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Context-free graphs and their transition groups

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Abstract

Starting from context-free inverse graphs, we introduce a new class of groups and study their structural properties. We establish closure properties, show that their coword problems are context-free, analyze torsion elements, and realize them as subgroups of the asynchronous rational group. Context-freeness is preserved under a generalized free product of graphs, and using this construction, we provide examples of groups that are not residually finite or not polycontext-free, making them relevant for testing the Lehnert and Brough conjectures. Moreover, we investigate how small local modifications of a graph affect the global structure of the transition group, showing that for locally quasi-transitive graphs with infinite orbits, the transition group decomposes into a highly structured quotient by a bounded torsion subgroup, showing strong global constraints induced by local graph properties.

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1 | INTRODUCTION

In combinatorial group theory, the relationship among graphs, formal languages, and groups has been significantly advanced through the investigation of the word problem. Given a group

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G generated by a finite symmetric set A, each element can be represented by many words over A. The *word problem* of G with respect to A is the collection of words representing the identity, or, equivalently, the words that label every closed path in the Cayley graph of G with respect to A. Throughout the vast literature, one of the main projects is to fully adapt the famous Chomsky hierarchy to finitely generated groups. The first efforts toward this goal have been made by Anisimov [2], who classified the class of all regular groups as that of all finite groups, and later by Muller and Schupp [26, 27], who have shown that the class of all context-free groups is that of all virtually free groups. Such results can be rephrased verbatim in the language of graph theory. Further developments of their renowned theorem were pursued by Ceccherini-Silberstein and Woess [12], and later by Rodaro [29] to generalize it in a more combinatorial setting to a certain class of digraphs. We also recall that Herbst [17] has proved that a group is one-counter if and only if it is virtually cyclic as an intermediate step in the characterization. Again, we have a generalization for this due to Holt, Owens, and Thomas [18], showing that finitely generated groups having an intersection of finitely many one-counter languages as word problem are virtually abelian.

The next step of the hierarchy is that of context-sensitive groups. Currently, many intermediate classes have been provided with examples of groups belonging to them: polycontext-free (i.e., its word problem is a finite intersection of context-free languages), cocontext-free, co-ET0L, and coindexed, among others. Several famous examples, such as bounded automata groups and $\mathbb{Z} * \mathbb{Z}^2$, were recently proven to belong to co-ET0L [1, 6]. On the other hand, the classes of polycontext-free and cocontext-free groups come with their universal container conjectures.

Lehnert Conjecture. *A finitely generated group is cocontext-free group if and only if it embeds into Thompson V.*

This conjecture is here restated in the formulation given by [7].

Brough Conjectures. *The following conjectures are due to Brough:*

(1) *A finitely generated group is polycontext-free if and only if it is a finitely generated subgroup of a virtual direct product of free groups.*
(2) *A finitely generated solvable group is polycontext-free if and only if it is virtually abelian.*
(3) *If a group G is polycontext-free, then G does not have arbitrarily large finite subgroups.*

The main contributions of this paper lie in the fruitful interplay between group theory and graph theory, which we use to provide new constructions. These techniques allow us to translate algebraic properties of groups into a combinatorial framework, which, in turn, enables a precise analysis of the associated formal languages. The deep interaction between graph structures, context-free languages, and groups is well established in the literature, as demonstrated by the aforementioned works in [12, 26, 27, 29] and further evidenced by [23, 33]. In particular, starting from the class of tree-like graphs given by context-free inverse graphs Γ (i.e., deterministic involutive labeled digraphs with finitely many end-cone types; see Section 3), we define a new group, called the *context-free transition group* $\mathcal{G}(\Gamma)$. This group is generated by the permutations of the vertices induced by the labeled directed edges of Γ. We then extend this construction to a finite collection of disjoint context-free graphs $\Gamma^{(1)}, \dots, \Gamma^{(k)}$, and denote the class of all such groups by *CF-TR*. Although this class has a concrete combinatorial description, its algebraic properties remain largely unexplored, making it an intriguing subject of study in its own right. The paper is organized as follows. Sections 2 and 3 are dedicated to give some background on inverse graphs

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and to recall the characterization of context-free inverse graphs. In Section 4, we introduce our class of groups and prove some closure properties that allow us to describe any finitely generated subgroup of a virtual direct product of free groups as a transition group. Moreover, we prove the following result that frames our research in the context of the Lehnert conjecture.

Theorem 4.11. *Let $\Gamma^{(1)}, \dots, \Gamma^{(k)}$ be context-free inverse A-digraphs that are also complete, then the word problem of $\mathcal{G}(\Gamma^{(1)} \sqcup \dots \sqcup \Gamma^{(k)})$ is a cocontext-free language.*

It is well known that Grigorchuk's group and $\mathbb{Z} * \mathbb{Z}^2$ are natural candidates for disproving Lehnert's conjecture (see [8]). Thus, CF-TR groups could provide counterexamples to the conjecture, should one of these candidates admit a CF-TR realization. The first candidate, however, is not a transition group, by the result proved in Section 5.

Corollary 5.8. *An infinite group $\mathcal{G}\left(\Gamma^{(1)} \sqcup \cdots \sqcup \Gamma^{(k)}\right) \in \mathbf{CF-TR}$ does not contain an infinite finitely generated torsion subgroup.*

In the same section, in particular, in Proposition 5.4, we show some upper bound of the order of an element with respect to its length. As a consequence, we reprove the solvability of the torsion problem, which is already known for the broader class of cocontext-free groups (see [19]). In Section 6, we introduce the subclass of poly-inverse-context-free groups and establish some of their structural properties. In particular, we show that they can be realized as transition groups of context-free graphs.

Theorem 6.1. *Let G be a group whose word problem $WP(G; A)$ is the intersection of k context-free languages accepted by rooted graphs:*

$$WP(G; A) = \bigcap_{i=1}^{k} L(\Gamma^{(i)}, x_i)$$

Then, $G \simeq \mathcal{G}(\Gamma^{(1)} \sqcup \dots \sqcup \Gamma^{(k)})$ is in CF-TR.

By assuming the graphs to be highly symmetric, that is, quasi-transitive, we are able to characterize their transition groups.

Theorem 6.2. *Let G be a group. The following are equivalent:*

(1) *$G = \mathcal{G}(\Gamma^{(1)} \sqcup \cdots \sqcup \Gamma^{(k)})$ for some quasi-transitive context-free A-graphs $\Gamma^{(i)}$;*
(2) *G is virtually a f.g. subgroup of a direct product of free groups;*
(3) *G belongs to Poly-QT-ICF, that is, the word problem is the intersections of finitely many languages accepted by graphs that are quasi-transitive and quasi-isometric to a tree.*

These last theorems fit into the framework of Brough's conjectures. In [15], the rational group $\mathcal{R}$ of homeomorphisms of the Cantor set, described by sequential transducers (see the beginning of Section 7), has been introduced. Interesting examples of groups embedding into $\mathcal{R}$ include the Thompson group V and any hyperbolic group [3]. In this paper, we prove the following embedding theorem.

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Theorem 7.7. *The transition group* $\mathcal{G}\left(\Gamma^{(1)} \sqcup \cdots \sqcup \Gamma^{(k)}\right)$ *of a collection of context-free graphs* $\Gamma^{(1)}, \ldots, \Gamma^{(k)}$ *is rational.*

In Section 8, we investigate the effect of small local modifications of a graph on its transition group. More broadly, this concerns the extent to which local changes can influence the global algebraic structure. A key result concerns the transition group G of a locally quasi-transitive context-free inverse graph Γ. Roughly speaking, Theorem 8.6 shows that G decomposes into a well-understood part and a "small" torsion part: there is a short exact sequence

$$\mathbb{1} \longrightarrow \mathscr{B}_F \longrightarrow G \longrightarrow H \longrightarrow \mathbb{1},$$

where H is virtually a finitely generated subgroup of a direct product of free groups, and $\mathscr{B}_F$ is a torsion normal subgroup with elements of controlled order. In other words, G is an extension of a highly structured group by a torsion subgroup, showing how local perturbations imposes strong global constraints on the transition group.

Motivated by the fact that the direct product of two transition groups is again a transition group (see Section 4), one may naturally ask whether the same holds for the free product. Although this question remains open, and a positive answer could solve Lehnert's conjecture, in Section 9, we introduce a notion of free product for context-free graphs and show that the resulting product graph remains context-free (see Theorem 9.5). Building on this construction and partially leveraging the techniques from Section 8, Section 10 presents three notable examples of context-free transition groups.

(1) A nonresidually finite example.
(2) The group $\mathbb{Z} \wr \mathbb{Z}$ that is nonfinitely presented, solvable and not polycontext-free. In particular, this shows that the class of polycontext-free and *CoCF* is different.
(3) The group $C_2 \wr \mathbb{Z}$ that has an infinite torsion subgroup.

2 | INVOLUTIVE AND INVERSE LABELED DIGRAPHS

Let A be a finite *alphabet*. We assume throughout that A is *involutively closed*, meaning that $a \in A$ implies $a^{-1} \in A$. The free monoid on A is denoted as A^*, with identity 1, and A^+ denotes the free semigroup. A word $w \in A^*$ is said to be *reduced* if it contains no factor of the form uu^{-1} for some $u \in A$. Every word $w \in A^*$ has a unique reduced form $\overline{w}$, obtained iteratively by deleting all such factors. The set $R(A)$ of reduced words is a regular language. The free group F_A can be identified with $R(A)$, where the product is given by $u \cdot v = \overline{uv}$. With a slight abuse of notation, we will denote the identity of F_A by 1, while for a generic group G, we will use the notation $\mathbb{1}_G$. We use the following notion of digraph, in the sense of Serre.

Definition 2.1 (Inverse graph). An *A-digraph* is a tuple $\Gamma = (V, E, \iota, \tau, \lambda)$ consisting of a vertex set V, a set of directed edges E, maps $\iota, \tau : E \to V$ assigning to each edge its initial and terminal vertex, and a labeling map $\lambda : E \to A$. We write $v_1 \xrightarrow{a} v_2$ to denote an edge $e \in E$ such that $\iota(e) = v_1$, $\tau(e) = v_2$, and $\lambda(e) = a$. An *involutive A-digraph* is a tuple $\Gamma = (V, E, \iota, \tau, \lambda, \cdot^{-1})$ such that $(V, E, \iota, \tau, \lambda)$ is an A-digraph and $\cdot^{-1} : E \to E$ is an involution, that is, $(e^{-1})^{-1} = e$ for all $e \in E$, satisfying:

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(1) $\lambda(e^{-1}) = \lambda(e)^{-1}$ for all $e \in E$;
(2) $\iota(e^{-1}) = \tau(e)$ and $\tau(e^{-1}) = \iota(e)$ for all $e \in E$;
(3) $e \neq e^{-1}$ for all $e \in E$.

Moreover, we say that Γ is:

- *deterministic* if for every $v \in V$ and $a \in A$, there is at most one outgoing edge $v \xrightarrow{a} v'$;
- *complete* if for every $v \in V$ and $a \in A$, there is at least one outgoing edge labeled by a.

An involutive A-digraph is called *inverse* if it is deterministic and connected.

A subgraph $\Gamma' = (V', E', \iota, \tau, \lambda, \cdot^{-1})$ of Γ satisfies $V' \subseteq V$ and $E' \subseteq E$, and $e^{-1} \in E$ for all $e \in E'$, and we write $\Gamma' \subseteq \Gamma$. From now on, except for Section 9, we will refer to inverse A-digraphs simply as A-graphs, and whenever the alphabet A does not need to be specified, we will simply write graphs.

Definition 2.2. A *walk* in Γ is a finite or infinite sequence of edges $p = e_1 e_2 \cdots e_k \cdots$ such that $\tau(e_i) = \iota(e_{i+1})$ for all i. For a finite walk $p = e_1 \cdots e_k$, the *label* is $\lambda(p) = \lambda(e_1) \cdots \lambda(e_k) \in A^+$. The walk is said to be *reduced* if it contains no subwalk of the form ee^{-1}. We denote the initial and terminal vertices of p by $\iota(p)$ and $\tau(p)$; when Γ is deterministic and $x \in V$, a word $w \in A^*$ determines at most one walk $x \xrightarrow{w} y$. In case the walk p has the property $\iota(p) = \tau(p)$, p is called a *circuit*.

When we fix a vertex x_0 in Γ, the pair (Γ, x_0) is called a *rooted graph*. In this case, we see (Γ, x_0) as a *language acceptor* by using the distinguished vertex x_0 as an initial and final state. In this case, the language accepted by (Γ, x_0) is given by the set $L(\Gamma, x_0)$ formed by the labels of all circuits with initial vertex x_0. Note that $L(\Gamma, x_0)$ together with the usual operation of concatenation is also a submonoid of the free monoid A^*. More generally, an *A-automaton* has an initial state and a set $F \subseteq V(\Gamma)$ of final states, and the language $L(\Gamma, x_0, F)$ is the set formed by the labels of the walks p with $\iota(p) = x_0, \tau(p) \in F$. In case Γ is finite, $L(\Gamma, x_0, F)$ belongs to the class of regular languages [20].

The class of A-graphs forms a category where a morphism $\varphi : \Gamma^{(1)} \to \Gamma^{(2)}$ between the graphs $\Gamma^{(1)} = (V_1, E_1, \iota_1, \tau_1, \lambda_1, \cdot^{-1})$ to $\Gamma^{(2)} = (V_2, E_2, \iota_2, \tau_2, \lambda_2, \cdot^{-1})$ is a pair of maps $\varphi_e : E_1 \to E_2$, $\varphi_v : V_1 \to V_2$ such that:

- φ respects the involution: $\varphi_e(g^{-1}) = \varphi_e(g)^{-1}$ for all $g \in E_1$;
- φ commutes with the incidence and labeling maps:

$$\tau_2(\varphi_e(g)) = \varphi_v(\tau_1(g)), \quad \iota_2(\varphi_e(g)) = \varphi_v(\iota_1(g)), \quad \lambda_2(\varphi_e(g)) = \lambda_1(g).$$

Given an (inverse) subgraph $\Lambda \subseteq \Gamma^{(1)}$, its image $\varphi(\Lambda)$ is the (inverse) subgraph of $\Gamma^{(2)}$ with vertices $\varphi_v(V(\Lambda))$ and edges $\varphi_e(E(\Lambda))$. An isomorphism (resp. monomorphism) is a morphism φ such that both φ_v and φ_e are bijective (resp. injective).

Metric structure. If Γ is connected, we endow $V(\Gamma)$ with the usual walk metric

$$d(u, v) = \min\{\ell(p) : p \text{ is a walk from } u \text{ to } v\},$$

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where $\ell(p)$ denotes the length (number of edges) of p. A walk realizing the minimum is called a *geodesic*. For $v \in V(\Gamma)$ and $n \geqslant 0$, we denote by

$$D_n(v) = \{ u \in V(\Gamma) : d(v,u) \leqslant n \} \qquad \text{and} \qquad S_n(v) = \{ u \in V(\Gamma) : d(v,u) = n \}$$

the *disk* (ball) and the *sphere* of radius n, respectively, centered at v. For a connected subgraph $Y \subseteq \Gamma$, the diameter is

$$\delta_\Gamma(Y) = \max\{d(y_1, y_2) : y_1, y_2 \in V(Y)\}.$$

An interesting feature of inverse graphs is given by the following.

Lemma 2.3. *Let Γ be an A-graph. For any vertex $v \in V(\Gamma)$ and word $u \in A^*$, there is at most one walk $v \xrightarrow{u} y$ labeled by u. Moreover, if such a walk exists, then there is also a reduced walk $v \xrightarrow{\overline{u}} y$ labeled by the free reduction $\overline{u}$ of u.*

2.1 | Basic facts about morphisms of graphs

The condition of determinism and connectedness of an inverse graph has some consequences when we consider morphisms in the category of A-graphs. For instance, a morphism $\psi : \Gamma \to \Lambda$ is completely determined by the vertex map ψ_v. In particular, ψ is a monomorphism if for any $y \in V(\Lambda)$, the preimage $\psi_v^{-1}(y)$ contains at most one element and checking if two morphisms are equal is just a matter of verifying if they agree on a single vertex.

Lemma 2.4. *Let φ_1, φ_2 be two morphisms from the graph $\Gamma^{(1)}$ to the graph $\Gamma^{(2)}$, then $\varphi_1 = \varphi_2$ if and only if $\varphi_1(x) = \varphi_2(x)$ for some $x \in V(\Gamma)$.*

Proof. For any walk $x \xrightarrow{u} y$, the images under φ_1 and φ_2 are $\varphi_1(x) \xrightarrow{u} \varphi_1(y), \varphi_2(x) \xrightarrow{u} \varphi_2(y)$. Determinism of $\Gamma^{(2)}$ and $\varphi_1(x) = \varphi_2(x)$ implies $\varphi_1(y) = \varphi_2(y)$. Since $\Gamma^{(1)}$ is connected, φ_1 and φ_2 coincide on all vertices, and hence also on all edges. □

Another interesting feature of graphs, and particularly rooted graphs, is the relationship between accepted languages and morphisms. This connection is contained in the following proposition belonging to folklore, see, for instance, [29, Proposition 1].

Proposition 2.5. *Let $(\Gamma, x_0), (\Lambda, y_0)$ be two rooted graphs, then $L(\Gamma, x_0) \subseteq L(\Lambda, y_0)$ if and only if there is a morphism $\psi : \Gamma \to \Lambda$ with $\psi(x_0) = y_0$. Moreover, $L(\Gamma, x_0) = L(\Lambda, y_0)$ if and only if there is an isomorphism $\psi : \Gamma \to \Lambda$.*

Note that by the previous proposition, we have that, up to isomorphism, there is a unique rooted graph accepting the language $L(\Gamma, x_0)$. The set of automorphisms of the graph Γ will be denoted by $\mathrm{Aut}(\Gamma)$. This is a group acting on Γ in a canonical way. In case the quotient graph $\Gamma/\mathrm{Aut}(\Gamma)$ is finite, we say that Γ is *quasi-transitive*. Observe that this last condition is equivalent to the finiteness of $V(\Gamma)/\mathrm{Aut}(\Gamma)$. Any morphism $\varphi : \Gamma \to \Lambda$ between two graphs induces an injection $\varphi : \mathrm{Star}(\Gamma, v) \to \mathrm{Star}(\Gamma', \varphi(v))$ between the star sets in the obvious ways. Thus, morphisms

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of graphs are *immersions* of graphs as defined by Stallings [32]. In particular, if the graphs are complete, by Lemma 2.3, every morphism between two complete A-graphs is a *covering*, that is, an immersion of graphs $f : \Gamma \to \Lambda$ so that f maps the sets $V(\Gamma)$ and $E(\Gamma)$, respectively, onto $V(\Lambda)$ and $E(\Lambda)$. One central feature of coverings is the possibility of lifting walks, this is contained in the following characterization.

Proposition 2.6 [16, Proposition 2.3]. *Let $\varphi : \Gamma' \to \Gamma$ be a covering of graphs. Then, for each vertex $v \in V(\Gamma)$ and vertex $v' \in \varphi^{-1}(v)$, every walk p in Γ with $\iota(p) = v$ lifts to a unique walk p' starting at v', that is, $\varphi(p') = p$.*

3 | CONTEXT-FREE LANGUAGES AND GRAPHS

There are several ways to define context-free languages, either via the notion of grammar, or using the notion of pushdown automaton. For the sake of conciseness, we briefly recall the main notions that are used in the paper, and we refer the reader to [20] for further details.

Definition 3.1. A *context-free grammar* is a tuple (V, Σ, P, S) where V is a finite set of nonterminal symbols (or variables), Σ is the finite alphabet of terminal symbols that is disjoint from V, $S \in V$ is the starting symbol, and $P \subseteq V \times (V \cup \Sigma)^*$ is a finite set called productions.

Usually, productions are denoted by $X \Rightarrow \alpha$ with $X \in V$ and $\alpha \in (V \cup \Sigma)^*$. We extend $\Rightarrow$ by putting $\alpha X \gamma \Rightarrow \alpha\beta\gamma$ whenever $(X \Rightarrow \beta) \in P$, and we denote by $\Rightarrow^*$ the reflexive and transitive closure of $\Rightarrow$. The language generated by the grammar (V, Σ, P, S) is $L(S) = \{u \in \Sigma^* : S \Rightarrow^* u\}$, and in this case, we say that it is a *context-free language*. An L language with a context-free complement $\Sigma^* \setminus L$ is called *cocontext-free*, we will denote the class of context-free languages as *CoCF*.

In their paper, Muller and Schupp called a finitely generated group G presented by $\langle A \mid \mathcal{R}\rangle$ *context-free* if the word problem $WP(G; A) = \{w \in A^* \mid \pi(w) = \mathbb{1}_G\}$ is a context-free language, where $\pi : A^* \to G$ denotes the natural projection. With our notation, this language coincides with the set $L(\mathrm{Cay}(G; A), \mathbb{1}_G)$ of words labeling a circuit starting and ending at the group identity.

Definition 3.2. A rooted graph (Γ, x_0) is called *context-free* if $L(\Gamma, x_0)$ is a context-free language.

This definition is equivalent to a more geometric one that involves the notion of end-cones. We briefly recall it. Henceforth, we denote by $d(x, y)$ the usual distance in the graph Γ, and put $\|v\|_{x_0} = d(x_0, v)$. In [27], Muller and Schupp introduced the central notion of end-cone together with the idea of end-isomorphism of such subgraphs. For a vertex v with $\|v\|_{x_0} = n$, the *end-cone at v* is the subgraph $\Gamma(v, x_0)$ that is the connected component of $\Gamma \setminus D_{n-1}(x_0)$ containing v, and we denote by $\Delta(v, x_0) = \{y \in \Gamma(v, x_0) : \|y\|_{x_0} = n\}$ the set of *frontier vertices* of $\Gamma(v, x_0)$, that is, the vertices of $\Gamma(v, x_0)$ at distance n from the root x_0. An *end-isomorphism* $\psi : \Gamma(v, x_0) \to \Gamma(w, x_0)$ between the end-cones $\Gamma(v, x_0), \Gamma(w, x_0)$ is an isomorphism of graphs with the property of preserving their frontier vertices: $\psi(\Delta(v, x_0)) = \Delta(w, x_0)$.

Definition 3.3. The rooted graph (Γ, x_0) is called *a context-free graph in the sense of Muller–Schupp* if up to end-isomorphism, there are finitely many end-cones, that is, there are graphs $(\Gamma_1, \Delta_1), \ldots, (\Gamma_N, \Delta_N)$ such that for any end-cone $\Gamma(v, x_0)$ of Γ, there is an end-isomorphism $\psi : \Gamma(v, x_0) \to \Gamma_j$ for some $j \in [1, N]$.

The previous two notions are equivalent.

Proposition 3.4 [29, Proposition 5]. *For a graph* Γ*, the following conditions are equivalent:*

(1) *for any nonempty finite set* $F \subseteq V(\Gamma)$*, the language* $L(\Gamma, x_0, F)$ *is deterministic context-free;*
(2) *the language* $L(\Gamma, x_0)$ *is deterministic context-free, hence* (Γ, x_0) *is context-free;*
(3) *the rooted graph* (Γ, x_0) *is context-free in the sense of Muller–Schupp;*
(4) *for any vertex* $y_0 \in V(\Gamma)$*, the graph* (Γ, y_0) *is context-free in the sense of Muller–Schupp.*

The languages accepted by context-free graphs can be characterized more precisely through the construction of suitable pushdown automata, called inverse PDA. In [29], such languages are called *inverse-context-free*, and we will use this term throughout the paper.

4 | TRANSITION GROUPS OF GRAPHS AND COCONTEXT-FREE GROUPS

In this section, we introduce a new subclass of cocontext-free groups. For an A-graph Γ, the set of *everywhere-circuits* is

$$\mathcal{L}(\Gamma) = \{u \in A^* : \forall p \in V(\Gamma) \text{ there is a circuit } p \xrightarrow{u} p \text{ in } \Gamma\} = \bigcap_{p \in V(\Gamma)} L(\Gamma, p) \tag{1}$$

It is easy to see that if Γ is an A-graph that is also complete, then $\mathcal{L}(\Gamma)$ is a normal subgroup of F_A. The following is a key notion in our paper.

Definition 4.1. For a complete A-graph Γ, each letter $a \in A$ defines a permutation on the set of vertices $a : V(\Gamma) \to V(\Gamma)$. The *transition group* $\mathcal{G}(\Gamma)$ of the graph Γ is the group generated by composing all these permutations.

Remark 4.2. Note that this definition is a specialization of the well-known notion of transition monoid for an automaton to the case of inverse automata. This is also isomorphic to the syntactic monoid of the language $L = L(\Gamma, x_0)$ (where x_0 is any chosen vertex of Γ). Indeed, it is not difficult to see that if one considers the syntactic congruence $\sim_\Gamma$ on A^* defined by $u \sim_L v$ if, for all $x, y \in A^*$, we have $xuy \in L$ if and only if $xvy \in L$, then since Γ is inverse, we have that $A^*/_{\sim_L}$ is isomorphic to $\mathcal{G}(\Gamma)$. If we remove the completeness condition, the syntactic monoid is an inverse monoid, see [14, 30].

The group $\mathcal{G}(\Gamma)$ may be equivalently obtained by considering the quotient of the free group F_A generated by A by the normal subgroup $\mathcal{L}(\Gamma)$. There is a natural right action of $\mathcal{G}(\Gamma)$ on the set of vertices $V(\Gamma)$: for any $g \in \mathcal{G}(\Gamma)$ and vertex $v \in V(\Gamma)$, $v \cdot g \in V(\Gamma)$ is the vertex for which there is a walk $v \xrightarrow{w} (v \cdot g)$ in Γ for any $w \in A^*$ that represents g. We sometimes also put $v \cdot w = v \cdot g$ for any $w \in A^*$ representing g in $\mathcal{G}(\Gamma)$.

Example 4.3. For any group G on the symmetric generating set A, it is straightforward to check that the transition group of the Cayley graph $\mathrm{Cay}(G; A)$ is isomorphic to G. In particular, if Γ is either a finite n-cycle or an infinite line and by suitably labeling all edges either by a or a^{-1} from

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a single-letter alphabet $A = \{a\}$, we can turn Γ into a complete graph so that its transition group is readily seen to be either isomorphic to $\mathbb{Z}/n\mathbb{Z}$ or $\mathbb{Z}$, so that all these groups are transition groups of context-free graphs.

More in general, the rooted graph (Γ, p) may be naturally identified with the Schreier graph of the stabilizer subgroup $H = \mathcal{G}(\Gamma)_p = \{g \in \mathcal{G}(\Gamma) : g(p) = p\}$. This fact follows from Proposition 2.5 and the equality $L(\Gamma, p) = L(\mathrm{Sch}(H, A), H)$. On the other hand, given a group G generated by A and a subgroup H, the Schreier graph of H, that is, the graph which vertices are right coset of H and direct edges are labeled by elements in A, is an example of a complete A-graph. In view of this, one can provide an algebraic characterization of transition groups of a graph.

Proposition 4.4. *A group G is a transition group for a complete inverse A-graph Γ if and only G admits a core-free subgroup H whose Schreier graph is isomorphic to Γ.*

Proof. Let p be any vertex of Γ and let $H := G_p$ be the stabilizer of p. Since for any $g \in G$ the conjugate $G_p^g = G_{p \cdot g}$, the normal core of H (the intersection of all conjugates of H) coincides with the intersection of all vertex stabilizers in Γ. By the definition of a transition group, this intersection is trivial. Moreover, it is easy to see that the map $p \cdot g \mapsto H \cdot g$ is a well-defined isomorphism from Γ to $\mathrm{Sch}(H, A)$. Conversely, consider the Schreier graph $\mathrm{Sch}(H, A)$ of H. As previously discussed, it is a complete A-graph on which G acts naturally. Let w be a word in $\mathcal{L}(\mathrm{Sch}(H, A))$. By definition, w represents an element of G that stabilizes all vertices of the Schreier graph; that is, $Hg \cdot \overline{w} = Hg$ for all $g \in G$. Therefore, $\overline{w}$ belongs to the normal core of H, which is trivial, so $w \in WP(G, A)$. This shows that $\mathcal{L}(\mathrm{Sch}(H, A)) \subseteq WP(G, A)$. The reverse inclusion is straightforward; hence, G is indeed the transition group of $\mathrm{Sch}(H, A)$. □

Remark 4.5. By the previous proposition, we immediately deduce that the growth of the group $\mathcal{G}(\Gamma)$ is bounded below by the growth of the graph Γ.

We actually aim to work in a slightly more general setting. Let $\{\Gamma^{(1)}, \dots, \Gamma^{(k)}\}$ be a finite collection of complete A-graphs. As before, we can consider the permutations of vertices induced by the letters of A. This allows us to define the *transition group* associated with the collection as

$$\mathcal{G}(\Gamma^{(1)} \sqcup \dots \sqcup \Gamma^{(k)}) := {}^{F_A}\!/_{\mathcal{L}(\Gamma^{(1)}) \cap \dots \cap \mathcal{L}(\Gamma^{(k)})}.$$

It is convenient to view a transition group over a finite union of graphs as a subgroup of the direct product of the corresponding transition groups over single graphs. More precisely, we have the following fact.

Lemma 4.6. *Let $\Gamma^{(1)}, \dots, \Gamma^{(k)}$ be complete A-graphs. Then, $\mathcal{G}(\Gamma^{(1)} \sqcup \dots \sqcup \Gamma^{(k)}) \hookrightarrow \mathcal{G}(\Gamma^{(1)}) \times \dots \times \mathcal{G}(\Gamma^{(k)})$, that is, the transition group of the union is isomorphic to a subgroup of the direct product of the individual transition groups.*

Proof. Let $G = \mathcal{G}(\Gamma^{(1)} \sqcup \dots \sqcup \Gamma^{(k)})$. Define the map $\psi : G \to \mathcal{G}(\Gamma^{(1)}) \times \dots \times \mathcal{G}(\Gamma^{(k)})$ by putting $g = [u] \mapsto (\pi_1(u), \dots, \pi_k(u))$, where $\pi_i : A^* \to \mathcal{G}(\Gamma^{(i)})$ are the natural projections. By definition, $WP(G; A) = \bigcap_{i=1}^k \mathcal{L}(\Gamma^{(i)})$, so ψ is well defined. Moreover, ψ is injective: if $\pi_i(u) = \mathbb{1}$ for all i, then $u \in \bigcap_i \mathcal{L}(\Gamma^{(i)}) = WP(G; A)$; hence, $g = \mathbb{1}$ in G. □

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4.1 | Transition groups of context-free graphs

We now show that the transition group of any complete context-free graph belongs to *CoCF*. Henceforth, whenever we speak about transition groups associated with context-free graphs, we will implicitly assume, unless explicitly stated otherwise, that the graphs under consideration are complete.

Before proving the next lemma, we need to recall the following definition. For a language $L \subseteq B^* \subseteq A^*$, the *shift* of L, is the language $S_B(L) = \{xy \in B^* \mid yx \in L,\ x \in B^*,\ y \in B^*\}$ formed by all the words that may be obtained by a cyclic shift of some word of L. If $B = A$ and it is clear from the context, we drop the subscript A and we simply refer to $S(L)$. It is a well-known fact (see, for instance, Ex. 6.4 of [20]) that for a context-free language L, $S(L)$ is also context-free. For the next lemma, we need the following definition. Let $\pi = v \xrightarrow{u^\omega} \dots$ be an infinite walk, for some $u \in A^*$. We say that a vertex p is *colored by a prefix* u' of u if $u = u'u''$ and

$$\pi = v \xrightarrow{u^s} v' \xrightarrow{u'} p \xrightarrow{u''} p' \xrightarrow{u^\omega} \dots$$

for some integer $s \geqslant 0$.

Lemma 4.7. *Let $\mathcal{G}(\Gamma)$ be the transition group of a complete context-free graph Γ, and let $u \in A^*$ represent a nontrivial element of $\mathcal{G}(\Gamma)$. Then, there exist an end-cone $\Gamma(v, x_0)$ and a cyclic shift $\mathring{u} \in S(\{u\})$ such that the infinite walk $\pi = v \xrightarrow{\mathring{u}^\omega} \cdots$ is contained in $\Gamma(v, x_0)$. Moreover, for every vertex p on π whose color is a prefix u' of u, say $u = u'u''$, one has $p \cdot (u''u') \neq p$. In particular, $v \cdot \mathring{u} \neq v$.*

Proof. Since u represents a nontrivial element of $\mathcal{G}(\Gamma)$, there exists a vertex $q_1 \in V(\Gamma)$ such that $q_1 \cdot u \neq q_1$. Because Γ is complete, we may consider the infinite one-sided walk $\pi' = q_1 \xrightarrow{u^\omega} \cdots$. Choose a vertex v on π' that has minimal distance from the basepoint x_0 among all vertices of π'. Then the whole tail of π' starting at v is contained in the end-cone $\Gamma(v, x_0)$. Now write $u = rs$ in such a way that the tail of π' starting at v is labeled by the cyclic shift $\mathring{u} = sr \in S(\{u\})$, so we obtain an infinite walk $\pi = v \xrightarrow{\mathring{u}^\omega} \cdots$ contained in $\Gamma(v, x_0)$. Let p be a vertex on π whose color is a prefix u' of u, with $u = u'u''$. If one had $p \cdot (u''u') = p$, then, by determinism of Γ, the corresponding segment of π' labeled by u would also close up, implying $q_1 \cdot u = q_1$, a contradiction. Hence, $p \cdot (u''u') \neq p$. Taking $p = v$, whose color is the prefix r of $u = rs$, we obtain $v \cdot \mathring{u} \neq v$. □

For the next theorem, we need to know that the union of finitely many context-free languages is context-free, and for a deterministic context-free language, the complement is also deterministic context-free, see [20].

Theorem 4.8. *Let Γ be a context-free A-graph that is also complete, then $\mathcal{L}(\Gamma) \in$ **CoCF**.*

Proof. Let $\{\Gamma_1, \dots, \Gamma_N\}$ be the end-cones types of Γ and let Δ_i be the corresponding frontier vertices. For each $q \in \Delta_i$, we denote by

$$N(q) = \{u \in A^* : q \xrightarrow{u} p \text{ is a walk in } \Gamma_i \text{ with } p \neq q\}$$

the set of words that do not label a circuit at q in Γ_i. We claim:

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$$\mathcal{L}(\Gamma)^c = \bigcup_{i=1}^{N} \left(\bigcup_{q \in \Delta_i} S(N(q)) \right).$$

Let us prove that the right-hand side is contained in the left-hand side. Note that every walk $q \xrightarrow{u} p$ in Γ_i that does not label a circuit at q has the same property in Γ in any subgraph that is end-isomorphic to Γ_i, that is, $u \in \mathcal{L}(\Gamma)^c$. Moreover, take the word yx such that $xy \in N(q)$. Since $xy \neq \mathbb{1}_G$ implies $yx \neq \mathbb{1}_G$, we deduce that $yx \in S(N(q))$ satisfies $yx \in \mathcal{L}(\Gamma)^c$.

Let us now prove the other inclusion. So, let $u \in \mathcal{L}(\Gamma)^c$, by Lemma 4.7, there exists an end-cone $\Gamma(v, x_0)$ s.t. the infinite walk $v \xrightarrow{\mathring{u}^\omega} \ldots \subseteq \Gamma(v, x_0)$ with $\mathring{u} \in S(\{u\})$ and $v \cdot \mathring{u} \neq v$. Hence, if $\psi : \Gamma(v, x_0) \to \Gamma_i$ in the end-isomorphism that specifies the end-cone type, we have that $u \in S(N(\psi(v)))$, and this concludes the proof of the other inclusion. To show that $\bigcup_{i=1}^{N} \left(\bigcup_{q \in \Delta_i} S(N(q)) \right)$ is context-free, it is enough to show that each $N(q)$ is context-free, since context-free languages are closed under the cyclic-shift operator, and a finite union of context-free languages is context-free. Fix $q \in \Delta_i$. Define a deterministic A-graph $\widehat{\Gamma}_i$ as follows. Add to Γ_i a new vertex $\perp$; for every $q' \in \Delta_i$ and $a \in A$, if there is no edge $q' \xrightarrow{a}$ in Γ_i, add the edge $q' \xrightarrow{a} \perp$; finally add the loops $\perp \xrightarrow{a} \perp$, for all $a \in A$. The graph $\widehat{\Gamma}_i$ is no longer inverse, but it is deterministic. Consider the A-automaton

$$(\widehat{\Gamma}_i, q, \{q, \perp\}).$$

A word $u \in A^*$ is accepted if either $q \xrightarrow{u} q$ in Γ_i, or $q \xrightarrow{u} \perp$ in $\widehat{\Gamma}_i$. Hence, the nonaccepted words are exactly those labeling a walk $q \xrightarrow{u} p$ in Γ_i with $p \neq q$. Thus,

$$L(\widehat{\Gamma}_i, q, \{q, \perp\})^c = N(q).$$

Therefore, by using the fact that deterministic context-free is closed by taking the complement, to conclude that $N(q)$ is context-free, it is enough to prove that $L(\widehat{\Gamma}_i, q, \{q, \perp\})$ is deterministic context-free. Since Γ_i is an inverse context-free graph, by [29, Proposition 5], the rooted graph (Γ_i, q) is isomorphic to the rooted configuration graph $(\Gamma(M_i), (q_0, \varepsilon))$ of an inverse PDA M_i. Thus, computations of M_i starting from (q_0, ε) correspond to walks in Γ_i starting from q. Since Δ_i is finite, after a finite control refinement, equivalently after unfolding finitely many stack words into the control, we may assume that every $r \in \Delta_i$ is represented by a configuration (σ_r, ε). This preserves determinism and the rooted labeled graph up to isomorphism. We define a deterministic PDA $\widehat{M}_i$. On the part corresponding to Γ_i, it simulates M_i. Add one new state $s_\perp$. For every $r \in \Delta_i$ and $a \in A$, if there is no edge $r \xrightarrow{a}$ in Γ_i, add the transition $(\sigma_r, \varepsilon) \xrightarrow{a} (s_\perp, \varepsilon)$. For every $a \in A$, add also $(s_\perp, \varepsilon) \xrightarrow{a} (s_\perp, \varepsilon)$. No other transition is added. The new a-transition is inserted only when the corresponding a-transition is missing, hence $\widehat{M}_i$ is deterministic. We take as final states $\sigma_q, s_\perp$, and we accept with empty stack; this matches the automaton above because q is represented by (σ_q, ε), and $s_\perp$ is reached only from configurations (σ_r, ε), $r \in \Delta_i$. Hence, $L(\widehat{M}_i) = L(\widehat{\Gamma}_i, q, \{q, \perp\})$ is deterministic context-free, and this concludes the proof. □

Remark 4.9. In the previous proof, we have shown that each language $N(q)$ is a deterministic context-free language. If $S(N(q))$ were deterministic for all $q \in \Delta_i$, then, by taking the complement, we would deduce that $\mathcal{L}(\Gamma)$ is an intersection of finitely many context-free languages, that

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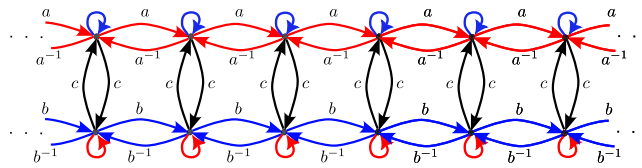


FIGURE 1 The graph Ω. The loops in red are labeled by a, a^{-1}, in blue by b, b^{-1}.

is, $\mathcal{L}(\Gamma)$ is polycontext-free. However, it seems that the cyclic closure of a context-free language is generally far from being deterministic, making the class of transition groups of context-free graphs quite different from that of polycontext-free groups. In fact, we are going to find an example of a transition group that is not polycontext-free (see Definition 10.5).

As an immediate consequence of the previous proposition, we obtain the following theorem.

Theorem 4.10. *Let* Γ *be a context-free* A*-graph that is also complete. Then, the transition group* $\mathcal{G}(\Gamma)$ *is CoCF.*

In the theorem above, the connectedness of Γ is not strictly necessary and since *CoCF* languages are closed under finite intersections, we have the following.

Theorem 4.11. *If* $\Gamma^{(1)}, \dots, \Gamma^{(k)}$ *are context-free* A*-graphs that are complete, then the transition group* $\mathcal{G}(\Gamma^{(1)} \sqcup \dots \sqcup \Gamma^{(k)})$ *is CoCF.*

Definition 4.12 (*CF-TR* groups). In view of Theorem 4.11, we may define the subclass of CoCF groups formed by all the transition groups of a finite disjoint collection of context-free A-graphs, denoted by CF-TR.

Question 4.13. There are several natural questions for such a class of groups.

(1) What is the relationship with the class of polycontext-free groups?
(2) Is the class CF-TR strictly included into CoCF?

4.2 | An example

As a concrete example of a transition group of a context-free graph, we consider the A-graph Ω on the alphabet $A = \{a, a^{-1}, b, b^{-1}, c, c^{-1}\}$ (see Figure 1). The set of vertices $V(\Omega) = \{p_i, q_i, i \in \mathbb{Z}\}$ and the set $E(\Omega)$ of edges is given by

$$p_i \xrightarrow{a} p_{i+1},\ p_{i+1} \xrightarrow{a^{-1}} p_i,\ q_i \xrightarrow{b} q_{i+1},\ q_{i+1} \xrightarrow{b^{-1}} q_i, \quad \forall i \in \mathbb{Z}$$

$$p_i \xrightarrow{b} p_i,\ p_i \xrightarrow{b^{-1}} p_i,\ q_i \xrightarrow{a} q_i,\ q_i \xrightarrow{a^{-1}} q_i, p_i \xrightarrow{x} q_i, q_i \xrightarrow{x} p_i, x \in \{c, c^{-1}\} \quad \forall i \in \mathbb{Z}$$

We sometimes identify the generic vertex p_n with the pair $(n, 0)$, $n \in \mathbb{Z}$, while q_m with the pair $(m, 1)$, $m \in \mathbb{Z}$.

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Note that Ω is clearly quasi-transitive and it is quasi-isometric to a tree; thus, by [29, Theorem 2], it is a context-free graph. The next proposition characterizes the transition group defined by Ω.

Proposition 4.14. *The transition group of Ω has the following properties:*

- *the subgroup $\langle a, b\rangle$ generated by a, b of the transition group $\mathcal{G}(\Omega)$ is isomorphic to $\mathbb{Z}^2$;*
- *$\mathcal{G}(\Omega)$ is isomorphic to $G = \langle a, b, c \mid c^2 = 1, ab = ba, ca = bc\rangle$, so it is a semidirect product $\mathbb{Z}^2 \rtimes C_2$;*
- *the action on $V(\Omega)$ is given by the formula $(x, y) \cdot a^n b^m c^j = (x + ym + (1 - y)n,\ y + j \mod 2)$ where $a^n b^m c^j$ with $m, n \in \mathbb{Z}$, $j \in \{0, 1\}$ is the standard normal form of the semidirect product.*

Proof. The formula of the action can be obtained by observing that for a generic vertex $(x, y) \in V(\Omega)$, we have

$$(x, y) \cdot a^n = (x + (1 - y)n, y),\ (x, y) \cdot b^m = (x + ym, y),\ (x, y) \cdot c^j = (x,\ y + j \mod 2).$$

Note that for any vertex v in Ω, we have $v \cdot ab = v \cdot ba$; hence, a, b commute in $\mathcal{G}(\Omega)$ ($ab = ba$ in $\mathcal{G}(\Omega)$). Similarly, one can check that for any vertex v, we have $v \cdot c^2 = v$ and $v \cdot (ca) = v \cdot (bc)$. Hence, in $\mathcal{G}(\Omega)$, we also have the equalities $c^2 = 1, ca = bc$ and also $ac = cb$ as a consequence of the involution $c^2 = 1$. This means that there is an epimomorphism $\phi : G \to \mathcal{G}(\Omega)$ that maps $a^n b^m c^j \in G$ with $m, n \in \mathbb{Z}$, $j \in \{0, 1\}$ into the corresponding element in $\mathcal{G}(\Omega)$. It suffices to show that ϕ is injective: consider two elements $a^n b^m c^j, a^{n'} b^{m'} c^{j'}$, then for any vertex $(x, y) \in V(\Omega)$, the equation $(x, y) \cdot a^n b^m c^j = (x, y) \cdot a^{n'} b^{m'} c^{j'}$ implies $j = j'$ and if we choose a vertex with $y = 0$, then we conclude that $n = n'$, while if we choose $y = 1$, we get $m = m'$. □

4.3 | Closure properties

In this section, we show that this class is closed under taking finitely generated subgroups, direct products, and finite extensions. We fix H a subgroup of G that is finitely generated by $B \subseteq A^*$ and we set $n := \max_{b \in B} \ell(b)$ with $\ell(b)$ the length in terms of A.

Lemma 4.15. *Let $g \in H$ be a nontrivial element represented by $u \in B^* \subseteq A^*$. Then, there exist an end-cone $\Gamma(v, x_0)$ and a vertex $\widetilde{v} \in \Gamma(v, x_0)$ with $d(v, \widetilde{v}) \leqslant n$ such that the infinite walk $\widetilde{v} \xrightarrow{\mathring{u}^\omega} \ldots \subseteq \Gamma(v, x_0)$ for some $\mathring{u} \in S_B(\{u\})$ and $\widetilde{v} \cdot \mathring{u} \neq \widetilde{v}$.*

Proof. By Lemma 4.7, there exists an end-cone $\Gamma(v, x_0)$ and a cyclic shift $\widehat{u} \in S(\{u\})$ such that the infinite walk $\pi = v \xrightarrow{\widehat{u}^\omega} \ldots$ is contained in $\Gamma(v, x_0)$. By the definition of n, possibly removing a prefix of π of length at most n, we may choose a vertex $\widetilde{v} \in \pi$ such that $d(v, \widetilde{v}) \leqslant n$, colored by a prefix u' of u, with $u = u'u''$, and set $\mathring{u} = u''u' \in S_B(\{u\})$. By construction, $\mathring{u}$ satisfies the required properties, namely, $\widetilde{v} \xrightarrow{\mathring{u}^\omega} \ldots \subseteq \Gamma(v, x_0)$ and $\widetilde{v} \cdot \mathring{u} \neq \widetilde{v}$. □

Proposition 4.16. *The class of CF-TR is closed by taking finitely generated subgroups.*

Proof. Suppose first that we are dealing with one context-free graph, so that H is a finitely generated subgroup of $\mathcal{G}(\Gamma)$, where Γ is a graph that is also context-free. Let $B \subseteq A^*$ be the involutive

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closed generating set and let $\Delta_1, \dots, \Delta_N$ be the frontier vertices of the end-cone types $\{\Gamma_1, \dots, \Gamma_N\}$. We set $\Delta_i^n := \{v \in \Gamma_i \mid d(x,v) \leqslant n \text{ for some } x \in \Delta_i\}$. With a slight abuse of notation, we identify the frontier vertices of Δ_i with the frontier vertices of an end cone $\Gamma(v, x_0)$ of Γ that is end-isomorphic to Γ_i. We do the same for the set Δ_i^n. By definition of transition group, we need to show that H is the transition group of a finite union of complete context-free B-graphs Θ_B. We claim

$$\Theta_B = \bigcup_{i=1}^{N} \bigcup_{v \in \Delta_i^n} (\Gamma^B, v)$$

where (Γ^B, v) is given by the vertices y of Γ such that there exists $w \in B^*$ and $v \xrightarrow{w} y$. The edges are $y_1 \xrightarrow{w} y_2$ whenever $w \in B$ and there exists a walk labeled by w from y_1 to y_2 in Γ. It is not difficult to see that since B is involutive closed and Γ is inverse, then also (Γ^B, v) is inverse. Moreover, it is complete and context-free.

Let us show that the complement of the word problem of H and $\mathcal{G}(\Theta_B)$ coincides, as this will, of course, imply that their word problems coincide. Let $g \in \mathcal{G}(\Theta_B)$ be a nontrivial element, then there exists $\widetilde{v} \in (\Gamma^B, v)$ for some $v \in \Delta_i^n$ such that $\widetilde{v} \cdot g \neq \widetilde{v}$. It follows that $\widetilde{v} \xrightarrow{g} v'$ with $\widetilde{v} \neq v'$ is also a walk in Γ, and so, $g \neq \mathbb{1}_H$ in H.

Conversely, suppose that $\mathbb{1}_H \neq g \in H$, by Lemma 4.15, there exists an end-cone $\Gamma(x, x_0)$ and a vertex $v \in \Gamma(x, x_0)$ with $d(x,v) \leqslant n$ such that $v \xrightarrow{\mathring{g}^\omega} \dots \subseteq \Gamma(x, x_0)$ and $v \cdot \mathring{g} \neq v$ for some $\mathring{g} \in S_B(\{u\})$. Now $\Gamma(x, x_0)$ is end-isomorphic to some Γ_i; hence, we can find $\widetilde{v} \in \Delta_i^n$ such that $\widetilde{v} \xrightarrow{\mathring{g}^\omega} \dots \subseteq \Gamma_i$ and $\widetilde{v} \cdot \mathring{g} \neq \widetilde{v}$. A fortiori, $\widetilde{v} \xrightarrow{\mathring{g}^\omega} \dots \subseteq (\Gamma, \widetilde{v})$, whence $\widetilde{v} \xrightarrow{\mathring{g}^\omega} \dots \subset (\Gamma^B, \widetilde{v})$. Now in $(\Gamma^B, \widetilde{v})$, we can read g on a walk $v' \xrightarrow{u} \widetilde{v} \xrightarrow{s} v''$ such that $\widetilde{v} \xrightarrow{s} v''$ is a subwalk of $\widetilde{v} \xrightarrow{\mathring{g}^\omega} \dots$ and where $v' \neq v''$; otherwise, determinism would imply $\widetilde{v} \cdot \mathring{g} = \widetilde{v}$. In particular, g is nontrivial in $\mathcal{G}(\Theta_B)$. The previous argument may easily be adapted when considering a finite family of disjoint context-free graphs. □

Note that the previous proposition gives a constructive way to compute context-free graphs defining a finitely generated subgroup of a transition group of a context-free graphs Γ.

Proposition 4.17. *The class of CF-TR is closed by taking direct products.*

Proof. Let $\mathcal{G}(\Gamma_i)$ be a transition group of a context-free A_i-graph with $i = 1, 2$. We can assume that $A_1 \cap A_2$ is empty, otherwise we change accordingly the labels in A_2. Now we consider $\widehat{\Gamma}_1 \sqcup \widehat{\Gamma}_2$ where $\widehat{\Gamma}_i$ is a $A_1 \cup A_2$-graph in which for every vertex there are $|A_j|$ loops each labeled by an element in A_j with $j \neq i$. It is straightforward that $\mathcal{G}(\widehat{\Gamma}_1 \sqcup \widehat{\Gamma}_2)$ is the desired direct product. □

The technique to prove the last closure property is inspired by [19] and, in particular, it is based on the *inflated graph* construction introduced in [29] combined to the strategy in the proof of Proposition 4.16.

Definition 4.18. Let H be a group generated by the finite set A that is a finite index subgroup of G and let T be a right transversal for G with respect to H containing the identity. For each

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$y \in A \cup T - \{\mathbb{1}_G\}$ and $t \in T$, we fix a word $h_{t,y} \in A^*$ such that $ty =_G h_{t,y}\widetilde{t}$ for some $\widetilde{t} \in T$. We define the *Schreier automaton* $\mathcal{A}$ over $A \cup T$ to be the finite state automaton with set of states T, initial state $\mathbb{1}_G \in T$ and $t \xrightarrow{y|h_{t,y}} \widetilde{t}$ when $ty =_G h_{t,y}\widetilde{t}$.

Starting from a Schreier automaton, we can build one that is inverse, that is,

$$t \xrightarrow{y|h_{t,y}} \widetilde{t} \iff \widetilde{t} \xrightarrow{y^{-1}|h_{t,y}^{-1}} t,$$

without changing the groups involved H and G. The way we do so is by enlarging the alphabet for the labels from $A \cup T$ to $A \cup \{T, T^{-1}\}$ (remember A is already inverse by assumption) and by adding the inverse transitions $\widetilde{t} \xrightarrow{y^{-1}|h_{t,y}^{-1}} t$. The fact that we are still dealing with H and G follows from the equation $\widetilde{t}y^{-1} =_G h_{t,y}^{-1}t$, which is true if and only if $ty =_G h_{t,y}\widetilde{t}$. We call it the *inverse Schreier automaton* and with a slight abuse of notation we denote it again by $\mathcal{A}$.

Definition 4.19. Let Γ be a complete A-graph and let $H = \mathcal{G}(\Gamma)$ be the associated transition group. Consider G to be a group with H as a finite index subgroup and $\mathcal{A}$ the corresponding Schreier automaton over $A \cup T$. The *inflated graph* $\mathcal{A} \ltimes (\Gamma, x)$ with $x \in V(\Gamma)$ is the rooted $A \cup T$-graph defined as follows: the set of vertices is $T \times V(\Gamma)$, a generic edge is $(t, p) \xrightarrow{y} (t', p')$ with $t \xrightarrow{y|u} t' \in \mathcal{A}$ and $p \xrightarrow{u} p'$ a walk in Γ, and the root is $(\mathbb{1}_G, x)$.

Intuitively, the inflated graph simulates a cascade composition of the Schreier automaton $\mathcal{A}$ with the graph Γ: each transition of $\mathcal{A}$ labeled by $y \mid u$ triggers the traversal of the word u inside Γ. This construction is not genuinely new, and it can be viewed as a composition of sequential machines (transducers) in disguise, where the graph Γ itself is interpreted as a transducer. It is straightforward to see that $\mathcal{A} \ltimes \Gamma$ is inverse and if Γ is context-free, so is $\mathcal{A} \ltimes \Gamma$ (see the proof of [29, Lemma 11]).

Proposition 4.20. *The class of CF-TR is closed under taking finite-index overgroups.*

Proof. Let G be a finitely generated group and let $H = \mathcal{G}(\Gamma)$ be a finite-index subgroup of G. Let T be a transversal of G with respect to H containing the identity, so that $H = \langle A \rangle$ and $G = \langle A \cup T \rangle$, and let $\mathcal{A}$ be the corresponding Schreier automaton.

Let $\{\Delta_1, \dots, \Delta_N\}$ be the frontier vertices. Arguing as in Proposition 4.16, identify the frontier vertices of each Δ_i with the frontier vertices of an end-cone $\Gamma(v, x_0)$ of Γ end-isomorphic to Γ_i. Set

$$\Theta_{\mathcal{A}} := \bigcup_{i=1}^{N} \bigcup_{v \in \Delta_i} (\mathcal{A} \ltimes (\Gamma, v), (v, \mathbb{1}_G)).$$

We prove that $G \simeq \mathcal{G}(\Theta_{\mathcal{A}})$. Let $B := A \cup T$. For $t \in T$ and $u \in B^*$, write $t \xrightarrow{u|h_t(u)} \tau_t(u)$ for the walk in the Schreier automaton $\mathcal{A}$. Thus, $\tau_t(u) \in T$, $h_t(u) \in H$, and $tu =_G h_t(u)\tau_t(u)$. We claim that, for every inflated component and every vertex (p, t),

$$(p, t) \cdot u = (p \cdot h_t(u), \tau_t(u)).$$

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The claim follows by induction on $|u|$. The case $u = \varepsilon$ is immediate. For the induction step, write $u = wx$, with $x \in B$. If $t \xrightarrow{w|h_t(w)} \tau_t(w)$ and $\tau_t(w) \xrightarrow{x|h_{\tau_t(w)}(x)} \tau_{\tau_t(w)}(x)$, then, by the definition of $\mathcal{A} \ltimes (\Gamma, v)$, we have

$$(p, t) \cdot wx = (p \cdot h_t(w)h_{\tau_t(w)}(x), \tau_{\tau_t(w)}(x)).$$

This is exactly $(p \cdot h_t(u), \tau_t(u))$. We first show that the action of B^* on $\Theta_{\mathcal{A}}$ factors through G. Let $u \in B^*$ be such that $u =_G 1$. Then, $tu =_G t$ for every $t \in T$. Hence, $\tau_t(u) = t$ and $h_t(u) =_H 1$. Since $H = \mathcal{G}(\Gamma)$, the word $h_t(u)$ fixes every vertex of Γ. Therefore, $(p, t) \cdot u = (p, t)$ for every vertex (p, t) of every inflated component. Thus, every word trivial in G is trivial in $\mathcal{G}(\Theta_{\mathcal{A}})$, and the action is well defined as an action of G.

It remains to prove faithfulness. Let $g \neq 1$ in G, and let $u \in B^*$ represent g. If there exists $t \in T$ such that $\tau_t(u) \neq t$, then, for every admissible p, the formula above gives $(p, t) \cdot u = (p \cdot h_t(u), \tau_t(u)) \neq (p, t)$. Hence, g acts nontrivially on $\Theta_{\mathcal{A}}$. Assume instead that $\tau_t(u) = t$ for every $t \in T$. Taking $t = \mathbb{1}_G$, we get $u =_G h_{\mathbb{1}_G}(u) \in H$. Since $g \neq 1$ in G, the element $h_{\mathbb{1}_G}(u)$ is nontrivial in H. By Lemma 4.7, there exist $i \in \{1, \dots, N\}$, $v \in \Delta_i$, and $p' \in V(\Gamma, v)$ such that $p' \cdot h_{\mathbb{1}_G}(u) \neq p'$. Hence,

$$(p', \mathbb{1}_G) \cdot u = (p' \cdot h_{\mathbb{1}_G}(u), \mathbb{1}_G) \neq (p', \mathbb{1}_G).$$

Thus, g acts nontrivially on $\Theta_{\mathcal{A}}$. Therefore, the induced action of G on $\Theta_{\mathcal{A}}$ is faithful, and $G \simeq \mathcal{G}(\Theta_{\mathcal{A}})$. □

Remark 4.21. This means that any finitely generated subgroup of a virtual direct product of free groups is CF-TR. In particular, finitely generated virtually abelian groups are in CF-TR.

5 | DETECTING TORSION

We give some structural results focusing on the existence of torsion elements and on bounds on the order of such elements. Henceforth, we consider a context-free A-graph Γ with N end-cone types $\Gamma_1, \dots, \Gamma_N$ with the corresponding frontier vertices $\Delta_1, \dots, \Delta_N$ with $\Delta_i \subseteq \Gamma_i$. For any end-cone type Γ_i that is end-isomorphic to some end-cone $\Gamma(v, x_0)$ with $\|v\|_{x_0} = n$, there are $k_i \leqslant 2|A||\Delta_i|$ second-level end-cones $\Gamma(q_1, x_0), \dots, \Gamma(q_{k_i}, x_0)$ contained in $\Gamma(v, x_0)$ with $\|q_j\|_{x_0} = n + 1$. Roughly speaking, we are taking the end-cones of Γ_i whose frontier vertices are at distance one from Δ_i. Put $\gamma = \max\{k_i, i \in [1, N]\}$ and $\delta = \max\{|\Delta_i|, i \in [1, N]\}$.

We say that two end-cones $\Gamma(p, x_0), \Gamma(q, x_0)$ are *nested* if $\Gamma(p, x_0) \subseteq \Gamma(q, x_0)$ and there is an end-isomorphism $\psi : \Gamma(q, x_0) \to \Gamma(p, x_0)$ with $\psi(q) = p$. We have the following lemma.

Lemma 5.1. *Let $\Gamma(p, x_0), \Gamma(q, x_0)$ be two distinct end-isomorphic nested end-cones of Γ. Then, for any walk $q \xrightarrow{u} p \subseteq \Gamma(q, x_0)$, there is an infinite walk $q \xrightarrow{u^\omega} \dots \subseteq \Gamma(q, x_0)$ that is not a circuit.*

Proof. Let $\psi : \Gamma(q, x_0) \to \Gamma(p, x_0)$ be the end-isomorphism with $\psi(q) = p$ and consider a geodesic $q \xrightarrow{g} \psi(q) \subseteq \Gamma(q, x_0)$ connecting q with $\psi(q) = p$.

Since ψ preserves the labels, for each $k \in \mathbb{N}$, one gets a walk

$$\psi^k(q) \xrightarrow{u} \psi^{k+1}(q)$$

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included in $\psi^k(\Gamma(q,x_0)) \subseteq \Gamma(\psi(q),x_0)$, which does not contain any vertex in $\Delta(q,x_0)$, and, in particular, not q. This means that no finite concatenation of this walks can be a circuit. □

A subgraph Π of Γ is called *g-periodic* if Π is equal to the subgraph induced by the circuit $\pi = y \xrightarrow{g^k} y$ for some integer $k \geqslant 1$. The *order* $o(g,\Pi)$ is the smallest integer m such that Π is equal to the subgraph induced by the vertices and edges of the circuit $\pi = y \xrightarrow{g^m} y$. A *clip* is a walk $y_1 \xrightarrow{u} y_2 \subseteq \Gamma(y_1,x_0)$ with $y_1,y_2 \in \Delta(y_1,x_0)$. For a clip p, we denote by $(\chi(p),\subseteq)$ the poset of all subwalks of p that are clips ordered by inclusion.

Lemma 5.2. *For a clip $p = y_1 \xrightarrow{u} y_2$, let $h(\chi(p))$ be the length of a maximal chain in $\chi(p)$. Then, $h(\chi(p)) \geqslant \max\{\|x\|_{x_0} - \|y_1\|_{x_0} : x \in p\} + 1$.*

Proof. Let $k = \max\{\|x\|_{x_0} - \|y_1\|_{x_0} : x \in p\}$. We proceed by induction on k. The case $k = 0$ is trivial. Assume now $k > 0$, and let x be a vertex of p with $\|x\|_{x_0} - \|y_1\|_{x_0} = k$. Since p is a clip contained in $\Gamma(y_1,x_0)$ and reaches x, there is a second-level end-cone $\Gamma(q_1,x_0)$ of $\Gamma(y_1,x_0)$ containing x such that p enters it at some frontier vertex q_1, reaches x, and leaves it at some frontier vertex q_2. Let $p' = q_1 \xrightarrow{u'} q_2$ be the corresponding subwalk of p. Then, p' is a clip contained in $\Gamma(q_1,x_0)$, and $p' \subsetneq p$. Since $\Gamma(q_1,x_0)$ is a second-level end-cone, we have $\|x\|_{x_0} - \|q_1\|_{x_0} = k-1$, so $\max\{\|z\|_{x_0} - \|q_1\|_{x_0} : z \in p'\} = k-1$. By the induction hypothesis, $h(\chi(p')) \geqslant k$. Since $p' \subsetneq p$, every maximal chain in $\chi(p')$ extends to a chain in $\chi(p)$ by adding p. Hence, $h(\chi(p)) \geqslant h(\chi(p')) + 1 \geqslant k+1$. □

Lemma 5.3. *For any g-periodic subgraph Π of a context-free graph Γ, we have*

$$o(g,\Pi) \leqslant \begin{cases} \delta\,\gamma^{\delta^2 N|g|+1} & \text{if } \gamma > 1, \\ \delta\,(\delta^2 N|g|+1) & \text{if } \gamma = 1. \end{cases}$$

Proof. Let Π be the subgraph induced by the walk $\pi = y \xrightarrow{g^\ell} y$, where $\ell = o(g,\Pi)$. Note that, without loss of generality, by cyclically shifting g if necessary, we may assume that $\pi \subseteq \Gamma(y,x_0)$, so that π is a clip. For any clip $\lambda \in \chi(\pi)$ with $\lambda = y_1 \xrightarrow{u} y_2$ and $y_1,y_2 \in \Delta(y_1,x_0)$, we say that λ is *colored* by (x_1,x_2,Γ_i,g') if the following conditions hold:

- $\lambda \subseteq \Gamma(y_1,x_0)$ and there exists an end-isomorphism $\psi : \Gamma(y_1,x_0) \to \Gamma_i$ such that $x_1 = \psi(y_1), x_2 = \psi(y_2) \in \Delta_i$;
- g' is a prefix of g, and there exists an integer $s \geqslant 0$ such that $y \xrightarrow{g^s g'} y_1$.

Let us prove first the following claim:

$$\text{if } \lambda_1,\lambda_2 \in \chi(\pi), \lambda_1 \subsetneq \lambda_2, \text{then } \lambda_1,\lambda_2 \text{ have different colors.} \tag{2}$$

Suppose by contradiction that claim (2) does not hold, and λ_1,λ_2 have the same color (x_1,x_2,Γ_i,g'). Therefore, λ_1 and λ_2 can be factored as follows:

$$\lambda_2 = y_1 \xrightarrow{g''} q_1 \xrightarrow{g^m} q_2 \xrightarrow{g'} y_1' \xrightarrow{g''u_2} y_2' \xrightarrow{u_3} y_2, \qquad \lambda_1 = y_1' \xrightarrow{g''u_2} y_2',$$

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for some $m \geqslant 0$, where $g'g'' = g$. Since λ_1 and λ_2 have the same color, there exists an end-isomorphism $\psi : \Gamma(y_1, x_0) \to \Gamma(y_1', x_0)$ between two nested end-cones $\Gamma(y_1', x_0) \subseteq \Gamma(y_1, x_0)$. Hence, by Lemma 5.1, there exists an infinite walk $y_1 \xrightarrow{(g''g^m g')^\omega} \ldots = y_1 \xrightarrow{(g''g')^\omega} \ldots$ that is not a circuit, and this implies that Π is infinite, a contradiction.

By the determinism and the minimality of ℓ, each vertex $q \in V(\Pi)$ occurs at most $|g|$ times in the vertices appearing in the description of the walk $\pi = y \xrightarrow{g^\ell} y$. Hence, the length of π is at most $|g|\,|V(\Pi)|$, from which we deduce $|g|\ell \leqslant |g|\,|V(\Pi)|$, that is, $\ell \leqslant |V(\Pi)|$. Consider the function

$$M(g) := \begin{cases} \delta\,\gamma^{C(g)+1} & \text{if } \gamma > 1, \\ \delta\,(C(g)+1) & \text{if } \gamma = 1, \end{cases}$$

where $C(g)$ denotes the number of possible colors. We claim that $|V(\Pi)| \leqslant M(g)$. Suppose, for the sake of contradiction, that $|V(\Pi)| > M(g)$. Now, by the tree-like structure of Γ, the number of vertices $z \in \Gamma(y, x_0)$ such that $\|z\|_{x_0} - \|y\|_{x_0} \leqslant n$ is at most $\delta \sum_{i=0}^{n} \gamma^i$, which is equal to $\delta \frac{\gamma^{n+1}-1}{\gamma-1}$ if $\gamma > 1$, and to $\delta(n+1)$ if $\gamma = 1$. Hence, if all vertices of Π satisfied $\|z\|_{x_0} - \|y\|_{x_0} \leqslant C(g)$, then we would have $|V(\Pi)| \leqslant M(g)$. Therefore, the assumption $|V(\Pi)| > M(g)$ implies that there exists a vertex $z \in V(\Pi)$ such that $\|z\|_{x_0} - \|y\|_{x_0} > C(g)$. Then, by Lemma 5.2, we conclude that the length of a maximal chain $\lambda_1 \subseteq \lambda_2 \subseteq \ldots \subseteq \lambda_m$ in $\chi(\pi)$ is strictly greater than $C(g)+1$, that is, $m > C(g)$. Therefore, by the pigeonhole principle, there exist $\lambda_i \subsetneq \lambda_j$ with the same color, contradicting our claim (2). The lemma now follows by the upper bound $C(g) \leqslant \delta^2 N |g|$. □

The following proposition provides information about the order of an element.

Proposition 5.4. *The following facts hold:*

- *let $\Gamma(p, x_0), \Gamma(q, x_0)$ be two distinct end-isomorphic nested end-cones of Γ, and let $q \xrightarrow{u} p$ be a walk included in $\Gamma(q, x_0)$. Then, the element $u \in \mathcal{G}(\Gamma)$ has infinite order.*
- *Let $g \in \mathcal{G}(\Gamma)$ be an element of finite order. Then, the order $o(g)$ of g satisfies*

$$o(g) \leqslant \begin{cases} \delta^{N\delta|g|}\gamma^{(\delta^2 N|g|+1)N\delta|g|} = \exp(O(|g|^2)) & \text{if } \gamma > 1, \\ \left(\delta(\delta^2 N|g|+1)\right)^{N\delta|g|} = \exp(O(|g|\log|g|)) & \text{if } \gamma = 1. \end{cases}$$

Proof. The first claim follows from Lemma 5.1: since $q \xrightarrow{u^\omega} \ldots$ is an infinite walk that is not a circuit, we have $u^i \cdot q \neq u^j \cdot q$ for all $i \neq j$. Let us now prove the bound on the torsion element g. Note that $o(g)$ is upper bounded by the least common multiple of all possible values $o(g, \Pi)$, where Π ranges over all g-periodic subgraphs. By Lemma 5.3, these subgraphs are, up to end-isomorphism, finitely many; denote this finite family by $C = \{\Pi_1, \ldots, \Pi_k\}$. Each $\Pi \in C$ is entirely contained in some end-cone Γ_i and is the subgraph induced by a circuit $y \xrightarrow{g_1^\ell} y$, where g_1 is a cyclic shift of g. Since there are at most $|g|$ cyclic shifts of g, there can be at most $|g|$ g-periodic graphs $\Pi \subseteq \Gamma_i$ with $y \in \Delta_i \cap V(\Pi)$. Therefore, we have $|C| \leqslant N\delta|g|$. Now, the least common multiple of the integers

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$o(g, \Pi)$, with $\Pi \in C$, is bounded by their product. Hence, by Lemma 5.3,

$$o(g) \leqslant \begin{cases} \left(\delta\, \gamma^{\delta^2 N|g|+1}\right)^{N\delta|g|} & \text{if } \gamma > 1, \\ \left(\delta(\delta^2 N|g|+1)\right)^{N\delta|g|} & \text{if } \gamma = 1. \end{cases}$$

This yields

$$o(g) \leqslant \begin{cases} \delta^{N\delta|g|}\gamma^{(\delta^2 N|g|+1)N\delta|g|} = \exp(O(|g|^2)) & \text{if } \gamma > 1, \\ \left(\delta(\delta^2 N|g|+1)\right)^{N\delta|g|} = \exp(O(|g|\log|g|)) & \text{if } \gamma = 1, \end{cases}$$

as claimed. □

Remark 5.5. Note that in the case of a finite union of context-free graphs $\{\Gamma^{(1)}, \dots, \Gamma^{(k)}\}$, the first part of the proposition clearly remains valid. As for the second part, instead of considering the parameters δ and N, one must take $\max_{i\in\{1,\dots,k\}} \delta_i$ and $\max_{i\in\{1,\dots,k\}} N_i$, where δ_i and N_i denote the respective parameters of $\Gamma^{(i)}$.

Question 5.6. Is there a CF-TR group realizing the upper bound provided in the second point of Proposition 5.4?

As a consequence of the previous proposition, we have the following.

Proposition 5.7. *Let* $\{\Gamma^{(1)}, \dots, \Gamma^{(k)}\}$ *be a collection of context-free graphs, and let* $G = \mathcal{G}\left(\Gamma^{(1)} \sqcup \cdots \sqcup \Gamma^{(k)}\right) \in$ **CF-TR** *be the associated transition group. Then, the following are equivalent:*

(1) *G contains a copy of* $\mathbb{Z}$*;*
(2) *G is infinite;*
(3) *at least one of the graphs* $\Gamma^{(1)}, \dots, \Gamma^{(k)}$ *is infinite.*

Proof. (1) ⇒ (2) is trivial. For (2) ⇒ (3), note that if all $\Gamma^{(i)}$ are finite, then G is finite. For (3) ⇒ (1), assume without loss of generality that $\Gamma^{(1)}$ is infinite. Since there are finitely many end-cone types but infinitely many end-cones, the pigeonhole principle guarantees the existence of at least two nested end-cones. By Proposition 5.4, this produces an element of infinite order. □

Another consequence of the previous facts is the following.

Corollary 5.8. *An infinite group* $\mathcal{G}\left(\Gamma^{(1)} \sqcup \cdots \sqcup \Gamma^{(k)}\right) \in$ **CF** − **TR** *does not contain an infinite finitely generated torsion subgroup.*

Proof. This follows from the closure property under taking finitely generated subgroups, as stated in Proposition 4.16 and Proposition 5.7. □

It remains an open problem whether the Grigorchuk group belongs to the class *CoCF* (see [13]), and more generally, whether there are infinite torsion groups in the class *CoCF*. However, using the previous corollary, we can show the following.

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Corollary 5.9. *No infinite torsion group can be CF-TR. In particular, the Grigorchuk group is not CF-TR.*

Proposition 5.4 provides an alternative proof of the decidability of the torsion problem for the transition groups of context-free graphs by checking whether $g^k = \mathbb{1}_G$ for some $k \leqslant \exp(O(|g|^2))$. This decidability result can also be deduced from the corresponding result for the class of cocontext-free groups (see [19, Theorem 9]).

Theorem 5.10. *With the above notation, for a given element $g \in \mathcal{G}\big(\Gamma^{(1)} \sqcup \cdots \sqcup \Gamma^{(k)}\big)$, there is an algorithm to check whether $o(g)$ is finite or not. In other words, the torsion problem is solvable for CF-TR groups.*

We know that for F a free group of rank at least 2, the group $F \times F$ is in *CF-TR*. Indeed, F is in *CF-TR* simply by considering the Cayley graph on free generators and, by Proposition 4.17, the class is closed under direct products. This provides an example of a *CF-TR* group with unsolvable conjugacy problem (see, e.g., [25, Theorem 9, Ch.3]) and unsolvable generalized word problem [24].

6 | POLYCONTEXT-FREE GROUPS AND COCONTEXT-FREE TRANSITION GROUPS

In this section, we address the first question of Problem 4.13 and, at the end, discuss some conjectures on polycontext-free groups. We first focus on groups whose word problem is the intersection of finitely many inverse-context-free languages (denoted as *Poly-ICF*), that is, intersections of context-free languages accepted by graphs. The following theorem shows that this class is contained in *CF-TR*; it corresponds to the special case of intersections of deterministic context-free languages, which are contained in *Co-CF*.

Theorem 6.1. *Let G be a group whose word problem $WP(G;A)$ is the intersection of k context-free languages accepted by rooted graphs:*

$$WP(G;A) = \bigcap_{i=1}^{k} L(\Gamma^{(i)}, x_i)$$

Then, $G \simeq \mathcal{G}(\Gamma^{(1)} \sqcup \ldots \sqcup \Gamma^{(k)})$ is in CF-TR.

Proof. Since both groups are generated by the same set A, it is enough to prove that

$$WP(G;A) = WP\big(\mathcal{G}(\Gamma^{(1)} \sqcup \cdots \sqcup \Gamma^{(k)}); A\big).$$

Let $w \in WP(G;A)$. We show that w fixes every vertex of each $\Gamma^{(i)}$. Fix $i \in [1,k]$ and let $y \in V(\Gamma^{(i)})$. Choose a word $u \in A^*$ labeling a path $x_i \xrightarrow{u} y$ in $\Gamma^{(i)}$. Since $w \in WP(G;A)$, also $uwu^{-1} \in WP(G;A)$; hence, $uwu^{-1} \in L(\Gamma^{(i)}, x_i)$. Thus, uwu^{-1} fixes x_i, which implies that the path $x_i \xrightarrow{u} y \xrightarrow{w} y \cdot w \xrightarrow{u^{-1}} x_i$ is a circuit. Therefore, since $\Gamma^{(i)}$ is inverse, we deduce $y \cdot w = y$. Since y was arbitrary, w fixes every vertex of $\Gamma^{(i)}$, and since this holds for every i, we obtain $w \in WP(\mathcal{G}(\Gamma^{(1)} \sqcup \cdots \sqcup \Gamma^{(k)}); A)$. Conversely, let $w \in WP(\mathcal{G}(\Gamma^{(1)} \sqcup \cdots \sqcup \Gamma^{(k)}); A)$. Then, w fixes every

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vertex of $\Gamma^{(1)} \sqcup \cdots \sqcup \Gamma^{(k)}$, in particular each root x_i. Hence, $w \in L(\Gamma^{(i)}, x_i)$ for every $i \in [1, k]$, so $w \in \bigcap_{i=1}^{k} L(\Gamma^{(i)}, x_i) = WP(G; A)$. □

An interesting question is whether there exist examples of *CF-TR* groups that are not polycontext-free. The next result shows that any such example must be non-quasi-transitive (for a specific example, see Section 10). In this regard, we introduce the class *Poly-QT-ICF*, that is, intersections of finitely many languages accepted by graphs that are quasi-transitive and quasi-isometric to a tree. Here, we get a characterization that can be seen as a kind of converse to the previous result.

Theorem 6.2. *Let G be a group. The following are equivalent:*

(1) $G = \mathcal{G}(\Gamma^{(1)} \sqcup \cdots \sqcup \Gamma^{(k)})$ *for some quasi-transitive context-free A-graphs $\Gamma^{(i)}$;*
(2) *G is virtually a f.g. subgroup of a direct product of free groups;*
(3) *G belongs to Poly-QT-ICF.*

Proof. (1) ⇒ (3). Fix a generic A-graph $\Gamma^{(i)}$ of the family $\{\Gamma^{(1)}, \dots, \Gamma^{(k)}\}$. Since it is quasi-transitive, there is a finite set $F_i \subseteq V(\Gamma^{(i)})$ of representatives of the equivalence relation given by the orbits of $\mathrm{Aut}(\Gamma_i)$. Since by Proposition 2.5, we have that for any $\varphi \in \mathrm{Aut}(\Gamma^{(i)})$, $L(\Gamma^{(i)}, v) = L(\Gamma^{(i)}, \varphi(v))$, we deduce

$$\mathcal{L}(\Gamma^{(i)}) = \bigcap_{p \in V(\Gamma^{(i)})} L(\Gamma^{(i)}, p) = \bigcap_{\varphi \in \mathrm{Aut}(\Gamma^{(i)}), p \in F_i} L(\Gamma^{(i)}, \varphi(p)) = \bigcap_{p \in F_i} L(\Gamma^{(i)}, p)$$

Since the word problem $WP(G; A)$ for $G = \mathcal{G}(\Gamma^{(1)} \sqcup \dots \sqcup \Gamma^{(k)})$ is the intersection $\bigcap_{i=1}^{k} \mathcal{L}(\Gamma^{(i)})$, we have that $\mathcal{G}(\Gamma^{(1)} \sqcup \dots \sqcup \Gamma^{(k)})$ is a polycontext-free group. In particular,

$$WP(G; A) = \bigcap_{i=1}^{k} \left(\bigcap_{p \in F_i} L(\Gamma^{(i)}, p) \right)$$

is the intersection of finitely many context-free languages accepted by quasi-transitive A-graphs; hence, G belongs to Poly-QT-ICF.

(2) ⇔ (3). By [29, Theorem 3], the fact that $WP(G; A)$ is the intersection of finitely many context-free languages accepted by quasi-transitive A-graphs is equivalent to the fact that G is virtually a finitely generated subgroup of a direct product of free groups.

(2) ⇒ (1). By using [29, Proposition 13] if G is virtually a finitely generated subgroup of the direct product of free groups, then the word problem $WP(G; A) = \bigcap_{i=1}^{k} L(\Gamma^{(i)}, x_i)$ is the intersection of finitely many inverse-context-free languages accepted by graphs that are quasi-transitive. Hence, by Theorem 6.1, we have that the group $G \simeq \mathcal{G}(\Gamma^{(1)} \sqcup \dots \sqcup \Gamma^{(k)})$ is the transition group of the disjoint union of finitely many context-free A-graphs that are quasi-transitive. □

The previous theorem provides another characterization of groups that are virtually finitely generated subgroups of direct products of free groups. In particular, [29, Theorem 3] shows that this class coincides with groups whose word problem lies in *Poly-QT-ICF*. Note that the example in Subsection 4.2 belongs to *Poly-QT-ICF*, since a quick inspection of Figure 1 shows that Ω is quasi-transitive.

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Following [10], for $c = \langle c_0, \dots, c_s \rangle \in \mathbb{Z}^{s+1}$ with $s \geqslant 1$, $c_0, c_s \neq 0$, and $\mathrm{GCD}(c_0, \dots, c_s) = 1$, the group $G(c)$ is defined by

$$G(c) = \langle a, b \mid \mathfrak{R}_c \rangle, \quad \mathfrak{R}_c = \{[b, b^{a^i}]\,(i \in \mathbb{Z}),\ b^{c_0}(b^a)^{c_1} \cdots (b^{a^s})^{c_s}\},$$

using the usual notation $x^y := y^{-1}xy$ and $[x, y] := x^{-1}y^{-1}xy$. These are called *Gc-groups*, and a *Gc*-group is *proper* if it is not virtually abelian. We conclude this section with some remarks on Brough Conjecture (2), which asserts that a finitely generated polycontext-free group is solvable if and only if it is virtually abelian. In this direction, we have the following proposition that holds in general for *CoCF* groups.

Proposition 6.3. *If G is a finitely generated solvable group in CoCF, then one of the following holds:*

- *G is virtually abelian;*
- *G contains the free abelian group $\mathbb{Z}^\infty$ of countably infinite rank;*
- *G has arbitrarily large finite subgroups.*

Proof. Combining results from [9, 10, Theorem 2.12] shows that a solvable finitely generated group G that is not virtually abelian either contains

(1) $\mathbb{Z}^\infty$, or
(2) a proper *Gc*-group, or
(3) a finitely generated subgroup H with an infinite normal torsion subgroup U, such that H/U is either free abelian or a proper *Gc*-group.

Since G is *CoCF*, by [9, Proposition 4.13], a *Gc*-group is polycontext-free or cocontext-free if and only if it is virtually abelian. Hence, case (2) is not possible. Let us now analyze case (3) and assume that G contains a subgroup H with an infinite normal torsion subgroup U. Suppose that $U = \langle g_i : i \in \mathbb{N} \rangle$ and consider the subgroups $U_n = \langle g_i : i \leqslant n \rangle \leqslant U$. Then, each $U_n \leqslant U_{n+1}$ is a finitely generated torsion solvable group, hence finite. Since U is infinite, the orders of the subgroups U_n are unbounded. Therefore, G has arbitrarily large finite subgroups. □

The previous proposition shows that the obstruction is, in fact, more general and only depends on the *CoCF* hypothesis. We now restrict back to the class *CF-TR*, which is the setting relevant for the present paper. In Brough Conjecture (3), it is also conjectured that a polycontext-free group does not have arbitrarily large finite subgroups. In the intersection of CF-TR with polycontext-free groups, we have the following fact.

Corollary 6.4. *If it is true that a group G that is both polycontext-free and CF-TR does not have arbitrarily large finite subgroups, then the following are equivalent for a finitely generated group:*

(1) *G is virtually abelian;*
(2) *G is a solvable polycontext-free transition group in CF-TR.*

Proof. (1) ⇒ (2) Since $\mathbb{Z}/n\mathbb{Z}$ and $\mathbb{Z}$ can be realized as transition groups of context-free graphs by Example 4.3, then Proposition 4.17 and Proposition 4.20 imply that finitely generated virtually abelian groups belong to the class *CF-TR*; moreover, they are polycontext-free by [9]. (2) ⇒ (1) The converse implication follows from Proposition 6.3 and the fact that no group containing $\mathbb{Z}^\infty$ is

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polycontext-free, since such a group contains free abelian subgroups of rank k for all $k \in \mathbb{N}$, see [9, Lemma 4.6]. □

The previous discussion raises the following natural questions, which will be addressed in the following sections.

Question 6.5.

- Is it possible to realize a solvable transition group associated to a context-free graph containing $\mathbb{Z}^\infty$ as a subgroup? See Proposition 10.6.
- Is it true that a transition group associated to a context-free graph does not have arbitrarily large finite subgroups? By the argument at the end of the proof of Proposition 6.3, this is equivalent to not having an infinite torsion subgroup. See the Section 8 and Proposition 10.7.

7 | RATIONALITY OF *CF-TR*

Before turning to the questions discussed in the previous section, we first examine a property common to all currently known examples of *CoCF* groups and shared by our class. A *sequential transducer* is a quintuple $T = (Q, q_0, \Sigma, \delta, o)$, where

(1) Q is a finite set *states*;
(2) $q_0 \in Q$ is the *initial state*;
(3) δ is a function $Q \times \Sigma \to Q$ called *transition function*;
(4) o is a function $Q \times \Sigma \to \Sigma^*$ called *output function*.

See, for instance, [5]. For the sake of brevity, we denote $\delta(q, \sigma) = \widetilde{q}$ and $o(q, \sigma) = w$ simply with the diagram $q \xrightarrow{\sigma|w} \widetilde{q}$. It is straightforward that one can extend o and δ to finite and infinite words. More precisely, if $\widehat{\Sigma} = \Sigma^* \cup \Sigma^\omega$, we have $\widehat{o} : Q \times \widehat{\Sigma} \to \widehat{\Sigma}$ and $\widehat{\delta} : Q \times \widehat{\Sigma} \to Q$ simply by the formula

$$\widehat{o}(q, \sigma_1\sigma_2) = o(q, \sigma_1)o(\delta(q, \sigma_1), \sigma_2) \text{ and } \widehat{\delta}(q, \sigma_1\sigma_2) = \delta(\delta(q_1, \sigma_1), \sigma_2).$$

Furthermore, fixed q_0, we have the functions $q_0 * \colon \widehat{\Sigma} \to \widehat{\Sigma}$ and $q_0 \cdot : \Sigma^* \to Q$ with

$$q_0 * w = \widehat{o}(q_0, w) \text{ and } q_0 \cdot w = \widehat{\delta}(q_0, w).$$

A homeomorphism $\varphi : \Sigma^\omega \to \Sigma^\omega$ is *rational* if there exists a sequential transducer $(Q, q_0, \Sigma, \delta, o)$ such that $\varphi(\eta) = q_0 * \eta$ for all $\eta \in \Sigma^\omega$.

Definition 7.1 [15]. The set of rational homeomorphisms over an alphabet Σ is a group $\mathcal{R}_\Sigma$ (called the *rational group*).

Any two rational groups over two different alphabets are isomorphic (so it makes sense to simply write $\mathcal{R}$).

In [4], it is shown that the rational group $\mathcal{R}$ is simple and nonfinitely generated. We say that a group that embeds in any rational group is *rational*. From a topological point of view, Σ^ω is a Cantor space where the metric is given by the longest common prefix. In this way, there is a well-defined notion of bi-lipschitz homeomorphism. A transducer associated to a homeomorphism is said to be bi-lipschitz if the homeomorphism is bi-lipschitz. The family of groups that can be

represented by bi-lipschitz transducers includes Thompson-like groups, self-similar groups and hyperbolic groups (see [3]). We want to construct a bi-lipschitz transducer for each element of our group belonging to *CF-TR*, so that any such group embeds in $\mathcal{R}$. To fix the notation, let $\{\Gamma_0, \dots, \Gamma_N\}$ be the representative of each end-isomorphim class of the context-free graph (Γ, x_0), and let $\{\Delta_0, \dots, \Delta_N\}$ be the corresponding frontier vertices. We put $\Gamma_0 = \Gamma$. As we have discussed at the end of Lemma 5.1, for each end-cone $\Gamma(x, x_0)$ of a context-free graph (Γ, x_0), we have a certain number of second-level end-cones which are end-isomorphic to some representative among $\{\Gamma_0, \dots, \Gamma_N\}$. This means that each Γ_i has $k(i)$ second-level end-cones that we denote by $\Gamma_1^{(i)}, \dots, \Gamma_{k(i)}^{(i)}$. Each subgraph $\Gamma_j^{(i)}$ is end-isomorphic to some end-cone, say $\Gamma_h \in \{\Gamma_0, \dots, \Gamma_N\}$. Let us denote by

$$\Psi(i, j) : \Gamma_j^{(i)} \to \Gamma_h$$

such an end-isomorphism, and put $\psi(i, j) = h$, where $i \in [0, N]$ and $j \in [1, k(i)]$. We also denote by $\Delta_j^{(i)}$ the frontier vertices of $\Gamma_j^{(i)}$. We now define the alphabet on which our transducer will act:

$$\Lambda = \{(\Gamma_j^{(i)}, v) : v \in \Delta_j^{(i)},\ i \in [1, N],\ j \in [1, k(i)]\} \cup \{\Gamma_j^{(i)} : i \in [1, N],\ j \in [1, k(i)]\} \cup \{\Gamma_0, (\Gamma_0, x_0)\}.$$

We say that a letter $\lambda \in \Lambda$ is *final* if it is of the form $\lambda = (\Gamma_j^{(i)}, v)$; in this case we put $\mathrm{cone}(\lambda) = \Gamma_j^{(i)}$ and $\mathrm{vert}(\lambda) = v$. Moreover, for any $\Gamma_j^{(i)}, \Gamma_k^{(h)}$, we say that $\Gamma_j^{(i)} > \Gamma_k^{(h)}$ if $\Gamma_k^{(h)}$ is a second-level end-cone of $\Psi(i, j)(\Gamma_j^{(i)})$. Note that in this case, we have $h = \psi(i, j)$. A word $w \in \Lambda^*$ is called *well-formed* if either $w = (\Gamma_0, x_0)$, or

$$w = \Gamma_0 \Gamma_{j_1}^{(i_1)} \cdots \Gamma_{j_{\ell-1}}^{(i_{\ell-1})} (\Gamma_{j_\ell}^{(i_\ell)}, v)$$

with $v \in \Delta_{j_\ell}^{(i_\ell)}$ and $\Gamma_0 > \Gamma_{j_1}^{(i_1)} > \cdots > \Gamma_{j_\ell}^{(i_\ell)}$. We denote by $\mathcal{F}(\Lambda)$ the set of well-formed words on Λ. Roughly speaking, a well-formed word $w = \Gamma_0 \Gamma_{j_1}^{(i_1)} \cdots \Gamma_{j_{\ell-1}}^{(i_{\ell-1})} (\Gamma_{j_\ell}^{(i_\ell)}, v)$ represents a vertex $x \in \Gamma$, while the sequence

$$\Gamma_0 > \Gamma_{j_1}^{(i_1)} > \cdots > \Gamma_{j_\ell}^{(i_\ell)}$$

represents the sequence of end-cones traversed by a geodesic connecting x_0 to x. More precisely, an *end-geodesic* is a sequence of end-cones $\Gamma(x_0, x_0) \supset \Gamma(x_1, x_0) \supset \cdots \supset \Gamma(x_\ell, x_0)$ such that $\|x_{i+1}\|_{x_0} = \|x_i\|_{x_0} + 1$ for $i \in [0, \ell - 1]$. Note that each $\Gamma(x_{i+1}, x_0)$ is a second-level end-cone of $\Gamma(x_i, x_0)$. A *rooted end-geodesic* is an end-geodesic together with a choice of a vertex x_ℓ in the last end-cone, that is, $(\Gamma(x_0, x_0) \supset \Gamma(x_1, x_0) \supset \cdots \supset \Gamma(x_\ell, x_0), x_\ell)$. Henceforth, when we write $\Gamma(x_0, x_0) \supset \Gamma(x_1, x_0) \supset \cdots \supset \Gamma(x_\ell, x_0)$, we always mean a rooted end-geodesic whose root is the last vertex x_ℓ in the sequence. We have the following lemma.

Lemma 7.2. *There is a one-to-one correspondence between the following objects:*

- *rooted end-geodesics*

$$(\Gamma(x_0, x_0) \supset \Gamma(x_1, x_0) \supset \cdots \supset \Gamma(x_\ell, x_0), x_\ell),$$

and

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- *well-formed words*

$$w = \Gamma_0 \Gamma_{j_1}^{(i_1)} \cdots \Gamma_{j_{\ell-1}}^{(i_{\ell-1})} (\Gamma_{j_\ell}^{(i_\ell)}, v_\ell) \in \mathcal{F}(\Lambda),$$

such that the maps defined inductively by $\varphi_0 = \mathrm{id}$ *and*

$$\varphi_s = (\Psi(i_{s-1}, j_{s-1}) \circ \varphi_{s-1})|_{\Gamma(x_s, x_0)}, \qquad s \in [1, \ell],$$

are well-defined end-isomorphisms

$$\varphi_s : \Gamma(x_s, x_0) \to \Gamma_{j_s}^{(i_s)},$$

and satisfy $\varphi_\ell(x_\ell) = v_\ell$.

Proof. We prove the first direction by induction on s. For $s = 0$, the identity map φ_0 satisfies the required conditions. Since w is well formed, $\Gamma_{j_1}^{(i_1)}$ is a second-level end-cone of Γ_0. Hence, necessarily $\Gamma(x_1, x_0) = \Gamma_{j_1}^{(i_1)}$, and necessarily $\varphi_1 = \varphi_0|_{\Gamma(x_1, x_0)}$. Assume that we have already constructed a rooted end-geodesic

$$\Gamma(x_0, x_0) \supset \Gamma(x_1, x_0) \supset \cdots \supset \Gamma(x_{s-1}, x_0)$$

and end-isomorphisms φ_i, $i \in [1, s-1]$, satisfying the statement of the lemma. Consider the end-isomorphism $\Psi(i_{s-1}, j_{s-1}) : \Gamma_{j_{s-1}}^{(i_{s-1})} \to \Gamma_h$, where $h = \psi(i_{s-1}, j_{s-1})$, and put $\phi_s = \Psi(i_{s-1}, j_{s-1}) \circ \varphi_{s-1} : \Gamma(x_{s-1}, x_0) \to \Gamma_h$. Since w is well formed, $\Gamma_{j_s}^{(i_s)}$ is a second-level end-cone of Γ_h. Therefore,

$$\Gamma(x_s, x_0) = \phi_s^{-1}(\Gamma_{j_s}^{(i_s)})$$

is a second-level end-cone of $\Gamma(x_{s-1}, x_0)$. The map φ_s is then necessarily given by $\varphi_s = \phi_s|_{\Gamma(x_s, x_0)}$. Thus, $\Gamma(x_s, x_0)$ and φ_s are uniquely determined, and if $s = \ell$, then $\varphi_\ell(x_\ell) = v_\ell$ by construction.

For the converse direction, we again argue by induction on s. Let

$$\Gamma(x_0, x_0) \supset \Gamma(x_1, x_0) \supset \cdots \supset \Gamma(x_\ell, x_0)$$

be a rooted end-geodesic, and let φ_0 be the identity map. For $s = 0$, the corresponding well-formed word is (Γ_0, x_0). Assume that for some $s \in [0, \ell - 1]$, we have already constructed the well-formed word

$$\Gamma_0 \Gamma_{j_1}^{(i_1)} \cdots \Gamma_{j_{s-1}}^{(i_{s-1})} (\Gamma_{j_s}^{(i_s)}, v_s)$$

associated with the rooted end-geodesic

$$\Gamma(x_0, x_0) \supset \Gamma(x_1, x_0) \supset \cdots \supset \Gamma(x_s, x_0),$$

together with end-isomorphisms φ_i, $i \in [0, s]$, satisfying the statement of the lemma. Let $\Gamma_{j_{s+1}}^{(i_{s+1})}$ be the second-level end-cone

$$\Psi(i_s, j_s) \circ \varphi_s(\Gamma(x_{s+1}, x_0))$$

of $\Gamma_{j_s}^{(i_s)}$, and let $v_{s+1} = \Psi(i_s, j_s) \circ \varphi_s(x_{s+1})$. Then

$$\Gamma_0 \Gamma_{j_1}^{(i_1)} \cdots \Gamma_{j_s}^{(i_s)} (\Gamma_{j_{s+1}}^{(i_{s+1})}, v_{s+1})$$

is the required well-formed word associated with

$$\Gamma(x_0, x_0) \supset \Gamma(x_1, x_0) \supset \cdots \supset \Gamma(x_{s+1}, x_0).$$

Taking $s = \ell - 1$ yields the required well-formed word

$$\Gamma_0 \Gamma_{j_1}^{(i_1)} \cdots \Gamma_{j_{\ell-1}}^{(i_{\ell-1})} (\Gamma_{j_\ell}^{(i_\ell)}, v_\ell). \qquad \square$$

Note that once the finitely many end-isomorphisms $\Psi(i, j)$ are fixed, the maps φ_s are uniquely determined recursively, so no additional choices are involved in the construction. By Lemma 7.2, for each well-formed word

$$w = \Gamma_0 \Gamma_{j_1}^{(i_1)} \cdots \Gamma_{j_{\ell-1}}^{(i_{\ell-1})} (\Gamma_{j_\ell}^{(i_\ell)}, v_\ell) \in \mathcal{F}(\Lambda),$$

we may associate a vertex $x_\ell = \varphi_\ell^{-1}(v_\ell)$. Conversely, given a vertex x_ℓ in Γ, there is a unique rooted end-geodesic $\Gamma(x_0, x_0) \supset \Gamma(x_1, x_0) \supset \cdots \supset \Gamma(x_\ell, x_0)$, and hence a unique well-formed word associated with it, which we denote by $w[x_\ell]$. Therefore, there is a one-to-one correspondence between vertices of Γ and well-formed words. For any word $w \in \Lambda^*$, let $w\Lambda^\omega$ be the clopen set consisting of all infinite words starting with w. Therefore, as a direct consequence of Lemma 7.2, we can conclude the following.

Lemma 7.3. *The map $\Xi : V(\Gamma) \to \mathcal{F}(\Lambda)\Lambda^\omega$, defined by $\Xi(x) = w[x]\Lambda^\omega$ for all $x \in V(\Gamma)$, is a bijection.*

This is a key point that will be used in proving that the action of each letter $a \in A$ on Γ is rational. We now define the sequential transducer $\mathcal{T}_\Gamma$. This transducer acts on Λ and has set of states

$$Q_\Gamma = \left\{ (\Gamma_j^{(i)}, a) : i \in [1, N], j \in [1, k(i)], a \in A \right\} \cup \{(\Gamma_0, a) : a \in A\} \cup \{E\}.$$

We now describe the set of transitions. The state E is a sink state, which means that we have the transitions $E \xrightarrow{\lambda|\lambda} E$ for all $\lambda \in \Lambda$. For each fixed $a \in A$, we now describe a transducer that mimics the action of the letter a when we are in a vertex x encoded in the closed set $\Xi(x)$. This is done by first checking whether the word in the input encodes a well-formed word representing a vertex x, and then changing the last letters to match the new vertex $x \cdot a$ in the new encoding in the output. The transitions are defined as follows. We start by describing how the initial state acts:

$$\begin{cases} (\Gamma_0, a) \xrightarrow{\lambda|1} (\Gamma_0, a) & \text{if } \lambda = \Gamma_0 \\ (\Gamma_0, a) \xrightarrow{\lambda|\lambda} E & \text{in the other cases when } \lambda \text{ is not final,} \end{cases} \tag{3}$$

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where the case when λ is final is treated later in (6) and (7). The transition of a generic state $(\Gamma_j^{(i)}, a)$ with input λ depends on whether either $\lambda = \Gamma_s^{(h)}$ or $\lambda = (\Gamma_s^{(h)}, v)$ for some $\Gamma_j^{(i)} > \Gamma_s^{(h)}$. Let us first assume that $\lambda = \Gamma_s^{(h)}$ with $\Gamma_j^{(i)} > \Gamma_s^{(h)}$, in this case, we have the following transition:

$$(\Gamma_j^{(i)}, a) \xrightarrow{\lambda|\lambda'} (\Gamma_s^{(h)}, a), \text{ where } \lambda' = \Gamma_j^{(i)}, \textbf{if } \lambda = \Gamma_s^{(h)} \text{ and } \Gamma_j^{(i)} > \Gamma_s^{(h)}; \tag{4}$$

In case λ is final, we are in three cases (5)–(7) described below and the action depends on the end-cone $\Gamma_s^{(h)}$ and the vertex v such that $\lambda = (\Gamma_s^{(h)}, v)$. In the first case, when applying the letter a to the vertex v, we get back to the previous end-cone:

$$\begin{aligned}&(\Gamma_j^{(i)}, a) \xrightarrow{\lambda|\lambda'} E, \lambda' = (\Gamma_j^{(i)}, z), \textbf{ if } \lambda = (\Gamma_s^{(h)}, v), \Gamma_j^{(i)} > \Gamma_s^{(h)},\\ &\text{and } v \xrightarrow{a} \Psi(i,j)(z) \subseteq \Psi(i,j)(\Gamma_j^{(i)}) \text{ is an edge in } \Psi(i,j)(\Gamma_j^{(i)}) \text{ with } z \in \Delta_j^{(i)}.\end{aligned} \tag{5}$$

The second case is when applying the letter a to the vertex v, we are staying in the same frontier vertices of the end-cone $\Gamma_s^{(h)}$:

$$\begin{aligned}&(\Gamma_j^{(i)}, a) \xrightarrow{\lambda|\lambda'\lambda''} E, \text{ where } \lambda' = \Gamma_j^{(i)}, \lambda'' = (\Gamma_s^{(h)}, z), \textbf{ if } \lambda = (\Gamma_s^{(h)}, v), \Gamma_j^{(i)} > \Gamma_s^{(h)}\\ &\text{and } v \xrightarrow{a} z \subseteq \Gamma_s^{(h)} \text{ is an edge in } \Gamma_s^{(h)}, \text{ with } z \in \Delta_s^{(h)}.\end{aligned} \tag{6}$$

The last case occurs when applying the letter a to the vertex v, we are moving to a frontier vertex on a second-level end-cone $\Gamma_t^{(m)}$ of $\Gamma_s^{(h)}$:

$$\begin{aligned}&(\Gamma_j^{(i)}, a) \xrightarrow{\lambda|\lambda'\lambda''\lambda'''} E, \text{ where } \lambda' = \Gamma_j^{(i)}, \lambda'' = \Gamma_s^{(h)}, \lambda''' = (\Gamma_t^{(m)}, z) \textbf{ if } \lambda = (\Gamma_s^{(h)}, v),\\ &\Gamma_j^{(i)} > \Gamma_s^{(h)} > \Gamma_t^{(m)}, \text{ and } \Psi(s,h)(v) \xrightarrow{a} z \text{ is a walk in } \Psi(s,h)(\Gamma_s^{(h)}) = \Gamma_m, \text{ with } z \in \Delta_t^{(m)}.\end{aligned} \tag{7}$$

If the previous condition that $\lambda = \Gamma_s^{(h)}$ or $\lambda = (\Gamma_s^{(h)}, v)$ with $\Gamma_j^{(i)} > \Gamma_s^{(h)}$ does not hold, we add the transition

$$(\Gamma_j^{(i)}, a) \xrightarrow{\lambda|\lambda'\lambda} E, \lambda' = \Gamma_j^{(i)} \text{ whether } \lambda \text{ is final or not.} \tag{8}$$

We will prove that each state (Γ_0, a) represents the action of the letter $a \in A$ of each vertex of Γ. We have the following lemma.

Lemma 7.4. *For any word $w = \lambda_0 \dots \lambda_k \in \Lambda^*$ formed by nonfinal letters, with $\lambda_0 = \Gamma_0$ and satisfying $\lambda_0 > \dots > \lambda_k$, we have:*

$$(\Gamma_0, a) * w = \lambda_0 \dots \lambda_{k-1}, \quad (\Gamma_0, a) \cdot w = (\lambda_k, a)$$

for any letter $a \in A$.

Proof. It follows by induction on the length of w, where the base of induction is a consequence of (3), while the general case uses the transitions described in (4). □

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Lemma 7.5. *For any infinite word $\eta \notin w\Lambda^\omega$ for all $w \in \mathcal{F}(\Lambda)$, which is not well formed, we have $(\Gamma_0, a) * \eta = \eta$ for any letter $a \in A$.*

Proof. Let $w = \lambda_0 \dots \lambda_k$ be a (possibly empty) maximal prefix of η formed by nonfinal letters, with $\lambda_0 = \Gamma_0$ and $\lambda_0 > \dots > \lambda_k$. If w is empty, this means that $\lambda_0 \neq \Gamma_0, (\Gamma_0, x_0)$, hence by the transition described in (3), we deduce $(\Gamma_0, a) * \eta = \eta$. Otherwise, w is nonempty, and if k is finite, then $\lambda_k \not> \lambda_{k+1}$. By Lemma 7.4, we have $(\Gamma_0, a) * w = \lambda_0 \dots \lambda_{k-1}$ and $(\Gamma_0, a) \cdot w = (\lambda_k, a)$. Hence, using the condition $\lambda_k \not> \lambda_{k+1}$ and by transition (8), we conclude that $(\Gamma_0, a) * w\lambda_{k+1} = w\lambda_{k+1}$ and $(\Gamma_0, a) \cdot w\lambda_{k+1} = E$, that is, $(\Gamma_0, a) * \eta = \eta$. If k is infinite, meaning $w = \eta$, by Lemma 7.4, we may conclude our claim $(\Gamma_0, a) * \eta = \eta$. □

We recall that for a vertex $x \in \Gamma$ and a letter $a \in A$, we denote by $x \cdot a$ the vertex such that $x \xrightarrow{a} (x \cdot a)$ is an edge in Γ. We have the following lemma.

Lemma 7.6. *Let $a \in A$ and $x \in V(\Gamma)$. Then, $(\Gamma_0, a) * w[x]\eta = w[x \cdot a]\eta$ for all $\eta \in \Lambda^\omega$.*

Proof. Let

$$w[x] = \lambda_0 \dots \lambda_\ell = \Gamma_0 \Gamma_{j_1}^{(i_1)} \dots \Gamma_{j_s}^{(i_s)} \dots \Gamma_{j_{\ell-1}}^{(i_{\ell-1})} (\Gamma_{j_\ell}^{(i_\ell)}, v_\ell)$$

be the well-formed word associated to the (unique) rooted end-geodesic $\Gamma(x_0, x_0) \supset \Gamma(x_1, x_0) \dots \supset \Gamma(x_\ell, x_0)$ with $x_\ell = x$, for which there are end-isomorphisms $\varphi_s : \Gamma(x_s, x_0) \to \Gamma_{j_s}^{(i_s)}$ with $\varphi_\ell(x_\ell) = v_\ell$ and satisfying $(\Psi(i_{s-1}, j_{s-1}) \circ \varphi_{s-1})|_{\Gamma(x_s, x_0)} = \varphi_s$ for $s \in [1, \ell]$. By Lemma 7.4, we have

$$(\Gamma_0, a) * \lambda_0 \dots \lambda_{\ell-1} = \lambda_0 \dots \lambda_{\ell-2}, \quad (\Gamma_0, a) \cdot \lambda_0 \dots \lambda_{\ell-1} = (\lambda_{\ell-1}, a). \tag{9}$$

Now, since λ_ℓ is final and $\lambda_{\ell-1} > \Gamma_{j_\ell}^{(i_\ell)}$, we are in one of the three possible transitions (5)–(7). Put $\lambda_\ell = (\Gamma_s^{(h)}, v_\ell)$ and $\lambda_{\ell-1} = \Gamma_j^{(i)}$, where $\Gamma_j^{(i)} > \Gamma_s^{(h)}$.

- Suppose that condition (5) holds. In this case, $v_\ell \xrightarrow{a} \Psi(i, j)(z)$ is an edge of $\Gamma_h = \Psi(i, j)(\Gamma_j^{(i)})$ with $z \in \Delta_j^{(i)}$, and so, by (9) and by the definition, we get

$$(\Gamma_0, a) * w\eta = \lambda_0 \dots \lambda_{\ell-2}(\Gamma_j^{(i)}, z)\eta, \quad \forall \eta \in \Lambda^\omega.$$

  We claim that $\Xi(x \cdot a) = \lambda_0 \dots \lambda_{\ell-2}(\Gamma_j^{(i)}, z)\Lambda^\omega$. Now, since $\Psi(i, j)^{-1}(v_\ell) \xrightarrow{a} z$ is an edge of $\Gamma_j^{(i)}$ and $\varphi_\ell(x) = v_\ell$, we deduce that also $x \xrightarrow{a} \varphi_{\ell-1}^{-1}(z)$ is an edge in $\Gamma(x_{\ell-1}, x_0)$ with $x \cdot a = \varphi_{\ell-1}^{-1}(z) \in \Delta(x_{\ell-1}, x_0)$. Therefore, by Lemma 7.2, the well-formed word $\lambda_0 \dots \lambda_{\ell-2}(\Gamma_j^{(i)}, z)$ is associated to the rooted end-geodesic

$$\Gamma(x_0, x_0) \supset \Gamma(x_1, x_0) \dots \supset \Gamma(x_{\ell-2}, x_0) \supset \Gamma(x \cdot a, x_0)$$

  and this concludes the proof of our claim.
- Suppose that condition (6) holds. In this case, we have

$$(\Gamma_0, a) * w\eta = \lambda_0 \dots \lambda_{\ell-1}(\Gamma_{j_\ell}^{(i_\ell)}, z)\eta, \quad \forall \eta \in \Lambda^\omega,$$

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where $v_\ell \xrightarrow{a} z$ is an edge of $\Gamma^{(i_\ell)}_{j_\ell}$ with $z \in \Delta^{(i_\ell)}_{j_\ell}$. Now, since $\varphi_\ell(x) = v_\ell$ and $\varphi_\ell^{-1}(v_\ell) \xrightarrow{a} \varphi_\ell^{-1}(z)$ is an edge in $\Gamma(x_\ell, x_0)$, we conclude that $x \cdot a = \varphi_\ell^{-1}(z)$. Hence, $\lambda_0 \dots \lambda_{\ell-1}(\Gamma^{(i_\ell)}_{j_\ell}, z)$ is the well-formed word associated to the rooted end-geodesic $\Gamma(x_0, x_0) \supset \Gamma(x_1, x_0) \dots \supset \Gamma(x_{\ell-1}, x_0) \supset \Gamma(x \cdot a, x_0)$, that is, $(\Gamma_0, a) * w[x]\eta = w[x \cdot a]\eta$ for all $\eta \in \Lambda^\omega$.

- Suppose that condition (7) holds. Using the definition, we have

$$(\Gamma_0, a) * w\eta = \lambda_0 \dots \lambda_{\ell-1}\Gamma^{(i_\ell)}_{j_\ell}(\Gamma^{(m)}_t, z)\eta, \quad \forall \eta \in \Lambda^\omega,$$

where $\Gamma^{(i_\ell)}_{j_\ell} > \Gamma^{(m)}_t$ and $\Psi(j_\ell, i_\ell)(v_\ell) \xrightarrow{a} z$ is an edge in Γ_m with $z \in \Delta^{(m)}_t$. Since $\Gamma^{(m)}_t$ is a second-level end-cone of Γ_m and $\phi = \Psi(j_\ell, i_\ell) \circ \varphi_\ell : \Gamma(x, x_0) \to \Gamma_m$ is an end-isomorphism, we deduce that $\phi^{-1}(\Gamma^{(m)}_t)$ is a second-level end-cone of $\Gamma(x, x_0)$ containing $\phi^{-1}(z)$, and since $x \xrightarrow{a} \phi^{-1}(z)$ is an edge in $\Gamma(x, x_0)$, we have $\phi^{-1}(\Gamma^{(m)}_t) = \Gamma(x \cdot a, x_0)$. Take $\varphi_{\ell+1} = \phi|_{\Gamma(x\cdot a, x_0)}$. We clearly have $(\Psi(j_\ell, i_\ell) \circ \varphi_\ell)\big|_{\Gamma(x\cdot a, x_0)} = \varphi_{\ell+1}$. Thus, by Lemma 7.2, we have that the rooted end-geodesic

$$\Gamma(x_0, x_0) \supset \Gamma(x_1, x_0) \dots \supset \Gamma(x_\ell, x_0) \supset \Gamma(x \cdot a, x_0)$$

is associated to the well-formed word $\lambda_0 \dots \lambda_{\ell-1}\Gamma^{(i_\ell)}_{j_\ell}(\Gamma^{(m)}_t, z)$ and this is our claim $(\Gamma_0, a) * w[x]\eta = w[x \cdot a]\eta$ for all $\eta \in \Lambda^\omega$. □

Theorem 7.7. *The transition group $\mathcal{G}\big(\Gamma^{(1)} \sqcup \dots \sqcup \Gamma^{(k)}\big)$ of a collection of context-free graphs $\Gamma^{(1)}, \dots, \Gamma^{(k)}$ is rational.*

Proof. Since rationality is closed under direct products and passing to subgroups, by Lemma 4.6, it is enough to consider the case of a single graph Γ. As a shorthand notation, put $\gamma_a = (\Gamma_0, a)$ for $a \in A$. By Lemma 7.6 and the fact that Γ is an inverse graph, we have

$$(\gamma_{a^{-1}}\gamma_a) * w[x]\eta = w[x]\eta \qquad \text{and} \qquad (\gamma_a\gamma_{a^{-1}}) * w[x]\eta = w[x]\eta$$

for all $\eta \in \Lambda^\omega$. Moreover, by Lemma 7.5, for all $\xi \in \Lambda^\omega \setminus \mathcal{F}(\Lambda)\Lambda^\omega$, we have

$$(\gamma_{a^{-1}}\gamma_a) * \xi = \xi \qquad \text{and} \qquad (\gamma_a\gamma_{a^{-1}}) * \xi = \xi.$$

Therefore, each state γ_a of $\mathcal{T}_\Gamma$ gives rise to a rational bijection $\gamma_a *\colon \Lambda^\omega \to \Lambda^\omega$ with inverse $\gamma_{a^{-1}} *$. Thus, by [3, Proposition 1.3], each state γ_a is a rational homeomorphism of Λ^ω. Let $H(\Gamma) = \langle \gamma_a : a \in A \rangle \leqslant \mathcal{R}_\Lambda$. We use composition in $H(\Gamma)$ from right to left, consistently with the action notation. Consider the epimorphisms

$$\varphi : F_A \to \mathcal{G}(\Gamma), \qquad \psi : F_A \to H(\Gamma),$$

defined on generators by $\varphi(a) = a$ and $\psi(a) = \gamma_a$. Let $u = a_1 \cdots a_p \in A^*$. By Lemma 7.6, and by induction on p, we have

$$(\gamma_{a_p} \cdots \gamma_{a_1}) * w[x]\eta = w[x \cdot u]\eta, \qquad \forall x \in V(\Gamma),\ \forall \eta \in \Lambda^\omega.$$

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We now show that $\ker(\varphi) = \ker(\psi)$. Suppose first that $\varphi(u) = 1$. Then, $x \cdot u = x$ for all $x \in V(\Gamma)$. Hence,

$$w[x \cdot u]\eta = w[x]\eta \qquad \forall x \in V(\Gamma),\ \forall \eta \in \Lambda^\omega.$$

By the previous equality, this means that

$$(\gamma_{a_p} \cdots \gamma_{a_1}) * w[x]\eta = w[x]\eta \qquad \forall x \in V(\Gamma),\ \forall \eta \in \Lambda^\omega.$$

Thus, $\psi(u)$ acts trivially on $\mathcal{F}(\Lambda)\Lambda^\omega$. On the complement of $\mathcal{F}(\Lambda)\Lambda^\omega$, the action is trivial by Lemma 7.5. Therefore, $\psi(u) = 1$. Conversely, suppose that $\psi(u) = 1$. Then,

$$(\gamma_{a_p} \cdots \gamma_{a_1}) * w[x]\eta = w[x]\eta \qquad \forall x \in V(\Gamma),\ \forall \eta \in \Lambda^\omega.$$

Using again the equality $(\gamma_{a_p} \cdots \gamma_{a_1}) * w[x]\eta = w[x \cdot u]\eta$, we obtain

$$w[x \cdot u]\eta = w[x]\eta \qquad \forall x \in V(\Gamma),\ \forall \eta \in \Lambda^\omega.$$

Since well-formed words form a prefix code, this implies $w[x \cdot u] = w[x]$ for all $x \in V(\Gamma)$, hence $x \cdot u = x$ for all $x \in V(\Gamma)$. Therefore, $\varphi(u) = 1$. We have proved that $\ker(\varphi) = \ker(\psi)$, hence

$$H(\Gamma) \simeq F_A / \ker(\psi) = F_A / \ker(\varphi) \simeq \mathcal{G}(\Gamma).$$

Hence, $\mathcal{G}(\Gamma)$ embeds into the rational group $\mathcal{R}_\Lambda$, which concludes the proof. □

Note that, despite we already have a tree structure associated to the graph, namely, the tree of end-cones, the transducer acts on Λ^ω that contains a copy of such tree as a subtree.

Remark 7.8. The transducer that represents any element g in a CF-TR group is bi-lipschitz. It suffices to notice that any element in $\mathcal{F}(\Lambda)$ is shortened or lengthened by at most $|u|$ letters.

8 | PERTURBING A GRAPH: A STRUCTURAL RESULT

Another natural question about these groups is how much small local changes in the graph affect the associated transition groups. In this section, we investigate what happens to the transition group when we locally perturb a context-free graph (Γ, x_0). The idea is to consider a finite subset $F \subseteq V(\Gamma)$ and to associate with the graph $\Lambda_F = \Gamma \setminus F$ a group called the boundary group $\mathcal{B}_F$ with respect to F, and then to give a structural result connecting this group to the transition group $G = \mathcal{G}(\Gamma)$.

Definition 8.1 (Boundary group). Consider the following subset of A^*

$$\mathcal{O}_F = \{u \in A^* : x \xrightarrow{u} x' \text{ is a walk in } \Lambda_F \text{ for some } x \neq x'\}^{\mathbf{c}}$$

of the words u such that all walks labeled by u are either a circuit in Λ_F, or intersect F. It is easy to verify that $\mathcal{O}_F$ is a submonoid of A^* satisfying the property that for any $u \in A^*$ and $w \in \mathcal{O}_F$, we have $w^{-1} \in \mathcal{O}_F$ and $u^{-1}wu \in \mathcal{O}_F$. Therefore, if $\pi : A^* \to G$ is the natural map, the image $\mathcal{B}_F = \pi(\mathcal{O}_F)$ is a normal subgroup of G called the *boundary group with respect to* F.

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We can bound the supports of elements of $\mathcal{O}_F$ in the following way.

Lemma 8.2. *Let $F \subseteq V(\Gamma)$ be finite and let $u \in \mathcal{O}_F$. Then $|\operatorname{supp}(u)| \leqslant |F|(|u|+1)$.*

Proof. Write $u = a_1 \cdots a_m$, with $m = |u|$, and put $u_i = a_1 \cdots a_i$ for $0 \leqslant i < m$. Let $\operatorname{supp}(u)$ be the support of u. For every $y \in \operatorname{supp}(u)$, the walk $y \xrightarrow{u} y \cdot u$ is not a circuit; otherwise, $y \cdot u = y$, contradicting $y \in \operatorname{supp}(u)$. Since $u \in \mathcal{O}_F$, this walk is not contained in Λ_F; hence, it intersects F. Therefore, each $y \in \operatorname{supp}(u)$ gives a pair $(f, i) \in F \times \{0, \dots, m\}$ with $y \cdot u_i = f \in F$ and $i \in [0, m]$ is the index for which $u_i = a_1 \dots a_i$ is minimal among the prefixes of u with the property $y \cdot u_i \in F$ ($u_0 = \varepsilon$ is the empty word). The assignment $y \mapsto (f, i)$ is injective on $\operatorname{supp}(u)$: if $y \cdot u_i = z \cdot u_i$, then $y = z$ by injectivity of the action of u_i. Hence, $|\operatorname{supp}(u)| \leqslant |F|(m+1)$. □

The bound of the supports provides a bound on the orders. First, we recall that *Landau's function* $g : \mathbb{N} \to \mathbb{N}$ defines $g(n)$ to be the largest order of an element of the symmetric group S_n (see [28]).

Proposition 8.3. *For every finite $F \subseteq V(\Gamma)$, the boundary group $\mathscr{B}_F$ is a torsion normal subgroup of G. Moreover, every $h \in \mathscr{B}_F$ satisfies*

$$o(h) \leqslant g(|F|(|h|+1)) \asymp \exp\left(\sqrt{|F||h|\log(|F||h|)}\right),$$

where $|\cdot|$ is the word metric with respect to A, and g is Landau's function.

Proof. Normality follows from the definition of $\mathscr{B}_F$. Let $h \in \mathscr{B}_F$. By Lemma 8.2, we have $|\operatorname{supp}(h)| \leqslant |F|(|h|+1)$. Hence, h acts as a permutation of the finite set $\operatorname{supp}(h)$ and fixes its complement pointwise. Therefore, $o(h) \leqslant g(|\operatorname{supp}(h)|) \leqslant g(|F|(|h|+1))$. The displayed asymptotic estimate is Landau's estimate for g. □

In view of the structural result contained in Proposition 6.2, an interesting case is when the graph Λ_F can be obtained by the union of connected components each isomorphic to a quasi-transitive a finite subset.

Definition 8.4. A graph (Γ, x_0) is called *locally-quasi-transitive* if there is a finite set $F \subseteq V(\Gamma)$ and finitely many quasi-transitive graphs Θ_i, $i \in [1, k]$ together with finite sets $F_i \subseteq V(\Theta_i)$, such that each connected component of Λ_F is isomorphic to a connected component of $\Theta_i \setminus F_i$ for some $i \in [1, k]$.

Lemma 8.5. *With the above notation, suppose that $\{O_1, \dots, O_m\}$ is a set of* infinite orbits *such that*

$$V(\Theta_i \setminus F_i) \subseteq \bigcup_{j=1}^{m} O_j$$

for each $i \in [1, k]$. Let R_i be a set of representative vertices of the orbits $\{O_1, \dots, O_m\}$ belonging to Θ_i. Then,

$$\mathcal{O}_F = \bigcap_{i \in [1,k]} \bigcap_{p \in R_i} L(\Theta_i, p).$$

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Proof. We prove that $\mathcal{O}_F^c = \bigcup_{i\in[1,k]} \bigcup_{p\in R_i} L(\Theta_i, p)^c$. Let us first prove the inclusion

$$\bigcup_{i\in[1,k]} \bigcup_{p\in R_i} L(\Theta_i, p)^c \subseteq \mathcal{O}_F^c.$$

Take $u \in L(\Theta_i, p)^c$ for some $i \in [1, k]$ and $p \in R_i$. Then, there is a walk $p \xrightarrow{u} p'$ in Θ_i with $p' \neq p$. Since p belongs to one of the infinite orbits, say O_j, and Θ_i is locally finite, we may choose a vertex $q \in O_j \cap (\Theta_i \setminus F_i)$ sufficiently far from F_i. Because q and p lie in the same orbit, there is an isomorphism sending p to q, and therefore, the walk $p \xrightarrow{u} p'$ is sent to a walk $q \xrightarrow{u} q'$ in $\Theta_i \setminus F_i$. Moreover, $q' \neq q$, since otherwise by applying the inverse isomorphism, we would get $p' = p$, a contradiction. Hence, $u \in \mathcal{O}_F^c$. Conversely, let $u \in \mathcal{O}_F^c$. Then, there exist $i \in [1, k]$ and a walk $x \xrightarrow{u} x'$ entirely contained in $\Theta_i \setminus F_i$, with $x' \neq x$. By hypothesis, $V(\Theta_i \setminus F_i) \subseteq \bigcup_{j=1}^m O_j$, so x belongs to one of the infinite orbits, say O_j. Let $p \in R_i$ be the chosen representative of this orbit in Θ_i. Since x and p belong to the same orbit, there is an isomorphism ϕ sending x to p. Applying ϕ to the walk $x \xrightarrow{u} x'$, we obtain a walk $p \xrightarrow{u} \phi(x')$ in Θ_i. Since $x' \neq x$, also $\phi(x') \neq p$, and thus, this walk is not a circuit. Therefore, $u \in L(\Theta_i, p)^c$, and this proves the other inclusion. □

We are now in position to prove the following structural theorem.

Theorem 8.6. *Let (Γ, x_0) be a context-free graph that is also locally-quasi-transitive and let $G = \mathcal{G}(\Gamma)$ be its transition group and $g(\cdot)$ the Landau's function. Then, there is a finite set $F \subseteq V(\Gamma)$ and a short exact sequence*

$$\mathbb{1} \to \mathcal{B}_F \to G \to H \to \mathbb{1},$$

where H is a group that is virtually a finitely generated subgroup of the direct product of free groups, F is chosen according to Definition 8.4, and $\mathcal{B}_F$ is a torsion normal subgroup where the order of each element h is bounded above by $o(h) \leqslant g(|F|\ (|h| + 1))$. In particular, we have $H \simeq G/_{\mathcal{B}_F}$.

Proof. Choose F and the graphs $\Theta_i \setminus F_i$ as in Definition 8.4, so that $\Gamma \setminus F = \bigcup_{i=1}^k (\Theta_i \setminus F_i)$. By enlarging the finite sets F_i if necessary, we may assume that each graph $\Theta_i \setminus F_i$ is contained in the union of the infinite orbits of Θ_i. Indeed, any finite orbit in a quasi-transitive graph is contained in a finite connected component, and there are only finitely many such components. Absorbing these components into the sets F_i, and correspondingly into F, does not change the form of the locally-quasi-transitive decomposition. Let $\Theta = \bigcup_{i=1}^k \Theta_i$. Since Θ differs from Γ only by finitely many end-cones, and since there are only finitely many end-cone types up to end-isomorphism, it follows that Θ is again context-free. Now Lemma 8.5 applies directly, and if R_i is a set of representatives of the infinite orbits in Θ_i, then

$$\mathcal{O}_F = \bigcap_{i\in[1,k]} \bigcap_{p\in R_i} L(\Theta_i, p).$$

In particular, $\mathcal{O}_F \subseteq \mathcal{L}(\Theta)$. Set $H = \mathcal{G}(\Theta)$. By Proposition 6.2, the group H is virtually a finitely generated subgroup of a direct product of free groups. Since $\mathcal{O}_F \subseteq \mathcal{L}(\Theta)$, the natural map $\pi : A^* \to G = \mathcal{G}(\Gamma)$ induces a well-defined epimorphism $\psi : \mathcal{G}(\Gamma) \to \mathcal{G}(\Theta)$ whose kernel is precisely

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$\ker(\psi) = \pi(\mathcal{O}_F) = \mathscr{B}_F$. This yields the required short exact sequence

$$\mathbb{1} \to \mathscr{B}_F \to G \to H \to \mathbb{1},$$

and, in particular, the isomorphism $H \simeq G/\mathscr{B}_F$. Finally, by Proposition 8.3, the subgroup $\mathscr{B}_F$ is torsion, and every element $h \in \mathscr{B}_F$ satisfies $o(h) \leqslant g\big(|F|\,(|h|+1)\big)$. □

It is worth mentioning that in the settings of the previous proposition, the boundary group $\mathscr{B}_F$ is generated by a polycontext-free language. This setting is going to be a key ingredient in Section 10 where we will address the second point of Question 6.5 (see Proposition 10.7).

9 | FREE PRODUCT OF CONTEXT-FREE GRAPHS

The aim of this section is to introduce the operation of free product of graphs and proving that the property of being context-free is preserved under this operation. This is useful in constructing examples of context-free graphs from simpler ones.

As this construction is much more general than our setting, only for this section, we will drop the convention A-graph and specify the type (involutive, deterministic, etc.) of digraph involved in each step, though our digraph will always be connected.

Definition 9.1 (Free product of graphs). Let $\Theta_1 = \{(\Gamma^{(i)}, \gamma_0^i), i = 1, \dots, k\}$ be a collection of inverse A_1-digraphs, and similarly, let $\Theta_2 = \{(\Lambda^{(j)}, \lambda_0^j), j = 1, \dots, m\}$ be a collection of inverse A_2-digraphs, with $A_1 \cap A_2 = \emptyset$. For each $i \in [1,k]$, $j \in [1,m]$, we consider the functions $\psi_i^1 : V(\Gamma^{(i)}) \setminus \{\gamma_0^i\} \to \{\lambda_0^1, \dots, \lambda_0^m\}$ and $\psi_j^2 : V(\Lambda^{(j)}) \setminus \{\lambda_0^j\} \to \{\gamma_0^1, \dots, \gamma_0^k\}$; the gluing map is the collection Ψ of all such functions. We first consider the infinite disjoint union of copies of the graphs Θ_1, Θ_2 where the set of vertices are words alternating between vertices of the graphs Θ_1 and Θ_2. More precisely, we consider the union of inverse $A_1 \cup A_2$-digraphs $(\Theta_1 \sqcup \Theta_2)^{\oplus}$ whose set of vertices $\mathcal{V}$ is formed by alternating sequences of vertices of Θ_1, Θ_2 that are either of the form $\gamma_1\lambda_1\gamma_2\lambda_2 \dots \gamma_\ell\lambda_\ell$, or $\lambda_1\gamma_1\lambda_2\gamma_2 \dots \lambda_\ell\gamma_\ell$, with $\gamma_n \in \bigsqcup_i V(\Gamma^{(i)}), \lambda_n \in \bigsqcup_j V(\Lambda^{(j)})$, $n \in [1,\ell]$ and $\ell \in \mathbb{N}$. The edges are of the form

$$\begin{cases} \gamma_1\lambda_1\gamma_2\lambda_2 \dots \gamma_\ell\lambda_\ell \xrightarrow{a} \gamma_1\lambda_1\gamma_2\lambda_2 \dots \gamma_\ell\lambda_h, a \in A_2, \text{ whenever } (\lambda_\ell \xrightarrow{a} \lambda_h) \in E(\Lambda^{(j)}) \\ \lambda_1\gamma_1\lambda_2\gamma_2 \dots \lambda_\ell\gamma_\ell \xrightarrow{a} \lambda_1\gamma_1\lambda_2\gamma_2 \dots \lambda_\ell\gamma_h, a \in A_1, \text{ whenever } (\gamma_\ell \xrightarrow{a} \gamma_h) \in E(\Gamma^{(i)}). \end{cases}$$

Roughly speaking, we are taking infinitely many copies of the graphs $\Gamma^{(i)}, \Lambda^{(j)}$ where we simply rename each vertex by adding a prefix that is a word that is alternating in the sense explained above. Now, the free product $\Theta_1 \star_\Psi \Theta_2$ is the union of involutive $A_1 \cup A_2$-digraphs obtained from $(\Theta_1 \sqcup \Theta_2)^{\oplus}$ by identifying the vertices of $\mathcal{V}$ via the transitive closure of the following relation:

$$\gamma_0^1 v^* = \lambda_0^1 v^*,\ v^*\gamma_i u^* = v^*\psi_j^1(\gamma_i)u^*,\ v^*\lambda_i u^* = v^*\psi_j^2(\lambda_i)u^*$$

where u^*, v^* are generic alternating words on Θ_1, Θ_2 possibly empty. Henceforth, we will consider the rooted inverse digraph $(\Theta_1 \star_\Psi \Theta_2, \gamma_0^1)$, which is the connected component of $\Theta_1 \star_\Psi \Theta_2$ containing $\gamma_0^1 = \lambda_0^1$.

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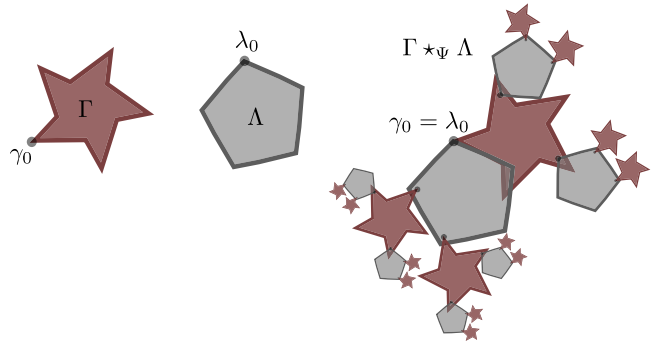


FIGURE 2 A depiction of the free product of two rooted graphs $(\Gamma, \gamma_0), (\Lambda, \lambda_0)$ with gluing maps $\psi^1(v) = \lambda_0$ for all $v \in \Gamma \setminus \{\gamma_0\}$ and $\psi^2(v) = \gamma_0$ for all $v \in \Lambda \setminus \{\lambda_0\}$.

Roughly speaking, the rooted graph $(\Theta_1 \star_\Psi \Theta_2, \gamma_0^1)$ is obtained from the graphs $\Gamma^{(1)} \sqcup \Lambda^{(1)}$ by first identifying $\gamma_0^1 = \lambda_0^1$, and then by iteratively gluing at a vertex with representative $v^*\gamma_i$, with $\gamma_i \notin \{\gamma_0^1, \dots, \gamma_0^k\}$, the graph $(\Lambda^{(s)}, \lambda_0^s)$ with $\lambda_0^s = \psi_j^1(\gamma_i)$, so that the two vertices $v^*\gamma_i, v^*\lambda_0^s$ identify. In case the vertex ends with a vertex from the Λ graphs such as $v^*\lambda_i$, then we glue the graph $(\Gamma^{(s)}, \gamma_0^s)$ by making the identification $\psi_j^2(\lambda_i) = \gamma_0^s$. Let us call a vertex $v^*\gamma_i$, where the last vertex belongs to some graph of the collection Θ_1, a vertex colored by 1, while if it ends by a vertex from the collection Θ_2, we say that it is colored by 2. To have an idea of what kind of graph we may generate see Figure 2 in which the free product of two rooted graphs $(\Gamma, \gamma_0), (\Lambda, \lambda_0)$ is depicted, where the gluing maps are $\psi^1(v) = \lambda_0$ for all $v \in \Gamma \setminus \{\gamma_0\}$ and $\psi^2(v) = \gamma_0$ for all $v \in \Lambda \setminus \{\lambda_0\}$. We will see later that if Γ, Λ are two Cayley graphs of two groups G_1, G_2, then $(\Gamma, \gamma_0) \star_\Psi (\Lambda, \lambda_0)$ is the Cayley graph of the free product $G_1 \star G_2$, hence the name. The following proposition is an easy consequence of the definition and the condition $A_1 \cap A_2 = \emptyset$.

Proposition 9.2. *If Θ_1 (Θ_2) is a family of deterministic A_1-digraphs (A_2-digraphs), then $\Theta_1 \star_\Psi \Theta_2$ is a deterministic $A_1 \cup A_2$-digraph. In particular, if Θ_1 (Θ_2) is a family of inverse A_1-digraphs (A_2-digraphs), then the rooted graph $(\Theta_1 \star_\Psi \Theta_2, \gamma_0^1)$ is an inverse $A_1 \cup A_2$-digraph.*

We now make some geometric considerations on this construction. For simplicity of notation, let $\alpha = \gamma_0^1 = \lambda_0^1$ and put $\Omega = \Theta_1 \star_\Psi \Theta_2$. Now, consider a geodesic $\alpha \xrightarrow{w} v^*\gamma_i$ connecting α to some vertex $v^*\gamma_i$ of (Ω, α), colored by 1 and not belonging to $\{\gamma_0^1, \dots, \gamma_0^k\}$. Now w ends with a word colored by 1; thus, it may be uniquely factor as an alternating product $w = z_1x_1z_2x_2 \dots z_kx_k$ with $z_t \in A_2^*$ and $x_t \in A_1^*$. Let $\Lambda^{(s)}$ be the graph of Θ_2 with $\lambda_0^s = \psi_j^1(\gamma_i)$ so that in Ω, the vertex $v^*\gamma_i$ is identified with $v^*\lambda_0^s$.

We define the *out-cone* $\Omega^+(v^*\gamma_i)$ to be the inverse $A_1 \cup A_2$-digraph induced by all vertices of the form $v^*\gamma_i u^*$ (identified in the product according to the gluing maps Ψ). Note that by the definitions, the vertex $v^*\gamma_i$ is a cut-point, so this graph only depends on $\psi_j^1(\gamma_i) = \lambda_0^s$; hence, there are finitely many inverse $A_1 \cup A_2$-digraphs $\{\Lambda^{(s)}(\uparrow_\Psi)\}_s$ rooted in λ_0^s, which are end-isomorphic to the out-cones $(\Omega^+(v^*\gamma_i), v^*\gamma_i)$. A symmetric construction can be done with $\Omega^+(v^*\lambda_i) \simeq \Gamma^{(h)}(\uparrow_\Psi)$. The following characterization is straightforward (see Figure 3 for a graphical representation).

Lemma 9.3. *Put $(\Omega, \alpha) = (\Theta_1 \star_\Psi \Theta_2, \gamma_0^1)$, then for any $\gamma_i \notin \{\gamma_0^1, \dots, \gamma_0^k\}, \lambda_i \notin \{\lambda_0^1, \dots, \lambda_0^m\}$, we have*

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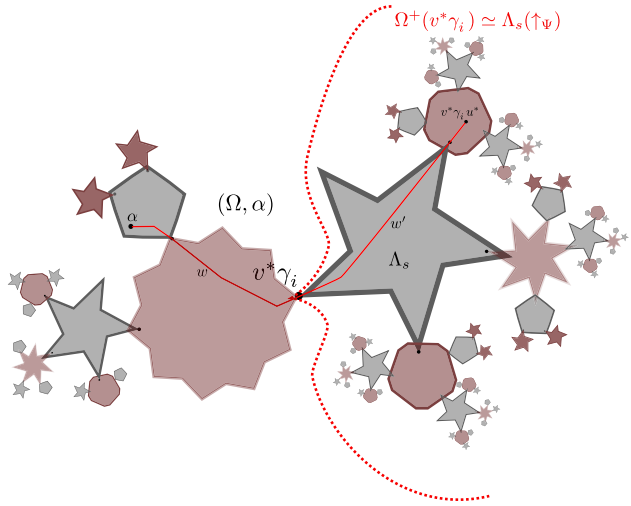


FIGURE 3 In the dashed region the out-cone $\Omega^+(\upsilon^*\gamma_i)$, and in red the geodesic $\alpha \xrightarrow{w} \upsilon^*\gamma_i \xrightarrow{w'} \upsilon^*\gamma_i u^*$.

$$V(\Omega^+(\upsilon^*\gamma_i)) = \{\upsilon^*\gamma_i\} \cup \{\upsilon^*\gamma_i u^* \ : \ \textit{there is a geodesic } \alpha \xrightarrow{w} \upsilon^*\gamma_i \xrightarrow{w'} \upsilon^*\gamma_i u^* \textit{ with } w' \in A_2^+(A_1 \cup A_2)^*\}$$

$$V(\Omega^+(\upsilon^*\lambda_i)) = \{\upsilon^*\lambda_i\} \cup \{\upsilon^*\lambda_i u^* \ : \ \textit{there is a geodesic } \alpha \xrightarrow{w} \upsilon^*\lambda_i \xrightarrow{w'} \upsilon^*\lambda_i u^* \textit{ with } w' \in A_1^+(A_1 \cup A_2)^*\}.$$

We now describe how to obtain the end-cones of Ω from the end-cones of the graphs in $\Theta_1 \cup \Theta_2$. We have the following lemma which, for simplicity, we state for a generic vertex colored by 2, but it clearly holds for any vertex of Ω.

Lemma 9.4. *With the above notation, for a given vertex $\upsilon^*\gamma_i\lambda_t$ of Ω colored by 2, the end-cone $\Omega(\upsilon^*\gamma_i\lambda_t, \alpha)$ is isomorphic to the disjoint union of inverse $A_1 \cup A_2$-digraphs*

$$\Lambda^{(s)}(\lambda_t, \lambda_0^s) \cup \bigsqcup_{\substack{z \in V(\Lambda^{(s)}(\lambda_t, \lambda_0^s)) \\ \gamma_0^h = \psi_s^2(z)}} \Gamma^{(h)}(\uparrow_\Psi)$$

and then, by identifying each vertex $z \in V(\Lambda^{(s)}(\lambda_t, \lambda_0^s))$ with $\gamma_0^h = \psi_s^2(z)$.

Proof. Set $n = \|\upsilon^*\gamma_i\lambda_t\|_\alpha$. Then, by Lemma 9.3, we may conclude that the end-cone $\Omega(\upsilon^*\gamma_i\lambda_t, \alpha)$ is contained in the out-cone $\Omega^+(\upsilon^*\gamma_i) \simeq \Lambda^{(s)}(\uparrow_\Psi)$. Moreover, the frontier vertices of $\Omega(\upsilon^*\gamma_i\lambda_t, \alpha)$ are in one-to-one correspondence with the frontier vertices of $\Lambda^{(s)}(\lambda_t, \lambda_0^s)$. Therefore, $\Omega(\upsilon^*\gamma_i\lambda_t, \alpha)$ may be obtained from $\Lambda^{(s)}(\lambda_t, \lambda_0^s)$ by gluing at each vertex υ of $\Lambda^{(s)}(\lambda_t, \lambda_0^s)$ the corresponding out-cone $\Gamma^{(h)}(\uparrow_\Psi)$ by identifying z with $\gamma_0^h = \psi_s^2(z)$. See Figure 4 for a graphical representation on how to obtain $\Omega(\upsilon^*\gamma_i\lambda_t, \alpha)$ from $\Lambda^{(s)}(\lambda_t, \lambda_0^s)$. □

9.1 | Context-free graphs and free product

In this section, we prove that the free product of graphs preserves the property of being context-free. For this purpose, we need to slightly adapt the notion of isomorphism and end-cone of an inverse A_1-digraph to include the notion of vertex coloring. A *vertex-coloring of a graph* Γ is

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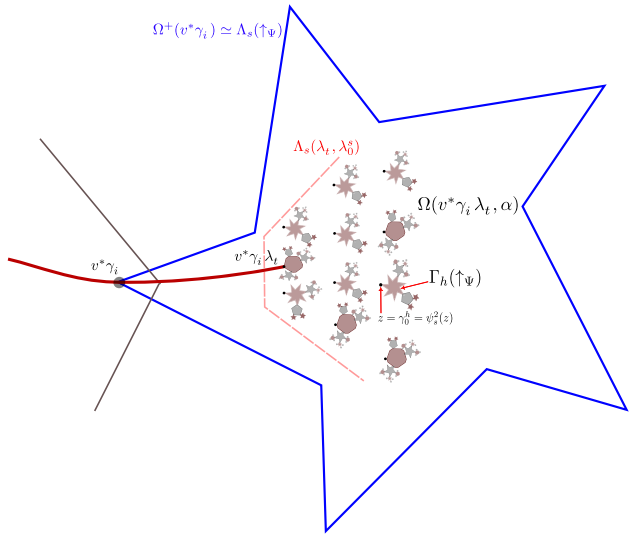


FIGURE 4 A depiction of the end-cone $\Omega(v^*\gamma_i\lambda_t, \alpha)$ that is obtained by gluing at each vertex z of the end-cone $\Lambda^{(s)}(\lambda_t, \lambda_0^s)$ of $\Lambda^{(s)}$ (in dashed red lines) the out-cone $\Gamma^{(h)}(\uparrow_\Psi)$ according to the value of gluing map $\gamma_0^h = \psi_s^2(z)$.

just a map $\chi : V(\Gamma) \to C$ onto some set of colors C. An *isomorphism* $\psi : \Gamma^{(1)} \to \Gamma^{(2)}$ between two inverse A-digraphs with color maps $\chi_1 : V(\Gamma) \to C$, $\chi_2 : V(\Gamma^{(2)}) \to C$ is an isomorphism of inverse A-digraphs that preserves the colorings: $\chi_1 = \chi_2 \circ \psi$. For technical reasons that will be clear later, we call a coloring of a rooted inverse A-digraph (Γ, γ_0) any map $\chi : V(\Gamma) \setminus \{\gamma_0\} \to C$, and we use the notation $(\Gamma, \gamma_0; \chi)$ to stress the coloring function. For such colored rooted inverse A-digraph $(\Gamma, \gamma_0; \chi)$, we may regard an end-cone $\Gamma(v, \gamma_0)$ as a colored inverse A-digraph with coloring map $\chi\,|_{V(\Gamma(v,\gamma_0))}$ given by the restriction of χ to the set of vertices of the end-cone $\Gamma(v, \gamma_0)$. We therefore say that the colored inverse A-digraph $(\Gamma, \gamma_0; \chi)$ is *context-free* if there are finitely many end-cones up to end-isomorphism (in the aforementioned sense). We stick to the notation introduced in Definition 9.1, and let us suppose that Θ_1 (Θ_2) is a collection of inverse A_1-digraphs (A_2-digraphs) with $A_1 \cap A_2 = \emptyset$ and with gluing maps Ψ. We say that $\Gamma^{(j)} \in \Theta_1$ ($\Lambda^{(j)} \in \Theta_2$) is context-free if the colored rooted inverse A_1-digraph $(\Gamma^{(j)}, \gamma_0^j; \psi_j^1)$ (A_2-digraph $(\Lambda^{(j)}, \lambda_0^j; \psi_j^2)$) is context-free. If all the colored inverse A_i-digraphs of $\Theta_1 \cup \Theta_2$ are context-free, we say that $(\Theta_1, \Theta_2, \Psi)$ is a *context-free ensemble*. We have the following theorem.

Theorem 9.5. *For a context-free ensemble* $(\Theta_1, \Theta_2, \Psi)$*, the free-product* $(\Theta_1 \star_\Psi \Theta_2, \gamma_0^1)$ *is a context-free inverse* $A_1 \cup A_2$*-digraph.*

Proof. By Lemma 9.4, using the same notation, we have that an end-cone $\Omega(v^*\gamma_i\lambda_t, \alpha)$ is obtained from the disjoint union

$$\Lambda^s(\lambda_t, \lambda_0^s) \cup \biguplus_{\substack{z \in V(\Lambda^s(\lambda_t, \lambda_0^s)) \\ \gamma_0^h = \psi_s^2(z)}} \Gamma^{(h)}(\uparrow_\Psi)$$

an then, by identifying z with $\psi_s^2(z)$. We prove that we have finitely many end-cones up to end-isomorphism (in the usual sense). Now, let $\Omega(v^*\gamma_i\lambda_\ell, \alpha)$ be another end-cone for which

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there is an end-isomorphism $\varphi : \Lambda^s(\lambda_t, \lambda_0^s) \to \Lambda^s(\lambda_\ell, \lambda_0^s)$ of end-cones of the same inverse A_2-digraph $\Lambda^{(s)}$ that preserves the coloring. Since the coloring map ψ_s^2 is commuting with φ, for any $z \in V(\Lambda^s(\lambda_t, \lambda_0^s))$, we have that $\psi_s^2(z) = \psi_s^2(\varphi(z)) = \gamma_0^h$, so both z and $\varphi(z)$ have the same color; hence, in both the vertex z and $\varphi(z)$, we are gluing the same graph $\Gamma^{(h)}(\uparrow_\Psi)$. Therefore, it is straightforward to show that φ may be extended

$$\widehat{\varphi} : \left(\Lambda^s(\lambda_t, \lambda_0^s) \cup \biguplus_{\substack{z \in V(\Lambda^s(\lambda_t, \lambda_0^s)) \\ \gamma_0^h = \psi_s^2(z)}} \Gamma^{(h)}(\uparrow_\Psi) \right) \to \left(\Lambda^s(\lambda_\ell, \lambda_0^s) \cup \biguplus_{\substack{z \in V(\Lambda^s(\lambda_\ell, \lambda_0^s)) \\ \gamma_0^h = \psi_s^2(z)}} \Gamma^{(h)}(\uparrow_\Psi) \right)$$

by locally extending φ from each disjoint copy $\Gamma^{(h)}(\uparrow_\Psi)$ associated to z, to the same graph $\Gamma^{(h)}(\uparrow_\Psi)$ associated to $\varphi(z)$. Now, since we are performing the identification $z = \psi_s^2(z)$ in the domain of $\widehat{\varphi}$ and this is the same as identifying $\varphi(z)$ with $\psi_s^2(\varphi(z))$, we have that $\widehat{\varphi}$ induces an isomorphism $\phi : \Omega(v^*\gamma_i\lambda_t, \alpha) \to \Omega(v^*\gamma_i\lambda_\ell, \alpha)$. Note that the frontier vertices of $\Omega(v^*\gamma_i\lambda_t, \alpha)$ are (up to renaming) the same as $\Lambda^s(\lambda_t, \lambda_0^s))$, and similarly for $\Omega(v^*\gamma_i\lambda_\ell, \alpha)$. Hence, ϕ is mapping the frontier vertices of $\Omega(v^*\gamma_i\lambda_t, \alpha)$ into the frontier vertices of $\Omega(v^*\gamma_i\lambda_\ell, \alpha)$; hence, they are end-isomorphic. We have just shown that the end-isomorphism classes of the end-cones of Ω of vertices colored by 2 depend on the end-isomorphism (preserving the coloring) classes of the graphs $\{\Lambda^{(1)}, \dots, \Lambda^{(m)}\}$ of Θ_2, which are finitely many. Similarly, one may show the same for the end-cones of the vertices colored by 1, and this concludes the proof of the theorem. □

10 | SOME EXAMPLES USING THE FREE PRODUCT OF GRAPHS

In this section, we use the free product of context-free graphs to construct some examples of interesting *CF-TR* groups. In the first example, the antenna graph, we show that in general transition groups of context-free graphs are not residually-finite. In the second one, the comb graph, we prove that the associated transition group is $\mathbb{Z} \wr \mathbb{Z}$, so a nonfinitely presented group. Hence, it is a solvable cocontext-free group that is not polycontext-free, showing that these classes of groups are different and solving the first point of Question 6.5. The third example is $C_2 \wr \mathbb{Z}$ and it addresses the second point of Question 6.5: it provides an infinite torsion subgroup, so that the group has arbitrarily large finite subgroups.

Definition 10.1 (The Antenna graph). Let $A_1 = \{c, c^{-1}\}$, $A_2 = \{a, a^{-1}, b, b^{-1}\}$. Consider the rooted A_1-graph (Γ, γ) that is the line

$$\gamma_i \xrightarrow{c} \gamma_{i+1},\ \gamma_{i+1} \xrightarrow{c^{-1}} \gamma_i, \quad \forall i \in \mathbb{Z}, \text{ where the root } \gamma = \gamma_0$$

and the two rooted A_2-graphs $(\Lambda_a, \lambda_a), (\Lambda_b, \lambda_b)$ that are lines with loops defined by

$$\lambda_{a,i} \xrightarrow{b} \lambda_{a,i},\ \lambda_{a,i} \xrightarrow{b^{-1}} \lambda_{a,i},\ \lambda_{a,i} \xrightarrow{a} \lambda_{a,i+1},\ \lambda_{a,i+1} \xrightarrow{a^{-1}} \lambda_{a,i}, \quad \forall i \in \mathbb{Z},\ \lambda_{a,0} = \lambda_a$$

$$\lambda_{b,i} \xrightarrow{a} \lambda_{b,i},\ \lambda_{b,i} \xrightarrow{a^{-1}} \lambda_{b,i},\ \lambda_{b,i} \xrightarrow{b} \lambda_{b,i+1},\ \lambda_{b,i+1} \xrightarrow{b^{-1}} \lambda_{b,i}, \quad \forall i \in \mathbb{Z},\ \lambda_{b,0} = \lambda_b$$

see the upper part of Figure 5. The gluing map Ψ is given by the functions defined as follows:

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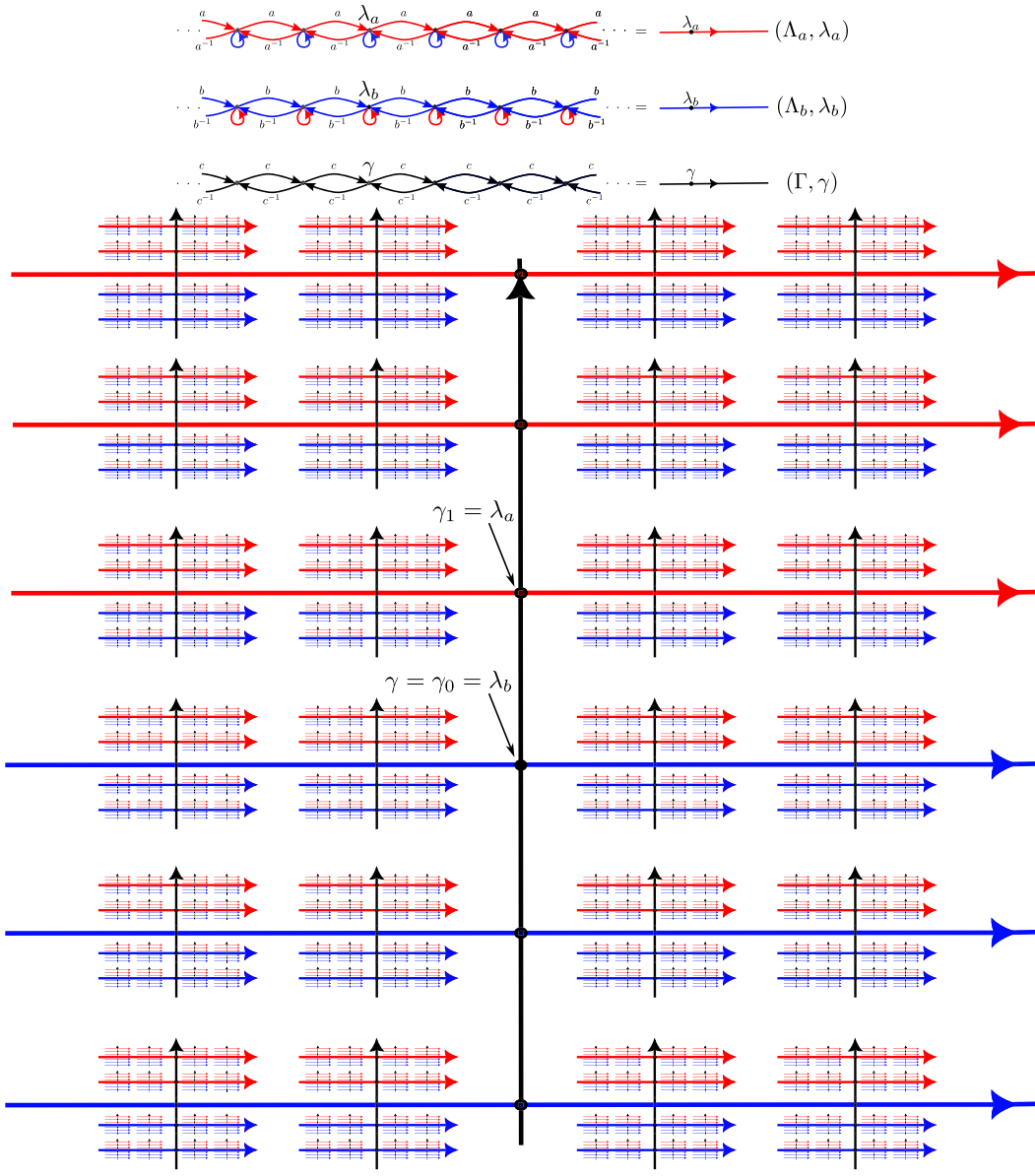


FIGURE 5 On the upper part the graphs Θ_1, Θ_2, in the lower picture the antenna graph $(\Theta_1 \star_\Psi \Theta_2, \gamma)$.

$$\psi^1 : V(\Gamma) \setminus \{\gamma\} \to \{\lambda_a, \lambda_b\},\ \psi^1(\gamma_i) = \begin{cases} \lambda_b & \text{if } i \leqslant 0 \\ \lambda_a & \text{otherwise} \end{cases}$$

and

$$\psi_a^2 : V(\Lambda_a) \setminus \{\lambda_a\} \to \{\gamma\},\ \psi_b^2 : V(\Lambda_b) \setminus \{\lambda_b\} \to \{\gamma\},$$

which are constant maps. By putting $\Theta_1 = \{(\Gamma, \gamma)\}$ and $\Theta_2 = \{(\Lambda_a, \lambda_a), (\Lambda_b, \lambda_b)\}$, the rooted $A_1 \cup A_2$-graph $(\mathfrak{A}, \gamma) = (\Theta_1 \star_\Psi \Theta_2, \gamma)$ is called the antenna graph, see Figure 5. Moreover, it is easy to see that $(\Theta_1, \Theta_2, \Psi)$ is a context-free-ensemble; thus, by Theorem 9.5, $(\mathfrak{A}, \gamma)$ is a context-free graph.

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Note that with the convention of the free product that we have given, the root γ of $\Theta_1 \star_\Psi \Theta_2$ is identified with λ_b.

By the tree structure, the set of vertices can be identified using words in $\gamma(A_1 \cup A_2)^*$ by associating to a vertex the unique geodesic connecting γ to such vertex. It is easy to see that the action of the alphabet $A_1 \cup A_2$ on the set of vertices is defined as follows. Suppose that we are in the vertex $v = wx^s$, $s \neq 0$, in case $x \in A_1 \cup A_2$ we have

$$\begin{aligned} &wc^s \cdot b^t = \begin{cases} wc^s, & \text{if } s > 0 \\ wc^s b^t, & \text{otherwise} \end{cases}, \; wc^s \cdot a^t = \begin{cases} wc^s a^t, & \text{if } s \geqslant 0 \\ wc^s, & \text{otherwise} \end{cases} \\ &wa^s \cdot b^t = wa^s, \; wb^s \cdot a^t = wb^s, \; wa^s \cdot c^t = wa^s c^t, \; wb^s \cdot c^t = wb^s c^t, \\ &\gamma \cdot a^s = \gamma, \gamma \cdot b^s = \gamma b^s, \gamma \cdot c^t = c^t. \end{aligned} \tag{10}$$

We now prove that $\mathcal{G}(\mathfrak{A})$ is not residually finite, but first we need the following lemma.

Lemma 10.2. *For each $k, s \in \mathbb{Z}$ and $t, p \geqslant 0$, we have $\left[c^{-t} a^s c^t, c^p b^k c^{-p}\right] = \mathbb{1}_{\mathcal{G}(\mathfrak{A})}$.*

Proof. This is equivalent to proving that $g = \left[a^s, c^\alpha b^k c^{-\alpha}\right] = \mathbb{1}_{\mathcal{G}(\mathfrak{A})}$ for every $\alpha \geqslant 0$. We may assume $k, s \neq 0$, and also $\alpha > 0$, since the relation $[a, b] = \mathbb{1}_{\mathcal{G}(\mathfrak{A})}$ is immediate. We use the notation introduced above. We can rewrite g as

$$g = a^s \cdot \left((c^\alpha b^k c^{-\alpha}) a (c^\alpha b^k c^{-\alpha})^{-1}\right)^{-s} = \left((a^s c^\alpha) b (a^s c^\alpha)^{-1}\right)^k \cdot (c^\alpha b c^{-\alpha})^{-k}.$$

Also, $\gamma \cdot g = \gamma$. For $g \in \mathcal{G}(\mathfrak{A})$, we denote by $\mathrm{supp}(g)$ the support of g, that is, the set of vertices $v \in V(\mathfrak{A})$ such that $v \cdot g \neq v$. Note that for any vertex $v \in \mathrm{supp}(a)$, we have $v \cdot (c^\alpha b^t c^{-\alpha}) = v$ for every $t \in \mathbb{Z}$ and $\alpha \geqslant 0$. Therefore, we have the following equality:

$$\mathrm{supp}\left((c^\alpha b^k c^{-\alpha}) a (c^\alpha b^k c^{-\alpha})^{-1}\right) = \mathrm{supp}(a)(c^\alpha b^k c^{-\alpha})^{-1} = \mathrm{supp}(a).$$

Hence, if $v \notin \mathrm{supp}(a)$, then v is fixed by g (fixed by the two factors in the decomposition of g stated above). Similarly, since $v \cdot c^{-\alpha} a^t \in \{wc^m : m < 0\}$ for $v \in \mathrm{supp}(b)$ and every $t \in \mathbb{Z}$, we have

$$\mathrm{supp}(a^s c^\alpha b c^{-\alpha} a^{-s}) = \mathrm{supp}(b) c^{-\alpha} a^s \subseteq \{wc^m : m < 0\},$$

$$\mathrm{supp}(c^\alpha b c^{-\alpha}) = \mathrm{supp}(b) c^{-\alpha} \subseteq \{wc^m : m < 0\}.$$

Thus, if $v \notin \{wc^m : m < 0\}$, then v is also fixed by g. Now, since $\mathrm{supp}(a) \cap \{wc^m : m < 0\} = \emptyset$, we conclude that any vertex is fixed by g, and this concludes the proof. □

Lemma 10.3. *For each $k, s \in \mathbb{Z}$ and $t, p \geqslant 0$, we have $\left[c^{-t} a^s c^t, c^p b^k c^{-p}\right] = \mathbb{1}_{\mathcal{G}(\mathfrak{A})}$.*

Proof. If $s = 0$ or $k = 0$, the claim is immediate. If $t = p = 0$, then $[a^s, b^k] = \mathbb{1}_{\mathcal{G}(\mathfrak{A})}$ follows directly from (10). Assume $t + p > 0$. For the right action, we use $\mathrm{supp}(r^{-1} h r) = \mathrm{supp}(h) \cdot r$. Indeed, $v \cdot r^{-1} h r \neq v$ if and only if $v \cdot r^{-1} \in \mathrm{supp}(h)$. By (10), and writing vertices by their unique geodesic representatives from γ, we have $\mathrm{supp}(a^s) \subseteq \{wa^q : q \neq 0\} \cup \{wc^q : q \geqslant 0\}$ and $\mathrm{supp}(b^k) \subseteq \{wb^q : q \neq 0\} \cup \{wc^q : q \leqslant 0\}$. Hence, $\mathrm{supp}(c^{-t} a^s c^t) = \mathrm{supp}(a^s) \cdot c^t$ is contained in $\{wa^q c^t : q \neq 0\} \cup \{wc^q : q \geqslant t\}$, while $\mathrm{supp}(c^p b^k c^{-p}) = \mathrm{supp}(b^k) \cdot c^{-p}$ is contained in $\{w' b^q c^{-p} : q \neq 0\} \cup \{w' c^q : q \leqslant -p\}$. These two sets are disjoint. Indeed, by uniqueness of geodesic representatives, a word of

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the form $wa^q c^t$, with $q \neq 0$, cannot be equal to a word of the form $w' b^{q'} c^{-p}$, with $q' \neq 0$. Moreover, a word of the first c-part has final exponent $q \geqslant t$, while a word of the second c-part has final exponent $q \leqslant -p$; these inequalities are incompatible since $t, p \geqslant 0$ and $t + p > 0$. Thus, two elements have disjoint supports, and therefore, they commute. □

Theorem 10.4. *The group $G = \mathcal{G}(\mathfrak{A})$ defined by the antenna graph $\mathfrak{A}$ is not residually finite.*

Proof. Consider the element

$$h = \left[a, c^{-1}bc\right] = ac^{-1}bca^{-1}c^{-1}b^{-1}c,$$

and let $\varphi : G \to H$ be an homomorphism with H a finite group.

Note that $h \neq \mathbb{1}_{\mathcal{G}(\mathfrak{A})}$ since we have

$$\gamma \cdot ac^{-1}bca^{-1}c^{-1}b^{-1}c = \gamma \cdot (c^{-1}bca^{-1}c^{-1}b^{-1}c) = \gamma c^{-1}bca^{-1}c^{-1}b^{-1}c \neq \gamma.$$

Since H is finite, there is some positive $\alpha > 0$ such that $\varphi(c)^\alpha = \varphi(c)^{-1}$. By Lemma 10.3, we have $[a, c^\alpha bc^{-\alpha}] = \mathbb{1}_{\mathcal{G}(\mathfrak{A})}$, hence

$$\mathbb{1}_H = \varphi([a, c^\alpha bc^{-\alpha}]) = [\varphi(a), \varphi(c)^\alpha \varphi(b)\varphi(c)^{-\alpha}] = \left[\varphi(a), \varphi(c)^{-1}\varphi(b)\varphi(c)\right] =$$
$$= \varphi\left(\left[a, c^{-1}bc\right]\right) = \varphi(h).$$

Hence, there is no homomorphism φ to a finite group such that $\varphi(h) \neq \mathbb{1}_H$ as desired. □

In light of Theorem 6.2 and the preceding theorem, we observe that, if Brough Conjecture (1) regarding polycontext-free groups is true, then the class of all polycontext-free groups would be strictly contained within the class *CF-TR*. Although we are not able to show that the class of polycontext-free is included into our class, in the next example, we show that these two classes are different. This is done by defining the following context-free graph.

Definition 10.5 (The comb graph). Let $A_1 = \{c, c^{-1}\}$ and $A_2 = \{a, a^{-1}\}$. We define three rooted graphs:

- (Γ, z_0) is the rooted A_1-graph whose underlying graph is the line

$$z_i \xrightarrow{c} z_{i+1},\ z_{i+1} \xrightarrow{c^{-1}} z_i, \quad \forall i \in \mathbb{Z}.$$

- (Λ_a, λ_a) is the rooted A_2-graph consisting of the single vertex λ_a with loops labeled by a, a^{-1}.
- (Λ, x_0) is the rooted A_2-graph whose underlying graph is the line

$$x_i \xrightarrow{a} x_{i+1},\ x_{i+1} \xrightarrow{a^{-1}} x_i, \quad \forall i \in \mathbb{Z}.$$

See the upper part of Figure 6. The gluing map Ψ is given by the following constant maps:

$$\psi_1 : V(\Gamma) \setminus \{z_0\} \to \{\lambda_a\}, \qquad \psi_1(z) = \lambda_a,$$

$$\psi_2 : V(\Lambda) \setminus \{x_0\} \to \{z_0\}, \qquad \psi_2(x_i) = z_0.$$

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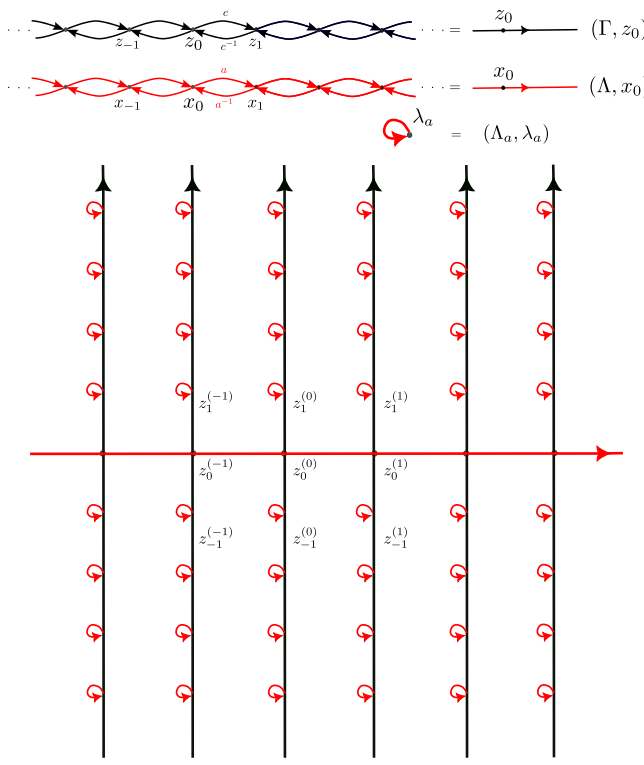


FIGURE 6 On the upper part the graphs Θ_1, Θ_2, and in the lower picture the comb graph $(\Theta_1 \star_\Psi \Theta_2, \gamma)$.

By putting $\Theta_1 = \{(\Gamma, z_0)\}$ and $\Theta_2 = \{(\Lambda, x_0), (\Lambda_a, \lambda_a)\}$, we see that $(\Theta_1, \Theta_2, \Psi)$ is a context-free ensemble. Thus, the free product $(\mathfrak{C}, \gamma) = (\Theta_1 \star_\Psi \Theta_2, z_0)$ defines a context-free graph, called the comb graph; see Figure 6. Note that, with our convention for the free product, the root of this graph is z_0, which is also identified with x_0.

In $\mathfrak{C}$, there are infinitely many copies of the c-line; we denote by $z_i^{(j)}$ the ith vertex of the jth copy of this line. Graphically, two vertices $z_i^{(j)}$ and $z_i^{(k)}$ are at the same "i-th level," but they belong to the jth and kth c-lines, respectively. The a-line intersects the c-lines at the vertices $z_0^{(j)}$.

For the next proposition, we recall the notion of standard action of $\mathbb{Z} \wr \mathbb{Z}$ on $\mathbb{Z} \times \mathbb{Z}$. In general, if $(f, b)(g, b') = (f \cdot g^b, bb')$ is the operation on $A \wr B$, where $g^b(x) = g(b^{-1}(x))$ for any $x \in B$. The standard action on $A \times B$ of $A \wr B$ is given by $(a, b)(f, b') = (af(b), bb')$. It is standard in the sense that the (unrestricted, permutational) wreath product $\mathrm{Sym}(A) \wr \wr_B \mathrm{Sym}(B)$ is the largest group acting on $A \times B$ preserving the block system $A \times \{b\}$, this is well studied in relation with imprimitive permutation groups, and $A \wr B$ naturally embeds in $\mathrm{Sym}(A) \wr \wr_B \mathrm{Sym}(B)$.

Proposition 10.6. *The transition group $G = \mathcal{G}(\mathfrak{C})$ is isomorphic to $\mathbb{Z} \wr \mathbb{Z}$. In particular, G is not finitely presented, solvable, and contains a copy of infinite-rank free abelian group $\mathbb{Z}^\infty$. Hence, it is not polycontext-free.*

Proof. Consider the action described above of $\mathbb{Z} \wr \mathbb{Z}$ on $\mathbb{Z} \times \mathbb{Z}$. The graph in Figure 6 is the corresponding Schreier graph (see [22]). Since the action is faithful, the stabilizer of any vertex is

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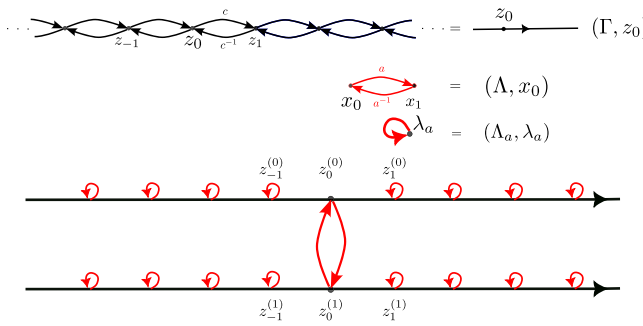


FIGURE 7 The context-free graph $\mathfrak{T}$ obtained by taking the free product of the graphs in the upper part, where the gluing maps are defined by taking $\psi^1(z_i) = \lambda_a$ for all $z_i \neq z_0$, and $\psi^2(x_1) = z_0$.

core-free. Hence, Proposition 4.4 applies, and therefore, $G \simeq \mathbb{Z} \wr \mathbb{Z}$. It is known that $\mathbb{Z} \wr \mathbb{Z}$ is solvable, not finitely presented, and contains the infinite-rank free abelian subgroup $\bigoplus_{n\in\mathbb{Z}} \mathbb{Z}$. In particular, it is not polycontext-free. □

We finally address the second point of Question 6.5 regarding the possibility for a transition group to have arbitrarily large finite subgroups. We prove that such examples may exist by showing a group in *CF-TR* having an infinite torsion subgroup. The associated graph $\mathfrak{T}$ on the alphabet $A = \{a, c, a^{-1}, c^{-1}\}$ is depicted in Figure 7. Using the notation in Subsection 8, we notice that $\mathfrak{T}$ is locally-quasi-transitive with infinite orbits by taking $F = D_1(z_0{}^{(0)})$. We have the following proposition.

Proposition 10.7. *The transition group $\mathcal{G}(\mathfrak{T})$ is isomorphic to $C_2 \wr \mathbb{Z}$. In particular, it is metabelian and not polycontext-free. Moreover, the boundary subgroup $\mathscr{B}_F$ is the base group of the wreath product; in particular,*

$$\mathscr{B}_F \cong \bigoplus_{n\in\mathbb{Z}} C_2,$$

so it is an infinite normal elementary abelian 2-subgroup of $\mathcal{G}(\mathfrak{T})$ with $\mathcal{G}(\mathfrak{T})/\mathscr{B}_F \cong \mathbb{Z}$.

Proof. Let $\chi_c : A^* \to \mathbb{Z}$ be the homomorphism defined by $\chi_c(c) = 1$, $\chi_c(c^{-1}) = -1$, and $\chi_c(a) = \chi_c(a^{-1}) = 0$. Note that $\mathfrak{T}$ is locally quasi-transitive and that

$$\mathfrak{T} \setminus F = \Lambda_1 \setminus \{y_1, y_2\},$$

where $F = \{y_1, y_2\}$ and Λ_1 is the transitive graph consisting of two c-lines, each carrying a- and a^{-1}-loops at every vertex. By Lemma 8.5, the language $\mathcal{O}_F$ consists of those words labeling a circuit at every vertex of Λ_1; hence, $\mathcal{O}_F = \{u \in A^* : \chi_c(u) = 0\}$. Let $u \in \mathcal{O}_F$. Then, for every vertex $z_i^{(j)}$ of $\mathfrak{T}$, with $j \in \{0, 1\}$, one has either

$$z_i^{(j)} \cdot u = z_i^{(j)} \qquad \text{or} \qquad z_i^{(j)} \cdot u = z_i^{(j+1 \bmod 2)}.$$

The second case occurs precisely when the path labeled by u crosses the edge

$$z_0^{(j)} \xrightarrow{x} z_0^{(j+1 \bmod 2)}, \qquad x \in \{a, a^{-1}\},$$

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an odd number of times. In particular, $z_i^{(j)} \cdot u^2 = z_i^{(j)}$ for every vertex $z_i^{(j)}$. Hence, every element of $\mathscr{B}_F$ has order 2. Moreover, $\mathscr{B}_F$ is infinite. Indeed, for each $n \in \mathbb{Z}$, let $h_n := c^{-n}ac^n \in \mathscr{B}_F$. These elements are pairwise distinct, since for every $j \in \{0,1\}$, one has

$$z_m^{(j)} \cdot h_n = \begin{cases} z_m^{(j+1 \bmod 2)} & \text{if } m = n, \\ z_m^{(j)} & \text{if } m \neq n. \end{cases}$$

Therefore, $\mathscr{B}_F$ is an infinite abelian torsion group of order 2. Let

$$C_2 \wr \mathbb{Z} = \langle \alpha, t \mid \alpha^2 = 1,\ [t^{-m}\alpha t^m, t^{-n}\alpha t^n] = 1\ \forall m, n \in \mathbb{Z} \rangle.$$

Since $\mathcal{G}(\mathfrak{T})$ is generated by a and c, with $a^2 = 1$, and since all conjugates $c^{-n}ac^n$ belong to the abelian subgroup $\mathscr{B}_F$, the assignment $\alpha \mapsto a, t \mapsto c$ extends to a surjective homomorphism $C_2 \wr \mathbb{Z} \twoheadrightarrow \mathcal{G}(\mathfrak{T})$. This epimorphism cannot be proper, since every proper quotient of $C_2 \wr \mathbb{Z}$ is virtually cyclic. For completeness, note that if $C_2 \wr \mathbb{Z} = B \rtimes \mathbb{Z}, B = \bigoplus_{n\in\mathbb{Z}} C_2$, then every nontrivial normal subgroup intersects B nontrivially, and every nonzero shift-invariant quotient of B is finite. Thus, every proper quotient is finite-by-cyclic. Therefore, $\mathcal{G}(\mathfrak{T}) \cong C_2 \wr \mathbb{Z}$. Under this identification, $\mathscr{B}_F$ is precisely the base group, hence $\mathscr{B}_F \cong \bigoplus_{n\in\mathbb{Z}} C_2$ and $\mathcal{G}(\mathfrak{T})/\mathscr{B}_F \cong \mathbb{Z}$. □

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REFERENCES

1. R. Al Kohli, *Groups defined by language theoretic classes*, Ph.D. thesis, University of St Andrews, 2024, DOI: 10.17630/sta/870.
2. A. V. Anisimov, *Über Gruppen-Sprachen*, Kibernetika **4** (1971), 18–24.
3. J. Belk, C. Bleak, and F. Matucci, *Rational embeddings of hyperbolic groups*, J. Comb. Algebra **5** (2021), no. 2, 123–183, DOI 10.4171/JCA/52.
4. J. Belk, J. Hyde, and F. Matucci, *On the asynchronous rational group*, Groups Geom. Dyn. **13** (2019), no. 4, 1271–1284, DOI 10.4171/GGD/523.
5. J. Berstel, *Transductions and context-free languages*, Vieweg+Teubner Verlag, Wiesbaden, 1979.
6. A. Bishop and M. Elder, *Bounded automata groups are co-ET0L*, In Conference Proceedings of LATA (2019), DOI 10.1007/978-3-030-13435-8_6.
7. C. Bleak, F. Matucci, and M. Neunhöffer, *Embeddings into Thompson's group V and coCF groups*, J. London Math. Soc. **94** (2016), no. 2, 583–597, DOI 10.1112/jlms/jdw044.
8. C. Bleak and O. Salazar-Díaz, *Free products in R. Thompson's group V*, Trans. Am. Math. Soc. **365** (2013), no. 11, 5967–5997, DOI 10.1090/S0002-9947-2013-05823-0.
9. T. Brough, *Groups with poly-context-free word problem*, Groups Complex. Cryptol. **6** (2014), no. 1, 9–29, 10.1515/gcc-2014-0002.
10. T. Brough and D. F. Holt, *Finitely generated soluble groups and their subgroups*, Comm. Algebra **41** (2013) no. 5, 1790–1799, DOI 10.1080/00927872.2011.651758.
11. M. Carter, S. Tornier, and G. A. Willis, *On free products of graphs*, Australas. J Comb. **78** (2020), no. 1, 154–176.
12. T. Ceccherini-Silberstein and W. Woess, *Context-free pairs of groups I: context-free pairs and graphs*, Eur. J. Comb. **33** (2012), 1449–1466, DOI 10.1016/j.ejc.2012.03.011.
13. L. Ciobanu, M. Elder, and M. Ferov, *Applications of L systems to group theory*, Int. J. Algebra Comput. **28** (2018), no. 2, 309–329, DOI 10.1142/S0218196718500145.
14. S. Eilenberg, *Automata, languages, and machines, volume A*, Pure and Applied Mathematics, Academic Press, New York, 1976.
15. R. I. Grigorchuk, V. Nekrashevych, and V. Sushchanskiĭ, *Automata, dynamical systems, and groups*, Proc. Steklov Inst. Math **231** (2000), no. 4, 128–203.
16. C. Groothuis and J. Meakin, *Graph immersions, inverse monoids and deck transformations*, J. Aust. Math. Soc. **111** (2020), 37–55, DOI 10.1017/S1446788720000087.
17. T. Herbst, *On a subclass of context-free groups*, RAIRO Theor. Inform. Appl. **25** (1991), no. 3, 255–272, DOI 10.1051/ita/1991250302551.
18. D. F. Holt, M. D. Owens, and R. M. Thomas, *Groups and semigroups with a one-counter word problem*, J. Aust. Math. Soc. **85** (2008), 197–209, DOI 10.1017/S1446788708000864.
19. D. F. Holt, S. Rees, C. E. Röver, and R. M. Thomas, *Groups with context-free co-word problem*, J. London Math. Soc. **71** (2005), no. 2, 643–657, DOI 10.1112/S002461070500654X.
20. J. E. Hopcroft and J. D. Ullman, *Introduction to automata theory, languages and computation*, Addison-Wesley Series in Computer Science, Addison-Wesley Publishing Co., Reading, MA, 1979, x+418 pp.
21. M. Kutrib and A. Malcher, *Reversible pushdown automata*, J. Comput. Syst. Sci. **78** (2012), 1814–1827, DOI 10.1016/j.jcss.2011.12.004.
22. A. Le Boudec and N. Matte Bon, *Growth of actions of solvable groups*, arXiv:2205.11924.
23. C. Lindorfer and W. Woess, *The language of self-avoiding walks*, Combinatorica **40** (2020), no. 5, 691–720, DOI 10.1007/s00493-020-4184-z.
24. K. A. Mihailova, *The occurrence problem for direct products of groups*, Dokl. Akad. Nauk SSSR **119** (1958), 1103–1105.
25. C. F. Miller III, *On group-theoretic decision problems and their classification*, Annals of Mathematics Studies, vol. 68, Princeton University Press, Princeton, NJ, 1971.
26. D. E. Muller and P. E. Schupp, *Groups, the theory of ends, and context-free languages*, J. Comput. System Sci. **26** (1983), no. 3, 295–310.
27. D. E. Muller and P. E. Schupp, *The theory of ends, pushdown automata, and second-order logic*, Theoret. Comput. Sci. **37** (1985), no. 1, 51–75.

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28. J.-L. Nicolas, *On Landau's function g(n)*, in R. L. Graham, J. Nešetřil, eds., The mathematics of Paul Erdös I. Algorithms and Combinatorics, vol 13. Springer, Berlin, Heidelberg, 1997.
29. E. Rodaro, *Generalizations of the Muller-Schupp theorem and tree-like inverse graphs*, J. London Math. Soc. **109** (2024), no. 5, DOI 10.1112/jlms.12903.
30. J. Sakarovitch, *Elements of automata theory*, Thomas R, trans., Cambridge University Press, 2009.
31. J.-P. Serre, *Trees*, Springer Monographs in Mathematics, Springer, Berlin, Heidelberg, 2003, x+142 pp.
32. J. R. Stallings, *Topology of finite graphs*, Invent. Math. **71** (1983), 551–565.
33. W. Woess, *Graphs and groups with tree-like properties*, J. Combin. Theory Ser. B **47** (1989), no. 3, 361–371, 10.1016/0095-8956(89)90034-8.

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