

Monomial characters of finite solvable groups

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Abstract

We give new evidence to the fact that the structure of a solvable group can be controlled by irreducible monomial characters. In particular we inspect the role of monomial characters in the Isaacs–Navarro–Wolf Conjecture and in Gluck’s Conjecture.

1 Introduction

Let G be a finite group and denote by $\text{Irr}(G)$ the set of irreducible complex characters of G . As is well known, the structure of G is strongly influenced by properties of the set $\text{Irr}(G)$. In [PL16], [PL17], [CL17], [CL18] and [CY20], the authors showed that certain aspects of the structure of a solvable group G can be controlled by only considering monomial characters, that is, characters induced by linear ones. Following this idea, we propose refinements of two well known open conjectures in character theory of finite groups introduced by Isaacs, Navarro and Wolf and by Gluck respectively.

Recall that an element $g \in G$ is *non-vanishing* if $\chi(g) \neq 0$ for every $\chi \in \text{Irr}(G)$. We denote by $\mathcal{N}(G)$ the set of non-vanishing elements of G . Let $\text{Irr}_m(G)$ be the set of irreducible monomial characters of G . We define an element $g \in G$ to be *monomial-non-vanishing* if $\chi(g) \neq 0$ for every $\chi \in \text{Irr}_m(G)$ and denote by $\mathcal{N}_m(G)$ the set of such elements. Notice that $\mathcal{N}(G) \subseteq \mathcal{N}_m(G)$ while there are solvable groups for which the inclusion is strict (the smallest such group is $\text{SL}_2(3)$). The Isaacs–Navarro–Wolf Conjecture, introduced in [INW99], states that the set $\mathcal{N}(G)$ is contained in the Fitting subgroup $\mathbf{F}(G)$ for every solvable group G . In this paper, we prove that $\mathcal{N}_m(G) \subseteq \mathbf{F}(G)$ in all the cases in which the Isaacs–Navarro–Wolf Conjecture is known to hold. We conjecture that this fact holds for every solvable group.

Conjecture A. *If G is a finite solvable group, then $\mathcal{N}_m(G) \subseteq \mathbf{F}(G)$.*

In [Glu85], Gluck proved that the index of the Fitting subgroup of an arbitrary finite group G is bounded by a polynomial function of the largest character degree $b(G)$ of G . In particular it is conjectured that $|G : \mathbf{F}(G)| \leq b(G)^2$ for every solvable group G . We define $b_m(G) := \{\chi(1) \mid \chi \in \text{Irr}_m(G)\}$ and observe that $b_m(G) \leq b(G)$. Moreover $b(G) - b_m(G)$ can be arbitrarily large (see Example 3.3). In Section 3, we verify the stronger bound $|G : \mathbf{F}(G)| \leq b_m(G)^2$ in all the cases in

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which Gluck's Conjecture is known to hold. As before, we conjecture that this fact holds for all solvable groups.

Conjecture B. *If G is a finite solvable group, then $|G : \mathbf{F}(G)| \leq b_m(G)^2$.*

To conclude, we show that the characterization of the normality of a Sylow p -subgroup given in [MN12] can be checked by considering only monomial characters in the case of solvable groups.

Theorem C. *Let G be a finite solvable group, p a prime and consider $P \in \text{Syl}_p(G)$. Then $P \trianglelefteq G$ if and only if p does not divide $\chi(1)$, for every $\chi \in \text{Irr}_m(G \mid 1_P)$.*

To conclude, we make a remark on the proportion of monomial characters of finite solvable groups. The above results might lead to the belief that finite solvable groups have a large number of monomial characters. However this does not need to be the case in general. For instance, consider $G := \text{SL}_2(3)$ and define G_n to be the direct product of n copies of G . Noticing that

$$\frac{|\text{Irr}_m(G_n)|}{|\text{Irr}(G_n)|} = \left(\frac{|\text{Irr}_m(G)|}{|\text{Irr}(G)|} \right)^n$$

by [vdW88], we deduce that

$$\lim_{n \rightarrow \infty} \frac{|\text{Irr}_m(G_n)|}{|\text{Irr}(G_n)|} = 0$$

since G is a non-monomial group. This shows that there exist solvable groups with an arbitrarily small proportion of monomial characters.

2 The Isaacs-Navarro-Wolf Conjecture

In [INW99], the Isaacs-Navarro-Wolf Conjecture is proved for groups of odd order and, more generally, for solvable groups with abelian Sylow 2-subgroups. In [MW04] the authors show that $\mathcal{N}(G) \subseteq \mathbf{F}_{10}(G)$ for every solvable group G and extend the previous result to every solvable group G in which every Fitting factor $\mathbf{F}_{i+1}(G)/\mathbf{F}_i(G)$ has an abelian Sylow 2-subgroup for every $1 \leq i \leq 9$ (see [MW04, Corollary 2.3]). Here, we define $\mathbf{F}_{i+1}(G)$ recursively by setting $\mathbf{F}_{i+1}(G)/\mathbf{F}_i(G) := \mathbf{F}(G/\mathbf{F}_i(G))$ and where $\mathbf{F}_0(G) := \mathbf{F}(G)$. Notice that [MW04, Corollary 2.3] is a consequence of a theorem on orbits of linear groups [MW04, Theorem E]. This latter result has been improved by Yang [Yan09, Theorem 3.5] with consequences on non-vanishing elements. In particular, the Isaacs-Navarro-Wolf Conjecture holds for solvable groups in which every Fitting factor $\mathbf{F}_{i+1}(G)/\mathbf{F}_i(G)$ has an abelian Sylow 2-subgroup, for every $1 \leq i \leq 7$. We are going to show that Conjecture A holds under the same hypothesis. We start with the following preliminary result which should be compared with [INW99, Lemma 2.3]. In what follows, $\Phi(G)$ denotes the Frattini subgroup of G .

Lemma 2.1. *Let $N \trianglelefteq G$ and $x \in \mathcal{N}_m(G)$. If N is abelian and $N \cap \Phi(G) = 1$, then x fixes a character in each G -orbit on $\text{Irr}(N)$.*

Proof. Fix $\lambda \in \text{Irr}(N)$. By hypothesis N has a complement in $T := G_\lambda$ according to [Hup67, III.4.4]. Thus λ extends to $\widehat{\lambda} \in \text{Irr}(T)$ by [Hup98, 19.12] and $\chi := \widehat{\lambda}^G \in \text{Irr}_m(G)$ by the Clifford correspondence. Since $x \in \mathcal{N}_m(G)$, there exists $g \in G$ such that $gxg^{-1} \in T$. In particular λ^g is fixed by x . \square

Recall that a chief factor K/L of G is said to be *non-Frattini* if $(K/L) \cap \Phi(G/L) = 1$. In this case, if K/L is abelian, then K/L has a complement in G/L .

Proposition 2.2. *Let G be solvable and let $x \in \mathcal{N}_m(G)$. Then the order of $\mathbf{F}(G)x$ in $G/\mathbf{F}(G)$ is a power of 2. In particular Conjecture A holds for groups of odd order.*

Proof. Let u be the $2'$ -part of x . We show that $u \in \mathbf{F}(G)$. By [DH92, 13.8], it is enough to fix a non-Frattini chief factor K/L of G and prove that $u \in \mathbf{C}_G(K/L) =: C$. Since $1 \neq \mathbf{F}(G) \leq C$, induction on $|G|$ yields $\bar{u} \in \mathbf{F}(\bar{G})$, where $\bar{G} := G/C$. Furthermore \bar{G} acts faithfully and irreducibly on $\text{Irr}(K/L)$. Consider the quotient G/L and observe that $Lx \in \mathcal{N}_m(G/L)$ and that K/L satisfies the hypothesis of Lemma 2.1 in G/L . It follows that Lx fixes a point in every (G/L) -orbit on $\text{Irr}(K/L)$, and so that \bar{x} fixes a point in every \bar{G} -orbit on $\text{Irr}(K/L)$. In particular \bar{u} fixes a point in each \bar{G} -orbit on $\text{Irr}(K/L)$ and, by [INW99, Theorem 4.2], we conclude that $\bar{u}^2 = 1$. This shows that $u \in \mathbf{F}(G)$. \square

Next, we need analogues of [INW99, Theorem 4.4] and of [Yan09, Theorem 5.2] for monomial characters.

Proposition 2.3. *Let G be a solvable group and assume there exists some $x \in \mathcal{N}_m(G) \setminus \mathbf{F}(G)$. Then there exists a subgroup $\mathbf{F}(G) \leq N \trianglelefteq G$ such that, if $\bar{G} := G/N$, then \bar{x} is an involution of $\mathbf{F}(\bar{G})$ and $\bar{x} \notin \bar{A}$, for every abelian normal subgroup $\bar{A} \trianglelefteq \bar{G}$.*

Proof. Let M be a maximal element of the non-empty set $\{N \trianglelefteq G \mid Nx \notin \mathbf{F}(G/N)\}$. Replacing G with G/M , we may assume that $Nx \in \mathbf{F}(G/N)$, for every $1 < N \trianglelefteq G$. Since $x \notin \mathbf{F}(G)$, by [DH92, 13.8], there exists a non-Frattini chief factor K/L of G such that $x \notin C := \mathbf{C}_G(K/L)$. We claim that C is the normal subgroup we are looking for. Let $\bar{G} := G/C$ and observe that $1 \neq \bar{x} \in \mathbf{F}(\bar{G})$ and that \bar{G} acts faithfully and irreducibly on $\text{Irr}(K/L)$. By Lemma 2.1 we deduce that Lx fixes a point in each (G/L) -orbit on $\text{Irr}(K/L)$. It follows that \bar{x} fixes a point in each \bar{G} -orbit on $\text{Irr}(K/L)$ and we conclude by [INW99, Theorem 4.2]. \square

Proposition 2.4. *Let G be a solvable group. Then there exists $\mu \in \text{Irr}_m(\mathbf{F}_8(G))$ such that $\mu^G \in \text{Irr}_m(G)$. In particular $\mathcal{N}_m(G) \subseteq F_8(G)$.*

Proof. As a consequence of [Isa08, Problem 1D.15], we deduce that $\mathbf{F}_i(G/\Phi(G)) = \mathbf{F}_i(G)/\Phi(G)$ for any $i \geq 1$. Then, proceeding by induction on the order of G , we may assume $\Phi(G) = 1$. Set $F := \mathbf{F}_8(G)$. Notice that $\text{Irr}(\mathbf{F}(G))$ is a completely reducible and faithful $(G/\mathbf{F}(G))$ -module and that, by [Yan09, 3.5], there exists $\lambda \in \text{Irr}(\mathbf{F}(G))$ such that $T := G_\lambda \leq F$. Using [Hup67, III.4.4] and [Hup98, 19.12], we deduce that λ extends to a linear character $\hat{\lambda} \in \text{Irr}(T \mid \lambda)$. By the Clifford correspondence $\mu := \hat{\lambda}^F$ has the required properties. \square

As a consequence we obtain the result mentioned at the beginning of the section.

Corollary 2.5. *Conjecture A holds for every solvable group G in which $\mathbf{F}_{i+1}(G)/\mathbf{F}_i(G)$ has an abelian Sylow 2-subgroup, for every $1 \leq i \leq 7$.*

Proof. Let $x \in \mathcal{N}_m(G)$ and suppose that $x \notin \mathbf{F}(G)$. For $i \geq 1$, set $F_i := \mathbf{F}_i(G)$ and notice that, by Proposition 2.4, there exists $1 \leq i \leq 7$ such that $x \in F_{i+1} \setminus F_i$. Replacing G with G/F_{i-1} , we may assume that $x \in F_2 \setminus F_1$ and that F_2/F_1 has abelian Sylow 2-subgroups. Let $F_1 \leq N \trianglelefteq G$ be as

in Proposition 2.3 and set $\overline{G} := G/N$. Then \overline{x} lies in a Sylow 2-subgroup of \overline{F}_2 and, since such a subgroup is abelian and normal in \overline{G} , we have a contradiction. \square

We end this section by mentioning two other cases in which Conjecture A holds. First, suppose that $G/\mathbf{F}(G)$ is supersolvable. Under this hypothesis the Isaacs–Navarro–Wolf Conjecture holds by the results of [He12].

Proposition 2.6. *Conjecture A holds whenever $G/\mathbf{F}(G)$ is supersolvable.*

Proof. By induction on $|G|$ we may assume $\Phi(G) = 1$. Now, $F := \mathbf{F}(G)$ is abelian and has a supersolvable complement in G . By [INW99, Theorem B] and [Hup98, 24.3], we deduce that $\mathcal{N}_m(G/F) \subseteq \mathbf{Z}(\mathbf{F}(G/F)) =: Z/F$ and that $\mathcal{N}_m(G) \subseteq Z$. Let $A := H \cap Z$ and observe that $Z = F \rtimes A$. The abelian group A acts faithfully on the completely reducible A -module $\text{Irr}(F)$. Noticing that, in this situation, complete reducibility coincide with the condition $(|G|, |\text{Irr}(F)|) = 1$, we can find $\lambda \in \text{Irr}(F)$ such that $A_\lambda = 1$ [INW99, Lemma 3.1]. In particular $Z_\lambda = F$ and $\vartheta := \lambda^Z \in \text{Irr}(Z)$. Since ϑ vanishes on $Z \setminus F$, and so do all its conjugates, we conclude that every character $\chi \in \text{Irr}(G \mid \vartheta)$ vanishes on $Z \setminus F$. On the other hand, observe that λ extends to $\widehat{\lambda} \in \text{Irr}(G_\lambda)$ and that $\chi := \widehat{\lambda}^G \in \text{Irr}_m(G)$. Since χ lies over ϑ , it follows that χ vanishes on $Z \setminus F$ and so $\mathcal{N}_m(G) \subseteq F$. \square

Finally, assume that 4 does not divide the degree of any irreducible monomial character.

Lemma 2.7. *Let G be a finite solvable group and let p be a prime number such that p^2 does not divide $\chi(1)$ for every $\chi \in \text{Irr}_m(G)$. Then the Sylow p -subgroup of $\mathbf{F}_{i+1}(G)/\mathbf{F}_i(G)$ has order 1 or p , for every $i \geq 1$.*

Proof. First, notice that it's enough to show the result for the Sylow p -subgroup S/F of F_2/F , where $F := \mathbf{F}(G)$ and $F_2 := \mathbf{F}_2(G)$. Furthermore, since $\mathbf{F}_i(G/\Phi(G)) = \mathbf{F}_i(G)/\Phi(G)$, we may assume $\Phi(G) = 1$. Now, G splits over the abelian normal subgroup F and every $\lambda \in \text{Irr}(F)$ has an extension $\widehat{\lambda} \in \text{Irr}(G_\lambda)$. By the Clifford correspondence $\widehat{\lambda}^G \in \text{Irr}_m(G)$ and so p^2 does not divide $|G : G_\lambda|$. In particular $|S : S_\lambda|$ is either 1 or p . Next observe that, since $\text{Irr}(F)$ is a completely reducible and faithful (S/F) -module, by [Wol99, Theorem A] there exists $\lambda_1 \in \text{Irr}(F)$ such that $|S : F|^{1/2} \leq |S : S_{\lambda_1}|$. In particular S/F is abelian. This, together with [INW99, Lemma 3.1], implies that there exists $\lambda_2 \in \text{Irr}(F)$ such that $F = S_{\lambda_2}$ and we conclude by the previous discussion. \square

Corollary 2.8. *Conjecture A holds whenever G is a solvable group such that 4 does not divide $\chi(1)$, for all $\chi \in \text{Irr}_m(G)$.*

Proof. This follows by Corollary 2.5 and Lemma 2.7. \square

Notice that Lemma 2.7 is an adaptation of [LNW14, Lemma 2.1]. One may wonder if the main result of that paper holds by considering only monomial characters.

Problem 2.9. Let G be a finite solvable group and let p be a prime number such that p^2 does not divide $\chi(1)$ for every $\chi \in \text{Irr}_m(G)$. Is it true that $|G : \mathbf{F}(G)|_p \leq p^2$?

To end this section we mention that, by introducing the ideas used in this section in the setting of Brauer non-vanishing elements, one can prove a monomial version of the main result of [DPS17].

3 Gluck's Conjecture

Gluck's Conjecture has been proved by Espuelas for groups of odd order [Esp91, Theorem 3.2], by Dolfi and Jabara for solvable groups with abelian Sylow 2-subgroups [DJ07, Theorem 1] and by Yang for solvable groups in which 3 does not divide $|G : \mathbf{F}(G)|$ [Yan11, Theorem 2.5]. Other partial results are obtained in [CHMN15]. Moreover, in [MW04, Corollary 2.7] the authors prove that $|G : \mathbf{F}(G)| \leq b(G)^3$ for every solvable group G . We show that Conjecture B holds in all the above mentioned situations. First, we collect some results on actions of solvable linear groups. For every integer n , we denote by $\pi(n)$ the set of prime divisors of n .

Theorem 3.1. *Let V be a finite faithful completely reducible G -module for a finite π -solvable group with $\pi = \pi(|V|)$. Define $\gamma := 1/2$ if one of the following holds:*

- (i) G is solvable and $V \rtimes G$ has abelian Sylow 2-subgroup;
- (ii) G is a solvable $3'$ -group;
- (iii) G is solvable and V is primitive with either $|V| \neq 3^4$ or $|V| = 3^4$ and G is not conjugate to a subgroup of $\text{GL}(V)$ of order 1152;
- (iv) G is π -solvable and $64 \cdot 81$ doesn't divide $|V|$.

Otherwise define $\gamma := 2/3$ if none of the above conditions is satisfied. Then there exists $v \in V$ such that $|\mathbf{C}_G(v)| \leq |G|^\gamma$.

Proof. See [CHMN15, Lemma 11], [DJ07, Theorem 2], [Esp91, Theorem 3.1], [MW04, Corollary 2.6] and [Yan11, Corollary 2.4]. □

In the next proposition, we denote by $\mathbf{F}^*(G)$ the generalised Fitting subgroup of G .

Proposition 3.2. *Let G be a π -solvable group where $\pi := \pi(|\mathbf{F}^*(G/\Phi(G))|)$ and define the module $V := \text{Irr}(\mathbf{F}^*(G/\Phi(G)))$. If γ is as in Theorem 3.1 and $\alpha := 1/(1 - \gamma)$, then*

$$|G : \mathbf{F}(G)| \leq b_m(G)^\alpha$$

where $b_m(G)$ is the largest irreducible monomial character degree of G .

Proof. Since, by hypothesis, $\mathbf{F}^*(G/\Phi(G)) = \mathbf{F}(G/\Phi(G)) = \mathbf{F}(G)/\Phi(G)$, we may assume $\Phi(G) = 1$. Now, $F := \mathbf{F}(G)$ is abelian and $V = \text{Irr}(F)$ is a finite faithful completely reducible (G/F) -module. By Theorem 3.1 there exists $\lambda \in V$ such that $|G_\lambda : F| \leq |G : F|^\gamma$. Let $\widehat{\lambda} \in \text{Irr}(G_\lambda)$ be an extension of λ and consider $\chi := \widehat{\lambda}^G \in \text{Irr}_m(G)$. Then

$$|G : F| = |G : F|^\alpha |G : F|^{-\alpha\gamma} \leq |G : F|^\alpha |G_\lambda : F|^{-\alpha} = |G : G_\lambda|^\alpha = \chi(1)^\alpha \leq b_m(G)^\alpha$$

as claimed above. □

We remark that Conjecture B holds also when $|V| = 3^4$ and G is conjugate to a subgroup of $\text{GL}(V)$ of order 1152. In this case $\text{GL}(V)$ has three conjugacy classes of subgroups of order 1152 and the subgroups of one of these classes do not have orbits of size at least $\sqrt{1152}$. However, using GAP [GAP19] one can check that the result holds for these subgroups as well.

As we have already mentioned, there exist solvable groups in which $b(G)$ is arbitrarily larger than $b_m(G)$.

Example 3.3. Let q and p be odd primes such that $p \equiv -1 \pmod{q}$ and consider P an extraspecial p -group of order p^3 and exponent p . Let $C \leq \text{Aut}(P)$ be a cyclic group of order q acting trivially on $\mathbf{Z}(P)$ and define $G := P \rtimes C$. By elementary character theory, we deduce that the only irreducible character degrees of G are 1, q and p and that no monomial character can have degree p . This shows that $b_m(G) = q < p = b(G)$. Notice that the prime p can be chosen to be arbitrarily large by Dirichlet's theorem on arithmetic progressions.

4 A characterization of normal Sylow p -subgroups

In this final section, we consider the characterization given by Malle and Navarro in [MN12] and obtain Theorem C. We prove a slightly more general result which also implies [PL16, Theorem 1.3] and [PL17, Theorem 1.1 (2)].

Theorem 4.1. *Let G be a finite group, p a prime number and P a Sylow p -subgroup. Set*

$$G_m^*(p) := \bigcap_{\chi} \text{Ker}(\chi),$$

where χ runs over all characters in $\text{Irr}_m(G \mid 1_P)$ which vanish on some p -element of G . If $G_m^(p)$ is solvable, then it has a normal Sylow p -subgroup.*

Proof. Let G be a minimal counterexample. Set $L := G_m^*(p)$ and notice that $L \trianglelefteq G$. Let Q be a Sylow p -subgroup of L . If M is a minimal normal subgroup of G with $M \leq L$, then $L/M = (G/M)_m^*(p)$ and we obtain $QM \trianglelefteq L$, which implies $QM \trianglelefteq G$. Now, if M is a p -group, then it follows that $Q = QM \trianglelefteq L$ and we are done. In particular $\mathbf{O}_p(L) = 1$ and, as L is solvable, we deduce that M is an abelian p' -subgroup. Furthermore, by the Frattini argument, $G = M \rtimes N$ and $L = M \rtimes (N \cap L)$, where $N := \mathbf{N}_G(Q)$. Consider $M < K \leq MQ$ such that K/M is a chief factor of G and set $H := K \cap Q$. Observe that $H \neq 1$ and that $\mathbf{C}_H(M) \leq \mathbf{O}_p(L) = 1$. Therefore H acts faithfully on $\text{Irr}(M)$ and, by [INW99, Lemma 3.1], there exists $\lambda \in \text{Irr}(M)$ such that $H_\lambda = 1$. As a consequence $K_\lambda = M$ and $\vartheta := \lambda^K$ is an irreducible character of K vanishing on the normal subset $K \setminus M$. By Clifford's theorem, every character $\chi \in \text{Irr}(G \mid \vartheta)$ vanishes on $K \setminus M$ and, in particular, on $H \setminus 1$.

Recall that G splits over M and let $\widehat{\lambda}$ be an extension of λ to $T := G_\lambda$. Consider $\alpha_0 := \widehat{\lambda}_{T \cap N}$ and let $\alpha \in \text{Irr}(T/M)$ be the character corresponding to α_0 via the isomorphism $T/M \simeq T \cap N$. Then, $\beta := \widehat{\lambda}\alpha^{-1}$ is a linear character of T lying over λ and $\chi := \beta^G \in \text{Irr}_m(G)$ by the Clifford correspondence. Notice that χ lies over 1_P by the MacKay formula. On the other hand, observe that χ lies over β , hence over λ and so over $\vartheta = \lambda^K$. By the previous paragraph, we deduce that $L = G_m^*(p) \leq \text{Ker}(\chi)$. This implies that $M \leq \text{Ker}(\vartheta)$ and so $\vartheta \in \text{Irr}(K/M)$. Since K/M is abelian, we conclude that $1 = \vartheta(1) = |K : M|$ a contradiction. \square

As an immediate consequence, we obtain the following strong form of Theorem C.

Corollary 4.2. *Let G be a finite solvable group, p a prime and consider $P \in \text{Syl}_p(G)$. Then the following are equivalent:*

- (i) $P \trianglelefteq G$;
- (ii) p does not divide $\chi(1)$, for every $\chi \in \text{Irr}_m(G \mid 1_P)$;
- (iii) $\chi(x) \neq 0$, for every $\chi \in \text{Irr}_m(G \mid 1_P)$ and $x \in P$.

Proof. First notice that (i) implies (ii) and that (ii) implies (iii). Then we conclude by Theorem 4.1. \square

The above result does not hold for arbitrary finite groups. For instance, consider $G = A_5$, $p = 2$ and observe that the irreducible monomial characters of G lying over the principal character of a Sylow p -subgroup are the trivial character and the character of degree 5.

Remark 4.3. At the end of [MN12], the authors conjecture a refinement of their main result, suggesting that the normality of a Sylow p -subgroup P of G should be controlled by those irreducible constituents of $(1_P)^G$ appearing with p' -multiplicity. This result has recently been proved in [GLLV22]. It would be interesting to understand whether this stronger characterisation can be adapted to the framework considered in this paper.

References

- [CHMN15] J. P. Cossey, Z. Halasi, A. Maróti, and H. N. Nguyen. On a conjecture of Gluck. *Math. Z.*, 279(3-4):1067–1080, 2015.
- [CL17] X. Chen and M. L. Lewis. Itô’s theorem and monomial Brauer characters. *Bull. Aust. Math. Soc.*, 96(3):426–428, 2017.
- [CL18] X. Chen and M. L. Lewis. Itô’s theorem and monomial Brauer characters II. *Bull. Aust. Math. Soc.*, 97(2):215–217, 2018.
- [CY20] X. Chen and Y. Yang. Normal p -complements and monomial characters. *Monatsh. Math.*, 193(4):807–810, 2020.
- [DH92] K. Doerk and T. Hawkes. *Finite soluble groups*, volume 4 of *De Gruyter Expositions in Mathematics*. Walter de Gruyter & Co., Berlin, 1992.
- [DJ07] S. Dolfi and E. Jabara. Large character degrees of solvable groups with abelian Sylow 2-subgroups. *J. Algebra*, 313(2):687–694, 2007.
- [DPS17] S. Dolfi, E. Pacifici, and L. Sanus. Nonvanishing elements for Brauer characters. *J. Aust. Math. Soc.*, 102(1):96–107, 2017.
- [Esp91] A. Espuelas. Large character degrees of groups of odd order. *Illinois J. Math.*, 35(3):499–505, 1991.
- [GAP19] The GAP Group. *GAP – Groups, Algorithms, and Programming, Version 4.10.1*, 2019.
- [GLLV22] E. Giannelli, S. Law, J. Long, and C. Vallejo. Sylow branching coefficients and a conjecture of Malle and Navarro. *Bull. Lond. Math. Soc.*, 54(2):552–567, 2022.
- [Glu85] D. Gluck. The largest irreducible character degree of a finite group. *Canad. J. Math.*, 37(3):442–451, 1985.

- [He12] L. He. Notes on non-vanishing elements of finite solvable groups. *Bull. Malays. Math. Sci. Soc.* (2), 35(1):163–169, 2012.
- [Hup67] B. Huppert. *Endliche Gruppen. I*. Die Grundlehren der Mathematischen Wissenschaften, Band 134. Springer-Verlag, Berlin-New York, 1967.
- [Hup98] B. Huppert. *Character theory of finite groups*, volume 25 of *De Gruyter Expositions in Mathematics*. Walter de Gruyter & Co., Berlin, 1998.
- [INW99] I. M. Isaacs, G. Navarro, and T. R. Wolf. Finite group elements where no irreducible character vanishes. *J. Algebra*, 222(2):413–423, 1999.
- [Isa08] I. M. Isaacs. *Finite group theory*, volume 92 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2008.
- [LNW14] M. L. Lewis, G. Navarro, and T. R. Wolf. p -parts of character degrees and the index of the Fitting subgroup. *J. Algebra*, 411:182–190, 2014.
- [MN12] G. Malle and G. Navarro. Characterizing normal Sylow p -subgroups by character degrees. *J. Algebra*, 370:402–406, 2012.
- [MW04] A. Moretó and T. R. Wolf. Orbit sizes, character degrees and Sylow subgroups. *Adv. Math.*, 184(1):18–36, 2004.
- [PL16] L. Pang and J. Lu. Finite groups and degrees of irreducible monomial characters. *J. Algebra Appl.*, 15(4):1650073, 4, 2016.
- [PL17] L. Pang and J. Lu. Finite groups and degrees of irreducible monomial characters II. *J. Algebra Appl.*, 16(12):1750231, 5, 2017.
- [vdW88] R. W. van der Waall. Direct products and monomial characters. *Publ. Math. Debrecen*, 35(1-2):149–153, 1988.
- [Wol99] T. R. Wolf. Large orbits of supersolvable linear groups. *J. Algebra*, 215(1):235–247, 1999.
- [Yan09] Y. Yang. Orbits of the actions of finite solvable groups. *J. Algebra*, 321(7):2012–2021, 2009.
- [Yan11] Y. Yang. Large character degrees of solvable $3'$ -groups. *Proc. Amer. Math. Soc.*, 139(9):3171–3173, 2011.

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