# The McKay Conjecture and central isomorphic character triples

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**Abstract**

We refine the reduction theorem for the McKay Conjecture proved by Isaacs, Malle and Navarro. Assuming the inductive McKay condition, we obtain a strong version of the McKay Conjecture that gives central isomorphic character triples.

## 1 Introduction

The McKay Conjecture is one of the major problems in representation theory of finite groups. It states that, if $p$ is a prime number and $P$ is a Sylow $p$-subgroup of a finite group $G$, then

$$|\mathrm{Irr}_{p'}(G)| = |\mathrm{Irr}_{p'}(\mathbf{N}_G(P))|,$$

where for any finite group $X$ we denote by $\mathrm{Irr}_{p'}(X)$ the set of irreducible complex characters of $G$ whose degree is not divisible by $p$. In [IMN07] Isaacs, Malle and Navarro prove a reduction theorem for the McKay Conjecture and show that the conjecture holds for every finite group with respect to the prime $p$ provided that the so-called inductive McKay condition holds for every finite non-abelian simple group with respect to the prime $p$.

The inductive McKay condition requires the existence of a bijection as the one predicted by the McKay Conjecture which gives central isomorphic character triples and is compatible with the action of automorphisms. Although this condition was originally thought for quasi-simple groups, it can be stated for arbitrary finite groups.

**Conjecture A.** *Let $G \trianglelefteq A$ be finite groups, $p$ a prime and $P$ a Sylow $p$-subgroup of $G$. Then there exists an $\mathbf{N}_A(P)$-invariant subgroup $\mathbf{N}_G(P) \leq M \leq G$, with $M < G$ whenever $P$ is not normal in $G$, and an $\mathbf{N}_A(P)$-equivariant bijection*

$$\Omega : \mathrm{Irr}_{p'}(G) \to \mathrm{Irr}_{p'}(M)$$

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*such that*

$$(A_\chi, G, \chi) \geq_c (M\mathbf{N}_A(P)_\chi, M, \Omega(\chi)),$$

*for every* $\chi \in \mathrm{Irr}_{p'}(G)$.

Observe that the above statement could equivalently be stated by taking $M = \mathbf{N}_G(P)$. However this additional flexibility is fundamental when proving the result for quasi-simple groups. It's also worth noting that, by using [Spä18, Theorem 2.16], it's no loss of generality to assume $A = G \rtimes \mathrm{Aut}(G)$.

The reduction theorem of Isaacs, Malle and Navarro can now be stated by saying that if Conjecture A holds for every universal covering group of finite non-abelian simple groups, then the McKay Conjecture holds for every finite group.

The first attempt to prove a reduction theorem for the local-global conjectures was made in [Dad97] in the context of Dade's Projective Conjecture. According to Dade's philosophy, there should exist a refinement of the conjecture that is strong enough to hold for every finite group when verified for quasi-simple groups. In the case of Dade's Projective Conjecture such a refinement should be found in the inductive form of Dade's conjecture [Dad97, 5.8] (see also [Spä17, Conjecture 1.2] and the introduction of [Ros22]). The aim of this paper is to show that Conjecture A provides the sought refinement in the case of the McKay Conjecture. We recall that a group $S$ is said to be *involved* in $G$ if there exists $N \trianglelefteq K \leq G$ such that $S \simeq K/N$.

**Theorem B.** *Let $G$ be a finite group and $p$ a prime. Suppose that Conjecture A holds at the prime $p$ for the universal covering group of every non-abelian simple group involved in $G$. Then Conjecture A holds for $G$ at the prime $p$.*

The above result is inspired by [NS14, Theorem 7.1] and provides evidence for the validity of Dade's philosophy. Furthermore, these kinds of results are extremely important in representation theory of finite groups. For instance, [NS14, Theorem 7.1] is used to obtain a reduction theorem for Brauer's Height Zero Conjecture (see [NS14, Theorem A]).

By work of Malle and Späth [MS16], Conjecture A is known to hold at the prime $p = 2$ for every universal covering group of finite non-abelian simple groups. For odd primes $p$, Conjecture A is know for almost all quasi-simple groups except possibly in certain cases when considering groups of Lie type $\mathbf{D}$ and ${}^2\mathbf{D}$ (see [Mal08], [Spä12], [CS13], [CS17a], [CS17b], [CS19] but also [Tay18]). The verification of Conjecture A for these remaining open cases has been initiated in [Spä21] and [Spä23]. In particular, the results of [Spä23] complete the verification for the prime $p = 3$. As a consequence of these results and of Theorem B, we obtain the following corollary.

**Corollary C.** *Conjecture A holds for $p = 2$ and $p = 3$.*

The paper is structured as follows: in Section 2 we introduce some preliminary results on character triples while in Section 3, assuming the inductive McKay condition, we obtain good bijections for groups whose quotient over the centre is isomorphic to a direct product of non-abelian simple groups. In the final section we prove Theorem B by inspecting the structure of a minimal counterexample.

## 2 Preliminaries on character triples

Let $G$ be a finite group, $N \trianglelefteq G$ and $\vartheta \in \mathrm{Irr}(N)$. If $\vartheta$ is $G$-invariant, then $(G, N, \vartheta)$ is a character triple. We are going to use the partial order relation $\geq_c$ on character triples as defined in [Nav18, Definition 10.14] and [Spä18, Definition 2.7]. Recall that this definition requires the existence of a pair of projective representation $(\mathcal{P}, \mathcal{P}')$ associated with two character triples $(G, N, \vartheta)$ and $(H, M, \varphi)$ satisfying certain properties. Whenever we want to specify the choice of the projective representations we say that $(G, N, \vartheta) \geq_c (H, M, \varphi)$ via $(\mathcal{P}, \mathcal{P}')$ or that $(\mathcal{P}, \mathcal{P}')$ gives $(G, N, \vartheta) \geq_c (H, M, \varphi)$. In this case we say that $(G, N, \vartheta)$ and $(H, M, \varphi)$ are central isomorphic character triples. First, we need the following version of [NS14, Theorem 3.14] which shows the compatibility of the order relation $\geq_c$ with the Clifford correspondence.

**Lemma 2.1.** *Let $N \trianglelefteq G$, $\widetilde{G} \leq G$, $H \leq G$ and set $M := N \cap H$, $\widetilde{H} := \widetilde{G} \cap H$, $\widetilde{M} := \widetilde{G} \cap M$ and $\widetilde{N} := \widetilde{G} \cap N$. Assume that $G = NH$, $H = \widetilde{H}M$ and $\mathbf{C}_G(N) \leq H$. Let $\widetilde{\vartheta} \in \mathrm{Irr}(\widetilde{N})$ and $\widetilde{\varphi} \in \mathrm{Irr}(\widetilde{M})$ such that $\vartheta := \widetilde{\vartheta}^N \in \mathrm{Irr}(N)$, $\varphi := \widetilde{\varphi}^M \in \mathrm{Irr}(M)$ and $(\widetilde{G}, \widetilde{N}, \widetilde{\vartheta}) \geq_c (\widetilde{H}, \widetilde{M}, \widetilde{\varphi})$. Assume that induction of characters gives bijections $\mathrm{Ind}_{\widetilde{J}}^{J} : \mathrm{Irr}(\widetilde{J} \mid \widetilde{\vartheta}) \to \mathrm{Irr}(J \mid \vartheta)$ and $\mathrm{Ind}_{\widetilde{J}\cap H}^{J\cap H} : \mathrm{Irr}(\widetilde{J} \cap H \mid \widetilde{\varphi}) \to \mathrm{Irr}(J \cap H \mid \varphi)$, for every $N \leq J \leq G$ where $\widetilde{J} := J \cap \widetilde{G}$, then $(G, N, \vartheta) \geq_c (H, M, \varphi)$.*

*Proof.* Consider a pair of projective representations $(\widetilde{\mathcal{P}}, \widetilde{\mathcal{P}}')$ associated to $(\widetilde{G}, \widetilde{N}, \widetilde{\vartheta}) \geq_c (\widetilde{H}, \widetilde{M}, \widetilde{\varphi})$. Arguing as in the proof of [NS14, Theorem 3.14], we construct the induced projective representations $\mathcal{P} := (\widetilde{\mathcal{P}})^G$ of $G$ and $\mathcal{P}' := (\widetilde{\mathcal{P}}')^H$ of $H$ associated respectively to $\vartheta$ and $\varphi$. Then $(\mathcal{P}, \mathcal{P}')$ is associated to $(G, N, \vartheta) \geq_c (H, M, \varphi)$. $\square$

Next, we recall that the strong isomorphism of character triples associated to central isomorphic character triples (see [Spä18, Theorem 2.2] and [Nav18, Theorem 10.13 and Problem 10.4]) is compatible with the order relation $\geq_c$.

**Proposition 2.2.** *Let $(G, N, \vartheta) \geq_c (H, M, \varphi)$ via $(\mathcal{P}, \mathcal{P}')$. For $N \leq J \leq G$, consider the associated $\mathbf{N}_H(J)$-equivariant bijection $\sigma_J : \mathrm{Irr}(J \mid \vartheta) \to \mathrm{Irr}(J \cap H \mid \varphi)$ (see [Spä18, Theorem 2.2 and Corollary 2.4 (c)]). Then*

$$\left(\mathbf{N}_G(J)_\psi, J, \psi\right) \geq_c \left(\mathbf{N}_H(J)_\psi, J \cap H, \sigma_J(\psi)\right),$$

*for every $\psi \in \mathrm{Irr}(J \mid \vartheta)$.*

*Proof.* By [Spä18, Theorem 2.2] or [Nav18, Theorem 10.13] there exists a projective representation $\mathcal{Q}$ of $J$ with $N \leq \mathrm{Ker}(\mathcal{Q})$ such that $\mathcal{P}_J \otimes \mathcal{Q}$ and $\mathcal{P}'_{J\cap H} \otimes \mathcal{Q}_{J\cap H}$ afford respectively $\psi$ and $\sigma_J(\psi)$. By [Nav18, Theorem 5.5] there exists a projective representation $\mathcal{D}$ of $\mathbf{N}_G(J)_\psi$ such that $\mathcal{D}_J = \mathcal{P}_J \otimes \mathcal{Q}$ and, arguing as in [NS14, p. 707], relying on the proof of [Nav98, Theorem 8.16] we can find a projective representation $\widehat{\mathcal{Q}}$ of $\mathbf{N}_G(J)_\psi$ satisfying $\mathcal{D} = \mathcal{P}_{\mathbf{N}_G(J)_\psi} \otimes \widehat{\mathcal{Q}}$. Set $\mathcal{D}' := \mathcal{P}'_{\mathbf{N}_H(J)_\psi} \otimes \widehat{\mathcal{Q}}_{\mathbf{N}_H(J)_\psi}$. Then $(\mathcal{D}, \mathcal{D}')$ is associated to $(\mathbf{N}_G(J)_\psi, J, \psi) \geq_c (\mathbf{N}_H(J)_\psi, J \cap H, \sigma_J(\psi))$. $\square$

We also need another basic observation that follows directly from the definition of $\geq_c$.

**Lemma 2.3.** *Let $(G, N, \vartheta) \geq_c (H, M, \varphi)$. Then $(J, N, \vartheta) \geq_c (J \cap H, M, \varphi)$, for every $N \leq J \leq G$.*

Given a bijection between characters sets which is compatible with $\geq_c$, we now show how to obtain another bijection lying over the starting one and with similar compatibility properties. To do so we apply Lemma 2.1 and Proposition 2.2.

**Proposition 2.4.** *Let $K \trianglelefteq A$ and $A_0 \leq A$ such that $A = KA_0$. For every $H \leq A$ set $H_0 := H \cap A_0$. Let $\mathcal{S} \subseteq \mathrm{Irr}(K)$ and $\mathcal{S}_0 \subseteq \mathrm{Irr}(K_0)$ be $A_0$-stable subsets and assume there exists an $A_0$-equivariant bijection*

$$\Psi : \mathcal{S} \to \mathcal{S}_0$$

*such that*

$$(A_\vartheta, K, \vartheta) \geq_c \left(A_{0,\vartheta}, K_0, \Psi(\vartheta)\right),$$

*for every $\vartheta \in \mathcal{S}$. Then, for every $K \leq J \leq A$, there exists an $\mathbf{N}_{A_0}(J)$-equivariant bijection*

$$\Phi : \mathrm{Irr}(J \mid \mathcal{S}) \to \mathrm{Irr}(J_0 \mid \mathcal{S}_0)$$

*such that*

$$(\mathbf{N}_A(J)_\chi, J, \chi) \geq_c (\mathbf{N}_{A_0}(J)_\chi, J_0, \Phi(\chi)),$$

*for every $\chi \in \mathrm{Irr}(K \mid \mathcal{S})$. Moreover, if $\mathcal{S} \subseteq \mathrm{Irr}_{p',Q}(K)$, $\mathcal{S}_0 \subseteq \mathrm{Irr}_{p',Q}(K_0)$ and $\mathbf{N}_A(Q) \leq A_0$ for some $Q \in \mathrm{Syl}_p(J)$, then $\Phi$ is an $\mathbf{N}_A(Q, J)$-equivariant bijection*

$$\Phi : \mathrm{Irr}_{p'}(J \mid \mathcal{S}) \to \mathrm{Irr}_{p'}(J_0 \mid \mathcal{S}_0).$$

*Proof.* Consider an $\mathbf{N}_{A_0}(J)$-transversal $\mathbb{S}$ in $\mathcal{S}$ and define $\mathbb{S}_0 := \{\Psi(\vartheta) \mid \vartheta \in \mathbb{S}\}$. Since $\Psi$ is $A_0$-equivariant, it follows that $\mathbb{S}_0$ is an $\mathbf{N}_{A_0}(J)$-transversal in $\mathcal{S}_0$. For every $\vartheta \in \mathbb{S}$, with $\vartheta_0 := \Psi(\vartheta) \in \mathbb{S}_0$, we fix a pair of projective representations $(\mathcal{P}^{(\vartheta)}, \mathcal{P}_0^{(\vartheta_0)})$ giving $(A_\vartheta, K, \vartheta) \geq_c (A_{0,\vartheta}, K_0, \vartheta_0)$. Now, let $\mathbb{T}$ be an $\mathbf{N}_{A_0}(J)$-transversal in $\mathrm{Irr}(J \mid \mathcal{S})$ such that every character $\chi \in \mathbb{T}$ lies above a character $\vartheta \in \mathbb{S}$ (this can be done by the choice of $\mathbb{S}$). Moreover, as $A = KA_0$, we have $J = KJ_0$ and therefore every $\chi \in \mathbb{T}$ lies over a unique $\vartheta \in \mathbb{S}$ by Clifford's theorem.

For $\chi \in \mathbb{T}$ lying over $\vartheta \in \mathbb{S}$, let $\varphi \in \mathrm{Irr}(J_\vartheta \mid \vartheta)$ be the Clifford correspondent of $\chi$ over $\vartheta$. Set $\vartheta_0 := \Psi(\vartheta) \in \mathbb{S}_0$ and consider the $\mathbf{N}_{A_0}(J)_\vartheta$-equivariant bijection $\sigma_{J_\vartheta} : \mathrm{Irr}(J_\vartheta \mid \vartheta) \to \mathrm{Irr}(J_{0,\vartheta} \mid \vartheta_0)$ induced by our choice of projective representations $(\mathcal{P}^{(\vartheta)}, \mathcal{P}_0^{(\vartheta_0)})$. Let $\varphi_0 := \sigma_{J_\vartheta}(\varphi)$. Since $\Psi$ is $A_0$-equivariant, we deduce that $J_{0,\vartheta} = J_{0,\vartheta_0}$ and therefore $\Phi(\chi) := \varphi_0^{J_0}$ is irreducible by the Clifford correspondence. Then, we define

$$\Phi\left(\chi^x\right) := \Phi(\chi)^x,$$

for every $\chi \in \mathbb{T}$ and $x \in \mathbf{N}_{A_0}(J)$. This defines an $\mathbf{N}_{A_0}(J)$-equivariant bijection $\Psi : \mathrm{Irr}(J \mid \mathcal{S}) \to \mathrm{Irr}(J_0 \mid \mathcal{S}_0)$.

We now prove the statement on character triples. By hypothesis we know that

$$(A_\vartheta, K, \vartheta) \geq_c \left(A_{0,\vartheta}, K_0, \vartheta_0\right)$$

and Proposition 2.2 implies

$$(\mathbf{N}_{A_\vartheta}(J)_\varphi, J_\vartheta, \varphi) \geq_c \left(\mathbf{N}_{A_{0,\vartheta}}(J)_\varphi, J_{0,\vartheta}, \varphi_0\right).$$

Noticing that $\mathbf{N}_{A_\vartheta}(J)_\varphi = \mathbf{N}_{A_\vartheta}(J)_\chi$ and that $\mathbf{N}_A(J)_\chi = J\mathbf{N}_{A_\vartheta}(J)_\chi$ it follows from Lemma 2.1 that

$$(\mathbf{N}_A(J)_\chi, J, \chi) \geq_c (\mathbf{N}_{A_0}(J)_\chi, J_0, \Phi(\chi)) .$$

The last part of the statement follows immediately by Clifford theory. $\square$

The final result of this section allows to construct centrally ordered character triples when dealing with a situation similar to the one described in Gallagher's theorem.

**Proposition 2.5.** *Let $N \trianglelefteq G$ and $H \leq G$ with $G = NH$ and set $M := N \cap H$. Let $K \trianglelefteq G$ with $K \leq M$ and consider a $G$-invariant $\zeta \in \mathrm{Irr}_{p'}(N)$ such that $\zeta_K \in \mathrm{Irr}(K)$. Let $\overline{G} := G/K$, $\overline{N} := N/K$, $\overline{H} := H/K$ and $\overline{M} := M/K$ and suppose that $(\overline{G}, \overline{N}, \overline{\chi}) \geq_c (\overline{H}, \overline{M}, \overline{\psi})$, for some $\overline{\chi} \in \mathrm{Irr}_{p'}(\overline{N})$ and $\overline{\psi} \in \mathrm{Irr}_{p'}(\overline{M})$. Then*

$$(G, N, \chi\zeta) \geq_c (H, M, \psi\zeta_M),$$

*where $\chi \in \mathrm{Irr}(N)$ and $\psi \in \mathrm{Irr}(M)$ are the lifts respectively of $\overline{\chi}$ and $\overline{\psi}$.*

*Proof.* Let $(\overline{\mathcal{P}}, \overline{\mathcal{P}}')$ be a pair or projective representations associated to $(\overline{G}, \overline{N}, \overline{\chi}) \geq_c (\overline{H}, \overline{M}, \overline{\psi})$ and consider the corresponding lifts $\mathcal{P}$ and $\mathcal{P}'$. Let $\mathcal{Q}$ be a projective representation of $G$ associated to $\zeta$ as in [Nav18, Definition 5.2]. Then $\mathcal{P} \otimes \mathcal{Q}$ and $\mathcal{P}' \otimes \mathcal{Q}_H$ are projective representations of $G$ and $H$ associated respectively to $\chi\zeta$ and $\psi\zeta_M$. Since $\mathbf{C}_G(N)K/K \leq \mathbf{C}_{G/K}(N/K)$, we conclude from the assumption that the pair $(\mathcal{P} \otimes \mathcal{Q}, \mathcal{P}' \otimes \mathcal{Q}_H)$ gives $(G, N, \chi\zeta) \geq_c (H, M, \psi\zeta_M)$. $\square$

## 3 The inductive condition

Our aim in this section will be to show how to obtain good bijections for groups whose quotient over the centre is isomorphic to a direct product of (not necessarily isomorphic) non-abelian simple groups whose universal covering groups satisfy Conjecture A. This is done in Corollary 3.4 which will be the main result of this section. Observe that Corollary 3.4 is a slight generalization of [Nav18, Theorem 10.25] and of [Spä18, Corollary 3.14].

**Lemma 3.1.** *Let $S$ be a non-abelian simple group of order divisible by $p$ and whose universal covering group satisfies Conjecture A for the prime number $p$. Consider a non-negative integer $n$ and let $\widetilde{X} := X^n$ be the universal covering group of $\widetilde{S} := S^n$. Let $\widetilde{P}$ be a Sylow $p$-subgroup of $\widetilde{X}$ and set $\widetilde{\Gamma} := \mathrm{Aut}(\widetilde{X})_{\widetilde{P}}$. Then, there exists a $\widetilde{\Gamma}$-invariant subgroup $\mathbf{N}_{\widetilde{X}}(\widetilde{P}) \leq \widetilde{M} < \widetilde{X}$ and a $\widetilde{\Gamma}$-equivariant bijection*

$$\widetilde{\Omega} : \mathrm{Irr}_{p'}\left(\widetilde{X}\right) \to \mathrm{Irr}_{p'}\left(\widetilde{M}\right)$$

*such that*

$$\left(\widetilde{X} \rtimes \widetilde{\Gamma}_{\widetilde{\vartheta}}, \widetilde{X}, \widetilde{\vartheta}\right) \geq_c \left(\widetilde{M} \rtimes \widetilde{\Gamma}_{\widetilde{\vartheta}}, \widetilde{M}, \widetilde{\Omega}\left(\widetilde{\vartheta}\right)\right),$$

*for every $\widetilde{\vartheta} \in \mathrm{Irr}_{p'}(\widetilde{X})$.*

*Proof.* This is [Spä18, Theorem 3.12]. $\square$

Now, proceeding as the proof of [Nav18, Theorem 10.25] we obtain the following result. Notice that this is just a version of [Nav18, Theorem 10.25] adapted to the more general case where $M$ does not need to coincide with the normaliser of a Sylow $p$-subgroup.

**Proposition 3.2.** *Let $K \trianglelefteq A$ be finite groups with $K = [K, K]$ and $K/\mathbf{Z}(K) \simeq S^n$ for a non-abelian simple group $S$ whose universal covering group satisfies Conjecture A. Let $P_0$ be a Sylow $p$-subgroup of $K$. Then there exists an $\mathbf{N}_A(P_0)$-invariant subgroup $\mathbf{N}_K(P_0) \leq M \leq K$, with $M < K$ whenever $P_0$ is not normal in $K$, and an $\mathbf{N}_A(P_0)$-equivariant bijection*

$$\Omega : \mathrm{Irr}_{p'}(K) \to \mathrm{Irr}_{p'}(M)$$

*such that*

$$(A_\vartheta, K, \vartheta) \geq_c (M\mathbf{N}_A(P_0)_\vartheta, M, \Omega(\vartheta)),$$

*for every $\vartheta \in \mathrm{Irr}_{p'}(K)$.*

*Proof.* If $p$ divides the order of $S$ then the result follows from the proof of [Nav18, Theorem 10.25] by applying Lemma 3.1. Furthermore, in this case we have $M < K$. On the other hand if $p$ does not divide the order of $S$ then $P_0 \trianglelefteq \mathbf{Z}(K)$ and the result follows by choosing $M = K$. □

Finally, we consider the case where $K$ is not necessarily perfect. To do so, we have to deal with characters of central products (we refer the reader to [IMN07, Section 5] and [Nav18, Section 10.3] for the relevant notation). First, we need a lemma.

**Lemma 3.3.** *Let $(G, N, \vartheta) \geq_c (H, M, \varphi)$ and consider $C \leq \mathbf{C}_G(N)$. Let $\nu \in \mathrm{Irr}(C \cap N)$ be the unique irreducible constituent of $\vartheta_{C\cap N}$ and $\varphi_{C\cap N}$. Then*

$$\left(\mathbf{N}_G(C)_{\vartheta\cdot\psi}, N \cdot C, \vartheta \cdot \psi\right) \geq_c \left(\mathbf{N}_H(C)_{\varphi\cdot\psi}, M \cdot C, \varphi \cdot \psi\right),$$

*for every $\psi \in \mathrm{Irr}(C \mid \nu)$.*

*Proof.* First recall that $C \leq \mathbf{C}_G(N) \leq H$ and observe that $N \cdot C$ and $M \cdot C$ are central products. By the assumption and applying Proposition 2.2 with $J := N \cdot C$ we obtain

$$\left(\mathbf{N}_G(C)_{\vartheta\cdot\psi}, N \cdot C, \vartheta \cdot \psi\right) \geq_c \left(\mathbf{N}_H(C)_{(\sigma_{N\cdot C}\vartheta\cdot\psi)}, M \cdot C, \sigma_{N\cdot C}(\vartheta \cdot \psi)\right).$$

To conclude, notice that [IMN07, Lemma 5.1] implies that $\sigma_{N\cdot C}(\vartheta \cdot \psi) = \varphi \cdot \psi$. □

We are now ready to prove the main result of this section.

**Corollary 3.4.** *Let $K \trianglelefteq A$ be finite groups such that $K/\mathbf{Z}(K)$ is a direct product of non-abelian simple groups whose universal covering groups satisfy Conjecture A. Let $P_0$ be a Sylow $p$-subgroup of $K$. Then there exists an $\mathbf{N}_A(P_0)$-invariant subgroup $\mathbf{N}_K(P_0) \leq M \leq K$, with $M < K$ whenever $P_0$ is not normal in $K$, and an $\mathbf{N}_A(P_0)$-equivariant bijection*

$$\Omega : \mathrm{Irr}_{p'}(K) \to \mathrm{Irr}_{p'}(M)$$

*such that*

$$(A_\vartheta, K, \vartheta) \geq_c (M\mathbf{N}_A(P_0)_\vartheta, M, \Omega(\vartheta)),$$

*for every $\vartheta \in \mathrm{Irr}_{p'}(K)$.*

*Proof.* By hypothesis there exist non-isomorphic non-abelian simple groups $S_1,\dots,S_\ell$ that satisfy the inductive McKay condition and non-negative integers $n_1,\dots,n_\ell$ such that $K/\mathbf{Z}(K)\simeq S_1^{n_1}\times\cdots\times S_\ell^{n_\ell}$. Consider the subgroups $\mathbf{Z}(K)\leq K_{0,i}\leq K$ such that $K_{0,i}/\mathbf{Z}(K)\simeq S_i^{n_i}$ and observe that $K_i:=[K_{0,i},K_{0,i}]$ is a perfect normal subgroup of $A$ with $K_i/\mathbf{Z}(K_i)\simeq S_i^{n_i}$, for $i=1,\dots,\ell$. If $K_0:=\mathbf{Z}(K)$, then $K=K_0\cdot\ldots\cdot K_\ell$ is a central product of the subgroups $K_i$ and $Z:=\cap_{i=0}^{\ell}K_i$ satisfies $Z=\mathbf{Z}([K,K])=\mathbf{Z}(K_i)$, for all $i=1,\dots,\ell$.

Let $\vartheta\in\mathrm{Irr}_{p'}(K)$ and consider the unique irreducible constituent $\nu\in\mathrm{Irr}(Z)$ of $\vartheta_Z$. By [IMN07, Lemma 5.1] there exist unique characters $\vartheta_i\in\mathrm{Irr}_{p'}(K_i\mid\nu)$ such that $\vartheta=\vartheta_0\cdot\ldots\cdot\vartheta_\ell$. Set $Q_i:=P_0\cap K_i\in\mathrm{Syl}_p(K_i)$ and $A_i:=\mathbf{N}_A(Q_i)$. By Proposition 3.2, for every $i=1,\dots,\ell$, there exists an $A_i$-invariant subgroup $\mathbf{N}_{K_i}(Q_i)\leq M_i\leq K_i$ and a $A_i$-equivariant bijection

$$\Omega_i:\mathrm{Irr}_{p'}(K_i)\to\mathrm{Irr}_{p'}(M_i)$$

such that

$$(A_{\vartheta_i},K_i,\vartheta_i)\geq_c\left(M_iA_{i,\vartheta_i},M_i,\Omega_i(\vartheta_i)\right),$$

for every $\vartheta_i\in\mathrm{Irr}_{p'}(K_i)$. For $i=0$, set $M_0:=K_0$ and let $\Omega_0$ be the identity map on $\mathrm{Irr}(K_0)$. Now, the subgroup $M:=M_0\cdot\ldots\cdot M_\ell$ is the central product of the $M_i$'s and has the required properties. Moreover, the map

$$\begin{aligned}\Omega:\mathrm{Irr}_{p'}(K)&\to\mathrm{Irr}_{p'}(M)\\ \vartheta_0\cdot\ldots\cdot\vartheta_\ell&\mapsto\Omega_0(\vartheta_0)\cdot\ldots\cdot\Omega_\ell(\vartheta_\ell)\end{aligned}$$

is a well defined $\mathbf{N}_A(P_0)$-equivariant bijection. It remains to check the statement on character triples. To do so, we are going to prove that

$$\begin{aligned}&(A_{\vartheta_0\cdot\ldots\cdot\vartheta_\ell},K_0\cdot\ldots\cdot K_\ell,\vartheta_0\cdot\ldots\cdot\vartheta_\ell)\geq_c\\ &\qquad((M_0\cdot\ldots\cdot M_\ell)\mathbf{N}_A(Q_0,\dots,Q_\ell)_{\vartheta_0\cdot\ldots\cdot\vartheta_\ell},M_0\cdot\ldots\cdot M_\ell,\Omega_0(\vartheta_0)\cdot\ldots\cdot\Omega_\ell(\vartheta_\ell))\end{aligned}\tag{3.1}$$

by induction on $\ell\geq 1$. Let $\ell=1$. By the previous section we know that

$$(A_{\vartheta_1},K_1,\vartheta_1)\geq_c\left(M_1A_{1,\vartheta_1},M_1,\Omega_1(\vartheta_1)\right)$$

and applying Lemma 3.3 with $C:=K_0$ we deduce

$$(A_{\vartheta_0\cdot\vartheta_1},K_0\cdot K_1,\vartheta_0\cdot\vartheta_1)\geq_c\left(M_1A_{1,\vartheta_0\cdot\vartheta_1},K_0\cdot M_1,\vartheta_0\cdot\Omega_1(\vartheta_1)\right),$$

here we used the fact that $A_{\vartheta_0\cdot\vartheta_1}\leq A_{\vartheta_0}\cap A_{\vartheta_1}$. Because $K_0=M_0$, $\Omega_0(\vartheta_0)=\vartheta_0$ and $M_1A_{1,\vartheta_0\cdot\vartheta_1}=(M_0\cdot M_1)\mathbf{N}_A(Q_0,Q_1)_{\vartheta_0\cdot\vartheta_1}$ it follows that (3.1) holds for $\ell=1$. Consider now $\ell>1$. The inductive hypothesis yields

$$\begin{aligned}&\left(A_{\vartheta_0\cdot\ldots\cdot\vartheta_{\ell-1}},K_0\cdot\ldots\cdot K_{\ell-1},\vartheta_0\cdot\ldots\cdot\vartheta_{\ell-1}\right)\geq_c\\ &\quad\left((M_0\cdot\ldots\cdot M_{\ell-1})\mathbf{N}_A(Q_0,\dots,Q_{\ell-1})_{\vartheta_0\cdot\ldots\cdot\vartheta_{\ell-1}},M_0\cdot\ldots\cdot M_{\ell-1},\Omega_0(\vartheta_0)\cdot\ldots\cdot\Omega_{\ell-1}(\vartheta_{\ell-1})\right).\end{aligned}$$

Noticing that $M_\ell\leq K_\ell\leq\mathbf{N}_A(Q_0,\dots,Q_{\ell-1})_{\vartheta_0\cdot\ldots\cdot\vartheta_\ell}$ and applying Lemma 3.3 we deduce

$$\begin{aligned}&(A_{\vartheta_0\cdot\ldots\cdot\vartheta_\ell},K_0\cdot\ldots\cdot K_\ell,\vartheta_0\cdot\ldots\cdot\vartheta_\ell)\geq_c\\ &\qquad((M_0\cdot\ldots\cdot M_\ell)\mathbf{N}_A(Q_0,\dots,Q_{\ell-1})_{\vartheta_0\cdot\ldots\cdot\vartheta_\ell},M_0\cdot\ldots\cdot M_{\ell-1}\cdot K_\ell,\\ &\qquad\qquad\Omega_0(\vartheta_0)\cdot\ldots\cdot\Omega_{\ell-1}(\vartheta_{\ell-1})\cdot\vartheta_\ell).\end{aligned}\tag{3.2}$$

On the other hand the fact that

$$(A_{\vartheta_\ell}, K_\ell, \vartheta_\ell) \geq_c \left(M_\ell A_{\ell,\vartheta_\ell}, M_\ell, \Omega_\ell(\vartheta_\ell)\right)$$

together with Lemma 2.3 implies

$$((M_0 \cdot \ldots \cdot M_\ell)\mathbf{N}_A(Q_0, \ldots, Q_{\ell-1})_{\vartheta_0 \cdot \ldots \cdot \vartheta_\ell}, K_\ell, \vartheta_\ell) \geq_c \\ ((M_0 \cdot \ldots \cdot M_\ell)\mathbf{N}_A(Q_0, \ldots, Q_\ell)_{\vartheta_0 \cdot \ldots \cdot \vartheta_\ell}, M_\ell, \Omega_\ell(\vartheta_\ell)) \,.$$

We now apply Lemma 3.3 with $C \coloneqq M_0 \cdot \ldots \cdot M_{\ell-1}$ and $\psi = \Omega_0(\vartheta_0) \cdot \ldots \cdot \Omega_{\ell-1}(\vartheta_{\ell-1})$ to obtain

$$((M_0 \cdot \ldots \cdot M_\ell)\mathbf{N}_A(Q_0, \ldots, Q_{\ell-1})_{\vartheta_0 \cdot \ldots \cdot \vartheta_\ell}, M_0 \cdot \ldots \cdot M_{\ell-1} \cdot K_\ell, \\ \Omega_0(\vartheta_0) \cdot \ldots \cdot \Omega_{\ell-1}(\vartheta_{\ell-1}) \cdot \vartheta_\ell) \geq_c \\ ((M_0 \cdot \ldots \cdot M_\ell)\mathbf{N}_A(Q_0, \ldots, Q_\ell)_{\vartheta_0 \cdot \ldots \cdot \vartheta_\ell}, M_0 \cdot \ldots \cdot M_\ell, \Omega_0(\vartheta_0) \cdot \ldots \cdot \Omega_\ell(\vartheta_\ell)) \,. \quad (3.3)$$

Now (3.1) follows from (3.2) and (3.3). □

# 4 The reduction

In this final section we prove Theorem B. To do so, proceeding as in [NS14, Section 7], we analyse the structure of a minimal counterexample to Theorem B.

**Lemma 4.1.** *Let $G \trianglelefteq A$ be a minimal counterexample to Theorem B with respect to $|G : \mathbf{Z}(G)|$. Let $K \trianglelefteq A$, $K \leq G$ such that $|G : K| < |G : \mathbf{Z}(G)|$ and consider an $A$-invariant $\zeta \in \mathrm{Irr}_{p'}(K)$. Then there exists an $\mathbf{N}_A(P)$-equivariant bijection*

$$\Upsilon_\zeta : \mathrm{Irr}_{p'}(G \mid \zeta) \to \mathrm{Irr}_{p'}(K\mathbf{N}_G(P) \mid \zeta)$$

*such that*

$$(A_\tau, G, \tau) \geq_c \left(K\mathbf{N}_A(P)_\tau, K\mathbf{N}_G(P), \Upsilon_\zeta(\tau)\right),$$

*for every $\tau \in \mathrm{Irr}_{p'}(G \mid \zeta)$.*

*Proof.* Arguing as in the proof of [NS14, Lemma 7.3], by choosing a projective representation of $A$ associated to $\zeta$ we construct groups $\overline{P} \leq \overline{G} \trianglelefteq \overline{A}$ such that $\overline{G}$ is a central extension of $G/K$ and $\overline{P}$ is a Sylow $p$-subgroup of $\overline{G}$. Since $|\overline{G} : \mathbf{Z}(\overline{G})| < |G : \mathbf{Z}(G)|$, the inductive hypothesis yields an $\mathbf{N}_{\overline{A}}(\overline{P})$-invariant subgroup $\mathbf{N}_{\overline{G}}(\overline{P}) \leq \overline{M} \leq \overline{G}$, with $\overline{M} < \overline{G}$ whenever $\overline{P}$ is not normal in $\overline{G}$, and an $\mathbf{N}_{\overline{A}}(\overline{P})$-equivariant bijection

$$\overline{\Omega} : \mathrm{Irr}_{p'}\left(\overline{G}\right) \to \mathrm{Irr}_{p'}\left(\overline{M}\right) \quad (4.1)$$

which gives central isomorphic character triples. Notice that Conjecture A also holds for every $\mathbf{N}_{\overline{G}}(\overline{P}) \leq \overline{X} \leq \overline{G}$ since $|\overline{X} : \mathbf{Z}(\overline{X})| \leq |\overline{G} : \mathbf{Z}(\overline{G})|$. In particular, for $\overline{X} = \overline{M}$ we obtain an $\overline{M}\mathbf{N}_{\overline{A}}(\overline{P})$-invariant subgroup $\mathbf{N}_{\overline{M}}(\overline{P}) \leq \overline{M}_1 \leq \overline{M}$, with $\overline{M}_1 < \overline{M}$ whenever $\overline{P}$ is not normal in $\overline{M}$, and an $M\mathbf{N}_{\overline{A}}(\overline{P})$-equivariant bijection

$$\overline{\Omega}_1 : \mathrm{Irr}_{p'}\left(\overline{M}\right) \to \mathrm{Irr}_{p'}\left(\overline{M}_1\right)$$

that gives central isomorphic character triples. Iterating this argument we obtain a sequence of $\mathbf{N}_{\overline{A}}(\overline{P})$-invariant subgroups $\overline{G} \geq \overline{M} \geq \overline{M}_1 \geq \cdots \geq \overline{M}_i$ and $\mathbf{N}_{\overline{A}}(\overline{P})$-equivariant bijections bijections

$$\overline{\Omega}_i : \mathrm{Irr}_{p'}\left(\overline{M}_{i-1}\right) \to \mathrm{Irr}_{p'}\left(\overline{M}_i\right)$$

inducing central isomorphic character triples. By the properties of the subgroups $\overline{M}_i$ we deduce that there exists some $t \geq 1$ such that $\overline{M}_t = \mathbf{N}_{\overline{G}}(\overline{P})$. Then, since the relation $\geq_c$ is transitive, this argument shows that it is no loss of generality to assume $\overline{M} = \mathbf{N}_{\overline{G}}(\overline{P})$ in (4.1). We can now proceed as in the proof of [NS14, Lemma 7.3] by applying [Spä18, Lemma 2.17] and Proposition 2.5 in place of [NS14, Corollary 4.5] and [NS14, Theorem 4.6] respectively. □

**Proposition 4.2.** *Let $G \trianglelefteq A$ be a minimal counterexample to Theorem B with respect to $|G : \mathbf{Z}(G)|$. Let $K \trianglelefteq A$, $K \leq G$ such that $|G : K| < |G : \mathbf{Z}(G)|$. Then there exists an $\mathbf{N}_A(P)$-equivariant bijection*

$$\Upsilon_K : \mathrm{Irr}_{p'}(G) \to \mathrm{Irr}_{p'}(K\mathbf{N}_G(P))$$

*such that*

$$(A_\tau, G, \tau) \geq_c (K\mathbf{N}_A(P)_\tau, K\mathbf{N}_G(P), \Upsilon_K(\tau)),$$

*for every $\tau \in \mathrm{Irr}_{p'}(G)$.*

*Proof.* This follows from the proof of [NS14, Proposition 7.4] by replacing [NS14, Theorem 3.14] and [NS14, Lemma 7.3] respectively with Lemma 2.1 and Lemma 4.1. □

As an immediate consequence we obtain that, for a minimal counterexample $G$, we have $G = K\mathbf{N}_G(P)$ for any $K \trianglelefteq A$ with $K \leq G$ and $|G : K| < |G : \mathbf{Z}(G)|$.

**Corollary 4.3.** *Let $G \trianglelefteq A$ be a minimal counterexample to Theorem B with respect to $|G : \mathbf{Z}(G)|$. Let $K \trianglelefteq A$, $K \leq G$ such that $|G : K| < |G : \mathbf{Z}(G)|$. Then $G = K\mathbf{N}_G(P)$.*

*Proof.* Suppose that $G_1 := K\mathbf{N}_G(P)$ is a proper subgroup of $G$. Then every non-abelian simple group involved in $G$ is also involved in $G_1$ and $|G_1 : \mathbf{Z}(G_1)| < |G : \mathbf{Z}(G)|$. Set $A_1 := K\mathbf{N}_A(P)$. By the minimality of $G$ we can find an $\mathbf{N}_{A_1}(P)$-equivariant bijection

$$\Omega_1 : \mathrm{Irr}_{p'}(G_1) \to \mathrm{Irr}_{p'}(\mathbf{N}_{G_1}(P))$$

such that

$$\left(A_{1,\vartheta}, G_1, \vartheta\right) \geq_c (\mathbf{N}_{A_1}(P)_\vartheta, \mathbf{N}_{G_1}(P), \Omega_1(\vartheta)),$$

for every $\vartheta \in \mathrm{Irr}_{p'}(G_1)$. Notice that $\mathbf{N}_{G_1}(P) = \mathbf{N}_G(P)$ and that $\mathbf{N}_{A_1}(P) = \mathbf{N}_A(P)$. Then, applying Proposition 4.2 and composing the obtained bijection with $\Omega_1$ we obtain a contradiction. This proves that $G = K\mathbf{N}_G(P)$. □

Next we want to show that, if $G$ is a minimal counterexample and $K \trianglelefteq A$ with $K \leq G$ such that $K$ has a Sylow $p$-subgroup which is central in $G$, then $K \leq \mathbf{Z}(G)$. To do so we use the following result.

**Theorem 4.4.** *Let $A$ be a finite group and $L, K \trianglelefteq A$ such that $K \leq L$ and $L/K$ is a $p$-group. Let $P$ be a $p$-subgroup of $L$ such that $L = KP$ and $P_0 := P \cap K \leq \mathbf{Z}(L)$. Then there exists an $\mathbf{N}_A(P)$-equivariant bijection*

$$\Lambda_P : \mathrm{Irr}_{p'}(L) \to \mathrm{Irr}_{p'}(\mathbf{N}_L(P))$$

*such that*

$$(A_\vartheta, L, \vartheta) \geq_c (\mathbf{N}_A(P)_\vartheta, \mathbf{N}_L(P), \Lambda_P(\vartheta)),$$

*for every $\vartheta \in \mathrm{Irr}_{p'}(L)$.*

*Proof.* This follows directly from [NS14, Corollary 5.14]. □

**Proposition 4.5.** *Let $G \trianglelefteq A$ be a minimal counterexample to Theorem B with respect to $|G : \mathbf{Z}(G)|$. Let $K \trianglelefteq A$, $K \leq G$ and suppose that $P_0 := P \cap K \leq \mathbf{Z}(G)$. Then $K \leq \mathbf{Z}(G)$.*

*Proof.* For the sake of contradiction assume $K \nleq \mathbf{Z}(G)$. Then $|G : K\mathbf{Z}(G)| < |G : \mathbf{Z}(G)|$ and Corollary 4.3 implies $G = K\mathbf{Z}(G)\mathbf{N}_G(P) = K\mathbf{N}_G(P)$. Recall that $A = G\mathbf{N}_A(P)$ by the Frattini argument. Then $A = K\mathbf{N}_A(P)$ and the subgroup $L := KP$ is normal in $A$ and satisfies $P_0 := K \cap P \leq \mathbf{Z}(L)$ by hypothesis. Now, Theorem 4.4 yields an $\mathbf{N}_A(P)$-equivariant bijection

$$\Lambda_P : \mathrm{Irr}_{p'}(L) \to \mathrm{Irr}_{p'}(\mathbf{N}_L(P))$$

such that

$$(A_\vartheta, L, \vartheta) \geq_c (\mathbf{N}_A(P)_\vartheta, \mathbf{N}_L(P), \Lambda_P(\vartheta)),$$

for every $\vartheta \in \mathrm{Irr}_{p'}(L)$. Finally, after noticing that $\mathrm{Irr}_{p'}(G) \subseteq \mathrm{Irr}(G \mid \mathrm{Irr}_{p',P}(L))$ and that $\mathrm{Irr}_{p'}(\mathbf{N}_G(P)) \subseteq \mathrm{Irr}(\mathbf{N}_G(P) \mid \mathrm{Irr}_{p',P}(\mathbf{N}_L(P)))$, we obtain a contradiction by applying Proposition 2.4 . □

We are finally ready to prove Theorem B.

*Proof of Theorem B.* Suppose, for the sake of a contradiction, that the result fails to hold and consider a counterexample $G \trianglelefteq A$ minimal with respect to $|G : \mathbf{Z}(G)|$. By Corollary 4.3 (applied with $K := \mathbf{Z}(G)\mathbf{O}_p(G)$) it follows that $\mathbf{O}_p(G) \leq \mathbf{Z}(G)$. Furthermore, as $\mathbf{O}_{p'}(G)\mathbf{Z}(G) \cap P \leq \mathbf{Z}(G)$, Proposition 4.5 yields $\mathbf{O}_{p'}(G) \leq \mathbf{Z}(G)$. As a consequence $\mathbf{Z}(G) = \mathbf{F}(G) < \mathbf{F}^*(G)$, where $\mathbf{F}^*(G) = \mathbf{F}(G)\mathbf{E}(G)$ is the generalized Fitting subgroup of $G$ which is the product of the Fitting subgroup $\mathbf{F}(G)$ and the layer $K := \mathbf{E}(G)$. Observe that $P_0 := P \cap K$ is a Sylow $p$-subgroup of $K$. Since $K \nleq \mathbf{Z}(G)$, Proposition 4.5 shows that $P_0 \nleq \mathbf{Z}(G)$ and, as $\mathbf{Z}(K) \leq \mathbf{F}(G) = \mathbf{Z}(G)$, we obtain $P_0 \nleq \mathbf{Z}(K)$. Now, we can apply Corollary 3.4 and find an $\mathbf{N}_A(P_0)$-invariant subgroup $\mathbf{N}_K(P_0) \leq M < K$ and an $\mathbf{N}_A(P_0)$-equivariant bijection

$$\Omega_K : \mathrm{Irr}_{p'}(K) \to \mathrm{Irr}_{p'}(M)$$

such that

$$(A_\vartheta, K, \vartheta) \geq_c (M\mathbf{N}_A(P_0)_\vartheta, M, \Omega_K(\vartheta)),$$

for every $\vartheta \in \mathrm{Irr}_{p'}(K)$. Next, observe that $\mathbf{N}_A(P) \leq \mathbf{N}_A(P_0)$, that $G = K\mathbf{N}_G(P_0)$ by Corollary 4.3 and that $\mathrm{Irr}_{p'}(G) \subseteq \mathrm{Irr}(G \mid \mathrm{Irr}_{p',P}(K))$ and $\mathrm{Irr}_{p'}(M\mathbf{N}_G(P_0)) \subseteq \mathrm{Irr}(M\mathbf{N}_G(P_0) \mid \mathrm{Irr}_{p',P}(M))$. Applying Proposition 2.4 we obtain an $\mathbf{N}_A(P)$-equivariant bijection

$$\Pi : \mathrm{Irr}_{p'}(G) \to \mathrm{Irr}_{p'}(M\mathbf{N}_G(P_0))$$

such that

$$(A_\chi, G, \chi) \geq_c (M\mathbf{N}_A(P_0)_\chi, M\mathbf{N}_G(P_0), \Pi(\chi)),$$

for every $\chi \in \mathrm{Irr}_{p'}(G)$. Finally, by the minimality of $G$, it follows that Theorem B holds for $G_1 := M\mathbf{N}_G(P_0)$ (recall that $M < K$). Then, if $A_1 := M\mathbf{N}_A(P_0)$, there exists an $\mathbf{N}_{A_1}(P)$-invariant subgroup $\mathbf{N}_{G_1}(P) \leq M_1 \leq G_1$ and an $\mathbf{N}_{A_1}(P)$-equivariant bijection

$$\Delta : \mathrm{Irr}_{p'}(G_1) \to \mathrm{Irr}_{p'}(M_1)$$

such that

$$\left(A_{1,\vartheta}, G_1, \vartheta\right) \geq_c (M_1\mathbf{N}_{A_1}(P)_\vartheta, M_1, \Delta(\vartheta)),$$

for every $\mathrm{Irr}_{p'}(G_1)$. Since $\mathbf{N}_{A_1}(P) = \mathbf{N}_A(P)$ and $\mathbf{N}_G(P) \leq M_1 \leq G_1 < G$, the composition of $\Delta$ and $\Pi$ yields a bijection satisfying the requirement of Conjecture A. This is a contradiction and the proof is now complete. □

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