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L^p gradient estimates and Calderón–Zygmund inequalities under Ricci lower bounds

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Abstract. In this paper, we investigate the validity of first and second order L^p estimates for the solutions of the Poisson equation depending on the geometry of the underlying manifold. We first present L^p estimates of the gradient under the assumption that the Ricci tensor is lower bounded in a local integral sense, and construct the first counterexample showing that they are false, in general, without curvature restrictions. Next, we obtain L^p estimates for the second order Riesz transform (or, equivalently, the validity of L^p Calderón–Zygmund inequalities) on the whole scale $1 < p < +\infty$ by assuming that the injectivity radius is positive and that the Ricci tensor is either pointwise lower bounded, or non-negative in a global integral sense. When $1 < p \leq 2$, analogous L^p bounds on higher even order Riesz transforms are obtained provided that also the derivatives of Ricci are controlled up to a suitable order.

1. Introduction

The purpose of this paper is to prove some regularity results (see Section 2 for the precise statements) concerning solutions to the Poisson equation on Riemannian manifolds under comparatively weak assumptions on their geometry. We also show that certain regularity results may be strongly influenced by the geometry at infinity of the manifold. One recurrent theme in our investigation is to prove (at least some of) our results under the assumption that the Ricci curvature satisfies appropriate L^p lower bounds in place of the pointwise bounds that commonly appear in the literature.

In order to place our research in perspective, we begin by making some comments that may help the reader orienting in this fascinating field of research.

Given a function f in $L^p(\mathbb{R}^n)$, where $1 < p < \infty$, and a distributional solution u of the Poisson equation $\Delta u = f$, it is well known that $\partial_j \partial_\ell u$ belongs to $L^p(\mathbb{R}^n)$ for every pair of integers j and ℓ in $\{1, \dots, n\}$, and

$$\|\partial_j \partial_\ell u\|_p \leq C \|f\|_p, \tag{1.1}$$

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where C does not depend on f. This regularity result may be reformulated as a boundedness result in $L^p(\mathbb{R}^n)$ for the so called second order Riesz transform, as follows. For j and ℓ as above, consider the operator $\mathcal{R}_{j,\ell}$ defined, at least formally, by

$$(\mathcal{R}_{j,\ell} f)\hat{}(\xi) = \frac{\xi_j\, \xi_\ell}{|\xi|^2}\, \widehat{f}(\xi).$$

The operator $\mathcal{R}_{j,\ell}$ is a paradigmatic example of Calderón–Zygmund singular integral operator, and acts on f by convolution with a specific principal value distribution, viz. the inverse Fourier transform of the function $\xi \mapsto \xi_j\, \xi_\ell/|\xi|^2$. Such operators are known to be bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, and of weak type $(1,1)$, see [21]. By virtue of the very special structure of the Euclidean space, this is equivalent to saying that the operator $\nabla^2(-\Delta)^{-1}$, where ∇^2 denotes the second covariant derivative associated to the Euclidean metric, extends to a bounded operator from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n; T_2\mathbb{R}^n)$, the space of all L^p sections of the second order covariant tensors on $\mathbb{R}^n$, endowed with the standard metric. The operator $\nabla^2(-\Delta)^{-1}$ will henceforth be called *second order Riesz transform*, and denoted by $\mathcal{R}^2$. More generally, for each positive integer k, one can consider the kth order Riesz transform $\nabla^k(-\Delta)^{-k/2}$, denoted by $\mathcal{R}^k$, which is bounded from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n; T_2\mathbb{R}^n)$, $1 < p < \infty$, and of weak type $(1,1)$.

The Riesz potential Δ^{-1} is unbounded on $L^p(\mathbb{R}^n)$, so that one cannot expect that a distributional solution of (1.1) with L^p datum f belongs to $L^p(\mathbb{R}^n)$. A simple scaling argument shows that both the estimates

$$\|u\|_p \leq C\|f\|_p \quad \text{and} \quad \||\nabla u|\|_p \leq C\|f\|_p$$

fail. However, Δ^{-1} is a smoothing operator. Indeed, if $n \geq 3$, then the Hardy–Littlewood–Sobolev inequality implies that Δ^{-1} maps $L^p(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n)$, where $1/r = 1/p - 2/n$. Thus, distributional solutions u of the Poisson equation (1.1) belong to $L^r(\mathbb{R}^n)$, hence locally to $L^p(\mathbb{R}^n)$. This, in turn, implies that u is locally (but not globally) in the Sobolev space $W^{2,p}(\mathbb{R}^n)$.

Recall that $-\Delta$ generates a Markovian semigroup, so that its L^p spectrum is contained in the closure of the right half plane. In particular, for every $\tau > 0$, the operator $\tau\mathcal{I} - \Delta$ is invertible in $L^p(\mathbb{R}^n)$, $1 < p < \infty$, or equivalently,

$$\|u\|_p \leq C\|\tau u - \Delta u\|_p \tag{1.2}$$

whenever the right-hand side is finite. In other words, solutions to the modified Poisson equation $\Delta u - \tau u = f$, with datum f in $L^p(\mathbb{R}^n)$, are in $L^p(\mathbb{R}^n)$. It is convenient to introduce the kth order *local Riesz transform* $\mathcal{R}^k_\tau := \nabla^k(\tau\mathcal{I} - \Delta)^{-k/2}$. Then the estimate (1.1) may be reformulated by saying that $\mathcal{R}^2_\tau$ is bounded from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n; T_2\mathbb{R}^n)$. Furthermore, observe that the L^p boundedness of the first order Riesz transform $\nabla(-\Delta)^{-1/2}$, and the moment inequality, see Proposition 6.6.4 in [18] (which we can apply, for $-\Delta$ is a sectorial operator on $L^p(\mathbb{R}^n)$), imply the gradient estimate

$$\||\nabla u|\|_p \leq C\|(-\Delta)^{1/2}u\|_p \leq C\|u\|_p^{1/2}\,\|\Delta u\|_p^{1/2} \leq C(\|u\|_p + \|\Delta u\|_p). \tag{1.3}$$

This and (1.1) then yield the bound

$$\|u\|_{W^{2,p}(\mathbb{R}^n)} \leq C(\|u\|_p + \|\Delta u\|_p). \tag{1.4}$$

It is natural to speculate how the scenario described above has to be modified as we progressively move away from the familiar Euclidean space, by replacing $\mathbb{R}^n$ with a complete noncompact n-dimensional Riemannian manifold M, and the Laplace operator by the Laplace–Beltrami operator, which we henceforth denote by Δ. Clearly, the definitions of Riesz transform and local Riesz transform of order k extend in an obvious way to this more general setting. They will be denoted by $\mathcal{R}^k_\tau$ and $\mathcal{R}^k$, respectively.

Simple examples that illustrate how subtle the influence of the geometry at infinity of M on the estimates discussed above can be are the hyperbolic space $\mathbb{H}^n$ and the connected sum $\mathbb{R}^n \sharp \mathbb{R}^n$ of two copies of $\mathbb{R}^n$. It is worth observing that both $\mathbb{H}^n$ and $\mathbb{R}^n \sharp \mathbb{R}^n$ have bounded geometry in the strongest possible sense.

Since the bottom of the L^2 spectrum of Δ is strictly negative and its L^1 spectrum is contained in the left half plane, Δ is invertible on $L^p(\mathbb{H}^n)$, $1 < p < \infty$, so that a distributional solution u of the Poisson equation $\Delta u = f$, with f in $L^p(\mathbb{H}^n)$, automatically belongs to $L^p(\mathbb{H}^n)$. Since the first order Riesz transform is bounded from $L^p(\mathbb{H}^n)$ to $L^p(\mathbb{H}^n; T_1\mathbb{H}^n)$ (see [2, 29]), we can argue as in (1.3), and conclude that

$$\|u\|_{W^{2,p}(\mathbb{H}^n)} \le C \|f\|_p, \tag{1.5}$$

an estimate which has no analogue in $\mathbb{R}^n$.

Coulhon and Duong [10] proved that the first order Riesz transform $\mathcal{R}^1$ is unbounded on $L^p(\mathbb{R}^n \sharp \mathbb{R}^n)$ for $p > n$. In fact, they considered the case $n \ge 3$, but their argument can be adapted to the case where $n = 2$. Thus, in particular, $\mathcal{R}^1$ is unbounded from $L^p(\mathbb{R}^2 \sharp \mathbb{R}^2)$ to $L^p(\mathbb{R}^2 \sharp \mathbb{R}^2; T_1(\mathbb{R}^2 \sharp \mathbb{R}^2))$ for all $p > 2$, a fact alien to $\mathbb{R}^n$. For an interesting generalization to manifolds with finitely many Euclidean ends, see [8].

Suppose now that (M, g) is an n-dimensional Riemannian manifold and that $1 < p < \infty$, and consider the problem of determining (geometric) assumptions under which the analogues of (1.1), (1.2), (1.3), (1.4) and (1.5) hold on M. It may be worth warning the reader that people in harmonic analysis and in global analysis quite often use different terminologies to denote the same object: in particular, the former speak about the L^p boundedness of local Riesz transforms, whereas the latter prefer to refer to the L^p Calderón–Zygmund inequalities

$$\big\||\nabla^2 u|\big\|_p \le C\,\big[\|u\|_p + \|\Delta u\|_p\big], \quad \forall u \in C^\infty_c(M). \tag{1.6}$$

An account of this latter approach can be found in the survey paper [28]. The equivalence between the L^p boundedness of the second order Riesz transform and the validity of an L^p Calderón–Zygmund inequality will be formalised in Proposition 2.4. The two formulations will be used interchangeably in the rest of the paper.

First we look at (1.3). A special case of a celebrated result of D. Bakry [3] states that if the Ricci curvature of M is bounded from below, then the first order local Riesz transform is bounded on $L^p(M)$ for every $p \in (1, \infty)$; equivalently, there exists a constant C such that

$$\||\nabla u|\|_p \le C\,\big[\|(-\Delta)^{1/2} u\|_p + \|u\|_p\big], \quad \forall u \in C^\infty_c(M). \tag{1.7}$$

Thus, much as in (1.3), we obtain the gradient estimate

$$\||\nabla u|\|_p \le C\,\big[\|u\|_p + \|\Delta u\|_p\big], \quad \forall u \in C^\infty_c(M). \tag{GE(p)}$$

In the case where $p > 2$, this result was also obtained via probabilistic arguments by Cheng, Thalmaier and Thompson [9]. To the best of our knowledge, it is not known whether the first order local Riesz transform is bounded from $L^p(M)$ to $L^p(M; TM)$, $1 < p < 2$, on any complete Riemannian manifold M. However, Coulhon and Duong [11] proved that if $p \in (1, 2]$, then the L^p gradient estimates (GE(p)) holds on any geodesically complete manifold. A much simpler proof thereof may be found in Lemma 1.6 of [20]. We emphasise that the multiplicative estimate

$$\| |\nabla u| \|_p \leq C \, \|u\|_p^{1/2} \, \|\Delta u\|_p^{1/2}, \quad \forall f \in C_c^\infty(M),$$

fails if $p > 2$ and $M = \mathbb{R}^2 \sharp \mathbb{R}^2$ (see the second remark after Theorem 4.1 in [11]), although M has Ricci curvature bounded from below, whence Bakry's estimate (1.7) and the moment inequality, Proposition 6.6.4 in [18], imply that (GE(p)) holds for every $p \in (1, \infty)$. This result illustrates how sensitive of the geometry of the underlying manifold these inequalities may be.

It is natural to speculate whether the gradient estimates (GE(p)) hold for some $p > 2$ under the sole assumption that M is geodesically complete. One of our main contributions (see Theorem B in Section 2) is to exhibit, for each $p > 2$ and each positive integer $n \geq 2$, an n-dimensional Riemannian manifold M that does not support the gradient estimate (GE(p)). According to what has been discussed above, the curvature of these manifolds is necessarily lower unbounded. However, as we will explain in Remark 5.1, it is possible to construct examples where the negative part of the curvature grows as slowly as desired.

Note that, as a consequence, both $\mathcal{R}^1$ and $\mathcal{R}^1_\tau$, for any $\tau > 0$, are unbounded on $L^p(M)$.

We also prove that if $p_0 > n$, and the Ricci curvature is bounded from below in an appropriate local $L^{p_0/2}$ integral sense (see Definition 2.1 in Section 2), then (GE(p)) holds for all $p \in (1, p_0)$ (see Theorem A in Section 2). Our condition is trivially satisfied if we assume standard pointwise lower bounds for the Ricci curvature, so that our result extends [9] (which, as mentioned above, can also be obtained as an easy consequence of the L^p boundedness of the first order local Riesz transform, proved in [3]). If, instead, p_0 is as above, M has positive injectivity radius and non-negative Ricci curvature in a global $L^{p_0/2}$ integral sense (see Definition 2.1 in Section 2), then (GE(p)) holds for all $p \in (1, \infty)$ (see Theorem B in Section 2).

Our next set of results is concerned with Riesz transforms of even order. We prove the following:

(1) if M has positive injectivity radius and the Ricci curvature is (pointwise) bounded from below, then $\mathcal{R}^2_\tau$ is bounded from $L^p(M)$ to $L^p(M; T_2M)$ for every $p \in (1, \infty)$ and $\tau > 0$;

(2) if M has positive injectivity radius and non-negative Ricci curvature in the global $L^{p_0/2}$ sense for some $p_0 > n$, then $\mathcal{R}^2_\tau$ is bounded from $L^p(M)$ to $L^p(M; T_2M)$ for every $p \in (1, \infty)$ and $\tau > 0$;

(3) if M has spectral gap and its and Ricci curvature is (pointwise) bounded from below, then $\mathcal{R}^2$ is bounded from $L^p(M)$ to $L^p(M; T_2M)$ for every $p \in (1, 2]$. As a consequence of this and the Federer–Fleming inequality, the analogue of (1.5) holds on M;

(4) if $\ell \geq 1$, the Ricci tensor of M and its derivatives up to the order $2\ell - 2$ are uniformly bounded, and M has positive injectivity radius, then $\mathcal{R}_\tau^{2\ell}$ is bounded from $L^p(M)$ to $L^p(M; T_{2\ell}M)$ for every $p \in (1, 2]$.

Note that (1) above was known under an additional pointwise upper bound on the Ricci curvature, thanks to work of Güneysu and the third author [16]. Subsequently, Baumgarth–Devyver–Güneysu [4] proved that for $p < 2$ one can replace the positivity of the injectivity radius with a bound on the whole Riemann tensor and its derivatives, as a consequence of some estimates on the covariant Riesz transforms. Finally, a very recent and far reaching result due to Cao–Cheng–Thalmaier [6] states that $\mathcal{R}_\tau^2$ is bounded from $L^p(M)$ to $L^p(M; T_2M)$ when $1 < p \leq 2$ under the sole assumption of Ricci curvature bounded from below. There is no hope to extend this result to $p > 2$ in full generality. Indeed, it is known [20, 24] (see also [13]) that, for every $p > 2$, there exists a complete Riemannian manifold (M, g) satisfying Sect ≥ 0 (in fact, Sect > 0 if $p > m$) on which $\mathcal{R}_\tau^2$ is unbounded in L^p for every positive τ. Apart from the case of Ricci-bounded geometry alluded to above, the only further set of assumptions ensuring the validity of (1.6) when $p > 2$ are given in Theorem 1.2 of [6]. The manifolds considered therein must satisfy (Kato type) conditions on the curvature and its derivatives but, on the other hand, could have zero injectivity radius. Finally, in a different direction, let us recall that $\mathcal{R}_\tau^2$ is bounded from $L^2(M)$ to $L^2(M; T_2M)$ also on manifolds whose curvature is very negative, i.e., explodes polynomially to $-\infty$ in an asymptotic sense [25].

Concerning (3) above, it was known under the additional assumption that M has positive injectivity radius. Indeed, $\mathcal{R}^2$ was known to be bounded from $L^p(M)$ to $L^p(M; T_2M)$ for $1 < p < 2$ [26]. Then the Federer–Fleming inequality and Bakry's estimate allow to conclude. In a related direction, let us also point out that the study of the L^p boundedness properties of $\mathcal{R}^2$ on complete manifolds whose full curvature tensor decays quadratically has been announced in [7]. Finally, note that (4) was known under the additional assumption that M has spectral gap (in which case an endpoint estimate for $p = 1$ was also provided).

In this paper, we do not consider Riesz transforms of odd order ≥ 3. We believe that it is an interesting problem to find geometric conditions on M under which either $\mathcal{R}_\tau^{2k+1}$ or $\mathcal{R}^{2k+1}$ is bounded on L^p, for some positive integer k. A neat result by Anker [2] shows that if M is a symmetric space of the noncompact type, then the Riesz transforms of any order are bounded on L^p, $1 < p < \infty$.

The paper is organised as follows. In Section 2, we give a precise statement of the main results. In Section 3, we prove the L^p gradient estimate (GE(p)) under local uniform L^q Ricci bounds. The proof for large p is based upon a related L^∞ estimate [12] and a covering argument. The whole range $p > 2$ is obtained via interpolation. In Section 4, the estimates (GE(p)) are proved under global L^q Ricci bounds, by exploiting the local expression in $W^{1,p}$-harmonic coordinates. To this end, the positivity of the injectivity radius is required. In Section 5, we exhibit the (as far as we know) first examples in the literature of complete Riemannian manifolds which do not support (GE(p)) for large p. Such examples are obtained through a suitable conformal deformation of the Euclidean plane. Harmonic coordinates with a uniform $W^{1,q}$ bound are also the key to prove the L^p boundedness of the second order Riesz transform in the case of lower bounded Ricci curvature and positive injectivity radius. This is the content of Section 6.

Note that $W^{1,q}$-harmonic estimates for large enough q imply a $C^{0,\alpha}$ control on the metric coefficients. This is an improvement on previously known bounds of the second order Riesz transform [16], which relied on the existence of uniform $C^{1,\alpha}$-harmonic coordinates, and thus required stronger geometric assumptions. At the end of this section, we also show how to pass from L^p boundedness of the local Riesz transform to global $W^{2,p}$ estimates under the assumption that the underlying manifold has a spectral gap. Finally, in Section 7 we deal with the L^p boundedness of higher even order local Riesz transforms. Namely, we use a trick which consists in considering the Cartesian product of M with a hyperbolic plane. This allows to reduce the problem to previously known bounds for the (global) Riesz transforms on manifolds with a spectral gap.

2. Assumptions and main results

All over this paper, $M = (M, g)$ denotes a smooth complete non-compact n-dimensional Riemannian manifold without boundary, and $p \in (1, \infty)$.

Throughout this paper, C will denote a positive constant, whose value may change from place to place. In each result, the constant C will depend only on the geometric bounds assumed there, i.e., on n, p, the curvature bound, and possibly the injectivity radius i and the spectral gap, whenever these last two quantities are relevant. Given a symmetric 2-tensor field T, we have denoted by $\min T$ its lowest eigenvalue.

In the literature, one can find two notions of integral curvature bounds, one of global nature and one of uniform local nature.

Definition 2.1. Suppose that $K \geq 0$, $R > 0$ and $1 < p < +\infty$. Set

$$\varrho_K(x) := (\min \mathrm{Ric} + (n-1)K^2)_-(x) \tag{2.1}$$

(where f_- denotes the negative part of f),

$$k(x, p, R, K) := R^2 \frac{\|\varrho_K\|_{L^p(B_R(x))}}{\mu(B_R(x))^{1/p}} \quad \text{and} \quad k(p, R, K) := \sup_{x \in M} k(x, p, R, K).$$

We say that

- M has Ricci curvature bounded from below by $-(n-1)K^2$ in the global L^p sense if $\varrho_K \in L^p(M)$;
- M has an $\varepsilon > 0$-amount of Ricci curvature below $-(n-1)K^2$ in the L^p sense at the scale R if $k(p, R, K) < \varepsilon$.

Our first main contribution is the following.

Theorem A. *Suppose that $n < p_0 < +\infty$. There exists a constant $\varepsilon = \varepsilon(p_0, n, K) > 0$ such that if $k(p_0/2, 1, K) \leq \varepsilon$ for some $K \geq 0$, then the L^p gradient estimate* (GE(p)) *holds on M for every $1 < p \leq p_0$.*

Remark 2.2. Note that $\rho_K(x) = 0$ if and only if $\mathrm{Ric}(x) \geq -(n-1)K^2 g_x$ where the inequality is intended in the sense of quadratic forms. In particular, if the Ricci curvature satisfies the lower bound $\mathrm{Ric} \geq -(n-1)K^2 g$, then $k(p, R, K) = 0$ for all $R > 0$ and for all $p \in (1, +\infty)$. Consequently, Theorem A provides yet another alternative proof of the result by Cheng, Thalmaier and Thompson, [9], using only PDEs methods. On the other hand, the integral bounds we assume are in general weaker than the usual pointwise bounds; see Remark 3.3 below.

Remark 2.3. If (M, g) is a complete Riemannian manifold supporting an L^p gradient estimate for some $p \in (1, +\infty)$, then (GE(p)) extends with the same constant to all functions in $H^{2,p}(M)$. Indeed, if $u \in H^{2,p}(M) = \{f \in L^p(M) : \Delta_{\mathrm{distr}} f \in L^p(M)\}$, by a result of Milatovic (Appendix of [17]), there exists a sequence $\{u_k\} \subseteq C_c^\infty(M)$ such that $u_k \to u$ with respect to the $H^{2,p}$ norm. Applying (GE(p)) to u_k, we deduce that ∇u_k is Cauchy and thus converges in the space of L^p vector fields. Testing ∇u_k against a smooth and compactly supported vector field and taking the limit shows in fact that ∇u_k converges in L^p norm to the weak gradient ∇u.

We also obtain the following variant of Theorem A in the case of global L^q lower Ricci bounds.

Theorem B. *Suppose $r_{\mathrm{inj}}(M) > 0$ and non-negative Ricci curvature in the global $L^{q/2}$ sense for some $n < q < +\infty$. Then, for every $1 < p < +\infty$, (GE(p)) holds on M.*

While several counterexamples to the validity of the L^p Calderón–Zygmund inequalities have been found in recent years, [16, 23, 25, 30], in the case of L^p gradient estimates the literature is lacking: see Section 9 in [28] for an extensive account of the topic. As mentioned in the introduction, using a sequence of conformal deformations on separated balls of the Euclidean plane, we are able to construct a complete Riemannian manifold on which the L^p gradient estimate fails for every $2 < p < +\infty$.

Theorem C. *Suppose that n is an integer ≥ 2. For any $p > 2$, there exists a complete n-dimensional Riemannian manifold M where the L^p gradient estimate (GE(p)) fails.*

As we will explain in Section 5, the examples in Theorem C shows that the result of Cheng, Thalmaier and Thompson [9] on L^p gradient estimates under Ricci lower bounds is, in fact, optimal with respect to pointwise bounds.

The next contributions of the paper will concern Riesz transforms of even order $2k \geq 2$. As announced in the introduction, adopting a different point of view, all the next theorems can be restated in term of Calderón–Zygmund inequalities, as a consequence of the following proposition, whose proof is deferred to Section 6.

Proposition 2.4. *Let $1 < p < \infty$, $\tau > 0$ and let $k \geq 1$ be an integer. The local Riesz transform $\mathcal{R}_\tau^{2k}$ of order $2k$ is bounded from $L^p(M)$ to $L^p(M; T_{2k}M)$ if and only if the L^p Calderón–Zygmund inequality or order $2k$*

$$\big\| |\nabla^{2k} u| \big\|_p \leq C\,[\|u\|_p + \|\Delta^k u\|_p], \quad \forall u \in \mathrm{Dom}_{L^p}(\Delta^k), \tag{2.2}$$

holds on M, where

$$\mathrm{Dom}_{L^p}(\Delta^k) = \{u \in L^p(M) : \Delta^k u \in L^p(M)\}$$

is the domain of the Laplacian in L^p.

Moreover, when $k = 1$, the latter assertions are also equivalent to

$$\big\| |\nabla^2 u| \big\|_p \leq C\,[\|u\|_p + \|\Delta u\|_p], \quad \forall u \in C_c^\infty(M).$$

First, we prove the L^p boundedness of the local second order Riesz transform (respectively, the validity of the L^p Calderón–Zygmund inequality), on manifolds with positive injectivity radius and a lower bound on the Ricci curvature.

Theorem D. *Suppose that* $r_{\mathrm{inj}}(M) > 0$. *Then* $\mathcal{R}^2_\tau$ *is bounded for every* $\tau > 0$ *from* $L^p(M)$ *to* $L^p(M; T_2M)$

(i) *for every* $1 < p < +\infty$, *if* $\mathrm{Ric} \geq -(n-1)K^2$ *for some* $K \geq 0$;

(ii) *for every* $1 < p \leq q$, *if the Ricci curvature is non-negative in the global* $L^{q/2}$ *sense for some* $q > n$.

In particular, as explained in [30], we have the validity of a new density result in Sobolev spaces.

Corollary E. *Under the assumptions of Theorem* D, $C^\infty_c(M)$ *is dense in* $W^{2,p}(M)$ *for the corresponding ranges of* p.

Remark 2.5. As it happens for the Calderón–Zygmund inequality of Theorem D, also the density result in Corollary E was already known when $1 \leq p \leq 2$ in the wider class of complete manifolds with a pointwise lower Ricci bound (indeed, a controlled growth of the negative part of the Ricci curvature is allowed in this case). See [20] and references therein.

Remark 2.6. We also prove that if M has a spectral gap, i.e., if the bottom of the L^2 spectrum of $-\Delta$ is strictly positive, then one has the estimate

$$\|u\|_{L^p} \leq C\|\Delta u\|_{L^p}, \quad \forall u \in C^\infty_c(M),$$

for every $1 < p < +\infty$. As a consequence, whenever the local Riesz transform $\mathcal{R}^2_\tau$ is bounded in L^p and we have a spectral gap, we obtain the following strong $W^{2,p}$ estimate:

$$\|u\|_{W^{2,p}} \leq C\|\Delta u\|_{L^p}, \quad \forall u \in C^\infty_c(M),$$

which, in particular, includes the L^p boundedness of the global Riesz transform $\mathcal{R}^2$.

It is worth noting that Calderón–Zygmund estimates can be derived for higher order derivatives up to imposing more stringent conditions on the geometry of the underlying manifold.

As recalled above, it was proved in [26] that the Riesz transform $\mathcal{R}^{2\ell}$ is bounded in $L^p(M)$ in the range $1 < p \leq 2$, provided the geometry is bounded at the order $2\ell - 2$ and M has a spectral gap. We shall show how to remove the latter condition.

Theorem F. *Suppose that* ℓ *is a positive integer. Let* $\tau > 0$. *Assume that* $r_{\mathrm{inj}}(M) > 0$ *and that the covariant derivatives of the Ricci tensor are uniformly bounded up to the order* $2\ell - 2$. *Then* $\mathcal{R}^{2\ell}_\tau$ *is bounded from* $L^p(M)$ *to* $L^p(M; T_{2\ell}M)$ *for every* $p \in (1, 2]$.

3. Gradient estimates: Local uniform L^q Ricci bounds

This section is devoted to proving Theorem A. Preliminarily, we point out the following facts, which will be repeatedly used in the sequel.

Remark 3.1. As noted in Section 2.3 in [27] for the case $K = 0$, smallness of $k(q, R_0, K)$ at a fixed scale R_0 implies a control on $k(q, R, K)$ for all scales $R > 0$. This is a consequence of a volume comparison result contained in Lemma 10 of [5]. Indeed, if $q > n/2$, there exists $\varepsilon = \varepsilon(n, q, K) > 0$ such that if $k(q, R_2, K) < \varepsilon$, then for every $0 < R_1 < R_2$ one has

$$k(q, R_1, K) \leq 4\Big(\frac{R_1}{R_2}\Big)^2\Big(\frac{v_K(R_2)}{v_K(R_1)}\Big)^{1/q} k(q, R_2, K),$$

where $v_K(R)$ is the volume of the geodesic ball of radius R in the n-dimensional space form of constant curvature K. Since $v_K(R_1) \sim R_1^n$, $k(q, R_1, K) \to 0$ as $R_1 \to 0$, i.e., $k(q, R_1, K)$ can be made arbitrarily small. See Corollary 13 in [5].

Note also that $k(p, r, K) \leq k(q, r, K)$ whenever $p \leq q$.

Under the assumption that $k(p/2, 1, K)$ is small, we first prove a local L^p gradient estimate, which is obtained integrating a local gradient estimate proved in [12]. In what follows, we use the notation

$$\|u\|^*_{L^p(\Omega)} = \Big(\fint_\Omega |u|^p\Big)^{1/p} = \Big(\frac{1}{\mathrm{vol}(\Omega)}\int_\Omega |u|^p\Big)^{1/p}.$$

Lemma 3.2. *Let $p > n$. There exist $\varepsilon = \varepsilon(n, p, K) > 0$, $C(n, p) > 1$ and $0 < R_0 \leq 1$ such that if $k(p/2, 1, K) \leq \varepsilon$, then*

$$\sup_{B_{R/2}(x)} |\nabla u|^2 \leq CR^{-2}\big[(\|u\|^*_{L^2(B_R(x))})^2 + (\|\Delta u\|^*_{L^p(B_R(x))})^2\big] \tag{3.1}$$

for all $0 < R \leq R_0$, for all $x \in M$, and for all smooth functions u on $B_1(x)$. Moreover, there exists a constant $D(n, p) > 0$ such that

$$\big\||\nabla u|\big\|^p_{L^p(B_{R/2}(x))} \leq DR^{-p}\big(\|u\|^p_{L^p(B_R(x))} + \|\Delta u\|^p_{L^p(B_R(x))}\big) \tag{3.2}$$

for all $x \in M$, $0 < R \leq R_0$ and all smooth functions u on $B_1(x)$.

Proof. By Theorem 1.9 in [12], there exists a constant $\varepsilon_0(n, p) > 0$ independent of R_0 such that if $k(p/2, R_0, 0) \leq \varepsilon_0$, then (3.1) holds for all $0 < R \leq R_0$. By Remark 3.1, we know that if $k(p/2, 1, K) \leq \varepsilon$, then $k(p/2, R, K) \lesssim R^{2-n/2p}$ as $R \to 0$, and since $\varrho_0(x) \leq \varrho_K(x) + (n-1)|K|$, we have

$$k(p/2, R, 0) \leq k(p/2, R, K) + (n-1)|K|R^2.$$

Hence, if we take R_0 small enough, then $k(p/2, R_0, 0) \leq \varepsilon_0$, which concludes the first part of the lemma. The constant R_0 depends on K, n, ε and ε_0.

From (3.1) we have

$$\sup_{B_{R/2}(x)} |\nabla u|^p \leq C^{p/2} R^{-p} 2^{p/2-1}\big[(\|u\|^*_{L^2(B_R(x))})^p + (\|\Delta u\|^*_{L^p(B_R(x))})^p\big].$$

By Hölder's inequality,

$$\Big(\fint_{B_R(x)} u^2\Big)^{p/2} \leq \fint_{B_R(x)} u^p,$$

whence

$$\int_{B_{R/2}(x)} |\nabla u|^p \le C^{p/2} R^{-p} 2^{p/2-1} \frac{\mathrm{vol}(B_{R/2}(x))}{\mathrm{vol}(B_R(x))} \Big(\int_{B_R(x)} |u|^p + \int_{B_R(x)} |\Delta u|^p \Big).$$

To conclude the proof of (3.2), recall that, as a consequence of the volume comparison, (M, g) satisfies a uniform local volume doubling property, i.e., there exists $C > 0$ such that

$$\mathrm{vol}(B_{R/2}(x)) \le C\, \mathrm{vol}(B_R(x))$$

for all $x \in M$ and $0 < R \le R_0$. See Lemma 10 and subsequent results in [5]. This completes the proof of Lemma 3.2. ■

We are now ready to prove the global L^p gradient estimate.

Proof of Theorem A. We start by noting that the local L^{p_0} gradient estimate (3.2), $p_0 > n$, extends to the whole manifold using a uniformly locally finite covering of M. The existence of such covering is a formal consequence of the local volume doubling inequality, which, as we have recalled above, holds under local integral Ricci bounds. Thus, let $u \in C_c^\infty(M)$ and $\Omega = \mathrm{supp}(u)$, and let $0 < R \le R_0$ be small enough such that $2R \le 1$. Here R_0 is the radius appearing in Lemma 3.2. By local volume doubling, there exist $x_1, \dots, x_h \in M$ such that

(i) $\Omega \subseteq \bigcup_{i=1}^h B_{R/2}(x_i)$;

(ii) every $x \in \Omega$ intersects at most N balls $B_R(x_i)$.

Then,

$$\begin{aligned}\int_M |\nabla u|^{p_0} &\le \sum_{i=1}^h \int_{B_{R/2}(x_i)} |\nabla u|^{p_0} \le D R^{-p} \sum_{i=1}^h \Big(\int_{B_R(x_i)} |u|^{p_0} + \int_{B_R(x_i)} |\Delta u|^{p_0} \Big) \\ &\le D R^{-p_0} \int_M \sum_{i=1}^h 1_{B_R(x_i)} \big(|u|^{p_0} + |\Delta u|^{p_0}\big) \le D R^{-p_0} N \Big(\int_M |u|^{p_0} + \int_M |\Delta u|^{p_0} \Big),\end{aligned}$$

which proves the gradient estimate (GE(p)) with $p = p_0 > n$.

Recall that if $p \in (1, 2]$, then L^p gradient estimates always holds on complete Riemannian manifolds [11]. We now interpolate between this and the result for $p > n$ obtained in the first part of the proof.

It is well known that the heat semigroup is strongly continuous and contractive on $L^p(M)$ for all $p \in [1, +\infty)$, see Theorem IV.8 in [15]. By the Hille–Yosida theorem, -1 is in the resolvent set of its infinitesimal generator $-\Delta$. Then $-\Delta + \mathcal{I}$ is (surjective and) invertible in $L^p(M)$. Therefore $(-\Delta + \mathcal{I})^{-1}$ is bounded on $L^p(M)$ and its range is contained in the domain of Δ. Now, suppose that $2 < p \le n$. Choose $q > n$ and θ in $(0, 1)$, so that $1/p = \theta/q + (1-\theta)/2$.

On the one hand, by the first part of the proof, the operator $\nabla(-\Delta + \mathcal{I})^{-1}$ extends to a bounded operator from $L^q(M)$ to $L^q(M; T_1 M)$. On the other hand,

$$\big\| |\nabla(-\Delta + \mathcal{I})^{-1} f| \big\|^2_{L^2(M)} = \big((-\Delta + \mathcal{I})^{-1} f \,, \Delta(-\Delta + \mathcal{I})^{-1} f \big)_{L^2(M)}.$$

Since both $(-\Delta + \mathcal{I})^{-1}$ and $\Delta(-\Delta + \mathcal{I})^{-1}$ extend to bounded operators on $L^2(M)$, the operator $\nabla(-\Delta + \mathcal{I})^{-1}$ extends to a bounded operator from $L^2(M)$ to $L^2(M; T_1 M)$.

By the Riesz–Thorin theorem, $\nabla(-\Delta + \mathcal{I})^{-1}$ extends to a bounded linear operator from $L^p(M)$ to $L^p(M; T_1 M)$. As a consequence, the L^p gradient estimate holds on M.

The proof of the theorem is complete. ∎

Remark 3.3. As alluded to in the introduction, the integral curvature bounds assumed here are weaker than the classical pointwise bounds. An easy example of a Riemannian manifold (M, g) satisfying $\inf_M \min \mathrm{Ric} = -\infty$, but with $k(p, 1, 0)$ arbitrarily small, can be constructed as follows. We let $M = \mathbb{R}^2$ endowed with the conformally flat metric $g = e^{2\varphi} dx^2$, where φ is a smooth non-positive function. In the following, the sub/superscript e denotes the objects taken with respect to the Euclidean metric. In particular, $\mathrm{vol}_g(K) \leq \mathrm{vol}_e(K)$ for any measurable set $K \subset \mathbb{R}^2$, and $B^g_R(w) \supseteq B^e_R(w)$ for any $R > 0$ and $w \in \mathbb{R}^2$. Suppose now that $\operatorname{supp} \varphi \in \bigcup_{n \in \mathbb{N}} B^e_{1/2}((4n, 0))$. This guarantees that $B^g_1(w) \subseteq B^e_2(w)$ for any $w \in \mathbb{R}^2$. Moreover, given $w \in \mathbb{R}^2$, let n_w be the unique integer (if any) such that $B^e_{1/2}((4n_w, 0))$ intersects $B^e_1(w)$. Then

$$\mathrm{vol}_g B^g_1(w) \geq \mathrm{vol}_g B^e_1(w) \geq \mathrm{vol}_g(B^e_1(w) \setminus B^e_{1/2}(4n_w, 0)) = \frac{3}{4}\pi. \tag{3.3}$$

Fix $a \in (2 - 2/p, 2)$ and $\phi_0 \in C^\infty_c(B^e_{1/2}(0, 0))$. We define $\varphi(x, y) = \sum_{n \in \mathbb{N}} \phi_n(x, y)$, where $\phi_n(x, y) = n^{-a} \phi_0(n(x - 4n, y))$ if $n \geq 1$. On the one hand, since $\Delta_e \phi_0$ attains positive values and since $\Delta_e \phi_n(x, y) = n^{2-a} \Delta_e \phi_0(n(x - 4n, y))$, we have that $\mathrm{Ric}_g = -2\Delta_e \varphi$ is lower unbounded. On the other hand,

$$\begin{aligned} \int_{B^g_1(w)} ((\min \mathrm{Ric})_-)^p \, d\mu_g &= 2^p \int_{B^g_1(w)} ((\Delta_e \varphi)_+)^p \, d\mu_g \leq 2^p \int_{B^e_2(w)} ((\Delta_e \phi_{n_w})_+)^p \, dx^2 \\ &= 2^p \, n_w^{2p-pa-2} \int_{B^e_1(0,0)} ((\Delta_e \phi_0)_+)^p \, dx^2 \leq 2^p \int_{B^e_1(0,0)} ((\Delta_e \phi_0)_+)^p \, dx^2, \end{aligned}$$

which is uniformly bounded independently from w. Moreover, choosing an appropriate ϕ_0, we can assume that the right-hand side of the estimate above is arbitrarily small. Together with the uniform volume lower bound (3.3), this proves that $k(p, 1, 0) < +\infty$ and can be made arbitrarily small.

4. Gradient estimates: Global L^q Ricci bounds

Preliminarily, we recall the following.

Definition 4.1. Let (M, g) be an n-dimensional Riemannian manifold. Let $n < q < +\infty$. The $W^{1,q}$ harmonic radius at x, denoted by $r_{W^{1,q}}(x)$, is the supremum of all $R > 0$ such that there exists a coordinate chart $\phi\colon B_R(x) \to \mathbb{R}^n$ satisfying

(a) $2^{-1} [\delta_{ij}] \leq [g_{ij}] \leq 2 [\delta_{ij}]$;

(b) $R^{1-n/q} \, \|\partial_k g_{ij}\|_{L^q(B_R(x))} \leq 1$;

(c) ϕ is a harmonic map.

The following result encloses in a single statement classical contributions by Anderson and Cheeger, [1], and a more recent contribution by Hiroshima, [19].

Theorem 4.2. *Fix $n \in \mathbb{N}$, $q > n$, $K \geq 0$ and $i > 0$. Let (M, g) be a complete, n-dimensional Riemannian manifold satisfying $r_{\text{inj}} \geq i$ and either of the following assumptions*:

(a) $\operatorname{Ric} \geq -(n-1)K^2$, *or*

(b) Ric *is non-negative in the global $L^{q/2}$ sense, i.e.,* $\lambda = \|(\min \operatorname{Ric})_-\|_{L^{q/2}(M)} < +\infty$.

Then, $r_{W^{1,q}}(z) \geq \bar{r}$ independently of $z \in M$, where $\bar{r} = \bar{r}(n, q, K, i, \lambda) > 0$.

We note that, by the Sobolev embedding, we have for free a $C^{0,\alpha}$ control on the metric coefficients within the ball $B_{\bar{r}/2}(z)$.

Finally, we observe the inclusions $B^e_{\bar{r}/8} \subseteq \phi(B_{\bar{r}/4}(z)) \subseteq B^e_{\bar{r}/2}$, where $B^e \subseteq \mathbb{R}^n$ denotes the Euclidean ball centered at the origin. Since, inside $B^e_{\bar{r}/8}$, the Euclidean and the Riemannian measures are mutually controlled by absolute constants, in performing integrations in local coordinates, the chosen measure is irrelevant.

Remark 4.3. We have already observed that complete manifolds with Ricci lower bounds, in the uniform local integral sense, enjoy the uniform local volume doubling property at any fixed scale. In the class of manifolds with positive injectivity radius, the same is true if we consider the case of global L^q conditions. This follows from the Croke isoperimetric estimate and from volume comparison. In particular, at a sufficiently small scale, we have the existence of the covering with finite intersection multiplicity as in the proof of Theorem A; see e.g. Proposition 1.5 in [19]. Conversely, if one assumes a priori that $r_{W^{1,q}}(M) := \inf_{x \in M} r_{W^{1,q}}(x) > 0$, then the double sided Euclidean control of the volume of the balls at a small scale implies the uniform volume doubling property, and hence the covering property.

In view of Remark 4.3 and of Theorem 4.2, we obtain that Theorem B is a direct consequence of the next result. Recall that, if $(x^1, \ldots, x^n)$ is a system of harmonic coordinates, then

$$(\nabla u)^j = g^{jk} \partial_k u \quad \text{and} \quad \Delta u = g^{ij} \partial^2_{ij} u,$$

where $g = [g_{ij}]$ and $g^{-1} = [g^{ij}]$ are, respectively, the matrix of the metric coefficients and its inverse.

Theorem 4.4. *Suppose $r_{W^{1,q}}(M) = \bar{r} > 0$ for some $q > n$. Then for every $1 < p < +\infty$, the L^p gradient estimate* (GE(p)) *holds on M.*

Proof. Fix $0 < r < \bar{r}/16$. Since the metric coefficients in $W^{1,q}$-harmonic coordinates are uniformly $C^{0,\alpha}$-controlled, there exists an absolute constant $C > 1$ such that, for any $u \in C^\infty_c(M)$ and $0 < R \leq r$,

$$C^{-1} \big\| |\nabla^e u| \big\|_{L^p(B^e_R)} \leq \big\| |\nabla u| \big\|_{L^p(B_{2R}(x))} \leq C \big\| |\nabla^e u| \big\|_{L^p(B^e_{4R})}$$

and

$$\|g^{ij} \partial^2_{ij} u\|_{L^p(B^e_R)} \leq C \|\Delta u\|_{L^p(B_{2R}(x))}.$$

On the other hand, by the Euclidean estimates of the gradient, Theorem 9.11 in [14], there exists an absolute constant $C = C(n, p, R) > 0$ such that

$$C^{-1} \big\| |\nabla^e u| \big\|_{L^p(B^e_{2r})} \leq \|u\|_{L^p(B^e_{4r})} + \|g^{ij} \partial^2_{ij} u\|_{L^p(B^e_{4r})}.$$

Hence,

$$\||\nabla u|\|_{L^p(B_r(x))} \le C\, \||\nabla^e u|\|_{L^p(B^e_{2r})} \le C\big(\|u\|_{L^p(B^e_{4r})} + \|g^{ij}\partial^2_{ij} u\|_{L^p(B^e_{4r})}\big)$$
$$\le C\big(\|u\|_{L^p(B_{8r}(x))} + \|\Delta u\|_{L^p(B_{8r}(x))}\big).$$

Since, thanks to the uniform local doubling condition, M has a countable covering by balls $\{B_r(x_j)\}$ such that $\{B_{8r}(x_j)\}$ has finite intersection multiplicity, the global L^p estimate follows by adding the local inequalities. ∎

5. Counterexamples to L^p gradient estimates

In this section, we prove Theorem C.

Proof of Theorem C. First we prove the result in the case where $n = 2$. Take $(\Sigma, g) = (\mathbb{R}^2, \lambda^2 dx^2)$, where dx^2 is the usual Euclidean metric on $\mathbb{R}^2$ and $\lambda \in C^\infty(\Sigma)$ is such that $0 < \lambda \le 1$. As above, we denote by Δ and ∇ the Laplace–Beltrami operator and gradient with respect to the metric g, while we use Δ_e and ∇^e to denote the corresponding Euclidean differential operators. The spaces $L^p(\Sigma)$ are defined in terms of the Riemannian volume form $d\mu_g$, whereas $L^p(\mathbb{R}^2)$ are the spaces with respect to the Lebesgue measure dx^2.

For each non-negative integer m, consider the point x_m in $\mathbb{R}^2$, with coordinates $(m, 0)$. Take $\lambda(x) = 1$ for all $x \in \Sigma \setminus \bigcup_{m\in\mathbb{N}} B_{1/8}(x_m)$. Since (Σ, g) is isometric to $(\mathbb{R}^2, dx^2)$ outside of a countable union of bounded sets whose pairwise distance is uniformly lower bounded, it is a complete Riemannian manifold. Next, take $\varphi_0 \in C^\infty_c(\Sigma)$ such that

$$\begin{cases} \varphi_0(u, v) = u + 1 & \text{on } B_{1/4}(x_0), \\ \operatorname{supp}(\varphi_0) \Subset B_{1/2}(x_0), \end{cases}$$

and let $\varphi_m(u, v) = \varphi_0(u - m, v)$, for all positive integers m. Then, for every positive integer k, define

$$u_k := \sum_{m=0}^{k} 2^{-m}\, \varphi_m.$$

Clearly, $u_k \in C^\infty_c(\Sigma)$. Notice that

$$\|u_k\|^p_{L^p(\Sigma)} = \sum_{m=0}^{k} 2^{-mp} \int_\Sigma |\varphi_m|^p\, \lambda^2\, dx$$
$$\le \sum_{m=0}^{k} 2^{-mp}\, \|\varphi_m\|^p_{L^p(\mathbb{R}^2)} = \|\varphi_0\|^p_{L^p(\mathbb{R}^2)} \sum_{m=0}^{+\infty} 2^{-mp} < +\infty.$$

Now observe that $\Delta\varphi_m = \lambda^{-2}\Delta_e\varphi_m$. Hence

$$\|\Delta u_k\|^p_{L^p(\Sigma)} = \sum_{m=0}^{k} 2^{-mp} \int_\Sigma |\Delta\varphi_m|^p\, \lambda^2\, dx = \sum_{m=0}^{k} 2^{-mp} \int_\Sigma |\Delta_e\varphi_m|^p\, \lambda^{2(1-p)}\, dx.$$

Moreover, we have that $\Delta_e \varphi_m(u,v) = (\Delta_e \varphi_0)(u-m,v)$. Since $\Delta_e \varphi_0$ vanishes on $B_{1/4}(x_0)$, the function $\Delta_e \varphi_m$ vanishes on $B_{1/4}(x_m)$. This and the fact that the support of φ_0 is contained in $B_{1/2}(x_0)$ yield

$$\|\Delta u_k\|^p_{L^p(\Sigma)} = \int_{B_{1/2}(x_0)\setminus B_{1/4}(x_0)} |\Delta_e \varphi_0|^p \, \lambda^{2(1-p)} \, dx = \int_{B_{1/2}(x_0)\setminus B_{1/4}(x_0)} |\Delta_e \varphi_0|^p \, dx,$$

where the last equality holds because $\lambda = 1$ on $B_{1/2}(x_0) \setminus B_{1/4}(x_0)$. Altogether, we obtain that

$$\|\Delta u_k\|^p_{L^p(\Sigma)} \le \|\Delta_e \varphi_0\|^p_{L^p(\mathbb{R}^2)} \sum_{m=0}^{+\infty} 2^{-mp} < +\infty.$$

Now, recall that $p > 2$ is given. Choose $\beta > 1/(p-2)$, and consider $\lambda_\infty(x) := |x|^{2\beta}$ in $B_\delta(x_0)$ for some $0 \le \delta \ll 1/8$. Note that $|\nabla^e \varphi_0| = 1$ on $B_{1/8}(x_0)$, whence

$$\int_{B_\delta(x_0)} |\nabla^e \varphi_0|^p_e \, \lambda_\infty^{2-p} \, dx = 2\pi \int_0^\delta r^{1-2\beta(p-2)} \, dr = +\infty.$$

Here $|x| = r$ denotes the Euclidean distance from the origin. Then, for any $m \in \mathbb{N}$, we can find $\varepsilon_m > 0$ such that $\varepsilon_m \to 0$ as $m \to +\infty$, and

$$\int_{B_\delta(x_0)} |\nabla^e \varphi_0|^p \, (|x|^2 + \varepsilon_m)^{(2-p)\beta} \, dx \ge 2^{mp}.$$

For $x \in B_{1/8}(x_0)$ and $\varepsilon \in [0,1]$, we define the function $\lambda_\varepsilon \in C^\infty(B_{1/8}(x_0))$ by

$$\begin{cases} 0 < \lambda_\varepsilon \le 1, \\ \lambda_\varepsilon(x) = (|x|^2 + \varepsilon)^\beta \quad \text{if } x \in B_\delta(x_0), \\ \operatorname{supp}(1-\lambda_\varepsilon) \subseteq B_{1/8}(x_0). \end{cases}$$

Now define $\lambda \in C^\infty(\Sigma)$ by

$$\begin{cases} 0 < \lambda \le 1, \\ \lambda(x) = 1 \quad \text{if } x \in \Sigma \setminus \bigcup_{m\in\mathbb{N}} B_{1/8}(x_m), \\ \lambda(x) = \lambda_{\varepsilon_m}(x - x_m) \quad \text{if } x \in B_\delta(x_m). \end{cases}$$

Then, arguing much as above,

$$\begin{aligned} \| |\nabla u_k| \|^p_{L^p(\Sigma)} &= \int_\Sigma \sum_{m=0}^{k} \frac{|\nabla \varphi_m|^p}{2^{mp}} \lambda^2 \, dx \ge \sum_{m=0}^{k} 2^{-mp} \int_{B_\delta(x_0)} |\nabla \varphi_o|^p \, \lambda_m^2 \, dx \\ &= \sum_{m=0}^{k} 2^{-mp} \int_{B_\delta(x_0)} |\nabla^e \varphi_0|^p \, (|x|^2 + \varepsilon_m)^{(2-p)\beta} \, dx \ge k. \end{aligned}$$

Since $\{\|u_k\|_{L^p(\Sigma)}\}$ and $\{\|\Delta u_k\|_{L^p(\Sigma)}\}$ are bounded, the gradient estimate fails on Σ.

This concludes the proof of Theorem C in the case where $n = 2$.

Suppose now that $n \geq 3$. We proceed as in [20]. Let (Σ, g) be the Riemannian manifold considered above and let (N, h) be any $(n-2)$-dimensional closed Riemannian manifold. Consider the product manifold $M = \Sigma \times N$, and define

$$v_k(x, y) = u_k(x), \quad \forall (x, y) \in \Sigma \times N.$$

Clearly $\{v_k\} \subseteq C_c^\infty(M)$. It is straightforward to check that the sequences $\{\|v_k\|_{L^p(M)}\}$ and $\{\|\Delta v_k\|_{L^p(M)}\}$ are bounded, whereas $\{\||\nabla v_k|\|_{L^p(M)}\}$ is unbounded. Hence the gradient estimate fails on M.

This concludes the proof of Theorem C. ■

Remark 5.1. We observe that the choice of the sequence $\{x_m\}$ is quite arbitrary. In particular, let $\alpha: [0, +\infty) \to [0, +\infty)$ be an arbitrary increasing function such that $\alpha(t) \to \infty$ as $t \to +\infty$. If we choose x_m which diverges quick enough to infinity, we can make the lower bound on Ricci arbitrarily small so that

$$\mathrm{Ric}(x) \geq -\alpha(r(x)).$$

This shows that the result by Cheng, Thalmaier and Thompson, [9] is, in fact optimal with respect to pointwise lower bounds, as observed after the statement of Theorem C.

We also point out the following straightforward consequence of the proof of Theorem C.

Corollary 5.2. *For any $n \geq 2$ and $p > 2$, there exist a Riemannian manifold M and a function $v_\infty \in H^{2,p}(M)$ such that $v_\infty \notin W^{1,p}(M)$.*

Indeed, for $n > p$, it is enough to define

$$u_\infty = \sum_{m=0}^{+\infty} 2^{-m} \varphi_m;$$

then $u_\infty, \Delta u_\infty \in L^p(\Sigma)$, while $|\nabla u_\infty| \notin L^p(\Sigma)$. In particular, $u_\infty \in H^{2,p}(\Sigma)$, while $u_\infty \notin W^{1,p}(\Sigma)$. The case $2 < p \leq n$ can be dealt with the same trick as in the proof of Theorem C.

6. Calderón–Zygmund inequalities

We begin this section by proving the equivalence, stated in Proposition 2.4, between boundedness of the local Riesz transform and Calderón–Zygmund inequalities.

Proof of Proposition 2.4. Since $-\Delta$ generates a contraction semigroup on $L^p(M)$, the operator $-\Delta$ is sectorial in $L^p(M)$, and the resolvent operator $(-\Delta + \tau \mathcal{I})^{-1}$ is bounded on L^p, by the Hille–Yosida theorem. Hence so is $(-\Delta + \tau \mathcal{I})^{-k}$. Set $\psi(\lambda) := (\lambda^k + \tau)$ $(\lambda + \tau)^{-k}$. It is not hard to prove that both ψ and $1/\psi$ are in the extended Dunford class $\mathcal{E}_\theta$ for every θ in $(\pi/2, \pi)$. By the standard functional calculus for sectorial operators, $\psi((-\Delta))$ and $(1/\psi)((-\Delta))$ extend to bounded operators on $L^p(M)$.

Suppose first that $\mathcal{R}_\tau^{2k}$ is bounded on $L^p(M)$, i.e., there exists a constant C such that

$$\big\||\mathcal{R}_\tau^{2k} f|\big\|_{L^p(M)} \le C\|f\|_{L^p(M)}, \quad \forall f \in L^p(M).$$

In particular, if u is in $\mathrm{Dom}_{L^p}((-\Delta)^k)$, then the function $f := (-\Delta + \tau\mathcal{I})^k u$ is in $L^p(M)$, and

$$\big\||\nabla^{2k} u|\big\|_{L^p(M)} \le C\big\|(-\Delta + \tau\mathcal{I})^k u\big\|_{L^p(M)} \le C\left[\|\Delta^k u\|_{L^p(M)} + \|u\|_{L^p(M)}\right],$$

where C depends on τ. The last inequality is a straightforward consequence of the boundedness in L^p of $(1/\psi)((-\Delta))$.

Conversely, suppose that (2.2) holds. Consider f in $L^p(M)$. Since τ is in the resolvent set of $(-\Delta)$, the operator $(-\Delta + \tau\mathcal{I})^k$ maps $\mathrm{Dom}_{L^p}(\Delta^k)$ onto $L^p(M)$. Therefore, there exists u in $\mathrm{Dom}_{L^p}\big(\Delta^k\big)$ such that $u = (-\Delta + \tau\mathcal{I})^{-k} f$. Consequently, (2.2), with u as above, yields

$$\big\||\nabla^{2k} u|\big\|_{L^p(M)} \le C\left[\big\|(-\Delta + \tau\mathcal{I})^{-k} f\big\|_p + \big\|\Delta^k(-\Delta + \tau\mathcal{I})^{-k} f\big\|_p\right]. \tag{6.1}$$

Now, both $(-\Delta + \tau\mathcal{I})^{-k}$ and $\Delta^k(-\Delta + \tau\mathcal{I})^{-k}$ are bounded operators on $L^p(M)$, as $\big((-\Delta)\big)$ is. Furthermore, standard properties of sectorial operators imply that there exists a constant C such that

$$\big|\big|\big|(-\Delta + \tau\mathcal{I})^{-k}\big|\big|\big|_{L^p(M)} \le \big|\big|\big|(-\Delta + \tau\mathcal{I})^{-1}\big|\big|\big|^k_{L^p(M)} \le \frac{C}{\tau^k}, \quad \forall \tau > 0,$$

and

$$\big|\big|\big|\Delta^k(-\Delta + \tau\mathcal{I})^{-k}\big|\big|\big|_{L^p(M)} \le \big|\big|\big|\Delta(-\Delta + \tau\mathcal{I})^{-1}\big|\big|\big|^k_{L^p(M)} \le C, \quad \forall \tau > 0,$$

where $|||\cdot|||_{L^p(M)}$ denotes the operatorial norm in L^p. This and (6.1) yield

$$\big\||\nabla^{2k}(-\Delta + \tau\mathcal{I})^{-k} f|\big\|_{L^p(M)} \le C\,[\tau^{-k}\,\|f\|_p + \|f\|_p] \le C\max(1, \tau^{-k})\,\|f\|_p,$$

as required.

Suppose now that $k = 1$, and that (1.6) holds for all $u \in C_c^\infty(M)$. Let $f \in \mathrm{Dom}_{L^p}(\Delta) = H^{2,p}(M)$. Thanks to a density result by O. Milatovic (see Appendix A in [17]), we can take a sequence $u_j \in C_c^\infty(M)$ converging to f in $H^{2,p}(M)$ as $j \to \infty$ (see Remark 2.3). Hence, (1.6) and (GE(p)) (which holds due to Theorem 2 in [17]) implies that u_j is a Cauchy sequence in $W^{2,p}(M)$, hence it converges to some limit $u_\infty \in W^{2,p}(M)$. Finally, $u_\infty = f$ and (1.6), as $W^{2,p}(M)$ embeds continuously in $H^{2,p}(M)$. ∎

The rest of this section is devoted to prove Theorem D. As in Section 4, we use local estimates in $W^{1,q}$-harmonic coordinates and then glue them together thanks to the uniform local volume doubling condition.

The crucial ingredient is the following estimate of the first order term in the local expression of the Hessian of a smooth function. Recall that, if $(x^1, \dots, x^n)$ is a system of harmonic coordinates, then

$$\nabla^2_{ij} u = \mathrm{Hess}(u)_{ij} = \partial^2_{ij} u - \Gamma^k_{ij}\partial_k u,$$

where Γ^k_{ij} denote the Christoffel symbols.

Lemma 6.1. *Let $1 < p < +\infty$. Fix $z \in M$, $q > \max(n, p)$ and let $0 < r = \frac{1}{4} r_{W^{1,q}}(z)$. Finally, denote by Γ_{ij}^k the Christoffel symbols with respect to the $W^{1,q}$ harmonic coordinates system $\phi(x) = (x^1, \dots, x^n)$: $B_r(z) \to U \supseteq B^e_{r/2}$. Then, there exists a constant $C = C(n, p, q, r) > 0$ such that, for any $u \in C^\infty(M)$,*

$$C^{-1} \cdot \|\Gamma_{ij}^k \partial_k u\|_{L^p(B^e_{r/2})} \le \big\| |\operatorname{Hess}^e u| \big\|_{L^p(B^e_{r/2})} + \big\| |\nabla u| \big\|_{L^p(B_r(z))}.$$

Proof. We apply Hölder's inequality with conjugate exponents $t = q/(q-p)$ and $t' = q/p$ to get

$$\|\Gamma_{ij}^k \partial_k u\|_{L^p(B^e_{r/2})} \le \sum_k \|\Gamma_{ij}^k\|_{L^q(B_r(z))} \big\| |\nabla^e u| \big\|_{L^{pq/(q-p)}(B^e_{r/2})}, \quad \forall i, j = 1, \dots, n. \tag{6.2}$$

Next, we recall that the Christoffel symbols display a C^1 dependence on the metric coefficients in the form

$$\Gamma = \frac{1}{2} g^{-1} \cdot \partial g.$$

Since $\|g\|_{L^\infty}$, $\|g^{-1}\|_{L^\infty}$ and $\|\partial g\|_{L^q}$ are bounded inside $B_r(z)$ (with a bound depending only on n, q, r), we deduce that there exists a constant $C = C(n, q, r) > 0$ such that

$$\|\Gamma_{ij}^k\|_{L^q(B_r(z))} \le C. \tag{6.3}$$

It remains to take care of gradient term in (6.2). To this end, for the sake of clarity, we distinguish three cases according to the values of p.

Case $1 < p < n$. Since

$$\frac{pq}{q-p} < p^* := \frac{np}{n-p},$$

we can apply directly the Sobolev(–Kondrachov) embedding theorem and deduce that, for some constant $S = S(r, p, q, n) > 0$,

$$S^{-1} \cdot \big\| |\nabla^e u| \big\|_{L^{pq/(q-p)}(B^e_{r/2})} \le \big\| |\operatorname{Hess}^e u| \big\|_{L^p(B^e_{r/2})} + \big\| |\nabla^e u| \big\|_{L^p(B^e_{r/2})}.$$

On the other hand, observe that

$$\big\| |\nabla^e u| \big\|_{L^p(B^e_{r/2})} \le C \big\| |\nabla u| \big\|_{L^p(B_r(z))}$$

for some absolute constant $C > 0$, whence

$$\big\| |\nabla^e u| \big\|_{L^{\frac{pq}{q-p}}(B^e_{r/2})} \le C \Big(\big\| |\operatorname{Hess}^e u| \big\|_{L^p(B^e_{r/2})} + \big\| |\nabla u| \big\|_{L^p(B_r(z))} \Big). \tag{6.4}$$

Inserting (6.3) and (6.4) into (6.2), gives the desired inequality when $1 < p < n$.

Case $p = n$. Let $1 < \tilde{p} < n = p$ be defined by

$$\tilde{p} = \frac{nq}{2q - n}.$$

Since

$$\frac{nq}{q-n} = \frac{n\tilde{p}}{n - \tilde{p}} =: \tilde{p}^*,$$

we can apply the Sobolev embedding theorem and the Hölder inequality to deduce that, for some constant $S = S(r,q,n) > 0$,

$$S^{-1} \cdot \big\| |\nabla^e u| \big\|_{L^{nq/(q-n)}(B^e_{r/2})} \leq \big\| |\operatorname{Hess}^e u| \big\|_{L^{\tilde p}(B^e_{r/2})} + \big\| |\nabla^e u| \big\|_{L^{\tilde p}(B^e_{r/2})}$$
$$\leq |B^e_{r/2}|^{(n-\tilde p)/n\tilde p} \big(\| |\operatorname{Hess}^e u| \|_{L^n(B^e_{r/2})} + \| |\nabla^e u| \|_{L^n(B^e_{r/2})} \big).$$

The conclusion follows exactly as above.

Case $p > n$. In this case, we can use Morrey's and Hölder's inequalities to deduce that, for some constant $S = S(r,p,q,n) > 0$,

$$\| |\nabla^e u| \|_{L^{pq/(q-p)}(B^e_{r/2})} \leq |B^e_{r/2}|^{(q-p)/pq} \cdot \| |\nabla^e u| \|_{L^\infty(B^e_{r/2})}$$
$$\leq S |B^e_{r/2}|^{(q-p)/qp} \big(\| |\operatorname{Hess}^e u| \|_{L^p(B^e_{r/2})} + \| |\nabla^e u| \|_{L^p(B^e_{r/2})} \big).$$

The proof of the lemma is complete. ∎

We are now in the position to prove the following abstract result, which combined with Proposition 2.4, proves Theorem D.

Theorem 6.2. *Let $1 < p < +\infty$. Suppose that $r_{W^{1,q}}(M) > 0$ for some $q > \max(n,p)$. Then the L^p Calderón–Zygmund estimate (1.6) holds on M.*

Proof. Set $\bar r = r_{W^{1,q}}(M)/4$ and let $u \in C^\infty_c(M)$. We preliminarily observe that there exists a uniform constant $C > 0$ such that, for any $z \in M$,

$$\big\| |\nabla^e u| \big\|_{L^p(B^e_{\bar r})} \leq C \big\| |\nabla u| \big\|_{L^p(B_{2\bar r}(z))} \quad \text{and} \quad \| g^{ij} \partial^2_{ij} u \|_{L^p(B^e_{\bar r})} \leq C \| \Delta u \|_{L^p(B_{2\bar r}(z))}.$$

Using the Euclidean Calderón–Zygmund estimate (see Theorem 9.11 in [14]) joint with Lemma 6.1, we find a constant $C = C(n,p,\bar r) > 0$ such that, for any $z \in M$,

$$\big\| |\operatorname{Hess}(u)| \big\|_{L^p(B_{\bar r/4}(z))} \leq \big\| |\operatorname{Hess}^e u| \big\|_{L^p(B^e_{\bar r/2})} + \sum_{ij} \| \Gamma^k_{ij} \partial_k u \|_{L^p(B^e_{\bar r/2})}$$
$$\leq C \big(\| g^{ij} \partial^2_{ij} u \|_{L^p(B^e_{\bar r})} + \| u \|_{L^p(B^e_{\bar r})} + \big\| |\nabla u| \big\|_{L^p(B_{2\bar r}(z))} \big)$$
$$\leq C \big(\| \Delta u \|_{L^p(B_{2\bar r}(z))} + \| u \|_{L^p(B_{2\bar r}(z))} + \big\| |\nabla u| \big\|_{L^p(B_{2\bar r}(z))} \big).$$

Now, according to Remark 4.3, we cover M by a sequence of balls $\{B_{\bar r/4}(z_j)\}_{j\in\mathbb N}$ with the property that the covering $\{B_{2\bar r}(z_j)\}_{j\in\mathbb N}$ has finite intersection multiplicity. Summing up the local inequalities and using monotone and dominated convergence, we deduce the existence of a constant $C = C(n,p,K,i) > 0$ such that

$$C^{-1} \big\| |\operatorname{Hess}(u)| \big\|_{L^p} \leq \| \Delta u \|_{L^p} + \| u \|_{L^p} + \big\| |\nabla u| \big\|_{L^p}.$$

To conclude we apply the L^p gradient estimates of Theorem 4.4. Accordingly, there exists a constant $C = C(n,p,K) > 0$ such that

$$C^{-1} \big\| |\nabla u| \big\|_{L^p} \leq \| u \|_{L^p} + \| \Delta u \|_{L^p}.$$

This completes the proof. ∎

We conclude this section by showing how to pass from a Calderón–Zygmund inequality to a strong $W^{2,p}$ estimate when the underlying manifold has a spectral gap. The main tool is the following result.

Lemma 6.3. *Let $1 < p < \infty$. Suppose that M has spectral gap $b > 0$. Then, there exists a constant $C = C(n, p, b) > 0$ such that, for any $u \in C_c^\infty(M)$, it holds*

$$C^{-1}\|u\|_{L^p} \le \|\Delta u\|_{L^p}.$$

Proof. As M has spectral gap $b > 0$, the heat semigroup on M satisfies the following estimate:

$$|||e^{-t\mathcal{L}}|||_{L^2} \le e^{-bt},$$

where $\mathcal{L} = -\Delta$ (recall that $|||\cdot|||_{L^p}$ denotes the operatorial norm in L^p). On the other hand, the heat semigroup is contractive in L^p for all $1 \le p \le \infty$, i.e.,

$$|||e^{-t\mathcal{L}}|||_{L^1} \le 1 \quad \text{and} \quad |||e^{-t\mathcal{L}}|||_{L^\infty} \le 1.$$

Hence, an application of the Riesz–Thorin interpolation theorem implies that

$$|||e^{-t\mathcal{L}}|||_{L^p} \le e^{-bc_p t},$$

for all $1 < p < \infty$, with $c_p = 1 - |(p-2)/p|$. Accordingly, for any $u \in C_c^\infty(M)$ one has the representation formula

$$u = \int_0^\infty e^{-t\mathcal{L}}\,\mathcal{L}u\,dt,$$

which in turn implies

$$\|u\|_{L^p} \le \int_0^\infty e^{-bc_p t}\|\mathcal{L}u\|_{L^p}\,dt \le \frac{1}{bc_p}\|\Delta u\|_{L^p}. \qquad \blacksquare$$

Remark 6.4. In a first draft of this paper, Lemma 6.3 was proved under the additional assumption that $\mathrm{Ric} \ge -K^2$. However, thanks to a suggestion of an anonymous referee, this assumption was later removed.

A trivial consequence of Lemma 6.3 is the estimate

$$\|u\|_{L^p} + \|\Delta u\|_{L^p} \le C\|\Delta u\|_{L^p}, \quad \forall u \in C_c^\infty(M).$$

This latter gives improved versions of L^p gradient estimates and Calderón–Zygmund inequalities whenever M has a spectral gap.

For instance, we have the following.

Corollary 6.5. *Suppose that* $\mathrm{Ric} \ge -(n-1)K^2$ *and that M has spectral gap. Then, for any fixed $1 < p \le 2$, the strong $W^{2,p}$-estimate*

$$(W(2,p)) \qquad \big\||\nabla u|\big\|_{L^p} + \big\||\nabla^2 u|\big\|_{L^p} \le C\|\Delta u\|_{L^p}, \quad \forall u \in C_c^\infty(M),$$

holds on M for some $C > 0$. Consequently, the whole $W^{2,p}$ norm of u can be bounded in terms of the L^p norm of its Laplacian.

Proof. Since $\mathrm{Ric} \geq -(n-1)K^2$ and $1 < p \leq 2$, by [6], there exists a constant $C > 0$ such that, for every $u \in C_c^\infty(M)$,

$$C^{-1} \big\| |\operatorname{Hess}(u)| \big\|_{L^p} \leq \|u\|_{L^p} + \|\Delta u\|_{L^p}.$$

On the other hand, the L^p gradient estimates state that, for a suitable constant $C > 0$,

$$C^{-1} \big\| |\nabla u| \big\|_{L^p} \leq \|u\|_{L^p} + \|\Delta u\|_{L^p}.$$

Summarising,

$$C^{-1} \|u\|_{W^{2,p}} \leq \|u\|_{L^p} + \|\Delta u\|_{L^p}.$$

An application of Lemma 6.3 yields the desired strong $W^{2,p}$-estimate. ∎

This corollary improves a result contained in [26] by removing the injectivity radius assumption. Similarly, we have the following straightforward consequences of Theorem A and Theorem 6.2, respectively.

Corollary 6.6. *Suppose that $n < p_0 < +\infty$ and that M has spectral gap $b > 0$. There exists a constant $\varepsilon = \varepsilon(p_0, n, K) > 0$ such that if $k(p_0/2, 1, K) \leq \varepsilon$ for some $K \geq 0$, then the L^p gradient estimate*

$$\| |\nabla u| \|_p \leq C \, \|\Delta u\|_p, \quad \forall u \in C_c^\infty(M),$$

holds on M for every $1 < p \leq p_0$.

Corollary 6.7. *Let $1 < p < +\infty$. Suppose that $r_{W^{1,q}}(M) > 0$ for some $q > \max(n, p)$ and that M has spectral gap. Then*

$$\big\| |\nabla^2 u| \big\|_p \leq C \, \|\Delta u\|_p, \quad \forall u \in C_c^\infty(M).$$

7. Higher order Calderón–Zygmund inequalities

We start by recalling the following consequence of Theorem 5.2 in [26], proved by the second named author joint with Mauceri and Vallarino.

Theorem 7.1. *Suppose that M has bounded geometry at the order $2\ell - 2 \in \mathbb{N}$, namely,*

$$|\nabla^j \operatorname{Ric}| \leq K, \ \forall j = 0, \dots, 2\ell - 2, \quad \text{and} \quad r_{\mathrm{inj}}(M) \geq i,$$

for some constants $K \geq 0$ and $i > 0$. Assume also that M has spectral gap $b > 0$. Then, for any $1 < p \leq 2$, there exists a constant $C = C(n, p, \ell, K, b, i) > 0$ such that the global Riesz transform $\mathcal{R}^{2\ell}$ of order 2ℓ is bounded from $L^p(M)$ to $L^p(M; T_{2\ell} M)$

Actually, the result in [26] is stronger, as it establishes that the global covariant Riesz transform $\mathcal{R}^{2\ell}$ is bounded as an operator from a certain Hardy space to L^1. Its L^p boundedness for $1 < p \leq 2$ then follows from an interpolation argument.

It is natural to speculate whether some of the assumptions in Theorem 7.1 can be removed. Our contribution is to allow b to be zero, at the expense of considering local Riesz transforms versus the global version thereof. This is the content of Theorem F that we are now going to prove.

Proof of Theorem F. All over this proof, we denote by $\mathcal{L} := -\Delta$ the positively defined Laplace–Beltrami operator of the underlying manifold. Suppose that (M, g) has bounded geometry at the order $2\ell - 2$. Take the standard hyperbolic plane $\mathbb{H}^2$, and consider the Riemannian product $(M \times \mathbb{H}^2, g + g_{\mathbb{H}^2})$. Then, denoting by $b_M = b$, $b_{\mathbb{H}^2}$ and $b_{M\times\mathbb{H}^2}$ the bottom of the L^2 spectrum of the (positive) Laplace–Beltrami operator on M, $\mathbb{H}^2$ and $M \times \mathbb{H}^2$, respectively, it holds

$$b_{M\times\mathbb{H}^2} = b_M + b_{\mathbb{H}^2} \geq b_{\mathbb{H}^2} = \frac{1}{4}.$$

Moreover,

$$|\nabla^j \operatorname{Ric}_N| \leq \max(1, K), \quad \text{for } j = 0, \dots, 2\ell - 2,$$

and also

$$r_{\mathrm{inj}}(M \times \mathbb{H}^2) \geq r_{\mathrm{inj}}(M) \geq i.$$

It follows from Theorem 7.1 and Proposition 2.4 that, if p is in $(1, 2)$, there exists a constant $C > 0$ such that

$$\big\| |\nabla_{M\times\mathbb{H}^2}|^{2\ell} w \big\|_{L^p(M\times\mathbb{H}^2)} \leq C\, \|\mathcal{L}^{\ell}_{M\times\mathbb{H}^2} w\|_{L^p(M\times\mathbb{H}^2)}, \quad \forall w \in \mathrm{Dom}_{L^p}(\mathcal{L}_{M\times\mathbb{H}^2}). \tag{7.1}$$

We apply this estimate to functions w of the form $\varphi \otimes \psi$, where $\varphi \in \mathrm{Dom}_{L^p}(\mathcal{L}_M)$ and ψ belongs to $C^\infty_c(\mathbb{H}^2)$. Since

$$\mathcal{L}^{\ell}_{M\times\mathbb{H}^2}(\varphi \otimes \psi) = \sum_{j=0}^{\ell} \binom{\ell}{j} (\mathcal{L}^j_M \varphi) \otimes (\mathcal{L}^{\ell-j}_{\mathbb{H}^2} \psi)$$

and

$$\big\|\nabla^{2\ell}_{M\times\mathbb{H}^2}(\varphi \otimes \psi)\big\|^2_{M\times\mathbb{H}^2} = \sum_{j=0}^{2\ell} 2 \binom{\ell}{j} \big\|(\nabla^j_M \varphi) \otimes (\nabla^{2\ell-j}_{\mathbb{H}^2} \psi)\big\|^2_{M\times\mathbb{H}^2},$$

by (7.1) we see that

$$\begin{aligned}
\big\| |\nabla^{2\ell}_M \varphi| \big\|_{L^p(M)} \|\psi\|_{L^p(\mathbb{H}^2)} &= \big\| |(\nabla^{2\ell}_M \varphi) \otimes \psi| \big\|_{L^p(M\times\mathbb{H}^2)} \\
&\leq \big\| |\nabla^{2\ell}_{M\times\mathbb{H}^2}(\varphi \otimes \psi)| \big\|_{L^p(M\times\mathbb{H}^2)} \leq C \|\mathcal{L}^{\ell}_{M\times\mathbb{H}^2}(\varphi \otimes \psi)\|_{L^p(M\times\mathbb{H}^2)} \\
&\leq C \sum_{j=0}^{\ell} \binom{\ell}{j} \|\mathcal{L}^j_M \varphi\|_{L^p(M)} \|\mathcal{L}^{\ell-j}_{\mathbb{H}^2} \psi\|_{L^p(\mathbb{H}^2)}.
\end{aligned}$$

Now, suppose that ψ does not vanish identically on $\mathbb{H}^2$. Then divide both sides of the previous inequality by $\|\psi\|_{L^p(\mathbb{H}^2)}$, and obtain that

$$\big\| |\nabla^{2\ell}_M \varphi| \big\|_{L^p(M)} \leq C\, \sigma_{p,\ell} \sum_{j=0}^{\ell} \binom{\ell}{j} \|\mathcal{L}^j_M \varphi\|_{L^p(M)}, \quad \forall \varphi \in L^p(M),$$

where

$$\sigma_{p,l} := \min_{0\leq j\leq l} \inf_{\psi\neq 0} \frac{\|\mathcal{L}^{l-j}_{\mathbb{H}^2} \psi\|_{L^p(\mathbb{H}^2)}}{\|\psi\|_{L^p(\mathbb{H}^2)}},$$

is a finite constant. Now, since $\mathcal{L}$ is sectorial on $L^p(M)$ (for $\mathcal{L}_M$ generates the contraction semigroup $\{\mathcal{H}_t\}$ on $L^p(M)$), the moment inequality (Theorem 6.6.4 in [18]) implies that

$$\|\mathcal{L}_M^j \varphi\|_{L^p(M)} \le C\, \|\varphi\|_{L^p(M)}^{1-j/\ell}\, \|\mathcal{L}_M^\ell \varphi\|_{L^p(M)}^{j/\ell},$$

so that

$$\begin{aligned}\sum_{j=0}^{\ell} \binom{\ell}{j} \|\mathcal{L}_M^j \varphi\|_{L^p(M)} &\le C\, \big(\|\varphi\|_{L^p(M)}^{1/l} + \|\mathcal{L}_M^\ell \varphi\|_{L^p(M)}^{1/\ell}\big)^\ell \\ &\le C\, 2^\ell \big(\|\varphi\|_{L^p(M)} + \|\mathcal{L}_M^\ell \varphi\|_{L^p(M)}\big).\end{aligned}$$

By combining the steps above, we find that there exists a constant $C > 0$ such that

$$\big\| |\nabla_M^{2\ell} \varphi| \big\|_{L^p(M)} \le C\, \big(\|\varphi\|_{L^p(M)} + \|\mathcal{L}_M^\ell \varphi\|_{L^p(M)}\big). \tag{7.2}$$

A further application of Proposition 2.4 concludes the proof. ■

Remark 7.2. (1) It is natural to speculate whether the Riesz transforms of higher odd order $\mathcal{R}_\tau^{2\ell-1}$ are bounded on $L^p(M)$ when $\ell \ge 2$.

(2) It should be possible to give an alternative proof to Theorem F using $C^{2\ell-1,\alpha}$ harmonic coordinates, which exist in our assumptions, see [1]. Such a proof would likely work also in the case $p > 2$, but it would be very technical and involved, due to the large number of terms of the coordinate expression of $\nabla^{2\ell}$ to deal with; compare for instance with the analogous result for the higher order density problem in [22]. For the sake of simplicity, we decided not to investigate such an approach in this paper.

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