Journal of Geometry and Physics 202 (2024) 105224

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/geomphys

The symplectic structure of a toric conic transform

Roberto Paoletti

Dipartimento di Matematica e Applicazioni, Università degli Studi di Milano-Bicocca, Via R. Cozzi 55, 20126 Milano, Italy

ARTICLE INFO

Article history:
Received 21 June 2022
Received in revised form 27 April 2024
Accepted 3 May 2024
Available online xxxx

Keywords:
Toric symplectic orbifold
Hamiltonian action
Contact lift
Marked moment polytope
Symplectic structure
Conic transform

ABSTRACT

Suppose that a compact r-dimensional torus T^r acts in a holomorphic and Hamiltonian manner on polarized complex d-dimensional projective manifold M, with nowhere vanishing moment map Φ. Assuming that Φ is transverse to the ray through a given weight $\boldsymbol{\nu}$, associated to these data there is a complex $(d-r+1)$-dimensional polarized projective orbifold $\widehat{M}_{\boldsymbol{\nu}}$ (referred to as the $\boldsymbol{\nu}$-th *conic transform* of M). Namely, $\widehat{M}_{\boldsymbol{\nu}}$ is a suitable quotient of the inverse image of the ray in the unit circle bundle of the polarization of M. With the aim to clarify the geometric significance of this construction, we consider the special case where M is toric, and show that $\widehat{M}_{\boldsymbol{\nu}}$ is itself a Kähler toric orbifold, whose (marked) moment polytope is obtained from the one of M by a certain 'transform' operation (depending on Φ and $\boldsymbol{\nu}$).

1. Introduction

Consider a d-dimensional connected projective manifold M, with complex structure J, and let (A, h) be a positive holomorphic line bundle on M. Thus A is ample, h is a Hermitian metric on it, and the unique covariant derivative ∇ on A compatible with both the metric and the complex structure has curvature $\Theta = -2\imath\,\omega$, with $\omega \in \Omega^2(M)$ a Kähler form on (M, J).

We shall denote by $A^\vee$ the dual line bundle to A, and by $X \subset A^\vee$ the unit circle bundle, with bundle projection $\pi : X \to M$. Then ∇ corresponds to a connection 1-form $\alpha \in \Omega^1(X)$, which is a contact form on X and satisfies

$$\mathrm{d}\alpha = 2\,\pi^*(\omega). \tag{1}$$

Let T^r be an r-dimensional compact torus, with Lie algebra $\mathfrak{t}^r$ and coalgebra $\mathfrak{t}^{r\vee}$. Furthermore, let $\mu^M : T^r \times M \to M$ be a holomorphic and Hamiltonian action of T^r on $(M, 2\,\omega, J)$, with moment map $\Phi : M \to \mathfrak{t}^{r\vee}$ (see e.g. [7] for general background on Hamiltonian actions and moment maps).

Any $\boldsymbol{\xi} \in \mathfrak{t}^r$ thus determines a Hamiltonian vector field $\boldsymbol{\xi}_M$ on M. As is well-known [10], Φ determines a natural lift of $\boldsymbol{\xi}_M$ to a contact vector field $\boldsymbol{\xi}_X = \boldsymbol{\xi}_X^\Phi$ on (X, α), given by

$$\boldsymbol{\xi}_X := \boldsymbol{\xi}_M^\sharp - \langle \Phi, \boldsymbol{\xi} \rangle\, \partial_\theta; \tag{2}$$

E-mail address: roberto.paoletti@unimib.it.

https://doi.org/10.1016/j.geomphys.2024.105224

here $V^\sharp$ is the horizontal lift to X, with respect to α, of a vector field V on M, and ∂_θ is the generator of the structure S^1-action on X, given by counterclockwise fiber rotation. The flow of $\boldsymbol{\xi}_X$ preserves the contact and CR structures of X, and the flows of $\boldsymbol{\xi}_X$ and $\boldsymbol{\xi}'_X$ commute, for any $\boldsymbol{\xi}, \boldsymbol{\xi}' \in \mathfrak{t}^r$.

We shall make the stronger hypothesis that μ^M lifts to a metric preserving line bundle action on A, and the induced action $\mu^X : T^r \times X \to X$ has the correspondence $\boldsymbol{\xi} \mapsto \boldsymbol{\xi}_X$ in (2) as its differential. We shall say that μ^X is the (contact and CR) lift of the holomorphic Hamiltonian action (μ^M, Φ).

For example, when $r = 1$ and μ^M is trivial, $\Phi : M \to \mathfrak{t}^{1\vee}$ is constant; choosing $\Phi = \imath$ in (2) yields the circle action ρ^X generated by $-\partial_\theta$, thus given by clockwise fiber rotation. If $\partial_\theta^{S^1}$ is the standard generator of the Lie algebra of S^1, $\partial_\theta = (\partial_\theta^{S^1})_X$ in (2) is the vector field on X generating the structure S^1-action given by counter-clockwise fiber rotation, while $-\partial_\theta$ is the vector field generating ρ^X (clockwise fiber rotation); we parametrize S^1 by $\theta \mapsto e^{\imath\theta}$.

Let us fix a non-zero weight $\boldsymbol{\nu} \in \mathfrak{t}^{r\vee}$. The results below rest on the following Basic Assumption on $(\Phi, \boldsymbol{\nu})$, henceforth referred to as BA 1.1.

Basic Assumption 1.1. The following holds:

1. $\boldsymbol{\nu}$ is primitive (or coprime);
2. Φ is nowhere vanishing, that is, $\mathbf{0} \notin \Phi(M)$;
3. Φ is transverse to the ray $\mathbb{R}_+ \cdot \boldsymbol{\nu}$.

Under these circumstances, a polarized Kähler orbifold $(\widehat{M}_{\boldsymbol{\nu}}, \widehat{\omega}_{\boldsymbol{\nu}}, \widehat{J}_{\boldsymbol{\nu}})$ can be constructed from the previous data, by taking a suitable quotient by a locally free action of T^r of a locus in X defined by $(\Phi, \boldsymbol{\nu})$; here $\widehat{\omega}_{\boldsymbol{\nu}}$ and $\widehat{J}_{\boldsymbol{\nu}}$ denote the (orbifold) symplectic and complex structures on $\widehat{M}_{\boldsymbol{\nu}}$ [16]. We refer to [16] (where $\widehat{M}_{\boldsymbol{\nu}}$ is denoted $N_{\boldsymbol{\nu}}$ and $\widehat{\omega}_{\boldsymbol{\nu}}$ by $\eta_{\boldsymbol{\nu}}$) for a discussion of the relevance of this geometric construction in geometric quantization; it generalizes the one of weighted projective spaces as quotients of an odd-dimensional sphere. Here we aim to clarify the relation between the symplectic structures of M and $\widehat{M}_{\boldsymbol{\nu}}$ in the toric setting: as we shall see, assuming that M is a toric manifold, $(\widehat{M}_{\boldsymbol{\nu}}, 2\widehat{\omega}_{\boldsymbol{\nu}})$ is a toric symplectic orbifold, and its marked moment polytope $\widehat{\Delta}_{\boldsymbol{\nu}}$ can be explicitly recovered from the moment polytope Δ of M (by [11] toric symplectic orbifolds are classified by marked convex rational simple polytopes).

Before stating the result precisely, let us briefly recall the geometric construction in point, referring to [13–16] for details. Let us set

$$M_{\boldsymbol{\nu}} := \Phi^{-1}(\mathbb{R}_+ \boldsymbol{\nu}) \subseteq M, \quad X_{\boldsymbol{\nu}} := \pi^{-1}(M_{\boldsymbol{\nu}}) \subseteq X. \tag{3}$$

Then, assuming BA 1.1, the following holds:

1. $M_{\boldsymbol{\nu}} \subseteq M$ is a connected and compact (real) submanifold, of codimension $r - 1$;
2. μ^X is locally free on $X_{\boldsymbol{\nu}}$.

We may and will assume without loss that μ^X is generically free on X (and $X_{\boldsymbol{\nu}}$). Then the quotient

$$\widehat{M}_{\boldsymbol{\nu}} := X_{\boldsymbol{\nu}}/T^r \tag{4}$$

(denoted $N_{\boldsymbol{\nu}}$ in [16]) is naturally a $(d - r + 1)$-dimensional complex orbifold, and comes equipped with a Kähler structure $(\widehat{M}_{\boldsymbol{\nu}}, \widehat{\omega}_{\boldsymbol{\nu}}, \widehat{J}_{\boldsymbol{\nu}})$ induced by (M, ω, J); here T^r acts on $X_{\boldsymbol{\nu}}$ by the restriction of μ^X to $X_{\boldsymbol{\nu}}$. We shall call $\widehat{M}_{\boldsymbol{\nu}}$ the $\boldsymbol{\nu}$-th *conic transform* of M; it depends on μ^X, hence on Φ.

We are interested in clarifying the geometry of $\widehat{M}_{\boldsymbol{\nu}}$ in the toric setting, thus assuming that M be toric, with structure action $\gamma^M : T^d \times M \to M$ and moment polytope $\Delta \subseteq \mathfrak{t}^\vee$.

Let us briefly recall the Delzant construction of M from Δ; obviously with no pretense of exhaustiveness, we refer to [1], [4], and [11] for more complete discussions. Since M is smooth, Δ is a Delzant polytope [4]. We shall denote by $\mathcal{F}(\Delta)$ the collection of all faces of Δ, by $\mathcal{F}_l(\Delta) \subseteq \mathcal{F}(\Delta)$ the subset of codimension-l faces, and specifically by $\mathcal{G}(\Delta) = \mathcal{F}_1(\Delta)$ the subset of facets. If $\mathcal{G}(\Delta) = \{F_1, \ldots, F_k\}$, then for every $j = 1, \ldots, k$ there exist unique

$$\boldsymbol{\upsilon}_j \in L(T^d) := \ker\left(\exp_{T^d}(2\pi\,\cdot)\right) \subset \mathfrak{t}^d = \mathrm{Lie}(T^d), \quad \lambda_j \in \mathbb{R}, \tag{5}$$

with $\boldsymbol{\upsilon}_j$ primitive, such that

$$\Delta = \bigcap_{j=1}^{k} \{\ell \in \mathfrak{t}^\vee : \ell(\boldsymbol{\upsilon}_j) \geq \lambda_j\}, \tag{6}$$

and for every $j = 1, \ldots, k$ the relative interior of F_j (open facet) is

$$F_j^0 = \{\ell \in \mathfrak{t}^\vee : \ell(\boldsymbol{\upsilon}_j) = \lambda_j, \ \ell(\boldsymbol{\upsilon}_{j'}) > \lambda_{j'} \ \forall\, j' \neq j\}. \tag{7}$$

Let us set

$$\omega_0 := \frac{\imath}{2}\sum_{j=1}^{k} \mathrm{d}z_j \wedge \mathrm{d}\overline{z}_j = \sum_{j=1}^{k} \mathrm{d}x_j \wedge \mathrm{d}y_j. \tag{8}$$

Then $(M, 2\omega)$ can be regarded as symplectic reduction of $(\mathbb{C}^k, 2\omega_0)$ under the action of a subtorus $N \leqslant T^k$, as follows. Denote the general element of T^k by $e^{\imath\boldsymbol{\vartheta}} = \left(e^{\imath\vartheta_1}, \ldots, e^{\imath\vartheta_k}\right)$; for $e^{\imath\boldsymbol{\vartheta}} \in T^k$ and $\mathbf{z} = (z_j)_{j=1}^k \in \mathbb{C}^k$, let us set $e^{\imath\boldsymbol{\vartheta}} \bullet \mathbf{z} := \left(e^{\imath\vartheta_j} z_j\right)_{j=1}^k$. For any choice of $\boldsymbol{\lambda} = (\lambda_j)_{j=1}^k \in \mathbb{R}^k$, the action

$$\Gamma^{\mathbb{C}^k} : \left(e^{\imath\boldsymbol{\vartheta}}, \mathbf{z}\right) \in T^k \times \mathbb{C}^k \mapsto e^{-\imath\boldsymbol{\vartheta}} \bullet \mathbf{z} \in \mathbb{C}^k \tag{9}$$

is then Hamiltonian on $\left(\mathbb{C}^k, 2\omega_0\right)$, with moment map

$$\Psi_{\boldsymbol{\lambda}} : \mathbf{z} \in \mathbb{C}^k \mapsto \imath \sum_{j=1}^{k} \left(|z_j|^2 + \lambda_j\right) \mathbf{e}_j^*, \tag{10}$$

where $(\mathbf{e}_j)_{j=1}^k$ is the canonical basis of $\mathbb{R}^k$ and $(\mathbf{e}_j^*)_{j=1}^k$ is the dual basis. The linear map $\mathbb{R}^k \to \mathbb{R}^d$ such that $\mathbf{e}_j \mapsto \mathbf{u}_j$ induces a short exact sequence of tori $0 \to N \to T^k \to T^d \to 0$; hence $\Gamma^{\mathbb{C}^k}$ restricts to a Hamiltonian action of N on $\left(\mathbb{C}^k, 2\omega_0\right)$, with a moment map $\Psi_{\boldsymbol{\lambda}}^N$ naturally induced from $\Psi_{\boldsymbol{\lambda}}$. Given that Δ is Delzant, N acts freely on $Z_\Delta := {\Psi_{\boldsymbol{\lambda}}^N}^{-1}(\mathbf{0})$; then $M = Z_\Delta/N$, with its symplectic structure 2ω, is the Marsden-Weinstein reduction of $\left(\mathbb{C}^k, 2\omega_0\right)$ for the action of N. By the arguments of [5], the standard complex structure J_0 of $\mathbb{C}^k$ descends to a compatible complex structure J on M, whence (M, ω, J) is a Kähler manifold.

Furthermore, Γ descends to a holomorphic and Hamiltonian action of $T^d = T^k/N$ on $(M, 2\omega, J)$, $\gamma^M : T^d \times M \to M$; the moment map $\Psi : M \to \mathfrak{t}^{d\vee}$ is obtained by descending to the quotient the restriction

$$\Psi_{\boldsymbol{\lambda}}|_{Z_\Delta} : Z_\Delta \to \mathfrak{n}^0 \cong \mathfrak{t}^{d\vee}.$$

In addition, if $\boldsymbol{\lambda} \in \mathbb{Z}^k$ this construction can be extended by the arguments of [5] so as to obtain an induced toric positive line bundle (A, h) on M, with curvature $\Theta = -2\imath\,\omega$ (§2.1.2); that (A, h) is toric means that γ^M lifts to a metric preserving line bundle action of T^d on A. Hence by restriction we obtain a contact CR action $\gamma^X : T^d \times X \to X$ lifting γ^M, where $X \subset A^\vee$ is the unit circle bundle.

In addition, we suppose given an effective holomorphic and Hamiltonian action $\mu^M : T^r \times M \to M$ of an r-dimensional compact torus T^r on M, with moment map $\Phi : M \to \mathfrak{t}^{r\vee}$ satisfying BA 1.1 for a certain $\boldsymbol{\nu} \in \mathfrak{t}^{r\vee}$, and commuting with γ^M. Thus μ^M factors through an injective group homomorphism $T^r \to T^k$, hence we may assume without loss of generality that $T^r \leqslant T^d$ and that μ^M is the restriction of γ^M to T^r; therefore, letting $\iota : \mathfrak{t}^r \hookrightarrow \mathfrak{t}^d$ be the Lie algebra inclusion,

$$\Phi = \iota^t \circ \Psi + \boldsymbol{\delta} \tag{11}$$

for some constant $\boldsymbol{\delta} \in \mathfrak{t}^{r\vee}$. Equivalently, given $\tilde{\boldsymbol{\delta}} \in \mathfrak{t}^{d\vee}$ such that $\boldsymbol{\delta} = \iota^t(\tilde{\boldsymbol{\delta}})$,

$$\Phi = \iota^t \circ (\Psi_{\tilde{\boldsymbol{\delta}}}), \qquad \text{where} \qquad \Psi_{\tilde{\boldsymbol{\delta}}} := \Psi + \tilde{\boldsymbol{\delta}}. \tag{12}$$

Let us assume that (μ^M, Φ) lifts to $\mu^X : T^r \times X \to X$ according to the previous procedure. While μ^M is the restriction of γ^M to T^r, μ^X is the restriction of γ^X only if $\boldsymbol{\delta} = \mathbf{0}$ in (11).

If μ^X exists, we can consider the conic transform $\widehat{M}_{\boldsymbol{\nu}}$ with respect to μ^X; as mentioned, $(\widehat{M}_{\boldsymbol{\nu}}, 2\,\widehat{\omega}_{\boldsymbol{\nu}})$ turns out to be a symplectic toric orbifold. Furthermore, its associated marked convex rational simple polytope $(\widehat{\Delta}_{\boldsymbol{\nu}}, \mathbf{s}_{\boldsymbol{\nu}})$ is obtained by applying a suitable 'transform' to Δ (depending on $\boldsymbol{\nu}$).

Since the situation is at its simplest when $r = 1$, we shall describe this case first. Thus $\mu^M : T^1 \times M \to M$ is a Hamiltonian action on $(M, 2\omega)$, with a nowhere vanishing moment map $\Phi : M \to \mathfrak{t}^{1\vee}$; the primitive integral weight $\boldsymbol{\nu} \in \mathfrak{t}^{1\vee}$ is uniquely determined by the condition that $M_{\boldsymbol{\nu}} \neq \emptyset$. Then $M_{\boldsymbol{\nu}} = M$, $X_{\boldsymbol{\nu}} = X$, μ^X is locally free, and $\widehat{M}_{\boldsymbol{\nu}} = X/T^1$ (the quotient is with respect to μ^X).

Let us choose a complementary torus T_c^{d-1} to T^1 in T^d, that is, $T^d \cong T_c^{d-1} \times T^1$. If $\mathfrak{t}_c^{d-1} \leqslant \mathfrak{t}^d$ is the Lie algebra of T_c^{d-1}, the corresponding lattices $L(T_c^{d-1}) \subset \mathfrak{t}_c^{d-1}$ and $L(T^1) \subset \mathfrak{t}^1$ are complementary in $L(T^d)$ (see §2.3 on how $\widehat{\Delta}_{\boldsymbol{\nu}}$ depends on T_c^{d-1}).

Let $\widetilde{\boldsymbol{\nu}} \in L(T^1)$ be the unique primitive lattice vector such that $\boldsymbol{\nu}(\widetilde{\boldsymbol{\nu}}) > 0$; since the weight lattice is the dual lattice to $L(T)$, primitivity implies $\boldsymbol{\nu}(\widetilde{\boldsymbol{\nu}}) = 1$, that is, $\boldsymbol{\nu} = \widetilde{\boldsymbol{\nu}}^* \in L(T^1)^\vee$ is the dual vector to $\widetilde{\boldsymbol{\nu}}$. Then $\boldsymbol{\delta} = \delta\,\boldsymbol{\nu} \in \mathfrak{t}^{1\vee}$, where $\delta = \boldsymbol{\delta}(\widetilde{\boldsymbol{\nu}}) \in \mathbb{Z}$ (notation as in (11)).

With Δ and $\boldsymbol{\upsilon}_j$ as in (6), for each $j = 1, \ldots, k$ there are unique $\boldsymbol{\upsilon}_j' \in \mathfrak{t}_c^{d-1}$ and $\rho_j \in \mathbb{Z}$ such that

$$\boldsymbol{\upsilon}_j = \boldsymbol{\upsilon}_j' + \rho_j \tilde{\boldsymbol{\nu}}. \tag{13}$$

For every $j = 1, \ldots, k$ let us define

$$\widehat{\boldsymbol{\upsilon}}_j := \boldsymbol{\upsilon}'_j - (\lambda_j + \rho_j\,\delta)\,\tilde{\boldsymbol{\nu}} \tag{14}$$

and

$$\widehat{\Delta}_{\boldsymbol{\nu}} := \bigcap_{j=1}^{k} \{\ell \in \mathfrak{t}^\vee : \ell\big(\widehat{\boldsymbol{\upsilon}}_j\big) \geq -\rho_j\}. \tag{15}$$

Thus $\widehat{\Delta}_{\boldsymbol{\nu}}$ is obtained from Δ by replacing each pair (ρ_j, λ_j) by the pair $(-(\lambda_j + \rho_j\,\delta), -\rho_j)$.

We shall see that $(\widehat{M}_{\boldsymbol{\nu}}, 2\,\widehat{\omega}_{\boldsymbol{\nu}})$ is a toric symplectic orbifold, and that $\widehat{\Delta}_{\boldsymbol{\nu}}$ is its moment polytope. To complete the combinatorial description of $(\widehat{M}_{\boldsymbol{\nu}}, 2\,\widehat{\omega}_{\boldsymbol{\nu}})$ following [11], we need to specify the corresponding marking of $\widehat{\Delta}_{\boldsymbol{\nu}}$, that is, the assignment to each of its facets $\widehat{F}_j$ of an appropriate integer $s_j \geq 1$. We shall denote the marking by $\mathbf{s}_{\boldsymbol{\nu}} = (s_j)_{j=1}^k \in \mathbb{N}^k$, and the marked polytope by the pair $(\widehat{\Delta}_{\boldsymbol{\nu}}, \mathbf{s}_{\boldsymbol{\nu}})$.

We premise a further piece of notation. Given a rank-r integral lattice $L \subset V$ in a real vector space, and a basis $(\ell_1, \ldots, \ell_r)$ of L, if $\ell \in L$ we shall denote by (ℓ) the greatest common divisor of the coefficients of ℓ in the given basis, that is,

$$(\ell) := \mathrm{G.C.D.}(\lambda_1, \ldots, \lambda_r) \quad \text{if} \quad \ell = \sum_{j=1}^{r} \lambda_j\,\ell_j. \tag{16}$$

The definition is well-posed, since (ℓ) is independent of the choice of a basis of L. Furthermore, the following holds:

1. ℓ is primitive in L if and only if $(\ell) = 1$;
2. if T is a (real) torus and $\boldsymbol{\xi} \neq \mathbf{0} \in L = L(T)$, then $e^{\vartheta\,\xi} = 1 \in T$ if and only if $e^{\imath\,\vartheta}$ is a $(\boldsymbol{\xi})$-th root of unity.

Let us define $\mathbf{s}_{\boldsymbol{\nu}} = (s_j) \in \mathbb{N}^k$ by setting

$$s_j := \big(\widehat{\boldsymbol{\upsilon}}_j\big) \qquad (j = 1, \ldots, k).$$

Theorem 1.1. *Under the above assumptions, thus with $r = 1$, $\big(\widehat{M}_{\boldsymbol{\nu}}, 2\,\widehat{\omega}_{\boldsymbol{\nu}}\big)$ is the symplectic toric orbifold with associated marked polytope $\big(\widehat{\Delta}_{\boldsymbol{\nu}}, \mathbf{s}_{\boldsymbol{\nu}}\big)$.*

The following consequence generalizes to conic transforms a well-known property of weighted projective spaces [8].

Corollary 1.1. *Under the previous assumptions (thus with $r = 1$),*

$$H^l(M, \mathbb{Q}) \cong H^l\big(\widehat{M}_{\boldsymbol{\nu}}, \mathbb{Q}\big) \qquad (l = 0, 1, \ldots).$$

Let us now consider a general $r \leq d$.

Let $\boldsymbol{\nu}^\perp \leqslant \mathfrak{t}^r$ be the kernel of $\boldsymbol{\nu}$, and $T^{r-1}_{\boldsymbol{\nu}^\perp} \leqslant T^r$ the corresponding subtorus. Under Basic Assumption 1.1, $T^{r-1}_{\boldsymbol{\nu}^\perp}$ acts locally freely on $M_{\boldsymbol{\nu}}$; then $\overline{M}_{\boldsymbol{\nu}} := M_{\boldsymbol{\nu}}/T^{r-1}_{\boldsymbol{\nu}^\perp}$, the Marsden-Weinstein reduction of M with respect $T^{r-1}_{\boldsymbol{\nu}^\perp}$, is a Kähler orbifold. The transversality requirement in Basic Assumption 1.1 can be conveniently reformulated as a transversality condition between $\Delta + \tilde{\boldsymbol{\delta}}$ and $\boldsymbol{\nu}^{\perp 0} \subseteq \mathfrak{t}^{d\vee}$ (the annihilator of $\boldsymbol{\nu}^\perp$), see §3.4. We shall for simplicity require that $T^{r-1}_{\boldsymbol{\nu}^\perp}$ acts freely on $M_{\boldsymbol{\nu}}$, which amounts to $\overline{\Delta}'_{\boldsymbol{\nu}} := (\Delta + \tilde{\boldsymbol{\delta}}) \cap \boldsymbol{\nu}^{\perp 0}$ being a Delzant polytope (see §3.5). Then $\overline{M}_{\boldsymbol{\nu}}$ is naturally a toric Kähler manifold, acted upon by the quotient torus $T^{d-r+1}_q := T^d/T^{r-1}_{\boldsymbol{\nu}^\perp}$; the associated moment polytope $\overline{\Delta}_{\boldsymbol{\nu}}$ can be identified with $\overline{\Delta}'_{\boldsymbol{\nu}}$ under the natural isomorphism between $\mathfrak{t}^{d-r+1}_q$ (the Lie algebra of T^{d-r+1}_q) and $\boldsymbol{\nu}^{\perp 0}$. The general case can then be reduced to the case $r = 1$, with M replaced by $\overline{M}_{\boldsymbol{\nu}}$.

Let us choose:

1. a complementary subtorus $\widehat{T}^1_{\boldsymbol{\nu}} \leqslant T^r$ to $T^{r-1}_{\boldsymbol{\nu}^\perp}$, so that exists a unique primitive $\widetilde{\boldsymbol{\nu}} \in L(\widehat{T}^1_{\boldsymbol{\nu}})$ with $\boldsymbol{\nu}(\widetilde{\boldsymbol{\nu}}) = 1$;
2. a complementary subtorus $T^{d-r}_c \leqslant T^d$ to T^r, with Lie algebra $\mathfrak{t}^{d-r}_c \leqslant \mathfrak{t}^d$, so that

$$T^d \cong T^{d-r}_c \times T^r \cong T^{d-r}_c \times \widehat{T}^1_{\boldsymbol{\nu}} \times T^{r-1}_{\boldsymbol{\nu}^\perp}, \tag{17}$$

$$\mathfrak{t}^d \cong \mathfrak{t}^{d-r}_c \times \widehat{\mathfrak{t}}^1_{\boldsymbol{\nu}} \times \mathfrak{t}^{r-1}_{\boldsymbol{\nu}^\perp}. \tag{18}$$

With notation as in (6) and (7), suppose that the k facets of Δ have been so numbered that $\mathcal{G}_{\boldsymbol{\nu}}(\Delta) := \{F_1, \ldots, F_l\} \subseteq \mathcal{G}(\Delta)$ is the subset of those facets of Δ such that $(F_j + \tilde{\boldsymbol{\delta}}) \cap \boldsymbol{\nu}^{\perp 0} \neq \emptyset$ (it then follows that $(F^0_j + \tilde{\boldsymbol{\delta}}) \cap \boldsymbol{\nu}^{\perp 0} \neq \emptyset$, see §3.1). For every $j = 1, \ldots, l$, let us decompose $\boldsymbol{\upsilon}_j$ according to (18):

$$\boldsymbol{\upsilon}_j = \boldsymbol{\upsilon}'_j + \rho_j\,\widetilde{\boldsymbol{\nu}} + \boldsymbol{\upsilon}''_j, \tag{19}$$

for unique $\boldsymbol{\upsilon}'_j \in L(T_c^{d-r})$, $\rho_j \in \mathbb{Z}$, $\boldsymbol{\upsilon}''_j \in L(T^{r-1}_{\boldsymbol{\nu}^\perp})$. If $\delta_j := \tilde{\boldsymbol{\delta}}(\boldsymbol{\upsilon}_j)$, $\overline{\Delta}_{\boldsymbol{\nu}}$ is canonically identifiable under the natural isomorphism $\mathfrak{t}_q^{d-r+1} \cong \boldsymbol{\nu}^{\perp 0}$ with the Delzant polytope

$$\boldsymbol{\nu}^{\perp 0} \supseteq \overline{\Delta}'_{\boldsymbol{\nu}} := \bigcap_{j=1}^{l} \{\gamma \in \boldsymbol{\nu}^{\perp 0} : \gamma(\boldsymbol{\upsilon}'_j + \rho_j\,\widetilde{\boldsymbol{\nu}}) \ge \lambda_j + \delta_j\}. \tag{20}$$

Let us set

$$\widehat{\boldsymbol{\upsilon}}_j := \boldsymbol{\upsilon}'_j - (\lambda_j + \delta_j)\;\tilde{\boldsymbol{\nu}}, \qquad s_j := \left(\widehat{\boldsymbol{\upsilon}}_j\right) \qquad (j = 1, \ldots, l). \tag{21}$$

$$\widehat{\Delta}'_{\boldsymbol{\nu}} := \bigcap_{j=1}^{l} \{\ell \in \boldsymbol{\nu}^{\perp 0} : \ell\left(\widehat{\boldsymbol{\upsilon}}_j\right) \ge -\rho_j\}. \tag{22}$$

Finally, let $\widehat{\Delta}_{\boldsymbol{\nu}} \subset \mathfrak{t}_q^{d-r+1}$ be the polytope corresponding to $\widehat{\Delta}'_{\boldsymbol{\nu}} \subset \boldsymbol{\nu}^{\perp 0}$, and let $\mathbf{s}_{\boldsymbol{\nu}} := (s_j) \in \mathbb{N}^l$.

Theorem 1.2. *Under Basic Assumption 1.1, suppose in addition that $\Delta + \tilde{\boldsymbol{\delta}}$ and $\boldsymbol{\nu}^{\perp 0}$ are transverse and that the intersection is Delzant. Then $\left(\widehat{M}_{\boldsymbol{\nu}}, 2\,\widehat{\omega}_{\boldsymbol{\nu}}\right)$ is the symplectic toric orbifold with associated marked polytope $\left(\widehat{\Delta}_{\boldsymbol{\nu}}, \mathbf{s}_{\boldsymbol{\nu}}\right)$.*

We have an analogue of Corollary 1.1, linking the cohomology groups of the symplectic reduction $\overline{M}_{\boldsymbol{\nu}}$ and of the conic transform $\widehat{M}_{\boldsymbol{\nu}}$. By the theory of [9], $H^l(\overline{M}_{\boldsymbol{\nu}}, \mathbb{Q})$ is tightly related to the equivariant cohomology of M for the action of $T^{r-1}_{\boldsymbol{\nu}^\perp}$.

Corollary 1.2. *Under the hypothesis of Theorem 1.2,*

$$H^l(\overline{M}_{\boldsymbol{\nu}}, \mathbb{Q}) \cong H^l\left(\widehat{M}_{\boldsymbol{\nu}}, \mathbb{Q}\right) \qquad (l = 0, 1, \ldots).$$

Remark 1.1. The reader may have wondered why, after introducing the Kähler structure ω, we refer the Hamiltonian structures to the form $2\,\omega$ (and similarly, for $\left(\widehat{M}_{\boldsymbol{\nu}}, 2\,\widehat{\omega}_{\boldsymbol{\nu}}\right)$). Needless to say, given a complex orbifold (R, J_R) and a 2-form γ on it, γ is Kähler on (R, J_R) if and only if so $2\,\gamma$. The emphasis on $2\,\omega$ is motivated on the one hand by the normalization $\Theta = -2\,\imath\,\omega$ (and the equivalent formula (1)), which is in line of the general conventions up to an occasional factor of π, and on the other by the formula (2) for the contact lift. In particular, suppose that $f \in \mathcal{C}^\infty(M)$ and $\upsilon_f \in \mathfrak{X}(M)$ is the Hamiltonian vector field of f with respect to $2\,\omega$. Let us lift υ_f to a vector field $\tilde{\upsilon} \in \mathfrak{X}(X)$ on X by the prescription (2), i.e.

$$\tilde{\upsilon} := \upsilon^\sharp - f\,\partial_\theta.$$

Then

$$L_{\tilde{\upsilon}}\alpha = \mathrm{d}\iota\left(\tilde{\upsilon}\right)\,\alpha + \iota\left(\tilde{\upsilon}\right)\,\mathrm{d}\alpha = -\pi^*(\mathrm{d}f) + \pi^*\left(\iota\left(\upsilon\right)\,(2\,\omega)\right) = 0,$$

so that with these choices $\tilde{\upsilon}$ is a contact vector field on (X, α). At then same time, one commonly adopts $\mathrm{d}V_M = \omega^{\wedge d}/d!$ as a volume form on M. Furthermore, when studying equivariant Szegö kernel scaling asymptotics, which gave the initial motivation for the conic transform construction, it is customary to refer the universal ψ_2 Heisenberg invariant in the leading exponent to ω.

Acknowledgements

The author is very grateful to the referee for many useful remarks and suggestions.

The author is a member of GNSAGA (Gruppo Nazionale per le Strutture Algebriche, Geometriche e le loro Applicazioni) of INdAM (Istituto Nazionale di Alta Matematica “Francesco Severi”), and thanks the group for its support.

The present work was partially funded by grant 2018-ATE-0259 of the Università degli Studi di Milano-Bicocca.

2. The case $r = 1$

2.1. Preliminaries

Before embarking on the proof of Theorem 1.1, we need to recall some basic constructions from toric geometry, referring to [1], [4] and [3] for details. We premise a digression on the geometric relation between Δ and $\widehat{\Delta}_{\boldsymbol{\nu}}$.

2.1.1. The transform of a polytope

Although not logically necessary, it is suggestive to describe the passage from Δ to $\widehat{\Delta}_\nu$ in terms of a general 'transform' operation on rational polytopes in a finite-dimensional real vector space with a full-rank lattice L, depending on the datum of a decomposition of L as the product of an oriented rank-1 sublattice and a complementary sublattice.

Let V be a d-dimensional real vector space, $L \subset V$ a full-rank lattice, $V^\vee$ the dual vector space, and $L^\vee$ the dual lattice. Suppose that $\Delta \subset V^\vee$ is d-dimensional rational simple convex polytope (terminology as in [11]). This means that there exist primitive $\mathbf{v}_i \in L$ and $\lambda_i \in \mathbb{R}$, $i = 1, \ldots, k$, such that

$$\Delta = \bigcap_{j=1}^{k} \left\{ \ell \in V^\vee \,:\, \ell(\mathbf{v}_j) \geq \lambda_j \right\}, \tag{23}$$

and that exactly d facets of Δ meet at each of its vertexes. In addition, we shall say that Δ is *integral* if $\lambda_j \in \mathbb{Z}$ for every j.

Suppose given:

1. a primitive lattice vector $\mathbf{v} \neq \mathbf{0} \in L$;
2. $\boldsymbol{\delta} \in \mathrm{span}(\mathbf{v})^\vee$ such that $\boldsymbol{\delta}(\mathbf{v}) \in \mathbb{Z}$ and
$$\ell(\mathbf{v}) + \boldsymbol{\delta}(\mathbf{v}) > 0 \qquad \forall\, \ell \in \Delta; \tag{24}$$
3. a complementary sublattice $L' \subset L$ to $\mathbb{Z} \cdot \mathbf{v}$, so that setting $V' := L' \otimes \mathbb{R}$ we have $V = V' \oplus \mathrm{span}(\mathbf{v})$ and dually $V^\vee = V'^\vee \oplus \mathrm{span}(\mathbf{v}^*)$, where $\mathbf{v}^* \in \mathrm{span}(\mathbf{v})^\vee$ is dual to $\mathbf{v}$.

Then we may uniquely extend $\boldsymbol{\delta}$ to $\tilde{\boldsymbol{\delta}} \in L^\vee \cap \mathrm{span}(\mathbf{v}^*) \subseteq V^\vee$ so that $\tilde{\boldsymbol{\delta}} = \delta\, \mathbf{v}^*$ with $\delta \in \mathbb{Z}$ (a different choice of $\tilde{\boldsymbol{\delta}}$ would result in a translation of the transformed polytope). By (24), $\Delta + \tilde{\boldsymbol{\delta}}$ lies in the open half-space $V^\vee_+ \subset V^\vee$ where pairing with $\mathbf{v}$ is positive.

Any $\ell \in V^\vee_+$ can be written uniquely as $\ell = \ell' + \ell(\mathbf{v})\, \mathbf{v}^*$, where $\ell' \in V'^\vee$ and $\ell(\mathbf{v}) > 0$. Let us define an involution $\varrho : V^\vee_+ \to V^\vee_+$ by setting

$$\varrho(\ell) := \frac{1}{\ell(\mathbf{v})}\, \ell' + \frac{1}{\ell(\mathbf{v})}\, \mathbf{v}^*. \tag{25}$$

Let us determine $\rho(\Delta + \tilde{\boldsymbol{\delta}})$. For each j, we can write uniquely $\mathbf{v}_j = \mathbf{v}'_j + \rho_j\, \mathbf{v}$ where $\mathbf{v}'_j \in L'$ and $\rho_j \in \mathbb{Z}$. Hence $\tilde{\boldsymbol{\delta}}(\mathbf{v}_j) = \delta\, \rho_j$. We have

$$\Delta + \tilde{\boldsymbol{\delta}} = \bigcap_{j=1}^{k} \left\{ \ell \in V^\vee \,:\, \ell(\mathbf{v}_j) = \ell'(\mathbf{v}'_j) + \rho_j\, \ell(\mathbf{v}) \geq \lambda_j + \delta\, \rho_j \right\}. \tag{26}$$

Since $\varrho = \varrho^{-1}$, by (25) and (26) we have

$$\begin{aligned} \widehat{\Delta} := \varrho(\Delta + \tilde{\boldsymbol{\delta}}) &= \bigcap_{j=1}^{k} \left\{ \ell \in V^\vee_+ \,:\, \varrho(\ell)(\mathbf{v}_j) = \frac{1}{\ell(\mathbf{v})} \left[\ell'(\mathbf{v}'_j) + \rho_j \right] \geq \lambda_j + \delta\, \rho_j \right\} \\ &= \bigcap_{j=1}^{k} \left\{ \ell \in V^\vee_+ \,:\, \ell'(\mathbf{v}'_j) - (\lambda_j + \delta\, \rho_j)\, \ell(\mathbf{v}) \geq -\rho_j \right\}. \end{aligned} \tag{27}$$

Thus $\widehat{\Delta}$ is the convex polytope obtained from Δ by replacing each primitive normal vector $\mathbf{v}_j = \mathbf{v}'_j + \rho_j\, \mathbf{v}$ with the integral vector $\widehat{\mathbf{v}}_j := \mathbf{v}'_j - (\lambda_j + \delta\, \rho_j)\, \mathbf{v}$, and each λ_j with $-\rho_j$. Clearly, $\widehat{\Delta}$ is rational; it is not claimed that each $\widehat{\mathbf{v}}_j$ be primitive, hence neither that $\hat{\Delta}$ be integral.

Furthermore, (27) shows that, if F_j is the facet of Δ normal to $\mathbf{v}_j$, then $\widehat{F}_j := \rho(F_j + \boldsymbol{\delta})$ is the facet of $\widehat{\Delta}$ normal to $\widehat{\mathbf{v}}_j$; this correspondence passes to intersection of facets, i.e. faces. Thus we have a bijection between the set of faces of each given dimension of Δ and $\widehat{\Delta}$, hence in particular between the families of their respective vertexes. In particular, the vertexes of $\widehat{\Delta}$ are the images by ϱ of the vertexes of Δ, and furthermore $\widehat{\Delta}$ is simple since so is Δ.

To make the construction more explicit, let us work in coordinates and take $L = \mathbb{Z}^d$ and $V = \mathbb{R}^d$; let $\mathcal{C}_d = (\boldsymbol{\epsilon}_1, \ldots, \boldsymbol{\epsilon}_d)$ denote the canonical basis, and choose $\mathbf{v} = \mathbf{e}_1$, $L' := \mathrm{span}(\boldsymbol{\epsilon}_2, \ldots, \boldsymbol{\epsilon}_d)$. We shall conveniently identify $V^\vee$ with $\mathbb{R}^d$ by means of the dual basis $\mathcal{C}^*_d = \left(\boldsymbol{\epsilon}^*_1, \ldots, \boldsymbol{\epsilon}^*_d\right)$. Then

$$V^\vee_+ = \left\{ (x_1, \mathbf{x}') \in \mathbb{R}^d = \mathbb{R} \times \mathbb{R}^{d-1} \,:\, x_1 > 0 \right\},$$

where $\mathbf{x}' = (x_2, \ldots, x_d)$ and $(x_1, \mathbf{x}')$ corresponds to $\sum_{j=1}^{d} x_j\, \mathbf{e}^*_j$. Then $\varrho : V^\vee_+ \to V^\vee_+$ in (25) is the involution given by

$$\varrho(x_1,\mathbf{x}')=\left(\frac{1}{x_1},\frac{1}{x_1}\mathbf{x}'\right).$$

In particular, ϱ is an inversion along the x_1-axis, and a positive dilation in $\{0\}\times\mathbb{R}^{d-1}$. If $a,c\in\mathbb{R}$ and $\mathbf{B}=(b_2,\ldots,b_d)\in\mathbb{R}^{d-1}$, let us consider the half-space $W\subset V$ given by

$$W=\left\{(x_1,\mathbf{x}')\in\mathbb{R}^d=\mathbb{R}\times\mathbb{R}^{d-1}\,:a\,x_1+\langle\mathbf{B},\mathbf{x}'\rangle\geq c\right\}.$$

Then

$$\varrho(W\cap V_+^\vee)=\left\{(x_1,\mathbf{x}')\in V_+^\vee\,:-c\,x_1+\langle\mathbf{B},\mathbf{x}'\rangle\geq -a\right\}$$

is again of the form $\tilde W\cap V_+^\vee$ for a new half-space $\tilde W$; it follows that ϱ transforms convex polytopes in $V_+^\vee$ in other convex polytopes in $V_+^\vee$. By the Fundamental Theorem of convex polytopes, therefore, if $\Delta\subset V_+^\vee$ is the (bounded and convex) polytope given by the convex hull of point $\mathbf{p}_1,\ldots,\mathbf{p}_r\in V_+^\vee$, then $\varrho(\Delta)\subset V_+^\vee$ is the convex hull of $\varrho(\mathbf{p}_1),\ldots,\varrho(\mathbf{p}_r)\in V_+^\vee$.

For example, let us take $M=\mathbb{P}^2$ with the Fubini-Study form; thus A is the hyperplane line bundle on $\mathbb{P}^2$ with the standard metric, $A^\vee$ is the tautological line bundle, and $X=S^3$ (the unit sphere in $\mathbb{C}^2$). Let us consider the toric action $\gamma^{\mathbb{P}^2}:T^2\times\mathbb{P}^2\to\mathbb{P}^2$ given by

$$\gamma^{\mathbb{P}^2}_{e^{\imath\boldsymbol{\vartheta}}}\left([z_0:z_1:z_2]\right):=\left[z_0:e^{-\imath\,\vartheta_1}\,z_1:e^{-\imath\,\vartheta_2}\,z_2\right]\quad\text{where}\quad e^{\imath\,\boldsymbol{\vartheta}}=\left(e^{\imath\,\vartheta_1},e^{\imath\,\vartheta_2}\right).$$

We shall identify $\mathfrak{t}\cong\mathfrak{t}^\vee\cong\mathbb{R}^2$ in the standard manner, and denote by $\mathcal{C}_2=(\mathbf{e}_1,\mathbf{e}_2)$ the canonical basis, with dual basis $\mathcal{C}_2^*=(\mathbf{e}_1^*,\mathbf{e}_2^*)$. A (normalized) moment map for $\gamma^{\mathbb{P}^2}$ can be taken to be

$$\Psi:[Z]\in\mathbb{P}^2\mapsto\left(\frac{|z_1|^2}{\|Z\|^2},\frac{|z_2|^2}{\|Z\|^2}\right)\in\mathbb{R}^2,$$

where $Z=(z_0,z_1,z_2)$, $[Z]=[z_0:z_1:z_2]$. Then $\Delta=\Psi(\mathbb{P}^2)$ is the triangle with vertexes $(0,0)$, $(1,0)$, $(0,1)$.

Let us consider the action $\mu^{\mathbb{P}^2}:T^1\times\mathbb{P}^2\to\mathbb{P}^2$ given by

$$\mu^{\mathbb{P}^2}_{e^{\imath\,\theta}}([Z]):=\left[z_0:e^{-\imath\,\theta}\,z_1:z_2\right],$$

with (normalized) everywhere positive moment map

$$\Phi^{\mathbb{P}^2}:[Z]\in\mathbb{P}^2\mapsto\frac{|z_1|^2}{\|Z\|^2}+1=\frac{|z_0|^2+2\,|z_1|^2+|z_2|^2}{\|Z\|^2}\in\mathbb{R}.$$

The linearization corresponding to $\Phi^{\mathbb{P}^2}$ yields the locally free action of T^1 on S^3 given by

$$\mu^{S^2}_{e^{\imath\,\theta}}(Z):=\left(e^{-\imath\,\theta}\,z_0,e^{-2\,\imath\,\theta}\,z_1,\,e^{-\imath\,\theta}\,z_2\right).$$

Then $\hat M=S^3/T^1$ (since $r=1$, there is no ambiguity in writing $\hat M=\hat M_{\boldsymbol{\nu}}$).

Thus $\left(\mu^{\mathbb{P}^2},\Phi^{\mathbb{P}^2}\right)$ is the Hamiltonian action obtained by restricting $\gamma^{\mathbb{P}^2}$ to the subgroup $T^1\cong T^1\times\{1\}$ of T^2 (with the injection $e^{\imath\,\theta}\mapsto\left(e^{\imath\,\theta},1\right)$), and by taking as moment map $\Psi_{\mathbf{e}_1^*}:=\Psi+\mathbf{e}_1^*$ (that is, $\tilde{\boldsymbol{\delta}}=\mathbf{e}_1^*$ in (12)). Hence,

$$\Psi_{\mathbf{e}_1^*}(\mathbb{P}^2)=\Delta+\mathbf{e}_1^*\subset V_+^\vee$$

is the triangle with vertexes at $(1,0)$, $(2,0)$, $(1,1)$. Thus, $\hat\Delta_{\mathbf{e}_1^*}=\varrho(\Delta+\mathbf{e}_1^*)$ is the triangle with vertexes at $\varrho(1,0)=(1,0)$, $\varrho(1,1)=(1,1)$, $\varrho(2,0)=\left(\frac{1}{2},0\right)$.

To determine the marking $\mathbf{s}$, we need to compute the normal vectors $\widehat{\boldsymbol{\upsilon}}_j$ in (14). In the notation (13), we have

$$\tilde{\boldsymbol{\nu}}=\mathbf{e}_1=(1,0),\quad\delta=1\quad\text{and}\quad\mathfrak{t}_c^1=\mathrm{span}(\mathbf{e}_2).$$

Furthermore,

$$\Delta=\{\langle\ell,\mathbf{e}_1\rangle\geq 0\}\cap\{\langle\ell,\mathbf{e}_2\rangle\geq 0\}\cap\{\langle\ell,-\mathbf{e}_1-\mathbf{e}_2\rangle\geq -1\}.$$

Hence we may take

$$\begin{array}{llll}\boldsymbol{\upsilon}_1=\mathbf{e}_1 & \boldsymbol{\upsilon}_1'=\mathbf{0} & \rho_1=1 & \lambda_1=0\\ \boldsymbol{\upsilon}_2=\mathbf{e}_2 & \boldsymbol{\upsilon}_2'=\mathbf{e}_2 & \rho_2=0 & \lambda_2=0\\ \boldsymbol{\upsilon}_3=-\mathbf{e}_1-\mathbf{e}_2 & \boldsymbol{\upsilon}_3'=-\mathbf{e}_2 & \rho_3=-1 & \lambda_3=-1.\end{array}$$

Thus applying (14) we have

$$\widehat{\boldsymbol{\upsilon}}_1 = -(0+1\cdot 1)\,\mathbf{e}_1 = (-1,0)$$
$$\widehat{\boldsymbol{\upsilon}}_2 = -(0+1\cdot 0)\,\mathbf{e}_1 + \mathbf{e}_2 = (0,1)$$
$$\widehat{\boldsymbol{\upsilon}}_2 = -(-1+1\cdot(-1))\,\mathbf{e}_1 - \mathbf{e}_2 = 2\,\mathbf{e}_1 - \mathbf{e}_2 = (2,-1).$$

Thus $s_j = 1$ for $j = 1, 2, 3$.

2.1.2. The toric line bundle A and its circle bundle

Let us review the construction of the positive toric line bundle (A, h) on M from the Delzant polytope $\Delta_{\boldsymbol{\lambda}}$, for $\boldsymbol{\lambda} \in \mathbb{Z}^k$, based on pairing the Delzant construction of M as a symplectic quotient of $\mathbb{C}^k$ with the construction of a polarization on the quotient in [5]. Consider the trivial line bundle $L := \mathbb{C}^k \times \mathbb{C}$, and define a Hermitian metric κ on L by setting

$$\kappa_{\mathbf{z}}\big((\mathbf{z}, w),\ (\mathbf{z}, v)\big) := w\,\overline{v}\,e^{-\|\mathbf{z}\|^2} \quad (\mathbf{z} \in \mathbb{C}^k,\ w, v \in \mathbb{C}).$$

The unit circle bundle $Y \subset L^\vee$ (that is, in L with the dual metric) is then

$$\begin{aligned} Y &:= \left\{(\mathbf{z}, w) \in \mathbb{C}^k \times \mathbb{C} \,:\, |w| = e^{-\frac{1}{2}\|\mathbf{z}\|^2}\right\} \\ &= \left\{\left(\mathbf{z}, e^{-\frac{1}{2}\|\mathbf{z}\|^2}\,e^{\imath\,\theta}\right) \,:\, \mathbf{z} \in \mathbb{C}^k,\ e^{\imath\,\theta} \in S^1\right\} \cong \mathbb{C}^k \times S^1. \end{aligned} \tag{28}$$

In the following we shall implicitly identify Y and $\mathbb{C}^k \times S^1$. In terms of the previous diffeomorphism, the unique compatible connection 1-form is

$$\beta := \frac{\imath}{2}\sum_{j=1}^{k}\left[z_j\,\mathrm{d}\overline{z}_j - \overline{z}_j\,\mathrm{d}z_j\right] + \mathrm{d}\theta. \tag{29}$$

Thus, $\beta(\partial_\theta) = 1$ and $\ker(\beta) \subset TY$ is the horizontal subspace.

If $f : \mathbb{C}^k \to \mathbb{C}$ is $\mathcal{C}^\infty$, the corresponding section σ_f of L has pointwise norm

$$\|\sigma_f(\mathbf{z})\|_\kappa = |f(\mathbf{z})|\,e^{-\frac{1}{2}\|\mathbf{z}\|^2}.$$

Applying this with $f = 1$ we obtain that, letting Θ_0 be the curvature of the unique compatible connection on L,

$$\Theta_0 = \overline{\partial}\,\partial\left(-\|\mathbf{z}\|^2\right) = \sum_{j=1}^{k}\mathrm{d}z_j \wedge \mathrm{d}\overline{z}_j = -2\,\imath\,\omega_0,$$

where $\omega_0 = (\imath/2)\sum_{j=1}^{k}\mathrm{d}z_j \wedge \mathrm{d}\overline{z}_j$ is the standard symplectic form on $\mathbb{C}^k$.

Given that $\boldsymbol{\lambda} \in \mathbb{Z}^k$, the Hamiltonian action $(\Gamma^{\mathbb{C}^k}, \Psi_{\boldsymbol{\lambda}})$ (see (9) and (10)) has the contact CR lift $\Gamma^Y_{\boldsymbol{\lambda}} : T^k \times Y \to Y$ given by

$$\Gamma^Y_{\boldsymbol{\lambda}} : \left(e^{\imath\,\boldsymbol{\vartheta}}, (\mathbf{z}, w)\right) \mapsto \left(\Gamma^{\mathbb{C}^k}(e^{\imath\,\boldsymbol{\vartheta}}, \mathbf{z}), e^{-\imath\,\langle \boldsymbol{\lambda}, \boldsymbol{\vartheta}\rangle}\,w\right) = \left(e^{-\imath\,\boldsymbol{\vartheta}} \cdot \mathbf{z}, e^{-\imath\,\langle \boldsymbol{\lambda}, \boldsymbol{\vartheta}\rangle}\,w\right). \tag{30}$$

This is the restriction a similarly defined metric preserving linearization $\Gamma^{L^\vee}_{\boldsymbol{\lambda}} : T^k \times L^\vee \to L^\vee$; dually, we also have a linearization $\Gamma^L_{\boldsymbol{\lambda}} : T^k \times L \to L$.

As in [5], we can take the quotient and obtain a positive line bundle (A, h) on $M = Z_\Delta/N$, by setting

$$A := L|_{Z_\Delta}/N = (Z_\Delta \times \mathbb{C})/N, \tag{31}$$

with associated unit circle bundle $X \subset A^\vee$ given by

$$X = Y|_{Z_\Delta}/N \cong (Z_\Delta \times S^1)/N. \tag{32}$$

2.1.3. The complexification $\widetilde{N}$ and its stable locus

Besides representing M as a Marsden-Weinstein reduction for the quotient of N, it is useful to consider its parallel description as a GIT quotient for the action of the complexification $\widetilde{N}$ ([4], [3]). In the following, for every compact group T, $\widetilde{T}$ is the complexification of T.

Every face $F \in \mathcal{F}(\Delta)$ of codimension c_F of Δ is uniquely an intersection of facets; hence there exists a unique increasing multi-index $I_F := \{i_1(F), \ldots, i_{c_F}(F)\} \subset \{1, \ldots, k\}^{c_F}$ such that $F = \bigcap_{j=i_1(F)}^{i_{c_F}(F)} F_{i_j(F)}$. Let us set

$$\mathbb{O}_F := \left\{\mathbf{z} = (z_j) \in \mathbb{C}^k \,:\, z_j = 0 \Leftrightarrow j \in I_F\right\} \qquad (F \in \mathcal{F}(\Delta)). \tag{33}$$

The following holds:

1. $\mathbb{O}_F$ is an orbit of $\tilde{T}^k$, and $\mathbb{O}_F \cong \mathbb{C}^{*k-c_F}$ equivariantly;
2. the stabilizer in T^k of every $\mathbf{z} \in \mathbb{O}_F$ is the subtorus T^k_F with Lie algebra

$$\mathfrak{t}^k_F := \imath\, \mathrm{span}_{\mathbb{R}}\{\mathbf{e}_j \,:\, j \in I_F\},$$

   and the corresponding statement holds for the stabilizer in the complexification, $\widetilde{T}^k_F \leqslant \tilde{T}^k$;
3. $\mathbb{C}_\Delta := \bigcup_{F\in\mathcal{F}(\Delta)} \mathbb{O}_F$ is the open subset of stable points for the action of $\widetilde{T}^k$ on $\mathbb{C}^k$ with the given linearization or, equivalently, the $\widetilde{T}^k$-saturation of Z_Δ;
4. $\widetilde{N}$ acts freely and properly on $\mathbb{C}_\Delta$;
5. $M = \mathbb{C}_\Delta/\widetilde{N}$ as a complex manifold;
6. for every $F \in \mathcal{F}(\Delta)$, $M^0_F := \mathbb{O}_F/\widetilde{N}$ is a $\widetilde{T}^d$-orbit and a complex submanifold of M of codimension c_F;
7. $M^0_F = \Psi^{-1}(F^0)$, where F^0 is the interior of F (recall that $\Psi : M \to \mathfrak{t}^{d\vee}$ is the moment map).

We have the following ([1], [4], [3]).

Lemma 2.1. *For any $F \in \mathcal{F}(\Delta)$, let $T^d_F \leqslant T^d$ be the subtorus with Lie subalgebra*

$$\mathfrak{t}^d_F := \mathrm{span}_{\mathbb{R}}\{\mathbf{u}_j \,:\, j \in I_J\}.$$

Then

1. *the isomorphism $T^k/N \cong T^d$ induces an isomorphism $\rho_F : T^k_F \cong T^d_F$;*
2. *T^d_F is the stabilizer in T^k of every $m \in M^0_F$.*
3. *$M_F := \Psi^{-1}(F)$ is the complex submanifold of fixed points of T^d_F;*
4. *M^0_F is the dense open subset of M_F of those points whose stabilizer is exactly T^d_F.*

Similar statements hold in the complexifications.

Let us denote by $P : Z_\Delta \to M$ and by $\tilde{P} : \mathbb{C}_\Delta \to M$ the projections. Then $\mathbb{C}_\Delta = \tilde{T}^k \cdot Z_\Delta$ and $P = \tilde{P}|_{Z_\Delta}$.

2.1.4. The lifted action of T^d on X

By passing to the quotient, $\Gamma^{\mathbb{C}^d}$ and $\Gamma^Y_{\boldsymbol{\lambda}}$ determine corresponding actions

$$\gamma^M : T^d \times M \to M, \quad \gamma^X : T^d \times X \to X;$$

given that $\Gamma^Y_{\boldsymbol{\lambda}}$ is the contact and CR lift of $(\Gamma^{\mathbb{C}^k}, \Psi_{\boldsymbol{\lambda}})$, γ^X is the contact and CR lift of (γ^M, Ψ).

Given $m \in M^0_F$, $T^d_F \leqslant T^d$ acts on $X_m = \pi^{-1}(m) \subset X$ by a character that we now specify. Let us choose $\mathbf{z} \in P^{-1}(m) \subset \mathbb{O}_F \cap Z_\Delta$. Since N acts freely on Z_Δ, the projection $L|_{Z_\Delta} \to A$ restricts to an isomorphism $L_{\mathbf{z}} \cong A_m$, which is equivariant with respect to the isomorphism $\rho_F : T^k_F \cong T^d_F$ in Lemma 2.1.

If $\imath\, \boldsymbol{\vartheta} \in \mathfrak{t}^k_F$ and $(\mathbf{z}, w) \in Y_{\mathbf{z}}$, then by (30)

$$\Gamma^Y_{\boldsymbol{\lambda}}\left(e^{\imath\,\boldsymbol{\vartheta}}, (\mathbf{z}, w)\right) = \left(\mathbf{z}, e^{-\imath\,\langle \boldsymbol{\lambda}, \boldsymbol{\vartheta}\rangle}\, w\right). \tag{34}$$

Since $\rho_F(e^{\imath\,\mathbf{e}_j}) = e^{\boldsymbol{\upsilon}_j}$ for $j \in I_F$, for $x = (m, \ell) \in X_m$ we have

$$\begin{aligned}\gamma^X\left(e^{\sum_{j\in I_F} \vartheta_j\, \boldsymbol{\upsilon}_j}, (m, \ell)\right) &= \left(m, e^{-\imath\,\langle \boldsymbol{\lambda}, \boldsymbol{\vartheta}\rangle}\, \ell\right) \\ &= e^{-\imath\,\langle \boldsymbol{\lambda}, \boldsymbol{\vartheta}\rangle}\, x = \chi_F\left(e^{\sum_{j\in I_F} \vartheta_j\, \boldsymbol{\upsilon}_j}\right)^{-1} x,\end{aligned} \tag{35}$$

where

$$\chi_F : e^{\sum_{j\in I_F} \vartheta_j\, \boldsymbol{\upsilon}_j} \in T^d_F \mapsto e^{\imath\, \sum_{j\in I_F} \vartheta_j\, \lambda_j} \in S^1. \tag{36}$$

We can reformulate this as follows.

Lemma 2.2. *Suppose that F is a face of Δ, that $m \in M^0_F$, and that $x \in X_m$. Then for every $t \in T^d_m$ we have $\gamma^X_t(x) = \rho^X_{\chi_F(t)}(x)$.*

2.1.5. The lifted action of T^1 on X

We have remarked that $\mu^M : T^1 \times M \to M$ is the restriction of $\gamma^M : T^d \times M \to M$ to T^1 while, on the other hand, $\mu^X : T^1 \times X \to X$ won't be the restriction of $\gamma^X : T^d \times X \to X$ to T^1, unless $\boldsymbol{\delta} = \mathbf{0} \in \mathfrak{t}^{1\vee}$ in (11). Since however both μ^X and the restriction of γ^X to T^1 lift μ^M, there is a character $\chi = \chi_{\boldsymbol{\delta}} : T^1 \to S^1$ such that

$$\mu^X_h(x) = \gamma^X_h \circ \rho^X_{\chi(h)}(x) \quad (x \in X,\, h \in T^1). \tag{37}$$

Let us make χ explicit. Since $\widetilde{\boldsymbol{\nu}}$ is primitive, the map $e^{\imath\,\vartheta} \in S^1 \mapsto e^{\vartheta\,\widetilde{\boldsymbol{\nu}}} \in T^1$ is an isomorphism of Lie groups.

Lemma 2.3. *We have $\delta \in \mathbb{Z}$ and $\chi(e^{\vartheta\,\widetilde{\boldsymbol{\nu}}}) = e^{\imath\,\delta\,\vartheta}$.*

Proof. Recall that, by choice of $\widetilde{\boldsymbol{\nu}}$, $\boldsymbol{\nu} = \widetilde{\boldsymbol{\nu}}^* \in \mathfrak{t}^{1\vee}$ is the dual basis to $\widetilde{\boldsymbol{\nu}}$, and so $\boldsymbol{\delta} = \delta\,\boldsymbol{\nu}$, where $\delta = \boldsymbol{\delta}(\widetilde{\boldsymbol{\nu}})$. Let us write $\widetilde{\boldsymbol{\nu}}^{\Phi}_X$ and $\widetilde{\boldsymbol{\nu}}^{\Psi}_X$ for the vector field on X induced by $\widetilde{\boldsymbol{\nu}}$ under μ^X and γ^X, respectively. In view of (2), we obtain

$$\begin{aligned} \widetilde{\boldsymbol{\nu}}^{\Phi}_X &= \widetilde{\boldsymbol{\nu}}^{\sharp}_M - \langle \Phi, \widetilde{\boldsymbol{\nu}}\rangle\,\partial_\theta \\ &= \widetilde{\boldsymbol{\nu}}^{\sharp}_M - \langle \Psi, \widetilde{\boldsymbol{\nu}}\rangle\,\partial_\theta - \delta\,\partial_\theta = \widetilde{\boldsymbol{\nu}}^{\Psi}_X - \delta\,\partial_\theta. \end{aligned} \tag{38}$$

Hence for every $x \in X$ and $e^{\vartheta\,\boldsymbol{\nu}} \in T^1$

$$\mu^X_{e^{\vartheta\,\widetilde{\boldsymbol{\nu}}}}(x) = e^{-\imath\,\delta\,\vartheta}\,\gamma^X_{e^{\vartheta\,\widetilde{\boldsymbol{\nu}}}}(x) = \gamma^X_{e^{\vartheta\,\widetilde{\boldsymbol{\nu}}}} \circ \rho^X_{e^{\imath\,\delta\,\vartheta}}(x). \tag{39}$$

Given that $\widetilde{\boldsymbol{\nu}} \in L(T^1)$, (39) implies $\rho^X_{e^{2\pi\,\imath\,\delta}} = \mathrm{id}_X$. Since ρ^X is free, this implies $\delta \in \mathbb{Z}$. Since (39) holds for any ϑ, the second claim follows as well. □

2.1.6. $\widehat{M}_{\boldsymbol{\nu}}$ and its Kähler structure

By assumption, $\mu^X : T^1 \times X \to X$ lifts $\mu^M : T^1 \times M \to M$. Let us set $\Phi^{\tilde{\boldsymbol{\nu}}} := \langle \Phi, \tilde{\boldsymbol{\nu}}\rangle$; as $\Phi^{\tilde{\boldsymbol{\nu}}} > 0$, μ^X is locally free by (2). Furthermore, since μ^M is holomorphic, μ^X preserves the CR structure of X. Hence the quotient $\widehat{M}_{\boldsymbol{\nu}} := X/\mu^X$ is a d-dimensional complex orbifold with complex structure $\widehat{J}_{\boldsymbol{\nu}}$ ([14], [16]). Furthermore, μ^X is effective, hence generically free; therefore the projection $\widehat{\pi}_{\boldsymbol{\nu}} : X \to \widehat{M}_{\boldsymbol{\nu}}$ is a principal V-bundle with structure group T^1.

We shall now see that $(\widehat{M}_{\boldsymbol{\nu}}, \widehat{J}_{\boldsymbol{\nu}})$ carries a Kähler structure $\widehat{\omega}_{\boldsymbol{\nu}}$, naturally induced from ω. Aside from slight changes in notation, the discussion is close to the ones in §2 of [14] and §5.3 of [16], so we'll be rather sketchy. To lighten notation, we shall adopt the following conventions.

1. if $(m, x, \widehat{m}) \in M \times X \times \widehat{M}_{\boldsymbol{\nu}}$, we shall write $m \leftarrow x \to \widehat{m}$ to mean $\pi(x) = m$ and $\widehat{\pi}_{\boldsymbol{\nu}}(x) = \widehat{m}$;
2. if $(m, \widehat{m}) \in M \times \widehat{M}_{\boldsymbol{\nu}}$, we shall write $m \sim \widehat{m}$ to mean that $m \leftarrow x \to \widehat{m}$ for some $x \in X$;
3. if $U \subseteq M$, we shall set $\widehat{U} := \widehat{\pi}_{\boldsymbol{\nu}}\left(\pi^{-1}(U)\right)$;
4. we shall generally omit symbols of pull-backs for functions, and denote by the same symbol a function $f : M \to \mathbb{C}$ and its pull-back $\pi^*(f) : X \to \mathbb{C}$;
5. similarly, if f is invariant and hence $\pi^*(f)$ descends to $\widehat{M}_{\boldsymbol{\nu}}$, we shall also denote by $f : \widehat{M}_{\boldsymbol{\nu}} \to \mathbb{C}$ the descended function.

Let the invariant differential 1-form $\widehat{\alpha}_{\boldsymbol{\nu}} \in \Omega^1(X)$ be defined by

$$\widehat{\alpha}_{\boldsymbol{\nu}} := \frac{1}{\Phi^{\tilde{\boldsymbol{\nu}}}}\,\alpha. \tag{40}$$

Then $\iota(\tilde{\boldsymbol{\nu}}_X)\,\widehat{\alpha}_{\boldsymbol{\nu}} = -1$, hence $\widehat{\alpha}_{\boldsymbol{\nu}}$ is a connection 1-form for $\widehat{\pi}_{\boldsymbol{\nu}}$. Hence there is a unique orbifold 2-form $\widehat{\omega}_{\boldsymbol{\nu}}$ on $\widehat{M}_{\boldsymbol{\nu}}$ such that $\mathrm{d}\widehat{\alpha}_{\boldsymbol{\nu}} = 2\,\widehat{\pi}^*_{\boldsymbol{\nu}}(\widehat{\omega}_{\boldsymbol{\nu}})$. Since by (40) $\ker(\alpha) = \ker(\widehat{\alpha}_{\boldsymbol{\nu}})$, π and $\widehat{\pi}_{\boldsymbol{\nu}}$ share the same horizontal bundle, i.e., $\mathcal{H}_x(\pi) = \mathcal{H}_x(\widehat{\pi}_{\boldsymbol{\nu}})$ for every $x \in X$. On the other hand, since $\Phi^{\tilde{\boldsymbol{\nu}}} > 0$ by (2) we have $\tilde{\boldsymbol{\nu}}_X(x) \notin \mathcal{H}(\pi)_x$ at every $x \in X$. Hence we can split the tangent bundle TX of X in the two alternative ways:

$$TX = \mathcal{H}(\pi) \oplus \mathrm{span}(\partial_\theta) = \mathcal{H}_x(\widehat{\pi}_{\boldsymbol{\nu}}) \oplus \mathrm{span}(\tilde{\boldsymbol{\nu}}_X). \tag{41}$$

In particular, if $m \leftarrow x \to \widehat{m}$ then there are complex linear isomorphisms

$$T_m M \cong \mathcal{H}_x(\pi) \cong T_{\widehat{m}}(\widehat{M}_{\boldsymbol{\nu}}), \tag{42}$$

where the latter denotes the uniformizing tangent space of $\widehat{M}_{\boldsymbol{\nu}}$ at $\tilde{m}$. Since

$$2\,\widehat{\pi}^*_{\boldsymbol{\nu}}(\widehat{\omega}_{\boldsymbol{\nu}}) = \mathrm{d}\widehat{\alpha}_{\boldsymbol{\nu}} = \frac{1}{\Phi^{\tilde{\boldsymbol{\nu}}}}\,2\,\pi^*(\omega) - \frac{1}{\Phi^{\tilde{\boldsymbol{\nu}}\,2}}\,\mathrm{d}\Phi^{\tilde{\boldsymbol{\nu}}} \wedge \alpha,$$

the triple $\left(T_{\widehat{m}}(\widehat{M}_{\boldsymbol{\nu}}), \widehat{J}_{\boldsymbol{\nu}}, \widehat{\omega}_{\boldsymbol{\nu}}\right)$ is isomorphic to $(T_m M,\, J_m,\, \omega_m/\Phi^{\tilde{\boldsymbol{\nu}}})$, so that $\widehat{\omega}_{\boldsymbol{\nu}}$ is a Kähler form on $\widehat{M}_{\boldsymbol{\nu}}$.

2.1.7. Horizontal and contact lifts with respect to $\widehat{\pi}_{\boldsymbol{\nu}}$

Since $\tilde{\boldsymbol{\nu}}$ is primitive, the map $e^{\imath\vartheta} \in S^1 \mapsto e^{\imath\vartheta\tilde{\boldsymbol{\nu}}} \in T^1$ is an isomorphism of Lie groups. Composing the latter with the effective action μ^X, we obtain an effective action of S^1 on X, which is free on a dense invariant subset. Therefore, there exists a dense (and smooth) open subset $\widehat{M}'_{\boldsymbol{\nu}} \subseteq \widehat{M}_{\boldsymbol{\nu}}$ over which $\widehat{\pi}_{\boldsymbol{\nu}}$ restricts to principal S^1-bundle. Let us set $X' := \widehat{\pi}_{\boldsymbol{\nu}}^{-1}(\widehat{M}'_{\boldsymbol{\nu}})$.

Given a smooth orbifold vector field $\boldsymbol{\upsilon}$ on $\widehat{M}_{\boldsymbol{\nu}}$, we shall say that a (smooth) vector field on X is the horizontal lift of $\boldsymbol{\upsilon}$ (with respect to $\widehat{\pi}_{\boldsymbol{\nu}}$) if it is horizontal (i.e., tangent to $\mathcal{H}(\pi) = \mathcal{H}(\widehat{\pi}_{\boldsymbol{\nu}})$) and $\widehat{\pi}_{\boldsymbol{\nu}}$-related to $\boldsymbol{\upsilon}$ over $\widehat{M}'_{\boldsymbol{\nu}}$.

Proposition 2.1. *Any smooth orbifold vector field $\boldsymbol{\upsilon}$ on $\widehat{M}_{\boldsymbol{\nu}}$ has a unique horizontal lift to X.*

We shall denote the horizontal lift in Proposition (2.1) by $\boldsymbol{\upsilon}^\flat$.

Proof. Any two horizontal lifts of $\boldsymbol{\upsilon}$ clearly coincide on X', hence everywhere in X. As to existence, obviously the horizontal lift exists over the smooth locus (i.e. on X'), so the point is to see that it has a smooth extension over the singular locus.

Suppose $\widehat{m} = \widehat{\pi}_{\boldsymbol{\nu}}(x) \in \widehat{M}_{\boldsymbol{\nu}}$, and let $F_1 \subset X$ be a slice for μ^X through x. Thus F_1 uniformizes an open neighborhood of $\widehat{m}$, and $\boldsymbol{\upsilon}$ corresponds to a vector field $\mathbf{v}_1$ on F_1, invariant under the action of the stabilizer subgroup T^1_x of x in T^1. Furthermore, a suitable invariant tubular neighborhood $U_1 \subseteq X$ of the T^1-orbit of x is equivariantly diffeomorphic to $T^1 \times_{T^1_x} F_1$. Hence we can push forward $\mathbf{v}_1$ (or, more precisely, $(\mathbf{0}, \mathbf{v}_1)$) under the local diffeomorphism $T^1_x \times F_1 \to U_1$, and obtain a smooth vector field $\boldsymbol{\upsilon}'_1$ on U_1 which is $\widehat{\pi}_{\boldsymbol{\nu}}$-related to $\boldsymbol{\upsilon}$ on $U_1 \cap X'$. Let $\boldsymbol{\upsilon}_1$ denote the horizontal component of $\boldsymbol{\upsilon}'_1$ with respect to $\widehat{\pi}_{\boldsymbol{\nu}}$, that is, its projection on $\mathcal{H}_x(\widehat{\pi}_{\boldsymbol{\nu}})$ along $\mathrm{span}(\tilde{\boldsymbol{\nu}}_X)$ in (41). Then $\boldsymbol{\upsilon}_1$ is a smooth vector field on U_1, horizontal and $\widehat{\pi}_{\boldsymbol{\nu}}$-related to $\boldsymbol{\upsilon}$ on $U_1 \cap X'$.

Another such vector field $\boldsymbol{\upsilon}_2$ similarly constructed on an invariant open set U_2 will necessarily coincide with $\boldsymbol{\upsilon}_1$ on $U_1 \cap U_2 \cap X'$, whence on all of $U_1 \cap U_2$ if the latter is non-empty. Hence by glueing these local constructions we obtain the desired lift. □

Suppose that f is a $\mathcal{C}^\infty$ real function on $\widehat{M}_{\boldsymbol{\nu}}$, and let $\boldsymbol{\upsilon}_f$ be its Hamiltonian orbifold vector field with respect to $2\widehat{\omega}_{\boldsymbol{\nu}}$. Let us define

$$\boldsymbol{\upsilon}^c_f := \boldsymbol{\upsilon}^\flat_f + f\,\tilde{\boldsymbol{\nu}}_X. \tag{43}$$

Proposition 2.2. *$\boldsymbol{\upsilon}^c_f$ is a contact vector field on $(X, \hat{\alpha}_{\boldsymbol{\nu}})$. If in addition the flow of $\boldsymbol{\upsilon}_f$ is holomorphic on $(\widehat{M}_{\boldsymbol{\nu}}, J_{\boldsymbol{\nu}})$, then the flow of $\boldsymbol{\upsilon}^c_f$ preserves the CR structure of X.*

Proof. We have (writing f for $\hat{\pi}^*_{\boldsymbol{\nu}}(f)$)

$$\iota(\boldsymbol{\upsilon}^c_f)\,\mathrm{d}\hat{\alpha}_{\boldsymbol{\nu}} = \iota(\boldsymbol{\upsilon}^c_f)\,2\,\hat{\pi}^*_{\boldsymbol{\nu}}\left(\hat{\omega}^*_{\boldsymbol{\nu}}\right) = \mathrm{d}f, \quad \mathrm{d}\left(\iota(\boldsymbol{\upsilon}^c_f)\,\hat{\alpha}_{\boldsymbol{\nu}}\right) = -\mathrm{d}f.$$

Hence $L_{\boldsymbol{\upsilon}^c_f}\hat{\alpha}_{\boldsymbol{\nu}} = \iota(\boldsymbol{\upsilon}^c_f)\,\mathrm{d}\hat{\alpha}_{\boldsymbol{\nu}} + \mathrm{d}\left(\iota(\boldsymbol{\upsilon}^c_f)\,\hat{\alpha}_{\boldsymbol{\nu}}\right) = 0$. This proves the first statement.

On the other hand, the flow of $\boldsymbol{\upsilon}^c_f$ preserves the horizontal tangent bundle and covers a holomorphic flow on $\widehat{M}_{\boldsymbol{\nu}}$; the second statement then follows in view of the unitary isomorphisms (42). □

By the same principle, we can consider lifts of Hamiltonian actions for $\widehat{\pi}_{\boldsymbol{\nu}}$ just as one does for π. Suppose given a holomorphic and Hamiltonian action $\varsigma^{\widehat{M}_{\boldsymbol{\nu}}}$ of a compact and connected Lie group G on $(\widehat{M}_{\boldsymbol{\nu}}, 2\widehat{\omega}_{\boldsymbol{\nu}})$ (in the orbifold sense, see [11]), with moment map $\Lambda : \widehat{M}_{\boldsymbol{\nu}} \to \mathfrak{g}^\vee$. Thus any $\boldsymbol{\xi} \in \mathfrak{g}$ determines an induced Hamiltonian (orbifold) vector field $\boldsymbol{\xi}_{\widehat{M}_{\boldsymbol{\nu}}}$ on $\widehat{M}_{\boldsymbol{\nu}}$. Applying (43) with $\boldsymbol{\upsilon} = \boldsymbol{\xi}_{\widehat{M}_{\boldsymbol{\nu}}}$, thus setting $\boldsymbol{\xi}_X := \boldsymbol{\xi}^c_{\widehat{M}_{\boldsymbol{\nu}}}$, we associate a contact and CR vector field on X to each $\boldsymbol{\xi} \in \mathfrak{g}$. A standard argument shows that this assignment defines an infinitesimal contact and CR action of $\mathfrak{g}$ on X. If this infinitesimal action is the differential of a Lie group action ς^X of G on X, we shall call the latter the (contact and CR) lift of $(\varsigma^{\widehat{M}_{\boldsymbol{\nu}}}, \Lambda)$.

When G acts on both M and $\widehat{M}_{\boldsymbol{\nu}}$, we have in principle two lifts in the picture and two different meanings for $\boldsymbol{\xi}_X$. We will clarify this point in the following section.

2.1.8. Transfering Hamiltonian actions from M to $\widehat{M}_{\boldsymbol{\nu}}$

Suppose that G is a connected compact Lie group and let $\Xi^M : G \times M \to M$ be a holomorphic and Hamiltonian action, with moment map $\Upsilon : M \to \mathfrak{g}^\vee$. Assume the following:

1. (Ξ^M, Υ) lifts to the contact CR action $\Xi^X : G \times X \to X$;
2. Ξ^M and μ^M commute.

Then one can see that Ξ^X commutes with μ^X; therefore Ξ^X descends to an action $\Xi^{\widehat{M}_{\boldsymbol{\nu}}} : G \times \widehat{M}_{\boldsymbol{\nu}} \to \widehat{M}_{\boldsymbol{\nu}}$.

Proposition 2.3. *Under the previous assumptions, $\Xi^{\widehat{M}_{\boldsymbol{\nu}}}$ is holomorphic and Hamiltonian on the Kähler orbifold $(\widehat{M}_{\boldsymbol{\nu}},\, 2\,\widehat{\omega}_{\boldsymbol{\nu}},\, \widehat{J}_{\boldsymbol{\nu}})$, with moment map*

$$\hat{\Upsilon}_{\boldsymbol{\nu}} := \frac{1}{\Phi^{\tilde{\boldsymbol{\nu}}}}\,\Upsilon.$$

Furthermore, Ξ^X is also the contact and CR lift of $\big(\Xi^{\widehat{M}_{\boldsymbol{\nu}}},\, \hat{\Upsilon}_{\boldsymbol{\nu}}\big)$.

Proof. Given that Ξ^M commutes with μ^M, it preserves $\Phi^{\tilde{\boldsymbol{\nu}}}$. Since Ξ^X preserves α and $\Phi^{\tilde{\boldsymbol{\nu}}}$, it generates a flow of contactomorphisms for $\widehat{\alpha}_{\boldsymbol{\nu}}$. Therefore, the flow of Ξ^X preserves $\widehat{\pi}_{\boldsymbol{\nu}}^*(\widehat{\omega}_{\boldsymbol{\nu}}) = \mathrm{d}\widehat{\alpha}_{\boldsymbol{\nu}}/2$. Since Ξ^X lifts $\Xi^{\widehat{M}_{\boldsymbol{\nu}}}$ by $\widehat{\pi}_{\boldsymbol{\nu}}$, we conclude that $\Xi^{\widehat{M}_{\boldsymbol{\nu}}}$ is a symplectic vector field for $\widehat{\omega}_{\boldsymbol{\nu}}$.

Since $\Upsilon : M \to \mathfrak{g}^\vee$ is G-equivariant by assumption and $\Phi^{\tilde{\boldsymbol{\nu}}}$ is G-invariant because Ξ^M and μ^M commute, $\hat{\Upsilon}_{\boldsymbol{\nu}} : \widehat{M}_{\boldsymbol{\nu}} \to \mathfrak{g}^\vee$ is G-equivariant. Thus it suffices to prove that $(\Xi^{\widehat{M}_{\boldsymbol{\nu}}}, \hat{\Upsilon}_{\boldsymbol{\nu}})$ is weakly Hamiltonian.

Suppose $\hat{m} = \widehat{\pi}_{\boldsymbol{\nu}}(x) \in \widehat{M}_{\boldsymbol{\nu}}$. Choose a slice $F \subset X$ at x for μ^X, and view it as the uniformizing open set of an open neighborhood of $\hat{m}$ in $\widehat{M}_{\boldsymbol{\nu}}$. We obtain a local action Ξ^F of G on F as follows. For any y in a neighborhood $F' \subseteq F$ of x and g in a neighborhood $G' \subset G$ of the identity e_G, there exists a unique $s(g, y) \in T^1$ such that $\mu^X_{s(g,y)} \circ \Xi^X_g(y) \in F$. Let us set

$$\Xi^F : (g, y) \in G' \times F' \mapsto \mu^X_{s(g,y)} \circ \Xi^X_g(y) \in F. \tag{44}$$

If $g_1,\ g_2 \in G'$ are sufficiently close to the identity,

$$\begin{aligned}
\Xi^F_{g_1} \circ \Xi^F_{g_2}(y) &= \Xi^F_{g_1}\left(\mu^X_{s(g_2,y)} \circ \Xi^X_{g_2}(y)\right)\\
&= \mu^X_{s(g_1,\Xi^F_{g_2}(y))} \circ \Xi^X_{g_1} \circ \mu^X_{s(g_2,y)} \circ \Xi^X_{g_2}(y)\\
&= \mu^X_{s(g_1,\Xi^F_{g_2}(y))} \circ \mu^X_{s(g_2,y)} \circ \Xi^X_{g_1} \circ \Xi^X_{g_2}(y)\\
&= \mu^X_{s(g_1\, g_2,y))} \circ \Xi^X_{g_1\, g_2}(y) = \Xi^F_{g_1\, g_2}(y).
\end{aligned}$$

Given $\boldsymbol{\xi} \in \mathfrak{g}$, the induced vector field $\boldsymbol{\xi}_F$ on F may be computed by considering the restricted local action of the 1-parameter subgroup $\tau \mapsto e^{\tau\,\boldsymbol{\xi}} \in G$, hence by differentiating at $\tau = 0$ the path $\Xi^F_{e^{\tau\,\boldsymbol{\xi}}}(y) = \mu^X_{s(e^{\tau\,\boldsymbol{\xi}},y)} \circ \mu^X_{e^{\tau\,\boldsymbol{\xi}}}(y)$. We conclude the following.

Lemma 2.4. *There exists a $\mathcal{C}^\infty$-function $\sigma : \mathfrak{g} \times F \to \mathbb{R}$ such that for any $\boldsymbol{\xi} \in \mathfrak{g}$ and $y \in F$*

$$\boldsymbol{\xi}_F(y) = \sigma(\boldsymbol{\xi}, y)\,\tilde{\boldsymbol{\nu}}_X(y) + \boldsymbol{\xi}_X(y).$$

Here $\boldsymbol{\xi}_X$ is as in (2), with Υ in place of Φ.

At any $y \in F$, we have a direct sum decomposition $T_yX = T_yF \oplus \operatorname{span}\big(\tilde{\boldsymbol{\nu}}_X(y)\big)$. Thus Lemma 2.4 may be reformulated as follows.

Corollary 2.1. *For any $y \in F$ and $\boldsymbol{\xi} \in \mathfrak{g}$, $\boldsymbol{\xi}_F(y)$ is the projection of $\boldsymbol{\xi}_X(y)$ on T_yF along $\operatorname{span}\big(\tilde{\boldsymbol{\nu}}_X(y)\big)$.*

By the commutativity of μ^X and Ξ^X, the stabilizer subgroup of x in T^1 acts on F preserving the previous direct sum of vector bundles on F. It follows that $\boldsymbol{\xi}_F$ is an invariant vector field on F, and the collection of all such is the induced vector field $\boldsymbol{\xi}_{\widehat{M}_{\boldsymbol{\nu}}}$ on $\widehat{M}_{\boldsymbol{\nu}}$.

Letting $\jmath : F \hookrightarrow X$ be the inclusion, let us set $\alpha_F := \jmath^*(\widehat{\alpha}_{\boldsymbol{\nu}})$ and $\omega_F := \mathrm{d}\alpha_F/2$. The collection of all pairs (F, ω_F) represents $\widehat{\omega}_{\boldsymbol{\nu}}$.

If $y \in F$ as above, $\iota\big(\boldsymbol{\xi}_F(y)\big)\,(\mathrm{d}\alpha_F)_y$ is the restriction to $T_yF \subset T_yX$ of $\iota\big(\boldsymbol{\xi}_F(y)\big)\,(\mathrm{d}\widehat{\alpha}_{\boldsymbol{\nu}})_y$. On the other hand, by Lemma 2.4 we have

$$\begin{aligned}
\iota\big(\boldsymbol{\xi}_F(y)\big)\,(\mathrm{d}\widehat{\alpha}_{\boldsymbol{\nu}})_y &= \iota\Big(\boldsymbol{\xi}_X(y) + \sigma(\boldsymbol{\xi}, y)\,\tilde{\boldsymbol{\nu}}_X(y)\Big)\,(\mathrm{d}\widehat{\alpha}_{\boldsymbol{\nu}})_y\\
&= \left[\iota(\boldsymbol{\xi}_X)\,(\mathrm{d}\widehat{\alpha}_{\boldsymbol{\nu}})\right]_y + \sigma(\boldsymbol{\xi}, y)\,\left[\iota(\tilde{\boldsymbol{\nu}}_X)\,(\mathrm{d}\widehat{\alpha}_{\boldsymbol{\nu}})\right]_y\\
&= -\left[\mathrm{d}\left(\iota(\boldsymbol{\xi}_X)\,\widehat{\alpha}_{\boldsymbol{\nu}}\right)\right]_y - \sigma(\boldsymbol{\xi}, y)\,\left[\mathrm{d}(\iota(\tilde{\boldsymbol{\nu}}_X)\,\widehat{\alpha}_{\boldsymbol{\nu}})\right]_y\\
&= \mathrm{d}_y\left(\frac{\Upsilon^{\boldsymbol{\xi}}}{\Phi^{\tilde{\boldsymbol{\nu}}}}\right) + \sigma(\boldsymbol{\xi}, y)\,\mathrm{d}_y(1)\\
&= \mathrm{d}_y\left(\frac{\Upsilon^{\boldsymbol{\xi}}}{\Phi^{\tilde{\boldsymbol{\nu}}}}\right) = \mathrm{d}_y\left(\widehat{\Upsilon}^{\boldsymbol{\xi}}\right).
\end{aligned}$$

We have used that both $\boldsymbol{\xi}_X$ and $\tilde{\boldsymbol{\nu}}_X$ are contact vector fields for $\widehat{\alpha}_{\boldsymbol{\nu}}$.

To prove the last statement of Proposition 2.3, we need to verify that $\boldsymbol{\xi}_X = \boldsymbol{\xi}^c_{\widehat{M}_{\boldsymbol{\nu}}}$ for every $\boldsymbol{\xi} \in \mathfrak{g}$. Since both $\boldsymbol{\xi}_X$ and $\boldsymbol{\xi}^c_{\widehat{M}_{\boldsymbol{\nu}}}$ lift $\boldsymbol{\xi}_X = \boldsymbol{\xi}^c_{\widehat{M}_{\boldsymbol{\nu}}}$ under $\widehat{\pi}_{\boldsymbol{\nu}}$, it suffices to show that the coefficient of $\boldsymbol{\xi}_X$ along $\tilde{\boldsymbol{\nu}}_X$ is $\widehat{\Upsilon}^{\boldsymbol{\xi}}$. Therefore, the equality

$$\boldsymbol{\xi}_X = \left(\boldsymbol{\xi}^\sharp_M - \widehat{\Upsilon}^{\boldsymbol{\xi}}\, \tilde{\boldsymbol{\nu}}^\sharp_M\right) + \widehat{\Upsilon}^{\boldsymbol{\xi}}\, \tilde{\boldsymbol{\nu}}_X, \tag{45}$$

implies that $\boldsymbol{\xi}^\sharp_M - \widehat{\Upsilon}^{\boldsymbol{\xi}}\, \tilde{\boldsymbol{\nu}}^\sharp_M$ is the horizontal lift (with respect to $\widehat{\pi}_{\boldsymbol{\nu}}$) of $\boldsymbol{\xi}_{\widehat{M}_{\boldsymbol{\nu}}}$, and that $\boldsymbol{\xi}_X = \boldsymbol{\xi}^c_{\widehat{M}_{\boldsymbol{\nu}}}$. $\square$

2.1.9. The torus $\widehat{T}^d$ and its action on $\widehat{M}_{\boldsymbol{\nu}}$

As in the Introduction, let $T^{d-1}_c \leqslant T^d$ be a complementary subtorus to T^1. Let us define a new torus

$$\widehat{T}^d := T^{d-1}_c \times S^1. \tag{46}$$

Since ρ^X (the action of S^1 on X with generator $-\partial_\theta$) and γ^X (the contact CR action of T^d on X) commute, the restriction of γ^X to T^{d-1}_c and ρ^X may be combined to yield a new action

$$\beta^X : \widehat{T}^d \times X \to X. \tag{47}$$

Since furthermore β^X commutes with $\mu^X : T^1 \times X \to X$, it descends to an action

$$\beta^{\widehat{M}_{\boldsymbol{\nu}}} : \widehat{T}^d \times \widehat{M}_{\boldsymbol{\nu}} \to \widehat{M}_{\boldsymbol{\nu}}. \tag{48}$$

In fact, (47) is the contact and CR lift of a Hamiltonian action β^M of $\widehat{T}^d$ on $(M, 2\omega)$. Given the decomposition $\mathfrak{t}^d = \mathfrak{t}^{d-1}_c \oplus \mathrm{span}(\tilde{\boldsymbol{\nu}})$ the moment map Ψ of γ^M may be written $\Psi = \Psi' + \Psi''$, where $\Psi' : M \to \mathfrak{t}^{d-1\vee}_c$, $\Psi'' : M \to \mathrm{span}(\tilde{\boldsymbol{\nu}})^\vee$. The restriction of γ^M to T^{d-1}_c is Hamiltonian, with moment map Ψ'. On the other, hand, with the usual identification of the Lie algebra and coalgebra of S^1 with $\imath\,\mathbb{R}$, ρ^X is the contact lift of the trivial action of S^1 on $(M, 2\omega)$ with constant moment map $\imath$. Therefore, β^X is the contact lift of the Hamiltonian action $\beta^M : \widehat{T}^d \times M \to M$ with moment map $\Xi = (\Psi', \imath)$. In view of Proposition 2.3, we conclude the following.

Proposition 2.4. *$\beta^{\widehat{M}_{\boldsymbol{\nu}}}$ in (48) is Hamiltonian, with moment map*

$$\hat{\Xi} := \left(\frac{\Psi'}{\Phi^{\tilde{\boldsymbol{\nu}}}}, \frac{\imath}{\Phi^{\tilde{\boldsymbol{\nu}}}}\right) : \widehat{M}_{\boldsymbol{\nu}} \to \widehat{\mathfrak{t}}^{d\vee} = \mathfrak{t}^{d-1\vee}_c \oplus \imath\,\mathbb{R}.$$

We now argue that $\beta^{\widehat{M}_{\boldsymbol{\nu}}}$ can be complexified to a holomorphic action of $\widehat{\mathbb{T}}^d$, the complexification of $\widehat{T}^d$, on $\widehat{M}_{\boldsymbol{\nu}}$. More generally, for any compact Lie group G we shall denote its complexification by $\mathbb{G}$.

To this end, we consider the complement of the zero section $A^\vee_0 \subset A^\vee$, and observe that all the actions involved on X uniquely extend to complexified actions on $A^\vee_0$. Thus μ^X extends to $\widetilde{\mu}^{A^\vee_0} : \mathbb{T}^1 \times A^\vee_0 \to A^\vee_0$, ρ^X to $\widetilde{\rho}^{A^\vee_0} : \mathbb{C}^* \times A^\vee_0 \to A^\vee_0$, γ^X to $\widetilde{\gamma}^{A^\vee_0} : \mathbb{T}^d \times A^\vee_0 \to A^\vee_0$, β^X to $\widetilde{\beta}^{A^\vee_0} : \widehat{\mathbb{T}}^d \times A^\vee_0 \to A^\vee_0$; clearly $\widehat{\mathbb{T}}^d = \mathbb{T}^{d-1}_c \times \mathbb{C}^*$.

In view of the discussion in §2, §3, and §5 of [16] (applied with $r = 1$), under Basic Assumption 1.1 the following holds:

1. $A^\vee_0 = \mathbb{T}^1 \cdot X$ (the $\widetilde{\mu}^{A^\vee_0}$-saturation of X);
2. $\widetilde{\mu}^{A^\vee_0}$ is proper and locally free;
3. there is a natural biholomorphism
$$X/T^1 \cong A^\vee_0/\mathbb{T}^1 \tag{49}$$
where the former quotient is taken with respect to μ^X, and the latter with respect to $\widetilde{\mu}^{A^\vee_0}$.

Since $\widetilde{\beta}^{A^\vee_0}$ commutes with $\widetilde{\mu}^{A^\vee_0}$, it descends to the quotient and we conclude the following.

Proposition 2.5. *$\beta^{\widehat{M}_{\boldsymbol{\nu}}}$ admits a unique holomorphic extension $\widetilde{\beta}^{\widehat{M}_{\boldsymbol{\nu}}} : \widehat{\mathbb{T}}^d \times \widehat{M}_{\boldsymbol{\nu}} \to \widehat{M}_{\boldsymbol{\nu}}$.*

We aim to relate the stabilizer of $m \in M$ under γ^M to the stabilizer of $\hat{m} \in \widehat{M}_{\boldsymbol{\nu}}$ under $\beta^{\widehat{M}_{\boldsymbol{\nu}}}$ if $m \leftarrow x \to \hat{m}$. More generally, we can consider the same issue for the complexified actions $\widetilde{\gamma}^M : \mathbb{T}^d \times M \to M$ and $\widetilde{\beta}^{\widehat{M}_{\boldsymbol{\nu}}}$; by Proposition 1.6 of [17], the stabilizer of m under $\widetilde{\gamma}^M$ is the complexification of the stabilizer under γ^M.

There is a dense open subset $M^0 \subset M$ where γ^M is free; then $\widetilde{\gamma}^M$ is free and transitive on M^0. Let us consider the corresponding open set

$$\widehat{M}^0 := \widehat{M^0} = \widehat{\pi}_{\boldsymbol{\nu}}\left(\pi^{-1}(M^0)\right) \subset \widehat{M}_{\boldsymbol{\nu}}. \tag{50}$$

Let us set $X^0 := \pi^{-1}(M^0)$.

Proposition 2.6. *Under the previous assumptions, the following holds.*

1. $\beta^{\widehat{M}_{\boldsymbol{\nu}}}$ *is free on* $\widehat{M}^0$.
2. $\widetilde{\beta}^{\widehat{M}_{\boldsymbol{\nu}}}$ *is free and transitive on* $\widehat{M}^0$.
3. $\widehat{\pi}_{\boldsymbol{\nu}}$ *restricts to a principal* S^1*-bundle* $X^0 \to \widehat{M}^0$.

In particular, $\widehat{M}^0$ *is smooth.*

Proof. Suppose $m \leftarrow x \to \hat{m}$ with $m \in M^0$, and that $(t, e^{\imath\theta}) \in \widehat{T}^d$ stabilizes $\hat{m}$. Hence there exists $h \in T^1$ such that

$$\gamma^X_t \circ \rho^X_{e^{\imath\theta}}(x) = \mu^X_h(x).$$

With χ as in (37) and Lemma 2.3, we conclude that

$$\gamma^X_{t\,h^{-1}} \circ \rho^X_{e^{\imath\theta}\,\chi(h)^{-1}}(x) = x \quad \Rightarrow \quad \gamma^M_{t\,h^{-1}}(m) = m.$$

Hence $t\,h^{-1} = 1 \in T^d$, and since T^{d-1}_c and T^1 are complementary subtori we conclude $t = h = 1$. Hence $e^{\imath\,\theta} = 1$ as well.

This proves the first statement; that $\widetilde{\beta}^{\widehat{M}_{\boldsymbol{\nu}}}$ is free on $\widehat{M}_0$ then follows either by a similar argument using complexifications, or else by appealing to Proposition 1.6 of [17].

We prove that $\widetilde{\beta}^{\widehat{M}_{\boldsymbol{\nu}}}$ is transitive on $\widehat{M}_0$. Suppose $\hat{m}_j \in \widehat{M}_0$, $j = 1, 2$. Then there exist $m_j \in M^0$ and $x_j \in \pi^{-1}(m_j) \subseteq X^0$, such that $m_j \leftarrow x_j \to \hat{m}_j$. There exists $t' \in T^d$ such that $\gamma^M_{t'}(m_1) = m_2$. We can factor t' uniquely as $t' = t\,h$, where $t \in T^{d-1}_c$ and $h \in T^1$. Lifting first this relation to X, and then descending to $\widehat{M}_{\boldsymbol{\nu}}$, this means that for some $e^{\imath\theta} \in S^1$ we have

$$\begin{aligned}\gamma^X_t \circ \gamma^X_h(x_1) = \rho^X_{e^{\imath\,\theta}}(x_2) &\Rightarrow \gamma^X_t \circ \rho^{-1}_{\chi(h)\,e^{\imath\,\theta}}(x_1) = \mu^X_{h^{-1}}(x_2)\\ &\Rightarrow \widetilde{\beta}^{\widehat{M}_{\boldsymbol{\nu}}}_{\hat{t}}(\hat{m}_1) = \hat{m}_2,\end{aligned} \tag{51}$$

where $\hat{t} := \big(t, \chi(h)^{-1}\,e^{-\imath\,\theta}\big) \in \widehat{T}^d$, and χ is as in (37).

Finally, since μ^X lifts the restriction of γ^M to T^1, and on the other hand γ^M is free on M^0, it follows that μ^X is free on X^0; the third statement follows. □

Corollary 2.2. $\widehat{M}_{\boldsymbol{\nu}}$ *(with the Hamiltonian action* $(\widetilde{\beta}^{\widehat{M}_{\boldsymbol{\nu}}}, \hat{\Xi})$*) is a symplectic toric orbifold and a complex toric variety.*

Let us consider a general triple $m \leftarrow x \to \hat{m}$, and denote by $T^d_m \leqslant T^d$ the stabilizer of m for γ^M, and by $\widehat{T}^d_{\hat{m}} \leqslant \widehat{T}^d$ the stabilizer of $\hat{m}$ for $\widetilde{\beta}^{\widehat{M}_{\boldsymbol{\nu}}}$. We want to describe the relation between T^d_m and $\widehat{T}^d_{\hat{m}}$.

Let $T^1_x \leqslant T^1$ be the stabilizer of x in T^1 under μ^X. Recall that μ^M is the restriction of γ^M, that is, $\mu^M = \gamma^M|_{T^1 \times M}$; hence we can unambiguously denote by $T^1_m = T^1 \cap T^d_m$ the μ^M-stabilizer of $m \in M$.

Lemma 2.5. *If* $m = \pi(x)$*, then* T^1_x *is a finite subgroup of* T^1_m.

Proof. Since μ^X is locally free, T^1_x is a discrete subgroup of T^1, hence finite. Furthermore, since μ^X lifts μ^M, which is the restriction of γ^M to T^1, we also have $T^1_x \leqslant T^d_m$. In fact, if $t \in T^1_x$ then $\mu^X_t(x) = x$ then

$$\mu^M_t(m) = \pi \circ \mu^X_t(x) = m,$$

hence $t \in T^d_m$. □

Thus $T^1_x \leqslant T^d_m$ is finite, and T^d_m is a subtorus of T^d [1]; hence T^d_m/T^1_x is a torus of the same dimension as T^d_m.

Proposition 2.7. *If* $m \leftarrow x \to \hat{m}$*, there is a natural isomorphism* $T^d_m/T^1_x \cong \widehat{T}^d_{\hat{m}}$*. In particular,* T^d_m *and* $\widehat{T}^d_{\hat{m}}$ *are tori of the same dimension.*

Proof. For every $m \in M$, there is a character $\delta_m : T^d_m \to S^1$ such that

$$\gamma^X_k(x) = \rho^X_{\delta_m(k)}(x) \quad (x \in \pi^{-1}(m),\ k \in T^d_m). \tag{52}$$

Let us factor $k = t\,h$ with $t \in T^{d-1}_c$, $h \in T^1$. Then (52) implies

$$\gamma^X_t \circ \rho^X_{\delta_m(k)^{-1}\,\chi(h)^{-1}}(x) = \mu^X_{h^{-1}}(x) \quad \Rightarrow \quad \widetilde{\beta}^{\widehat{M}_{\boldsymbol{\nu}}}_{\hat{k}}(\hat{m}) = \hat{m}, \tag{53}$$

where $\hat{k} := \big(t, \delta_m(k)^{-1}\,\chi(h)^{-1}\big) \in \widehat{T}^d$. Hence we obtain a Lie group homomorphism

$$\psi_m : k = t\,h \in T^d_m \mapsto \hat{k} := \left(t, \delta_m(k)^{-1}\,\chi(h)^{-1}\right) \in \widehat{T}^d_m. \tag{54}$$

Let us set $T^1_m := T^1 \cap T^d_m$. Then

$$\ker(\psi_m) = \left\{h \in T^1_m \,:\, \delta_m(k)\,\chi(h) = 1\right\}. \tag{55}$$

Lemma 2.6. $\ker(\psi_m) = T^1_x$.

Proof. By (37) and (52), we have for $h \in T^1_m$

$$\gamma^X_h(x) = \rho^X_{\delta_m(h)}(x) \quad \Rightarrow \quad \mu^X_h(x) = \rho^X_{\chi(h)\,\delta_m(h)}(x).$$

In other words, $\ker(\psi_m) = T^1_x$. $\square$

Let us prove that ψ_m is surjective. Suppose $(t, e^{\imath\theta}) \in \widehat{T}^d_{\hat{m}}$, i.e. $\widetilde{\beta}^{\widehat{M}}_{(t, e^{\imath\theta})}(\hat{m}) = \hat{m}$. There exists $h \in T^1$ such that

$$\gamma^X_t \circ \rho^X_{e^{\imath\theta}}(x) = \mu^X_{h^{-1}}(x) \Rightarrow \gamma^X_{t\,h} \circ \rho^X_{\chi(h)}(x) = \rho^X_{e^{-\imath\theta}}(x) \quad \Rightarrow \quad \gamma^M_{t\,h}(m) = m.$$

Thus $k := t\,h \in T^d_m$, whence $\gamma^X_k(x) = \rho^X_{\delta_m(k)}(x)$. We conclude $\rho^X_{\delta_m(k)\,\chi(h)}(x) = \rho^X_{e^{-\imath\theta}}(x)$, so that $e^{\imath\theta} = \delta_m(k)^{-1}\,\chi(h)^{-1}$. It follows that $(t, e^{\imath\theta}) = \psi_m(k)$. $\square$

2.2. The polytope $\widehat{\Delta}_{\boldsymbol{\nu}}$

By Proposition 2.4 and Corollary 2.2, $\widehat{M}_{\boldsymbol{\nu}}$ is a Kähler toric orbifold, and its associated convex rational simple polytope ([1], [11]) is

$$\widehat{\Delta}_{\boldsymbol{\nu}} := \Xi\left(\widehat{M}_{\boldsymbol{\nu}}\right) \subset \widehat{\mathfrak{t}}^{d\,\vee}. \tag{56}$$

We aim to describe the faces of $\widehat{\Delta}_{\boldsymbol{\nu}}$ in terms of the faces of Δ; to this end, we premise a few remarks.

Lemma 2.7. *Suppose that $R,\ S \subseteq M$; then $\widehat{R \cap S} \subseteq \widehat{R} \cap \widehat{S}$. If furthermore R and S are μ^M-invariant, then $\widehat{R \cap S} = \widehat{R} \cap \widehat{S}$.*

Proof. Suppose $\hat{m} \in \widehat{R \cap S}$. Then there exist $m \in R \cap S$, $x \in X$ such that $m \leftarrow x \rightarrow \hat{m}$; hence $\widehat{m} \in \widehat{R} \cap \widehat{S}$, so that and $\widehat{R \cap S} \subseteq \widehat{R} \cap \widehat{S}$.

Suppose that R and S are $\widehat{\mu}^M$-invariant, and that $\hat{m} \in \widehat{R} \cap \widehat{S}$. Hence there exist $m_1 \in R$, $m_2 \in S$ and $x_1,\, x_2 \in X$ such that $m_1 \leftarrow x_1 \rightarrow \hat{m}$ and $m_2 \leftarrow x_2 \rightarrow \hat{m}$. Hence $x_2 \in T^1 \cdot x_1$ (μ^X-orbit) and by the equivariance of π this implies $m_2 \in T^1 \cdot m_1$ (μ^M-orbit). Thus $m_1,\, m_2 \in R \cap S$ and so $\hat{m} \in \widehat{R \cap S}$. It follows that $\widehat{R \cap S} \supseteq \widehat{R} \cap \widehat{S}$. $\square$

Lemma 2.8. *If $R \subseteq M$, then $\widehat{\overline{R}} \subseteq \overline{\widehat{R}}$. If in addition $R \subseteq M$ is μ^M invariant, then $\widehat{\overline{R}} = \overline{\widehat{R}}$.*

Proof. We have by definition

$$\overline{\widehat{R}} = \widehat{\pi}_{\boldsymbol{\nu}}\left(\pi^{-1}\left(\overline{R}\right)\right) \supseteq \widehat{\pi}_{\boldsymbol{\nu}}\left(\pi^{-1}\left(R\right)\right) = \widehat{R};$$

hence $\widehat{\overline{R}}$ is closed and contains $\widehat{R}$, i.e. $\widehat{\overline{R}} \supseteq \overline{\widehat{R}}$.

Before considering the reverse inclusion, let us premise that - since $\pi : X \rightarrow M$ is an S^1-bundle - $\pi^{-1}(\overline{R}) = \overline{\pi^{-1}(R)}$.

Suppose that R is μ^M-invariant, and let $S \subseteq \widehat{M}_{\boldsymbol{\nu}}$ be closed with $\widehat{R} \subseteq S$. Then $\widehat{\pi}_{\boldsymbol{\nu}}^{-1}(S) \supseteq \widehat{\pi}_{\boldsymbol{\nu}}^{-1}(\widehat{R})$.

Claim 2.1. *Given that R is μ^M-invariant, $\widehat{\pi}_{\boldsymbol{\nu}}^{-1}(\widehat{R}) = \pi^{-1}(R)$.*

Proof of Claim 2.1. By construction, $\widehat{\pi}_{\boldsymbol{\nu}}^{-1}(\widehat{R})$ is the union of all μ^X-orbits passing through points of $\pi^{-1}(R)$. By equivariance, every μ^X-orbit projects down to M under π to a μ^M-orbit, and each such orbit through a point of R is entirely contained in R. Therefore, all μ^X-orbits through points of $\pi^{-1}(R)$ are entirely contained in $\pi^{-1}(R)$, whence the claim. $\square$

Thus $\widehat{\pi}_{\boldsymbol{\nu}}^{-1}(S) \supseteq \pi^{-1}(R)$, hence

$$\widehat{\pi}_{\boldsymbol{\nu}}^{-1}(S) \supseteq \overline{\pi^{-1}(R)} = \pi^{-1}\left(\overline{R}\right) \quad \Rightarrow \quad S \supseteq \widehat{\overline{R}}.$$

We conclude that $\widehat{\overline{R}} \subseteq \overline{\widehat{R}}$ when R is μ^M-invariant, which completes the proof of Lemma 2.8. $\square$

Let as above $\mathcal{G}(\Delta) = \{F_1, \ldots, F_k\}$ be the collection of the facets of Δ. Recalling that $\Psi : M \to \mathfrak{t}^{d\vee}$ is the moment map for γ^M, for each j let $M_j := \Psi^{-1}(F_j)$ (see §2.1.3). Then M_j is a complex submanifold of codimension 1 of M. We shall set

$$\widehat{M}_j := \pi_{\boldsymbol{\nu}}\left(\pi^{-1}(M_j)\right), \quad \widehat{M}_j^0 := \widehat{M_j^0} = \pi_{\boldsymbol{\nu}}\left(\pi^{-1}(M_j^0)\right);$$

then $\widehat{M}_j$ is a complex suborbifold of $\widehat{M}_{\boldsymbol{\nu}}$ of codimension 1, and $\widehat{M}_j^0$ is open and dense in $\widehat{M}_j$. Furthermore, M_j is the fixed locus of the 1-parameter subgroup $f_j : \tau \mapsto e^{\tau\,\boldsymbol{\upsilon}_j}$, where $\boldsymbol{\upsilon}_j$ is as in (5) and (6), while M_j^0 is the locus of those points in M whose stabilizer subgroup in T^d is exactly $f_j(S^1)$.

Since M_j and M_j^0 are μ^M-invariant for every j, in light of Proposition 2.7, Lemma 2.7, and Lemma 2.8 we conclude the following.

Corollary 2.3. *For every $j, l \in \{1, \ldots, k\}$, the following holds:*

1. $\widehat{M_j \cap M_l} = \widehat{M}_j \cap \widehat{M}_l$;
2. $\widehat{M}_j = \overline{\widehat{M}_j^0}$;
3. $\widehat{M}_j^0 \cap \widehat{M}_l^0 = \emptyset$ *if* $j \neq l$;
4. *for every* $j = 1, \ldots, k$, $\bigcup_{j=1}^k \widehat{M}_j^0 \subset \widehat{M}_{\boldsymbol{\nu}}$ *is the locus of points with a* 1*-dimensional stabilizer subgroup in* $\widehat{T}^d$.

Let us set $\widehat{F}_j := \Xi(\widehat{M}_j)$ We can use the conclusions of Corollary 2.3 to relate the faces of Δ and $\widehat{\Delta}_{\boldsymbol{\nu}}$.

Proposition 2.8. *Let $\mathcal{F}_l(\widehat{\Delta}_{\boldsymbol{\nu}})$ and $\mathcal{G}(\widehat{\Delta}_{\boldsymbol{\nu}}) = \mathcal{F}_1(\widehat{\Delta}_{\boldsymbol{\nu}})$ be, respectively, the collections of codimension-l faces and of facets of $\widehat{\Delta}_{\boldsymbol{\nu}}$. Then there are bijective correspondences*

1. $F_j = \Psi(M_j) \in \mathcal{G}(\Delta) \mapsto \widehat{F}_j := \Xi(\widehat{M}_j) \in \mathcal{G}(\widehat{\Delta}_{\boldsymbol{\nu}})$;
2. $\bigcap_{j=i_1}^{i_l} F_{i_j} \in \mathcal{F}_l(\Delta) \mapsto \bigcap_{j=i_1}^{i_l} \widehat{F}_{i_j} \in \mathcal{F}_l(\widehat{\Delta}_{\boldsymbol{\nu}})$;
3. *for every* $j = 1, \ldots, k$, *the relative interior of* F_j *is* $F_j^0 = \Psi(M_j^0)$;
4. *for every* $j = 1, \ldots, k$, *the relative interior of* $\widehat{F}_j$ *is* $\widehat{F}_j^0 = \Xi(\widehat{M}_j^0)$

Proof. By the theory in [1] and [11], the moment maps Ψ and Ξ are quotients by the respective actions, i.e., the fibers are orbits. Furthermore, the relative interior of F_j is $F_j^0 = \Psi(M_j^0)$.

By Corollary 2.3, the orbifolds $\widehat{M}_j^0$ are the connected components of the locus of points in $\hat{M}_{\boldsymbol{\nu}}$ having 1-dimensional stabilizer subgroup in $\widehat{T}^d$. Therefore, $\overline{\Xi(\widehat{M}_j^0)} = \Xi(\widehat{M}_j)$ is a facet of $\widehat{\Delta}_{\boldsymbol{\nu}}$, and these are all the (distinct) faces of $\widehat{\Delta}_{\boldsymbol{\nu}}$.

One argues similarly for the other faces. □

Let us next determine the normal vectors to the facets $\widehat{F}_j$ of $\widehat{\Delta}_{\boldsymbol{\nu}}$. This amounts to determining the stabilizer subgroup in $\widehat{T}^d$ of the points in each relative interior $\widehat{F}_j^0$ (recall that $\widehat{T}^d$ acts on $\widehat{M}_{\boldsymbol{\nu}}$ by $\beta^{\widehat{M}_{\boldsymbol{\nu}}}$ in (48)).

Proposition 2.9. *The stabilizer subgroup for $\beta^{\widehat{M}_{\boldsymbol{\nu}}}$ of any $\hat{m} \in \widehat{F}_j$ is the* 1*-parameter subgroup of $\widehat{T}^d$ generated by $\widehat{\boldsymbol{\upsilon}}_j := \boldsymbol{\upsilon}'_j - (\lambda_j + \rho_j\,\delta)\,\partial_\theta^{S^1} \in \widehat{\mathfrak{t}}^d$.*

Remark 2.1. Given Lemma 2.3, $\hat{\boldsymbol{\upsilon}}_j \in L(\widehat{T}^d)$; it is not claimed that $\hat{\boldsymbol{\upsilon}}_j$ is primitive.

Proof. Assume $m \leftarrow x \rightarrow \widehat{m}$ with $m \in F_j^0$, whence $\widehat{m} \in \widehat{F}_j^0$. The stabilizer subgroup of m is then the 1-parameter subgroup $\tau \mapsto e^{\tau\,\boldsymbol{\upsilon}_j}$, where $\boldsymbol{\upsilon}_j$ is as in (7). Hence $\mu^M_{e^{\tau\,\boldsymbol{\upsilon}_j}}(m) = m$ for every $\tau \in \mathbb{R}$, and by Lemma 2.2

$$\gamma^X_{e^{\tau\,\boldsymbol{\upsilon}_j}}(x) = \rho^X_{e^{\tau\,\lambda_j}}(x) \quad (\tau \in \mathbb{R}). \tag{57}$$

In view of (13), (37), and Lemma 2.3, (57) may be rewritten as follows

$$\begin{aligned} x &= \gamma^X_{e^{\tau\,(\boldsymbol{\upsilon}'_j + \rho_j \tilde{\boldsymbol{\nu}})}} \circ \rho^X_{e^{-\tau\,\lambda_j}}(x) = \gamma^X_{e^{\tau\,\boldsymbol{\upsilon}'_j}} \circ \rho^X_{e^{-\tau\,\lambda_j}} \circ \gamma^X_{\rho_j \tilde{\boldsymbol{\nu}}}(x) \\ &= \gamma^X_{e^{\tau\,\boldsymbol{\upsilon}'_j}} \circ \rho^X_{e^{-\tau\,\lambda_j}} \circ \rho^X_{e^{-\tau\,\rho_j\,\delta}} \circ \mu^X_{e^{\tau\,\rho_j \tilde{\boldsymbol{\nu}}}}(x) \\ &= \gamma^X_{e^{\tau\,\boldsymbol{\upsilon}'_j}} \circ \rho^X_{e^{-\tau\,(\lambda_j + \rho_j\,\delta)}} \circ \mu^X_{e^{\tau\,\rho_j \tilde{\boldsymbol{\nu}}}}(x) \\ &= \mu^X_{e^{\tau\,\rho_j \tilde{\boldsymbol{\nu}}}} \circ \beta^X_{e^{\tau\,[\boldsymbol{\upsilon}'_j - (\lambda_j + \rho_j\,\delta)\,\partial_\theta^{S^1}]}}(x) \quad (\tau \in \mathbb{R}), \end{aligned} \tag{58}$$

where β^X was introduced in (47). Passing to the quotient, we can reformulate (58) in terms of $\beta^{\widehat{M}_{\boldsymbol{\nu}}}$:

$$\widehat{m} = \beta^{\widehat{M}_{\boldsymbol{\nu}}}_{e^{\tau\,[\boldsymbol{v}'_j-(\lambda_j+\rho_j\,\delta)\,\partial^{S^1}_\theta]}}(\widehat{m}) \quad (\tau \in \mathbb{R}). \quad \square \tag{59}$$

Given Proposition 2.9, the facets of $\widehat{\Delta}_{\boldsymbol{\nu}}$ are defined by equations of the form

2.3. Proof of Theorem 1.1

We can now combine the previous results to a proof of Theorem 1.1. Let us premise a piece of notation. We shall denote by $\mathrm{d}^{S^1}\theta$ the dual basis in $\mathrm{Lie}(S^1)^\vee$ to $\partial^{S^1}_\theta$. Assuming $m \leftarrow x \rightarrow \hat{m}$, by Proposition 2.4

$$\hat{\Xi}(\hat{m}) = \frac{\Psi'(m)}{\Phi^{\tilde{\boldsymbol{\nu}}}(m)} + \frac{1}{\Phi^{\tilde{\boldsymbol{\nu}}}(m)}\,\mathrm{d}^{S^1}\theta = \frac{\Psi'(m) + \mathrm{d}^{S^1}\theta}{\Psi^{\tilde{\boldsymbol{\nu}}}(m) + \delta}. \tag{60}$$

Proof of Theorem 1.1. We have $\widehat{\Delta}_{\boldsymbol{\nu}} = \hat{\Xi}(\widehat{M}_{\boldsymbol{\nu}})$. Assume $\rho \in \widehat{\Delta}_{\boldsymbol{\nu}}$, and choose a triple $m \leftarrow x \rightarrow \hat{m}$ with $\rho = \hat{\Xi}(\hat{m})$. Then $\langle \Psi(m), \boldsymbol{v}_j\rangle \geq \lambda_j$ for every $j = 1, \ldots, k$. With $\boldsymbol{v}_j$ as in (13) this yields for every j

$$\langle \Psi'(m), \boldsymbol{v}'_j\rangle - (\lambda_j + \rho_j\,\delta) \geq -\rho_j\left(\Psi^{\tilde{\boldsymbol{\nu}}}(m) + \delta\right) = -\Phi^{\tilde{\boldsymbol{\nu}}}(m)\,\rho_j. \tag{61}$$

In view of Proposition 2.9 and (60), dividing (61) by $\Phi^{\tilde{\boldsymbol{\nu}}}(m) > 0$, one gets

$$\left\langle \Xi'(\hat{m}), \widehat{\boldsymbol{v}}_j\right\rangle \geq -\rho_j. \tag{62}$$

Hence, every $\rho \in \widehat{\Delta}_{\boldsymbol{\nu}}$ satisfies $\left\langle \rho, \widehat{\boldsymbol{v}}_j\right\rangle \geq -\rho_j$ for every j. Furthermore, the previous argument also shows that the inequalities are all strict if and only if $\rho \in \widehat{M}^0$ (notation as in (50)), and that on the other hand equality holds for exactly one j if and only if ρ belongs to the corresponding facet $\hat{F}_j$.

Since we know already that $\widehat{\Delta}_{\boldsymbol{\nu}}$ is a rational convex polytope and the $\widehat{F}_j$'s are its facets, we conclude that

$$\widehat{\Delta}_{\boldsymbol{\nu}} = \bigcap_{j=1}^{k}\left\{\rho \in \widehat{\mathfrak{t}}^\vee_{\boldsymbol{\nu}} \,:\, \rho\left(\widehat{\boldsymbol{v}}_j\right) \geq -\rho_j\right\},$$

and in particular that each $\widehat{\boldsymbol{v}}_j$ is inward-pointing.

The previous discussion completes the proof that shape of $\widehat{\Delta}_{\boldsymbol{\nu}}$ is as claimed in the statement of Theorem 1.1, except that $\widehat{\Delta}_{\boldsymbol{\nu}}$ is realized in the Lie coalgebra of $\widehat{T}^d$ rather than T^d. To obtain the corresponding statement of Theorem 1.1 we need only compose with the isomorphism $T^d \cong T^{d-1}_c \times T^1 \rightarrow T^{d-1}_c \times S^1$ given by the product of the identity and $e^{\vartheta\,\boldsymbol{\nu}} \mapsto e^{\imath\,\vartheta}$.

It remains to determine the marking of $\widehat{\Delta}_{\boldsymbol{\nu}}$, that is, the assignment to each facet $\widehat{F}_j$ of the order $s_j \geq 1$ of the structure group G_j of an arbitrary $\hat{m} \in \widehat{M}^0_j$. By construction, given $m \leftarrow x \rightarrow \hat{m}$, up to isomorphism G_j may be identified with the stabilizer subgroup $T^1_x \leqslant T^1$ of x under μ^X. Now if $\mu^X_{e^{\vartheta\,\boldsymbol{\nu}}}(x) = x$ then $\mu^M_{e^{\vartheta\,\boldsymbol{\nu}}}(m) = m$ by equivariance of π. Since $m \in M^0_j$, this means that for some (unique) $e^{\imath\,\vartheta'} \in S^1$

$$e^{\vartheta\,\tilde{\boldsymbol{\nu}}} = e^{\vartheta'\,\boldsymbol{v}_j} = e^{\vartheta'\,(\boldsymbol{v}'_j + \rho_j\,\tilde{\boldsymbol{\nu}})} \quad \Rightarrow \quad e^{\vartheta'\,\boldsymbol{v}'_j} = e^{(\vartheta - \rho_j\,\vartheta')\,\tilde{\boldsymbol{\nu}}} \in T^{d-1}_c \cap T^1 = (1) \tag{63}$$

(notation as in (13)).

In particular, since $\tilde{\boldsymbol{\nu}}$ is primitive, we see from (63) that $e^{\imath\,\vartheta} = e^{\imath\,\rho_j\,\vartheta'} \in S^1$. Let us distinguish the following cases, depending on the relation between $\widetilde{\boldsymbol{\nu}}$ and $\boldsymbol{v}_j$.

Case 1. Suppose $\rho_j = 0$, that is, $\boldsymbol{v}_j = \boldsymbol{v}'_j = \widehat{\boldsymbol{v}}_j \in L(T^{d-1}_c) \subset \mathfrak{t}^{d-1}_c$. Then $e^{\imath\,\vartheta} = 1$ (and since $\boldsymbol{v}_j$ is primitive we obtain from (63) that $e^{\imath\,\vartheta'} = 1$ as well). Thus G_x is trivial in this case, that is, $s_j = 1 = (\widehat{\upsilon}_j)$.

At the opposite extreme, suppose that $\boldsymbol{v}_j \wedge \widetilde{\boldsymbol{\nu}} = 0$. Then $\boldsymbol{v}'_j = 0$ and $\boldsymbol{v}_j = \pm\widetilde{\boldsymbol{\nu}}$ by primitivity.

Case 2. Assume first $\boldsymbol{v}_j = \widetilde{\boldsymbol{\nu}}$, that is, $\rho_j = 1$; thus $\widehat{\boldsymbol{v}}_j = -(\lambda_j + \delta)\,\widetilde{\boldsymbol{\nu}}$. As $m \in F_j$,

$$0 < \Phi^{\widetilde{\boldsymbol{\nu}}}(m) = \Psi^{\widetilde{\boldsymbol{\nu}}}(m) + \delta = \Psi^{\boldsymbol{v}_j}(m) + \delta = \lambda_j + \delta. \tag{64}$$

On the other hand, by (37), Lemma 2.3, and Lemma 2.2

$$\mu^X_{e^{\vartheta\,\tilde{\boldsymbol{\nu}}}}(x) = \gamma^X_{e^{\vartheta\,\tilde{\boldsymbol{\nu}}}} \circ \rho^X_{e^{\imath\delta\vartheta}}(x) = \gamma^X_{e^{\vartheta\,\boldsymbol{v}_j}} \circ \rho^X_{e^{\imath\delta\vartheta}}(x) = e^{-\imath\,\vartheta\,(\lambda_j+\delta)}\,x. \tag{65}$$

Hence if $\widetilde{\boldsymbol{\nu}} = \boldsymbol{v}_j$, then $\delta + \lambda_j > 0$, and G_x is isomorphic to the group of $(\lambda_j + \delta)$-th roots of unity, that is, $s_j = \lambda_j + \delta = (\widehat{\boldsymbol{v}}_j)$.

Case 3. If $\boldsymbol{v}_j = -\widetilde{\boldsymbol{\nu}}$, then $\rho_j = -1$; hence $\widehat{\boldsymbol{v}}_j = -(\lambda_j - \delta)\,\widetilde{\boldsymbol{\nu}}$. In place of (64) we have

$$0 < \Phi^{\widetilde{\boldsymbol{\nu}}}(m) = \Psi^{\widetilde{\boldsymbol{\nu}}}(m) + \delta = \Psi^{-\boldsymbol{v}_j}(m) + \delta = -\lambda_j + \delta, \tag{66}$$

and in place of (65) we obtain

$$\mu^X_{e^{\vartheta}\,\tilde{\boldsymbol{\nu}}}(x) = \gamma^X_{e^{\vartheta}\,\tilde{\boldsymbol{\nu}}} \circ \rho^X_{e^{\imath\delta\vartheta}}(x) = \gamma^X_{e^{-\vartheta}\,\boldsymbol{\upsilon}_j} \circ \rho^X_{e^{\imath\delta\vartheta}}(x) = e^{-\imath\,\vartheta\,(-\lambda_j+\delta)}\,x. \tag{67}$$

Hence $\delta - \lambda_j > 0$ and G_x is isomorphic to the group of $(\delta-\lambda_j)$-th roots of unity. In particular, $s_j = \delta - \lambda_j = (\widehat{\boldsymbol{\upsilon}}_j)$.

Case 4. Suppose that $\boldsymbol{\upsilon}_j \notin L(T^1) \cup L(T^{d-1}_c)$, that is, $\rho_j\,\boldsymbol{\upsilon}'_j \neq \mathbf{0}$. By (63) $e^{\imath\,\vartheta'}$ is a $(\boldsymbol{\upsilon}'_j)$-th root of unity, and $e^{\imath\,\vartheta} = e^{\imath\,\vartheta'\,\rho_j}$. We have

$$\begin{aligned}\mu^X_{e^{\vartheta}\,\tilde{\boldsymbol{\nu}}}(x) &= \gamma^X_{e^{\vartheta}\,\tilde{\boldsymbol{\nu}}} \circ \rho^X_{e^{\vartheta}\,\delta}(x)\\ &= \gamma^X_{e^{\vartheta'}\,\boldsymbol{\upsilon}_j} \circ \rho^X_{e^{\vartheta'}\,\rho_j\,\delta}(x) = e^{-\imath\,\vartheta'\,(\lambda_j+\rho_j\,\delta)}\,x.\end{aligned} \tag{68}$$

Thus, $e^{\vartheta}\,\tilde{\boldsymbol{\nu}} \in G_x$ if and only if $e^{\imath\,\vartheta} = e^{\imath\,\rho_j\,\vartheta'}$, where $e^{\imath\,\vartheta'}$ is both a $(\boldsymbol{\upsilon}'_j)$-th root of unity (by (63) and a $(\lambda_j+\rho_j\,\delta)$-th root of unity (by (68)), i.e. a $G.C.D.\big((\boldsymbol{\upsilon}'_j), \lambda_j+\rho_j\,\delta\big)$-th root of unity.

Since $\boldsymbol{\upsilon}_j$ is primitive, $G.C.D.\big((\boldsymbol{\upsilon}'_j), \rho_j\big) = 1$; therefore also

$$G.C.D.\Big(G.C.D.\big((\boldsymbol{\upsilon}'_j), \lambda_j+\rho_j\,\delta\big), \rho_j\Big) = 1.$$

Hence we may assume that $e^{\imath\,\vartheta}$ is a primitive $G.C.D.\big((\boldsymbol{\upsilon}'_j), \lambda_j+\rho_j\,\delta\big)$-th root of unity. In other words, G_x is isomorphic to the group of $G.C.D.\big((\boldsymbol{\upsilon}'_j), \lambda_j+\rho_j\,\delta\big)$-th roots of unity, whence by Proposition 2.9

$$s_j = G.C.D.\big((\boldsymbol{\upsilon}'_j), \lambda_j+\rho_j\,\delta\big) = \big(\widehat{\boldsymbol{\upsilon}}_j\big)\,.$$

The proof of Theorem 1.1 is complete. □

Proof of Corollary 1.1. Since J and $\widehat{J}_{\boldsymbol{\nu}}$ are torus invariant complex structures on M and $\widehat{M}_{\boldsymbol{\nu}}$, respectively, by Theorem 9.1 of [11] both M and $\widehat{M}_{\boldsymbol{\nu}}$ have structures of complex toric varieties (of course in the case of M this is our starting assumption); furthermore, the corresponding fans $\mathrm{Fan}(M)$ and $\mathrm{Fan}(\widehat{M}_{\boldsymbol{\nu}})$ are defined by their respective polytopes, Δ and $\widehat{\Delta}_{\boldsymbol{\nu}}$. Since Δ and $\widehat{\Delta}_{\boldsymbol{\nu}}$ are simple and compact, $\mathrm{Fan}(M)$ and $\mathrm{Fan}(\widehat{M}_{\boldsymbol{\nu}})$ are simplicial and complete.

Hence the Betti numbers β_j and $\widehat{\beta}_j$ of M and $\widehat{M}_{\boldsymbol{\nu}}$ are determined by the collection of the all the numbers d_r and $\widehat{d}_r$ of r-dimensional cones in $\mathrm{Fan}(M)$ and $\mathrm{Fan}(\widehat{M}_{\boldsymbol{\nu}})$, respectively (§4.5 of [2]). Thus it suffices to prove that $d_r = \widehat{d}_r$ for any r.

On the other hand, in order to determine the fan Fan_Γ associated to a polytope Γ we may assume without loss that Γ contains the origin in its interior; in this case, furthermore, the cones in Fan_Γ are the cones over the faces of the polar polytope Γ^0 to Γ (§1.5 of [2]). Hence we need to show the polar polytopes of (suitable translates of) Δ and $\widehat{\Delta}_{\boldsymbol{\nu}}$ share the same number of faces in each dimension. However, for any d-dimensional polytope Γ in a d-dimensional real vector space, containing the origin in its interior, there is an order-reversing bijection between the faces of Γ and those of Γ^0, with corresponding faces F and F^* having dimensions adding up to $d-1$ ([2]). Thus the statement follows from Proposition 2.8. □

It is in order to briefly digress on how $\widehat{\Delta}_{\boldsymbol{\nu}}$ in Theorem 1.1 depends on the choice of $T^{d-1}_c \leqslant T^d$.

Suppose first that $\boldsymbol{\delta} = \mathbf{0}$ in (11), so that μ^X is the restriction of γ^X to T^1. Let S^{d-1}_c, $T^{d-1}_c \leqslant T^d$ be different complementary subtori to T^1, so that $T^d \cong S^{d-1}_c \times T^1 \cong T^{d-1}_c \times T^1$; thus projecting onto T^{d-1}_c along T^1 yields an isomorphism $P : S^{d-1}_c \cong T^{d-1}_c$. Let us choose an isomorphism of the standard torus $T^{d-1}_{st} = (S^1)^{d-1}$ with S^{d-1}_c, so that (composing with P) $T^{d-1}_{st} \cong S^{d-1}_c \cong T^{d-1}_c$.

We obtain actions φ^X and ψ^X of T^{d-1}_{st} on X, by composing the previous isomorphisms with the restrictions of γ^X to S^{d-1}_c and T^{d-1}_c respectively; then $\varphi^X \neq \psi^X$ (unless $S^{d-1}_c = T^{d-1}_c$) and, by construction, the two actions differ by the composition of a character $T^{d-1}_{st} \to T^1$ with μ^X. On the other hand, since φ^X and ψ^X commute with μ^X, they descend to symplectic actions $\varphi^{\widehat{M}_{\boldsymbol{\nu}}}$ and $\psi^{\widehat{M}_{\boldsymbol{\nu}}}$ of T^{d-1}_{st} on $(\widehat{M}_{\boldsymbol{\nu}}, 2\,\widehat{\omega}_{\boldsymbol{\nu}})$; in fact, by the previous remark and the construction of $\widehat{M}_{\boldsymbol{\nu}}$ as a quotient, $\varphi^{\widehat{M}_{\boldsymbol{\nu}}} = \psi^{\widehat{M}_{\boldsymbol{\nu}}}$.

Thus φ^X and ψ^X are different contact lifts to X of the same symplectic action of T^{d-1}_{st} on $\widehat{M}_{\boldsymbol{\nu}}$; hence they correspond to different Hamiltonian structures for the latter action, whose moment maps differ by a translation in $\mathfrak{t}^{d-1\vee}_{st}$.

Let us consider the action of $\widehat{T}^d := T^{d-1}_{st} \times S^1$ on $\widehat{M}_{\boldsymbol{\nu}}$ given by the product of $\varphi^{\widehat{M}_{\boldsymbol{\nu}}} = \psi^{\widehat{M}_{\boldsymbol{\nu}}}$ and $\rho^{\widehat{M}_{\boldsymbol{\nu}}}$, where the latter is the action of S^1 on $\widehat{M}_{\boldsymbol{\nu}}$ obtained by descending ρ^X. Let us adopts the previous choices of Hamiltonian structures (for the second factor, we use the same Hamiltonian structure as in the proof of Theorem 1.1, see Propositions 2.3 and 2.4). The corresponding moment maps then differ by a translation by a constant in $\mathfrak{t}^{d-1\vee}_c \times \{0\}$; hence so do the corresponding moment polytopes, say $\widehat{\Delta}^{\psi}_{\boldsymbol{\nu}}$ and $\widehat{\Delta}^{\varphi}_{\boldsymbol{\nu}}$.

The previous considerations may be extended to the case where $\boldsymbol{\delta} \neq \mathbf{0}$, and therefore $\varphi^{\widehat{M}_{\boldsymbol{\nu}}} \neq \psi^{\widehat{M}_{\boldsymbol{\nu}}}$. In fact, if $\boldsymbol{\delta} \neq \mathbf{0}$, then φ^X and ψ^X differ by the composition of a morphism $T^{d-1}_{st} \to T^1$, say of the form $e^{\imath\,\boldsymbol{\vartheta}} \mapsto e^{\imath\,\langle \mathbf{a}, \boldsymbol{\vartheta}\rangle\,\tilde{\boldsymbol{\nu}}}$ where $\mathbf{a} \in \mathbb{Z}^{d-1}$, with γ^X. Hence, passing to the quotient, in view of (37) the induced actions $\varphi^{\widehat{M}_{\boldsymbol{\nu}}}$ and $\psi^{\widehat{M}_{\boldsymbol{\nu}}}$ will now differ by the composition of a character $T^{d-1}_{st} \to S^1$ of the form $e^{\imath\,\vartheta} \mapsto e^{-\imath\,\delta\,\langle \mathbf{a},\vartheta\rangle}$ with $\rho^{\widehat{M}_{\boldsymbol{\nu}}}$. Identifying the coalgebra of T^{d-1}_{st} with $\imath\,\mathbb{R}^{d-1}$, the corresponding moment maps Φ^{ψ} and Φ^{φ} for $\psi^{\widehat{M}_{\boldsymbol{\nu}}}$ and $\varphi^{\widehat{M}_{\boldsymbol{\nu}}}$ are related by a relation of the form $\Phi^{\psi} = \Phi^{\varphi} - \delta\,\mathbf{a}\,\Gamma$, where $\Gamma : \widehat{M}_{\boldsymbol{\nu}} \to \imath\,\mathbb{R}$ is the moment map for $\rho^{\widehat{M}_{\boldsymbol{\nu}}}$ (recall from Proposition 2.4 that $\Gamma(\hat{m}) = \imath\,\Phi^{\tilde{\boldsymbol{\nu}}}(m)^{-1}$ if $m \leftarrow x \to \hat{m}$).

It follows that the two cones are related by a transformation in $\imath\,\mathbb{R}^d \times \imath\,\mathbb{R}$ of the form $\imath\,(\mathbf{x}, y) \mapsto \imath\,(\mathbf{x} - y\,\delta\,\mathbf{a}, y)$, followed perhaps by a translation.

3. The case of arbitrary *r*

We shall now remove the restriction that $r = 1$, and allow any value $1 \le r \le d$. Before dealing directly with the geometric situation, we shall dwell on some handy technical results.

3.1. Preliminaries on transversality of polytopes

Definition 3.1. Let V be a finite dimensional real vector space, $\Gamma \subset V$ a convex polytope, $W \subseteq V$ an affine subspace. We shall say that Γ and W *meet transversely*, or that they are transverse to each other, if W is transverse to the relative interior F^0 of every face F of Γ.

In the hypothesis of Definition 3.1, let us set $\Gamma_W := \Gamma \cap W$. Clearly, Γ_W is a convex polytope in W.

Let $\mathcal{F}(\Gamma)$ be the collection of faces of Γ and $\mathcal{G}(\Gamma) = \{F_1, \ldots, F_s\} \subseteq \mathcal{F}(\Gamma)$ be the subset of its facets. For each $j = 1, \ldots, s$ let $\ell_j \in V^{\vee}$ be an inward normal covector to F_j, so that

$$\Gamma = \bigcap_{j=1}^{s} \left\{ v \in V \,:\, \ell_j(v) \ge \lambda_j \right\}$$

for certain $\lambda_j \in \mathbb{R}$; the j-th facet is thus

$$F_j := \Gamma \cap \left\{ p \in V \,:\, \ell_j(p) = \lambda_j \right\}. \tag{69}$$

If $L \in \mathcal{F}(\Gamma)$, there exists a unique subset $I_L \subseteq \{1, \ldots, s\}$ such that $L = \bigcap_{j \in I_L} F_j$.

We are interested in *simple* polytopes (meaning that exactly n facets of Γ meet at each vertex, where $n = \dim_{\mathbb{R}}(V)$ [11]; if Γ is simple, then every codimension-k face $L \in \mathcal{F}(\Gamma)$ is the intersection of exactly k facets, that is, $|I_L|$ equals the codimension of L.

Proposition 3.1. *In the setting of Definition 3.1, suppose that Γ and W meet transversely. The following holds.*

1. *If $F \subseteq L$ are faces of Γ and $W \cap F^0 \neq \emptyset$, then $W \cap L^0 \neq \emptyset$.*
2. *If $W \cap \Gamma \neq \emptyset$, then $W \cap \Gamma^0 \neq \emptyset$*

More precisely, regarding 1. we shall show that for any $p \in W \cap F^0$ and any open neighborhood W' of p in W one has $W' \cap L^0 \neq \emptyset$. Similarly, regarding 2. we shall show that for any $p \in \Gamma_W$ and any open neighborhood W' of p in W one has $W' \cap \Gamma^0 \neq \emptyset$.

Proof of 1. Since $W \subseteq V$ is an affine subspace, it is a translate of a vector subspace $\widehat{W} \subseteq V$. Suppose $p \in F^0 \cap W$. Then $W = p + \widehat{W}$, and by transversality the map $\rho : (w, q) \in \widehat{W} \times F^0 \mapsto w + q \in V$ is submersive, hence open. We have $p = \rho(\mathbf{0}, p)$, hence the image of an arbitrary small neighborhood of $(\mathbf{0}, p)$ in $\widehat{W} \times F^0$ contains an open neighborhood of p.

Since $p \in F \subseteq L$, we can find points $p' \in L^0$ arbitrarily close to p. For any such p', therefore, there exist $w \in \widehat{W}$, $w \sim \mathbf{0}$, and $q \in F^0$, $q \sim p$, such that $p' = w + q$.

Claim 3.1. *With the previous choices, $w + p \in W \cap L^0$.*

Proof of Claim 3.1. Clearly, $w + p \in W$ by construction. Let us prove that $w + p \in L^0$, i.e. that $\ell_j(w + p) = \lambda_j$ if $j \in I_L$ and $\ell_j(w + p) > \lambda_j$ if $j \notin I_L$.

Since $F \subseteq L$, we have $I_L \subseteq I_F$.

If $j \in I_L$, we have $\ell_j(p') = \ell_j(q) = \ell(p) = \lambda_j$ since p, p', $q \in L$. Therefore, $\ell_j(w) = \ell_j(p' - q) = \lambda_j - \lambda_j = 0$. Thus $\ell_j(w + p) = 0 + \lambda_j = \lambda_j$ for every $j \in I_L$, so that $w + p \in L$.

If $j \in I_F \setminus I_L$, we have $\ell_j(p') > \lambda_j$ since $p' \in L^0$, and $\ell_j(p) = \ell_j(q) = \lambda_j$ since $p, q \in F$. Therefore, $\ell_j(w) = \ell_j(p' - q) = \ell_j(p') - \ell_j(q) > 0$, and so $\ell_j(w + p) = \ell_j(w) + \lambda_j > \lambda_j$.

Let $\delta := \min\{\ell_j(p) - \lambda_j \,:\, j \notin I_F\}$; thus $\delta > 0$, since $p \in F^0$. If $j \notin I_F$, therefore, $\ell_j(p + w) \ge \lambda_j + \delta/2 > \lambda_j$ as $w \sim \mathbf{0}$. $\square$

Thus for every $p \in W \cap F^0$ we have found points $p + w$ arbitrarily close to p in $W \cap L^0$, and this completes the proof of 1. □

Proof of 2. This is a slight modification of the previous argument. Suppose $p \in \Gamma \cap W$. If $p \in \Gamma^0$, there is nothing to prove. Otherwise, $p \in F^0$ for some face $F \in \mathcal{F}(\Gamma)$. We can find $p' \in \Gamma^0$ arbitrarily close to p, and therefore - by the previous considerations - for any such p' there exist $w \in \widehat{W}$ with $w \sim \mathbf{0}$ and $q \in F^0$ with $q \sim p$ such that $p' = w + q$.

If $j \in I_F$, $\ell_j(p) = \ell_j(q) = \lambda_j$ since $p, q \in F$, and $\ell_j(p') > \lambda_j$ since $p' \in \Gamma^0$. Therefore $\ell_j(w) = \ell_j(p') - \ell_j(q) > 0$. Hence $\ell_j(p + w) = \lambda_j + \ell_j(w) > \lambda_j$ for all $j \in I_F$.

Let $\delta := \min\{\ell_j(p) - \lambda_j : j \notin I_F\}$; then $\delta > 0$ since $p \in F^0$. Since $w \sim \mathbf{0}$, $\ell_j(p + w) \geq \lambda_j + \delta/2 > \lambda_j$ for all $j \notin I_F$.

Thus $\ell_j(p + w) > \lambda_j$ for every $j = 1, \ldots, r$, i.e. $p + w \in W \cap \Gamma^0$. □

Corollary 3.1. *Under the hypothesis of Proposition 3.1, if L is a face of Γ and $\Gamma_W \cap L \neq \emptyset$, then $\Gamma_W \cap L^0 \neq \emptyset$.*

Proof. If $p \in W \cap L$, there is a face $F \in \mathcal{F}(\Gamma)$ with $F \subseteq L$ and $p \in W \cap F^0$. Hence $W \cap L^0 \neq \emptyset$ by Proposition 3.1. □

Proposition 3.2. *Under the hypothesis of Proposition 3.1, the following holds:*

1. *the facets of Γ_W are the non-empty intersections of W with the facets of Γ;*
2. *$\Gamma_W^0 = W \cap \Gamma^0$;*
3. *if Γ is simple, then the codimension-k faces of Γ_W are the non-empty intersections of W with the codimension-k faces of Γ;*
4. *if F is a face of Γ such that $F_W := F \cap W \neq \emptyset$, then the relative interior of F_W is $F_W^0 = F^0 \cap W$;*
5. *if Γ is simple, then so is Γ_W.*

Proof of 1. Let F_j be the j-th facet of Γ as in (69), and suppose $F_j \cap W \neq \emptyset$. Then $F_j^0 \cap W \neq \emptyset$ by Corollary 3.1. Since W meets F_j^0 transversely by assumption, $F_j^0 \cap W$ has codimension one in W and ℓ_j restricts to a non-constant affine linear functional on W. Thus, if $p \in F_j^0$ then every neighborhood of p in W intersects both Γ^0 and Γ^c. It follows that $F_j \cap W$ is a facet of Γ_W.

Conversely, let F be a facet of Γ_W, and let $p \in F^0$. Since $p \notin \Gamma^0$ (for else $p \in \Gamma_W^0$), there exists $j \in \{1, \ldots, s\}$ such that $p \in F_j$, and therefore $F_j^0 \cap W \neq \emptyset$. By the above $F_j \cap W$ is a facet of Γ_W. Since a small neighborhood of p in W meets no facet of Γ_W other that F, we may slightly perturb p in F^0 and assume that $p \in F^0 \cap F_j^0$ and therefore $F^0 \cap W' = F_j^0 \cap W'$ for some open neighborhood W' of p in W. This forces $F = F_j \cap W$. □

Proof of 2. Since every face is the intersection of the facets containing it, by 1. we have

$$\Gamma_W^0 = (W \cap \Gamma) \setminus \bigcup_{j=1}^{s} (W \cap F_j) = W \cap \left(\Gamma \setminus \bigcup_{j=1}^{s} F_j \right) = W \cap \Gamma^0. \quad \square$$

Proof of 3. and 4. If F is a codimension-k face of Γ such that $F \cap W \neq \emptyset$, let us choose $p \in F^0 \cap W \neq \emptyset$. Since Γ is simple, $I_F = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, s\}$ and there is a neighborhood W' of p in W such that

$$W' \cap F = W' \cap F^0 = W' \cap \bigcap_{j=1}^{k} F_{i_j}.$$

By transversality, $F^0 \cap W$ has codimension k in W and furthermore each F_{i_j} is a facet of Γ_W by 1. Thus $F_W := F \cap W$ is a face of Γ_W, since it is a non-empty intersection of facets, and it has codimension-k in W, since it has a non-empty open subset which has codimension k. Furthermore, it is given by the intersection of the k facets $F_{i_j} \cap \Gamma_W$ of Γ_W. It then follows that

$$F_W^0 = \bigcap_{j=1}^{k} (W \cap F_{i_j}) \setminus \bigcup_{i \notin I_F} (W \cap F_i) = W \cap \left(\bigcap_{j=1}^{k} F_{i_j} \setminus \bigcup_{i \notin I_F} F_i \right) = W \cap F^0.$$

Conversely, suppose that $K \subset \Gamma_W$ is a face, and suppose $p \in K^0$. Since $p \notin \Gamma^0$, there exists a unique face F of Γ such that $p \in F^0$. By the previous discussion, $F_W = F \cap W$ is also a face of Γ_W, and $p \in F^0 \cap W = F_W^0$. Since distinct faces of Γ_W have disjoint relative interiors, $K = F_W$. □

Proof of 5. By 3., every codimension-k face of Γ_W is the intersection of k facets, and this means that Γ_W is simple. □

3.2. The reduction $\overline{M}_{\boldsymbol{\nu}}$ and the circle bundle $Y_{\boldsymbol{\nu}}$

Definition 3.2. Given a subspace $\mathfrak{a} \subseteq \mathfrak{t}^{r\vee}$, we shall denote by $\mathfrak{a}^{\perp} \subseteq \mathfrak{t}^{r}$ its annihilator in $\mathfrak{t}^r$. Given $\boldsymbol{\rho} \in \mathfrak{t}^{r\vee}$ we shall also set $\boldsymbol{\rho}^{\perp} := \mathrm{span}(\boldsymbol{\rho})^{\perp}$.

Definition 3.3. A vector subspace $\mathfrak{c} \subseteq \mathfrak{t}^d$ is *integral* if $\mathfrak{c} \cap L(T)$ is a full-rank lattice in $\mathfrak{c}$ or, equivalently, if $\mathfrak{c}$ is the Lie subalgebra of a closed embedded torus in T^d.

Thus $\boldsymbol{\nu}^{\perp} \subseteq \mathfrak{t}^r \subseteq \mathfrak{t}^d$ is a vector subspace of dimension $r-1$; since $\boldsymbol{\nu} \in L(T)^{\vee}$, furthermore, $\boldsymbol{\nu}^{\perp}$ is integral. Let $T^{r-1}_{\boldsymbol{\nu}^{\perp}} \leqslant T^r \leqslant T^d$ be the (closed) torus with Lie algebra $\boldsymbol{\nu}^{\perp}$; equivalently, if $\chi_{\boldsymbol{\nu}}(e^{\boldsymbol{\xi}}) := e^{2\pi\imath\langle \boldsymbol{\nu},\boldsymbol{\xi}\rangle}$ is the character of T^r defined by $\boldsymbol{\nu}$, then $T^{r-1}_{\boldsymbol{\nu}^{\perp}} = \ker(\chi_{\boldsymbol{\nu}})^0$ (the connected component of the kernel).

We are interested in $\widehat{M}_{\boldsymbol{\nu}} = X_{\boldsymbol{\nu}}/T^r$, the action being μ^X (notation as in (3) and (4)). The latter quotient may be performed in stages. Namely, under Basic Assumption 1.1, T^r acts in a locally free manner on $X_{\boldsymbol{\nu}}$, whence *a fortiori* so does $T^{r-1}_{\boldsymbol{\nu}^{\perp}}$. We first form the partial quotient $Y_{\boldsymbol{\nu}} := X_{\boldsymbol{\nu}}/T^{r-1}_{\boldsymbol{\nu}^{\perp}}$, where $T^{r-1}_{\boldsymbol{\nu}^{\perp}}$ acts on $X_{\boldsymbol{\nu}}$ by μ^X; next, μ^X descends to a residual locally free action of $T^1_{\boldsymbol{\nu}} := T^r/T^{r-1}_{\boldsymbol{\nu}^{\perp}}$ on $Y_{\boldsymbol{\nu}}$,

$$\mu^{Y_{\boldsymbol{\nu}}} : T^1_{\boldsymbol{\nu}} \times Y_{\boldsymbol{\nu}} \to Y_{\boldsymbol{\nu}}. \tag{70}$$

Then

$$\widehat{M}_{\boldsymbol{\nu}} = X_{\boldsymbol{\nu}}/T^r = Y_{\boldsymbol{\nu}}/T^1_{\boldsymbol{\nu}}. \tag{71}$$

$Y_{\boldsymbol{\nu}}$ inherits a natural contact structure. Let $\jmath_{\boldsymbol{\nu}} : X_{\boldsymbol{\nu}} \hookrightarrow X$ be the inclusion. Then $\jmath_{\boldsymbol{\nu}}^*(\alpha)$ is $T^{r-1}_{\boldsymbol{\nu}^{\perp}}$-invariant. Furthermore, (writing Φ for $\Phi \circ \pi$ with abuse of notation) by definition of $X_{\boldsymbol{\nu}}$ we have $\Phi \circ \jmath_{\boldsymbol{\nu}} = \lambda\,\boldsymbol{\nu}$ for some $\mathcal{C}^{\infty}$ function $\lambda : X_{\boldsymbol{\nu}} \to \mathbb{R}_+$. Hence, if $\boldsymbol{\xi} \in \mathfrak{t}^{r-1}_{\boldsymbol{\nu}^{\perp}} = \boldsymbol{\nu}^{\perp}$ then $\boldsymbol{\xi}^{\Phi}_X$ satisfies

$$\boldsymbol{\xi}^{\Phi}_X\big|_{X_{\boldsymbol{\nu}}} = \boldsymbol{\xi}^{\sharp}_M\Big|_{X_{\boldsymbol{\nu}}} - \lambda\,\langle \boldsymbol{\nu}, \boldsymbol{\xi}\rangle\;\partial_{\vartheta}|_{X_{\boldsymbol{\nu}}} = \boldsymbol{\xi}^{\sharp}_M\Big|_{X_{\boldsymbol{\nu}}} \in \ker\left(\jmath_{\boldsymbol{\nu}}^*(\alpha)\right).$$

In other words, $[\iota(\boldsymbol{\xi}^{\Phi}_X)\,\alpha] \circ \jmath_{\boldsymbol{\nu}} = 0$.

We conclude the following. Let $q_{\boldsymbol{\nu}} : X_{\boldsymbol{\nu}} \to Y_{\boldsymbol{\nu}}$ be the projection. Thus we have arrows

$$Y_{\boldsymbol{\nu}} \xleftarrow{q_{\boldsymbol{\nu}}} X_{\boldsymbol{\nu}} \overset{\jmath_{\boldsymbol{\nu}}}{\hookrightarrow} X.$$

Lemma 3.1. *There exists a differential 1-form (in the orbifold sense) $\alpha_{\boldsymbol{\nu}} \in \Omega^1(Y_{\boldsymbol{\nu}})$, such that $q_{\boldsymbol{\nu}}^*(\alpha_{\boldsymbol{\nu}}) = \jmath_{\boldsymbol{\nu}}^*(\alpha)$.*

Under the stronger condition that $T^{r-1}_{\boldsymbol{\nu}^{\perp}}$ acts freely on $M_{\boldsymbol{\nu}}$, the quotient

$$\overline{M}_{\boldsymbol{\nu}} := M_{\boldsymbol{\nu}}/T^{r-1}_{\boldsymbol{\nu}^{\perp}} \tag{72}$$

is smooth; furthermore, the action of $T^{r-1}_{\boldsymbol{\nu}^{\perp}}$ on $X_{\boldsymbol{\nu}}$ induced by μ^X is also free, and therefore $Y_{\boldsymbol{\nu}}$ is non-singular. In addition, ρ^X (the action on X generated by $-\partial_{\theta}$) descends to a free S^1-action $\rho^{Y_{\boldsymbol{\nu}}} : S^1 \times Y_{\boldsymbol{\nu}} \to Y_{\boldsymbol{\nu}}$, and we also have

$$\overline{M}_{\boldsymbol{\nu}} = Y_{\boldsymbol{\nu}}/S^1. \tag{73}$$

In addition, $\alpha_{\boldsymbol{\nu}}$ in Lemma 3.1 is a connection form for $\rho^{Y_{\boldsymbol{\nu}}}$.

Furthermore, $\overline{M}_{\boldsymbol{\nu}}$ inherits a complex structure $\overline{J}_{\boldsymbol{\nu}}$ and a compatible symplectic structure $\overline{\omega}_{\boldsymbol{\nu}}$; the triple $(\overline{M}_{\boldsymbol{\nu}}, 2\,\overline{\omega}_{\boldsymbol{\nu}}, \overline{J}_{\boldsymbol{\nu}})$ is a Hodge manifold. More precisely, $(\overline{M}_{\boldsymbol{\nu}}, 2\,\overline{\omega}_{\boldsymbol{\nu}})$ is the Marsden-Weinstein reduction of $(M, 2\,\omega)$ under the restriction of the Hamiltonian action (μ^M, Φ) to $T^{r-1}_{\boldsymbol{\nu}^{\perp}} \leqslant T^r$, and $\overline{J}_{\boldsymbol{\nu}}$ is determined from J as in [5].

Let $\mathfrak{t}^1_{\boldsymbol{\nu}}$ denote the Lie algebra of $T^1_{\boldsymbol{\nu}}$, so that

$$\mathfrak{t}^1_{\boldsymbol{\nu}} \cong \mathfrak{t}^r/\boldsymbol{\nu}^{\perp}, \quad \mathfrak{t}^{1\vee}_{\boldsymbol{\nu}} \cong \mathrm{span}(\boldsymbol{\nu}).$$

By definition of $M_{\boldsymbol{\nu}}$, restricting Φ yields a map $\Phi' : M_{\boldsymbol{\nu}} \to \mathrm{span}(\boldsymbol{\nu}) \cong \mathfrak{t}^{1\vee}_{\boldsymbol{\nu}}$, which descends to a non-vanishing $T^1_{\boldsymbol{\nu}}$-equivariant map $\overline{\Phi} : \overline{M}_{\boldsymbol{\nu}} \to \mathfrak{t}^{1\vee}_{\boldsymbol{\nu}}$.

Let $\overline{\pi}_{\boldsymbol{\nu}} : Y_{\boldsymbol{\nu}} \to \overline{M}_{\boldsymbol{\nu}}$ and $\widehat{\pi}_{\boldsymbol{\nu}} : Y_{\boldsymbol{\nu}} \to \widehat{M}_{\boldsymbol{\nu}}$ be the projections. From the previous discussion and the theory in [5] one obtains the following.

Proposition 3.3. *There is a positive holomorphic line bundle $(A_{\boldsymbol{\nu}}, h_{\boldsymbol{\nu}})$ on $\overline{M}_{\boldsymbol{\nu}}$, such that:*

1. *$Y_{\boldsymbol{\nu}}$ is the unit circle bundle in $A^{\vee}_{\boldsymbol{\nu}}$;*

2. *$\alpha_{\boldsymbol{\nu}}$ is the normalized connection form associated to the unique compatible covariant derivative on $A_{\boldsymbol{\nu}}$;*
3. *$\mathrm{d}\alpha_{\boldsymbol{\nu}} = 2\,\overline{\pi}_{\boldsymbol{\nu}}^*(\overline{\omega}_{\boldsymbol{\nu}})$;*
4. *$\mu^{Y_{\boldsymbol{\nu}}}$ in (70) descends to an action $\mu^{\overline{M}_{\boldsymbol{\nu}}} : T^1_{\boldsymbol{\nu}} \times \overline{M}_{\boldsymbol{\nu}} \to \overline{M}_{\boldsymbol{\nu}}$, which is holomorphic for $\overline{J}_{\boldsymbol{\nu}}$ and symplectic for $\overline{\omega}_{\boldsymbol{\nu}}$;*
5. *$\mu^{\overline{M}_{\boldsymbol{\nu}}}$ is Hamiltonian for $2\,\overline{\omega}_{\boldsymbol{\nu}}$, with moment map $\overline{\Phi}$;*
6. *$\mu^{Y_{\boldsymbol{\nu}}}$ is the contact lift of $(\mu^{\overline{M}_{\boldsymbol{\nu}}}, \overline{\Phi})$.*

In other words, the description of $\widehat{M}_{\boldsymbol{\nu}}$ can be abstractly reduced to the case $r = 1$, with M replaced by $\overline{M}_{\boldsymbol{\nu}}$ and X by $Y_{\boldsymbol{\nu}}$. We need to describe how to transfer the toric structure to this quotient picture.

3.3. The toric structure of $\overline{M}_{\boldsymbol{\nu}}$

W aim to verify that the toric setting is preserved in the reduction process of Proposition 3.3. To this end, let us consider the saturation

$$\tilde{M}_{\boldsymbol{\nu}} := \mathbb{T}^{r-1}_{\boldsymbol{\nu}^\perp} \cdot M_{\boldsymbol{\nu}};$$

thus $\tilde{M}_{\boldsymbol{\nu}}$ is the set of (semi)-stable points in M for the action of the complexification $\mathbb{T}^{r-1}_{\boldsymbol{\nu}^\perp}$ of $T^{r-1}_{\boldsymbol{\nu}^\perp}$, with respect to the linearization induced by Φ (hence to the lift μ^X).

Therefore, $m \in \tilde{M}_{\boldsymbol{\nu}}$ if and only if there exists a $T^{r-1}_{\boldsymbol{\nu}^\perp}$-invariant holomorphic section σ of $A^{\otimes k}$, for some $k \geq 1$, such that $\sigma(m) \neq 0$. Equivalently, $m \in \tilde{M}_{\boldsymbol{\nu}}$ if and only if for some, and therefore for any, $x \in \pi^{-1}(m) \subseteq X$ there exists a CR function $\tilde{\sigma} \in H(X)_k$ which is $T^{r-1}_{\boldsymbol{\nu}^\perp}$-invariant under μ^X, and satisfies $\tilde{\sigma}(x) \neq 0$. Here, $T^{r-1}_{\boldsymbol{\nu}^\perp}$-invariance means that

$$\tilde{\sigma}^g = \tilde{\sigma} \quad \forall\, g \in T^{r-1}_{\boldsymbol{\nu}^\perp} \quad \text{where} \quad \tilde{\sigma}^g := \tilde{\sigma} \circ \mu^X_{g^{-1}}. \tag{74}$$

It is convenient to emphasize the holomorphic structure. Recall that $\widetilde{\gamma}^M : \mathbb{T}^d \times M \to M$ denotes the complexification of $\gamma^M : T^d \times M \to M$; similarly, we have complexified bundle actions $\widetilde{\gamma}^{A_0^\vee} : \mathbb{T}^d \times A_0^\vee \to A_0^\vee$ (the complexification of γ^X) and $\widetilde{\mu}^{A_0^\vee} : \mathbb{T}^r \times A_0^\vee \to A_0^\vee$ (the complexification of μ^X). Accordingly, we have associated linear representations of $\mathbb{T}^r$ and $\mathbb{T}^d$ on each space of global holomorphic sections $H^0(M, A^{\otimes k})$, $k = 0, 1, 2, \ldots$. In fact, $H^0(M, A^{\otimes k})$ is canonically isomorphic with the space of holomorphic functions on $A_0^\vee$ that are homogeneous of degree k, $\mathcal{H}_k(A_0^\vee) \subset \mathcal{O}(A_0^\vee)$, and given $\hat{\sigma} \in \mathcal{H}_k(A_0^\vee)$ and $g \in \mathbb{T}^r$ we set

$$\hat{\sigma}^g := \hat{\sigma} \circ \widetilde{\mu}^{A_0^\vee}_{g^{-1}}. \tag{75}$$

The correspondences

$$\hat{\sigma} \in \mathcal{H}_k(A_0^\vee) \mapsto \tilde{\sigma} \in H(X)_k \mapsto \sigma \in H^0(M, A^{\otimes k})$$

are natural and equivariant isomorphisms. Therefore, $m \in \tilde{M}_{\boldsymbol{\nu}}$ if and only if for some, and therefore for any, $\ell \in A_0^\vee$ lying over m there exists $\hat{\sigma} \in \mathcal{H}_k(A_0^\vee)$ which is $\mathbb{T}^{r-1}_{\boldsymbol{\nu}^\perp}$-invariant under (75), and satisfies $\hat{\sigma}(\ell) \neq 0$.

Lemma 3.2. *$\tilde{M}_{\boldsymbol{\nu}}$ is $\mathbb{T}^d$-invariant, that is, $\widetilde{\gamma}^M_t(\tilde{M}_{\boldsymbol{\nu}}) = \tilde{M}_{\boldsymbol{\nu}}\ \forall\, t \in \mathbb{T}^d$.*

Proof. All actions involved commute. Suppose $m \in \tilde{M}_{\boldsymbol{\nu}}$ and let $\hat{\sigma} \in \mathcal{H}_k(A_0^\vee)$ satisfy $\hat{\sigma}^g = \hat{\sigma}$ for all $g \in \mathbb{T}^r$, and be such that $\tilde{\sigma}(\ell) \neq 0$ for some (hence any) $\ell \in A_0^\vee$ lying over m. Then for any $t \in \mathbb{T}^d$ we have

$$0 \neq \hat{\sigma}(\ell) = \hat{\sigma} \circ \widetilde{\gamma}^{A_0^\vee}_{t^{-1}} \circ \widetilde{\gamma}^{A_0^\vee}_t(\ell). \tag{76}$$

Clearly, $\hat{\sigma} \circ \widetilde{\gamma}^{A_0^\vee}_{t^{-1}} \in \mathcal{H}_k(A_0^\vee)$; furthermore, by the assumed invariance of $\hat{\sigma}$ we have for every $g \in \mathbb{T}^{r-1}_{\boldsymbol{\nu}^\perp}$

$$\hat{\sigma} \circ \widetilde{\gamma}^{A_0^\vee}_{t^{-1}} \circ \widetilde{\mu}^{A_0^\vee}_{g^{-1}} = \hat{\sigma} \circ \widetilde{\mu}^{A_0^\vee}_{g^{-1}} \circ \widetilde{\gamma}^{A_0^\vee}_{t^{-1}} = \hat{\sigma} \circ \widetilde{\gamma}^{A_0^\vee}_{t^{-1}}. \tag{77}$$

Since $\widetilde{\gamma}^{A_0^\vee}_t(\ell)$ lies over $\tilde{\gamma}^M_t(m)$, (76) and (77) imply that $\tilde{\gamma}^M_t(m) \in \tilde{M}_{\boldsymbol{\nu}}$. $\square$

Recall that M^0 is the dense open subset where $\tilde{\gamma}^M$ is free and transitive. Since $\tilde{M}_{\boldsymbol{\nu}}$ and M^0 are open and dense in M, $\tilde{M}_{\boldsymbol{\nu}} \cap M^0 \neq \emptyset$. Therefore Lemma 3.2 implies the following.

Corollary 3.2. $\tilde{M}_{\boldsymbol{\nu}} \supseteq M^0$.

As is well-known, we have a natural identification [9], [12], [5]

$$\overline{M}_{\boldsymbol{\nu}} = M_{\boldsymbol{\nu}}/T^{r-1}_{\boldsymbol{\nu}^\perp} \cong \tilde{M}_{\boldsymbol{\nu}}/\mathbb{T}^{r-1}_{\boldsymbol{\nu}^\perp}, \tag{78}$$

which will be left implicit in the following. Accordingly, we shall set

$$\overline{M}^0_{\boldsymbol{\nu}} := M^0/\mathbb{T}^{r-1}_{\boldsymbol{\nu}^\perp} \subseteq \overline{M}_{\boldsymbol{\nu}}, \tag{79}$$

an open and dense subset of $\overline{M}_{\boldsymbol{\nu}}$.

Let us define the quotient tori

$$T^{d-r+1}_q := T^d/T^{r-1}_{\boldsymbol{\nu}^\perp}, \quad \mathbb{T}^{d-r+1}_q := \mathbb{T}^d/\mathbb{T}^{r-1}_{\boldsymbol{\nu}^\perp}; \tag{80}$$

clearly $\mathbb{T}^{d-r+1}_q$ is the complexification of T^{d-r+1}_q. Then there are induced quotient actions

$$\gamma^{\overline{M}_{\boldsymbol{\nu}}} : T^{d-r+1}_q \times \overline{M}_{\boldsymbol{\nu}} \to \overline{M}_{\boldsymbol{\nu}}, \quad \widetilde{\gamma}^{\overline{M}_{\boldsymbol{\nu}}} : \mathbb{T}^{d-r+1}_q \times \overline{M}_{\boldsymbol{\nu}} \to \overline{M}_{\boldsymbol{\nu}},$$

and $\widetilde{\gamma}^{\overline{M}_{\boldsymbol{\nu}}}$ is the complexification of $\gamma^{\overline{M}_{\boldsymbol{\nu}}}$. Furthermore, $T^1_{\boldsymbol{\nu}} \leqslant T^{d-r+1}_q$ is a 1-dimensional subtorus (notation as in (70) and Proposition 3.3), and the action $\mu^{\overline{M}_{\boldsymbol{\nu}}}$ in Proposition 3.3 is the restriction of $\gamma^{\overline{M}_{\boldsymbol{\nu}}}$ to $T^1_{\boldsymbol{\nu}}$.

Proposition 3.4. *$\widetilde{\gamma}^{\overline{M}_{\boldsymbol{\nu}}}$ is free and transitive on $\overline{M}^0_{\boldsymbol{\nu}}$.*

Before giving the proof let us interject some pieces of notation.

1. Let us choose a complementary subtorus $\widehat{T}^1_{\boldsymbol{\nu}} \leqslant T^r$ to $T^{r-1}_{\boldsymbol{\nu}^\perp}$, so that $T^r \cong \widehat{T}^1_{\boldsymbol{\nu}} \times T^{r-1}_{\boldsymbol{\nu}^\perp}$; projecting yields an isomorphism $\widehat{T}^1_{\boldsymbol{\nu}} \cong T^1_{\boldsymbol{\nu}}$. Having chosen $\widehat{T}^1_{\boldsymbol{\nu}}$, there is a unique primitive $\widetilde{\boldsymbol{\nu}} \in L(\widehat{T}^1_{\boldsymbol{\nu}})$ such that $\boldsymbol{\nu}(\widetilde{\boldsymbol{\nu}}) = 1$. Correspondingly, we have isomorphisms $T^r \cong \widehat{T}^1_{\boldsymbol{\nu}} \times T^{r-1}_{\boldsymbol{\nu}^\perp}$, $L(T^r) \cong \mathbb{Z}\,\widetilde{\boldsymbol{\nu}} \oplus L(T^{r-1}_{\boldsymbol{\nu}^\perp})$, and dually $L(T^r)^\vee \cong \mathbb{Z}\,\boldsymbol{\nu} \oplus L(T^{r-1}_{\boldsymbol{\nu}^\perp})^\vee$.
2. Let us choose a complementary subtorus $T^{d-r}_c \leqslant T^d$ to T^r, so that
$$T^d \cong T^{d-r}_c \times \widehat{T}^1_{\boldsymbol{\nu}} \times T^{r-1}_{\boldsymbol{\nu}^\perp}, \quad \mathbb{T}^d \cong \mathbb{T}^{d-r}_c \times \widehat{\mathbb{T}}^1_{\boldsymbol{\nu}} \times \mathbb{T}^{r-1}_{\boldsymbol{\nu}^\perp}. \tag{81}$$
Then
$$L(T^d) \cong L(T^{d-r}_c) \oplus \mathbb{Z}\,\widetilde{\boldsymbol{\nu}} \oplus L(T^{r-1}_{\boldsymbol{\nu}^\perp}) \tag{82}$$
and dually
$$L(T^d)^\vee \cong L(T^{d-r}_c)^\vee \oplus \mathbb{Z}\,\boldsymbol{\nu} \oplus L(T^{r-1}_{\boldsymbol{\nu}^\perp})^\vee. \tag{83}$$
3. Projection induces isomorphisms
$$T^{d-r+1}_c := T^{d-r}_c \times \widehat{T}^1_{\boldsymbol{\nu}} \cong T^{d-r+1}_q, \quad \mathbb{T}^{d-r+1}_c := \mathbb{T}^{d-r}_c \times \widehat{\mathbb{T}}^1_{\boldsymbol{\nu}} \cong \mathbb{T}^{d-r+1}_q; \tag{84}$$
we shall denote by $\mathbb{T}^{d-r}_q \leqslant \mathbb{T}^{d-r+1}_q$ the image of $\mathbb{T}^{d-r}_c$, so that $\mathbb{T}^{d-r+1}_q \cong \mathbb{T}^{d-r}_q \times \mathbb{T}^1_{\boldsymbol{\nu}}$.
4. If $t \in \mathbb{T}^{d-r+1}_c$, we shall denote by $\overline{t} \in \mathbb{T}^{d-r+1}_q$ its image, and for any $m \in \tilde{M}_{\boldsymbol{\nu}}$ we shall denote by $\overline{m} \in \overline{M}_{\boldsymbol{\nu}}$ its projection.

Proof. Suppose $\overline{m'}, \overline{m''} \in \overline{M}^0_{\boldsymbol{\nu}}$, and choose $m', m'' \in \tilde{M}^0_{\boldsymbol{\nu}}$ lying over them. Then there exists a unique $t \in \mathbb{T}^d$ such that $m'' = \tilde{\gamma}^M_t(m')$. Let us factor $t = t_1 t_2 t_3$ according to (81), that is, $t_1 \in \mathbb{T}^{d-r}_c$, $t_2 \in \widehat{\mathbb{T}}^1_{\boldsymbol{\nu}}$, $t_3 \in \mathbb{T}^{r-1}_{\boldsymbol{\nu}^\perp}$. Hence

$$m'' = \tilde{\gamma}^M_{t_1} \circ \tilde{\gamma}^M_{t_2} \circ \tilde{\gamma}^M_{t_3}(m) \quad \Rightarrow \quad \overline{m''} = \tilde{\gamma}^{\overline{M}_{\boldsymbol{\nu}}}_{\overline{t}_1} \circ \tilde{\gamma}^{\overline{M}_{\boldsymbol{\nu}}}_{\overline{t}_2}(\overline{m'}) = \tilde{\gamma}^{\overline{M}_{\boldsymbol{\nu}}}_{\overline{t}_1 \overline{t}_2}(\overline{m'}).$$

This establishes that $\widetilde{\gamma}^{\overline{M}_{\boldsymbol{\nu}}}$ is transitive on $\overline{M}^0_{\boldsymbol{\nu}}$.

Suppose $\overline{m} \in \overline{M}^0_{\boldsymbol{\nu}}$, $\overline{t} \in \mathbb{T}^{d-r+1}_q$ and $\overline{m} = \tilde{\gamma}^{\overline{M}_{\boldsymbol{\nu}}}_{\overline{t}}(\overline{m})$. Let us choose $m \in \tilde{M}^0_{\boldsymbol{\nu}}$ lying over m and $t \in \mathbb{T}^{d-r+1}_c$ lying over $\overline{t}$. Then there exists $t' \in \mathbb{T}^{r-1}_{\boldsymbol{\nu}^\perp}$ such that $m = \tilde{\gamma}^M_{t\,t'}(m)$; hence $t\,t' = 1$ and therefore $t = t' = 1$, so $\overline{t} = 1$. In conclusion, $\widetilde{\gamma}^{\overline{M}_{\boldsymbol{\nu}}}$ is free on $\overline{M}^0_{\boldsymbol{\nu}}$. $\square$

Corollary 3.3. *$(\overline{M}_{\boldsymbol{\nu}}, \overline{J}_{\boldsymbol{\nu}})$ is a toric projective manifold.*

We can similarly recover the structure of a toric symplectic manifold, as follows. Let us choose $\tilde{\boldsymbol{\delta}} \in \mathfrak{t}^{d\,\vee}$ such that $\boldsymbol{\delta} = \iota^t(\tilde{\boldsymbol{\delta}})$ as in (11) and (12).

Definition 3.4. Given a subspace $\mathfrak{b} \subseteq \mathfrak{t}^d$, we shall denote by $\mathfrak{b}^0 \subseteq \mathfrak{t}^{d\vee}$ its annihilator in $\mathfrak{t}^{d\vee}$.

Hence $\boldsymbol{\nu}^{\perp 0} \subseteq \mathfrak{t}^{d\vee}$ is a vector subspace of dimension $d - r + 1$, and

$$\boldsymbol{\nu}^{\perp 0} = (\iota^t)^{-1}\big(\mathrm{span}(\boldsymbol{\nu})\big). \tag{85}$$

Thus with notation as in (12)

$$M_{\boldsymbol{\nu}} = \Psi_{\tilde{\boldsymbol{\delta}}}^{-1}(\boldsymbol{\nu}^{\perp 0}); \tag{86}$$

hence $\Psi_{\tilde{\boldsymbol{\delta}}}\big|_{M_{\boldsymbol{\nu}}}$ is an equivariant map $M_{\boldsymbol{\nu}} \to \boldsymbol{\nu}^{\perp 0} \cong \mathfrak{t}_q^{d-r+1\vee}$. Therefore, $\Psi_{\tilde{\boldsymbol{\delta}}}\big|_{M_{\boldsymbol{\nu}}}$ passes to the quotient and yields a T_q^{d-r+1}-equivariant map $\overline{\Psi}_{\tilde{\boldsymbol{\delta}}} : \overline{M}_{\boldsymbol{\nu}} \to \mathfrak{t}_q^{d-r+1\vee}$, which is a moment map for $\gamma^{\overline{M}_{\boldsymbol{\nu}}}$.

We conclude the following.

Lemma 3.3. *$(\overline{M}_{\boldsymbol{\nu}}, 2\,\omega_{\boldsymbol{\nu}}, \overline{\Psi}_{\tilde{\boldsymbol{\delta}}})$ is a symplectic toric orbifold [11]. Furthermore, the moment map $\overline{\Phi}$ in Proposition 3.3 is induced by $\overline{\Psi}_{\tilde{\boldsymbol{\delta}}}$.*

Remark 3.1. Distinct choices of $\tilde{\boldsymbol{\delta}}$ determine distinct moment maps $\overline{\Psi}_{\tilde{\boldsymbol{\delta}}}$, differing by a constant in $\mathfrak{t}^{r0} \subseteq \boldsymbol{\nu}^{\perp 0} \cong \mathfrak{t}_q^{d-r+1\vee}$.

3.4. The reduced moment polytope $\overline{\Delta}_{\boldsymbol{\nu}}$

We aim to clarify the relation between the moment polytope $\overline{\Delta}_{\boldsymbol{\nu}}$ of $\overline{M}_{\boldsymbol{\nu}}$ and the moment polytope Δ of M, and to interpret properties of γ and Ψ in terms of Δ and $\overline{\Delta}_{\boldsymbol{\nu}}$.

In view of (86) and the identification $\mathfrak{t}_q^{d-r+1\vee} \cong \boldsymbol{\nu}^{\perp 0}$,

$$\overline{\Delta}_{\boldsymbol{\nu}} = \overline{\Psi}_{\tilde{\boldsymbol{\delta}}}(\overline{M}_{\boldsymbol{\nu}}) = \Psi_{\tilde{\boldsymbol{\delta}}}(M_{\boldsymbol{\nu}}) = (\Delta + \tilde{\boldsymbol{\delta}}) \cap \boldsymbol{\nu}^{\perp 0}. \tag{87}$$

With notation as in (11), (12), and in view of Definition 3.1, we have the following.

Proposition 3.5. *Suppose that $\Phi(m) \neq \mathbf{0}$ for every $m \in M$, and $\Phi^{-1}(\mathbb{R}_+ \cdot \boldsymbol{\nu}) \neq \emptyset$. The following conditions are equivalent:*

1. *Φ is transverse to $\mathbb{R}_+ \cdot \boldsymbol{\nu}$;*
2. *$\Psi_{\tilde{\boldsymbol{\delta}}}$ is transverse to $\boldsymbol{\nu}^{\perp 0}$;*
3. *Ψ is transverse to $\boldsymbol{\nu}^{\perp 0} - \tilde{\boldsymbol{\delta}}$;*
4. *$\boldsymbol{\nu}^{\perp 0}$ and $\Delta + \tilde{\boldsymbol{\delta}}$ meet transversely in $\mathfrak{t}^{d\vee}$;*
5. *$\boldsymbol{\nu}^{\perp 0} - \tilde{\boldsymbol{\delta}}$ and Δ meet transversely in $\mathfrak{t}^{d\vee}$;*
6. *if $T_1, \ldots, T_a \leqslant T^d$ are the (distinct) compact tori stabilizing some point of $M_{\boldsymbol{\nu}}$, then the projection $\pi_q : T^d \to T_q^{d-r+1}$ restricts to a finite map $T_j \to \pi_q(T_j)$ for $j = 1, \ldots, a$;*
7. *$T_j \cap T^{r-1}_{\boldsymbol{\nu}^\perp}$ is finite for $j = 1, \ldots, a$.*

The proof of Proposition 3.5 rests on the following property of the moment map Ψ of a toric symplectic manifold (see [1], [4], [11]). Let Δ be the moment polytope and F be a face of Δ. If $\boldsymbol{\xi} \in F^0$ $m \in \Psi^{-1}(\boldsymbol{\xi})$, then

$$\mathrm{d}_m \Psi(T_m M) = T_{\boldsymbol{\xi}} F^0. \tag{88}$$

Proof. To begin with, by the hypothesis and the convexity of $\Phi(M)$ [6], $\Phi^{-1}(\mathbb{R}_+ \cdot \boldsymbol{\nu}) = \Phi^{-1}\big(\mathrm{span}(\boldsymbol{\nu})\big)$.

The equivalence of 1. and 2. follows from (85). That 2. is equivalent to 3. and that 4. is equivalent to 5. is obvious, as is the equivalence of 6. and 7., given that $T_q^{d-r+1} = T^d / T^{r-1}_{\boldsymbol{\nu}^\perp}$. On the other hand, 7. is equivalent to $T^{r-1}_{\boldsymbol{\nu}^\perp}$ acting locally freely on $M_{\boldsymbol{\nu}}$, and this condition is equivalent to 1.

Thus it suffices to show that 2. is equivalent to 4. Let us adopt $\Psi_{\tilde{\boldsymbol{\delta}}}$ as moment map for the action of T^d on $(M, 2\,\omega)$, so that $\Delta_{\tilde{\boldsymbol{\delta}}} := \Delta + \tilde{\boldsymbol{\delta}}$ is the corresponding moment polytope. By definition, $\Psi_{\tilde{\boldsymbol{\delta}}}$ is transverse to $\boldsymbol{\nu}^{\perp 0}$ if and only if for every $m \in \Psi_{\tilde{\boldsymbol{\delta}}}^{-1}(\boldsymbol{\nu}^{\perp 0})$ we have

$$\mathrm{d}_m \Psi_{\tilde{\boldsymbol{\delta}}}(T_m M) + \boldsymbol{\nu}^{\perp 0} = \mathfrak{t}^{d\vee}. \tag{89}$$

Let F be a face of $\Delta_{\tilde{\boldsymbol{\delta}}}$ such that $F^0 \cap \boldsymbol{\nu}^{\perp 0} \neq \emptyset$. If $\boldsymbol{\xi} \in F^0 \cap \boldsymbol{\nu}^{\perp 0}$ and $m \in \Psi_{\tilde{\boldsymbol{\delta}}}^{-1}(\boldsymbol{\xi})$ then by (88) and (89) (with Ψ replaced by $\Psi_{\tilde{\boldsymbol{\delta}}}$)

$$T_{\boldsymbol{\xi}} F^0 + \boldsymbol{\nu}^{\perp 0} = \mathrm{d}_m \Psi_{\tilde{\boldsymbol{\delta}}}(T_m M) + \boldsymbol{\nu}^{\perp 0} = \mathfrak{t}^{d\vee}.$$

Hence 2. implies 4. The argument for the reverse implication is similar. $\square$

Proposition 3.6. *With the notation in Proposition 3.5, the closed subtori $\overline{T}_j := \pi_q(T_j) \leqslant T_q^{d-r+1}$, $j = 1, \ldots, a$, are the subgroups that appear as stabilizer subgroups of points in $\overline{M}_{\boldsymbol{\nu}}$.*

Proof. If $T_j \leqslant T^d$ is the stabilizer subgroup of $m \in M_{\boldsymbol{\nu}}$, by equivariance of the projection $M_{\boldsymbol{\nu}} \to \overline{M}_{\boldsymbol{\nu}}$ clearly $\overline{T}_j \leqslant T_q^{d-r+1}$ is a closed subtorus stabilizing $\overline{m}$ (the image of m in $\overline{M}_{\boldsymbol{\nu}}$). Thus, if $S \leqslant T_q^{d-r+1}$ is the stabilizer of $\overline{m}$, then $\overline{T}_j \leqslant S$, and S is a torus [11].

Given the isomorphism $T^d \cong T_c^{d-r+1} \times T_{\boldsymbol{\nu}^\perp}^{r-1} \cong T_q^{d-r+1} \times T_{\boldsymbol{\nu}^\perp}^{r-1}$, we can lift S to a subgroup $S' = S \times \{1\}$ of T^d. Again by equivariance, for every $s \in S'$ there exist finitely many $t \in T_{\boldsymbol{\nu}}^{r-1}$ such that $\mu_{st}^M(m) = m$. The collection $\tilde{S}$ of all such pairs $(s,t) \in S' \times T_{\boldsymbol{\nu}}^{r-1}$ is a closed subgroup of T^d of the same dimension as S, stabilizing m (hence contained in T_j) and projecting onto S in T_q^{d-r+1}; conversely, any element of T^d stabilizing m must have this form and therefore $\tilde{S} = T_j$. It follows that $S = \overline{T}_j$.

Hence the subgroups $\overline{T}_j \leqslant T_q^{d-r+1}$ are all the stabilizer subgroups of points in $\overline{M}_{\boldsymbol{\nu}}$. □

Let F be a facet of Δ, so that $F + \tilde{\boldsymbol{\delta}}$ is a facet of $\Delta + \tilde{\boldsymbol{\delta}}$. Let $M_F := \Psi^{-1}(F) = \Psi_{\tilde{\boldsymbol{\delta}}}^{-1}(F + \tilde{\boldsymbol{\delta}})$, $M_F^0 := \Psi^{-1}(F^0)$. If $\boldsymbol{\upsilon}$ is an inward primitive normal vector to F, then M_F^0 is the locus of points in M having stabilizer the 1-dimensional torus $S_F \leqslant T^d$ generated by $\boldsymbol{\upsilon}$. Furthermore, M_F^0 is open and dense in the 1-codimensional complex submanifold M_F of F.

Proposition 3.7. *Assume that the equivalent conditions of Proposition 3.5 are satisfied, and let F be a facet of Δ. Then:*

1. *$(F + \tilde{\boldsymbol{\delta}}) \cap \boldsymbol{\nu}^{\perp 0} = \Psi_{\tilde{\boldsymbol{\delta}}}(M_F \cap M_{\boldsymbol{\nu}})$, and in particular $M_F \cap M_{\boldsymbol{\nu}} \neq \emptyset$ if and only if $(F + \tilde{\boldsymbol{\delta}}) \cap \boldsymbol{\nu}^{\perp 0} \neq \emptyset$;*
2. *if $M_F \cap M_{\boldsymbol{\nu}} \neq \emptyset$, then the intersection is transverse in M.*

Proof of 1. Suppose $m \in M_F \cap M_{\boldsymbol{\nu}}$. Then $\Psi(m) \in F$ (since $m \in M_F$), whence $\Psi_{\tilde{\boldsymbol{\delta}}}(m) \in F + \tilde{\boldsymbol{\delta}}$; on the other hand $\Phi(m) \in \mathbb{R}\,\boldsymbol{\nu}$ (since $m \in M_{\boldsymbol{\nu}}$), hence $\Psi_{\tilde{\boldsymbol{\delta}}}(m) \in \boldsymbol{\nu}^{\perp 0}$. Thus, $\Psi_{\tilde{\boldsymbol{\delta}}}(m) \in (F + \tilde{\boldsymbol{\delta}}) \cap \boldsymbol{\nu}^{\perp 0}$.

Conversely, suppose $\gamma \in (F + \tilde{\boldsymbol{\delta}}) \cap \boldsymbol{\nu}^{\perp 0}$. Thus there exists $m \in M$ such that $\gamma = \Psi_{\tilde{\boldsymbol{\delta}}}(m) \in F + \tilde{\boldsymbol{\delta}}$ (i.e., $m \in M_F$), and $\Psi_{\tilde{\boldsymbol{\delta}}}(m) \in \boldsymbol{\nu}^{\perp 0}$, i.e. $m \in M_{\boldsymbol{\nu}}$. Thus $m \in M_F \cap \boldsymbol{\nu}^{\perp 0}$, whence $\gamma \in \Psi_{\tilde{\boldsymbol{\delta}}}(M_F \cap M_{\boldsymbol{\nu}})$. □

Before giving the proof of 2., a remark is in order. The holomorphic and Hamiltonian action $(\gamma^M, \Psi_{\tilde{\boldsymbol{\delta}}})$ of T^d on $(M, 2\omega)$ restricts to a holomorphic and Hamiltonian action (λ^M, Λ) of $T_{\boldsymbol{\nu}^\perp}^{r-1}$, where the moment map $\Lambda : M \to \mathfrak{t}_{\boldsymbol{\nu}^\perp}^{r-1\vee}$ is induced by $\Psi_{\tilde{\boldsymbol{\delta}}}$ in the standard manner. Then $M_{\boldsymbol{\nu}} = \Lambda^{-1}(\mathbf{0})$ and the transversality hypothesis in Proposition 3.5 are equivalent to the condition that $\mathbf{0}$ be a regular value of Λ, or - still equivalently - that $T_{\boldsymbol{\nu}^\perp}^{r-1}$ act locally freely on $M_{\boldsymbol{\nu}}$.

Proof of 2. M_F is a Kähler submanifold of $(M, J, 2\omega)$. It is furthermore T^d-invariant, hence Λ restricts to a moment map $\Lambda_F : M_F \to \mathfrak{t}_{\boldsymbol{\nu}^\perp}^{r-1\vee}$ for the action of $T_{\boldsymbol{\nu}^\perp}^{r-1}$ on M_F. Transversality of M_F and $M_{\boldsymbol{\nu}}$ is then equivalent to $\mathbf{0}$ being a regular value for Λ_F, hence to $T_{\boldsymbol{\nu}^\perp}^{r-1}$ acting locally freely on $\Lambda_F^{-1}(0)$. However $T_{\boldsymbol{\nu}^\perp}^{r-1}$ does act locally freely on $\Lambda_F^{-1}(0) = M_{\boldsymbol{\nu}} \cap M_F$, since it acts locally freely on all of $M_{\boldsymbol{\nu}}$. □

Assume that the conditions in Proposition 3.5 are satisfied, and let F be a facet of Δ such that $F_{\boldsymbol{\nu}} := (F + \tilde{\boldsymbol{\delta}}) \cap \boldsymbol{\nu}^{\perp 0} \neq \emptyset$. Then $F_{\boldsymbol{\nu}}$ is a facet of $\overline{\Delta}_{\boldsymbol{\nu}}$. Furthermore, if $\overline{M}_F := (M_F \cap M_{\boldsymbol{\nu}})/T_{\boldsymbol{\nu}^\perp}^{r-1} \subseteq \overline{M}_{\boldsymbol{\nu}}$, we can draw the following conclusion from the previous discussion.

Corollary 3.4. *$\overline{M}_F$ is a complex suborbifold of $\overline{M}_{\boldsymbol{\nu}}$, and $\overline{M}_F = \overline{\Psi}_{\tilde{\boldsymbol{\delta}}}^{-1}(F_{\boldsymbol{\nu}})$.*

3.5. Smoothness conditions on $\overline{\Delta}_{\boldsymbol{\nu}}$

Proposition 3.5 characterizes the transversality of Φ to $\mathbb{R}\,\boldsymbol{\nu}$ in terms of the mutual position of Δ and $\boldsymbol{\nu}^{\perp 0}$ in $\mathfrak{t}^{d\vee}$. This condition ensures that $M_{\boldsymbol{\nu}}$ is a submanifold, that $T_{\boldsymbol{\nu}}^{r-1}$ acts locally freely on it, and therefore that $\overline{M}_{\boldsymbol{\nu}}$ is a Kähler orbifold. Since our present focus is on the case where $\overline{M}_{\boldsymbol{\nu}}$ is a Kähler manifold, we want to similarly characterize this stronger condition using Δ and $\boldsymbol{\nu}^{\perp 0}$.

By the discussion in §3.1, (87), and Proposition 3.5, $\overline{\Delta}_{\boldsymbol{\nu}}$ is the convex polytope in $\boldsymbol{\nu}^{\perp 0}$ having as facets the non-empty intersections of $\boldsymbol{\nu}^{\perp 0}$ with the facets of $\Delta + \tilde{\boldsymbol{\delta}}$. Equivalently, it is the convex hull of the intersection of $\boldsymbol{\nu}^{\perp 0}$ with the $(d-r+1)$-codimensional (i.e., $(r-1)$-dimensional) faces of $\Delta + \tilde{\boldsymbol{\delta}}$. The connected component of such intersection are precisely vertices of $\overline{\Delta}_{\boldsymbol{\nu}}$; furthermore, if $F \subseteq \Delta$ is an $(r-1)$-dimensional face, then $(F + \tilde{\boldsymbol{\delta}}) \cap \boldsymbol{\nu}^{\perp 0} = (F^0 + \tilde{\boldsymbol{\delta}}) \cap \boldsymbol{\nu}^{\perp 0}$, since by transversality $\boldsymbol{\nu}^{\perp 0}$ must have empty intersection with any face of lesser dimension.

Let $\mathcal{G}(\Delta) = \{F_1, \dots, F_k\}$ be the collection of facets of Δ, so that the collection of facets of $\Delta + \tilde{\boldsymbol{\delta}}$ is $\mathcal{G}(\Delta + \tilde{\boldsymbol{\delta}}) = \{F_1 + \tilde{\boldsymbol{\delta}}, \dots, F_k + \tilde{\boldsymbol{\delta}}\}$. Thus for every $j = 1, \dots, s$ there exist unique $\boldsymbol{\upsilon}_j \in L(T^d)$ primitive and $\lambda_j \in \mathbb{R}$ such that

$$\Delta + \tilde{\boldsymbol{\delta}} = \bigcap_{j=1}^{k} \{\ell \in \mathfrak{t}^\vee : \ell(\boldsymbol{\upsilon}_j) \geq \lambda_j + \delta_j\}, \quad \delta_j := \tilde{\boldsymbol{\delta}}(\boldsymbol{\upsilon}_j). \tag{90}$$

Let us assume that the F_j's have been so numbered that $(F_j + \tilde{\boldsymbol{\delta}}) \cap \boldsymbol{\nu}^{\perp 0} \neq \emptyset$ for $j = 1, \dots, l$, and $(F_j + \tilde{\boldsymbol{\delta}}) \cap \boldsymbol{\nu}^{\perp 0} = \emptyset$ for $l + 1 \leq j \leq k$. Hence by (90) and the previous discussion (with the usual identification $\mathfrak{t}_q^{d-r+1\,\vee} \cong \boldsymbol{\nu}^{\perp 0}$)

$$\mathfrak{t}_q^{d-r+1\,\vee} \supset \overline{\Delta}_{\boldsymbol{\nu}} \cong \bigcap_{j=1}^{l} \{\gamma \in \boldsymbol{\nu}^{\perp 0} : \gamma(\boldsymbol{\upsilon}_j) \geq \lambda_j + \delta_j\}. \tag{91}$$

In view of (17), (82), and (84) there exist unique $\boldsymbol{\upsilon}'_j \in L(T_c^{d-r})$, $\rho_j \in \mathbb{Z}$, $\boldsymbol{\upsilon}''_j \in L(T^{r-1}_{\boldsymbol{\nu}^\perp})$ such that

$$\boldsymbol{\upsilon}_j = \boldsymbol{\upsilon}'_j + \rho_j\, \widetilde{\boldsymbol{\nu}} + \boldsymbol{\upsilon}''_j. \tag{92}$$

Therefore (92) may be rewritten as

$$\overline{\Delta}_{\boldsymbol{\nu}} \cong \bigcap_{j=1}^{l} \{\gamma \in \boldsymbol{\nu}^{\perp 0} : \gamma(\boldsymbol{\upsilon}'_j + \rho_j\, \widetilde{\boldsymbol{\nu}}) \geq \lambda_j + \delta_j\}. \tag{93}$$

Definition 3.5. For $I = \{i_1, \dots, i_a\} \subseteq \{1, \dots, l\}$, let $\mathfrak{s}_I := \mathrm{span}(\boldsymbol{\upsilon}_j : j \in I\} \subseteq \mathfrak{t}^d$ and let $S_I \leqslant T^d$ be the closed subtorus with Lie subalgebra $\mathfrak{s}_I$.

Suppose that I is such that $F := F_{i_1} \cap \ldots \cap F_{i_a}$ is a face of Δ; then, since Δ is a Delzant polytope, the sequence of normal vectors $(\boldsymbol{\upsilon}_{i_1}, \dots, \boldsymbol{\upsilon}_{i_a})$ is a primitive system in $L(T^d)$ (meaning that it can be extended to a lattice basis). Furthermore, S_I is the stabilizer subgroup of any $m \in \Psi^{-1}(F^0)$.

In particular, let F be a codimension-a face of Δ such that $(F + \tilde{\boldsymbol{\delta}}) \cap \boldsymbol{\nu}^{\perp 0} \neq \emptyset$ (whence $(F^0 + \tilde{\boldsymbol{\delta}}) \cap \boldsymbol{\nu}^{\perp 0} \neq \emptyset$ by Corollary 3.1). Then there is a unique $I_F = \{i_1, \dots, i_a\} \subseteq \{1, \dots, l\}$, such that F is the intersection of the facets $F_{i_1}, \dots, F_{i_a}$, and so $(\boldsymbol{\upsilon}_{i_1}, \dots, \boldsymbol{\upsilon}_{i_a})$ is a primitive system.

Lemma 3.4. *Under the previous assumption, and with notation (92), the following conditions are equivalent:*

1. *$T^{r-1}_{\boldsymbol{\nu}^\perp}$ acts freely on $\Psi_{\tilde{\boldsymbol{\delta}}}^{-1}\big((F^0 + \tilde{\boldsymbol{\delta}}) \cap \boldsymbol{\nu}^{\perp 0}\big)$;*
2. *$(\boldsymbol{\upsilon}'_{i_1} + \rho_{i_1}\, \widetilde{\boldsymbol{\nu}}, \dots, \boldsymbol{\upsilon}'_{i_a} + \rho_{i_a}\, \widetilde{\boldsymbol{\nu}})$ is a primitive system.*

Proof. Suppose $\gamma \in (F^0 + \tilde{\boldsymbol{\delta}}) \cap \boldsymbol{\nu}^{\perp 0}$, and choose $m \in M$ such that $\Psi_{\tilde{\boldsymbol{\delta}}}(m) = \gamma$. Then $\Psi(m) \in F^0$, and so the stabilizer subgroup of m is S_{I_F}. Hence m has trivial stabilizer in $T^{r-1}_{\boldsymbol{\nu}^\perp}$ if and only if $T^{r-1}_{\boldsymbol{\nu}^\perp} \cap S_{I_F}$ is trivial.

Suppose that 1. holds, and let $\vartheta_j \in \mathbb{R}$, $j = 1, \dots, a$, be such that

$$\exp\left(\sum_{j=1}^{a} \vartheta_j\, (\boldsymbol{\upsilon}'_{i_j} + \rho_{i_j}\, \widetilde{\boldsymbol{\nu}})\right) = 1.$$

Then

$$\exp\left(\sum_{j=1}^{a} \vartheta_j\, \boldsymbol{\upsilon}_{i_j}\right) = \exp\left(\sum_{j=1}^{a} \vartheta_j\, \boldsymbol{\upsilon}''_{i_j}\right) \in S_{I_F} \cap T^{r-1}_{\boldsymbol{\nu}^\perp} = (1).$$

Thus necessarily $\vartheta_j \in 2\pi\,\mathbb{Z}$ because $(\boldsymbol{\upsilon}_{i_j})_j$ is a primitive system. Hence 2. holds.

Conversely, assume that 2. holds. Suppose that $t \in S_{I_F} \cap T^{r-1}_{\boldsymbol{\nu}^\perp}$. There exist ϑ_j, $j = 1, \dots, a$, and $\boldsymbol{\xi} \in \mathfrak{t}^{r-1}_{\boldsymbol{\nu}^\perp}$ such that

$$t = \exp\left(\sum_{j=1}^{a} \vartheta_j\, \boldsymbol{\upsilon}_j\right) = \exp(\boldsymbol{\xi})$$

$$\Rightarrow \exp\left(\sum_{j=1}^{a} \vartheta_j\, (\boldsymbol{\upsilon}'_{i_j} + \rho_{i_j}\, \widetilde{\boldsymbol{\nu}})\right) = \exp\left(\boldsymbol{\xi} - \sum_{j=1}^{a} \vartheta_j\, \boldsymbol{\upsilon}''_j\right)$$

$$\Rightarrow \exp\left(\sum_{j=1}^{a} \vartheta_j \left(\boldsymbol{v}'_{i_j} + \rho_{i_j}\,\widetilde{\boldsymbol{v}}\right)\right) \in \left(T_c^{d-r} \times \widehat{T}^1_{\boldsymbol{v}}\right) \cap T^{r-1}_{\boldsymbol{v}^\perp} = (1)$$
$$\Rightarrow \vartheta_j \in 2\pi\,\mathbb{Z}, \quad \forall\, j = 1, \ldots, a \quad \Rightarrow \quad t = 1,$$

where we have made use of (17). Hence 2. implies 1. □

This can be strengthened as follows.

Proposition 3.8. *Under the previous assumptions, the following conditions are equivalent:*

1. *$T^{r-1}_{\boldsymbol{v}^\perp}$ acts freely on $M_{\boldsymbol{v}}$;*
2. *for every $(r-1)$-dimensional face F of Δ such that $(F + \tilde{\boldsymbol{\delta}}) \cap \boldsymbol{v}^{\perp 0} \neq \emptyset$, with $I_F = \{i_1, \ldots, i_{d-r+1}\} \subseteq \{1, \ldots, l\}$, the sequence*
$$(\boldsymbol{v}'_{i_1} + \rho_{i_1}\,\widetilde{\boldsymbol{v}}, \ldots, \boldsymbol{v}'_{i_{d-r+1}} + \rho_{i_{d-r+1}}\,\widetilde{\boldsymbol{v}})$$
is a primitive system;
3. *for every $(b+r-1)$-dimensional face F of Δ (with $b \geq 0$) such that $(F + \tilde{\boldsymbol{\delta}}) \cap \boldsymbol{v}^{\perp 0} \neq \emptyset$, with $I_F = \{i_1, \ldots, i_{d-b-r+1}\} \subseteq \{1, \ldots, l\}$, the sequence*
$$(\boldsymbol{v}'_{i_1} + \rho_{i_1}\,\widetilde{\boldsymbol{v}}, \ldots, \boldsymbol{v}'_{i_{d-b-r+1}} + \rho_{i_{d-b-r+1}}\,\widetilde{\boldsymbol{v}})$$
is a primitive system.

Proof. That 1. implies 2. follows immediately from Lemma 3.4. Suppose that 2. holds. Let $m \in M_{\boldsymbol{v}}$. If $m \in M^0$ (i.e., $\Psi(m) \in \Delta^0$), then T^d acts freely at m, hence so does $T^{r-1}_{\boldsymbol{v}^\perp}$. Otherwise, $\Psi(m) \in F^0$ for a unique face F of Δ, whence $\Psi_{\tilde{\boldsymbol{\delta}}}(m) \in (F^0 + \tilde{\boldsymbol{\delta}}) \cap \boldsymbol{v}^{\perp 0}$. Applying again Lemma 3.4, we conclude that $T^{r-1}_{\boldsymbol{v}^\perp}$ acts freely at m. Thus if 2. holds then $T^{r-1}_{\boldsymbol{v}^\perp}$ acts freely at every $m \in \Psi^{-1}_{\tilde{\boldsymbol{\delta}}}(\boldsymbol{v}^{\perp 0}) = M_{\boldsymbol{v}}$, i.e. 1. holds. That 3. implies 2. is obvious, since 2. is formally the special case of 3. with $b = 0$. Suppose that 2. holds, and let F be a $(b+r-1)$-dimensional face of Δ as in the statement of 3; then $F_{\boldsymbol{v}} := (F + \tilde{\boldsymbol{\delta}}) \cap \boldsymbol{v}^{\perp 0}$ is a b-dimensional face of $\overline{\Delta}_{\boldsymbol{v}}$. Therefore $F_{\boldsymbol{v}}$ contains a vertex γ of $\overline{\Delta}_{\boldsymbol{v}}$. Hence there exists an $(r-1)$-dimensional face F' of Δ, as in the statement of 2., such that $\{\gamma\} = (F' + \tilde{\boldsymbol{\delta}}) \cap \boldsymbol{v}^{\perp 0}$. The sequence of normal vectors corresponding to F' contains the sequence corresponding to F, and since a subsystem of a primitive system is necessarily also primitive, we conclude that 3. holds. □

3.6. Proof of Theorem 1.2

We can now build on the previous discussion to give the proof of the Theorem.

Proof of Theorem 1.2. Under the given assumptions on Δ and $\boldsymbol{v}^{\perp 0}$, $\overline{M}_{\boldsymbol{v}}$ is a toric manifold, acted upon by $T_q^{d-r+1} \cong T_c^{d-r+1} = T_c^{d-r} \times \widehat{T}^1_{\boldsymbol{v}}$ in (84), and with associated moment polytope $\overline{\Delta}_{\boldsymbol{v}}$. Furthermore, by (73) and the discussion in §3.2, $Y_{\boldsymbol{v}}$ is the unit circle bundle associated to the positive line bundle $(A_{\boldsymbol{v}}, h_{\boldsymbol{v}})$ on $\overline{M}_{\boldsymbol{v}}$. In addition, $\widehat{M}_{\boldsymbol{v}} = Y_{\boldsymbol{v}}/T^1_{\boldsymbol{v}}$ by (71), where $T^1_{\boldsymbol{v}}$ acts on $Y_{\boldsymbol{v}}$ by the contact lift $\mu^{Y_{\boldsymbol{v}}}$ of the Hamiltonian action $(\mu^{\overline{M}_{\boldsymbol{v}}}, \overline{\Phi})$ (Proposition 3.3).

We are therefore in the situation of Theorem 1.1, with the following replacements: M by $\overline{M}_{\boldsymbol{v}}$; T^d by $T_q^{d-r+1} \cong T_c^{d-r+1} = T_c^{d-r} \times \widehat{T}^1_{\boldsymbol{v}}$; Δ by $\overline{\Delta}_{\boldsymbol{v}}$; X by $Y_{\boldsymbol{v}}$; Ψ by $\overline{\Psi}_{\tilde{\boldsymbol{\delta}}}$; T^r by $T^1_{\boldsymbol{v}} \cong \widehat{T}^1_{\boldsymbol{v}}$; Φ by $\overline{\Phi}$; T_c^{d-1} by T_c^{d-r}. Furthermore, the constants λ_j are replaced by $\lambda_j + \delta_j$ for $j = 1, \ldots, l$ in view of (91), and the scalar δ is taken to vanish by Lemma 3.3 (once Ψ has been replaced by $\overline{\Psi}_{\tilde{\boldsymbol{\delta}}}$, no further translation is required).

The statement of Theorem 1.2 is now an immediate consequence of Theorem 1.1. □

Data availability

No data was used for the research described in the article.

References

[1] T. Delzant, Hamiltoniens périodiques et images convexes de l'application moment, Bull. Soc. Math. Fr. 116 (3) (1988) 315–339.
[2] W. Fulton, Introduction to toric varieties, in: The William H. Roever Lectures in Geometry, in: Annals of Mathematics Studies, vol. 131, Princeton University Press, Princeton, NJ, ISBN 0-691-00049-2, 1993, xii+157 pp.
[3] V. Guillemin, Kaehler structures on toric varieties, J. Differ. Geom. 40 (2) (1994) 285–309.

[4] V. Guillemin, Moment Maps and Combinatorial Invariants of Hamiltonian T^n-Spaces, Progress in Mathematics, vol. 122, Birkhäuser Boston, Inc., Boston, MA, ISBN 0-8176-3770-2, 1994, viii+150 pp.
[5] V. Guillemin, S. Sternberg, Geometric quantization and multiplicities of group representations, Invent. Math. 67 (3) (1982) 515–538.
[6] V. Guillemin, S. Sternberg, Convexity properties of the moment mapping, Invent. Math. 67 (3) (1982) 491–513.
[7] V. Guillemin, S. Sternberg, Symplectic Techniques in Physics, second edition, Cambridge University Press, Cambridge, ISBN 0-521-38990-9, 1990, xii+468 pp.
[8] T. Kawasaki, Cohomology of twisted projective spaces and lens complexes, Math. Ann. 206 (1973) 243–248.
[9] F. Kirwan, Cohomology of Quotients in Symplectic and Algebraic Geometry, Mathematical Notes, vol. 31, Princeton University Press, Princeton, NJ, ISBN 0-691-08370-3, 1984, i+211 pp.
[10] B. Kostant, Quantization and unitary representations. I. Prequantization, in: Lectures in Modern Analysis and Applications, III, in: Lecture Notes in Math., vol. 170, Springer, Berlin, 1970, pp. 87–208.
[11] E. Lerman, S. Tolman, Hamiltonian torus actions on symplectic orbifolds and toric varieties, Trans. Am. Math. Soc. 349 (10) (1997) 4201–4230.
[12] L. Ness, A stratification of the null cone via the moment map, with an appendix by David Mumford, Am. J. Math. 106 (6) (1984) 1281–1329.
[13] R. Paoletti, Asymptotics of Szegö kernels under Hamiltonian torus actions, Isr. J. Math. 191 (1) (2012) 363–403, https://doi.org/10.1007/s11856-011-0212-4.
[14] R. Paoletti, Lower-order asymptotics for Szegö and Toeplitz kernels under Hamiltonian circle actions, in: Recent Advances in Algebraic Geometry, in: London Math. Soc. Lecture Note Ser., vol. 417, Cambridge Univ. Press, Cambridge, 2015, pp. 321–369.
[15] R. Paoletti, Conic reductions for Hamitonian actions of $U(2)$ and its maximal torus, Rend. Circ. Mat. Palermo (2) 72 (4) (2023) 2515–2563.
[16] R. Paoletti, Polarized orbifolds associated to quantized Hamiltonian torus actions, J. Geom. Phys. 170 (2021) 104363.
[17] R. Sjamaar, Holomorphic slices, symplectic reduction and multiplicities of representations, Ann. Math. (2) 141 (1) (1995) 87–129.