# On the ranking of players in network games with local average*

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**Abstract.** We consider a network Nash equilibrium problem where players are represented as nodes of the graph of social relationships and each player's action is influenced by the actions of her social contacts. Within this model, we investigate the problem of ranking the importance of different players. To this end, we first consider some classic topological measures, commonly used to assess the importance of nodes of a graph, and combine them using the entropy weighting method to obtain a single topological measure. We then consider a different measure of importance, based on the social welfare function, computed at the Nash equilibrium, for the original network, as well as for all the networks obtained from the initial one by removing a single player. The two measures are then illustrated and compared with the help of small size test problems.

## 1 Introduction

Games played on networks are Nash equilibrium problems where players are modeled as nodes of a graph, and the social or economic relationship between any two players is represented by a (possibly direct) link connecting them. A main feature of this topic is that the graph-theoretical properties are deeply intermingled with the game-theoretical concepts. Indeed, in the seminal paper [3], the unique Nash equilibrium of a network game was expressed by using the power expansion of the adjacency matrix of the graph following the method in [6]. In the last fifteen years many scholars have devoted a large number of papers to theoretically extend the original model in [3] and provided a wealth of social and economic applications, see for instance [4, 2, 12, 13]. Most authors confined themselves to treat the case of quadratic utility functions and interior solutions, in order to obtain closed form expressions that could be analyzed in detail from the social and economic point of view. Only recently, the powerful theory of variational inequalities (VIs) has been used to analyze and solve network games, although the VI approach to Nash equilibrium was developed many decades ago in [11]. In this respect we mention the interesting paper [20], where the authors investigate some models of Network Games and provide a detailed analysis of monotonicity properties and stability of solutions, and

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the papers [21, 22, 23] where the authors use the VI approach to extend some results previously obtained in the case of interior solution.

It is important to point out that, in order to carry out a detailed investigation of how the structure of the graph of relationships influences the Nash equilibrium, it is useful to consider game classes where all players are influenced by their neighbors in the same direction, giving rise to the classes of games with strategic complements and games with strategic substitutes, which will be defined precisely below. Specific forms of the utility functions are also very appropriate to model conformity effects, where a player tends to imitate the actions of her peers, represented as her graph neighbors. This kind of conformity effect has been analyzed in a general economic model in [9] and framed in the Network Game Theory in [17].

In this paper, we apply the variational inequality approach to the game described in [17] and provide a ranking of players according to their contribution to the overall network utility, which is computed using the social welfare function at the Nash equilibrium. Because players are represented as nodes in the graph of relationships, they can also be ranked according to some classic topological measures of node importance, such as: *degree, betweenness, closeness.* We combine these three topological measures by using the entropy method of multicriterion analysis, and compare the single topological measure thus obtained with the social welfare based measure, with the help of two small scale examples.

The paper is structured as follows. The following Section 2 covers some basic concepts on network games and variational inequalities. In Section 3 we first present the model with the conformity (or local average) term and then investigate the monotonicity properties and the social welfare function in two subsections. In the short Section 4 we describe the four ranking criteria for players and, in Section 5, we outline the entropy method used to weight the three topological criteria. Section 6 is then devoted to illustrate the above mentioned concepts with the help of two small size test problems. In the last concluding section we summarize our results and mention possible developments.

## 2 Basics on network games and variational inequalities

In Network Games players are represented by the nodes (or vertices) of a graph $(V, E)$, where $V = \{v_1, \dots, v_n\}$ is the sets of nodes and $E$ is the set of directed edges (also called arcs or links), formed by ordered pairs of nodes $(v_i, v_j)$. There are neither multiple arcs connecting the same pair of nodes, nor loops in our model. If $(v, w)$ and $(w, v)$ are the same, the graph is said to be undirected. Two nodes $v_i$ and $v_j$ are said to be adjacent if they are connected by the edge $(v_i, v_j)$. The information about the adjacency of nodes can be stored in the adjacency matrix $G$ whose elements $g_{ij}$ are equal to 1 if $(v_i, v_j)$ is an edge, 0 otherwise. $G$ is thus a zero-diagonal matrix. Given a node $v$, the nodes connected to $v$ with an edge are called the *neighbors* of $v$ and are grouped in the set $N_v(G)$. The number of elements of $N_v(G)$ is the *degree* $d_v(G)$ of $v$. A *walk* in the graph is a finite sequence of the form $v_{i_0}, e_{j_1}, v_{i_1}, e_{j_2}, \dots, e_{j_k}, v_{j_k}$, which consists of alternating nodes and edges of the graph, such that $v_{i_{t-1}}$ and $v_{i_t}$ are end nodes of edge $e_{j_t}$. The *length* of a walk is the number of its edges. Let us remark that it is allowed to visit a node or go through an edge more than once. The indirect connections between any two nodes in the graph are described by means of the powers of the adjacency matrix $G$. Indeed, it can be proved that the element $g_{ij}^{[k]}$ of $G^k$ gives the number of walks of length $k$ between nodes $v_i$ and $v_j$.

In the sequel, the set of players will be denoted by $\{1, 2, \dots, n\}$ instead of $\{v_1, v_2, \dots, v_n\}$. We denote with $A_i \subset \mathbb{R}$ the action space of player $i$, while $A = A_1 \times \dots \times A_n$ is called the space of action profiles. For each $a = (a_1, \dots, a_n)$, $a_{-i} = (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n)$ and the notation $a = (a_i, a_{-i})$ will be used when we want to distinguish the action of player $i$ from the action of all the other players. Each player $i$ is endowed with a payoff function $u_i : A \to \mathbb{R}$ that she wishes to maximize. The notation $u_i(a, G)$ is often utilized when one wants to emphasize that the utility of player $i$ also depends on the actions taken by her neighbors in the graph.

The solution concept that we consider here is the Nash equilibrium of the game, that is, we seek an element $a^* \in A$ such that for each $i \in \{1, \dots, n\}$:

$$u_i(a_i^*, a_{-i}^*) \geq u_i(a_i, a_{-i}^*), \qquad \forall\, a_i \in A_i. \tag{1}$$

Depending on how variations of the actions of player's $i$ neighbors affect her marginal utility, the game can be classified as follows.

**Definition 2.1.** *The network game has the property of strategic complements if:*

$$\frac{\partial^2 u_i}{\partial a_j \partial a_i}(a) > 0, \qquad \forall\, (i,j):\ g_{ij} = 1,\ \forall\, a \in A.$$

**Definition 2.2.** *The network game has the property of strategic substitutes if:*

$$\frac{\partial^2 u_i}{\partial a_j \partial a_i}(a) < 0, \qquad \forall\, (i,j):\ g_{ij} = 1,\ \forall\, a \in A.$$

For the subsequent development it is important to recall (see, e.g., [19]) that if the $u_i$ are continuously differentiable functions on $A$ and $u_i(\cdot, a_{-i})$ are concave, the Nash equilibrium problem is equivalent to the variational inequality $VI(F, A)$: find $a^* \in A$ such that

$$[F(a^*)]^\top (a - a^*) \geq 0, \qquad \forall\, a \in A, \tag{2}$$

where

$$[F(a)]^\top := -\left(\frac{\partial u_1}{\partial a_1}(a), \dots, \frac{\partial u_n}{\partial a_n}(a)\right) \tag{3}$$

is also called the pseudo-gradient of the game. For an account of variational inequalities the interested reader can refer to [10, 16]. We recall here some useful monotonicity properties.

**Definition 2.3.** *A map $T : \mathbb{R}^n \to \mathbb{R}^n$ is said to be monotone on $A$ iff:*

$$[T(x) - T(y)]^\top (x - y) \geq 0, \qquad \forall\, x, y \in A.$$

*If the equality holds only when $x = y$, then $T$ is said to be strictly monotone.*
*$T$ is said to be $\beta$-strongly monotone on $A$ iff there exists $\beta > 0$ such that*

$$[T(x) - T(y)]^\top (x - y) \geq \beta \|x - y\|^2, \qquad \forall\, x, y \in A.$$

For linear operators on $\mathbb{R}^n$ the two concepts of strict and strong monotonicity coincide and are equivalent to the positive definiteness of the corresponding matrix. Conditions that ensure the unique solvability of a variational inequality problem are given by the following theorem (see, e.g., [10, 15]).

**Theorem 2.1.** *If $K \subset \mathbb{R}^n$ is a compact convex set and $T : \mathbb{R}^n \to \mathbb{R}^n$ is continuous on $K$, then the variational inequality problem $VI(F, K)$ admits at least one solution. In the case $K$ is unbounded, existence of a solution may be established under the following coercivity condition:*

$$\lim_{\|x\| \to +\infty} \frac{[T(x) - T(x_0)]^\top (x - x_0)}{\|x - x_0\|} = +\infty,$$

*for $x \in K$ and some $x_0 \in K$.*
*Furthermore, if $T$ is strictly monotone on $K$ the solution is unique.*

## 3 The network game with local average

We consider a social network with $n$ players with symmetric relationships, whose (symmetric) adjacency matrix $G$ has elements $g_{ij} \in \{0,1\}$. Let $d_i(G)$ represent the degree of player $i$, which we assume greater than zero for every $i$, i.e., there are no disconnected players in the network. We then introduce a second matrix $\hat{G}$ with elements:

$$\hat{g}_{ij} = \frac{g_{ij}}{d_i(G)}.$$

We then have that $0 \leq \hat{g}_{ij} \leq 1$, and $\sum_{j=1}^{n} \hat{g}_{ij} = 1$ for any $i = 1, \ldots, n$. Let us notice that, in general, $\hat{G}$ is not a symmetric matrix even if $G$ is so. For each player $i$, we define the average action of her neighbors as:

$$\bar{a}_i(G) = \frac{1}{d_i(G)} \sum_{j=1}^{n} g_{ij} a_j = \sum_{j=1}^{n} \hat{g}_{ij} a_j.$$

Each player is endowed with a utility function given by:

$$u_i(a, G) = \alpha_i a_i - \frac{1}{2} a_i^2 - \frac{\delta}{2} \left( a_i - \frac{1}{d_i(G)} \sum_{j=1}^{n} g_{ij} a_j \right)^2, \tag{4}$$

where $\delta > 0$, and $\alpha_i > 0$ for any $i = 1, \ldots, n$. We also assume that $a_i \in [0, L_i]$, for any $i = 1, \ldots, n$. The first term in the expression (4) represents the gain of player $i$ when her action is $a_i$ and the second term is the corresponding intrinsic cost. The third term models a conformity effect and, since $\delta > 0$, penalizes the utility of player $i$ when her action differs from the average action of her neighbors. The pseudo-gradient's components are:

$$F_i(a) = -\frac{\partial u_i}{\partial a_i} = -\alpha_i + (1+\delta) a_i - \delta \sum_{j=1}^{n} \hat{g}_{ij} a_j, \qquad \forall\, i = 1, \ldots, n,$$

which, in compact form, can be written as:

$$F(a) = (1+\delta) \left[ I - \frac{\delta}{1+\delta} \hat{G} \right] a - \alpha, \tag{5}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$.

In the case of interior solution, the first order conditions applied to the utility functions (4) are (see [17]):

$$a_i = \frac{\alpha_i}{1+\delta} + \frac{\delta}{1+\delta} \sum_{j=1}^{n} \hat{g}_{ij} a_j, \qquad \forall\, i = 1, \ldots, n.$$

The system in compact form is written as:

$$\left[ I - \frac{\delta}{1+\delta} \hat{G} \right] a = \frac{1}{1+\delta} \alpha,$$

which can be uniquely solved if the matrix $\left[ I - \frac{\delta}{1+\delta} \hat{G} \right]$ is nonsingular. In the general case, we have to solve the variational inequality $VI(F, A)$.

### 3.1 Monotonicity properties

The map $F$ is strictly monotone iff the matrix $I - \frac{\delta}{1+\delta}\hat{G}$ is positive definite. We then consider the symmetric part of $I - \frac{\delta}{1+\delta}\hat{G}$, that is, $I - \frac{\delta}{1+\delta}\left(\frac{\hat{G}+\hat{G}^\top}{2}\right)$, and check conditions for its smallest eigenvalue to be positive. Because the eigenvalues of $I - \frac{\delta}{1+\delta}\left(\frac{\hat{G}+\hat{G}^\top}{2}\right)$ can be written as $1 - \frac{\delta}{1+\delta}\lambda_i\left(\frac{\hat{G}+\hat{G}^\top}{2}\right)$, where $\lambda_i\left(\frac{\hat{G}+\hat{G}^\top}{2}\right)$ denotes the $i$-th eigenvalue of $\frac{\hat{G}+\hat{G}^\top}{2}$, the condition $\min_i\left\{1 - \frac{\delta}{1+\delta}\lambda_i(\frac{\hat{G}+\hat{G}^\top}{2})\right\} > 0$ can be written in an equivalent manner as

$$\frac{\delta}{1+\delta}\lambda_{max}\left(\frac{\hat{G}+\hat{G}^\top}{2}\right) < 1. \tag{6}$$

Let us recall that for a symmetric matrix, whose entries are all nonnegative real numbers, the Perron-Frobenius theorem entails that its greater eigenvalue coincides with the spectral radius. We then get that:

$$\frac{\delta}{1+\delta}\lambda_{max}\left(\frac{\hat{G}+\hat{G}^\top}{2}\right) < 1 \Longleftrightarrow \frac{\delta}{1+\delta}\rho\left(\frac{\hat{G}+\hat{G}^\top}{2}\right) < 1. \tag{7}$$

Now, let us notice that the condition which ensures the invertibility of $I - \frac{\delta}{1+\delta}\hat{G}$ is $\frac{\delta}{1+\delta}\rho(\hat{G}) < 1$. We then have to compare $\rho(\hat{G})$ with $\rho\left(\frac{\hat{G}+\hat{G}^\top}{2}\right)$.

**Lemma 3.1.** *Let $M$ be a square matrix with nonnegative entries. We then have:*

$$\rho(M) \le \rho\left(\frac{M+M^\top}{2}\right),$$

*where the equality holds if and only if $M$ and $M^\top$ share an eigenvalue corresponding to $\rho(M)$.*

*Proof.* See, e.g., [5] ex. 6.5, page 53, or [25]. □

### 3.2 Social Welfare

As usual in Game Theory, the social welfare $W$ is defined as the sum of the utilities of all players:

$$W(a,\hat{G}) := \sum_{i=1}^n u_i(a,\hat{G}) = \sum_{i=1}^n \alpha_i a_i - \frac{1}{2}\sum_{i=1}^n a_i^2 - \frac{\delta}{2}\sum_{i=1}^n\left(a_i - \sum_{j=1}^n \hat{g}_{ij}a_j\right)^2. \tag{8}$$

To obtain an expression of $W$ which can be easily analyzed we observe that:

$$\sum_{i=1}^n\left(\sum_{j=1}^n \hat{g}_{ij}a_j\right)^2 = \left(\hat{G}a\right)^\top \hat{G}a = a^\top\hat{G}^\top\hat{G}a,$$

and

$$\delta\sum_{i=1}^n a_i\sum_{j=1}^n \hat{g}_{ij}a_j = \delta a^\top\hat{G}a = \delta a^\top\left(\frac{\hat{G}+\hat{G}^\top}{2}\right)a,$$

whence:

$$W(a,\hat{G}) = -\frac{1}{2}a^\top[(1+\delta)I - \delta(\hat{G}+\hat{G}^\top - \hat{G}^\top G)]a + \alpha^\top a. \tag{9}$$

It is interesting to provide conditions which ensure the strict concavity of $W$. For this, let us notice that the quadratic form in $W$ is negative definite iff the matrix

$$(1+\delta)I - \delta(\hat{G} + \hat{G}^\top - \hat{G}^\top G)$$

is positive definite and, because $\hat{G}^\top G$ is positive definite, it is enough that $(1+\delta)I - \delta\left(\hat{G} + \hat{G}^\top\right)$ is positive definite which amounts to requiring that:

$$\rho(\hat{G} + \hat{G}^\top) < 1 + \frac{1}{\delta}.$$

Given that uniqueness of the Nash equilibrium holds if $\rho(\hat{G} + \hat{G}^\top) < 2(1 + \frac{1}{\delta})$, we get that if $\rho(\hat{G} + \hat{G}^\top) < 1 + \frac{1}{\delta}$ there exist a unique Nash equilibrium and a unique social optimum.

For the reader convenience we report here the expressions of the first and second derivatives of the social welfare, namely:

$$\begin{aligned}\frac{\partial W}{\partial a_l} &= -a_l + \alpha_l - \delta \sum_{i=1}^{n} \left(a_i - \sum_{j=1}^{n} \hat{g}_{ij} a_j\right) \left(\delta_{il} - \sum_{j=1}^{n} \hat{g}_{ij} \delta_{lj}\right) \\ &= \alpha_l - (1+\delta)a_l + \delta \sum_{i=1}^{n} (\hat{g}_{il} + \hat{g}_{li}) a_i - \delta \sum_{i=1}^{n} (\hat{g}_{il})^2 a_l - \delta \sum_{i=1}^{n} \sum_{j \neq i} \hat{g}_{ij} \hat{g}_{il} a_j,\end{aligned}$$

$$\frac{\partial^2 W}{\partial a_l^2} = -\left[1 + \delta + \delta \sum_{i=1}^{n} (\hat{g}_{il})^2\right],$$

$$\frac{\partial^2 W}{\partial a_k \partial a_l} = \delta \left(\hat{g}_{kl} + \hat{g}_{lk} - \sum_{i=1}^{n} \hat{g}_{ik} \hat{g}_{il}\right), \qquad l \neq k.$$

# 4 Ranking criteria

We now define some common topological measures used to assess the importance of a node $i$ in a network:

- *degree centrality*: $DC_i = d_i(G)$;
- *closeness centrality*: $CC_i = \frac{1}{\sum_{j \neq i} d_{ij}}$, where $d_{ij}$ is the shortest path length between $i$ and $j$;
- *betweenness centrality*: $BC_i = \sum_{s \neq i} \sum_{t \neq i} \frac{n_{st}(i)}{n_{st}}$, where $n_{st}$ is the number of shortest paths between $s$ and $t$ and $n_{st}(i)$ is the number of such paths that pass through node $i$.

Together with the topological measure of importance of a node, we also consider a different measure based on the Nash equilibrium of the game.

Specifically, for any node $i = 1, \ldots, n$, we measure the importance of $i$ as the relative variation of the social welfare computed at the Nash equilibrium after $i$ is removed from the network, that is

$$\textit{social welfare centrality:} \;\; SWC_i = \frac{W(NE(G)) - W(NE(G \setminus \{i\}))}{W(NE(G))}, \tag{10}$$

where $NE(G)$ is the Nash equilibrium in the network $G$, $NE(G \setminus \{i\})$ is the Nash equilibrium in the network $G$ where node $i$ has been removed, and $W$ is the social welfare function (9). Note that $SWC(i)$ can be negative if the total social welfare of the network increases after removing the node $i$ (see Example 6.2). This situation can be compared to the well-known Braess paradox [7], where the efficiency of a network improves due to the removal of a link. The use of a mix of topological indices to measure the importance of a nodes in real transport network has been recently applied to the case of the railway network of Shanghai in [26], using the entropy method described below.

# 5 The entropy method in multicriterion analysis

In multicriterion analysis, a decision maker has to make a choice among different alternatives, taking into account multiple, possibly conflicting, objectives. For instance, in a road network, there are multiple paths which connect an origin and a destination, and a driver usually consider travel time and travel cost as the two main criteria which influence her decision. How to compare the importance of travel time and travel cost can depend on personal preferences as well as on objective facts. In our model, we are given a social network, and we wish to assess the importance of its nodes. Because there can be different criteria to measure the importance of a node, an aggregate measure would be useful.

A large number of problems from economics or other fields of applied sciences present the feature of the above examples, of making a choice considering multiple objectives. In those cases, a simple aggregation method consists of assigning weights to each criterion and then compute the weighted sum. The process of assigning weights is often subjective and can lead to unsatisfactory results. In his influential work, M. Zeleny [27], proposed a method to assess the weights in an objective manner, still allowing for the possible intervention of a decision maker. To this end he first specified the concept of weight as follows:

*"A weight, assigned to the i-th attribute as a measure of its relative importance for a given decision problem, is directly related to the average intrinsic information generated by a given set of alternatives through the i-th attribute, as well as to its subjective assessment."*

Thus, a procedure to generate weights requires an operational definition of what the *average intrinsic information* is. To this end, Zeleny made use of the concept of Shannon entropy. We now recall the basic version of the entropy method (see also [24]).

Assume that we are given $m$ alternatives, $a_1, a_2, \dots, a_m$ and $n$ criteria $C_1, C_2, \dots, C_n$. The utility of applying criterion $C_j$ to the alternative $a_i$ is denoted with $a_{ij}$. In our model, the alternatives are the nodes of the network, and the criteria are the three topological measure of their importance. The matrix whose entries are the elements $a_{ij}$ is called the decision matrix. The coefficients $a_{ij}$ must be normalized so as to take on values in the interval $[0, 1]$. Once the weights $w_j$ are computed as explained below, for each alternative $a_i$ we evaluate the number:

$$R(a_i) = \sum_{j=1}^{n} w_j \frac{a_{ij}}{\sum_{i=1}^{m} a_{ij}}$$

which gives the importance of $a_i$. The steps to compute weights are as follows:

1. For each criterion $C_j$ compute the associated Shannon entropy:

$$E_j = -k \sum_{i=1}^{m} \frac{a_{ij}}{\sum\limits_{i=1}^{m} a_{ij}} \log \left( \frac{a_{ij}}{\sum\limits_{i=1}^{m} a_{ij}} \right),$$

where $k = 1/\log m$, to have $E_j \in [0,1]$.

2. It is well known that, for each $j$, the entropy is maximized when the values $\frac{a_{ij}}{\sum_{i=1}^{m} a_{ij}}$, $i = 1,\ldots,m$, are all equal, which is the case where no information is provided about the ranking of the alternatives. To rank the alternatives we thus compute:

$$D_j = 1 - E_j, \qquad \forall\, j = 1,\ldots,n.$$

3. Since we want that weights take on values in the interval $[0,1]$, we further normalize:

$$w_j = \frac{D_j}{\sum_{j=1}^{n} D_j}, \qquad \forall\, j = 1,\ldots,n.$$

# 6 Numerical examples

In this section, we show two numerical examples, in which the social welfare centrality measure is compared with three topological centrality measures (degree, closeness and betweenness) and their weighted sum obtained by exploiting the entropy weighting method. To find Nash equilibria, we reformulated the affine variational inequality (2) as an equivalent convex quadratic optimization problem (see [1]) and solved it with the function `quadprog` from MATLAB the optimization toolbox.

**Example 6.1.** *We consider a network game on a graph with 10 nodes, where the adjacency matrix $G$ has been randomly generated (see Fig. 1), and the game parameters are $\delta = 0.9/[\rho(\hat{G} + \hat{G}^\top) - 1]$, $\alpha_i = 5\,i$ and $L_i = 30$ for any $i = 1,\ldots,10$. Table 1 shows the normalized values of the degree, closeness and betweenness centrality (col. 2–4), their weighted sum with entropy weighting method (col. 5) and the social welfare centrality (col. 6). Table 2 shows the ranking of nodes according to the analyzed centrality measures. It is interesting noting that the ranking defined by the new measure is quite different from that provided by the other measures. In particular, node 1 is the most important according to topological measures, while it is the least important according to the social welfare measure. Moreover, Fig. 1 gives a graphical representation of the normalized values associated to the network nodes for each considered centrality measure.*

Table 1: Example 6.1: Normalized values of degree, closeness and betweenness centrality, their weighted sum, and social welfare centrality.

| | Topological measures | | | Weighted sum of | Social |
|---|---|---|---|---|---|
| Nodes | Degree | Closeness | Betweenness | topological measures | welfare |
| 1 | 0.1389 | 0.1157 | 0.2849 | 0.1733 | 0.0039 |
| 2 | 0.0556 | 0.0885 | 0.0161 | 0.0558 | 0.0062 |
| 3 | 0.0833 | 0.0940 | 0.0161 | 0.0675 | 0.0205 |
| 4 | 0.0833 | 0.0940 | 0.0457 | 0.0762 | 0.0449 |
| 5 | 0.0833 | 0.1003 | 0.0457 | 0.0784 | 0.0752 |
| 6 | 0.1389 | 0.1157 | 0.2097 | 0.1513 | 0.1001 |
| 7 | 0.0833 | 0.0836 | 0.0215 | 0.0654 | 0.1377 |
| 8 | 0.1389 | 0.1075 | 0.2231 | 0.1523 | 0.1654 |
| 9 | 0.0833 | 0.1003 | 0.0887 | 0.0910 | 0.2132 |
| 10 | 0.1111 | 0.1003 | 0.0484 | 0.0890 | 0.2328 |

Table 2: Example 6.1: Ranking of nodes according to degree, closeness and betweenness centrality, their weighted sum, and social welfare centrality.

| | Topological measures | | | Weighted sum of | Social |
|---|---|---|---|---|---|
| Rank | Degree | Closeness | Betweenness | topological measures | welfare |
| 1 | 1 | 1 | 1 | 1 | 10 |
| 2 | 6 | 6 | 8 | 8 | 9 |
| 3 | 8 | 8 | 6 | 6 | 8 |
| 4 | 10 | 5 | 9 | 9 | 7 |
| 5 | 3 | 9 | 10 | 10 | 6 |
| 6 | 4 | 10 | 4 | 5 | 5 |
| 7 | 5 | 3 | 5 | 4 | 4 |
| 8 | 7 | 4 | 7 | 3 | 3 |
| 9 | 9 | 2 | 2 | 7 | 2 |
| 10 | 2 | 7 | 3 | 2 | 1 |

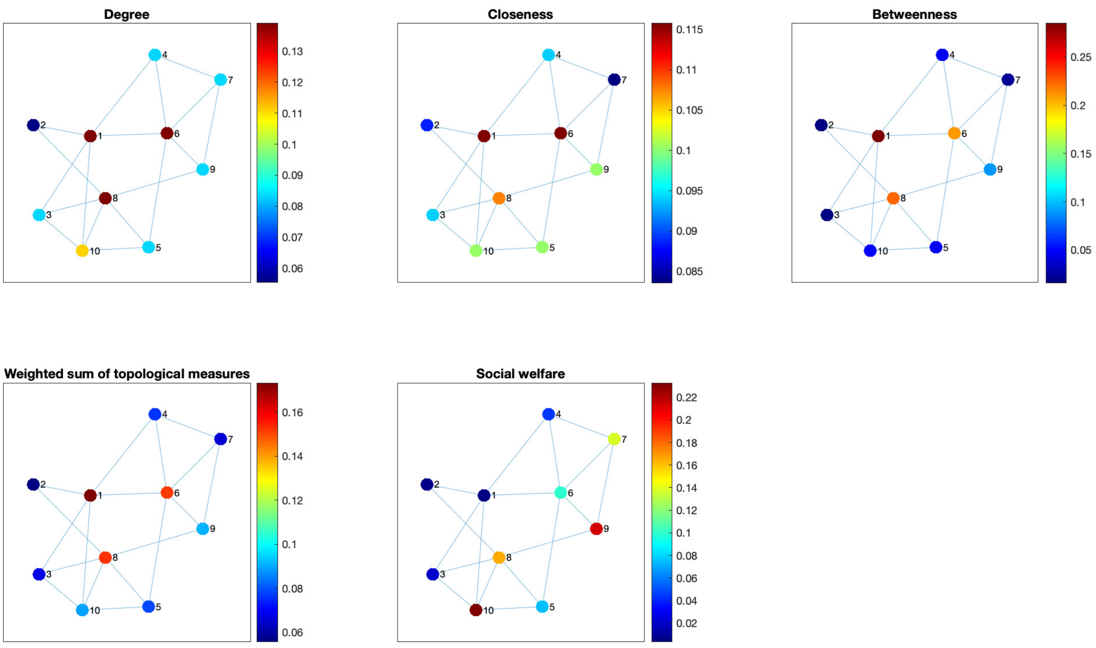


Figure 1: Example 6.1: Comparison between degree, closeness and betweenness centrality, their weighted sum, and social welfare centrality.

**Example 6.2.** *We consider a network game on a graph with 15 nodes, where the adjacency matrix $G$ has been randomly generated (see Fig. 2), and the game parameters are $\delta = 0.9/[\rho(\hat{G}+\hat{G}^\top)-1]$, $\alpha_i = 5\,i$ and $L_i = 40$ for any $i = 1, \ldots, 15$. Tables 3 and 4 show the normalized values of the five centrality measures and the corresponding ranking of nodes. Notice that nodes 1 and 2 have negative values according to the social welfare centrality measure. Moreover, node 1 is the most important according to degree and closeness centrality measures, while it is the least important according to the social welfare measure. Fig. 2 gives a graphical representation of the normalized values associated to the network nodes for each considered centrality measure.*

Table 3: Example 6.2: Normalized values of degree, closeness and betweenness centrality, their weighted sum, and social welfare centrality.

| | Topological measures | | | Weighted sum of | Social |
|---|---|---|---|---|---|
| Nodes | Degree | Closeness | Betweenness | topological measures | welfare |
| 1 | 0.0833 | 0.0727 | 0.1170 | 0.0905 | −0.0244 |
| 2 | 0.0521 | 0.0606 | 0.0208 | 0.0450 | −0.0135 |
| 3 | 0.0625 | 0.0661 | 0.0569 | 0.0620 | 0.0009 |
| 4 | 0.0521 | 0.0606 | 0.0208 | 0.0450 | 0.0116 |
| 5 | 0.0833 | 0.0727 | 0.0728 | 0.0763 | 0.0230 |
| 6 | 0.0417 | 0.0582 | 0.0391 | 0.0465 | 0.0353 |
| 7 | 0.0833 | 0.0727 | 0.1208 | 0.0918 | 0.0496 |
| 8 | 0.0625 | 0.0661 | 0.0847 | 0.0709 | 0.0685 |
| 9 | 0.0729 | 0.0693 | 0.0300 | 0.0579 | 0.0769 |
| 10 | 0.0312 | 0.0539 | 0.0194 | 0.0352 | 0.0891 |
| 11 | 0.0729 | 0.0661 | 0.0608 | 0.0667 | 0.1076 |
| 12 | 0.0729 | 0.0693 | 0.0600 | 0.0675 | 0.1218 |
| 13 | 0.0833 | 0.0727 | 0.1412 | 0.0983 | 0.1356 |
| 14 | 0.0625 | 0.0661 | 0.0581 | 0.0623 | 0.1513 |
| 15 | 0.0833 | 0.0727 | 0.0975 | 0.0843 | 0.1669 |

## 7 Conclusions

In this paper, we considered the problem of ranking the importance of players in a network game with conformity effect, where each player tends to use a strategy close to the average strategy of her peers. Our ranking was based on the use of the social welfare function computed at the Nash equilibrium for the original network and for all the networks resulting from the removal of a single player. This ranking was compared with the one obtained combining three standard topological measures of node importance in a graph with the entropy weighting method. In future work, we aim to apply this model to a class of environmental games described in [14]. Another interesting research direction is the introduction of additional shared constraints, which would yield to a generalized Nash equilibrium problem (see, e.g., [18] for some recent results on this topic).

Table 4: Example 6.2: Ranking of nodes according to degree, closeness and betweenness centrality, their weighted sum, and social welfare centrality.

| | Topological measures | | | Weighted sum of | Social |
|---|---|---|---|---|---|
| Rank | Degree | Closeness | Betweenness | topological measures | welfare |
| 1 | 1 | 1 | 13 | 13 | 15 |
| 2 | 5 | 5 | 7 | 7 | 14 |
| 3 | 7 | 7 | 1 | 1 | 13 |
| 4 | 13 | 13 | 15 | 15 | 12 |
| 5 | 15 | 15 | 8 | 5 | 11 |
| 6 | 9 | 9 | 5 | 8 | 10 |
| 7 | 11 | 12 | 11 | 12 | 9 |
| 8 | 12 | 3 | 12 | 11 | 8 |
| 9 | 3 | 8 | 14 | 14 | 7 |
| 10 | 8 | 11 | 3 | 3 | 6 |
| 11 | 14 | 14 | 6 | 9 | 5 |
| 12 | 2 | 2 | 9 | 6 | 4 |
| 13 | 4 | 4 | 2 | 2 | 3 |
| 14 | 6 | 6 | 4 | 4 | 2 |
| 15 | 10 | 10 | 10 | 10 | 1 |

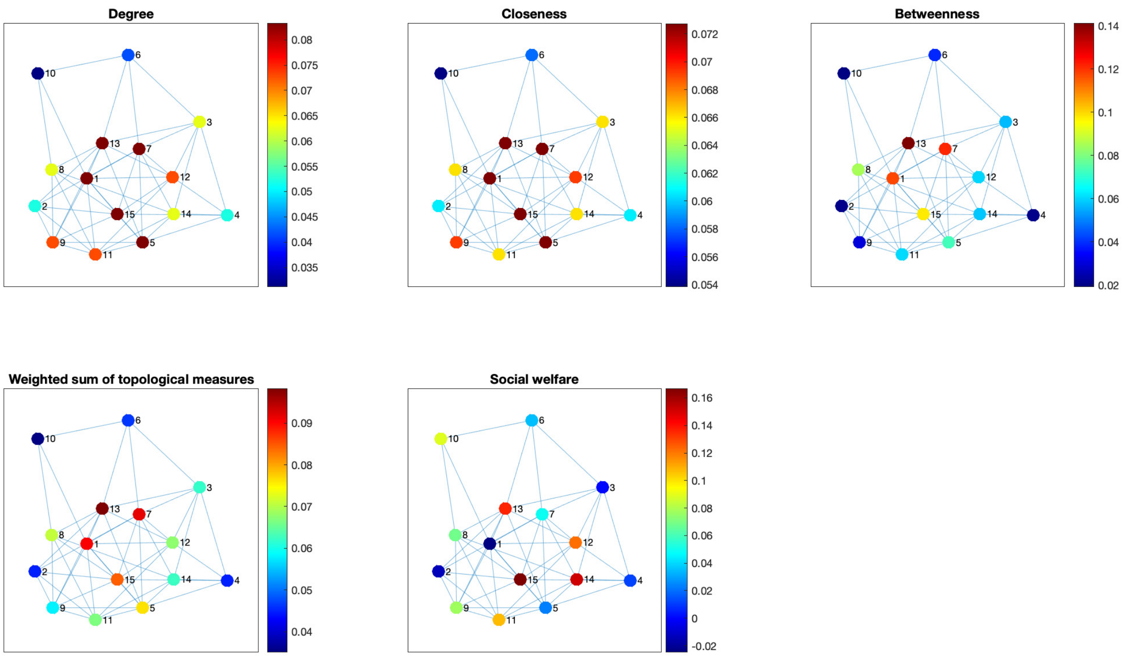


Figure 2: Example 6.2: Comparison between degree, closeness and betweenness centrality, their weighted sum, and social welfare centrality.

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