i$ such that $\mathcal{P}_i|_{[E_i]\times C_i} \simeq E_i$, under the identification $[E_i] \times C_i \simeq C_i$. This follows from a result of [20] if $g_i \geq 2$ and from the isomorphism $\mathcal{U}_{C_i}(r, d_i) \simeq C_i$ when $g_i = 1$. For $i = 1, 2$, consider the natural inclusion

$$\iota_i : \mathcal{U}_{C_i}(r, d_i) \times q_i \hookrightarrow \mathcal{U}_{C_i}(r, d_i) \times C_i,$$

and the pull back $\iota_i{}^*(\mathcal{P}_i)$ of the Poincaré bundle. Since $\mathcal{U}_{C_i}(r, d_i) \times q_i$ is isomorphic to $\mathcal{U}_{C_i}(r, d_i)$, $\iota_i{}^*(\mathcal{P}_i)$ can be seen as a vector bundle on $\mathcal{U}_{C_i}(r, d_i)$ of rank r whose fiber at $[E_i]$ is actually E_{i,q_i}.

Note that the product $\mathcal{U}_{C_1}(r, d_1) \times \mathcal{U}_{C_2}(r, d_2)$ is a smooth irreducible variety. Let p_1 and p_2 denote the projections of the product onto factors. We define on $\mathcal{U}_{C_1}(r, d_1) \times \mathcal{U}_{C_2}(r, d_2)$ the following sheaf:

$$\mathcal{F}: = \mathcal{H}om(p_1^*(\iota_1{}^*(\mathcal{P}_1)), p_2^*(\iota_2{}^*(\mathcal{P}_2))). \tag{3.1}$$

By construction, $\mathcal{F}$ is a vector bundle of rank r^2 whose fiber at the point $([E_1], [E_2])$ is $\mathrm{Hom}(E_{1,q_1}, E_{2,q_2})$. By taking the associated projective bundle we conclude the proof. □

Let C_1 and C_2 be smooth irreducible curves. We consider a nodal curve C with two smooth components and a single node p which is obtained by identifying the points $q_1 \in C_1$ and $q_2 \in C_2$. Let E_i be a stable vector bundle of rank r and degree d_i on C_i and consider a non-zero homomorphism $\sigma : E_{1,q_1} \to E_{2,q_2}$ between the fibres. Assume that the rank of σ is k, with $1 \leq k \leq r$. We can associate to these data a depth one sheaf on the nodal curve C, roughly speaking, by gluing the vector bundles E_1 and E_2 along the fibers (at q_1 and q_2 respectively) with the homomorphism σ, as follows:

Let j_p be the inclusion of p in C and let $j_i : C_i \to C$ be the inclusion of C_i in C for $i = 1, 2$. The sheaf $j_{i*}E_i$ is a depth one sheaf on C whose stalk at p is the stalk of E_i at q_i. Hence, there is a natural surjective map given by restriction onto the fiber of E_i at q_i, i.e. the map

$$\rho_i : j_{i*}E_i \to E_{i,q_i}.$$

The sheaf $j_{1*}(E_1) \oplus j_{2*}(E_2)$ is of depth one on C and we have a surjective map

$$\rho_1 \oplus \rho_2 : j_{1*}E_1 \oplus j_{2*}E_2 \to E_{1,q_1} \oplus E_{2,q_2}.$$

The sheaf $j_{p*}j_p{}^*j_{2*}(E_2)$ has depth one too, and it is a skyscraper sheaf over p whose stalk is E_{2,q_2}. So we have again a surjective map

$$\rho : j_{p*}j_p{}^*j_{2*}(E_2) \to E_{2,q_2}.$$

Let $\sigma : E_{1,q_1} \to E_{2,q_2}$ be a non-zero homomorphism and consider the induced surjective map

$$\sigma \oplus id : E_{1,q_1} \oplus E_{2,q_2} \to \mathrm{Im}(\sigma) \oplus E_{2,q_2}.$$

We have, moreover, the map

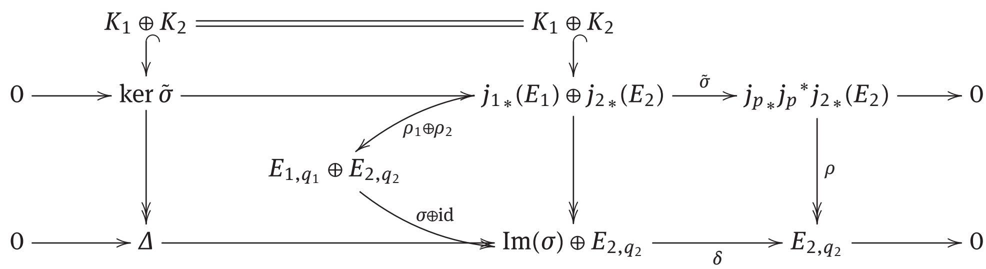
$$\delta : \mathrm{Im}(\sigma) \oplus E_{2,q_2} \to E_{2,q_2}$$

which sends (u, v) to $u - v$. We denote by $\Delta \subset \mathrm{Im}(\sigma) \oplus \mathrm{Im}(\sigma)$ the diagonal. By construction we have $\Delta \simeq \mathbb{C}_p^k$.

Finally we define the map of sheaves

$$\tilde{\sigma} : {j_1}_*(E_1) \oplus {j_2}_*(E_2) \to {j_p}_* {j_p}^* {j_2}_*(E_2)$$

by requiring that the following diagram commutes.



(3.2)

It follows immediately by construction that $\ker \tilde{\sigma}$ is a sheaf of depth one on C, which coincides with E_i on $C_i \setminus p$. One can easily see that the isomorphism class of $\ker \tilde{\sigma}$ does not depend on the isomorphism classes of the E_i. Moreover, the same happens if one uses $\sigma' = \lambda \sigma$ with $\lambda \in \mathbb{C}^*$, instead of σ.

From now on, we assume that the hypothesis of Lemma 3.1 holds. Let $\mathbb{P}(\mathcal{F})$ be the projective bundle on $\mathcal{U}_{C_1}(r, d_1) \times \mathcal{U}_{C_2}(r, d_2)$. We can conclude that the construction of $\ker \tilde{\sigma}$ depends on the data contained in $u = (([E_1], [E_2]), [\sigma]) \in \mathbb{P}(\mathcal{F})$ and not on the particular choices of E_1, E_2 and σ.

Definition 3.1. We denote by E_u the kernel of $\tilde{\sigma}$ defined by $u \in \mathbb{P}(\mathcal{F})$.

The above construction gives the following:

Proposition 3.2. *Let E_u be the sheaf defined by $u = (([E_1], [E_2]), [\sigma]) \in \mathbb{P}(\mathcal{F})$. Then E_u is a depth one sheaf on C with $\chi(E_u) = \chi(E_1) + \chi(E_2) - r$ and multirank (r, r). It is a vector bundle if and only if σ is an isomorphism. In this case, $E_{u|C_i} = E_i$.*

Proof. Let $\mathrm{Rk}(\sigma) = k$. Since E_u is a depth one sheaf, the stalk of E_u at the node p is isomorphic to $\mathcal{O}_p^a \oplus \mathcal{O}_{q_1}^b \oplus \mathcal{O}_{q_1}^c$ where $a + b = \mathrm{Rk}(E_u|C_1) = r$ and $a + c = \mathrm{Rk}(E_u|C_2) = r$ (see Section 2). From the diagram 3.2, it follows that the rank of the free part of the stalk of E_u in p is k, so $a = k$. Hence we have $E_u|_p \simeq \mathcal{O}_p^k \oplus \mathcal{O}_{q_1}^{r-k} \oplus \mathcal{O}_{q_2}^{r-k}$. In particular, E_u is a vector bundle if and only if $k = r$, i.e. exactly when σ is an isomorphism. □

In order to obtain a w-semistable sheaf, for some polarization w, the following condition is necessary:

Lemma 3.3. *Let $E = E_u$ be the sheaf defined by $u = (([E_1], [E_2]), [\sigma]) \in \mathbb{P}(\mathcal{F})$ and let k be the rank of σ. If E is w-semistable for some w, then the following conditions are satisfied:*

$$\chi(E)w_1 \le \chi(E_1) \le \chi(E)w_1 + k \quad \textit{and} \quad \chi(E)w_2 + r - k \le \chi(E_2) \le \chi(E)w_2 + r. \tag{3.3}$$

Proof. Assume that E is w-semistable for a polarization w. Let K_1 be the kernel of the map

$$\sigma \circ \rho_1 : {j_1}_* E_1 \to \mathrm{Im}\, \sigma,$$

and let K_2 be the kernel of the map $\rho_2 : {j_2}_* E_2 \to E_{2,q_2}$ as in diagram 3.2. Since K_i is a subsheaf of E, by w-semistability of E we have $\mu_w(K_i) \le \mu_w(E)$. We also have $\mu_w(K_1) = \frac{\chi(K_1)}{w_1 r} = \frac{\chi(E_1) - k}{w_1 r} \le \frac{\chi(E)}{r}$, which implies

$$\chi(E_1) \le \chi(E)w_1 + k.$$

By replacing $\chi(E_1) = \chi(E) - \chi(E_2) + r$ in the above inequality, we obtain

$$\chi(E_2) \ge \chi(E)w_2 + r - k.$$

Finally, we have $\mu_w(K_2) = \frac{\chi(K_2)}{w_2 r} = \frac{\chi(E_2) - r}{w_2 r} \le \frac{\chi(E)}{r}$, which implies

$$\chi(E_2) \le \chi(E)w_2 + r.$$

Again, by replacing $\chi(E_2) = \chi(E) - \chi(E_1) + r$ we obtain $\chi(E_1) \ge \chi(E)w_1$. □

Given $u = (([E_1], [E_2]), [\sigma])$ and E_u defined by u, we wonder if there exists a polarization w such that the above Conditions 3.3 hold. The answer depends only on numerical assumptions on $(\chi(E_1), \chi(E_2))$ and $\mathrm{Rk}\,\sigma$, as the following lemma shows.

Lemma 3.4. *Let $r \geq 2$ and $1 \leq k \leq r$ be integers. There exists a non-empty subset $\mathcal{W}_{r,k} \subset \mathbb{Z}^2$ such that for any pair $(\chi_1, \chi_2) \in \mathcal{W}_{r,k}$ we can find a polarization w satisfying the conditions*

$$\chi w_1 \leq \chi_1 \leq \chi w_1 + k \quad \text{and} \quad \chi w_2 + r - k \leq \chi_2 \leq \chi w_2 + r, \qquad \text{where } \chi = \chi_1 + \chi_2 - r. \tag{3.4}$$

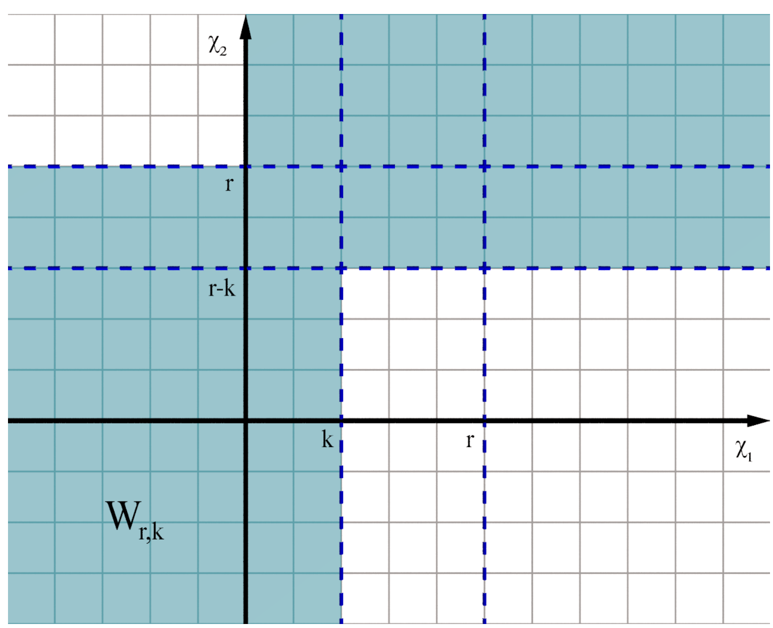
Proof. Note that if $\chi = 0$, i.e. $\chi_1 + \chi_2 = r$ and we assume that $0 \leq \chi_1 \leq r$, then any polarization w satisfies Conditions 3.4. We distinguish two cases according to the sign of χ. Assume that $\chi > 0$. Then there exists a polarization w satisfying Conditions 3.4, if and only if the following system has solutions:

$$\frac{\chi_1 - k}{\chi} \leq w_1 \leq \frac{\chi_1}{\chi}, \qquad \frac{\chi_2 - r}{\chi} \leq w_2 \leq \frac{\chi_2 + k - r}{\chi}, \qquad w_1 + w_2 = 1, \qquad 0 < w_i < 1, w_i \in \mathbb{Q}.$$

This occurs if and only if $\chi_1 > 0$ and $\chi_2 > r - k$. Likewise, if $\chi < 0$, then we have the system

$$\frac{\chi_1}{\chi} \leq w_1 \leq \frac{\chi_1 - k}{\chi}, \qquad \frac{\chi_2 - r + k}{\chi} \leq w_2 \leq \frac{\chi_2 - r}{\chi}, \qquad w_1 + w_2 = 1, \qquad 0 < w_i < 1, w_i \in \mathbb{Q},$$

which has solutions if and only if $\chi_1 < k$ and $\chi_2 < r$. □



Remark 3.1. Let $\mathcal{W}_r = \bigcap_{k=1}^{r} \mathcal{W}_{r,k}$. Note that it is a non-empty subset and it is actually $\mathcal{W}_{r,1}$. Moreover, if $(\chi_1, \chi_2) \in \mathcal{W}_r$, then by the proof of Lemma 3.4 it follows that we can find a polarization w which satisfies the Conditions 3.4 for all $k = 1, \dots, r$.

Assume that $\mathrm{Rk}\,\sigma = r$, i.e. E is a vector bundle. Then the necessary conditions of Lemma 3.3 are the same in Theorem 2.1. Hence, by the above theorem, they are also sufficient to give w-semistability of E. So we obtain the following:

Corollary 3.5. *Let $E = E_u$ be the sheaf defined by $u = (([E_1], [E_2]), [\sigma]) \in \mathbb{P}(\mathcal{F})$. Assume that $\mathrm{Rk}\,\sigma = r$ and $(\chi(E_1), \chi(E_2)) \in \mathcal{W}_{r,r}$. Then there exists a polarization w such that E is w-semistable. In particular, since the E_i are stable, then E is w-stable too.*

Unfortunately, when E_u fails to be a vector bundle, the necessary conditions of Lemma 3.3 are not enough to ensure w-semistability, see [25] for an example. Nevertheless, we are able to produce an open subset of $\mathcal{U}_{C_1}(r, d_1) \times \mathcal{U}_{C_1}(r, d_1)$ such that for every u over this open subset, the sheaf E_u is w-semistable.

We recall the following definition, see [16].

Definition 3.2. Let G be a vector bundle on a smooth curve. For every integer k we set

$$\mu_k(G) = \frac{\deg(G) + k}{\mathrm{Rk}(G)}.$$

A vector bundle G is called (m, k)-*semistable* (respectively *stable*) if for any subsheaf F we have

$$\mu_m(F) \leq \mu_{m-k}(G) \quad \text{(respectively } <).$$

Proposition 3.6. *Let $E = E_u$ be the sheaf defined by $u = (([E_1], [E_2]), [\sigma]) \in \mathbb{P}(\mathcal{F})$. Assume that $\mathrm{Rk}\,\sigma = k \leq r-1$. If $(\chi(E_1), \chi(E_2)) \in \mathcal{W}_{r,k}$, E_1 is $(0, k)$-semistable and E_2 is $(0, r)$-semistable, then there exists a polarization w such that E is w-semistable. Moreover, if E_1 is $(0, k)$-stable or E_2 is $(0, r)$- stable, then E is w-stable too.*

Proof. Since $(\chi(E_1), \chi(E_2)) \in \mathcal{W}_{r,k}$, by Lemma 3.4 there exists a polarization w such that the necessary Conditions 3.3 hold. We claim that if E_1 is $(0, k)$-semistable and E_2 is $(0, r)$-semistable, then E is w-semistable.

Let $F \subset E$ be a subsheaf; it is a sheaf of depth one too. Assume that F has multirank (s_1, s_2) and that at the node p the stalk of F is $\mathcal{O}_p^s \oplus \mathcal{O}_{q_1}^a \oplus \mathcal{O}_{q_2}^b$ with $s \geq 0$, $s_1 = s + a \leq r$ and $s_2 = s + b \leq r$. Since $\mathrm{Rk}\,\sigma = k$, by construction the free part of the stalk of E at p is $\mathcal{O}_p^k$. This implies that $0 \leq s \leq k$.

By construction, there exist two vector bundles $F_1 \subseteq E_1$ and $F_2 \subseteq E_2$ such that F is the kernel of the restriction of $\tilde{\sigma}$ to the subsheaf $j_{1*}(F_1) \oplus j_{2*}(F_2)$:

$$\tilde{\sigma}_{|j_{1*}(F_1) \oplus j_{2*}(F_2)} : j_{1*}(F_1) \oplus j_{2*}(F_2) \to j_{p*} j_p^* j_{2*}(E_2).$$

Proceding as in the diagram 3.2, we deduce that F fits into an exact sequence as follows:

$$0 \to G_1 \oplus G_2 \to F \to \mathbb{C}_p^s \to 0,$$

where G_1 is the kernel of $(\sigma \circ \rho_1)|_{F_1}$ and G_2 is the kernel of $\rho_2|_{F_2}$. Hence $G_i \subseteq K_i$. Note that if $s = 0$, then actually $F \simeq G_1 \oplus G_2$.

For any s, we compute the w-slope of F:

$$\mu_w(F) = \frac{\chi(F)}{w_1 s_1 + w_2 s_2} = \frac{\chi(G_1) + \chi(G_2) + s}{w_1 s_1 + w_2 s_2} = \frac{deg(G_1) + s_1(1 - g_1) + \deg(G_2) + s_2(1 - g_2) + s}{w_1 s_1 + w_2 s_2}.$$

Since E_1 is $(0, k)$-semistable, we have

$$\frac{deg(G_1)}{s_1} \leq \frac{d_1 - k}{r}.$$

Since E_2 is $(0, r)$-semistable, $E_2(-q_2)$ is $(0, r)$-semistable too, so we have

$$\frac{deg(G_2)}{s_2} \leq \frac{d_2 - 2r}{r}.$$

By replacing we obtain:

$$\begin{aligned} \mu_w(F) \leq \frac{1}{w_1 s_1 + w_2 s_2} \left[s_1 w_1 \left(\frac{(d_1 - k) + r(1 - g_1)}{w_1 r} \right) + s_2 w_2 \left(\frac{(d_2 - r) + r(1 - g_2)}{w_2 r} \right) + s - s_2 \right] = \\ = \frac{s_1 w_1}{w_1 s_1 + w_2 s_2} \mu_w(K_1) + \frac{s_2 w_2}{w_1 s_1 + w_2 s_2} \mu_w(K_2) + \frac{s - s_2}{w_1 s_1 + w_2 s_2}. \end{aligned} \tag{3.5}$$

By Lemma 3.3 we have $\mu_w(K_i) \leq \mu_w(E)$, so we obtain:

$$\mu_w(F) \leq \mu_w(E) + \frac{s - s_2}{w_1 s_1 + w_2 s_2}.$$

Since $s - s_2 \leq 0$, we have $\mu_w(F) \leq \mu_w(E)$.

Finally, if E_1 is $(0, k)$-stable or E_2 is $(0, r)$-stable, then the above inequality is strict. □

Note that, by definition, if E_i is $(0, r)$-stable, then it is also $(0, k)$-stable for all $k \leq r$.

Lemma 3.7. *Let $\mathcal{U}_{C_i}(r, d_i)$ be the moduli space of semistable vector bundles of rank r and degree d_i on a smooth curve C_i of genus g_i. If d_i and r are coprime and $g_i > r + 1$, then the locus of vector bundles of $\mathcal{U}_{C_i}(r, d_i)$ which are $(0, r)$-stable is a non-empty open subset of $\mathcal{U}_{C_i}(r, d_i)$.*

Proof. We consider the locus

$$Y = \{[E] \in \mathcal{U}_{C_i}(r, d_i) \mid E \text{ is not } (0, r) - \text{stable}\}$$

and the subset $Y_{a,s}$ of Y given by all stable vector bundles E which can be written as $0 \to F \to E \to Q \to 0$, where F is a subbundle of E with $\deg(F) = a$ and $\mathrm{Rk}(F) = s \leq r - 1$ and

$$\mu(E) - 1 = \mu_{-r}(E) \leq \mu(F) \leq \mu_0(E) = \mu(E).$$

A deformation argument (see the proof of Proposition 1.4 of [21]) shows that if $Y_{a,s} \neq \emptyset$, then for a general E in $Y_{a,s}$ both F and Q are stable. Moreover, since E is stable, we have $\mathrm{Hom}(Q, F) = 0$. Hence we can write

$$\begin{aligned}\dim Y_{a,s} &\leq \dim \mathcal{U}_{C_i}(s, a) + \dim \mathcal{U}_{C_i}(r - s, d_i - a) + \dim H^1(C_i, \mathcal{H}om(Q, F)) - 1 = \\ &= (g_i - 1)(r^2 - rs + s^2) + 1 + (d_i s - ar).\end{aligned}$$

Hence

$$\dim \mathcal{U}_{C_i}(r, d_i) - \dim Y_{a,s} \geq (g_i - 1)(rs - s^2) - (d_i s - ar).$$

Since $E \in Y$, we have $\mu_0(F) \geq \mu_{-r}(E)$, i.e.

$$\frac{a}{s} \geq \frac{d_i - r}{r},$$

which implies $d_i s - ar \leq rs$. Finally, if $g_i > 1 + r$, then for all $s \leq r - 1$ we have

$$\dim \mathcal{U}_{C_i}(r, d_i) - \dim Y_{a,s} \geq s[(g_i - 1)(r - s) - r] > 0,$$

which concludes the proof. □

4 Main results

In this section we prove our main results. We assume that the hypothesis of Lemma 3.1 is satisfied. Let $\mathbb{P}(\mathcal{F})$ be the projective bundle on $\mathcal{U}_{C_1}(r, d_1) \times \mathcal{U}_{C_2}(r, d_2)$. For $1 \leq k \leq r - 1$ let $\mathcal{B}_k$ be the subset of $\mathbb{P}(\mathcal{F})$ such that

$$\mathcal{B}_k \cap \pi^{-1}([E_1], [E_2]) = \{[\sigma] \in \mathbb{P}(\mathrm{Hom}(E_{1,q_1}, E_{2,q_2})) \mid \mathrm{Rk}(\sigma) \leq k\}.$$

It is a proper closed subvariety of $\mathbb{P}(\mathcal{F})$.

Definition 4.1. We denote by $\mathscr{U}$ the open subset given by the complement of $\mathcal{B}_{r-1}$ in $\mathbb{P}(\mathcal{F})$.

Remark 4.1. Note that $\dim \mathscr{U} = \dim \mathbb{P}(\mathcal{F}) = r^2(g_1 + g_2 - 1) + 1$. Denote by $\pi_{\mathscr{U}}$ the restriction of π to $\mathscr{U}$. By construction,

$$\pi_{\mathscr{U}} : \mathscr{U} \to \mathcal{U}_{C_1}(r, d_1) \times \mathcal{U}_{C_2}(r, d_2)$$

is a fiber bundle whose fibers are isomorphic to $\mathrm{PGL}(r)$. More precisely,

$$\pi_{\mathscr{U}}^{-1}([E_1], [E_2]) = \mathbb{P}(\mathrm{GL}(E_{1,q_1}, E_{2,q_2})).$$

For $\chi = d_1 + d_2 + r(1 - g_1 - g_2)$, let $\mathcal{U}_C(w, r, \chi)_{d_1,d_2}$ be the irreducible component of the moduli space of depth one sheaves on C of rank r and characteristic χ corresponding to the multidegree (d_1, d_2); see Section 2. Let $\mathcal{V}_C(w, r, \chi)_{d_1,d_2} \subset \mathcal{U}_C(w, r, \chi)_{d_1,d_2}$ be the subset parametrizing classes of vector bundles.

Theorem 4.1. *Let C be a nodal curve with a single node p and two smooth irreducible components C_i of genus $g_i \geq 1$. Fix $r \geq 2$. For any $d_i \in \mathbb{Z}$ we set $\chi_i = d_i + r(1 - g_i)$ and $\chi = d_1 + d_2 + r(1 - g_1 - g_2)$. Assume that r is coprime with both d_1 and d_2 and that $(\chi_1, \chi_2) \in \mathcal{W}_{r,r}$. Then there exists a polarization w such that the map*

$$\varphi\colon \mathbb{P}(\mathcal{F}) \dashrightarrow \mathcal{U}_C(w, r, \chi)_{d_1,d_2}$$

sending u to $[E_u]$ is birational. In particular, the restriction $\varphi|_{\mathscr{U}}$ is a an injective morphism and the image $\varphi(\mathscr{U})$ is contained in $\mathcal{V}_C(w, r, \chi)_{d_1,d_2}$.

Proof. Let $u = (([E_1], [E_2]), [\sigma]) \in \mathbb{P}(\mathcal{F})$ and consider the sheaf $E = E_u$ defined by u, as in Section 3. Since $(\chi_1, \chi_2) \in \mathcal{W}_{r,r}$, as a consequence of Lemma 3.4 and Corollary 3.5 there exists a polarization w such that E_u is w-semistable for every $u \in \mathscr{U}$. This gives a point in the moduli space $\mathcal{U}_C(w, r, \chi)_{d_1,d_2}$ and it shows that φ is well defined at least on $\mathscr{U}$.

We prove that $\varphi_{|\mathscr{U}}$ is injective. Let $u = (([E_1], [E_2]), [\sigma])$ and $u' = (([E_1'], [E_2']), [\sigma'])$ in $\mathscr{U}$ with $\varphi(u) = [E]$ and $\varphi(u') = [E']$. Assume that $\varphi(u) = \varphi(u')$. Since E and E' are both w-stable and are in the same S_w-equivalence class, they have to be isomorphic (see Section 2). Let $\tau : E \to E'$ be an isomorphism. This induces an isomorphism $\tau_i : E_i \to E_i'$. So we can assume that $E_i' = E_i$; thus $\sigma, \sigma' : E_{1,q_1} \to E_{2,q_2}$ and $\tau_i : E_i \to E_i$ are isomorphisms. As E_p (respectively E_p') is obtained by glueing E_{1,q_1} with E_{2,q_2} along the isomorphism σ (respectively along σ'), the τ_i have to satisfy a compatibility condition which is summarized in the following commutative diagram:

$$\begin{array}{ccc} E_{1,q_1} & \xrightarrow{\ \sigma\ } & E_{2,q_2} \\ {\scriptstyle (\tau_1)_{q_1}}\downarrow & & \downarrow{\scriptstyle (\tau_2)_{q_2}} \\ E_{1,q_1} & \xrightarrow[\ \sigma'\ ]{} & E_{2,q_2} \end{array}$$

Since E_i is stable we have $\mathrm{Hom}(E_i, E_i) \simeq \mathbb{C} \cdot \mathrm{id}_{E_i}$. Hence $(\tau_i)_{q_i}$ is the multiplication by some $\lambda_i \in \mathbb{C}^*$. In particular, σ' is a non-zero multiple of σ and thus $[\sigma] = [\sigma']$.

Now we prove that $\varphi_{|\mathscr{U}}$ is a morphism. It is enough to prove that φ is regular at u_0, for any $u_0 \in \mathscr{U}$. For this, we claim that there exists a non-empty open subset $W \subseteq \mathscr{U}$ with $u_0 \in W$ and a vector bundle $\mathcal{E}$ on $W \times C$ such that

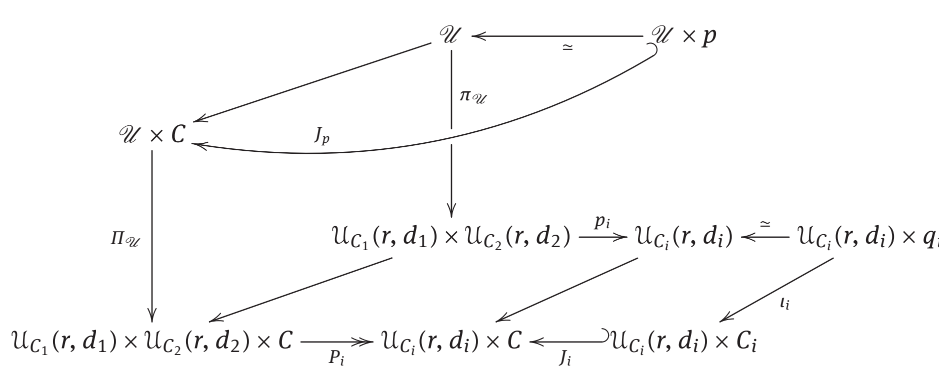
$$[\mathcal{E}|_{u \times C}] = \varphi(u) \quad \text{for all } u \in W.$$

Step 1: There exist two sheaves $\mathcal{Q}$ and $\mathcal{R}$ on $\mathscr{U} \times C$ such that for each $u = (([E_1], [E_2], [\sigma]) \in \mathscr{U}$ we have

$$\mathcal{Q}|_{u \times C} \simeq j_{1*}(E_1) \oplus j_{2*}(E_2), \quad \mathcal{R}_{|u \times C} \simeq j_{p*}(j_p{}^*(j_{2*}(E_2))),$$

where $j_p : p \hookrightarrow C$ and $j_i : C_i \hookrightarrow C$ are the natural inclusions.

Consider the diagram



(4.1)

where the morphisms which appear have been defined as

$$J_i = \mathrm{id}_{\mathcal{U}_{C_i}(r,d_i)} \times j_i, \qquad P_i = p_i \times \mathrm{id}_C, \qquad \Pi_{\mathscr{U}} = \pi_{\mathscr{U}} \times \mathrm{id}_C, \quad J_p = \mathrm{id}_{\mathscr{U}} \times j_p. \tag{4.2}$$

As before, we denote with $\mathcal{P}_i$ the Poincaré bundle on $\mathcal{U}_{C_i}(r, d_i) \times C_i$ and we set

$$\mathcal{Q}_i = \Pi^*_{\mathscr{U}}(P_i{}^*(J_{i*}(\mathcal{P}_i))), \quad \mathcal{Q} = \mathcal{Q}_1 \oplus \mathcal{Q}_2 \quad \text{and} \quad \mathcal{R} = J_{p*}(J_p{}^*(Q_2)).$$

Note that $\mathrm{Supp}(\mathcal{R}) = \mathscr{U} \times p$. Moreover, one can verify that if we identify $\mathscr{U} \times p$ with $\mathscr{U}$ we have

$$J_p^*(\mathcal{Q}_i) \simeq \pi^*_{\mathscr{U}}(p_i^*(\iota_i^*\mathcal{P}_i)), \tag{4.3}$$

where $\iota_i : \mathcal{U}_{C_i}(r, d_i) \times q_i \hookrightarrow \mathcal{U}_{C_i}(r, d_i) \times C_i$.

Step 2: There is an open subset $W \subset \mathscr{U}$ containing u_0 and a surjective map of sheaves

$$\mathcal{Q}_1 \oplus \mathcal{Q}_2|_{W\times C} \xrightarrow{\Sigma_W} \mathcal{R}|_{W\times C}$$

whose kernel is the desired vector bundle $\mathcal{E}$ on $W \times C$.

Let $\pi : \mathbb{P}(\mathcal{F}) \to \mathcal{U}_{C_1}(r, d_1)\times\mathcal{U}_{C_2}(r, d_2)$ be the projective bundle defined in Lemma 3.1. Consider on $\mathbb{P}(\mathcal{F})$ the tautological line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{F})}(-1)$ which is, by definition, the subsheaf of $\pi^*(\mathcal{F})$ whose fiber at $u \in \mathbb{P}(\mathcal{F})$ is

$$\mathrm{Span}(\sigma) \subset \mathrm{Hom}(E_{1,q_1}, E_{2,q_2}),$$

where $u = (([E_1], [E_2]), [\sigma])$. We can choose W to be an open subset of $\mathscr{U}$ containing the point u_0 and admitting a section $s \in \mathcal{O}_{\mathbb{P}(\mathcal{F})}(-1)(W)$ with $s(u) \neq 0$ for any $u \in W$.

In particular, s induces a map of sheaves

$$s : \pi^*_{\mathscr{U}} p_1^*(\iota_1^*(\mathcal{P}_1)))|_W \to \pi^*_{\mathscr{U}} p_2^*(\iota_2^*(\mathcal{P}_2)))|_W \tag{4.4}$$

such that $s_u : E_{1,q_1} \to E_{2,q_2}$ is an isomorphism and $[s_u] = [\sigma]$ in $\mathbb{P}(\mathrm{Hom}(E_{1,q_1}, E_{2,q_2}))$. We can also define a morphism of sheaves

$$s - \mathrm{id}_2 : \pi^*_{\mathscr{U}} p_1^*(\iota_1^*(\mathcal{P}_1)))|_W \oplus \pi^*_{\mathscr{U}} p_2^*(\iota_2^*(\mathcal{P}_2)))|_W \to \pi^*_{\mathscr{U}} p_2^*(\iota_2^*(\mathcal{P}_2)))|_W \tag{4.5}$$

where id_2 is the identity of $\pi^*_{\mathscr{U}} p_2^*(\iota_2^*(\mathcal{P}_2)))|_W$.

This allows us to define the map Σ_W we are looking for. Indeed, since $\mathrm{Supp}(\mathcal{R}|_{W\times C}) = W \times p$, it is enough to give the map on $W \times p$, which can be identified with W. Using the isomorphism 4.3, we have a diagram which defines Σ_W:

$$\begin{array}{ccc}
\mathcal{Q}_1 \oplus \mathcal{Q}_2|_{W\times C} & \xrightarrow{\Sigma_W} & \mathcal{R}|_{W\times C} \\
\downarrow{\scriptstyle |_{W\times p}} & & \downarrow{\scriptstyle |_{W\times p}} \\
J_p^*(\mathcal{Q}_1 \oplus \mathcal{Q}_2|_{W\times C}) & \xrightarrow{\Sigma_W|_{W\times p}} & J_p^*(\mathcal{R}|_{W\times C}) \\
\downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} \\
\pi^*_{\mathscr{U}} p_1^*(\iota_1^*(\mathcal{P}_1)))|_W \oplus \pi^*_{\mathscr{U}} p_2^*(\iota_2^*(\mathcal{P}_2)))|_W & \xrightarrow[s-\mathrm{id}_2]{} & \pi^*_{\mathscr{U}} p_2^*(\iota_2^*(\mathcal{P}_2)))|_W
\end{array}$$

By taking the kernel $\mathcal{E}$ of this map we conclude the second step of the proof of the claim. In particular, $\varphi_{|\mathscr{U}}$ is a morphism.

By construction, $\varphi(\mathscr{U})$ is contained in $\mathcal{V}_C(w, r, \chi)_{d_1,d_2}$ and it coincide with the open subset of w-semistable vector bundles whose restrictions are semistable. Moreover, $\mathcal{V}_C(w, r, \chi)_{d_1,d_2}$ is a dense open subset of $\mathcal{U}_C(w, r, \chi)_{d_1,d_2}$, see [23]. By Remark 4.1 we have

$$\dim(\varphi(\mathscr{U})) = \dim(\mathscr{U}) = r^2(g_1 + g_2 - 1) + 1,$$

which is the dimension of $\mathcal{U}_C(w, r, \chi)_{d_1,d_2}$, see Theorem 2.1. This implies that φ is a dominant map. Hence, by a generic smoothness argument, we can conclude that $\varphi_{|\mathscr{U}}$ is a birational morphism. □

Corollary 4.2. *Let C be a nodal curve with a single node p and two smooth irreducible components C_i of genus $g_i \geq 1$. Assume that the moduli space $\mathcal{U}_C(w, r, \chi)$ has an irreducible component corresponding to the bidegree (d_1, d_2) with d_1 and d_2 coprime with r. Then this component is birational to a projective bundle over the smooth variety $\mathcal{U}_{C_1}(r, d_1) \times \mathcal{U}_{C_2}(r, d_2)$.*

Note that φ provides a desingularization of the component $\mathcal{U}_C(w, r, \chi)_{d_1,d_2}$. If the genus of the curve C_i is big enough, we can be more precise about the domain of the rational map φ. If $g_i > r + 1$, then by Lemma 3.7 the locus of vector bundles of $\mathcal{U}_{C_i}(r, d_i)$ which are $(0, r)$-stable is a non-empty open subset of $\mathcal{U}_{C_i}(r, d_i)$; let us denote it by V_i.

Definition 4.2. We denote by $\mathscr{V}$ the open subset $\pi^{-1}(V_1 \times V_2)$ in $\mathbb{P}(\mathcal{F})$.

By construction, $\mathscr{V}$ is a projective bundle over $V_1 \times V_2$.

Theorem 4.3. *Assume that the hypothesis of Theorem 4.1 holds. Moreover, let $g_i > r + 1$ and $(\chi_1, \chi_2) \in \mathcal{W}_r$. Then there exists a polarization w such that the map φ sending u to $[E_u]$ is a birational map such that $\varphi|_{\mathscr{U} \cup \mathscr{V}}$ is a morphism.*

Proof. Since $(\chi_1, \chi_2) \in \mathcal{W}_r$, by Remark 3.1 there exists a polarization w such that the Conditions 3.4 hold for any $k = 1, \ldots, r$. In particular, as $\mathcal{W}_r \subset \mathcal{W}_{r,r}$, Theorem 4.1 holds: φ is a birational map which is defined on the open subset $\mathscr{U}$.

Assume that $u \in \mathscr{V}$ and $u \notin \mathcal{U}$. Then $u = (([E_1], [E_2]), [\sigma])$, with $([E_1], [E_2]) \in V_1 \times V_2$ and $\operatorname{Rk} \sigma \leq r - 1$. Since $[E_i] \in V_i$, Lemma 3.6 implies that E_u is w-semistable, hence φ is defined all over the open subset $\mathscr{V}$ too. To prove that $\varphi|_{\mathscr{V}}$ is a morphism, we can proceed as in the proof of Theorem 4.1, just by replacing $\mathscr{U}$ with $\mathscr{V}$ and $\mathcal{U}_{C_i}(r, d_i)$ with V_i. □

5 Fixed-determinant moduli space

Let C be a smooth curve of genus $g \geq 1$ and $L \in \operatorname{Pic}^d(C)$. We recall that the moduli space of semistable vector bundles of rank r and determinant L on C is denoted by $\mathcal{S}U_C(r, L)$ and it is an irreducible and projective variety. It is the fiber of the determinant map

$$\det\colon \mathcal{U}_C(r, d) \to \operatorname{Pic}^d(C).$$

In this section we investigate a similar subvariety of the moduli space $\mathcal{U}_C(w, r, \chi)_{d_1,d_2}$ for a nodal reducible curve with two irreducible components C_i. Fix a pair (L_1, L_2) with $L_i \in \operatorname{Pic}^{d_i}(C_i)$. Note that there exists a unique line bundle L on the nodal curve C whose restriction to the component C_i is L_i. Recall that $\mathcal{V}_C(w, r, \chi)_{d_1,d_2} \subset \mathcal{U}_C(w, r, \chi)_{d_1,d_2}$ is the open subset parametrizing w-semistable classes which are represented by vector bundles.

Definition 5.1. Let L be the line bundle on C that is induced by the pair (L_1, L_2). We define $\mathcal{S}U_C(w, r, L)$ as the closure of

$$\{[E] \in \mathcal{V}_C(w, r, \chi)_{d_1,d_2} \mid \det E = L\}$$

in $\mathcal{U}_C(w, r, \chi)_{d_1,d_2}$.

If we assume that r and d_i are coprime, then $\mathcal{S}U_{C_i}(r, L_i)$ is a smooth irreducible projective variety of dimension $(r^2 - 1)(g_i - 1)$. As in Lemma 3.1, we can define a vector bundle $\mathcal{F}_L$ on $\mathcal{S}U_{C_1}(r, L_1) \times \mathcal{S}U_{C_2}(r, L_2)$ just by restricting $\mathcal{F}$. Then we can consider the associated projective bundle $\mathbb{P}(\mathcal{F}_L)$ and

$$\mathscr{U}_L = \mathscr{U} \cap \mathbb{P}(\mathcal{F}_L),$$

a $\operatorname{PGL}(r)$-bundle on $\mathcal{S}U_{C_1}(r, L_1) \times \mathcal{S}U_{C_2}(r, L_2)$. We denote by φ_L the restriction of the morphism φ defined in Theorem 4.1 to $\mathscr{U}_L$. As a consequence of Theorem 4.1, we have the following:

Corollary 5.1. *Under the hypothesis of Theorem* 4.1, *the map*

$$\varphi_L : \mathbb{P}(\mathcal{F}_L) \dashrightarrow \mathcal{S}U_C(w, r, L)$$

is a birational map, whose restriction $\varphi_L|_{\mathcal{U}_L}$ *is an injective morphism.*

Proof. $\varphi_L|_{\mathcal{U}_L}$ is a morphism and its image is the set $\operatorname{Im}\varphi_L = \{E \in \mathcal{V}_C(w, r, \chi)_{d_1,d_2} \mid [E_{|C_i}] \in \mathcal{S}U_{C_i}(r, L_i)\}$. In particular, $\operatorname{Im}\varphi_L \subseteq \mathcal{S}U_C(w, r, L)$. Consider the map

$$\psi : \mathcal{V}_C(w, r, \chi)_{d_1,d_2} \to \mathrm{Pic}^{d_1}(C_1) \times \mathrm{Pic}^{d_2}(C_2),$$

sending E to $(\det(E|_{C_1}), \det(E|_{C_2}))$, which fits into the following commutative diagramm:

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\quad\varphi\quad} & \mathcal{V}_C(w, r, \chi)_{d_1,d_2} \\ {\scriptstyle \pi_{\mathcal{U}}}\downarrow & & \downarrow{\scriptstyle \psi} \\ \mathcal{U}_{C_1}(r, d_1) \times \mathcal{U}_{C_2}(r, d_2) & \xrightarrow[\det_1 \times \det_2]{} & \mathrm{Pic}^{d_1}(C_1) \times \mathrm{Pic}^{d_2}(C_2) \end{array} \tag{5.1}$$

It follows immediately that ψ is a surjective morphism and that $\operatorname{Im}\varphi_L \subset \psi^{-1}(L_1, L_2)$.

We claim that ψ has irreducible fibers of dimension $(r^2 - 1)(g_1 + g_2 - 1)$.

First we prove that any two fibers of ψ are isomorphic. If (L_1, L_2) and (L_1', L_2') are in $\mathrm{Pic}^{d_1}(C_1) \times \mathrm{Pic}^{d_2}(C_2)$, then there exist $\xi_i \in \mathrm{Pic}^0(C_i)$ such that $L_i \otimes \xi_i^r \simeq L_i'$. Let ξ be the unique line bundle on C such that $\xi_{|C_i} \simeq \xi_i$. The natural map

$$\psi^{-1}(L_1, L_2) \to \psi^{-1}(L_1', L_2')$$

sending E to $E \otimes \xi$ preserves w-semistability and gives an isomorphism of the fibers. In particular, with the fiber dimension theorem (see [13], p.95) this implies that any fiber has pure dimension $(r^2 - 1)(g_1 + g_2 - 1)$.

Finally we prove that any fiber is irreducible. Let $Y = \mathcal{V}_C(w, r, \chi)_{d_1,d_2} \setminus \varphi(\mathcal{U})$; it is a proper subvariety of $\mathcal{V}_C(w, r, \chi)_{d_1,d_2}$. Assume that the fiber of ψ over (L_1, L_2) is reducible, and let F_1 be the irreducible component containing $\varphi(\mathcal{U}_L)$. Then there exists an irreducible component $F_2 \subset Y$. So the restriction of ψ to Y is a surjective morphism whose fibers have dimension $(r^2-1)(g_1+g_2-1)$. This implies that $\dim Y = \dim \mathcal{V}_C(w, r, \chi)_{d_1,d_2}$, which is impossible.

This allows us to conclude that $\mathcal{S}U_C(w, r, L)$ is irreducible too and φ_L is a birational morphism. □

Theorem 5.2. *Under the hypothesis of Theorem* 4.1, $\mathcal{S}U_C(w, r, L)$ *is a rational variety.*

Proof. By hypothesis d_i and r are coprime, hence the moduli space $\mathcal{S}U_{C_i}(r, L_i)$ is rational for any line bundle $L_i \in \mathrm{Pic}^{d_i}(C_i)$, see [14], [17] and [19]. Since $\mathcal{U}_L$ is a $\mathbb{P}^{r^2-1}$-bundle over the product $\mathcal{S}U_{C_1}(r, L_1) \times \mathcal{S}U_{C_2}(r, L_2)$, it is a rational variety too. The assertion follows from Corollary 5.1. □

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