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ESSAYS ON GROUP CONTESTS

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To Riccardo, who first taught me fundamental economic principles.
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Declarations

Chapter 1 is a joint work with Mario Gilli and it was presented at The 10th annual conference "Contests: Theory and Evidence", June 2024, University of Reading (UK), and at The 2nd Milan Ph.D. Economics Workshop, September 2024, University of Milano-Bicocca, Milan (IT). An initial working paper version is available as: Gilli, Mario and Sorrentino, Andrea, The Set of Equilibria in Max-Min Two Groups Contests with Binary Actions and a Private Good Prize (June 12, 2024). University of Milan Bicocca Department of Economics, Management and Statistics Working Paper No. 539, available at SSRN: https://ssrn.com/abstract=4863239.

Chapter 2 is a joint work with Mario Gilli and it has been accepted for publication in *Mathematical Social Sciences* (3rd October, 2025). [1] An initial working paper version is available as: Mario, G. and Sorrentino, A., Characterization of the Set of Equilibria in Max-Min Group Contests Continuous Efforts and a Private Good Prize (July 09, 2024). University of Milan Bicocca Department of Economics, Management and Statistics (DEMS) Working Paper No. 541, available at SSRN: https://ssrn.com/abstract=4890131.

Chapter 3 was presented as a joint work with Davide Bosco and Mario Gilli at The 46th Tax Day, October 2024, Max Planck Institute for Tax Law and Public Finance, Munich (DE), The European Meeting on Game Theory 2025 (SING 20), June 2025, Maastricht University (NL), and at PET 2025 International Conference on Public Economic Theory, June 2025, ISEG - Lisbon School of Economics and Management (2025) and at the 20th BiGSEM Workshop, December 2025, Bielefeld University (DE). A working paper version is available as Bosco, Davide and Gilli, Mario and Sorrentino, Andrea, A Max-Min Two-Group Contest with Binary Actions and Incomplete Information à la Global Games (October 24, 2024). University of Milan Bicocca Department of Economics, Management and Statistics Working Paper No. 545, available at SSRN: https://ssrn.com/abstract=4997808.

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[1] Mario Gilli, Andrea Sorrentino, The set of pure-strategy equilibria in max–min two-group contests with a private good prize, Mathematical Social Sciences, Volume 138, 2025, 102471, ISSN 0165-4896, https://doi.org/10.1016/j.mathsocsci.2025.102471.

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Introduction

This thesis is conceived as a collection of essays on deterministic group contests with non-additive impact functions. Most of the work is devoted to the case in which there are perfect complementarities within groups, that is teammates' efforts are aggregated by the so-called weakest-link impact function. In particular, we build on Chowdhury, Lee and Topolyan (2016) [2] by introducing a private good prize along endogenous sharing rules and complete information as a natural extension of their "max-min group contests". We consider a binary action set, a discrete action set and the more usual continuous action set representing efforts. Equilibrium multiplicity – a natural of property of many models displaying strategic complementarities – proves to be robust to the introduction of a private good prize and endogenous sharing rules. In this regard, we give a characterization of the resulting indeterminacy of subgame perfect Nash equilibria in pure strategies. Despite being an uncommon assumption for contests, a binary action set allows to introduce incomplete information à la global games [3] in such models delivering informational realism and equilibrium selection results. Finally, it is shown that different properties for equilibrium existence and uniqueness hold in deterministic group contests with the best-shot impact function, binary actions and a club good prize with incomplete information à la global games.

Although my personal interest in these models has been mostly about the characterization of equilibria and the related game theoretic properties, the essays will give a picture of the nice economic applications these models have, spanning from R&D competition and political

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[2] Chowdhury, Subhasish M., Dongryul Lee, and Iryna Topolyan (2016). "The Max-Min Group Contest: Weakest-link (Group) All-Pay Auction". In: Southern Economic Journal 83.1, pp. 105–125. issn: 00384038, 23258012.

[3] In the original sense of Carlsson, Hans and Eric van Damme (Sept. 1993a). "Global Games and Equilibrium Selec- tion". In: Econometrica 61.5, pp. 989–1018. issn: 00129682, 14680262.

economy to sports.

The collection of essays is organized as follows. Chapter 1 characterizes the set of subgame perfect Nash equilibria in pure strategies in a deterministic two-group contest with the weakest-link impact function, that is a max-min group contest, as defined by Chowdhury, Lee, and Topolyan (2016), with binary actions and endogenous sharing rules under complete information. Chapter 2 considers a continuous action set instead. Chapter 3 introduces incomplete information à la global games in max-min group contests and characterizes the set of Bayes Nash equilibria. It was originally a joint work with Davide Bosco and Mario Gilli, but after some revisions both authors agreed to make it appear as a single-author contribution. Finally, Chapter 4 introduces incomplete information à la global games in deterministic group contests with the best-shot impact function and addresses the existence of Bayes Nash equilibria in (monotonic) switching strategies.

1 Equilibrium Indeterminacy in Max-Min Two-Group Contests with Binary Actions and a Private Good Prize

quod superest, vacuas auris < animumque sagacem >
semotum a curis adhibe veram ad rationem,
ne mea dona tibi studio disposta fideli,
intellecta prius quam sint, contempta relinquas.

(Titus Lucretius Carus)

Equilibrium Indeterminacy in Max-Min Two-Group Contests with Binary Actions and a Private Good Prize [*]

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Abstract

We consider a deterministic two-group contest with the weakest-link impact function, that is a max-min group contest, as defined by Chowdhury, Lee, and Topolyan (2016), with binary actions and a private good prize. We include two stages: the first one about the setting of a sharing rule parameter and the second one about simultaneous and independent within-group actions choices. The binary action set allows us to (i) characterize the full set of the second stage equilibrium actions; (ii) computationally characterize the set of within-group symmetric subgame perfect Nash equilibria in pure strategies in the entire game. We find conditions such that the set of within-group symmetric subgame perfect Nash equilibria in pure strategies has the cardinality of the continuum, which means that in equilibrium players' behavior is indeterminate. We check whether this indeterminacy result is due to discreteness or to binary choice. To this aim, we expand the set of second-stage actions from the binary case to any subset of the natural numbers with cardinality at least equal to three. Then, we propose a counterexample, proving that in this case there are no subgame perfect Nash equilibria in pure strategies.

JEL classification: D74, D71, C72

Keywords: Group contests; sharing rules; weakest-link; indeterminacy

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1.1 Introduction

Competition among groups of agents is widespread in many socioeconomic activities, spanning from rent-seeking situations, labor markets, investments in R&D, military conflicts, electoral competitions, and sports. In these situations, complementarities within groups are often a salient feature. For example, perfect complementarity within groups, the case of the weakest-link impact function, models settings where each individual has veto power on the total group bid to be provided, as it is the case for example in EU for many issues that require unanimity, so that a dissenting vote is enough to prevent the adoption of a policy.[1.1] An alternative situation where the weakest-link can be considered the best approximation to a real situation is a country defense system whose effectivity depends on the weakest component of the system, as well as production supply chains which are only as strong as their weakest link. These are settings where each agent is responsible for one link of a chain. But, also other examples of teamwork may involve group contests with the weakest-link technology. For example research contests between teams where each team consists of some experts in different fields and the expert's input from each field is indispensable to the success of the research. Also team competitions in soccer or basket or in bicycle races may be modeled as group contests with the weakest-link technology, since the teams' performances may critically depend on the weakest agent. More generally, in situations where the capability of a group depends upon each person's doing his duty, the conditions for applicability of the weakest-link rule are approximated. In other words, the weakest-link impact function/perfect complementarity case implies the interdependence among agents in a group so that the role of each member is an indispensable part of the group joint effort. Note that in comparison with the more common summation formula - perfect substitutability among members' effort - we expect that underprovision of effort tends to be considerably moderated when perfect complementarities/weakest-link is applicable.

In this paper we incorporate the weakest-link rule into a deterministic group contest with a private good as prize. Specifically, we consider a deterministic two-group contest under

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[1.1] See Gilli and Tedeschi (2022).

complete information where the effort choices made by the teammates are aggregated into group performance by the weakest-link technology, that is a "max-min two group contest", as defined by Chowdhury, D. Lee, and Topolyan (2016). We innovate considering private good prizes, so that the sharing issue within the winning group plays a crucial role. Therefore, we include two stages: the first one about the setting of a sharing rule parameter and the second one about simultaneous and independent actions choices. On the other hand, instead of a continuum effort set as in Chowdhury, D. Lee, and Topolyan (2016) and Gilli and Sorrentino (2025), we employ a binary action set. Moreover, contrary to what done in Chowdhury, D. Lee, and Topolyan (2016), we will focus on equilibria in pure strategies only. Although we recognize that the analysis of second-period equilibria in mixed strategies deserves attention in our model, being potentially object of future research, we think one of the main advantages of non-standard impact functions in deterministic group contests is that they allow for the existence of equilibria in pure strategies, which are more easily interpretable and testable.

The assumption of a binary action set is clearly very restrictive, and it is usually justified because of computational reasons. However, the assumption of a binary action set is also theoretically interesting to understand key trade-offs in strategic situations that are naturally binary, such as entering in a market, adopting a technology, voting yes/no, cooperating or not. We could also think of research groups where each member can apply or not for a grant, or the possible signature of an international agreement among countries belonging to two contrasting alliances. Indeed, in many settings, the choice faced by teammates can be easily conceived as a binary decision. More generally, for example, the important class of Global Games of Regime Change[1.2] has a binary action set, showing that this assumption, even if restrictive, may lead to important results.

In our setting, the binary action set allows us to innovate on the existing literature by (i) characterizing the full set of the second stage equilibrium actions in pure strategies; (ii) computationally characterizing in MATLAB the set of within-group symmetric subgame perfect Nash equilibria in pure strategies in the entire game. We find that the characterization of the full set of subgame perfect Nash equilibria in pure strategies would require hundreds

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[1.2] See Morris and Shin (2003).

of billions of iterations even in our simple model. Nonetheless, depending on the size of the private good prize with respect to groups' size, we obtain conditions such that the set of within-group symmetric subgame perfect Nash equilibria in pure strategies has the cardinality of the continuum, which means that in equilibrium players' behavior is indeterminate. This result is particularly interesting , because in Gilli and Sorrentino (2024), it is proved that games in the class of max-min group contests with continuous effort and a private good prize have no subgame perfect equilibria unless the choice of the sharing rule is restricted. In Section 1.6, we check whether the indeterminacy result is due to discreteness or to binary choice. To this aim, we expand the set of second-stage actions from the binary case to any subset of the natural numbers with cardinality at least equal to three. Then, we propose an example, proving that in this case there are no subgame perfect Nash equilibria in pure strategies, as in the continuum case. Likewise, by employing restricted sharing rules, we find conditions on the prize such that subgame perfect equilibria do exist. The basic reason is that once we introduce more than two effort choices, it is necessary to ensure that both upward and downward deviations are not profitable, not just either upward or downward ones, as in the binary action setup. Therefore, under the assumption of unrestricted sharing rules, the possibility of characterizing precisely the indeterminacy result in terms of continua of equilibria is due to the binary action set.

The paper is organized as follows. The next subsection quickly reviews the related literature, while Section 1.2 outlines the model. Section 1.3 characterizes the set of second-period equilibria for an exogenous profile of incentive schemes. Section 1.4 restricts the analysis to within-group symmetric (WGS) equilibria and endogenizes the incentive schemes. Section 1.5 provides the subgame perfect WGS equilibria of the entire game. Section 1.6 extends the analysis to a discrete number of effort levels. Section 1.7 concludes the paper.

1.1.1 Related Literature

To the best of our knowledge, Hirshleifer (1983) is the first paper to consider, for the case of private provision of a public good, the weakest-link as social composition function, called impact function in the literature on group contests.[1.3] Based on the works of Hirshleifer (1983) Hirshleifer (1985) many scholars studied the problem of the private provision of public goods considering the weakest-link technology both theoretically and experimentally: Harrison and Hirshleifer (1989), Vicary (1990), Sandler and Vicary (2001), Arce (2001), Vicary and Sandler (2002), Cornes and Hartley (2007), Lei et al. (2007). These papers deal with a general setting where self-interested strategic agents are called to contribute to the supply of a public good when the social composition function displays perfect complementarities, which is partially related to the literature on group contests. In particular, an important theoretical contribution to this literature is provided by Cornes and Hartley (2007), that shows that in the Bergstrom-Blume-Varian model of noncooperative voluntary contributions to a public good, the case with the weakest-link as social composition function has a continuum of Pareto ranked equilibria.

Similarly to the first seminal papers on private provision of public goods, most of the early group contest literature has assumed that a group's aggregate effort, the impact function, is a simple sum of individual bids, i.e. that individual contributions within a group are perfect substitutes. In particular, Nitzan (1991) which marks the origin of the literature on group contests with private good as prize, generalize Tullock (1980)'s rent-seeking model assuming that the players are members of two groups that compete for a private prize, which in turn is distributed within the winning group according to a sharing rule. S. Lee (1995) extends the Nitzan (1991) model analyzing the endogenous determination of intra-group sharing rules, thus considering a two-stage game, as we do in this paper. In the first stage, the groups play a simultaneous moves game, where each group decides its own sharing rule to maximize the group's utilitarian welfare. In the second stage, each member of every group decides the extent of contribution to the group's rent-seeking efforts, as in Nitzan (1991) model. The full game is then solved using subgame perfection. A general contribution to this literature is provided by Ueda (2002) that examines when and how in this class of models, some contestants decide to retire from rent-seeking. More recent works displaying restricted sharing rules and a probabilistic contest success function include Nitzan and Ueda (2018) and Kobayashi and Konishi (2021), where the former focuses on within-group heterogeneity, while the latter takes

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[1.3] Waerneryd (1998).

into account within-group complementarities via a CES effort aggregator.

The above papers consider a Tullock probability contest success function, and assume that within groups individual bids are perfect substitutes. Key contributions for group contests with non-standard impact functions under both complete information and incomplete information are, in the former case, D. Lee (2012), Chowdhury, D. Lee, and Sheremeta (2013), Kolmar and Rommeswinkel (2013) and Barbieri, Malueg, and Topolyan (2014) and, in the latter, Barbieri and Malueg (2016), Barbieri, Kovenock, et al. (2019) and Barbieri and Topolyan (2021). In particular, Chowdhury, D. Lee, and Sheremeta (2013) considers best-shot probabilistic group contests, while Barbieri, Malueg, and Topolyan (2014) analyzes best-shot deterministic group contests. In both papers, the authors consider group-specific public good prize, thus settings where sharing rules are irrelevant. Chowdhury, D. Lee, and Sheremeta (2013) fully characterizes the set of equilibria, as we do in this paper, and show that in any equilibrium at most one player in each group exerts strictly positive effort. On the other hand, Barbieri, Malueg, and Topolyan (2014) derive equilibria of a symmetric model in which multiple agents per group are active and total expected efforts vary across equilibria. Kolmar and Rommeswinkel (2013) analyze an $n-$group contest with a CES production function, thus allowing for complementarity in agents' effort within groups, however with a probabilistic contest success function and a group-specific public good prize. For this class of games, they characterize the Nash equilibria and carry out comparative-static exercises with respect to the elasticity of substitution among group members' efforts.

D. Lee (2012) is more directly related to this paper. The author analyzes n groups competing to win a group-specific public good prize, where individual players choose their effort levels simultaneously and independently, and each group's total effort is given by a weakest-link impact function, however with a probabilistic smooth success function. The author proves that there are multiple pure-strategy Nash equilibria in the game but there is a unique coalition-proof Nash equilibrium. Moreover, no free riding problem exists in equilibrium. This paper differs from our because of three aspects: n groups instead of 2, the group-specific public good prize, and the probabilistic success function, however its multiplicity result align with our indeterminacy result, suggesting that the perfect complementarity in effort within each group is crucial. However, our results on discrete (Section 1.6) and on continuous (Gilli and Sorrentino (2024)) action sets prove that behind our indeterminacy result there are more complex determinants than simply the weakest link impact function.

Our reference model is Chowdhury, D. Lee, and Topolyan (2016). In this paper, the authors investigate a deterministic group contest in which each group's effort is represented by the minimum among the effort levels exerted by the group members - a max-min group contests - and the prize is a group-specific public good. Hence, our model differs from theirs in two aspects only: the private good prize - thus the cruciality of the within group sharing rules - and the binary action set. Chowdhury, D. Lee, and Topolyan (2016) fully characterize the symmetric equilibria. In our paper, we instead exploit the simplicity of the binary action set to calculate all the possible pure strategy equilibria of the two stage game, showing the existence of a continuum of subgame perfect equilibria.

On the assumption of a binary action set, there is a consistent theoretical and experimental literature on binary contests, as described in the survey by Sheremeta (2018), regarding participation decisions, such as Rapoport and Bornstein (1987) and Bornstein (1992) for intergroup competition over a public good. There are also works on entry decisions and contests, as Boosey et al. (2020), innovation contests, as Hofstetter et al. (2018), remuneration or promotions, as Gehman and Grimes (2017), Ladley et al. (2015) and Glover and Kim (2021) and competition for status as Dubey and Geanakoplos (2010). Other contributions concerning binary decisions in contests include sabotage activities, as Harbring et al. (2007) and Gürtler et al. (2013).

1.2 A Binary Group Contest with a Finite Number of Agents

Consider a simple two-group model that sums up the main characteristics of group contests under complete information. The model is defined by the following elements:

1. two **groups**, denoted by $j \in \{1, 2\}$;

2. each group has $n_j \geq 4$ members in each group. The total number of agents is $N = n_1 + n_2$. As notation device, let us write ij for **agents** $i \in \{1, \ldots, n_j\}$ of group j;

3. the **choice** of member $i \in \{1, ..., n_j\}$ in group $j \in \{1, 2\}$, to increase the possibility of getting the prize, is denoted by $x_{ij} \in \{0, 1\}$. Let $\mathbf{x}_j$ be the vector of all j-group agents' efforts, and $\mathbf{x}$ the vector of all agents' efforts. Moreover, let define the share of active players i in group j as
$$\gamma_j = \frac{1}{n_j} \sum_{i=1}^{n_j} x_{ij} \in [0, 1] ;$$

4. a private **prize** worth v to be allocated to one of the groups;

5. the **impact function** of group j is given by the weakest-link technology
$$X_j = \min \{x_{ij} \in \{0, 1\}, i \in \{1, ..., n_j\}\} ;$$
note that with binary actions $x_{ij} \in \{0, 1\}$, the weakest-link impact function is equivalent to a Cobb-Douglas production function, since
$$\prod_{ij} x_{ij}^{\alpha_{ij}} \equiv \min \{x_{ij} \in \{0, 1\}, i \in \{1, ..., n_j\}\} \ \text{, with } 0 < \alpha_{ij} < 1 \ \forall ij \in \{1, \ldots, n_j\} \ .$$

6. The **contest success function** is given by the *all-pay auction:*
$$p_j (X_1, X_2) = \begin{cases} 1 & if \;\; X_j > X_{-j} \\ \frac{1}{2} & if \;\; X_j = X_{-j} \\ 0 & if \;\; X_j < X_{-j}; \end{cases}$$

7. the **sharing rule**, such that if group $j \in \{1, 2\}$ wins, then a member $i \in \{1, \ldots, n_j\}$ gets a share of the prize
$$q_{ij} \left(x_{1j}, ..., x_{n_j j}\right) =$$

$$= \begin{cases} \underbrace{(1-\alpha_j)}_{incentivation\ part} \dfrac{x_{ij}}{\sum_{i=1}^{n_j} x_{ij}} + \underbrace{\alpha_j}_{equalizing\ part} \dfrac{1}{n_j} & if \quad \sum_{i=1}^{n_j} x_{ij} > 0 \\ \dfrac{1}{n_j} & otherwise \end{cases}$$

where

- α_j is the share of the prize that the members of the winning team get independently of their effort: let us call α_j the **equalizing part** of the sharing rule, while $1-\alpha_j$ is called the **incentivation part**;
- we assume that $\alpha_j \in \mathbb{R}$**,** i.e. the sharing rule goes beyond pure redistribution by allowing groups to engage in cross-subsidization:
  - $\underline{\alpha_j < 0}$: conditioned on winning the prize v, a negative value means that group j collects $-\alpha_j \frac{v}{n_j}$ from each of its members, and then distributes $(1-\alpha_j)\, v$ among members in proportion to relative outlay: it is a penalization for group's non-active members;
  - $\underline{\alpha_j > 1}$: conditioned on winning the prize v, a value greater than 1 means that group j collects $-(1-\alpha_j)\, v$ from its members according to relative outlay, and then distributes $\alpha_j v$ equally among all its members: it is a premium for group's non-active members.

This unrestricted sharing rule was first introduced by Hwan Baik and S. Lee (1997) as a generalization of the restricted case where $\alpha_j \in [0, 1]$. Notwithstanding the latter is more common for applications, we argue that there benefits in considering the generalized version to explore the role of selective incentives on players' choices within groups. Moreover, in the next sections it will be transparent how our results hold under the assumption of a restricted sharing rule as well. Let us stress that the application of the sharing rule is conditioned on winning, hence rewards and punishment are applied only if a group wins. The implicit assumption is that screening between free riders and contributors within a group is performed only if the group gets the prize, otherwise it is too costly. For example, an internal auditing to share a bonus in a team may be performed only after an order is awarded to a company. Similarly, in a conflict the punishment of free riders might be possible only after winning, while a defeat precludes such a possibility.

8. The individual **costs of effort** $C_{ij}(x_{ij}) = x_{ij}$;
9. the **timing**, there are two stages:
   i. in the first stage, the groups choose the optimal sharing rule within each group α_j;
   ii. in the second stage all the members of the groups observe the first stage choices (α_1, α_2) and choose simultaneously and independently their effort x_{ij} and the prize is allocated to one of the two groups according to the contest success function.

   Note that this timing structure, used in most papers that endogenize the sharing rule, implies that the groups can precommit to an equilibrium sharing rule, without any subsequent renegotiation. Let us stress that this sequential structure means that, once established, the sharing rule cannot be object of negotiation within or across the groups, i.e. there is a commitment to respect the choices of the first stage. Of course, it is possible to consider different sequential structures, where in the first stage, the groups compete for a single rent, while in the second stage, the members of each group compete for the rent won by their group, as in Katz and Tokatlidu (1996), for instance. However, our timing structure is the one employed by many contributions on group contests with a private good prize such as Nitzan (1991), S. Lee (1995), and Nitzan and Ueda (2018), where the sharing rule is used as a commitment device.

As a consequence of these modeling characteristics, player ij has the expected **payoff**

$$\pi_{ij}\left(\alpha_j, \alpha_{-j}, x_{1j}, ..., x_{n_j j}, x_{1-j}, ..., x_{n_{-j}-j}\right) = p_j q_{ij} v - x_{ij} =$$

$$= \begin{cases} \left[(1-\alpha_j)\frac{x_{ij}}{\sum_i x_{ij}} + \alpha_j \frac{1}{n_j}\right] v - x_{ij} & if \quad \min\{\mathbf{x}_j\} > \min\{\mathbf{x}_{-j}\} \\ \frac{1}{2}\left[(1-\alpha_j)\frac{x_{ij}}{\sum_i x_{ij}} + \alpha_j \frac{1}{n_j}\right] v - x_{ij} & if \quad \min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} \\ -x_{ij} & if \quad \min\{\mathbf{x}_j\} < \min\{\mathbf{x}_{-j}\} \end{cases}$$

Now, we are able to provide a formal definition of a binary group contest.

DEFINITION 1.1. *A* Binary Max-Min Group Contest $BMMGC$ *is a two-stage game* $\langle \{1,2\}, N, S_j, B_{ij}, \pi_{ij} \rangle$ *defined by*

1. *the set of groups* $\{1,2\}$;
2. *the set of players* $N = \{1,...,n_1+n_2\}$;
3. *the set of first-period actions* $S_j = \mathbb{R}$: *for each group* j, *the choice of the share* α_j *in the sharing rule*;
4. *the set of second-period actions* $B_{ij} = \{0,1\}$: *for each player* ij, *the choice of the effort* x_{ij};
5. *the payoff functions for each player* $ij \in N$

$$\pi_{ij}(\boldsymbol{\alpha}, \mathbf{x}) = p_j q_{ij} v - x_{ij} =$$

$$= \begin{cases} \left[(1-\alpha_j)\frac{x_{ij}}{\sum_i x_{ij}(i)} + \alpha_j \frac{1}{n_j}\right] v - x_{ij} & if \quad \min\{\mathbf{x}_j\} > \min\{\mathbf{x}_{-j}\} \\ \frac{1}{2}\left[(1-\alpha_j)\frac{x_{ij}}{\sum_i x_{ij}} + \alpha_j \frac{1}{n_j}\right] v - x_{ij} & if \quad \min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} \\ -x_{ij} & if \quad \min\{\mathbf{x}_j\} < \min\{\mathbf{x}_{-j}\} \end{cases}$$

where $\boldsymbol{\alpha}$ *and* $\mathbf{x}$ *are, respectively, the vector of first and second period actions.*

The notation used in this paper is summed up in table 1.1.

Variable	Meaning
ij	agent i of group j
$\{1, ..., n_j\}$	set of agents in group j
x_{ij}	effort of agent i in group j
$X_j = \min\{x_{ij} \in \{0,1\}, i \in \{1, ..., n_j\}\}$	impact of effort of all agents in group j
$\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2)$	vector of efforts of all agents
$C_{ij}(x_{ij}) = x_{ij}$	cost of effort for agent i of group j
$p_j(X_1, X_2)$	probability of group j of winning the contest
$q_{ij}\left(x_{1j}, ..., x_{n_j j}\right)$	sharing rule for agent i of group j
$\alpha_j \in \mathbb{R}$	equalizing part of the sharing rule
$\boldsymbol{\alpha}$	vector of α_j for $j \in \{1, 2\}$
$\pi_{ij}(\boldsymbol{\alpha}, \mathbf{x})$	payoff function of agent i of group j
$\gamma_j = \frac{1}{n_j}\sum_{i=1}^{n_j} x_{ij} \in [0,1]$	share of active agents in group j

Table 1.1

1.3 The Set of Second-Period Equilibria

Without loss of generality, the equilibria are presented in terms of share of active agents in each group, i.e. as pairs

$$(\gamma_1, \gamma_2) \in [0,1] \times [0,1],$$

so that geometrically they can be represented in the unit square. Then, we will indicate by

$$\pi_{ij}(\gamma_1, \gamma_2 | \boldsymbol{\alpha})$$

the second-period payoff of player ij, as a function of (γ_1, γ_2) for a given $\alpha := (\alpha_1, \alpha_2)$.

Moreover, when $\gamma_j \in (0,1)$, denote by

$$\gamma_j^+ = \frac{1}{n_j}\left(\sum_{i=1}^{n_j} x_{ij} + 1\right) \in [0,1]$$

the share of active agents at a marginal increase and by

$$\gamma_j^- = \frac{1}{n_j}\left(\sum_{i=1}^{n_j} x_{ij} - 1\right) \in [0,1]$$

the share of active agents at a marginal decrease.

1.3.1 Characterization of the Set of Pure Strategy Nash Equilibria in the Second Period.

In this subsection, we characterize the full set of second-period Nash equilibria in pure strategies, to study the interplay of strong complementarities at play within groups, which favor the alignment of effort choice by teammates, and the selective incentives induced by the sharing rules.

PROPOSITION 1.1. *In the BMMGC, the set of the second period pure strategy Nash equilibria of the game is characterized as follows:*

1. *if* $v \geq 2\max\{n_1, n_2\}$, *then*

$$(\gamma_1^*, \gamma_2^*) = (1,1) \;\; for\ any \;\; (\alpha_1, \alpha_2) \in \mathbb{R} \times \mathbb{R};$$

2. *if* $v > 0$, *then*

$$(\gamma_1^{**}, \gamma_2^{**}) = (0,0) \;\; for\ any \;\; (\alpha_1, \alpha_2) \in \left[1 - \frac{2n_1}{(n_1-1)\,v}, \infty\right) \times \left[1 - \frac{2n_2}{(n_2-1)\,v}, \infty\right)$$

3. *if* $v > 0$, *then*

$$\left(\gamma_j^{***}, \gamma_{-j}^{***}\right) = (1,0) \;\; for\ any \;\; (\alpha_j, \alpha_{-j}) \in \left(-\infty, 2\left(1 - \frac{n_j}{v}\right)\right] \times \mathbb{R};$$

4. *if* $v > 0$, *then*

$$\left(\gamma_j^{****a}, \gamma_{-j}^{****a}\right) \in (0,1) \times \{0\} \quad with \; \sum_{i=1}^{n_j} x_{ij} \in \{2, \ldots, n_j - 2\}$$

$$\textit{for any}\quad (\alpha_j, \alpha_{-j}) \in \left[1 - \frac{2\left(n_j\gamma_j + 1\right)}{v}, 1 - \frac{2n_j\gamma_j}{v}\right] \times \left[1 - \frac{2n_{-j}}{\left(n_{-j} - 1\right)v}, \infty\right);$$

5. *if* $v > 0$, *then*

$$\left(\gamma_j^{****b}, \gamma_{-j}^{****b}\right) \in (0,1) \times \{0\} \quad \textit{with} \; \sum_{i=1}^{n_j} x_{ij} = 1$$

$$\textit{for any}\quad (\alpha_j, \alpha_{-j}) \in \left[1 - \frac{2\left(n_j\gamma_j + 1\right)}{v}, 1 - \frac{2n_j\gamma_j}{\left(1 - \gamma_j\right)v}\right] \times \left[1 - \frac{2n_{-j}}{\left(n_{-j} - 1\right)v}, \infty\right);$$

6. *if* $0 < v \leq 2$, *then*

$$\left(\gamma_j^{****c}, \gamma_{-j}^{****c}\right) \in (0,1) \times \{0\} \quad \textit{with} \; \sum_{i=1}^{n_j} x_{ij} = n_j - 1$$

$$\textit{for any}\quad (\alpha_j, \alpha_{-j}) \in \left[2\left(1 - \frac{n_j}{v}\right), 1 - \frac{2n_j\gamma_j}{v}\right] \times \left[1 - \frac{2n_{-j}}{\left(n_{-j} - 1\right)v}, \infty\right);$$

7. *if* $v > 0$, *then*

$$\left(\gamma_1^{*****a}, \gamma_2^{*****a}\right) \in (0,1) \times (0,1) \quad \textit{with} \; \sum_{i=1}^{n_j} x_{ij} \in \{2, \ldots, n_j - 2\}$$

$$\textit{for any}\quad (\alpha_1, \alpha_2) \in \left[1 - \frac{2\left(n_1\gamma_1 + 1\right)}{v}, 1 - \frac{2n_1\gamma_1}{v}\right] \times \left[1 - \frac{2\left(n_2\gamma_2 + 1\right)}{v}, 1 - \frac{2n_2\gamma_2}{v}\right];$$

8. *if* $v > 0$, *then*

$$\left(\gamma_1^{*****b}, \gamma_2^{*****b}\right) \in (0,1) \times (0,1) \quad \textit{with} \; \sum_{i=1}^{n_j} x_{ij} = 1$$

$$\textit{for any}\quad (\alpha_1, \alpha_2) \in$$

$$\left[1 - \frac{2\left(n_1\gamma_1 + 1\right)}{v}, 1 - \frac{2n_1\gamma_1}{\left(1 - \gamma_1\right)v}\right] \times \left[1 - \frac{2\left(n_2\gamma_2 + 1\right)}{v}, 1 - \frac{2n_2\gamma_2}{\left(1 - \gamma_2\right)v}\right];$$

9. *if* $0 < v \leq 2$, *then*

$$\left(\gamma_1^{*****c}, \gamma_2^{*****c}\right) \in (0,1) \times (0,1) \quad \textit{with} \; \sum_{i=1}^{n_j} x_{ij} = n_j - 1$$

$$for\ any \quad (\alpha_1, \alpha_2) \in \left[2\left(1-\frac{n_1}{v}\right), 1-\frac{2n_1\gamma_1}{v}\right] \times \left[2\left(1-\frac{n_2}{v}\right), 1-\frac{2n_2\gamma_2}{v}\right].$$

Proof. See Appendix 1.A.1 . □

REMARK 1.1. *Note that in the BMMGC there are even within-group asymmetric second-period equilibria in pure strategies in contrast with what found for the public good setting in Chowdhury, D. Lee, and Topolyan (2016). This property is due to the presence of selective incentives induced by the sharing rules sustaining heterogeneity in effort choices within groups. On the other hand, with a public good and perfect complementarities driven by the weakest-link impact function, the incentives are perfectly aligned within groups.*

Geometrically, if we represent in the space

$$(\gamma_1, \gamma_2) \in [0,1] \times [0,1]$$

the set of second-stage pure strategy Nash equilibria, depending on groups' sizes, we cover almost all points corresponding to rational numbers in the unit square, apart from the two segments

$$(\gamma_1, \gamma_2) \in \{1\} \times (0,1) \quad \text{and} \quad (\gamma_1, \gamma_2) \in (0,1) \times \{1\},$$

as represented in figure 2.1.

It is possible to give a behavioral interpretation to the equilibria above.

- $(\gamma_1, \gamma_2) = (1,1)$ to be an equilibrium requires that the prize is sufficiently high, so that a unilateral deviation to the zero effort choice is not profitable. Moreover, any α_j is admissible at equilibrium, since the sharing rule is irrelevant both at equilibrium, for it is within-group symmetric, and at the unique unilateral deviation for any player of both groups, which would trigger a defeat for the belonging group.

- $(\gamma_1, \gamma_2) = (0,0)$ to be an equilibrium requires that the equalizing part of the sharing rule is sufficiently high, so that the effort provision choice is not profitable.

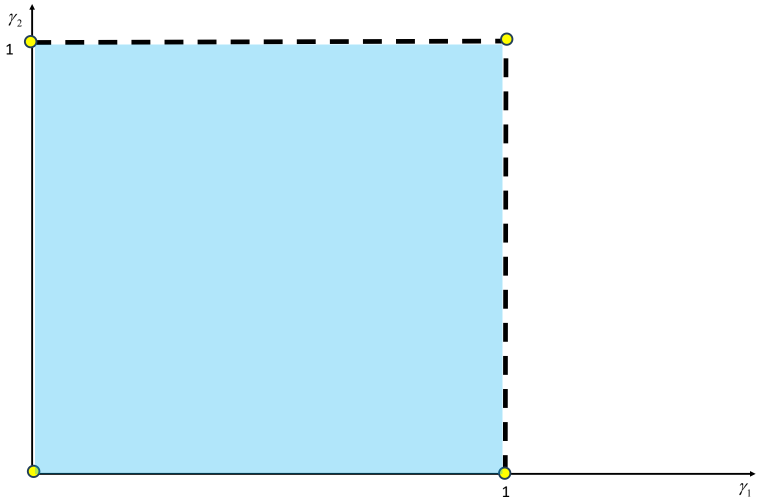


Figure 1.1: Geometric representation of the set of second-period pure strategy Nash equilibria.

- $(\gamma_j, \gamma_{-j}) = (1, 0)$ to be an equilibrium requires that the equalizing part of the sharing rule for the active group is sufficiently low, guaranteeing there are no incentives for forcing a tie between the two groups by free-riding; on the other hand, any α_{-j} is consistent with the equilibrium, for the zero-effort group lose with probability one both at equilibrium and at the unique unilateral deviation available to each to its members. Finally, note that in the restricted sharing rule case, i.e. $\alpha_j \in [0,1]\,\forall j \in \{1,2\}$, the prize has to be sufficiently high, so that $0 \leq 2\,(1 - n_j/v) < 1$.

- $(\gamma_j, \gamma_{-j}) \in (0,1) \times \{0\}$ with $\sum_{i=1}^{n_j} x_{ij} \in \{2, \ldots, n_j - 2\}$ to be an equilibrium requires the equalizing part of j-group sharing rule casting a balance between incentivization and equality within the group where some members provide effort. Note that a unilateral downward deviation of any active member would not change the probability of winning of her group, since at least two teammates are not exerting effort and already determine the groups' fate. On the other hand, the equalizing part of the sharing rule of group $-j$ has to be sufficiently high making upward deviations unprofitable for group members. Finally, note that in the restricted sharing rule case, i.e. $\alpha_j \in [0,1]\,\forall j \in \{0,1\}$, the prize has to be sufficiently high, so that $0 \leq 1 - (2n_j\gamma_j)/v < 1$.

- $(\gamma_j, \gamma_{-j}) \in (0,1) \times \{0\}$ with $\sum_{i=1}^{n_j} x_{ij} = 1$ to be an equilibrium requires almost same conditions on the equalizing parts of the sharing rules as for the strategy profile $(\gamma_j, \gamma_{-j}) \in (0,1) \times \{0\}$ with $\sum_{i=1}^{n_j} x_{ij} \in \{2, \dots, n_j - 2\}$. However, given that a unilateral deviation of the unique active member in group j would make the sharing rule uniform within her group at equilibrium, the upper boundary on the equalizing part of the sharing rule is different for group j. Finally, note that in the restricted sharing rule case, i.e. $\alpha_j \in [0,1] \, \forall j \in \{1,2\}$, the prize has to be sufficiently high, so that $0 \leq 1 - (2n_j\gamma_j) \, / \, (1-\gamma_j) \, v < 1$.

- $(\gamma_j, \gamma_{-j}) \in (0,1) \times \{0\}$ with $\sum_{i=1}^{n_j} x_{ij} = n_j - 1$ to be an equilibrium requires almost same conditions on the equalizing parts of the sharing rules as for the strategy profile $(\gamma_j, \gamma_{-j}) \in (0,1) \times \{0\}$ with $\sum_{i=1}^{n_j} x_{ij} \in \{2, \dots, n_j - 2\}$. However, given that a unilateral deviation of the unique inactive member in group j would make the sharing rule uniform within her group at equilibrium and discretely change the probability of winning for group j, the lower boundary on the equalizing part of the sharing rule is different for group j. Moreover, the value of the prize has to be sufficiently low, so that there is not an incentive for the unique inactive member to force a win for her group. Finally, note that in the restricted sharing rule case, i.e. $\alpha_j \in [0,1] \, \forall j \in \{1,2\}$, the prize has to be sufficiently high, so that $0 \leq 1 - (2n_j\gamma_j) \, / v < 1$. In particular, $n_j = 2$ and $v = 2$ have to hold.

- $(\gamma_1, \gamma_2) \in (0,1) \times (0,1)$ with $\sum_{i=1}^{n_j} x_{ij} \in \{2, \dots, n_j - 2\}$, see the comment for group j at point 4 .

- $(\gamma_1, \gamma_2) \in (0,1) \times (0,1)$ with $\sum_{i=1}^{n_j} x_{ij} = 1$, see the comment for group j at point 5.

- $(\gamma_1, \gamma_2) \in (0,1) \times (0,1)$ with $\sum_{i=1}^{n_j} x_{ij} = n_j - 1$, see the comment for group j at point 6.

We would like to highlight that larger groups should be disadvantaged at any within-group symmetric equilibrium in terms of per-capita valuation of the prize. However, Nash equilibrium as a solution concept does not prevent implausible expectations of a win by the disadvantaged group from holding. Equilibria in pure strategies in the second period are very diverse in terms of outcomes.

For each of these classes of equilibria, we can compute the continuation payoffs. However, the possible combinations of continuation payoffs associated to the possible values of the sharing rules would involve a huge amount of possible combinations: even if it would be computationally feasible to characterize the set of subgame perfect Nash equilibria in pure strategies, it would involve hundreds of billions of iterations as shown in figure 1.6, so that we decided to limit ourselves to the case of within-group symmetric (WGS) second stage equilibria.

1.3.2 The Set of Pure Strategy Within-Group Symmetric Nash Equilibria in the Second Period.

The following corollary follows immediately from proposition 1.

COROLLARY 1.1. *In the $BMMGC$, the set of the second period WGS pure strategy Nash equilibria of the game is the following:*

1. *if $v \geq 2\max\{n_1, n_2\}$, then*

$$\left(\gamma_1^*, \gamma_2^*\right) = (1, 1) \ \ for\ any\ \ (\alpha_1, \alpha_2) \in \mathbb{R} \times \mathbb{R};$$

2. *if $v > 0$, then*

$$\left(\gamma_1^{**}, \gamma_2^{**}\right) = (0, 0) \ \ for\ any\ \ (\alpha_1, \alpha_2) \in \left[1 - \frac{2n_1}{(n_1 - 1)\, v}, \infty\right) \times \left[1 - \frac{2n_2}{(n_2 - 1)\, v}, \infty\right)$$

3. *if $v > 0$, then*

$$\left(\gamma_j^{***}, \gamma_{-j}^{***}\right) = (1, 0) \ \ for\ any\ \ (\alpha_j, \alpha_{-j}) \in \left(-\infty, 2\left(1 - \frac{n_j}{v}\right)\right] \times \mathbb{R}.$$

The following result is useful to derive the subgame perfect equilibria of the game.

COROLLARY 1.2. *When*

$$v < \frac{2n_j\,(n_j - 2)}{n_j - 1}$$

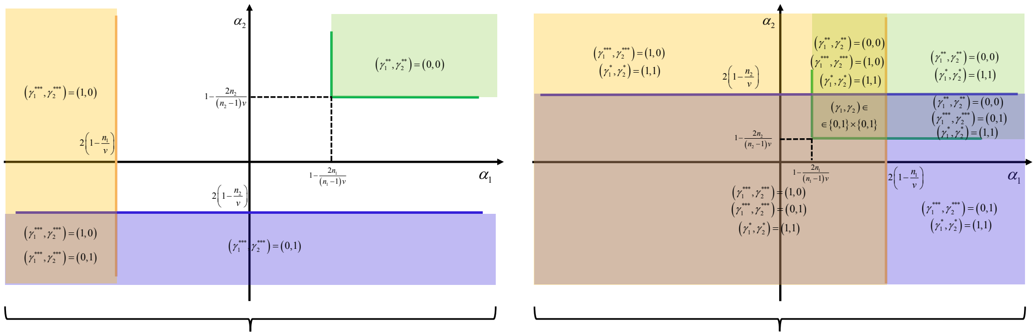
for some $j \in \{1, 2\}$, *then there exists a region of sharing rules* (α_1, α_2) *such that there exists no second stage pure strategy WGS equilibrium.*

Proof. See Appendix 1.A.2 . ☐

REMARK 1.2. *This result means that there are only two possible cases for the second stage, either multiple pure strategy WGS equilibria for some values of the sharing rules, when*

$$v \geq \max\left\{\frac{2n_1(n_1-2)}{n_1-1}, \frac{2n_2(n_2-2)}{n_2-1}\right\},$$

or possible values of the sharing rules such that there exists no pure strategy WGS equilibrium. The following figure represents two possible (extreme) situations.



$v < \min\left\{\frac{2n_1(n_1-2)}{n_1-1}, \frac{2n_2(n_2-2)}{n_2-1}\right\}$

$v \geq 2\max\{n_1, n_2\}$

Figure 1.2: Non existence and multiple WGS equilibria.

1.4 The Equilibrium Choice of the Sharing Rules in the First Period

To derive the optimal sharing rules, we consider their optimal choice in each group by an utilitarian ruler with payoff function [1.4]

$$\pi_j^C\left(\alpha_j, \alpha_{-j} | \gamma_1^e, \gamma_2^e\right) = \sum_{i=1}^{n_j} \pi_{ij}^C\left(\alpha_j, \alpha_{-j} | \gamma_1^e, \gamma_2^e\right).$$

Consider the agents' and groups' continuation payoffs associated to the different second period equilibria:

1. when

   $$\left(\gamma_1^*, \gamma_2^*\right) = (1, 1), \quad for \quad any \quad v \geq 2 \cdot max\{n_1, n_2\} \quad and \quad (\alpha_1, \alpha_2) \in \mathbb{R} \times \mathbb{R}$$

   then

   $$\pi_{ij}^C\left(\alpha_j, \alpha_{-j} | \gamma_1^*, \gamma_2^*\right) = \frac{1}{2}\frac{1}{n_j}v - 1 \text{ and } \pi_j^C\left(\alpha_j, \alpha_{-j} | \gamma_1^*, \gamma_2^*\right) = \frac{v}{2} - n_j \ .$$

2. when

   $$\left(\gamma_1^{**}, \gamma_2^{**}\right) = (0, 0), \quad for \quad any \quad \alpha_j \geq 1 - \frac{2n_j}{\left(n_j - 1\right)v} \quad with \quad j \in \{1, 2\} \ .$$

   then

   $$\pi_{ij}^C\left(\alpha_j, \alpha_{-j} | \gamma_1^{**}, \gamma_2^{**}\right) = \frac{1}{2}\frac{1}{n_j}v \quad \text{and} \quad \pi_j^C\left(\alpha_j, \alpha_{-j} | \gamma_1^{**}, \gamma_2^{**}\right) = \frac{v}{2} \ .$$

3. when

   $$\left(\gamma_j^{***}, \gamma_{-j}^{***}\right) = (0, 1), \quad for \quad any \quad \alpha_j \in \mathbb{R} \quad \text{and} \quad any \quad \alpha_{-j} \leq 2\left(1 - \frac{n_{-j}}{v}\right)$$

[1.4] Note that the optimal choice of an utilitarian ruler in the first period corresponds to the non-cooperative optimal choice of an individual player, for we limit ourselves to WGS equilibria in the second period.

then

$$\pi_{ij}^{C}\left(\alpha_j, \alpha_{-j} | \gamma_1^{***}, \gamma_2^{***}\right) = 0 \ , \ \ \pi_{i-j}\left(\alpha_j, \alpha_{-j} | \gamma_1^{***}, \gamma_2^{***}\right) = \frac{1}{n_{-j}} v - 1$$

$$\text{and} \ \ \pi_{j}^{C}\left(\alpha_j, \alpha_{-j} | \gamma_1^{***}, \gamma_2^{***}\right) = 0 \ , \ \ \pi_{-j}\left(\alpha_j, \alpha_{-j} | \gamma_1^{***}, \gamma_2^{***}\right) = v - n_{-j} \ .$$

Note that in order to find the set of first-period equilibria the continuation payoffs for both groups have to be specified at each element of the Cartesian product of α_1 and α_2 sustaining the set of second-period equilibria for both groups, so that we obtain:

1. if $v \geq 2 \cdot max\left\{n_1, n_2\right\}$, there are 7776 continuation-payoffs matrices;
2. if $v < 2 \cdot max\left\{n_1, n_2\right\}$ and $1 - \frac{2n_j}{(n_j-1)v} < 2\left(1 - \frac{n_j}{v}\right)$ with $j \in \{1, 2\}$, there are 96 continuation-payoffs matrices;
3. if $v < 2 \cdot max\left\{n_1, n_2\right\}$ and $1 - \frac{2n_j}{(n_j-1)v} < 2\left(1 - \frac{n_j}{v}\right)$ and $1 - \frac{2n_{-j}}{(n_{-j}-1)v} = 2\left(1 - \frac{n_{-j}}{v}\right)$ with $j \in \{1, 2\}$, there are 96 continuation-payoffs matrices;
4. if $v < 2 \cdot max\left\{n_1, n_2\right\}$ and $1 - \frac{2n_j}{(n_j-1)v} = 2\left(1 - \frac{n_j}{v}\right)$ for any $j \in \{1, 2\}$, there are 96 continuation-payoffs matrices.

On the other hand, if $1 - \frac{2n_j}{(n_j-1)v} > 2\left(1 - \frac{n_j}{v}\right) \Leftrightarrow v < \frac{2n_j(n_j-2)}{n_j-1}$, the continuation payoffs cannot be pinned down for some $(\alpha_1, \alpha_2) \in \mathbb{R} \times \mathbb{R}$, so that there exists no optimal sharing rule and thus no pure strategy subgame perfect equilibria.

As reported in figures 1.5 and 1.6, the cardinality of the set of continuation-payoffs matrices is obtained by taking the Cartesian product of the number of second-stage equilibria in each interval, determined by the equilibrium thresholds, over the (α_1, α_2) space. Given the magnitude of the cardinality of the set of continuation-payoffs matrices, even for the WGS case, we employ a simple recursive algorithm, written in MATLAB, to compute the optimal sharing rules. Thus, we might conclude with the following result. [1.5]

PROPOSITION 1.2. *In the BMMGC, in the first period, there is a continuum of optimal sharing rules such that:*

[1.5] All lines of code are available online at https://data.mendeley.com/datasets/5fj2fww2yv/1 .

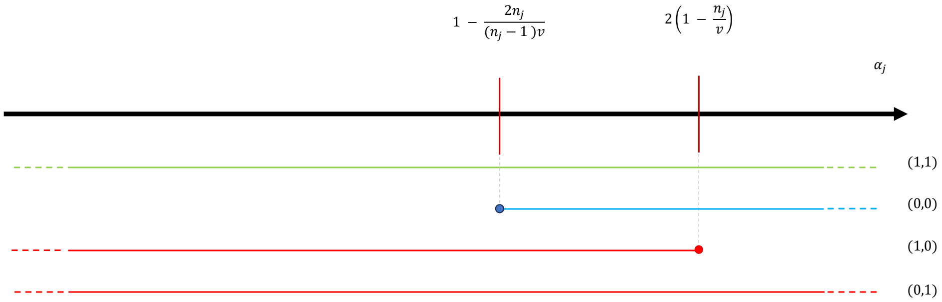


Figure 1.3: Intervals of α_j sustaining within-group symmetric second-period equilibria $\forall v \geq 2 \cdot max\left\{n_j, n_{-j}\right\}$.

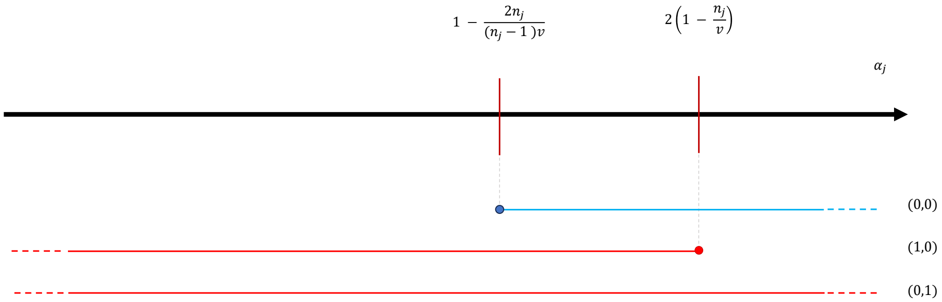


Figure 1.4: Intervals of α_j sustaining within-group symmetric second-period equilibria $\forall v < 2 \cdot max\left\{n_j, n_{-j}\right\}$.

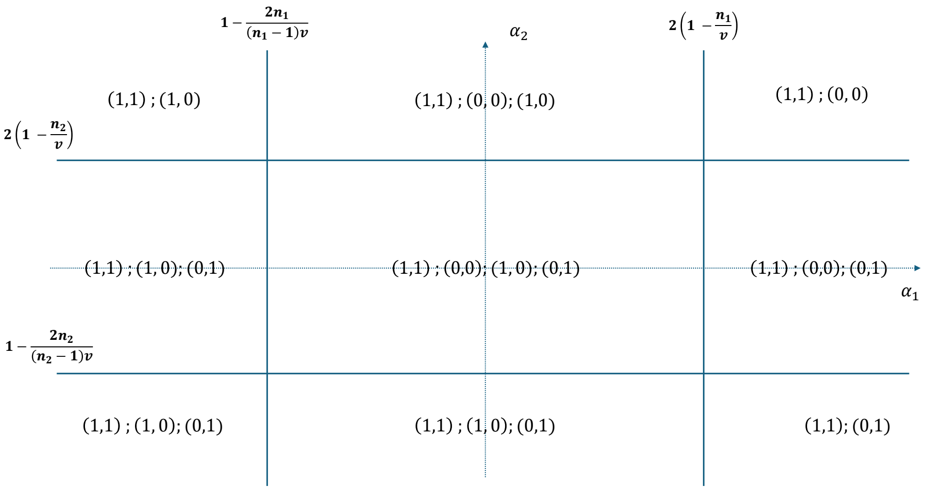


Figure 1.5: Second-period WGS pure Nash equilibria in $\alpha_1 \times \alpha_2$ space $\forall v \geq 2 \cdot max\,\{n_j, n_{-j}\}$. Note that the number of continuation-payoffs matrices is $2 \times 3 \times 2 \times 3 \times 4 \times 3 \times 3 \times 3 \times 2 = 7776$.

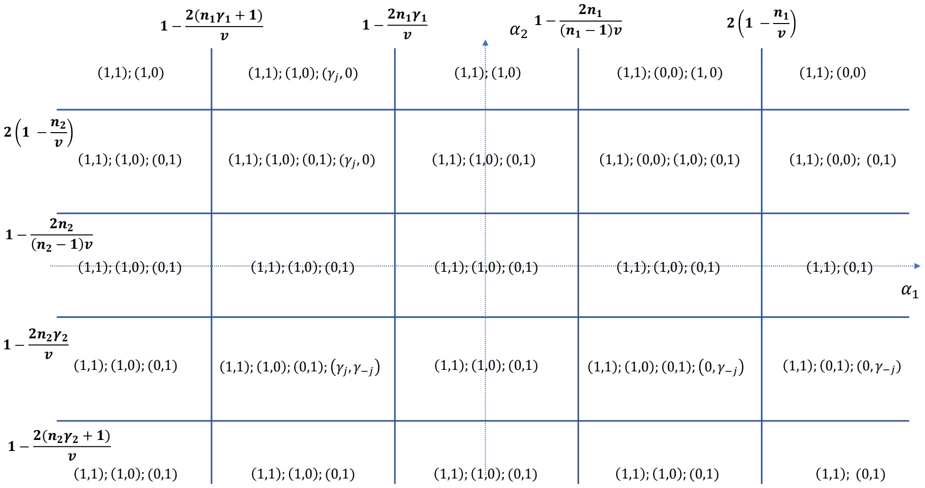


Figure 1.6: Second-period pure Nash equilibria in $\alpha_1 \times \alpha_2$ space $\forall v \geq 2 \cdot max\,\{n_j, n_{-j}\}$. Note that for just one asymmetric equilibrium (γ_j, γ_{-j}), the number of continuation-payoffs matrices is $2\times3\times2\times3\times2\times3\times4\times3\times4\times3\times3\times3\times3\times3\times2\times3\times4\times3\times4\times3\times3\times3\times3\times3\times2 = 352,640,000,000$.

- *if* $v \geq 2 \cdot max\,\{n_1, n_2\}$

$$(\alpha_1^*, \alpha_2^*) \in \mathbb{R} \times \mathbb{R};$$

- *if* $\frac{2 \cdot max\{n_1,n_2\} \cdot (max\{n_1,n_2\}-2)}{max\{n_1,n_2\}-1} \leq v < 2 \cdot max\{n_1, n_2\}$

$$(\alpha_1^*, \alpha_2^*) \subset \mathbb{R} \times \mathbb{R};$$

- *otherwise*

$$(\alpha_1^*, \alpha_2^*) \equiv \emptyset \times \emptyset \ .$$

REMARK 1.3. *Proposition 2 means that the set of optimal sharing rules computed over all continuation-payoffs matrices coincides with* $\mathbb{R} \times \mathbb{R}$ *for a sufficiently high prize, that is when* $v \geq 2 \cdot max\{n_1, n_2\}$*, the condition sustaining the most competitive second-period WGS pure Nash equilibrium* $(\gamma_j, \gamma_{-j}) = (1, 1)$*.*

REMARK 1.4. *It is possible to show that there is a continuum of optimal sharing rules even for the restricted sharing rule case, i.e.* $\alpha_j \in [0, 1]$*.*

1.5 The Set of Subgame Perfect Equilibria

From the previous results, it is straightforward to derive the following one.

PROPOSITION 1.3. *In the BMMGC, there is a continuum of within-group symmetric subgame perfect Nash equilibria in pure strategies.*

REMARK 1.5. *Even though the characterization of within-group asymmetric subgame perfect Nash equilibria is computationally demanding, as discussed in the previous section, it is straightforward to state that in the BMMGC there is a continuum of within-group asymmetric subgame perfect Nash equilibria in pure strategies as well.*

REMARK 1.6. *This result means that in the BMMGC there is indeterminacy, in the sense that in equilibrium anything is possible. Even equilibria in which the larger group wins hold, despite that group being disadvantaged in terms of per-capita valuation of the prize at any within-group symmetric equilibrium.*

REMARK 1.7. *The indeterminacy in the $BMMGC$ can be shown to extend to the restricted sharing rule case as well, namely for $\alpha_j \in [0,1] \forall j \in \{1,2\}$.*

As an illustration, we provide a few examples of pure strategy within-group symmetric subgame perfect equilibria:

1. if $v > 2\max\{n_1, n_2\}$,

   $$\left(\alpha_1^{SGP}, \alpha_2^{SGP}\right) \in \mathbb{R} \times \mathbb{R}$$

   $$\left(\gamma_1^{SGP}, \gamma_2^{SGP}\right) = (1,1) \;\; \forall \left(\alpha_1, \alpha_2\right) \in \mathbb{R} \times \mathbb{R}$$

   with

   $$\left(\alpha_1^{SGP}, \alpha_2^{SGP}\right) \in \mathbb{R} \times \mathbb{R} \;\text{ and }\; \left(\gamma_1^{SGP}, \gamma_2^{SGP}\right) = (1,1)$$

   as equilibrium outcomes.

   Comment: if players of both groups expect the equilibrium in which groups tie on the positive effort provision choice, then, clearly, any $\alpha_j \in \mathbb{R}$ is optimal in the first period, that is any incentivization scheme is optimal when expecting such a second-period equilibrium. Such a tie is particularly plausible for $n_1 = n_2 \geq 4$, in which case no group is advantaged in terms of per-capita valuation of the prize at equilibrium; ;

2. if $v > 2n_1$ and $n_2 < v < 2n_2$ such that $1 - \frac{2n_2}{(n_2-1)v} < 2\left(1 - \frac{n_2}{v}\right)$,

   $$\left(\alpha_1^{SGP}, \alpha_2^{SGP}\right) \in \left(-\infty, 1 - \frac{2n_1}{\left(n_1 - 1\right)v}\right) \times \mathbb{R}$$

   $$\begin{cases} \left(\gamma_1^{SGP}, \gamma_2^{SGP}\right) = (1,0) & \text{if } \left(\alpha_1, \alpha_2\right) \in \left(-\infty, 1 - \frac{2n_1}{(n_1-1)v}\right) \times \mathbb{R} \\ \left(\gamma_1^{SGP}, \gamma_2^{SGP}\right) = (1,0) & \text{if } \left(\alpha_1, \alpha_2\right) \in \left[1 - \frac{2n_1}{(n_1-1)v}, 2\left(1 - \frac{n_1}{v}\right)\right] \times \left(-\infty, 1 - \frac{2n_2}{(n_2-1)v}\right) \\ \left(\gamma_1^{SGP}, \gamma_2^{SGP}\right) = (0,0) & \text{if } \left(\alpha_1, \alpha_2\right) \in \left[1 - \frac{2n_1}{(n_1-1)v}, \infty\right) \times \left[1 - \frac{2n_2}{(n_2-1)v}, \infty\right) \\ \left(\gamma_1^{SGP}, \gamma_2^{SGP}\right) = (0,1) & \text{if } \left(\alpha_1, \alpha_2\right) \in \left[2\left(1 - \frac{n_1}{v}\right), \infty\right) \times \left[-\infty, 1 - \frac{2n_2}{(n_2-1)v}\right) \end{cases}$$

   with

   $$\left(\alpha_1^{SGP}, \alpha_2^{SGP}\right) \in \left(-\infty, 1 - \frac{2n_1}{\left(n_1 - 1\right)v}\right) \times \mathbb{R} \;\text{ and }\; \left(\gamma_1^{SGP}, \gamma_2^{SGP}\right) = (1,0)$$

as equilibrium outcomes.

Comment: if players from both groups expect the equilibrium in which group 1, being the smaller group, wins the with certainty, along the equilibrium implying a tie on the zero-effort choice, and the equilibrium in which group 2 wins, then the advantaged group, i.e. group 1, chooses α_1 guaranteeing the victory;

3. if $n_1 < v < 2n_1$ and $n_2 < v < 2n_2$ such that $1 - \frac{2n_1}{(n_1-1)v} = 2\left(1 - \frac{n_1}{v}\right)$ and $1 - \frac{2n_2}{(n_2-1)v} < 2\left(1 - \frac{n_2}{v}\right)$,

$$\left(\alpha_1^{SGP}, \alpha_2^{SGP}\right) \in$$

$$\left[1 - \frac{2n_1}{\left(n_1 - 1\right)v}, \infty\right) \times \left(2\left(1 - \frac{n_2}{v}\right), \infty\right)$$

$$\begin{cases} \left(\gamma_1^{SGP}, \gamma_2^{SGP}\right) = (0,0) & \text{if } (\alpha_1, \alpha_2) \in \left[1 - \frac{2n_1}{(n_1-1)v}, \infty\right) \times \left(2\left(1 - \frac{n_2}{v}\right), \infty\right) \\ \left(\gamma_1^{SGP}, \gamma_2^{SGP}\right) = (1,0) & \text{if } (\alpha_1, \alpha_2) \in \left(-\infty, 1 - \frac{2n_1}{(n_1-1)v}\right) \times \left(2\left(1 - \frac{n_2}{v}\right), \infty\right) \\ \left(\gamma_1^{SGP}, \gamma_2^{SGP}\right) = (0,1) & \text{if } (\alpha_1, \alpha_2) \in \mathbb{R} \times \left(-\infty, 2\left(1 - \frac{n_2}{v}\right)\right] \end{cases}$$

with

$$\left(\alpha_1^{SGP}, \alpha_2^{SGP}\right) \in$$

$$\left[1 - \frac{2n_1}{\left(n_1 - 1\right)v}, \infty\right) \times \left(2\left(1 - \frac{n_2}{v}\right), \infty\right) \text{ and } \left(\gamma_1^{SGP}, \gamma_2^{SGP}\right) = (0,0)$$

as equilibrium outcomes.

Comment: if the per-capita valuation of the prize is sufficiently low for both groups and if all players expect the equilibrium in which groups tie on the zero-effort choice, the equilibrium in which groups 1 wins, and the equilibrium in which groups 2 wins, then the optimal choice of sharing rules for both groups will select the equilibrium in which groups tie without exerting effort;

4. if $n_1 < v < 2n_1$ and $n_2 < v < 2n_2$ such that $1 - \frac{2n_j}{(n_j-1)v} = 2\left(1 - \frac{n_j}{v}\right) \forall j \in \{1,2\}$,

$$\left(\alpha_1^{SGP}, \alpha_2^{SGP}\right) \in \left(-\infty, 2\left(1 - \frac{n_1}{v}\right)\right] \times \left(-\infty, 2\left(1 - \frac{n_2}{v}\right)\right)$$

$$\begin{cases} \left(\gamma_1^{SGP}, \gamma_2^{SGP}\right) = (0,1) & \text{if } (\alpha_1, \alpha_2) \in \mathbb{R} \times \left(-\infty, 2\left(1 - \frac{n_2}{v}\right)\right] \\ \left(\gamma_1^{SGP}, \gamma_2^{SGP}\right) = (0,0) & \text{if } (\alpha_1, \alpha_2) \in \left[2\left(1 - \frac{n_1}{v}\right), \infty\right) \times \left(2\left(1 - \frac{n_2}{v}\right), \infty\right) \\ \left(\gamma_1^{SGP}, \gamma_2^{SGP}\right) = (1,0) & \text{if } (\alpha_1, \alpha_2) \in \left(-\infty, 2\left(1 - \frac{n_1}{v}\right)\right) \times \left(2\left(1 - \frac{n_2}{v}\right), \infty\right) \end{cases}$$

with

$$\left(\alpha_1^{SGP}, \alpha_2^{SGP}\right) \in$$

$$\left(-\infty, 2\left(1 - \frac{n_1}{v}\right)\right) \times \left(-\infty, 2\left(1 - \frac{n_2}{v}\right)\right] \text{ and } \left(\gamma_1^{SGP}, \gamma_2^{SGP}\right) = (0,1)$$

as equilibrium outcomes.

Comment: if the per-capita valuation of the prize is sufficiently low for both equally-sized groups and if all players expect the equilibrium in which groups tie on the zero-effort choice, the equilibrium in which group 1 wins and the equilibrium in which group 2 wins, then the optimal choice of sharing rules for both groups selects the equilibrium in which group 2 wins. Note that for the same expectations of second-period equilibria in the $\alpha_1 \times \alpha_2$ space, there is an equilibrium in which the optimal sharing rules select the equilibrium in which groups tie on the zero-effort choice.

REMARK 1.8. *The examples above can be rewritten making intervals for* (α_1^*, α_2^*) *open or closed, depending on which second-period equilibrium is assumed to be played at the internal thresholds* $1 - \frac{2n_j}{(n_j-1)v}, 2\left(1 - \frac{n_j}{v}\right)$, $\forall j = 1, 2$.

Given the indeterminacy result obtained, the predictive power of the model under consideration is limited. However, making assumptions about the players' expectations about the second-period equilibria, for instance regarding the impossibility of a larger groups to win with certainty, the span of equilibrium outcomes can be reduced. Moreover, more specific details about the competition of interest might deliver the appropriate equilibrium selection criterion for applications.

1.6 Extension to K Effort Levels

In this section we try to address some potential questions arising from the results obtained under Proposition 1, 2 and 3, which are in close relation with the assumptions under our binary group contest. In particular, we expand the set of second-stage actions from the binary case to any subset of the natural numbers with cardinality at least equal to three.

DEFINITION 1.2. *A* K-Actions Max-Min Group Contest *$KMMGC$ is a two-stage game* $\langle \{1, 2\}, N, S_j, K_{ij}, \pi_{ij} \rangle$ *defined by*

1. *the set of groups* $\{1, 2\}$;
2. *the set of players* $N = \{1, ..., n_1 + n_2\}$;
3. *the set of first period actions* $S_j = \mathbb{R}$: *for each group* j, *the choice of the share* α_j *in the sharing rule*;
4. *the set of second period actions* $K_{ij} = \{0, \dots, K\}$, *with* $K_{ij} \subset \mathbb{N}$ *and* $K \geq 2$: *for each player* ij, *the choice of the effort* x_{ij};
5. *the payoff functions for each player* $ij \in N$

$$\pi_{ij}(\boldsymbol{\alpha}, \mathbf{x}) = p_j q_{ij} v - x_{ij} =$$

$$= \begin{cases} \left[(1-\alpha_j)\frac{x_{ij}}{\sum_i x_{ij}} + \alpha_j \frac{1}{n_j}\right] v - x_{ij} & if \quad \min\{\mathbf{x}_j\} > \min\{\mathbf{x}_{-j}\} \\ \frac{1}{2}\left[(1-\alpha_j)\frac{x_{ij}}{\sum_i x_{ij}} + \alpha_j \frac{1}{n_j}\right] v - x_{ij} & if \quad \min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} \\ -x_{ij} & if \quad \min\{\mathbf{x}_j\} < \min\{\mathbf{x}_{-j}\} \end{cases}$$

where $\boldsymbol{\alpha}$ *and* $\mathbf{x}$ *are, respectively, the vector of first and second period actions.*

PROPOSITION 1.4. *In the KMMGC, for any* $\alpha_j \in \mathbb{R}$, *the set of the second-period pure strategy Nash equilibria of the game is characterized as follows:*

1. *if* $v > 0$, *then*

$$(\gamma_j, \gamma_{-j}) = (0, k) \;\; \textit{with } k \in \{1, \dots, K\} \;\; \textit{and } \mathbf{x}_{-j} = \mathbf{k}$$ [1.6]

for any $(\alpha_j, \alpha_{-j}) \in$

$$\begin{cases} \mathbb{R} \times \left[1 - \frac{n_{-j}\gamma_{-j}}{v} - \frac{n_{-j}(\gamma_{-j}+1)}{(n_{-j}-1)v}, 2\left(1 - \frac{n_{-j}\gamma_{-j}}{v}\right)\right] & \textit{if } v < n_{-j}\gamma_{-j} - \frac{n_{-j}(\gamma_{-j}-1)}{n_{-j}-1}; \\ \mathbb{R} \times \left[1 - \frac{n_{-j}\gamma_{-j}}{v} - \frac{n_{-j}(\gamma_{-j}+1)}{(n_{-j}-1)v}, 1 - \frac{n_{-j}\gamma_{-j}}{v} - \frac{n_{-j}(\gamma_{-j}-1)}{(n_{-j}-1)v}\right] & \textit{if } v \geq n_{-j}\gamma_{-j} - \frac{n_{-j}(\gamma_{-j}-1)}{n_{-j}-1}; \end{cases}$$

2. *if* $v > 0$, *then*

$$(\gamma_1, \gamma_2) = (0, 0)$$

for any

$$(\alpha_1, \alpha_2) \in \left[1 - \frac{2n_1}{(n_1 - 1)\, v}, \infty\right) \times \left[1 - \frac{2n_2}{(n_2 - 1)\, v}, \infty\right);$$

3. *if* $v \geq 2 \cdot max\,\{n_1\gamma_1, n_2\gamma_2\}$, *then*

$$(\gamma_1, \gamma_2) = (k, k) \;\; \textit{with } k \in \{1, \dots, K\} \;\; \textit{and } \mathbf{x}_j = \mathbf{k}$$

for any

$$(\alpha_1, \alpha_2) \in \left[1 - \frac{2n_1\gamma_1}{v} - \frac{2n_1\,(\gamma_1 + 1)}{(n_1 - 1)v}, \infty\right) \times \left[1 - \frac{2n_2\gamma_2}{v} - \frac{2n_2\,(\gamma_2 + 1)}{(n_2 - 1)v}, \infty\right);$$

4. *if* $v \geq max \left\{\frac{2n_1\left(\gamma_1(n_1\gamma_1-1)-(min\{\mathbf{x}_1\})^2\right)}{n_1\gamma_1-min\{\mathbf{x}_1\}-1}, \frac{2n_2\left(\gamma_2(n_2\gamma_2-1)-(min\{\mathbf{x}_2\})^2\right)}{n_2\gamma_j-min\{\mathbf{x}_2\}-1}\right\}$, *then*

$$(\gamma_1, \gamma_2) \;\; \textit{such that } \gamma_j \in (1, K)\,,\; min\,\{\mathbf{x}_1\} = min\,\{\mathbf{x}_2\} > 0, 2 \leq \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=min\{\mathbf{x}_j\}} \leq n_j - 1$$ [1.7]

for any

$$(\alpha_1, \alpha_2) \in$$

[1.6] $\mathbf{x}_{-j} = \mathbf{k}$ with $k \in \{1, \dots, K\}$ means that all agents in group $-j$ exert the same level of effort k .

[1.7] Note that $\mathbb{1}_{x_{ij}=min\{\mathbf{x}_j\}}$ stands for the Indicator function taking value 1 when $x_{ij} = min\{\mathbf{x}_j\}$ for any ij.

$$\left[max\left\{\frac{min\left\{\mathbf{x}_1\right\}}{\gamma_1 - min\left\{\mathbf{x}_1\right\}}\left(\frac{2n_1\gamma_1}{v} - 1\right), 1 - \frac{2n_1\gamma_1\left(n_1\gamma_1 + 1\right)}{\left(n_1\gamma_1 - min\left\{\mathbf{x}_1\right\}\right)v}\right\},\right.$$

$$\left.1 - \frac{2n_1\gamma_1\left(n_1\gamma_1 - 1\right)}{\left(n_1\gamma_1 - min\left\{\mathbf{x}_1\right\} - 1\right)v}\right] \times$$

$$\left[max\left\{\frac{min\left\{\mathbf{x}_2\right\}}{\gamma_2 - min\left\{\mathbf{x}_2\right\}}\left(\frac{2n_2\gamma_2}{v} - 1\right), 1 - \frac{2n_2\gamma_2\left(n_2\gamma_2 + 1\right)}{\left(n_2\gamma_2 - min\left\{\mathbf{x}_2\right\}\right)v}\right\},\right.$$

$$\left.1 - \frac{2n_2\gamma_2\left(n_2\gamma_2 - 1\right)}{\left(n_2\gamma_2 - min\left\{\mathbf{x}_2\right\} - 1\right)v}\right],$$

$$max\left\{\mathbf{x}_1\right\} = min\left\{\mathbf{x}_1\right\} + 1 \text{ and } max\left\{\mathbf{x}_2\right\} = min\left\{\mathbf{x}_2\right\} + 1;$$

5. *if* $v \geq max\left\{2n_j\gamma_j, \frac{2n_{-j}\left(\gamma_{-j}(n_{-j}\gamma_{-j}-1)-(min\{\mathbf{x}_{-j}\})^2\right)}{n_{-j}\gamma_{-j}-min\{\mathbf{x}_{-j}\}-1}\right\}$*, then*

$$\left(\gamma_j, \gamma_{-j}\right) \text{ such that } \gamma_j = k \in \{1, \ldots, K-1\} \text{ and } \mathbf{x}_j = \mathbf{k}, \gamma_{-j} \in (1, K),$$

$$min\left\{\mathbf{x}_j\right\} = min\left\{\mathbf{x}_{-j}\right\} > 0, 2 \leq \sum_{i=1}^{n_{-j}} \mathbb{1}_{x_{i-j}=min\left\{\mathbf{x}_{-j}\right\}} \leq n_{-j} - 1$$

for any

$$\left(\alpha_j, \alpha_{-j}\right) \in$$

$$\left[1 - \frac{2n_j\gamma_j}{v} - \frac{2n_j\left(\gamma_j + 1\right)}{(n_j - 1)v}, \infty\right) \times$$

$$\left[max\left\{\frac{min\left\{\mathbf{x}_{-j}\right\}}{\gamma_{-j} - min\left\{\mathbf{x}_{-j}\right\}}\left(\frac{2n_{-j}\gamma_{-j}}{v} - 1\right), 1 - \frac{2n_{-j}\gamma_j\left(n_{-j}\gamma_{-j} + 1\right)}{\left(n_j\gamma_j - min\left\{\mathbf{x}_{-j}\right\}\right)v}\right\},\right.$$

$$\left.1 - \frac{2n_{-j}\gamma_{-j}\left(n_{-j}\gamma_{-j} - 1\right)}{\left(n_{-j}\gamma_{-j} - min\left\{\mathbf{x}_{-j}\right\} - 1\right)v}\right],$$

$$max\left\{\mathbf{x}_{-j}\right\} = min\left\{\mathbf{x}_{-j}\right\} + 1;$$

6. *if* $v > 0$*, then*

$$\left(\gamma_j, \gamma_{-j}\right) \text{ such that } \gamma_j = 0, \gamma_{-j} \in (0, 1) \text{ and } \sum_{i=1}^{n_{-j}} \mathbb{1}_{x_{i-j}=0} = n_{-j} - 1$$

for any

$$\left(\alpha_j, \alpha_{-j}\right) \in \left[1 - \frac{2n_j}{\left(n_j - 1\right) v}, \infty\right) \times \left[1 - \frac{2\left(n_{-j}\gamma_{-j} + 1\right)}{v}, 1 - \frac{2n_{-j}}{\left(n_{-j} - 1\right) v}\right];$$

7. *if* $v > 0$, *then*

$$\left(\gamma_1, \gamma_2\right) \text{ such that } \gamma_j \in \left(0, 1\right), min\left\{\mathbf{x}_1\right\} = min\left\{\mathbf{x}_2\right\} = 0 \text{ and } \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=0} = n_j - 1$$

for any

$$\left(\alpha_1, \alpha_2\right) \in \left[1 - \frac{2\left(n_1\gamma_1 + 1\right)}{v}, 1 - \frac{2n_1}{\left(n_1 - 1\right) v}\right] \times \left[1 - \frac{2\left(n_2\gamma_2 + 1\right)}{v}, 1 - \frac{2n_2}{\left(n_2 - 1\right) v}\right] \; ;$$

8. *if* $v \leq 2n_{-j}\left(1 - \frac{\gamma_{-j}(2n_{-j} - n_{-j}\gamma_{-j} - 1)(n_{-j}\gamma_{-j} - 1)}{(n_{-j}\gamma_{-j} - 2)(n_{-j}\gamma_{-j} + 1)}\right)$, *then*

$$\left(\gamma_j, \gamma_{-j}\right) \text{ such that } \gamma_j = 0, \gamma_{-j} \in \left(0, 2\right), min\left\{\mathbf{x}_1\right\} = min\left\{\mathbf{x}_2\right\} = 0 \text{ and } \mathbb{1}_{x_{i-j}=0} = 1$$

for any

$$\left(\alpha_j, \alpha_{-j}\right) \in$$

$$\left[1 - \frac{2n_j}{\left(n_j - 1\right) v}, \infty\right) \times$$

$$\left[max\left\{\frac{2n_{-j}\left(v - n_{-j}\gamma_{-j} - 1\right)}{\left(2n_{-j} - n_{-j}\gamma_{-j} - 1\right) v}, 1 - \frac{2n_{-j}\gamma_{-j}\left(n_{-j}\gamma_{-j} + 1\right)}{\left(n_{-j}\gamma_{-j} - 1\right) v}\right\}, 1 - \frac{2n_{-j}\gamma_{-j}\left(n_{-j}\gamma_{-j} - 1\right)}{\left(n_{-j}\gamma_{-j} - 2\right) v}\right] \; ;$$

9. *if* $v \leq max\left\{2n_1\left(1 - \frac{\gamma_1(2n_1 - n_1\gamma_1 - 1)(n_1\gamma_1 - 1)}{(n_1\gamma_1 - 2)(n_1\gamma_1 + 1)}\right), 2n_2\left(1 - \frac{\gamma_2(2n_2 - n_2\gamma_2 - 1)(n_2\gamma_2 - 1)}{(n_2\gamma_2 - 2)(n_2\gamma_2 + 1)}\right)\right\}$, *then*

$$\left(\gamma_1, \gamma_2\right) \text{ such that } \gamma_j \in \left(0, 2\right), min\left\{\mathbf{x}_1\right\} = min\left\{\mathbf{x}_2\right\} = 0 \text{ and } \mathbb{1}_{x_{ij}=0} = 1$$

for any

$$\left(\alpha_1, \alpha_2\right) \in$$

$$\left[max\left\{\frac{2n_1\left(v - n_1\gamma_1 - 1\right)}{\left(2n_1 - n_1\gamma_1 - 1\right) v}, 1 - \frac{2n_1\gamma_1\left(n_1\gamma_1 + 1\right)}{\left(n_1\gamma_1 - 1\right) v}\right\}, 1 - \frac{2n_1\gamma_1\left(n_1\gamma_1 - 1\right)}{\left(n_1\gamma_1 - 2\right) v}\right] \times$$

$$\left[max \left\{ \frac{2n_2 \left(v - n_2\gamma_2 - 1\right)}{\left(2n_2 - n_2\gamma_2 - 1\right) v}, 1 - \frac{2n_2\gamma_2 \left(n_2\gamma_2 + 1\right)}{\left(n_2\gamma_2 - 1\right) v} \right\}, 1 - \frac{2n_2\gamma_2 \left(n_2\gamma_2 - 1\right)}{\left(n_2\gamma_2 - 2\right) v} \right] .$$

Proof. See Appendix 1.A.3 . □

REMARK 1.9. *Regarding the first stage, it is straightforward to highlight that it is not possible to express the continuation payoffs for any possible interval on* $\boldsymbol{\alpha}$*, since some intervals do not sustain any second-period equilibrium.*

Consider the following example which should clarify the argument above.

EXAMPLE 1.1. *Suppose* $K_{ij} = \{0, 1, 2\}$ *and* $v \geq 2 \cdot max \{n_1, n_2\}$*, and focus on within-group symmetric SGP equilibria. Consider*

$$(\alpha_1, \alpha_2) \in \mathbb{R} \times \left(-\infty, 1 - \frac{2n_2}{v} - \frac{3n_2}{\left(n_2 - 1\right) v}\right) :$$

then, no second-period equilibrium is sustained by these intervals on α*, so that we cannot specify second-period strategy profiles for any interval on* $\boldsymbol{\alpha} \in \mathbb{R} \times \mathbb{R}$*, and thus the groups' continuation payoffs.* [1.8]

Therefore, we can conclude with the following proposition.

PROPOSITION 1.5. *In the KMMGC, there are no subgame perfect Nash equilibria in pure strategies.*

Therefore, the discrete actions case does not represent a generalization of the binary actions case. The reason for this result is that in *KMMGC* the sharing rule parameters sustaining second-period equilibria must ensure that both upward and downward deviations are not profitable, for them to be equilibria, not just upward or downward ones, as in the binary actions setup. This difference translates into more restrictive conditions over the $\boldsymbol{\alpha}$ parameters, so that there are intervals over $\mathbb{R} \times \mathbb{R}$ not sustaining second-period equilibria.

[1.8] Note that similar examples can be built considering both within-group symmetric and within group asymmetric subgame-perfect Nash equilibria in pure strategies.

Nevertheless, it is immediate to verify that in the $KMMGC$ with restricted sharing rules, i.e. with $\alpha_j \in [0,1] \, \forall j \in \{1,2\}$, there is at least one second-period equilibrium in the entire $\alpha_1 \times \alpha_2$ space for some values of the prize v, so that it is possible to derive both the first-stage equilibria and the subgame perfect equilibria in pure strategies, as stated by the following proposition.

PROPOSITION 1.6. *In the $KMMGC$ with restricted sharing rules, i.e. with $\alpha_j \in [0,1] \, \forall j \in \{1,2\}$, for*

- $v \leq 2 \cdot min\left\{\frac{n_1}{n_1-1}, \frac{n_2}{n_2-1}\right\}$ *or*
- $2 \cdot max\left\{n_1 k, n_2 k\right\} \leq v \leq max\left\{2n_1 k + \frac{2n_1}{n_1-1}(k+1), 2n_2 k + \frac{2n_2}{n_2-1}(k+1)\right\} \forall k \in \{1, \ldots, K\}$,

there exist subgame perfect Nash equilibria in pure strategies.

Proof. See Appendix 1.A.4. □

From proposition 4 and 6 it is straightforward to derive two additional results.

COROLLARY 1.3. *In the KMMGC with restricted sharing rules, for $v \leq 2 \cdot min\left\{\frac{n_1}{n_1-1}, \frac{n_2}{n_2-1}\right\}$, there is a unique within-group symmetric second-stage equilibrium, i.e. $(\gamma_1, \gamma_2) = (0,0)$ for any $(\alpha_1, \alpha_2) \in [0,1] \times [0,1]$, so that there exist a continuum of within-group symmetric subgame perfect Nash equilibria in pure strategies such that:*

$$\left(\alpha_1^{SGP}, \alpha_2^{SGP}\right) \in [0,1] \times [0,1]$$

$$\left(\gamma_1^{SGP}, \gamma_2^{SGP}\right) = (0,0) \;\; \forall \left(\alpha_1, \alpha_2\right) \in [0,1] \times [0,1]$$

with

$$\left(\alpha_1^{SGP}, \alpha_2^{SGP}\right) \in [0,1] \times [0,1] \;\; \textit{and} \;\; \left(\gamma_1^{SGP}, \gamma_2^{SGP}\right) = (0,0)$$

as equilibrium outcomes.

COROLLARY 1.4. *In the KMMGC with restricted sharing rules, for $v < 2 \cdot min\left\{\frac{n_1}{n_1-1}, \frac{n_2}{n_2-1}\right\}$, there is a unique second-stage equilibrium, i.e. $(\gamma_1, \gamma_2) = (0,0)$ for any $(\alpha_1, \alpha_2) \in [0,1] \times [0,1]$,* *so that there exist a continuum of subgame perfect Nash equilibria in pure strategies such that:*

$$\left(\alpha_1^{SGP}, \alpha_2^{SGP}\right) \in [0,1] \times [0,1]$$

$$\left(\gamma_1^{SGP}, \gamma_2^{SGP}\right) = (0,0) \;\; \forall \left(\alpha_1, \alpha_2\right) \in [0,1] \times [0,1]$$

with

$$\left(\alpha_1^{SGP}, \alpha_2^{SGP}\right) \in [0,1] \times [0,1] \;\; \textit{and} \;\; \left(\gamma_1^{SGP}, \gamma_2^{SGP}\right) = (0,0)$$

as equilibrium outcomes.

REMARK 1.10. *Retrieving the full set of subgame perfect Nash equilibria in pure strategies in the KMMGC with restricted sharing rules for a given action set $K_{ij} \subset \mathbb{N}$ is possible but computationally convoluted, as it involves the direct inspection of all possible continuation-payoffs matrices, as done for the binary case.*

1.7 Conclusions

In this paper, we innovate on the existing literature by characterizing the entire set of second-period pure strategy Nash equilibria in binary max-min group contests with a private good prize. Differently from what obtained with a public good prize in our reference paper, i.e. Chowdhury, D. Lee, and Topolyan (2016), we find within-group asymmetric equilibria as well. Moreover, we computationally characterize the set of within-group symmetric subgame perfect pure strategy Nash equilibria. Depending on the size of the private good prize with respect to groups' size, we find conditions such that both the set of first-period equilibria in pure strategies and the set of within-group symmetric subgame perfect Nash equilibria in pure strategies have the cardinality of the continuum, i.e. there is indeterminacy. Our results show the interplay between the complementarities induced by the weakest-link impact function and the selective incentives set by the endogenous sharing rule. Then, we check whether this paper's results are due to discreteness or to binary choice. To this aim, in Section 1.6 we expand the set of second-stage actions from the binary case to any subset of the natural numbers with cardinality at least equal to three. In this case, we can find a counterexample, proving that in this case there are no subgame perfect Nash equilibria in pure strategies. The reason is that once we introduce more than two effort choices, it is necessary to ensure that both upward and downward deviations are not profitable, not just either upward or downward ones, as in the binary action setup. We claim that the indeterminacy result extends to the restricted sharing rule case which is more common in applications. Finally, we stress that additional assumptions stemming from the details of the competition between groups considered might deliver the equilibrium selection needed to make meaningful predictions about outcomes.

1.8 References

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1.A Proofs

1.A.1 Proof Proposition 1.1

The results are derived by direct inspection.

1. Suppose
$$(\gamma_1, \gamma_2) = (1, 1).$$
Then
$$x_{ij} = \min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} = 1 \text{ and } \sum_{i=1}^{n_j} x_j(i) = n_j$$
so that
$$\pi_{ij}(\gamma_1, \gamma_2 | \boldsymbol{\alpha}) = \frac{1}{2}\frac{1}{n_j}v - 1.$$
If agent ij deviates to $x_{ij} = 0$, then
$$x_{ij} = \min\{\mathbf{x}_j\} = 0 < \min\{\mathbf{x}_{-j}\} = 1 \text{ and } \sum_{i=1}^{n_j} x_j(i) = n_j - 1$$
so that the deviation payoff is
$$\pi_{ij}^{D}(\gamma_1, \gamma_2 | \boldsymbol{\alpha}) = 0.$$
Hence for any player ij there is no incentive to deviate if and only if $v \geq 2n_j$.

2. Suppose
$$(\gamma_1, \gamma_2) = (0, 0).$$
Then
$$x_{ij} = \min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} = 0 \text{ and } \sum_{i=1}^{n_j} x_j(i) = 0$$
so that
$$\pi_{ij}(\gamma_1, \gamma_2 | \boldsymbol{\alpha}) = \frac{1}{2}\frac{1}{n_j}v.$$
If agent ij deviates to $x_{ij} = 1$, then
$$x_{ij} = 1 > \min\{\mathbf{x}_j\} = 0 = \min\{\mathbf{x}_{-j}\} \text{ and } \sum_{i=1}^{n_j} x_j(i) = 1$$

so that the deviation payoff is

$$\pi_{ij}^{D}\left(\gamma_j^{+},\gamma_{-j}|\boldsymbol{\alpha}\right)=\frac{1}{2}\left[\left(1-\alpha_j\right)+\alpha_j\frac{1}{n_j}\right]v-1.$$

Hence, for any player ij there is no incentive to deviate if and only if

$$\frac{1}{2}\frac{1}{n_j}v\geq\frac{1}{2}\left[\left(1-\alpha_j\right)+\alpha_j\frac{1}{n_j}\right]v-1\Leftrightarrow\left[\left(1-\alpha_j\right)+\alpha_j\frac{1}{n_j}-\frac{1}{n_j}\right]v\leq 2\Leftrightarrow$$

$$\Leftrightarrow\left[\left(1-\alpha_j\right)\left(1-\frac{1}{n_j}\right)\right]v\leq 2\Leftrightarrow\left[\left(1-\alpha_j\right)\left(\frac{n_j-1}{n_j}\right)\right]v\leq 2\Leftrightarrow$$

$$\Leftrightarrow 1-\alpha_j\leq\frac{2n_j}{\left(n_j-1\right)v}\Leftrightarrow\alpha_j\geq 1-\frac{2n_j}{\left(n_j-1\right)v}.$$

3. Suppose

$$\left(\gamma_j,\gamma_{-j}\right)=\left(1,0\right).$$

Then

$$x_{ij}=\min\left\{\mathbf{x}_j\right\}=1>\min\left\{\mathbf{x}_{-j}\right\}=x_{i-j}=0,\ \sum_{i=1}^{n_j}x_{ij}=n_j\ \text{and}\ \sum_{i=1}^{n_{-j}}x_{i-j}=0$$

so that

$$\pi_{ij}\left(\gamma_j,\gamma_{-j}|\boldsymbol{\alpha}\right)=\frac{1}{n_{-j}}v-1\ \ \text{and}\ \ \pi_{i-j}\left(\gamma_j,\gamma_{-j}\right)=0.$$

If agent ij deviates to $x_{ij}=0$, then

$$\min\left\{\mathbf{x}_j\right\}=0=\min\left\{\mathbf{x}_{-j}\right\}=0=x_{ij},\ \sum_{i=1}^{n_j}x_j\left(i\right)=n_j-1\ \text{and}\ \sum_{i=1}^{n_{-j}}x_{i-j}=0$$

so that the deviation payoff is

$$\pi_{ij}^{D}\left(\gamma_j^{-},\gamma_{-j}|\boldsymbol{\alpha}\right)=\frac{1}{2}\frac{1}{n_j}\alpha_j v\leq\pi_{ij}\left(\gamma_j,\gamma_{-j}\right)=\frac{1}{n_j}v-1\Leftrightarrow$$

$$\Leftrightarrow\alpha_j\leq 2\left(1-\frac{n_j}{v}\right);$$

hence no agent of group j has an incentive to deviate if and only if $\alpha_j\leq 2\left(1-\frac{n_j}{v}\right)$. On the other hand, if agent $i-j$ deviates to $x_{i-j}=1$, then

$$\min\left\{\mathbf{x}_j\right\}=1>\min\left\{\mathbf{x}_{-j}\right\}=0,\ \sum_{i=1}^{n_j}x_{ij}=n_j\ \text{and}\ \sum_{i=1}^{n_{-j}}x_{i-j}=1$$

so that the deviation payoff is

$$\pi^{D}_{i-j}\left(\gamma_j,\gamma^{+}_{-j}|\boldsymbol{\alpha}\right)=-1\leq\pi_{i-j}\left(\gamma_j,\gamma_{-j}\right)=0,$$

hence no agent of group j has an incentive to deviate.

4. Suppose

$$\left(\gamma_j,\gamma_{-j}\right)\text{ such that }\gamma_j\in(0,1)\text{ with }\sum_{i=1}^{n_j}x_{ij}\in\{2,\ldots,n_j-2\}\text{ and }\gamma_{-j}=0\ .$$

Then

$$\min\left\{\mathbf{x}_j\right\}=\min\left\{\mathbf{x}_{-j}\right\}=0\text{ and }\sum_{i=1}^{n_j}x_{ij}\in\{2,\ldots,n_j-2\},\sum_{i=1}^{n_{-j}}x_{i-j}=0\ .$$

Suppose $x_{ij}=1$, then

$$\pi_{ij}\left(\gamma_j,\gamma_{-j}|\boldsymbol{\alpha}\right)=\frac{1}{2}\left[\left(1-\alpha_j\right)\frac{1}{n_j\gamma_j}+\alpha_j\frac{1}{n_j}\right]v-1\ .$$

If agent ij deviates to $x_{ij}=0$, then

$$\min\left\{\mathbf{x}_j\right\}=\min\left\{\mathbf{x}_{-j}\right\}=0\text{ and }\sum_{i=1}^{n_j}x_{ij}=n_j\gamma_j-1\in\{1,\ldots,n_j-3\}$$

so that the deviation payoff is

$$\pi^{D}_{ij}\left(\gamma^{-}_j,\gamma_{-j}|\boldsymbol{\alpha}\right)=\frac{1}{2}\left[\alpha_j\frac{1}{n_j}\right]v;$$

hence agent ij has no incentive to deviate if and only if

$$\frac{1}{2}\left[\left(1-\alpha_j\right)\frac{1}{n_j\gamma_j}+\alpha_j\frac{1}{n_j}\right]v-1\geq\frac{1}{2}\left[\alpha_j\frac{1}{n_j}\right]v\Leftrightarrow\frac{1}{2}\left(1-\alpha_j\right)\frac{1}{n_j\gamma_j}v\geq1\Leftrightarrow$$

$$\Leftrightarrow\left(1-\alpha_j\right)\geq\frac{2n_j\gamma_j}{v}\Leftrightarrow\alpha_j\leq1-\frac{2n_j\gamma_j}{v}.$$

Suppose $x_{ij}=0$, then

$$\pi_{ij}\left(\gamma_j,\gamma_{-j}|\boldsymbol{\alpha}\right)=\frac{1}{2}\alpha_j\frac{1}{n_j}v\ .$$

If agent ij deviates to $x_{ij} = 1$, then

$$\min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} = 0 \text{ and } \sum_{i=1}^{n_j} x_{ij} = n_j\gamma_j + 1 \in \{3, \ldots, n_j - 1\}$$

so that the deviation payoff is

$$\pi_{ij}^{D}\left(\gamma_j^{+}, \gamma_{-j}|\boldsymbol{\alpha}\right) = \frac{1}{2}\left[(1-\alpha_j)\frac{1}{n_j\gamma_j+1} + \alpha_j\frac{1}{n_j}\right]v - 1;$$

agent ij has no incentive to deviate if and only if

$$\frac{1}{2}\alpha_j\frac{1}{n_j}v \geq \frac{1}{2}\left[(1-\alpha_j)\frac{1}{n_j\gamma_j+1} + \alpha_j\frac{1}{n_j}\right]v - 1 \Leftrightarrow \frac{1}{2}(1-\alpha_j)\frac{1}{n_j\gamma_j+1}v \leq 1 \Leftrightarrow$$

$$\Leftrightarrow (1-\alpha_j) \leq \frac{2(n_j\gamma_j+1)}{v} \Leftrightarrow \alpha_j \geq 1 - \frac{2(n_j\gamma_j+1)}{v} .$$

Moreover,

$$\pi_{i-j}(\gamma_j, \gamma_{-j}) = \frac{1}{2}\frac{1}{n_{-j}}v .$$

If agent $i-j$ deviates to $x_{i-j} = 1$, then

$$x_{i-j} = 1 > \min\{\mathbf{x}_j\} = 0 = \min\{\mathbf{x}_{-j}\} \text{ and } \sum_{i=1}^{n_{-j}} x_{i-j} = 1$$

so that the deviation payoff is

$$\pi_{i-j}^{D}\left(\gamma_j, \gamma_{-j}^{+}|\boldsymbol{\alpha}\right) = \frac{1}{2}\left[(1-\alpha_{-j}) + \alpha_{-j}\frac{1}{n_{-j}}\right]v - 1;$$

hence, for any player $i-j$ there is no incentive to deviate if and only if

$$\frac{1}{2}\frac{1}{n_{-j}}v \geq \frac{1}{2}\left[(1-\alpha_{-j}) + \alpha_{-j}\frac{1}{n_{-j}}\right]v - 1 \Leftrightarrow$$

$$\Leftrightarrow \left[(1-\alpha_{-j}) + \alpha_{-j}\frac{1}{n_{-j}} - \frac{1}{n_{-j}}\right]v \leq 2 \Leftrightarrow$$

$$\Leftrightarrow \left[(1-\alpha_{-j})\left(1 - \frac{1}{n_{-j}}\right)\right]v \leq 2 \Leftrightarrow \left[(1-\alpha_{-j})\left(\frac{n_{-j}-1}{n_{-j}}\right)\right]v \leq 2 \Leftrightarrow$$

$$\Leftrightarrow 1-\alpha_{-j} \leq \frac{2n_{-j}}{(n_{-j}-1)v} \Leftrightarrow \alpha_{-j} \geq 1 - \frac{2n_{-j}}{(n_{-j}-1)v} .$$

Thus,

$$(\gamma_j, \gamma_{-j}) \text{ such that } \gamma_j \in (0,1) \text{ with } \sum_{i=1}^{n_j} x_{ij} \in \{2, \ldots, n_j - 2\} \text{ and } \gamma_{-j} = 0$$

is a Nash equilibrium if and only if

$$\alpha_j \in \left[1 - \frac{2\left(n_j \gamma_j + 1\right)}{v}, 1 - \frac{2 n_j \gamma_j}{v}\right] \text{ and } \alpha_{-j} \geq 1 - \frac{2 n_{-j}}{\left(n_{-j} - 1\right) v} \ .$$

5. Suppose

$$(\gamma_j, \gamma_{-j}) \text{ such that } \gamma_j \in (0,1) \text{ with } \sum_{i=1}^{n_j} x_{ij} = 1 \text{ and } \gamma_{-j} = 0 \ .$$

Then

$$\min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} = 0 \text{ and } \sum_{i=1}^{n_j} x_{ij} = 1 \ .$$

Suppose $x_{ij} = 1$, then

$$\pi_{ij}\left(\gamma_j, \gamma_{-j} | \boldsymbol{\alpha}\right) = \frac{1}{2}\left[\left(1 - \alpha_j\right) \frac{1}{n_j \gamma_j} + \alpha_j \frac{1}{n_j}\right] v - 1 \ .$$

If agent ij deviates to $x_{ij} = 0$, then

$$\min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} = 0 \text{ and } \sum_{i=1}^{n_j} x_{ij} = 0$$

so that the deviation payoff is

$$\pi_{ij}^{D}\left(\gamma_j^{-}, \gamma_{-j} | \boldsymbol{\alpha}\right) = \frac{1}{2}\left[\frac{1}{n_j}\right] v;$$

hence, agent ij has no incentive to deviate if and only if

$$\frac{1}{2}\left[\left(1 - \alpha_j\right) \frac{1}{n_j \gamma_j} + \alpha_j \frac{1}{n_j}\right] v - 1 \geq \frac{1}{2}\left[\frac{1}{n_j}\right] v \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{2}\left[\frac{1}{n_j \gamma_j} - \alpha_j \left(\frac{1 - \gamma_j}{n_j \gamma_j}\right)\right] v - 1 \geq \frac{1}{2}\left[\frac{1}{n_j}\right] v \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{2}\left[\frac{1}{n_j \gamma_j}\right] v - 1 - \frac{1}{2}\left[\frac{1}{n_j}\right] v \geq \frac{1}{2}\left[\alpha_j \left(\frac{1 - \gamma_j}{n_j \gamma_j}\right)\right] v \Leftrightarrow$$

$$\Leftrightarrow \frac{\left(1 - \gamma_j\right) v - 2 n_j \gamma_j}{2 n_j \gamma_j} \geq \alpha_j \left(\frac{1 - \gamma_j}{2 n_j \gamma_j}\right) v \Leftrightarrow \alpha_j \leq 1 - \frac{2 n_j \gamma_j}{\left(1 - \gamma_j\right) v} \ .$$

Suppose $x_{ij} = 0$, then

$$\pi_{ij}\left(\gamma_j, \gamma_{-j} | \boldsymbol{\alpha}\right) = \frac{1}{2}\alpha_j \frac{1}{n_j} v \ .$$

If agent ij deviates to $x_{ij} = 1$, then

$$\min\left\{\mathbf{x}_j\right\} = \min\left\{\mathbf{x}_{-j}\right\} = 0 \text{ and } \sum_{i=1}^{n_j} x_{ij} = 2$$

so that the deviation payoff is

$$\pi_{ij}^{D}\left(\gamma_j^{+}, \gamma_{-j} | \boldsymbol{\alpha}\right) = \frac{1}{2}\left[\left(1-\alpha_j\right)\frac{1}{n_j\gamma_j+1} + \alpha_j\frac{1}{n_j}\right]v - 1;$$

hence, agent ij has no incentive to deviate if and only if

$$\frac{1}{2}\alpha_j\frac{1}{n_j}v \geq \frac{1}{2}\left[\left(1-\alpha_j\right)\frac{1}{n_j\gamma_j+1} + \alpha_j\frac{1}{n_j}\right]v - 1 \Leftrightarrow$$

$$\frac{1}{2}\left(1-\alpha_j\right)\frac{1}{\gamma_j+1}v \leq 1 \Leftrightarrow$$

$$\Leftrightarrow \left(1-\alpha_j\right) \leq \frac{2\left(n_j\gamma_j+1\right)}{v} \Leftrightarrow \alpha_j \geq 1 - \frac{2\left(n_j\gamma_j+1\right)}{v} \ .$$

Moreover,

$$\pi_{i-j}\left(\gamma_j, \gamma_{-j} | \boldsymbol{\alpha}\right) = \frac{1}{2}\frac{1}{n_{-j}} v \ .$$

If agent $i-j$ deviates to $x_{i-j} = 1$, then

$$x_{i-j} = 1 > \min\left\{\mathbf{x}_j\right\} = 0 = \min\left\{\mathbf{x}_{-j}\right\} \text{ and } \sum_{i=1}^{n_{-j}} x_{-j}\left(i\right) = 1$$

so that the deviation payoff is

$$\pi_{i-j}^{D}\left(\gamma_j, \gamma_{-j}^{+} | \boldsymbol{\alpha}\right) = \frac{1}{2}\left[\left(1-\alpha_{-j}\right) + \alpha_{-j}\frac{1}{n_j}\right]v - 1 \ .$$

Hence, for any player $i-j$ there is no incentive to deviate if and only if

$$\frac{1}{2}\frac{1}{n_{-j}}v \geq \frac{1}{2}\left[\left(1-\alpha_{-j}\right) + \alpha_{-j}\frac{1}{n_{-j}}\right]v - 1 \Leftrightarrow$$

$$\Leftrightarrow \left[\left(1-\alpha_{-j}\right) + \alpha_{-j}\frac{1}{n_{-j}} - \frac{1}{n_{-j}}\right]v \leq 2 \Leftrightarrow$$

$$\Leftrightarrow \left[(1-\alpha_{-j})\left(1-\frac{1}{n_{-j}}\right)\right] v \leq 2 \Leftrightarrow \left[(1-\alpha_{-j})\left(\frac{n_{-j}-1}{n_{-j}}\right)\right] v \leq 2 \Leftrightarrow$$

$$\Leftrightarrow 1-\alpha_{-j} \leq \frac{2n_{-j}}{(n_{-j}-1)\,v} \Leftrightarrow \alpha_{-j} \geq 1-\frac{2n_{-j}}{(n_{-j}-1)\,v} \ .$$

Thus,

$$(\gamma_j, \gamma_{-j}) \text{ such that } \gamma_j \in (0,1) \text{ with } \sum_{i=1}^{n_j} x_{ij} = 1 \text{ and } \gamma_{-j} = 0$$

is a Nash equilibrium if and only if

$$\alpha_j \in \left[1-\frac{2\,(n_j\gamma_j+1)}{v}, 1-\frac{2n_j\gamma_j}{(1-\gamma_j)\,v}\right] \text{ and } \alpha_{-j} \geq 1-\frac{2n_{-j}}{(n_{-j}-1)\,v} \ .$$

6. Suppose

$$(\gamma_j, \gamma_{-j}) \text{ such that } \gamma_j \in (0,1) \text{ with } \sum_{i=1}^{n_j} x_{ij} = n_j - 1 \text{ and } \gamma_{-j} = 0 \ .$$

Then

$$\min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} = 0 \text{ and } \sum_{i=1}^{n_j} x_{ij} = n_j - 1, \sum_{i=1}^{n_{-j}} x_{i-j} = 0 \ .$$

Suppose $x_{ij} = 1$, then

$$\pi_{ij}\left(\gamma_j, \gamma_{-j} | \boldsymbol{\alpha}\right) = \frac{1}{2}\left[(1-\alpha_j)\frac{1}{n_j\gamma_j} + \alpha_j\frac{1}{n_j}\right] v - 1 \ .$$

If agent ij deviates to $x_{ij} = 0$, then

$$\min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} = 0 \text{ and } \sum_{i=1}^{n_j} x_{ij} = n_j - 2$$

so that the deviation payoff is

$$\pi_{ij}^{D}\left(\gamma_j^{-}, \gamma_{-j} | \boldsymbol{\alpha}\right) = \frac{1}{2}\left[\alpha_j\frac{1}{n_j}\right] v;$$

hence, agent ij has no incentive to deviate if and only if

$$\frac{1}{2}\left[(1-\alpha_j)\frac{1}{n_j\gamma_j} + \alpha_j\frac{1}{n_j}\right] v - 1 \geq \frac{1}{2}\left[\alpha_j\frac{1}{n_j}\right] v \Leftrightarrow \frac{1}{2}(1-\alpha_j)\frac{1}{n_j\gamma_j} v \geq 1 \Leftrightarrow$$

$$\Leftrightarrow (1-\alpha_j) \geq \frac{2n_j\gamma_j}{v} \Leftrightarrow \alpha_j \leq 1-\frac{2n_j\gamma_j}{v} \ .$$

Suppose $x_{ij} = 0$, then

$$\pi_{ij}\left(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}\right) = \frac{1}{2}\alpha_j \frac{1}{n_j} v.$$

If agent ij deviates to $x_{ij} = 1$, then

$$\min\left\{\mathbf{x}_j\right\} = 1 > \min\left\{\mathbf{x}_{-j}\right\} = 0 \text{ and } \sum_{i=1}^{n_j} x_{ij} = n_j$$

so that the deviation payoff is

$$\pi_{ij}^D\left(\gamma_j^+, \gamma_{-j}\right) \frac{1}{n_j} v - 1;$$

hence, agent ij has no incentive to deviate if and only if

$$\frac{1}{2}\alpha_j \frac{1}{n_j} v \geq \frac{1}{n_j} v - 1 \Leftrightarrow \alpha_j \geq \frac{2n_j}{v}\left(\frac{1}{n_j} v - 1\right) \Leftrightarrow$$

$$\Leftrightarrow \alpha_j \geq 2\left(1 - \frac{n_j}{v}\right).$$

Note that $2\left(1 - \frac{n_j}{v}\right) v \leq 1 - \frac{2n_j\gamma_j}{v}$ if and only if $v \leq 2$. Moreover,

$$\pi_{i-j}\left(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}\right) = \frac{1}{2}\frac{1}{n_{-j}} v \ .$$

If agent $i - j$ deviates to $x_{i-j} = 1$, then

$$x_{i-j} = 1 > \min\left\{\mathbf{x}_j\right\} = 0 = \min\left\{\mathbf{x}_{-j}\right\} \text{ and } \sum_{i=1}^{n_j} x_j\left(i\right) = 1$$

so that the deviation payoff is

$$\pi_{i-j}^D\left(\gamma_j, \gamma_{-j}^+|\boldsymbol{\alpha}\right) = \frac{1}{2}\left[\left(1 - \alpha_{-j}\right) + \alpha_{-j}\frac{1}{n_{-j}}\right] v - 1;$$

hence, for any player $i - j$ there is no incentive to deviate if and only if

$$\frac{1}{2}\frac{1}{n_{-j}} v \geq \frac{1}{2}\left[\left(1 - \alpha_{-j}\right) + \alpha_{-j}\frac{1}{n_{-j}}\right] v - 1 \Leftrightarrow$$

$$\left[\left(1 - \alpha_{-j}\right) + \alpha_{-j}\frac{1}{n_{-j}} - \frac{1}{n_{-j}}\right] v \leq 2 \Leftrightarrow$$

$$\Leftrightarrow \left[\left(1 - \alpha_{-j}\right)\left(1 - \frac{1}{n_{-j}}\right)\right] v \leq 2 \Leftrightarrow \left[\left(1 - \alpha_{-j}\right)\left(\frac{n_{-j} - 1}{n_{-j}}\right)\right] v \leq 2 \Leftrightarrow$$

$$\Leftrightarrow 1-\alpha_{-j} \leq \frac{2n_{-j}}{\left(n_{-j}-1\right)v} \Leftrightarrow \alpha_{-j} \geq 1-\frac{2n_{-j}}{\left(n_{-j}-1\right)v} \ .$$

Thus,

$$\left(\gamma_j, \gamma_{-j}\right) \text{ such that } \gamma_j \in (0,1) \text{ with } \sum_{i=1}^{n_j} x_{ij} = n_j - 1 \text{ and } \gamma_{-j} = 0$$

is a Nash equilibrium if and only if

$$\alpha_j \in \left[2\left(1-\frac{n_j}{v}\right), 1-\frac{2n_j\gamma_j}{v}\right] \text{ and } \alpha_{-j} \geq 1-\frac{2n_{-j}}{\left(n_{-j}-1\right)v} \ \forall \ v \leq 2 \ .$$

7. Suppose

$$\left(\gamma_1, \gamma_2\right) \quad \text{such that } \gamma_j \in (0,1) \text{ with} \sum_{i=1}^{n_j} x_{ij} \in \{2, \ldots, n_j-2\} \ .$$

Then

$$\min\left\{\mathbf{x}_j\right\} = \min\left\{\mathbf{x}_{-j}\right\} = 0 \text{ and } \sum_{i=1}^{n_j} x_{ij} \in \{2, \ldots, n_j-2\} \ .$$

Suppose $x_{ij}=1$, then

$$\pi_{ij}\left(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}\right) = \frac{1}{2}\left[\left(1-\alpha_j\right)\frac{1}{n_j\gamma_j} + \alpha_j\frac{1}{n_j}\right]v - 1 \ .$$

If agent ij deviates to $x_{ij}=0$, then

$$\min\left\{\mathbf{x}_j\right\} = \min\left\{\mathbf{x}_{-j}\right\} = 0 \text{ and } \sum_{i=1}^{n_j} x_{ij} = n_j\gamma_j - 1 \in \{1, \ldots, n_j-3\}$$

so that the deviation payoff is

$$\pi_{ij}^{D}\left(\gamma_j^{-}, \gamma_{-j}|\boldsymbol{\alpha}\right) = \frac{1}{2}\left[\alpha_j\frac{1}{n_j}\right]v;$$

hence, agent ij has no incentive to deviate if and only if

$$\frac{1}{2}\left[\left(1-\alpha_j\right)\frac{1}{n_j\gamma_j} + \alpha_j\frac{1}{n_j}\right]v - 1 \geq \frac{1}{2}\left[\alpha_j\frac{1}{n_j}\right]v \Leftrightarrow \frac{1}{2}\left(1-\alpha_j\right)\frac{1}{n_j\gamma_j}v \geq 1 \Leftrightarrow$$

$$\Leftrightarrow \left(1-\alpha_j\right) \geq \frac{2n_j\gamma_j}{v} \Leftrightarrow \alpha_j \leq 1-\frac{2n_j\gamma_j}{v} \ .$$

Suppose $x_{ij}=0$, then

$$\pi_{ij}\left(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}\right) = \frac{1}{2}\alpha_j\frac{1}{n_j}v \ .$$

If agent ij deviates to $x_{ij} = 1$, then

$$\min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} = 0 \text{ and } \sum_{i=1}^{n_j} x_{ij} = n_j\gamma_j + 1 \in \{3, \ldots, n_j\}$$

so that the deviation payoff is

$$\pi_{ij}^D\left(\gamma_j^+, \gamma_j|\boldsymbol{\alpha}\right) = \frac{1}{2}\left[(1-\alpha_j)\frac{1}{n_j\gamma_j+1} + \alpha_j\frac{1}{n_j}\right]v - 1;$$

hence, agent ij has no incentive to deviate if and only if

$$\frac{1}{2}\alpha_j\frac{1}{n_j}v \geq \frac{1}{2}\left[(1-\alpha_j)\frac{1}{n_j\gamma_j+1} + \alpha_j\frac{1}{n_j}\right]v - 1 \Leftrightarrow$$

$$\frac{1}{2}(1-\alpha_j)\frac{1}{n_j\gamma_j+1}v \leq 1 \Leftrightarrow$$

$$\Leftrightarrow (1-\alpha_j) \leq \frac{2(n_j\gamma_j+1)}{v} \Leftrightarrow \alpha_j \geq 1 - \frac{2(n_j\gamma_j+1)}{v} \; .$$

Thus,

$$(\gamma_1, \gamma_2) \;\; such \;\; that \;\; \gamma_j \in (0,1) \text{ with} \sum_{i=1}^{n_j} x_{ij} \in \{2, \ldots, n_j - 2\}$$

is a Nash equilibrium if and only if

$$\alpha_j \in \left[1 - \frac{2(n_j\gamma_j+1)}{v}, 1 - \frac{2n_j\gamma_j}{v}\right].$$

8. Suppose

$$(\gamma_1, \gamma_2) \quad \text{such that } \gamma_j \in (0,1) \;\; \text{with} \;\; \sum_{i=1}^{n_j} x_{ij} = 1 \; .$$

Then

$$\min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} = 0 \text{ and } \sum_{i=1}^{n_j} x_{ij} = 1 \; .$$

Suppose $x_{ij} = 1$, then

$$\pi_{ij}(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}) = \frac{1}{2}\left[(1-\alpha_j)\frac{1}{n_j\gamma_j} + \alpha_j\frac{1}{n_j}\right]v - 1 \; .$$

If agent ij deviates to $x_{ij} = 0$, then

$$\min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} = 0 \text{ and } \sum_{i=1}^{n_j} x_{ij} = 0$$

so that the deviation payoff is

$$\pi_{ij}^D\left(\gamma_j^-, \gamma_j|\boldsymbol{\alpha}\right) = \frac{1}{2}\left[\frac{1}{n_j}\right]v;$$

hence, agent ij has no incentive to deviate if and only if

$$\frac{1}{2}\left[(1-\alpha_j)\frac{1}{n_j\gamma_j} + \alpha_j\frac{1}{n_j}\right]v - 1 \geq \frac{1}{2}\left[\frac{1}{n_j}\right]v \Leftrightarrow$$

$$\frac{1}{2}\left[\frac{1}{n_j\gamma_j} - \alpha_j\left(\frac{1-\gamma_j}{n_j\gamma_j}\right)\right]v - 1 \geq \frac{1}{2}\left[\frac{1}{n_j}\right]v \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{2}\left[\frac{1}{n_j\gamma_j}\right]v - 1 - \frac{1}{2}\left[\frac{1}{n_j}\right]v \geq \frac{1}{2}\left[\alpha_j\left(\frac{1-\gamma_j}{n_j\gamma_j}\right)\right]v \Leftrightarrow$$

$$\Leftrightarrow \frac{(1-\gamma_j)v - 2n_j\gamma_j}{2n_j\gamma_j} \geq \alpha_j\left(\frac{1-\gamma_j}{2n_j\gamma_j}\right)v \Leftrightarrow \alpha_j \leq 1 - \frac{2n_j\gamma_j}{(1-\gamma_j)v}.$$

Suppose $x_{ij} = 0$, then

$$\pi_{ij}(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}) = \frac{1}{2}\alpha_j\frac{1}{n_j}v\ .$$

If agent ij deviates to $x_{ij} = 1$, then

$$\min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} = 0 \text{ and } \sum_{i=1}^{n_j} x_{ij} = 2$$

so that the deviation payoff is

$$\pi_{ij}^D\left(\gamma_j^+, \gamma_{-j}|\boldsymbol{\alpha}\right) = \frac{1}{2}\left[(1-\alpha_j)\frac{1}{n_j\gamma_j+1} + \alpha_j\frac{1}{n_j}\right]v - 1;$$

hence, agent ij has no incentive to deviate if and only if

$$\frac{1}{2}\alpha_j\frac{1}{n_j}v \geq \frac{1}{2}\left[(1-\alpha_j)\frac{1}{n_j\gamma_j+1} + \alpha_j\frac{1}{n_j}\right]v - 1 \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{2}(1-\alpha_j)\frac{1}{\gamma_j+1}v \leq 1 \Leftrightarrow$$

$$\Leftrightarrow (1-\alpha_j) \leq \frac{2\left(n_j\gamma_j+1\right)}{v} \Leftrightarrow \alpha_j \geq 1-\frac{2\left(n_j\gamma_j+1\right)}{v} \ .$$

Thus,

$$(\gamma_1, \gamma_2) \ \ such \ \ that \ \ \gamma_j \in (0,1) \quad \text{with} \quad \sum_{i=1}^{n_j} x_{ij} = 1$$

is a Nash equilibrium if and only if

$$\alpha_j \in \left[1-\frac{2\left(n_j\gamma_j+1\right)}{v}, 1-\frac{2n_j\gamma_j}{\left(1-\gamma_j\right)v}\right].$$

9. Finally, suppose

$$(\gamma_1, \gamma_2) \quad \text{such that } \gamma_j \in (0,1) \quad \text{with} \quad \sum_{i=1}^{n_j} x_{ij} = n_j - 1 \ .$$

Then

$$\min\left\{\mathbf{x}_j\right\} = \min\left\{\mathbf{x}_{-j}\right\} = 0 \text{ and } \sum_{i=1}^{n_j} x_{ij} = n_j - 1 \ .$$

Suppose $x_{ij} = 1$, then

$$\pi_{ij}\left(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}\right) = \frac{1}{2}\left[\left(1-\alpha_j\right)\frac{1}{n_j\gamma_j} + \alpha_j\frac{1}{n_j}\right]v - 1 \ .$$

If agent ij deviates to $x_{ij} = 0$, then

$$\min\left\{\mathbf{x}_j\right\} = \min\left\{\mathbf{x}_{-j}\right\} = 0 \text{ and } \sum_{i=1}^{n_j} x_{ij} = n_j - 2$$

so that the deviation payoff is

$$\pi_{ij}^D\left(\gamma_j^-, \gamma_{-j}|\boldsymbol{\alpha}\right) = \frac{1}{2}\left[\alpha_j\frac{1}{n_j}\right]v;$$

hence, agent ij has no incentive to deviate if and only if

$$\frac{1}{2}\left[\left(1-\alpha_j\right)\frac{1}{n_j\gamma_j} + \alpha_j\frac{1}{n_j}\right]v - 1 \geq \frac{1}{2}\left[\alpha_j\frac{1}{n_j}\right]v \Leftrightarrow$$

$$\frac{1}{2}\left(1-\alpha_j\right)\frac{1}{n_j\gamma_j}v \geq 1 \Leftrightarrow$$

$$\Leftrightarrow (1-\alpha_j) \geq \frac{2n_j\gamma_j}{v} \Leftrightarrow \alpha_j \leq 1-\frac{2n_j\gamma_j}{v} \ .$$

Suppose $x_{ij} = 0$, then

$$\pi_{ij}\left(\gamma_j, \gamma_{-j} | \boldsymbol{\alpha}\right) = \frac{1}{2}\alpha_j \frac{1}{n_j} v \ .$$

If agent ij deviates to $x_{ij} = 1$, then

$$\min\left\{\mathbf{x}_j\right\} = 1 > \min\left\{\mathbf{x}_{-j}\right\} = 0 \text{ and } \sum_{i=1}^{n_j} x_{ij} = n_j$$

so that the deviation payoff is

$$\pi_{ij}^D\left(\gamma_j^+, \gamma_{-j} | \boldsymbol{\alpha}\right) = \frac{1}{n_j} v - 1;$$

agent ij has no incentive to deviate if and only if

$$\frac{1}{2}\alpha_j \frac{1}{n_j} v \geq \frac{1}{n_j} v - 1 \Leftrightarrow \alpha_j \geq \frac{2n_j}{v}\left(\frac{1}{n_j} v - 1\right) \Leftrightarrow \alpha_j \geq 2\left(1 - \frac{n_j}{v}\right) .$$

Note that $2\left(1 - \frac{n_j}{v}\right) \leq 1 - \frac{2n_j\gamma_j}{v}$ if and only if $v \leq 2$. Thus,

$$\left(\gamma_1, \gamma_2\right) \ \textit{such that} \ \ \gamma_j \in (0,1) \text{ with} \sum_{i=1}^{n_j} x_{ij} = n_j - 1$$

is a Nash equilibrium if and only if

$$\alpha_j \in \left[2\left(1 - \frac{n_j}{v}\right), 1 - \frac{2n_j\gamma_j}{v}\right] \ \forall \ v \leq 2 \ .$$

Note that strategy profiles

$$\left(\gamma_j, \gamma_{-j}\right) \in (0,1) \times \{1\} \text{ with } \sum_{i=1}^{n_j} x_{ij} \in \{1, \ldots, n_j - 1\}$$

can be easily shown not be pure strategy Nash equilibria in the effort stage.

QED

1.A.2 Proof Corollary 1.2

From the previous result, it is immediate that when

$$1-\frac{2n_j}{(n_j-1)\,v}>2\left(1-\frac{n_j}{v}\right)\quad\text{and}\quad v<2\cdot max\{n_1,n_2\}$$

there exists a region of sharing rules (α_1,α_2) such that there exists no second stage pure strategy WGS equilibrium. Note that

$$1-\frac{2n_j}{(n_j-1)\,v}>2\left(1-\frac{n_j}{v}\right)\Leftrightarrow 1-\frac{2n_j}{(n_j-1)\,v}>2-\frac{2n_j}{v}\Leftrightarrow$$

$$\Leftrightarrow\frac{2n_j}{v}-\frac{2n_j}{(n_j-1)\,v}>1\Leftrightarrow 2n_j^2-4n_j>(n_j-1)\,v\Leftrightarrow$$

$$\Leftrightarrow v<\frac{2n_j\,(n_j-2)}{n_j-1}.$$

Since

$$\frac{2n_j\,(n_j-2)}{n_j-1}<2n_j$$

then

$$1-\frac{2n_j}{(n_j-1)\,v}>2\left(1-\frac{n_j}{v}\right)\Rightarrow v<2\cdot\max\{n_1,n_2\}.$$

QED

1.A.3 Proof Proposition 1.4

1. Suppose

   $$(\gamma_j,\gamma_{-j})=(0,k)\;\text{ with }k\in\{1,\ldots,K\}\;\text{ and }\mathbf{x}_{-j}=\mathbf{k}.$$

   Then

   $$x_{ij}=\min\{\mathbf{x}_j\}=0<\min\{\mathbf{x}_{-j}\}=k\text{ and }\sum_{i=1}^{n_j}x_{ij}=0,\sum_{i=1}^{n_{-j}}x_{i-j}=n_{-j}k\ ,$$

   so that

   $$\pi_{ij}\left(\gamma_j,\gamma_{-j}|\boldsymbol{\alpha}\right)=0\ .$$

If agent ij deviates to $x_{ij} = k$, $\forall k \in \{1, \ldots, K\}$, then

$$\min\{\mathbf{x}_j\} = 0 < \min\{\mathbf{x}_{-j}\} = k \text{ and } \sum_{i=1}^{n_j} x_{ij} = k, \sum_{i=1}^{n_{-j}} x_{ij} = n_{-j}k$$

so that the deviation payoff is

$$\pi_{ij}^{D}\left(\gamma_j', \gamma_{-j}|\boldsymbol{\alpha}\right) = -k \text{ .}^{1.\text{A9}}$$

Hence, for any player ij there is no incentive to deviate. On the other hand,

$$\pi_{i-j}\left(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}\right) = \frac{1}{n_{-j}}v - k \text{ .}$$

If agent $i-j$ deviates to any $k' \in \{1, \ldots, K\} - \{k\}$, then

$$min\{\mathbf{x}_j\} = 0 < min\{\mathbf{x}_{-j}\} \in \{1, \ldots, K\} \text{ and}$$

$$\sum_{i=1}^{n_j} x_{ij} = 0, \sum_{i=1}^{n_{-j}} x_{i-j} = (n_{-j} - 1)k + k'$$

so that the deviation payoff is

$$\pi_{i-j}^{D}\left(\gamma_j, \gamma_{-j}'|\boldsymbol{\alpha}\right) = \left[(1-\alpha_{-j})\frac{k'}{(n_{-j}-1)k + k'} + \frac{\alpha_{-j}}{n_{-j}}\right]v - k' \leq \frac{1}{n_{-j}}v - k \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \alpha_{-j} \geq 1 - \frac{n_{-j}k}{v} - \frac{n_{-j}k'}{(n_{-j}-1)v} & \text{if } k' > k \\ \alpha_{-j} \leq 1 - \frac{n_{-j}k}{v} - \frac{n_{-j}k'}{(n_{-j}-1)v} & \text{if } k' < k \end{cases}$$

Since the two conditions above have to be valid for any $k' > k$ and any $k' < k \in \{1, \ldots, K\} - \{k\}$, respectively, we get

$$\alpha_{-j} \in \left[1 - \frac{n_{-j}k}{v} - \frac{n_{-j}(k+1)}{(n_{-j}-1)v}, 1 - \frac{n_{-j}k}{v} - \frac{n_{-j}(k-1)}{(n_{-j}-1)v}\right] \text{ .}$$

Conversely, if agent $i-j$ deviates to $k' = 0$, then

$$min\{\mathbf{x}_j\} = min\{\mathbf{x}_{-j}\} = 0 \text{ and } \sum_{i=1}^{n_j} x_{ij} = 0, \sum_{i=1}^{n_{-j}} x_{i-j} = (n_{-j} - 1)k$$

[1.A9] We will denote $\gamma_j' = \frac{1}{n_j}\left(\sum_{i=1}^{n_j} x_{ij} - k + k'\right) \in [0, K]$ the average group effort at any deviation, both upwards and downwards.

so that

$$\pi^D_{i-j}\left(\gamma_j, \gamma'_{-j}|\boldsymbol{\alpha}\right) = \frac{1}{2}\frac{\alpha_{-j}}{n_{-j}}v \leq \frac{1}{n_{-j}}v - k \Leftrightarrow \alpha_{-j} \leq 2\left(1 - \frac{n_{-j}k}{v}\right) .$$

Note that

$$2\left(1 - \frac{n_{-j}\gamma_{-j}}{v}\right) \geq 1 - \frac{n_{-j}\gamma_{-j}}{v} - \frac{n_{-j}\left(\gamma_{-j} - 1\right)}{\left(n_{-j} - 1\right)v} \Leftrightarrow v \geq n_{-j}\gamma_{-j} - \frac{n_{-j}\left(\gamma_{-j} - 1\right)}{n_{-j} - 1} .$$

Hence,

$$\left(\gamma_j, \gamma_{-j}\right) = (0, k) \text{ with } k \in \{1, \ldots, K\} \text{ and } \mathbf{x}_{-j} = \mathbf{k}$$

is a Nash equilibrium if and only if

$$\left(\alpha_j, \alpha_{-j}\right) \in$$

$$\begin{cases} \mathbb{R} \times \left[1 - \frac{n_{-j}\gamma_{-j}}{v} - \frac{n_{-j}(\gamma_{-j}+1)}{(n_{-j}-1)v}, 2\left(1 - \frac{n_{-j}\gamma_{-j}}{v}\right)\right] & \text{if } v < n_{-j}\gamma_{-j} - \frac{n_{-j}(\gamma_{-j}-1)}{n_{-j}-1}; \\ \mathbb{R} \times \left[1 - \frac{n_{-j}\gamma_{-j}}{v} - \frac{n_{-j}(\gamma_{-j}+1)}{(n_{-j}-1)v}, 1 - \frac{n_{-j}\gamma_{-j}}{v} - \frac{n_{-j}(\gamma_{-j}-1)}{(n_{-j}-1)v}\right] & \text{if } v \geq n_{-j}\gamma_{-j} - \frac{n_{-j}(\gamma_{-j}-1)}{n_{-j}-1}; \end{cases}$$

2. Suppose

$$\left(\gamma_1, \gamma_2\right) = (0, 0)$$

Then,

$$x_{ij} = min\left\{\mathbf{x}_j\right\} = min\left\{\mathbf{x}_{-j}\right\} \text{ and } \sum_{i=1}^{n_j} x_{ij} = \sum_{i=1}^{n_{-j}} x_{i-j} = 0$$

so that

$$\pi_{ij}\left(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}\right) = \frac{1}{2}\left[\frac{1}{n_j}\right]v .$$

If agent ij deviates to $x_{ij} = k' = 1$,[1.A10] then

$$x_{ij} = min\left\{\mathbf{x}_j\right\} = min\left\{\mathbf{x}_{-j}\right\} \text{ and } \sum_{i=1}^{n_j} x_{ij} = k', \sum_{i=1}^{n_{-j}} x_{i-j} = 0$$

so that the deviation payoff is

$$\pi^D_{ij}\left(\gamma'_j, \gamma_{-j}|\boldsymbol{\alpha}\right) = \frac{1}{2}\left[\left(1 - \alpha_j\right)\frac{1}{1} + \frac{\alpha_j}{n_j}\right]v - 1 .$$

[1.A10] Note that any deviation to $k' > 1$ would be strictly payoff-dominated by $k' = 1$.

Hence, for any player ij there is no incentive to deviate if and only if

$$\frac{v}{2n_j} \geq \frac{1}{2}\left[(1-\alpha_j)\frac{1}{1}+\frac{\alpha_j}{n_j}\right]v-1 \Leftrightarrow \alpha_j \geq 1-\frac{2n_j}{(n_j-1)\,v} \ .$$

3. Suppose

$$(\gamma_1,\gamma_2)=(k,k) \ \text{ with } k\in\{1,\ldots,K\} \ \text{ and } \mathbf{x}_j=\mathbf{k} \ .$$

Then,

$$x_{ij}=min\,\{\mathbf{x}_j\}=min\,\{\mathbf{x}_{-j}\}\,,\sum_{i=1}^{n_j}x_{ij}=n_j\gamma_j \ \text{and} \ \sum_{i=1}^{n_{-j}}x_{i-j}=n_{-j}\gamma_{-j}$$

so that

$$\pi_{ij}\left(\gamma_j,\gamma_{-j}|\boldsymbol{\alpha}\right)=\frac{1}{2}\left[\frac{1}{n_j}\right]v-k \ .$$

If agent ij deviates to $x_{ij}=k'=0$, then

$$x_{ij}=min\,\{\mathbf{x}_j\}<min\,\{\mathbf{x}_{-j}\} \ \text{ and}$$

$$\sum_{i=1}^{n_j}x_{ij}=(n_j-1)\,\gamma_j \ \text{and} \ \sum_{i=1}^{n_{-j}}x_{i-j}=n_{-j}\gamma_{-j}$$

so that the deviation payoff is

$$\pi_{ij}^{D}\left(\gamma_j',\gamma_{-j}|\boldsymbol{\alpha}\right)=0\leq\frac{v}{2n_j}-k\Leftrightarrow v\geq 2n_jk \ .$$

Note that any deviation $0<k'<k$ is strictly payoff-dominated by $k'=0$, so that we can now take into account upward deviations only. If agent ij deviates to $x_{ij}=k'>k$, then

$$x_{ij}=min\,\{\mathbf{x}_j\}=min\,\{\mathbf{x}_{-j}\}\,,\sum_{i=1}^{n_j}x_{ij}=(n_j-1)\,\gamma_j+k' \ \text{and} \ \sum_{i=1}^{n_{-j}}x_{i-j}=n_{-j}\gamma_{-j}$$

so that the deviation payoff is

$$\pi_{ij}^{D}\left(\gamma_j',\gamma_{-j}|\boldsymbol{\alpha}\right)=\frac{1}{2}\left[(1-\alpha_j)\frac{k'}{(n_j-1)\,k+k'}+\frac{\alpha_j}{n_j}\right]v-k'\leq\frac{v}{2n_j}-k\Leftrightarrow$$

$$\Leftrightarrow \alpha_j\geq 1-\frac{2n_j\gamma_j}{v}-\frac{2n_j\,(\gamma_j+1)}{(n_j-1)\,v} \ .$$

Therefore,

$$(\gamma_1, \gamma_2) = (k, k) \text{ with } k \in \{1, \ldots, K\} \text{ and } \mathbf{x}_j = \mathbf{k}$$

is a Nash equilibrium for any

$$\alpha_j \in \left[1 - \frac{2n_j\gamma_j}{v} - \frac{2n_j\left(\gamma_j + 1\right)}{(n_j - 1)v}, \infty\right) \text{ and } v \geq 2n_j\gamma_j \ .$$

4. Suppose

$$(\gamma_1, \gamma_2) \text{ such that } \gamma_j \in (1, K), min\left\{\mathbf{x}_1\right\} = min\left\{\mathbf{x}_2\right\} > 0 \text{ and}$$

$$2 \leq \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij} = min\{\mathbf{x}_j\}} \leq n_j - 1 \ .$$

Then, if $x_{ij} = k \in \{min\left\{\mathbf{x}_j\right\}, \ldots, K\} \geq min\left\{\mathbf{x}_1\right\} = min\left\{\mathbf{x}_2\right\} > 0$,

$$x_{ij} \geq \min\left\{\mathbf{x}_j\right\} = \min\left\{\mathbf{x}_{-j}\right\} > 0 \text{ and } \sum_{i=1}^{n_j} x_{ij} = n_j\gamma_j, \sum_{i=1}^{n_{-j}} x_{i-j} = n_{-j}\gamma_{-j}$$

so that

$$\pi_{ij}\left(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}\right) = \frac{1}{2}\left[\left(1 - \alpha_j\right)\frac{k}{n_j\gamma_j} + \frac{\alpha_j}{n_j}\right]v - k \ .$$

If agent ij deviates to $k' \in \{min\left\{\mathbf{x}_j\right\}, \ldots, K\} - \{k\}$, then

$$x_{ij} \geq \min\left\{\mathbf{x}_j\right\} = \min\left\{\mathbf{x}_{-j}\right\} > 0 \text{ and } \sum_{i=1}^{n_j} x_{ij} = n_j\gamma_j - k + k', \sum_{i=1}^{n_{-j}} x_{i-j} = n_{-j}\gamma_{-j}$$

so that the deviation payoff is

$$\pi_{ij}^D\left(\gamma_j', \gamma_{-j}|\boldsymbol{\alpha}\right) = \frac{1}{2}\left[\left(1 - \alpha_j\right)\frac{k'}{n_j\gamma_j - k + k'} + \frac{\alpha_j}{n_j}\right]v - k' \ .$$

In contrast, if agent ij deviates to $k' \in \{0, \ldots, min\left\{\mathbf{x}_j\right\} - 1\}$, then

$$x_{ij} = \min\left\{\mathbf{x}_j\right\} < \min\left\{\mathbf{x}_{-j}\right\} \text{ and } \sum_{i=1}^{n_j} x_{ij} = n_j\gamma_j - k + k', \sum_{i=1}^{n_{-j}} x_{i-j} = n_{-j}\gamma_{-j}$$

so that the deviation payoff is

$$\pi_{ij}^D\left(\gamma_j', \gamma_{-j}|\boldsymbol{\alpha}\right) = -k' \ .$$

We select the payoff-dominant downward deviation, that is $k' = 0$. Hence, for any player ij

there is no incentive to deviate if and only if

a.

$$\frac{1}{2}\left[\left(1-\alpha_j\right)\frac{k}{n_j\gamma_j}+\frac{\alpha_j}{n_j}\right]v-k\geq\frac{1}{2}\left[\left(1-\alpha_j\right)\frac{k'}{n_j\gamma_j-k+k'}+\frac{\alpha_j}{n_j}\right]v-k'\Leftrightarrow$$

$$\Leftrightarrow\begin{cases}\alpha_j\geq 1-\frac{2n_j\gamma_j\left(n_j\gamma_j-k+k'\right)}{(n_j\gamma_j-k)v} & \text{if } k'>k\\ \alpha_j\leq 1-\frac{2n_j\gamma_j\left(n_j\gamma_j-k+k'\right)}{(n_j\gamma_j-k)v} & \text{if } k'<k\end{cases}\Leftrightarrow$$

$$\Leftrightarrow\begin{cases}\alpha_j\geq 1-\frac{2n_j\gamma_j(n_j\gamma_j-min\{\mathbf{x}_j\}+min\{\mathbf{x}_j\}+1)}{(n_j\gamma_j-min\{\mathbf{x}_j\})v} & \text{if } k'>k\\ \alpha_j\leq 1-\frac{2n_j\gamma_j(n_j\gamma_j-max\{\mathbf{x}_j\}+max\{\mathbf{x}_j\}-1)}{(n_j\gamma_j-max\{\mathbf{x}_j\})v} & \text{if } k'<k\end{cases}$$

and

b.

$$\frac{1}{2}\left[\left(1-\alpha_j\right)\frac{k}{n_j\gamma_j}+\frac{\alpha_j}{n_j}\right]v-k\geq 0\Leftrightarrow\begin{cases}\alpha_j\geq\frac{k}{\gamma_j-k}\left(\frac{2n_j\gamma_j}{v}-1\right) & \gamma_j>k\\ v\geq 2n_j\gamma_j & \gamma_j=k\\ \alpha_j\leq\frac{k}{\gamma_j-k}\left(\frac{2n_j\gamma_j}{v}-1\right) & \gamma_j<k\end{cases}$$

Note that an α_j preventing upward and downward deviations can exist if and only if the lower and upper bounds at point *a.* do not cross, that is for $max\left\{\mathbf{x}_j\right\}=min\left\{\mathbf{x}_j\right\}+1$. Therefore, combining these two sets of conditions above we get that for any player ij there is no incentive to deviate if and only if $\alpha_j\in$

$$\left[max\left\{\frac{min\left\{\mathbf{x}_j\right\}}{\gamma_j-min\left\{\mathbf{x}_j\right\}}\left(\frac{2n_j\gamma_j}{v}-1\right),1-\frac{2n_j\gamma_j\left(n_j\gamma_j+1\right)}{\left(n_j\gamma_j-min\left\{\mathbf{x}_j\right\}\right)v}\right\},\right.$$

$$\left.1-\frac{2n_j\gamma_j\left(n_j\gamma_j-1\right)}{\left(n_j\gamma_j-min\left\{\mathbf{x}_j\right\}-1\right)v}\right]$$

and $v\geq$

$$max\left\{\frac{2n_1\left(\gamma_1\left(n_1\gamma_1-1\right)-\left(min\left\{\mathbf{x}_1\right\}\right)^2\right)}{n_1\gamma_1-min\left\{\mathbf{x}_1\right\}-1},\frac{2n_2\left(\gamma_2\left(n_2\gamma_2-1\right)-\left(min\left\{\mathbf{x}_2\right\}\right)^2\right)}{n_2\gamma_j-min\left\{\mathbf{x}_2\right\}-1}\right\}.$$

Hence,

$$\left(\gamma_1,\gamma_2\right)\text{ such that }\gamma_j\in\left(1,K\right),min\left\{\mathbf{x}_1\right\}=min\left\{\mathbf{x}_2\right\}>0\text{ and}$$

$$2\leq\sum_{i=1}^{n_j}\mathbb{1}_{x_{ij}=min\{\mathbf{x}_j\}}\leq n_j-1$$

is a Nash equilibrium for any

$$\alpha_j \in$$

$$\left[max\left\{\frac{min\left\{\mathbf{x}_j\right\}}{\gamma_j - min\left\{\mathbf{x}_j\right\}}\left(\frac{2n_j\gamma_j}{v}-1\right), 1-\frac{2n_j\gamma_j\left(n_j\gamma_j+1\right)}{\left(n_j\gamma_j - min\left\{\mathbf{x}_j\right\}\right)v}\right\},\right.$$

$$\left.1-\frac{2n_j\gamma_j\left(n_j\gamma_j-1\right)}{\left(n_j\gamma_j - min\left\{\mathbf{x}_j\right\}-1\right)v}\right],$$

$$max\left\{\mathbf{x}_j\right\} = min\left\{\mathbf{x}_j\right\}+1 \text{ and}$$

$$v \geq max\left\{\frac{2n_1\left(\gamma_1\left(n_1\gamma_1-1\right)-\left(min\left\{\mathbf{x}_1\right\}\right)^2\right)}{n_1\gamma_1 - min\left\{\mathbf{x}_1\right\}-1}, \frac{2n_2\left(\gamma_2\left(n_2\gamma_2-1\right)-\left(min\left\{\mathbf{x}_2\right\}\right)^2\right)}{n_2\gamma_j - min\left\{\mathbf{x}_2\right\}-1}\right\} .$$

5. Suppose

$$\left(\gamma_j, \gamma_{-j}\right) \text{ such that } \gamma_j = k \in \{1,\dots,K-1\} \text{ and } \mathbf{x}_j = \mathbf{k}, \gamma_{-j} \in (1,K),$$

$$min\left\{\mathbf{x}_j\right\} = min\left\{x_{-j}\right\} > 0, 2 \leq \sum_{i=1}^{n_{-j}} \mathbb{1}_{x_{ij}=min\left\{\mathbf{x}_j\right\}} \leq n_j - 1 .$$

Then, the proof is a direct application of what shown at points 3. and 4., so that

$$\left(\gamma_j, \gamma_{-j}\right) \text{ such that } \gamma_j = k \in \{1,\dots,K-1\} \text{ and } \mathbf{x}_j = \mathbf{k}, \gamma_{-j} \in (1,K),$$

$$min\left\{\mathbf{x}_j\right\} = min\left\{x_{-j}\right\} > 0, 2 \leq \sum_{i=1}^{n_{-j}} \mathbb{1}_{x_{ij}=min\left\{\mathbf{x}_j\right\}} \leq n_j - 1$$

is a Nash equilibrium for any

$$\left(\alpha_j, \alpha_{-j}\right) \in$$

$$\left[1-\frac{2n_j\gamma_j}{v}-\frac{2n_j\left(\gamma_j+1\right)}{(n_j-1)v}, \infty\right) \times$$

$$\left[max\left\{\frac{min\left\{\mathbf{x}_{-j}\right\}}{\gamma_{-j} - min\left\{\mathbf{x}_{-j}\right\}}\left(\frac{2n_{-j}\gamma_{-j}}{v}-1\right), 1-\frac{2n_{-j}\gamma_j\left(n_{-j}\gamma_{-j}+1\right)}{\left(n_j\gamma_j - min\left\{\mathbf{x}_{-j}\right\}\right)v}\right\},\right.$$

$$\left.1-\frac{2n_{-j}\gamma_{-j}\left(n_{-j}\gamma_{-j}-1\right)}{\left(n_{-j}\gamma_{-j} - min\left\{\mathbf{x}_{-j}\right\}-1\right)v}\right],$$

$$max\left\{\mathbf{x}_{-j}\right\} = min\left\{\mathbf{x}_{-j}\right\}+1 \text{ and}$$

$$v \geq max\left\{2n_j\gamma_j, \frac{2n_{-j}\left(\gamma_{-j}\left(n_{-j}\gamma_{-j}-1\right)-\left(min\left\{\mathbf{x}_{-j}\right\}\right)^2\right)}{n_{-j}\gamma_{-j} - min\left\{\mathbf{x}_{-j}\right\}-1}\right\} .$$

6. Suppose

$$(\gamma_1, \gamma_2) \text{ such that } \gamma_j = 0, \gamma_{-j} \in (0, K), min\{\mathbf{x}_1\} = min\{\mathbf{x}_2\} \text{ and}$$

$$\sum_{i=1}^{n_{-j}} \mathbb{1}_{x_{i-j}=0} = n_{-j} - 1 \ .$$

Then,

$$x_{ij} = \min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} = 0 \text{ and } \sum_{i=1}^{n_j} x_{ij} = n_j\gamma_j, \sum_{i=1}^{n_{-j}} x_{i-j} = n_{-j}\gamma_{-j}$$

so that

$$\pi_{ij}(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}) = \frac{1}{2}\frac{v}{n_j} \ .$$

If agent ij deviates to $x_{ij} = 1$,[1.A11] then

$$\min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} = 0 \text{ and } \sum_{i=1}^{n_j} x_{ij} = n_j\gamma_j - k + 1, \sum_{i=1}^{n_{-j}} x_{i-j} = n_{-j}\gamma_{-j}$$

so that the deviation payoff is

$$\pi_{ij}^{D}(\gamma_j', \gamma_{-j}|\boldsymbol{\alpha}) = \frac{1}{2}\left[(1-\alpha_j)\frac{1}{1} + \frac{\alpha_j}{n_j}\right]v - 1 \ .$$

Hence for any player ij there is no incentive to deviate if and only if

$$\frac{1}{2}\frac{v}{n_j} \geq \frac{1}{2}\left[(1-\alpha_j) + \frac{\alpha_j}{n_j}\right]v - 1 \Leftrightarrow \alpha_j \geq 1 - \frac{2n_j}{(n_j-1)v}$$

On the other hand, consider $x_{i-j} = 0$, that is

$$x_{i-j} = \min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} = 0 \text{ and } \sum_{i=1}^{n_j} x_{ij} = n_j\gamma_j, \sum_{i=1}^{n_{-j}} x_{i-j} = n_{-j}\gamma_{-j}$$

so that

$$\pi_{i-j}(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}) = \frac{1}{2}\frac{\alpha_{-j}}{n_{-j}}v \ .$$

If agent $i-j$ deviates to $x_{i-j} = k' \in \{1, \ldots, K\}$, then

$$\min\{\mathbf{x}_{-j}\} = \min\{\mathbf{x}_j\} = 0 \text{ and } \sum_{i=1}^{n_j} x_{ij} = n_j\gamma_j, \sum_{i=1}^{n_{-j}} x_{i-j} = n_{-j}\gamma_{-j} + k'$$

[1.A11] Any deviation $k' > 1$ would not deliver a greater payoff than the one attained at $k' = 1$

so that the deviation payoff is

$$\pi^D_{i-j}\left(\gamma_j,\gamma'_{-j}|\boldsymbol{\alpha}\right)=\frac{1}{2}\left[\left(1-\alpha_{-j}\right)\frac{k'}{n_{-j}\gamma_{-j}+k'}+\frac{\alpha_{-j}}{n_{-j}}\right]v-k'\ .$$

Hence, for any player $i-j$ such that $x_{i-j}=0$ there is no incentive to deviate if and only if

$$\frac{1}{2}\frac{\alpha_{-j}}{n_{-j}}v\geq\frac{1}{2}\left[\left(1-\alpha_{-j}\right)\frac{k'}{n_{-j}\gamma_{-j}+k'}+\frac{\alpha_{-j}}{n_{-j}}\right]v-k'\Leftrightarrow\alpha_{-j}\geq1-\frac{2\left(n_{-j}\gamma_{-j}+k'\right)}{v}$$

Given that the condition must hold for all $k'\in\{1,\ldots,K\}$, it must be such that

$$\alpha_{-j}\geq1-\frac{2\left(n_{-j}\gamma_{-j}+1\right)}{v}\ .$$

Consider the unique player $i-j$ such that $x_{i-j}=k\in\{1,\ldots,K\}$, that is

$$x_{i-j}>min\left\{\mathbf{x}_j\right\}=min\left\{\mathbf{x}_{-j}\right\}=0\text{ and }\sum_{i=1}^{n_j}x_{ij}=n_j\gamma_j,\sum_{i=1}^{n_{-j}}x_{i-j}=n_{-j}\gamma_{-j}$$

so that

$$\pi_{i-j}\left(\gamma_j,\gamma_{-j}|\boldsymbol{\alpha}\right)=\frac{1}{2}\left[\left(1-\alpha_{-j}\right)\frac{k}{k}+\frac{\alpha_{-j}}{n_{-j}}\right]v-k$$

It is straightforward to see that the deviation $k'=1$ strictly payoff-dominates any positive effort level k for player $i-j$. On the other hand, if player $i-j$ deviates to $x_{i-j}=0$, then

$$x_{i-j}=min\left\{\mathbf{x}_j\right\}=min\left\{\mathbf{x}_{-j}\right\}=0\text{ and }\sum_{i=1}^{n_j}x_{ij}=0,\sum_{i=1}^{n_{-j}}x_{i-j}=0$$

so that the deviation payoff is

$$\pi^D_{i-j}\left(\gamma_j,\gamma'_{-j}|\boldsymbol{\alpha}\right)=\frac{1}{2}\frac{v}{n_{-j}}\ .$$

Hence, for any player $i-j$ there is no incentive to deviate if and only if

$$\frac{1}{2}\left[\left(1-\alpha_{-j}\right)\frac{1}{1}+\frac{\alpha_{-j}}{n_{-j}}\right]v-1\geq\frac{1}{2}\frac{v}{n_{-j}}\Leftrightarrow\alpha_{-j}\leq1-\frac{2n_{-j}}{\left(n_{-j}-1\right)v}$$

Therefore,

$$\left(0,\gamma_{-j}\right)\text{ such that }\gamma_{-j}\in(0,1)\text{ and }\sum_{i=1}^{n_{-j}}\mathbb{1}_{x_{i-j}=0}=n_{-j}-1$$

is a Nash equilibrium if and only if

$$(\alpha_j, \alpha_{-j}) \in \left[1 - \frac{2n_j}{(n_j - 1)\,v}, \infty\right) \times \left[1 - \frac{2\,(n_{-j}\gamma_{-j} + 1)}{v}, 1 - \frac{2n_{-j}}{(n_{-j} - 1)\,v}\right] .$$

7. Suppose

$$(\gamma_1, \gamma_2) \text{ such that } \gamma_j \in (0, K) \text{ and } \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=0} = n_j - 1 .$$

Then, the proof follows the corresponding arguments shown at point 6., so that

$$(\gamma_1, \gamma_2) \text{ such that } \gamma_j \in (0, 1) \text{ and } \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=0} = n_j - 1$$

is a Nash equilibrium for any $(\alpha_1, \alpha_2) \in$

$$\left[1 - \frac{2\,(n_1\gamma_1 + 1)}{v}, 1 - \frac{2n_1}{(n_1 - 1)\,v}\right] \times \left[1 - \frac{2\,(n_2\gamma_2 + 1)}{v}, 1 - \frac{2n_2}{(n_2 - 1)\,v}\right] .$$

8. Suppose

$$(\gamma_1, \gamma_2) \text{ such that } \gamma_j = 0, \gamma_{-j} \in (0, K)\,, min\,\{\mathbf{x}_1\} = min\,\{\mathbf{x}_2\} \text{ and } \mathbb{1}_{x_{i-j}=0} = 1 .$$

Then,

$$x_{ij} = \min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} = 0 \text{ and } \sum_{i=1}^{n_j} x_{ij} = n_j\gamma_j, \sum_{i=1}^{n_{-j}} x_{i-j} = n_{-j}\gamma_{-j}$$

so that

$$\pi_{ij}\left(\gamma_j, \gamma_{-j} | \boldsymbol{\alpha}\right) = \frac{1}{2}\frac{v}{n_j} .$$

If agent ij deviates to $x_{ij} = 1,$[1.A12] then

$$\min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} = 0 \text{ and } \sum_{i=1}^{n_j} x_{ij} = n_j\gamma_j - k + 1, \sum_{i=1}^{n_{-j}} x_{i-j} = n_{-j}\gamma_{-j}$$

so that the deviation payoff is

$$\pi_{ij}^{D}\left(\gamma_j', \gamma_{-j} | \boldsymbol{\alpha}\right) = \frac{1}{2}\left[(1 - \alpha_j)\,\frac{1}{1} + \frac{\alpha_j}{n_j}\right] v - 1 .$$

[1.A12] Any deviation $k' > 1$ would not deliver a greater payoff than the one attained at $k' = 1$

Hence for any player ij there is no incentive to deviate if and only if

$$\frac{1}{2}\frac{v}{n_j} \geq \frac{1}{2}\left[(1-\alpha_j)+\frac{\alpha_j}{n_j}\right]v-1 \Leftrightarrow \alpha_j \geq 1-\frac{2n_j}{(n_j-1)v}$$

On the other hand, consider $x_{i-j}=0$, that is

$$x_{i-j}=\min\{\mathbf{x}_j\}=\min\{\mathbf{x}_{-j}\}=0 \text{ and } \sum_{i=1}^{n_j} x_{ij}=n_j\gamma_j, \sum_{i=1}^{n_{-j}} x_{i-j}=n_{-j}\gamma_{-j}$$

so that

$$\pi_{i-j}\left(\gamma_j,\gamma_{-j}|\boldsymbol{\alpha}\right)=\frac{1}{2}\frac{\alpha_{-j}}{n_{-j}}v \ .$$

If agent $i-j$ deviates to $x_{i-j}=k\in\{1,\ldots,K\}$, then

$$\min\{\mathbf{x}_{-j}\}>\min\{\mathbf{x}_j\}=0 \text{ and } \sum_{i=1}^{n_j} x_{ij}=n_j\gamma_j, \sum_{i=1}^{n_{-j}} x_{i-j}=n_{-j}\gamma_{-j}+k$$

so that the deviation payoff is

$$\pi^D_{i-j}\left(\alpha,\gamma_j,\gamma'_{-j}\right)\left(\gamma_j,\gamma'_{-j}|\boldsymbol{\alpha}\right)=\left[(1-\alpha_{-j})\frac{k}{n_{-j}\gamma_{-j}+k}+\frac{\alpha_{-j}}{n_{-j}}\right]v-k \ .$$

Hence for the unique player $i-j$ such that $x_{i-j}=0$ there is no incentive to deviate if and only if

$$\frac{1}{2}\frac{\alpha_{-j}}{n_{-j}}v \geq \left[(1-\alpha_{-j})\frac{k}{n_{-j}\gamma_{-j}+k}+\frac{\alpha_{-j}}{n_{-j}}\right]v-k \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \alpha_{-j} \geq \frac{2n_{-j}k(v-n_{-j}\gamma_{-j}-k)}{(2n_{-j}k-n_{-j}\gamma_{-j}-k)v} & \text{if } 2n_{-j}k-n_{-j}\gamma_{-j}-k>0 \\ \alpha_{-j} \leq \frac{2n_{-j}k(v-n_{-j}\gamma_{-j}-k)}{(2n_{-j}k-n_{-j}\gamma_{-j}-k)v} & \text{if } 2n_{-j}k-n_{-j}\gamma_{-j}-k<0 \end{cases}$$

Consider any player $i-j$ such that $x_{i-j}>0$, that is

$$x_{i-j}>\min\{\mathbf{x}_j\}=\min\{\mathbf{x}_{-j}\}=0 \text{ and } \sum_{i=1}^{n_j} x_{ij}=n_j\gamma_j, \sum_{i=1}^{n_{-j}} x_{i-j}=n_{-j}\gamma_{-j}$$

so that

$$\pi_{i-j}\left(\gamma_j,\gamma_{-j}|\boldsymbol{\alpha}\right)=\frac{1}{2}\left[(1-\alpha_{-j})\frac{k}{n_{-j}\gamma_{-j}}+\frac{\alpha_{-j}}{n_{-j}}\right]-k \ .$$

If agent $i-j$ deviates to $x_{i-j} = k' \in \{0, \dots, K\}$, then

$$\min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} = 0 \text{ and } \sum_{i=1}^{n_j} x_{ij} = n_j\gamma_j, \sum_{i=1}^{n_{-j}} x_{i-j} = n_{-j}\gamma_{-j} - k + k'$$

so that the deviation payoff is

$$\pi_{i-j}^{D}\left(\gamma_j, \gamma'_{-j} | \boldsymbol{\alpha}\right) = \frac{1}{2}\left[\left(1-\alpha_{-j}\right)\frac{k'}{n_{-j}\gamma_{-j} - k + k'} + \frac{\alpha_{-j}}{n_{-j}}\right] v - k' \ .$$

Hence for any player $i-j$ there is no incentive to deviate if and only if

$$\frac{1}{2}\left[\left(1-\alpha_{-j}\right)\frac{k}{n_{-j}\gamma_{-j}} + \frac{\alpha_{-j}}{n_{-j}}\right] - k \geq \frac{1}{2}\left[\left(1-\alpha_{-j}\right)\frac{k'}{n_{-j}\gamma_{-j} - k + k'} + \frac{\alpha_{-j}}{n_{-j}}\right] v - k' \Leftrightarrow$$

$$\Leftrightarrow$$

$$\alpha_{-j} \geq 1 - \frac{2n_{-j}\gamma_{-j}\left(n_{-j}\gamma_{-j} - k + k'\right)}{\left(n_{-j}\gamma_{-j} - k\right) v} \qquad \forall k \in \{1, \dots, K\} \text{ and } k' \in \{k+1, \dots, K\}$$

$$\alpha_{-j} \leq 1 - \frac{2n_{-j}\gamma_{-j}\left(n_{-j}\gamma_{-j} - k + k'\right)}{\left(n_{-j}\gamma_{-j} - k\right) v} \qquad \forall k \in \{1, \dots, K\} \text{ and } k' \in \{0, \dots, k-1\}$$

Note that the bounds above are decreasing in both k and k', so that for the lower bound we set $k = 1$ and $k' = 2$, whereas for the upper bound we set $k = K$ and $k' = K - 1$. However, the bounds do not cross if and only if $K = 2$, so that $max\{\mathbf{x}_{-j}\} = 2$. Nonetheless, in the case $2n_{-j}k - n_{-j}\gamma_{-j} - k > 0$, which always holds for $max\{n_{-j}\gamma_{-j}\} \leq (n_{-j} - 1)\, 2$ and $k \geq 1$, we have

$$max\left\{\frac{2n_{-j}k\left(v - n_{-j}\gamma_{-j} - k\right)}{\left(2n_{-j}k - n_{-j}\gamma_{-j} - k\right) v}\right\} = \frac{2n_{-j}\left(v - n_{-j}\gamma_{-j} - 1\right)}{\left(2n_{-j} - n_{-j}\gamma_{-j} - 1\right) v} \ ,$$

so that, for $v \leq 2n_{-j}\left(1 - \frac{\gamma_{-j}(2n_{-j} - n_{-j}\gamma_{-j} - 1)(n_{-j}\gamma_{-j} - 1)}{(n_{-j}\gamma_{-j} - 2)(n_{-j}\gamma_{-j} + 1)}\right)$,

$$\left[\frac{2n_{-j}\left(v - n_{-j}\gamma_{-j} - 1\right)}{\left(2n_{-j} - n_{-j}\gamma_{-j} - 1\right) v}, \infty\right) \cap \left[1 - \frac{2n_{-j}\gamma_{-j}\left(n_{-j}\gamma_{-j} + 1\right)}{\left(n_{-j}\gamma_{-j} - 1\right) v}, 1 - \frac{2n_{-j}\gamma_{-j}\left(n_{-j}\gamma_{-j} - 1\right)}{\left(n_{-j}\gamma_{-j} - 2\right) v}\right] \neq \emptyset \ .$$

As a matter of fact,

$$\frac{2n_{-j}\left(v - n_{-j}\gamma_{-j} - 1\right)}{\left(2n_{-j} - n_{-j}\gamma_{-j} - 1\right) v} \leq 1 - \frac{2n_{-j}\gamma_{-j}\left(n_{-j}\gamma_{-j} - 1\right)}{\left(n_{-j}\gamma_{-j} - 2\right) v} \Leftrightarrow$$

$$\Leftrightarrow \frac{\left(v - n_{-j}\gamma_{-j} - 1\right)}{\left(2n_{-j} - n_{-j}\gamma_{-j} - 1\right)} \leq \frac{v}{2n_{-j}} - \frac{\gamma_{-j}\left(n_{-j}\gamma_{-j} - 1\right)}{\left(n_{-j}\gamma_{-j} - 2\right)} \Leftrightarrow$$

$$\Leftrightarrow (2n_{-j} - n_{-j} - 1)\left[(n_{-j}\gamma_{-j} - 2)\, v - 2n_{-j}\gamma_{-j}\,(n_{-j}\gamma_{-j} - 1)\right] \geq$$

$$(v - n_{-j}\gamma_{-j} - 1)\, 2n_{-j}\,(n_{-j}\gamma_{-j} - 1) \Leftrightarrow$$

$$\left[-4n^2_{-j}\gamma_{-j} + 2n_{-j}\gamma_{-j}\,(n_{-j}\gamma_{-j} + 1)\right](n_{-j}\gamma_{-j} - 1) \geq (v - 2n_{-j})\,(n_{-j}\gamma_{-j} - 2)\,(n_{-j}\gamma_{-j} + 1)$$

$$v \leq 2n_{-j}\left(1 - \frac{\gamma_{-j}\,(2n_{-j} - n_{-j}\gamma_{-j} - 1)\,(n_{-j}\gamma_{-j} - 1)}{(n_{-j}\gamma_{-j} - 2)\,(n_{-j}\gamma_{-j} + 1)}\right) .$$

Moreover, note that

$$\frac{\gamma_{-j}\,(2n_{-j} - n_{-j}\gamma_{-j} - 1)\,(n_{-j}\gamma_{-j} - 1)}{(n_{-j}\gamma_{-j} - 2)\,(n_{-j}\gamma_{-j} + 1)} < 1 \;\forall\; n_{-j}\gamma_{-j} \leq (n_{-j} - 1)\, 2 ,$$

so that

$$0 < v \leq 2n_{-j}\left(1 - \frac{\gamma_{-j}\,(2n_{-j} - n_{-j}\gamma_{-j} - 1)\,(n_{-j}\gamma_{-j} - 1)}{(n_{-j}\gamma_{-j} - 2)\,(n_{-j}\gamma_{-j} + 1)}\right) .$$

Therefore,

$$(\gamma_j, \gamma_{-j}) \text{ such that } \gamma_j = 0, \gamma_{-j} \in (0, 2)\,, min\,\{\mathbf{x}_1\} = min\,\{\mathbf{x}_2\} = 0 \text{ and } \mathbb{1}_{x_{i-j}=0} = 1$$

is a Nash equilibrium for any

$$(\alpha_j, \alpha_{-j}) \in$$

$$\left[1 - \frac{2n_j}{(n_j - 1)\, v}, \infty\right) \times$$

$$\left[max\left\{\frac{2n_{-j}\,(v - n_{-j}\gamma_{-j} - 1)}{(2n_{-j} - n_{-j}\gamma_{-j} - 1)\, v}, 1 - \frac{2n_{-j}\gamma_{-j}\,(n_{-j}\gamma_{-j} + 1)}{(n_{-j}\gamma_{-j} - 1)\, v}\right\}, 1 - \frac{2n_{-j}\gamma_{-j}\,(n_{-j}\gamma_{-j} - 1)}{(n_{-j}\gamma_{-j} - 2)\, v}\right]$$

and

$$v \leq 2n_{-j}\left(1 - \frac{\gamma_{-j}\,(2n_{-j} - n_{-j}\gamma_{-j} - 1)\,(n_{-j}\gamma_{-j} - 1)}{(n_{-j}\gamma_{-j} - 2)\,(n_{-j}\gamma_{-j} + 1)}\right) .$$

9. Suppose

$$(\gamma_1, \gamma_2) \text{ such that } \gamma_j \in (0, K) \text{ and } \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=0} = 1$$

Then, the proof follows the corresponding arguments shown at point 8., so that

$$(\gamma_1, \gamma_2) \text{ such that } \gamma_j \in (0, 2) \text{ and } \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=0} = 1$$

is a Nash equilibrium for any

$$(\alpha_1, \alpha_2) \in$$

$$\left[max\left\{\frac{2n_1\left(v-n_1\gamma_1-1\right)}{\left(2n_1-n_1\gamma_1-1\right)v}, 1-\frac{2n_1\gamma_1\left(n_1\gamma_1+1\right)}{\left(n_1\gamma_1-1\right)v}\right\}, 1-\frac{2n_1\gamma_1\left(n_1\gamma_1-1\right)}{\left(n_1\gamma_1-2\right)v}\right] \times$$
$$\left[max\left\{\frac{2n_2\left(v-n_2\gamma_2-1\right)}{\left(2n_2-n_2\gamma_2-1\right)v}, 1-\frac{2n_2\gamma_2\left(n_2\gamma_2+1\right)}{\left(n_2\gamma_2-1\right)v}\right\}, 1-\frac{2n_2\gamma_2\left(n_2\gamma_2-1\right)}{\left(n_2\gamma_2-2\right)v}\right] .$$

and

$$v \leq max\Bigg\{\{2n_1\left(1-\frac{\gamma_1\left(2n_1-n_1\gamma_1-1\right)\left(n_1\gamma_1-1\right)}{\left(n_1\gamma_1-2\right)\left(n_1\gamma_1+1\right)}\right),$$
$$2n_2\left(1-\frac{\gamma_2\left(2n_2-n_2\gamma_2-1\right)\left(n_2\gamma_2-1\right)}{\left(n_2\gamma_2-2\right)\left(n_2\gamma_2+1\right)}\right)\Bigg\} .$$

Finally, note that similar arguments prove that the following strategy profiles are not second-period Nash equilibria in pure strategies:

i. (γ_1, γ_2) s.t. $\gamma_j \in (1, K), min\{\mathbf{x}_1\} = min\{\mathbf{x}_2\} > 0$ and $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=min\{\mathbf{x}_j\}} = 1$. Suppose

$$(\gamma_1, \gamma_2) \text{ such that } \gamma_j \in (1, K), min\{\mathbf{x}_1\} = min\{\mathbf{x}_2\} > 0 \text{ and } \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=min\{\mathbf{x}_j\}} = 1 .$$

Then, if $x_{ij} = k \in \{min\{\mathbf{x}_j\}+1, \ldots, K\} > min\{\mathbf{x}_1\} = min\{\mathbf{x}_2\} > 0$,

$$x_{ij} > \min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} > 0 \text{ and } \sum_{i=1}^{n_j} x_j(i) = n_j\gamma_j, \sum_{i=1}^{n_{-j}} x_{i-j} = n_{-j}\gamma_{-j}$$

so that

$$\pi_{ij}\left(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}\right) = \frac{1}{2}\left[\left(1-\alpha_j\right)\frac{k}{n_j\gamma_j} + \frac{\alpha_j}{n_j}\right]v - k .$$

If agent ij deviates to $k' \in \{min\{\mathbf{x}_j\}, \ldots, K\} - \{k\}$, then

$$x_{ij} \geq \min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} > 0 \text{ and } \sum_{i=1}^{n_j} x_{ij} = n_j\gamma_j - k + k', \sum_{i=1}^{n_{-j}} x_{i-j} = n_{-j}\gamma_{-j}$$

so that the deviation payoff is

$$\pi_{ij}^D\left(\gamma_j', \gamma_{-j}|\boldsymbol{\alpha}\right) = \frac{1}{2}\left[\left(1-\alpha_j\right)\frac{k'}{n_j\gamma_j - k + k'} + \frac{\alpha_j}{n_j}\right]v - k' .$$

In contrast, if $x_{ij} = k \in \{min\{\mathbf{x}_j\}, \ldots, K\} \geq min\{\mathbf{x}_1\} = min\{\mathbf{x}_2\} > 0$, and agent ij deviates to $k' \in \{0, \dots, min\{\mathbf{x}_j\} - 1\}$, then

$$x_{ij} = \min\{\mathbf{x}_j\} < \min\{\mathbf{x}_{-j}\} \text{ and } \sum_{i=1}^{n_j} x_j(i) = n_j\gamma_j - k + k', \sum_{i=1}^{n_{-j}} x_{i-j} = n_{-j}\gamma_{-j}$$

so that the deviation payoff is

$$\pi_{ij}^{D}\left(\gamma_j', \gamma_{-j}|\boldsymbol{\alpha}\right) = -k' \ .$$

We select the payoff-dominant downward deviation, that is $k' = 0$. On the other hand, if the unique agent ij exerting effort $k = min\{\mathbf{x}_j\}$ deviates to $k' \in \{min\{\mathbf{x}_j + 1\}, \dots, K\}$, then

$$x_{ij} \geq \min\{\mathbf{x}_j\} > \min\{\mathbf{x}_{-j}\} \text{ and } \sum_{i=1}^{n_j} x_{ij} = n_j\gamma_j - k + k', \sum_{i=1}^{n_{-j}} x_{i-j} = n_{-j}\gamma_{-j}$$

so that the deviation payoff is

$$\pi_{ij}^{D}\left(\gamma_j', \gamma_{-j}|\boldsymbol{\alpha}\right) = \left[(1-\alpha_j)\frac{k'}{n_j\gamma_j - k + k'} + \frac{\alpha_j}{n_j}\right]v - k' \ .$$

Hence, for any player ij there is no incentive to deviate if and only if

a.

$$\frac{1}{2}\left[(1-\alpha_j)\frac{k}{n_j\gamma_j} + \frac{\alpha_j}{n_j}\right]v - k \geq \frac{1}{2}\left[(1-\alpha_j)\frac{k'}{n_j\gamma_j - k + k'} + \frac{\alpha_j}{n_j}\right]v - k' \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \alpha_j \geq 1 - \frac{2n_j\gamma_j\left(n_j\gamma_j - k + k'\right)}{(n_j\gamma_j - k)v} & \text{if } k' > k \\ \alpha_j \leq 1 - \frac{2n_j\gamma_j\left(n_j\gamma_j - k + k'\right)}{(n_j\gamma_j - k)v} & \text{if } k' < k \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \alpha_j \geq 1 - \frac{2n_j\gamma_j(n_j\gamma_j - min\{\mathbf{x}_j\} + min\{\mathbf{x}_j\} + 1)}{(n_j\gamma_j - min\{\mathbf{x}_j\})v} & \text{if } k' > k \\ \alpha_j \leq 1 - \frac{2n_j\gamma_j(n_j\gamma_j - min\{\mathbf{x}_j\} - 1 + min\{\mathbf{x}_j\})}{(n_j\gamma_j - min\{\mathbf{x}_j\} - 1)v} & \text{if } k' < k \end{cases}$$

b.

$$\frac{1}{2}\left[(1-\alpha_j)\frac{k}{n_j\gamma_j} + \frac{\alpha_j}{n_j}\right]v - k \geq 0 \Leftrightarrow \begin{cases} \alpha_j \geq \frac{k}{\gamma_j - k}\left(\frac{2n_j\gamma_j}{v} - 1\right) & \gamma_j > k \\ v \geq 2n_j\gamma_j & \gamma_j = k \\ \alpha_j \leq \frac{k}{\gamma_j - k}\left(\frac{2n_j\gamma_j}{v} - 1\right) & \gamma_j < k \end{cases}$$

c.

$$\frac{1}{2}\left[(1-\alpha_j)\frac{k}{n_j\gamma_j} + \frac{\alpha_j}{n_j}\right]v - k \geq \left[(1-\alpha_j)\frac{k'}{n_j\gamma_j - k + k'} + \frac{\alpha_j}{n_j}\right]v - k' \Leftrightarrow$$

$$\Leftrightarrow \alpha_j \lesseqgtr \frac{\left(2n_j\gamma_j k' - k\left(n_j\gamma_j - k + k'\right)\right)v + (k - k')\, 2n_j\gamma_j\left(n_j\gamma_j - k + k'\right)}{\left(2n_j\gamma_j k' - (\gamma_j + k)\left(n_j\gamma_j - k + k'\right)\right)v}$$

Note that the bounds at point *a.* are decreasing in both k and k', so that for the lower bound we set $k = min\left\{\mathbf{x}_j\right\}$ and $k' = min\left\{\mathbf{x}_j\right\} + 1$, whereas for the upper bound we set $k = \mathbf{x}_j$ and $k' = \mathbf{x}_j - 1$. However, the bounds do not cross if and only if $max\left\{\mathbf{x}_j\right\} = min\left\{\mathbf{x}_j\right\} + 1$. Consequently, we can substitute these values in the condition at point *c.* which holds for the unique player ij exerting effort $k = min\left\{\mathbf{x}_j\right\}$ and, for $v \geq 2n_j\gamma_j$ as resulting from condition at point b., we obtain

$$\alpha_j \geq \frac{\left(2n_j\gamma_j\left(min\left\{\mathbf{x}_j\right\}+1\right) - min\left\{\mathbf{x}_j\right\}\left(n_j\gamma_j+1\right)\right)v - 2n_j\gamma_j\left(n_j\gamma_j+1\right)}{\left(2n_j\gamma_j\left(min\left\{\mathbf{x}_j\right\}+1\right) - \left(\gamma_j + min\left\{\mathbf{x}_j\right\}\right)\left(n_j\gamma_j+1\right)\right)v} > 1 \ .$$

However, it follows

$$\left[1 - \frac{2n_j\gamma_j\left(n_j\gamma_j+1\right)}{\left(n_j\gamma_j - min\left\{\mathbf{x}_j\right\}\right)v}, 1 - \frac{2n_j\gamma_j\left(n_j\gamma_j-1\right)}{\left(n_j\gamma_j - min\left\{\mathbf{x}_j\right\}-1\right)v}\right] \cap$$

$$\cap \left[\frac{\left(2n_j\gamma_j\left(min\left\{\mathbf{x}_j\right\}+1\right) - min\left\{\mathbf{x}_j\right\}\left(n_j\gamma_j+1\right)\right)v - 2n_j\gamma_j\left(n_j\gamma_j+1\right)}{\left(2n_j\gamma_j\left(min\left\{\mathbf{x}_j\right\}+1\right) - \left(\gamma_j + min\left\{\mathbf{x}_j\right\}\right)\left(n_j\gamma_j+1\right)\right)v}, \infty\right) = \emptyset.$$

Therefore,

$$\left(\gamma_1, \gamma_2\right) \text{ such that } \gamma_j \in \left(1, K\right), min\left\{\mathbf{x}_1\right\} = min\left\{\mathbf{x}_2\right\} > 0 \text{ and } \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=min\left\{\mathbf{x}_j\right\}} = 1$$

is not a Nash equilibrium for any

$$\left(\alpha_1, \alpha_2\right) \in \mathbb{R} \times \mathbb{R} \ .$$

ii. $\left(\gamma_j, \gamma_{-j}\right)$ such that $\gamma_j = k \in \left\{1, \ldots, K-1\right\}$ and $\mathbf{x}_j = \mathbf{k}, \gamma_{-j} \in (1, K)$, $min\left\{\mathbf{x}_j\right\} = min\left\{\mathbf{x}_{-j}\right\} > 0$ and $\sum_{i=1}^{n_{-j}} \mathbb{1}_{x_{i-j}=min\left\{\mathbf{x}_{-j}\right\}} = 1$.

Then, the proof directly follows from what shown at point i.

iii. $\left(\gamma_1, \gamma_2\right)$ such that $\gamma_j = 0, \gamma_{-j} \in \left(0, K\right), min\left\{\mathbf{x}_1\right\} = min\left\{\mathbf{x}_2\right\}$ and $2 \leq \sum_{i=1}^{n_{-j}} \mathbb{1}_{x_{i-j}=0} \leq n_{-j} - 2$.

Suppose

$$\left(\gamma_1, \gamma_2\right) \text{ such that } \gamma_j = 0, \gamma_{-j} \in \left(0, K\right), min\left\{\mathbf{x}_1\right\} = min\left\{\mathbf{x}_2\right\} \text{ and}$$

$$2 \leq \sum_{i=1}^{n_{-j}} \mathbb{1}_{x_{i-j}=0} \leq n_{-j} - 2 \ .$$

Then,

$$x_{ij} = \min\left\{\mathbf{x}_j\right\} = \min\left\{\mathbf{x}_{-j}\right\} = 0 \text{ and } \sum_{i=1}^{n_j} x_{ij} = n_j\gamma_j, \sum_{i=1}^{n_{-j}} x_{i-j} = n_{-j}\gamma_{-j}$$

so that

$$\pi_{ij}\left(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}\right) = \frac{1}{2}\frac{v}{n_j} \ .$$

If agent ij deviates to $x_{ij} = 1$[1.A13], then

$$\min\left\{\mathbf{x}_j\right\} = \min\left\{\mathbf{x}_{-j}\right\} = 0 \text{ and } \sum_{i=1}^{n_j} x_{ij} = n_j\gamma_j - k + 1, \sum_{i=1}^{n_{-j}} x_{i-j} = n_{-j}\gamma_{-j}$$

so that the deviation payoff is

$$\pi_{ij}^{D}\left(\gamma_j', \gamma_{-j}|\boldsymbol{\alpha}\right) = \frac{1}{2}\left[\left(1-\alpha_j\right)\frac{1}{1} + \frac{\alpha_j}{n_j}\right]v - 1 \ .$$

Hence, for any player ij there is no incentive to deviate if and only if

$$\frac{1}{2}\frac{v}{n_j} \geq \frac{1}{2}\left[\left(1-\alpha_j\right) + \frac{\alpha_j}{n_j}\right]v - 1 \Leftrightarrow \alpha_j \geq 1 - \frac{2n_j}{(n_j-1)v} \ .$$

On the other hand,

$$x_{i-j} \geq \min\left\{\mathbf{x}_j\right\} = \min\left\{\mathbf{x}_{-j}\right\} = 0 \text{ and } \sum_{i=1}^{n_j} x_j\left(i\right) = n_j\gamma_j, \sum_{i=1}^{n_{-j}} x_{i-j} = n_{-j}\gamma_{-j}$$

so that

$$\pi_{i-j}\left(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}\right) = \frac{1}{2}\left[\left(1-\alpha_{-j}\right)\frac{k}{n_{-j}\gamma_{-j}} + \frac{\alpha_{-j}}{n_{-j}}\right]v - k \ .$$

If agent $i-j$ deviates to $x_{i-j} = k' \in \{0, \ldots, K\} - \{k\}$, then

$$\min\left\{\mathbf{x}_j\right\} = \min\left\{\mathbf{x}_{-j}\right\} = 0 \text{ and } \sum_{i=1}^{n_j} x_{ij} = n_j\gamma_j, \sum_{i=1}^{n_{-j}} x_{i-j} = n_{-j}\gamma_{-j} - k + k'$$

so that the deviation payoff is

$$\pi_{i-j}^{D}\left(\gamma_j, \gamma_{-j}'|\boldsymbol{\alpha}\right) = \frac{1}{2}\left[\left(1-\alpha_{-j}\right)\frac{k'}{n_{-j}\gamma_{-j} - k + k'} + \frac{\alpha_{-j}}{n_{-j}}\right]v - k' \ .$$

Hence, for any player ij there is no incentive to deviate if and only if

$$\frac{1}{2}\left[\left(1-\alpha_{-j}\right)\frac{k}{n_{-j}\gamma_{-j}} + \frac{\alpha_{-j}}{n_{-j}}\right]v - k \geq \frac{1}{2}\left[\left(1-\alpha_j\right)\frac{k'}{n_j\gamma_j - k + k'} + \frac{\alpha_j}{n_j}\right]v - k' \Leftrightarrow$$

[1.A13] Any deviation $k' > 1$ would not deliver a greater payoff than the one attained at $k' = 1$.

$$\Leftrightarrow \alpha_{-j} \geq 1 - \frac{2n_{-j}\gamma_{-j}\left(n_{-j}\gamma_{-j} - k + k'\right)}{\left(n_{-j}\gamma_{-j} - k\right)v} \qquad \forall k \in \{0,\ldots,K\} \text{ and } k' \in \{k+1,\ldots,K\}$$

$$\alpha_{-j} \leq 1 - \frac{2n_{-j}\gamma_{-j}\left(n_{-j}\gamma_{-j} - k + k'\right)}{\left(n_{-j}\gamma_{-j} - k\right)v} \qquad \forall k \in \{0,\ldots,K\} \text{ and } k' \in \{0,\ldots,k-1\}$$

Note that the bounds above are decreasing in both k and k', so that for the lower bound we set $k = 0$ and $k' = 1$, whereas for the upper bound we set $k = K$ and $k' = K - 1$. However, the bounds do not cross if and only if $K = 1$, contradicting the *KMMGC* model assumptions.

Therefore,

$$\left(\gamma_j, \gamma_{-j}\right) \text{ such that } \gamma_j = 0, \gamma_{-j} \in (0, K), min\{\mathbf{x}_1\} = min\{\mathbf{x}_2\} = 0 \text{ and}$$

$$2 \leq \sum_{i=1}^{n_{-j}} \mathbb{1}_{x_{i-j}=0} \leq n_{-j} - 2$$

is not a Nash equilibrium for any

$$\left(\alpha_j, \alpha_{-j}\right) \in \mathbb{R} \times \mathbb{R} \ .$$

iv. $\left(\gamma_j, \gamma_{-j}\right)$ such that $\gamma_j \in (0,1), min\{\mathbf{x}_1\} = min\{\mathbf{x}_2\} = 0$ and $2 \leq \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=0} \leq n_j - 2$.

Then, the proof directly follows from what shown at point iii.

QED

1.A.4 Proof Proposition 1.6

From Proposition 1.4 it follows that in the *KMMGC* with restricted sharing rules, i.e. with $\alpha_j \in [0,1]\,\forall j \in \{1,2\}$, the second-stage equilibria are:

1. if $v \geq n_{-j}\gamma_{-j} - \frac{n_{-j}(\gamma_{-j}-1)}{n_{-j}-1}$, then

$$\left(\gamma_j, \gamma_{-j}\right) = (0, k) \text{ with } k \in \{1,\ldots,K\} \text{ and } \mathbf{x}_{-j} = \mathbf{k}\ ^{1.A14}$$

for any

$$\left(\alpha_j, \alpha_{-j}\right) \in [0,1] \times \left[1 - \frac{n_{-j}\gamma_{-j}}{v} - \frac{n_{-j}(\gamma_{-j}+1)}{\left(n_{-j}-1\right)v}, 1 - \frac{n_{-j}\gamma_{-j}}{v} - \frac{n_{-j}(\gamma_{-j}-1)}{\left(n_{-j}-1\right)v}\right];$$

[1.A14] $\mathbf{x}_{-j} = \mathbf{k}$ with $k \in \{1,\ldots,K\}$ means that all agents in group $-j$ exert the same level of effort k .

2. if $v > 0$, then

$$(\gamma_1, \gamma_2) = (0, 0)$$

for any

$$(\alpha_1, \alpha_2) \in \left[max\left\{0, 1 - \frac{2n_1}{(n_1 - 1)\, v}\right\}, 1\right] \times \left[max\left\{0, 1 - \frac{2n_2}{(n_2 - 1)\, v}\right\}, 1\right];$$

3. if $v \geq 2 \cdot max\,\{n_1\gamma_1, n_2\gamma_2\}$, then

$$(\gamma_1, \gamma_2) = (k, k) \text{ with } k \in \{1, \ldots, K\} \text{ and } \mathbf{x}_j = \mathbf{k}$$

for any $(\alpha_1, \alpha_2) \in$

$$\left[max\left\{0, 1 - \frac{2n_1\gamma_1}{v} - \frac{2n_1\,(\gamma_1 + 1)}{(n_1 - 1)v}\right\}, 1\right] \times \left[max\left\{0, 1 - \frac{2n_2\gamma_2}{v} - \frac{2n_2\,(\gamma_2 + 1)}{(n_2 - 1)v}\right\}, 1\right];$$

4. if $v \geq max\left\{\frac{2n_1\gamma_1(n_1\gamma_1 - 1)}{n_1\gamma_1 - min\{\mathbf{x}_1\} - 1}, \frac{2n_2\gamma_2(n_2\gamma_2 - 1)}{n_2\gamma_2 - min\{\mathbf{x}_2\} - 1}\right\}$, then

$$(\gamma_1, \gamma_2) \text{ such that } \gamma_j \in (1, K)\,,\, min\,\{\mathbf{x}_1\} = min\,\{\mathbf{x}_2\} > 0, 2 \leq \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij} = min\{\mathbf{x}_j\}} \leq n_j - 1$$ [1.A15]

for any

$$(\alpha_1, \alpha_2) \in$$

$$\left[max\left\{0, \frac{min\,\{\mathbf{x}_1\}}{\gamma_1 - min\,\{\mathbf{x}_1\}}\left(\frac{2n_1\gamma_1}{v} - 1\right), 1 - \frac{2n_1\gamma_1\,(n_j\gamma_1 + 1)}{(n_1\gamma_1 - min\,\{\mathbf{x}_1\})\, v}\right\},\right.$$

$$\left. 1 - \frac{2n_1\gamma_1\,(n_1\gamma_1 - 1)}{(n_1\gamma_1 - min\,\{\mathbf{x}_1\} - 1)\, v}\right] \times$$

$$\left[max\left\{0, \frac{min\,\{\mathbf{x}_2\}}{\gamma_2 - min\,\{\mathbf{x}_2\}}\left(\frac{2n_2\gamma_2}{v} - 1\right), 1 - \frac{2n_2\gamma_2\,(n_2\gamma_2 + 1)}{(n_2\gamma_2 - min\,\{\mathbf{x}_2\})\, v}\right\},\right.$$

$$\left. 1 - \frac{2n_2\gamma_2\,(n_2\gamma_2 - 1)}{(n_2\gamma_2 - min\,\{\mathbf{x}_2\} - 1)\, v}\right],$$

$$max\,\{\mathbf{x}_1\} = min\,\{\mathbf{x}_1\} + 1 \text{ and } max\,\{\mathbf{x}_2\} = min\,\{\mathbf{x}_2\} + 1;$$

[1.A15] Note that $\mathbb{1}_{x_{ij} = min\{\mathbf{x}_j\}}$ stands for the Indicator function taking value 1 when $x_{ij} = min\{\mathbf{x}_j\}$ for any ij.

5. if $v \geq max\left\{2n_j\gamma_j, \frac{2n_{-j}\gamma_{-j}(n_{-j}\gamma_{-j}-1)}{n_{-j}\gamma_{-j}-min\{\mathbf{x}_{-j}\}-1}\right\}$, then

$$(\gamma_j,\gamma_{-j}) \text{ such that } \gamma_j = k \in \{1,\ldots,K-1\} \text{ and } \mathbf{x}_j = \mathbf{k}, \gamma_{-j} \in (1,K),$$

$$min\{\mathbf{x}_j\} = min\{\mathbf{x}_{-j}\} > 0, 2 \leq \sum_{i=1}^{n_{-j}} \mathbb{1}_{x_{i-j}=min\{\mathbf{x}_{-j}\}} \leq n_{-j}-1$$

for any

$$(\alpha_j,\alpha_{-j}) \in$$

$$\left[max\left\{0, 1-\frac{2n_j\gamma_j}{v}-\frac{2n_j(\gamma_j+1)}{(n_j-1)v}\right\}, 1\right] \times$$

$$\left[max\left\{0, \frac{min\{\mathbf{x}_{-j}\}}{\gamma_{-j}-min\{\mathbf{x}_{-j}\}}\left(\frac{2n_{-j}\gamma_{-j}}{v}-1\right), 1-\frac{2n_{-j}\gamma_{-j}(n_{-j}\gamma_{-j}+1)}{(n_{-j}\gamma_{-j}-min\{\mathbf{x}_{-j}\})v}\right\},\right.$$

$$\left. 1-\frac{2n_{-j}\gamma_{-j}(n_{-j}\gamma_{-j}-1)}{(n_{-j}\gamma_{-j}-min\{\mathbf{x}_{-j}\}-1)v}\right],$$

$$max\{\mathbf{x}_{-j}\} = min\{\mathbf{x}_{-j}\}+1;$$

6. if $v \geq \frac{2n_{-j}}{n_{-j}-1}$, then

$$(\gamma_j,\gamma_{-j}) \text{ such that } \gamma_j = 0, \gamma_{-j} \in (0,1) \text{ and } \sum_{i=1}^{n_{-j}} \mathbb{1}_{x_{i-j}=0} = n_{-j}-1$$

for any $(\alpha_j,\alpha_{-j}) \in$

$$\left[max\left\{0, 1-\frac{2n_j}{(n_j-1)v}\right\}, 1\right] \times \left[max\left\{0, 1-\frac{2(n_{-j}\gamma_{-j}+1)}{v}\right\}, 1-\frac{2n_{-j}}{(n_{-j}-1)v}\right];$$

7. if $v \geq 2 \cdot max\left\{\frac{n_1}{n_1-1}, \frac{n_2}{n_2-1}\right\}$, then

$$(\gamma_1,\gamma_2) \text{ such that } \gamma_j \in (0,1), min\{\mathbf{x}_1\} = min\{\mathbf{x}_2\} = 0 \text{ and } \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=0} = n_j-1$$

for any

$$(\alpha_1,\alpha_2) \in$$

$$\left[max\left\{0, 1-\frac{2(n_1\gamma_1+1)}{v}\right\}, 1-\frac{2n_1}{(n_1-1)v}\right] \times$$

$$\left[max\left\{0, 1-\frac{2(n_2\gamma_2+1)}{v}\right\}, 1-\frac{2n_2}{(n_2-1)v}\right].$$

Hence, for $v \leq 2 \cdot min\left\{\frac{n_1}{n_1-1}, \frac{n_2}{n_2-1}\right\}$ there always exists at least one second-stage equilibrium in the $\alpha_1 \times \alpha_2$ space, that is $(\gamma_1, \gamma_2) = (0, 0)$. On the other hand, for

$$2 \cdot max\left\{n_1 k, n_2 k\right\} \leq v \leq max\left\{2n_1 k + \frac{2n_1}{n_1 - 1}(k+1), 2n_2 k + \frac{2n_2}{n_2 - 1}(k+1)\right\}$$

$\forall k \in \{1, \ldots, K\}$, there always exists at least one second-stage equilibrium in the $\alpha_1 \times \alpha_2$ space, that is $(\gamma_1, \gamma_2) = (k, k)$ with $k \in \{1, \ldots, K\}$ and $\mathbf{x}_j = \mathbf{k}$. In both cases, the continuation-payoffs can be expressed, so that both the first-stage equilibria and the subgame perfect equilibria can be found. QED

1.B Representation of Second-Period Nash Equilibria in Pure Strategies

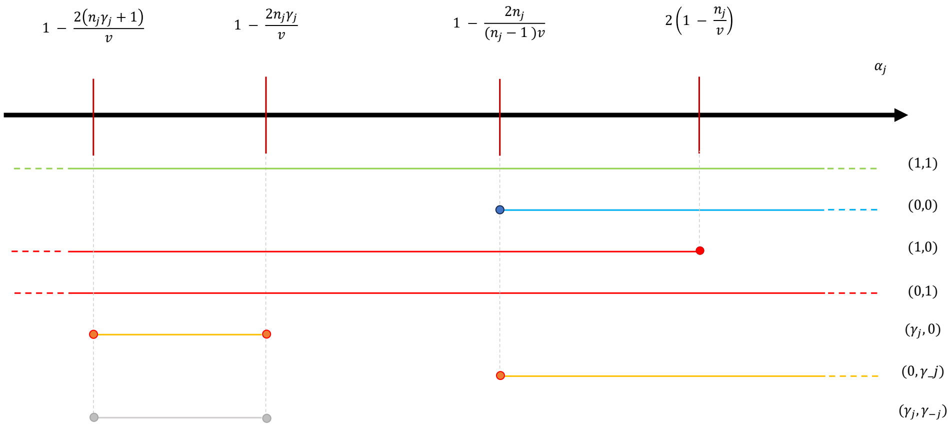

Figure 1.B7: Intervals of α_j sustaining second-period equilibria $\forall v \geq 2 \cdot max\left\{n_j, n_{-j}\right\}$.

2 The Set of Pure-Strategy Equilibria in Max-Min Two-Group Contests with a Private Good Prize

Ista, mi Lucili, condenda in animum sunt, ut contemnas voluptatem ex plurium assessione venientem. Multi te laudant: ecquid habes cur placeas tibi, si is es quem intellegant multi? Introrsus bona tua spectent. Vale.

(Lucius Annaeus Seneca)

The Set of Pure-Strategy Equilibria in Max-Min Two-Group Contests with a Private Good Prize *

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Abstract

We characterize the set of pure-strategy equilibria in a deterministic group contest with the weakest-link impact function and a private good prize, complementing the results obtained by Lee (2012) and Chowdhury, Lee, and Topolyan (2016). We consider a two-stage two-group model, where in the first stage the agents simultaneously choose the sharing rule, while in the second stage they choose efforts. We find that there are continua of subgame perfect equilibria, which means that in equilibrium players' behavior is indeterminate. By additional restrictions on the effort levels of each class of effort equilibria, we are able to computationally characterize the set of subgame perfect Nash equilibria in pure strategies.

JEL classification: D74; D71; C72

Keywords: Group contests; Sharing rules; Weakest-link; Indeterminacy

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2.1 Introduction

Competition among groups of agents takes place in many areas of socioeconomic activities, including rent-seeking, military conflict, labor markets, investment in R&D, electoral competition, litigation, and sports. Compactness and unity of a group crashing with the interests of an opposing group can be justified in terms of complementarities within the group. The scenario of perfect complementarity has been modeled by the so-called weakest-link impact function in the group contest literature. In many settings the performance of every individual in charge of a single link is crucial for the fate of the group: research teams where the specific expertise of every member is crucial in formulating a grant proposal to receive contestable funds; defensive lines in a country are as strong as their weakest points. Supply chains and logistics are another example, where processes might collapse even for the dysfunctioning of a single unit. Performance in team sports such as soccer, baseball, basketball, American football and cycling might depend on single athletes playing at their best in their unique role in the team. Under perfect complementarities, we expect alignment of choices within the group and we envisage free-riding issues plaguing groups to be less severe than in the perfect substitutability case which materializes in the simple summation formula. On the other hand, the nature of the prize contested influences the degree of complementarity within a group. Public goods favor common interests in the outcome of the competition, whereas private goods may elicit idiosyncratic incentives. Moreover, private goods raise the question of the optimal sharing of a prize within a group, thus of individual incentivization. Questions addressing how a bonus should be split among the members of a project team within a firm or how the monetary prize in a group sport competition should be assigned to the single athletes are of potential interest. At first glance, it is not clear the impact of selective incentives in the presence of complementarities, for they may promote heterogeneous choices within a group.

In this paper, we build on the contribution due to D. Lee (2012) and Chowdhury et al. (2016), by considering a deterministic two-group contest with the weakest-link, which the authors of the latter paper term “max-min group contest”, and a private good prize in place of a public good one. Thus, prize sharing within the winning group plays a crucial role, so that we consider two stages: the first one about the setting of a sharing rule parameter according to a utilitarian social welfare function and the second one about simultaneous and independent effort choices. We do focus on equilibria in pure strategies only, instead of both pure and mixed strategies as in Chowdhury et al. (2016). As in all-pay auctions with nonmonotonic payoffs analyzed by Chowdhury (2017), deterministic group contests with non-standard impact functions have equilibria in pure strategies, so that we maintain they deserve special attention. These contests display payoffs not strictly monotone in players' bids which enable the existence of pure strategy equilibria, as in the individualistic all-pay auction with a random threshold for winning and a bid-dependent prize value analyzed by Chowdhury (2017). We find that the result by Chowdhury et al. (2016) about the existence of only within-group symmetric Nash equilibria in pure strategies is robust to the introduction of a private good prize. The sharing rule at equilibrium casts a balance between equity and incentives, depending on whether it has to discourage wasteful effort or to avoid free riding. Accordingly, diverse second-stage equilibria in terms of outcomes hold. We can conclude that there are continua of subgame perfect equilibria, which means that in equilibrium players' behavior is indeterminate. In this case, by additional restrictions on the effort levels of each class of effort equilibria, we are able to computationally characterize the set of subgame perfect equilibria in pure strategies. In our model, an advantaged group is the smaller group, for its members have a higher per-capita evaluation of the prize at any within-group symmetric equilibrium. Indeterminacy of equilibrium outcomes is so severe that there are equilibria in which the larger group, despite being disadvantaged, wins with certainty. However, by requiring that groups expect the disadvantaged team cannot win and that the Pareto efficient equilibria are played in the second period, we are able to find a continuum of subgame perfect Nash equilibria in pure strategies in which the advantaged group wins with certainty. We finally highlight that more specific details of the competition setting might deliver assumptions which refine the set of both second-period equilibria and subgame perfect equilibria in pure strategies.

The paper proceeds as follows. The next subsection quickly reviews the related literature. In Section 2.2, we present the basic model. In Section 2.3, we characterize the set of pure strategy Nash equilibria of the second effort stage. In Section 2.4, we discuss the players' behavior in the first stage, while, in Section 2.5, we characterize the set of subgame perfect equilibria of the entire game. Finally, Section 2.6 presents the conclusions.

2.1.1 Related Literature

To the best of our knowledge, the first paper to use a weakest-link impact function in a strategic setting was Hirshleifer (1983), where the author showed that in this case, underprovision of the public good tends to be considerably moderated compared to the cases of perfect substitutability (summation) and of best-shot impact functions. Cornes and Hartley (2007) build more formally on the same approach, characterizing the set of equilibria. Both of these papers do not consider a group contest setting. Sheremeta (2010) analyzes a group contest model experimentally considering three possible impact functions - weakest-link, best-shot, and perfect substitutability - combining these impact functions with a Tullock contest success function. The results of the experiment confirm that in weakest-link contests there is almost no free riding and all players expend similar positive efforts. D. Lee (2012) considers a group contest with a group-specific public good prize, weakest-link and a general non-deterministic contests success function. Within a group contest model, Kolmar and Rommeswinkel (2013) generalizes previous works considering a CES impact function, so that the weakest-link is a particular case, and a Tullock contests success function. Sheremeta (2010), D. Lee (2012), and Kolmar and Rommeswinkel (2013) share the use of a non-deterministic contest success function, while Barbieri, Malueg, and Topolyan (2014), Chowdhury et al. (2016), Barbieri and Malueg (2016), Barbieri, Kovenock, et al. (2019), Barbieri and Topolyan (2021) and Barbieri and Topolyan (2024) consider a deterministic contest success function, so that they are more directly related to this paper. However, these papers consider (group-specific) public goods, so there is no need for a within groups sharing rule. Accordingly, our results with a private prize complement the results obtained by Chowdhury et al. (2016), since they depict the set of Nash equilibria in both pure and mixed strategies in a "max-min group contest" with a public good prize. The authors themselves envisage the possibility of introducing a private good prize in their theoretical framework, as a future line of research. In our paper we will adopt a standard sharing rule from the seminal contribution of Nitzan (1991). In that paper, the Tullock (1980) model is extended to incorporate two groups who compete by exerting non-refundable effort for a prize which is shared according to an exogenous sharing rule. S. Lee (1995) builds on the work due to Nitzan (1991) by endogenizing the choice of the sharing rule in a two-stage model solved by subgame perfection. We depart from both papers, for they share a summation impact function and a probabilistic contest success function. Ueda (2002) investigates how and when some contestants decide to withdraw from rent-seeking in such models. Hwan Baik and S. Lee (1997) extends the sharing rule parameters from the restricted, i.e. $[0, 1]$, to the unrestricted case, studying group formation choices.[2.1] More recent contributions on "restricted sharing rules" are Nitzan and Ueda (2018) and Kobayashi and Konishi (2021), which share the probabilistic contest-success function, but differ for the former focus on heterogeneous prize valuations, while the latter on effort complementarities within groups modeled by a CES effort aggregator.

Balart, Flamand, and Troumpounis (2016) analyze various degrees of privateness of the prize and its impact on optimal sharing rules. Balart, Flamand, Gürtler, et al. (2018) study a group contest where heterogeneous groups choose how to divide the prize sequentially and not simultaneously. Con-

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[2.1] Analysis of our model under this assumption is available at SSRN: http://dx.doi.org/10.2139/ssrn.4890131 .

versely, we study a model where sharing rule parameters are adopted simultaneously under complete information. Private information in group contests with private good prizes has been the focus of works such as Baik and S. Lee (2007), Nitzan and Ueda (2011), Baik (2016), and Baik and D. Lee (2012).

Prize allocation within groups under both complete and incomplete information and diverse assumptions about the cost of effort and heterogeneity is studied by Trevisan (2020). In our model, the unique source of heterogeneity is the number of players within each group. Recently, the trade-off between equality and effort incentivization in diverse teams has been tested experimentally in Ma and Trevisan (2024). Other experimental works include Cason et al. (2017), and Brookins et al. (2015) where complementarities are investigated.

2.2 A Continuous Effort Max-Min Two-Group Contest with a Finite Number of Agents

Consider a simple two-group model that sums up the main characteristics of group contests under complete information. The model is defined by the following elements:

1. two **groups**, denoted by $j \in \{1, 2\}$;
2. each group has $n_j \geq 4$ members in each group. The total number of agents is $N = n_1 + n_2$. As notation device, let us write ij for **agents** $i \in \{1, \ldots, n_j\}$ of group j;
3. the **effort** of member $i \in \{1, ..., n_j\}$ in group $j \in \{1, 2\}$, to increase the possibility of getting the prize, is denoted by $x_{ij} \in \mathbb{R}_+$. Let $\mathbf{x}_j$ be the vector of all agents' efforts of group j, and $\mathbf{x}$ the vector of all agents' efforts. Moreover, let define the average exerted effort in group j as
$$\gamma_j = \frac{1}{n_j} \sum_{i=1}^{n_j} x_{ij} \in \mathbb{R}_+;$$
4. a private **prize** worth v to be allocated to one of the groups;
5. the **impact function** of group j is given by the weakest-link technology
$$X_j = \min \{x_{ij} \in \mathbb{R}_+, i \in \{1, ..., n_j\}\};$$

6. the **contest success function** is given by the *all-pay auction:*

$$p_j\left(X_1, X_2\right) = \begin{cases} 1 & if \quad X_j > X_{-j} \\ \frac{1}{2} & if \quad X_j = X_{-j} \\ 0 & if \quad X_j < X_{-j}; \end{cases}$$

7. the **sharing rule**, such that if group $j \in \{1, 2\}$ wins, then a member $i \in \{1, \ldots, n_j\}$ gets a share of the prize

$$q_{ij}\left(x_{1j}, ..., x_{n_j j}\right) =$$

$$= \begin{cases} \underbrace{(1-\alpha_j)}_{incentivation\ part} \quad \frac{x_{ij}}{\sum_{i=1}^{n_j} x_{ij}} + \quad \underbrace{\alpha_j}_{equalizing\ part} \quad \frac{1}{n_j} & if \quad \sum_{i=1}^{n_j} x_{ij} > 0 \\ \frac{1}{n_j} & otherwise \end{cases}$$

where $\alpha_j \in [0, 1]$ is the share of the prize that the members of the winning team get independently of their effort.

In particular, $\alpha_j \in [0, 1]$ implies that the sharing rule amounts to pure within-group redistribution: [2.2]

- $\underline{\alpha_j \text{ towards } 0}$ means that the main bulk of the prize share is distributed according to each agent's effort, the incentivization part, while there is almost no uniform redistribution, the equalizing part;
- $\underline{\alpha_j \text{ towards } 1}$ means that the main bulk of the prize share is distributed according to a uniform redistribution, while almost no part is distributed as a reward for each agent's effort;
- more generally $\underline{\alpha_j \text{ from } 0 \text{ to } 1}$ is able to span the entire set of redistribution/rewarding rules, from no uniform redistribution to no effort rewarding.

8. the individual **costs of effort** $C_{ij}\left(x_{ij}\right) = x_{ij}$;

9. the **timing**, there are two stages:

   i. in the first stage, players non-cooperatively choose the equilibrium sharing rule within each group α_j: if the second-period equilibria are within-group symmetric, the non-cooperative optimal choice of the sharing rule is equivalent to the optimal choice according to an

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[2.2] In a previous version of the present paper we considered the unrestricted case $\alpha_j \in \mathbb{R}$ as well, available at SSRN: https://ssrn.com/abstract=4890131 or http://dx.doi.org/10.2139/ssrn.4890131.

utilitarian welfare function, for the latter is a simple linear transformation of the players' payoffs;

ii. in the second stage all the members of the groups observe the first stage choices (α_1, α_2) and choose simultaneously and independently their effort x_{ij} and the prize is allocated to one of the two groups according to the contest success function.

Note that this timing structure, used in most papers that endogenize the sharing rule, implies that the groups can precommit to an equilibrium sharing rule, without any subsequent renegotiation. Let us stress that this sequential structure means that, once established, the sharing rule cannot be object of negotiation within or across the groups, there is a commitment to respect the choices of the first stage. The same underlying assumption is present in many works on group contests with a public good prize, such as Nitzan (1991), S. Lee (1995) and Nitzan and Ueda (2018). [2.3]

As a consequence of these modeling characteristics, player ij has the **payoff**

$$\pi_{ij}\left(\alpha_j, \alpha_{-j}, x_{1j}, ..., x_{n_j j}, x_{1-j}, ..., x_{n_{-j}-j}\right) = p_j q_{ij} v - x_{ij} =$$

$$= \begin{cases} \left[(1-\alpha_j)\frac{x_{ij}}{\sum_i x_{ij}} + \alpha_j \frac{1}{n_j}\right] v - x_{ij} & if \quad \min\{\mathbf{x}_j\} > \min\{\mathbf{x}_{-j}\} \\ \frac{1}{2}\left[(1-\alpha_j)\frac{x_{ij}}{\sum_i x_{ij}} + \alpha_j \frac{1}{n_j}\right] v - x_{ij} & if \quad \min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} \\ -x_{ij} & if \quad \min\{\mathbf{x}_j\} < \min\{\mathbf{x}_{-j}\} \end{cases}$$

Now, we are able to provide a formal definition of the group contest under analysis.

DEFINITION 2.1. *A* Max-Min Group Contest *$CMMGC$ is a two-stage game $\langle\{1,2\}, N, S_j, E_{ij}, \pi_{ij}\rangle$ defined by*

1. *the set of groups $\{1,2\}$;*
2. *the set of players $N = \{1, ..., n_1 + n_2\}$;*
3. *the set of first-period actions $S_j = [0,1]$: for each group j, the choice of α_j in the sharing rule*;
4. *the set of second-period actions $E_{ij} = \mathbb{R}_+$: for each player ij, the choice of the effort x_{ij}. Note that because of the sequential structure of the game, each player ij second-period **strategies** are maps $[0,1] \times [0,1] \mapsto \mathbb{R}_+$;*

[2.3] Of course, it is possible to consider different sequential structures, as in Katz and Tokatlidu (1996), for instance.

5. *the payoff functions for each player $ij \in N$*

$$\pi_{ij}(\boldsymbol{\alpha}, \mathbf{x}) = p_j q_{ij} v - x_{ij} =$$

$$= \begin{cases} \left[(1-\alpha_j)\frac{x_{ij}}{\sum_i x_{ij}} + \alpha_j \frac{1}{n_j}\right] v - x_{ij} & if \quad \min\{\mathbf{x}_j\} > \min\{\mathbf{x}_{-j}\} \\ \frac{1}{2}\left[(1-\alpha_j)\frac{x_{ij}}{\sum_i x_{ij}} + \alpha_j \frac{1}{n_j}\right] v - x_{ij} & if \quad \min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} \\ -x_{ij} & if \quad \min\{\mathbf{x}_j\} < \min\{\mathbf{x}_{-j}\} \end{cases}$$

where $\boldsymbol{\alpha}$ and $\mathbf{x}$ are, respectively, the vector of first and second period actions.

The notation used in this paper is summed up in table 2.1.

Variable	Meaning
ij	agent i of group j
$\{1, ..., n_j\}$	set of agents in group j
x_{ij}	effort of agent i in group j
$X_j = \min\{x_{ij} \in \mathbb{R}_+, i \in \{1, ..., n_j\}\}$	impact of effort of all agents in group j
$\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2)$	vector of efforts of all agents
$C_{ij}(x_{ij}) = x_{ij}$	cost of effort for agent i of group j
$p_j(X_1, X_2)$	probability of group j of winning the contest
$q_{ij}(x_{1j}, ..., x_{n_j j})$	sharing rule for agent i of group j
$\alpha_j \in [0, 1]$	equalizing part of the sharing rule
$\boldsymbol{\alpha}$	vector of α_j for $j \in \{1, 2\}$
$\pi_{ij}(\boldsymbol{\alpha}, \mathbf{x})$	payoff function of agent i of group j
$\gamma_j = \frac{1}{n_j}\sum_{i=1}^{n_j} x_{ij} \in \mathbb{R}_+$	average effort in group j

Table 2.1

2.3 The Set of Second-Period Equilibria

Without loss of generality, the second-stage equilibria are presented in terms of average effort in each group, i.e. as pairs

$$(\gamma_1, \gamma_2) \in \mathbb{R}_+ \times \mathbb{R}_+,$$

so that geometrically they can be represented in the non-negative quadrant.

Moreover, denote by

$$\gamma_j' = \frac{1}{n_j}\left(\sum_{i=1}^{n_j} x_{ij} \pm e_{ij}\right) \in \mathbb{R}_+ \quad \forall e_{ij} \in \mathbb{R}_{++}$$

the average effort in group j at any individual deviation from x_{ij}.

2.3.1 Characterization of the Set of Pure Strategy Nash Equilibria in the Second Period.

In this subsection, we characterize the full set of second-period Nash equilibria in pure strategies, to study the interplay within and between groups of the selective incentives induced by the sharing rules, which affect the incentives to free ride, with the strong complementarities, which favor the alignment of effort choice by teammates. Since the payoff functions are not smooth, we approach the problem by direct consideration of all possible strategy profiles, checking whether there is an individual incentive to deviate.

Without loss of generality, let us assume $n_2 \geq n_1$. Then we are able to obtain the following result.

PROPOSITION 2.1. *In the CMMGC, the set of the second-period pure strategy Nash equilibria of the game is characterized as follows:*

1. *when*

$$(\alpha_1, \alpha_2) \in \{1\} \times \{1\}$$

then there exists a pure strategy equilibrium such that

$$(\gamma_1, \gamma_2) = (0, 0)\,;$$

2. *when*

$$e \in \left(0, \frac{v}{2n_2}\right] \text{ and } (\alpha_1, \alpha_2) \in$$

$$\left[max\left\{0, 1 - \frac{2n_1^2 e}{(n_1 - 1)\, v}\right\}, 1\right] \times \left[max\left\{0, 1 - \frac{2n_2^2 e}{(n_2 - 1)\, v}\right\}, 1\right],$$

there exists a continuum of pure strategy equilibria such that

$$(\gamma_1, \gamma_2) = (e, e) \text{ with } \mathbf{x}_j = \mathbf{e};$$

3. *when*

$$e \in \left(0, \frac{(n_2-1)\,v}{n_2^2}\right] \text{ and } (\alpha_1, \alpha_2) \in [0,1] \times \left\{1 - \frac{n_2^2 e}{(n_2-1)\,v}\right\}$$

   there exists a continuum of second-period pure strategy equilibria such that

$$(\gamma_1, \gamma_2) = (0, e) \quad \text{such that } \mathbf{x}_2 = \mathbf{e};$$

4. *when*

$$e \in \left(0, \frac{(n_1-1)\,v}{n_1^2}\right] \text{ and } (\alpha_1, \alpha_2) \in \left\{1 - \frac{n_1^2 e}{(n_1-1)\,v}\right\} \times [0,1]$$

   there exists a continuum of pure strategy equilibria such that

$$(\gamma_1, \gamma_2) = (e, 0) \quad \text{such that } \mathbf{x}_1 = \mathbf{e}.$$

Proof. See 2.A.1. □

It is possible to give a natural interpretation to the equilibria above.

- $(\gamma_1, \gamma_2) = (0, 0)$ to be an equilibrium requires that there should not be an incentive for players $i \in \{1, \ldots, n_j\} \,\forall j \in \{1, 2\}$ to provide positive effort: with $\alpha_j = 1 \;\forall j \in \{1, 2\}$ the prize is shared uniformly irrespective of the individual contributions;
- $(\gamma_1, \gamma_2) = (e, e)$ with $\mathbf{x}_j = \mathbf{e}$ to be an equilibrium needs both upward and downward deviations not being profitable. Accordingly, for low levels of e, the sharing rule has to be sufficiently egalitarian, i.e. α_j close to 1, ensuring that there are no benefits from exerting more effort. Given the weakest-link technology, the only downward deviation rationalizable is the zero effort provision choice leading to the group defeat, which is prevented by ensuring that the prize in case of a tie is sufficiently high, that is $v > 2n_2 e$;
- $(\gamma_1, \gamma_2) = (0, e)$ such that $\mathbf{x}_2 = \mathbf{e}$ to be an equilibrium needs both upward and downward deviations not being profitable for group 2, so that a unique α_2 casting a balance between incentivization and egalitarianism is found. On the other hand, any deviation for players of group 1 is strictly dominated for any value of α_1;
- similarly for $(\gamma_1, \gamma_2) = (e, 0)$ such that $\mathbf{x}_1 = \mathbf{e}$.

We further stress that Nash equilibrium as an equilibrium concept does not prevent implausible equilibria from holding. In particular, $(\gamma_1, \gamma_2) = (0, e)$ such that $\mathbf{x}_2 = \mathbf{e}$ implies that group 2, namely the disadvantaged one in terms of individual prize share v/n_2 at a symmetric equilibrium, wins with certainty and the equilibrium is sustained by such beliefs.

From Proposition 2.1, two corollaries immediately follow.

COROLLARY 2.1. *In the CMMGC, there are no asymmetric second-period pure strategy Nash equilibria.*

COROLLARY 2.2. *In the CMMGC, if*

$$n_1 \geq 4 \text{ and } n_1 \in \{n_2 - 1, n_2\}$$

then, when

$$e \in \left[\frac{(n_1 - 1)\, v}{2n_1^2}, \frac{v}{2n_2}\right] \text{ and } (\alpha_1, \alpha_2) \in [0,1] \times [0,1]\,,$$

there exists a continuum of pure strategy equilibria such that

$$\left(\gamma_1^b, \gamma_2^b\right) = (e, e) \text{ with } \mathbf{x}_j = \mathbf{e}.$$

Proof. See 2.A.2 . □

REMARK 2.1. *Note that when $n_1 \geq 4$ and $n_1 \in \{n_2 - 1, n_2\}$, all four types of effort equilibria exist. In figure 2.1, we represent the region of sharing rules with the consequent effort equilibria.*

Although the sharing rules do not affect players' payoffs at any within-group symmetric equilibrium, for they boil down to $1/n_j$, we can assess the trade-off between effort-incentivization and equity within groups by looking at how they shape the opportunity cost of a given level of effort:

- a fully equalizing sharing rule not only prevents any wasteful effort from being profitable at the zero-effort tying equilibrium, but it also avoids any upward deviation at any positive-effort tying equilibrium;
- a sufficiently equalizing sharing rule sustains equilibria in which groups tie on a positive level of effort, so that there are no substantial gains from providing effort within the group. Downward deviations are prevented if the prize is sufficiently high;

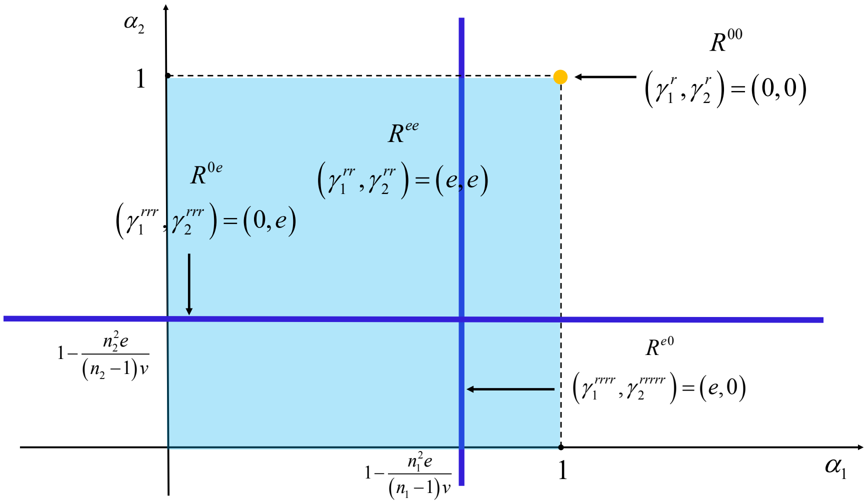

Figure 2.1: Sharing rules regions and effort equilibria. R^{00}, R^{ee}, R^{0e} and R^{e0} stand for the regions of existence of $(\gamma_1^r, \gamma_2^r) = (0,0)$, $(\gamma_1^{rr}, \gamma_2^{rr}) = (e,e)$, $(\gamma_1^{rrr}, \gamma_2^{rrr}) = (0,e)$ and $(\gamma_1^{rrrr}, \gamma_2^{rrrr}) = (e,0)$, respectively, in the $\alpha_1 \times \alpha_2$ space.

- a specific intermediate value of the equalizing part of the sharing rule sustains the equilibrium in which a group wins with certainty: the higher the effort level, the lower the equalizing part accruing to a downwardly deviating agent. However, a specific value of the sharing rule parameter is needed to prevent both upward and downward deviations from a given level of effort;

- finally, casting a balance between effort-incentivization and equity is immaterial for a group losing with certainty, for any individual deviation would not change the probability of winning due to the weakest-link impact function.

Hence, a fully equalizing sharing rule in both groups could guarantee a tie on the zero-effort equilibrium or on a positive-effort one. An intermediate value of the equalizing part of sharing rule, decreasing in the effort level, could ensure both a victory and a loss with certainty or with a one-half probability for both groups. Finally, a fully incentivizing sharing rule could lead to both a tie on a positive effort level and to a loss with certainty for the belonging group.

2.4 The Set of First-Period Equilibria

When $n_1 \geq 4$ and $n_1 \in \{n_2 - 1, n_2\}$, then for all values of (α_1, α_2) it is possible to express the continuation payoffs, so that it is possible to try to calculate the subgame perfect equilibria of the entire game. Accordingly, we assume $n_1 \geq 4$ and $n_1 \in \{n_2 - 1, n_2\}$ for the rest of the analysis.

From Corollary 2.1 we know there are no asymmetric second-period equilibria in $CMMGC$, so that, to derive the equilibrium sharing rules as commonly done in the literature,[2.4] we consider their optimal choice in each group according to an utilitarian welfare function

$$\pi_j^C \left(\alpha_j, \alpha_{-j} | \gamma_1^e, \gamma_2^e\right) = \sum_{i=1}^{n_j} \pi_{ij}^C \left(\alpha_j, \alpha_{-j} | \gamma_1^e, \gamma_2^e\right).$$

REMARK 2.2. *It is straightforward to check that the existence of only within-group symmetric equilibria in the second period guarantees that players in both groups attain ex post the same payoff. The expected payoff according to an utilitarian welfare function is a linear transformation of the generic individual player expected payoff, thus in each group the sharing rule preferred according to an utilitarian welfare function coincides with the sharing rule preferred by any of the members of the group. Accordingly, the non-cooperative equilibrium choice leads to the same sharing rule preferred according to an utilitarian welfare function. Then, conditional to any possible sharing rule* α_{-j} *chosen by the rival group* $-j$*, the desired utilitarian redistribution scheme and the non-cooperative choice by individual group members lead to the same best-response* $\alpha_j = BR_j\left(\alpha_{-j}\right)$*.*

In the first period, the players may have different expectations on the second-period equilibria belonging to the Cartesian product over $\alpha_1 \times \alpha_2$ space, as depicted in Figure 2.1. Nevertheless, some of these expectations might be implausible, for they entail that the disadvantaged group wins with certainty when $(\gamma_1, \gamma_2) = (0, e)$. For instance:

- if players in both groups expect the second-period equilibrium in which group 2 wins and the equilibrium in which groups tie on a positive level of effort in the $\alpha_1 \times \alpha_2$ space, then the best-response for group 2 is selecting α_2 ensuring $(\gamma_1, \gamma_2) = (0, e)$, making the first-period choice of group 1 irrelevant and ensuring group-2 victory. Such expectations are easily conceivable in the case $n_1 = n_2$, namely when no group is advantaged. However, they are quite unreasonable if $n_1 < n_2$, for they imply the advantaged group expecting to lose in the second-period;
- if players in both groups expect the second-period equilibrium in which group 1 wins, the equilibrium in which groups tie on a positive level of effort, and the equilibrium in which group 2 wins in the $\alpha_1 \times \alpha_2$ space, then best-responses of both groups select (α_1, α_2) ensuring the group-1 victory. However, note that such expectations about the second-period equilibria sustain the

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[2.4] See S. Lee (1995), for instance.

first-period equilibrium ensuring the victory of group 2 as well, which is the disadvantaged one for $n_1 < n_2$.

Other examples are available in an analytical form in Appendix 2.C.1.

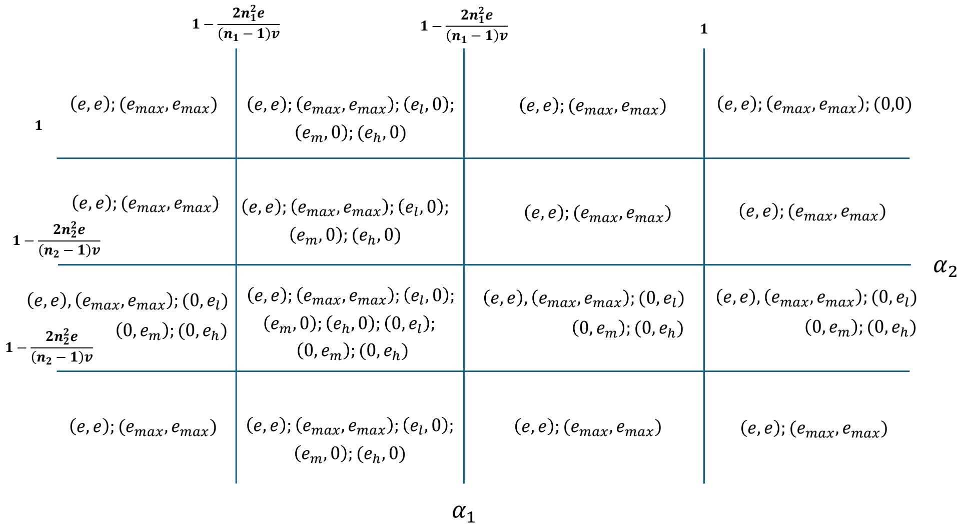

Figure 2.2: Second-period pure Nash equilibria in $\alpha_1 \times \alpha_2$ space with $n_1 \in \{n_2 - 1, n_2\}$.

Let us denote

- (e_{max}, e_{max}) for (e, e) such that $e = \frac{v}{2n_2}$;
- $(e_l, 0)$ any $(e, 0)$ such that $e \in \left(0, \frac{v}{2n_1}\right)$;
- $(e_m, 0)$ for $(e, 0)$ such that $e = \frac{v}{2n_1}$;
- $(e_h, 0)$ any $(e, 0)$ such that $e \in \left(\frac{v}{2n_1}, \frac{(n_1-1)v}{n_1^2}\right]$;
- $(0, e_l)$ any $(0, e)$ such that $e \in \left(0, \frac{v}{2n_2}\right)$;
- $(0, e_m)$ for $(0, e)$ such that $e = \frac{v}{2n_2}$;
- $(0, e_h)$ any $(0, e)$ such that $e \in \left(\frac{v}{2n_2}, \frac{(n_2-1)v}{n_2^2}\right]$.

Notwithstanding the presence of continua of equilibria in the second period, within each class of equilibria the continuation payoffs can be ordered. For instance, consider the (e, e) equilibrium which is sustained by any $(\alpha_1, \alpha_2) \in [0, 1] \times [0, 1]$. In this class of equilibria, the effort level e can take up to sixteen different values in the sixteen regions defined by the thresholds equilibrium conditions, as shown in figure 2.2. The same argument applies to equilibria $(e_l, 0), (e_m, 0), (e_h, 0)$ and $(0, e_l), (0, e_m), (0, e_h)$, where the effort level can take up to four different values in the four regions.

However, for computational reasons, we limit ourselves to the case where there is just one effort level over the sixteen regions determined by the equilibrium threshold conditions for each class of equilibria. [2.5]

Accordingly, we employ a simple algorithm in MATLAB to compute the set of first-period equilibria, and thus the set of subgame perfect Nash equilibria. [2.6]Nonetheless, given that the between-groups and within-groups symmetric equilibrium (e, e) is sustained by any $(\alpha_1, \alpha_2) \in [0,1] \times [0,1]$, even in the absence of the aforementioned restriction it is straightforward to derive the following proposition.

PROPOSITION 2.2. *In the CMMGC, there exists at least a continuation-payoffs matrix such that the equilibrium sharing rules are:*

$$(\alpha_1^*, \alpha_2^*) = [0,1] \times [0,1] \ .$$

As a way not to weigh down the exposition, we report in Appendix 2.B the results regarding the preference ordering according to utilitarian welfare functions.

Overall, the presence of multiple second-period equilibria in the same areas of the $\alpha_1 \times \alpha_2$ space is responsible for the crucial role of beliefs in determining the set of first-period equilibria, and thus equilibrium outcomes. Depending on the subset of second-period equilibria expected in the first-period, there are different best-responses selecting first-period equilibria. As highlighted in Section 2.3, Nash equilibrium as a solution concept does not prevent implausible second-period equilibria from holding, that is equilibria in which the disadvantaged group, in terms of per capita evaluation of the prize at a symmetric equilibrium, wins with certainty.

Therefore, the beliefs in the first period about the continuation game crucially affect the optimal choice of prize redistribution upon winning adopted. As already stressed, any sharing rule is *de facto* egalitarian at any equilibrium. However, optimal sharing rules select the second-stage equilibrium maximizing the expected continuation-payoffs which in turn depend on the subset of second-period equilibria expected, and they shape the opportunity cost of a deviation as described in Section 2.3. The indeterminacy result is twofold: first, for every subset of second-period equilibria expected, there are diverse best-responses in the first-period, and thus very different equilibrium outcomes; second, a continuum of sharing rules sustain the same second-period equilibrium, so that there is an indeterminate

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[2.5] It is possible to show that our simplifying assumption reduces the number of continuation payoff matrices from 30622000000000000000000 to 96000000.

[2.6] Codes and output are available at https://data.mendeley.com/datasets/ybjbvskvs3/1 .

number of optimal sharing rules.

For applications, “optimism” and “pessimism” in the form of plausible and implausible beliefs, i.e. ensuring the advantaged group not losing or losing with certainty, are crucial to determine equilibrium outcomes when there are within-group complementarities. Moreover, there are many (a continuum of) optimal ways in which the prize contested can be split among teammates, ensuring the same equilibrium outcomes. Therefore, there are many optimal ways in which the trade-off between incentivization and equity can be solved within groups when there are technology-driven complementarities. Finally, as shown by Balart, Chowdhury, et al. (2017), sharing rules à la Nitzan (1991) can be employed as a contest success function that is a convex combination of a standard Tullock CSF with the r exponent equal to 1 and a fair lottery in a individualistic contest. Accordingly, our group contest can be interpreted as a nested one where players of the winning group compete according to a contest success function whose level of noise is set in the first stage: a more egalitarian sharing rule translates into a contest success function with a higher level of noise for the within-group individualistic contest.

2.5 The Set of Subgame Perfect Equilibria

From previous results, it is immediate to derive the following proposition.[2.7]

PROPOSITION 2.3. *In the CMMGC, there are continua of subgame perfect Nash equilibria in pure strategies.*

Clearly, we reached an indeterminacy result, not just in terms of continuum of equilibria, but regarding equilibrium outcomes as well, so that the predictive power of our model is limited. The multiplicity of second-period Nash equilibria drives this result. Therefore, we try to refine the set of beliefs sustaining the set of second-period equilibria, by requiring that players expect that the smaller group is advantaged and that when there are more equilibria in the same subset of the $\alpha_1 \times \alpha_2$ space, then the Pareto dominant equilibrium is expected.

ASSUMPTION 2.1. *Let us assume:*

- $4 \leq n_1 < n_2$;
- *players in both groups expect that the disadvantaged group, i.e. group 2, cannot win with certainty;*
- *players in both groups expect the Pareto efficient effort equilibria* $(e_l, 0)$ *and* $(0, 0)$*, other than* (e, e) *in the remaining regions of the* $\alpha_1 \times \alpha_2$ *space.*

[2.7] In 2.C.2 we report some examples of subgame perfect equilibria.

Accordingly, we get the following result.

PROPOSITION 2.4. *Under Assumption 2.1, there is a continuum of subgame perfect Nash equilibria in pure strategies in which the advantaged group, namely group 1, wins with certainty, such that:*

$$e \in \begin{cases} \left[\frac{(n_1-1)v}{2n_1^2}, \frac{v}{2n_2}\right) & \textit{if } (\gamma_1,\gamma_2)=(e,e) \ \textit{ with } \mathbf{x}_j=\mathbf{e} \\ \left(0, \frac{v}{2n_1}\right) & \textit{if } (\gamma_1,\gamma_2)=(e,0) \end{cases},$$

$$\left(\alpha_1^{SGP}, \alpha_2^{SGP}\right) = \left\{1 - \frac{n_1^2 e}{(n_1-1)\,v}\right\} \times [0,1]$$

$$\left(\gamma_1^{SGP}, \gamma_2^{SGP}\right) =$$

$$\begin{cases} (e,e) \ \textit{ with } \mathbf{x}_j = \mathbf{e} & otherwise \\ (0,0) & if \quad (\alpha_1,\alpha_2) \in \{1\} \times \{1\} \\ (e,0) & if \quad (\alpha_1,\alpha_2) \in \left\{1 - \frac{n_1^2 e}{(n_1-1)v}\right\} \times [0,1] \end{cases}$$

with

$$\left(\alpha_1^{SGP}, \alpha_2^{SGP}\right) = \left\{1 - \frac{n_2^2 e}{(n_2-1)\,v}\right\} \times [0,1] \ \textit{ and } \ \left(\gamma_1^{SGP}, \gamma_2^{SGP}\right) = (e,0)$$

as equilibrium outcomes.

2.6 Conclusions

We complemented the results of Chowdhury et al. (2016) by characterizing the set of pure strategy equilibria in a deterministic two-group contest with the weakest-link impact function, continuous efforts and a private good prize. We find that the non-existence of within-group asymmetric Nash equilibria in pure strategies in the effort stage is robust to the introduction of a private good prize. Moreover, we are able to state that there are continua of subgame perfect equilibria, which means that in equilibrium players' behavior is indeterminate. By additional restrictions on the effort levels of each class of effort equilibria, we are able to computationally characterize the set of subgame perfect equilibria in pure strategies. Given the extent of the set of equilibria, indeterminacy of equilibrium behavior and outcomes affects the predictive feature of the model. As a matter of fact, Nash equilibrium as an equilibrium concept does not avoid implausible expectations about second-period equilibria from holding. Accordingly, we find equilibria in which players of the group advantaged in terms of per-capita prize valuation expect the other group to win in the second period. Selective incentives fail to force a unique equilibrium, for, proceeding backwardly, there are multiple equilibria in the second period stemming from the within-group perfect complementarity determined by the weakest-link impact function. This multiplicity marks a sharp difference with works on probabilistic group contests with additive impact function, private good prize and an endogenously determined sharing rule under complete information, such as S. Lee (1995), Hwan Baik and S. Lee (1997), and Nitzan and Ueda (2018). Second-period equilibria are of particular interest as they enable us to study the trade-off between effort incentivization and "egalitarianism" within groups. However, their characterization is incomplete, since we focused on equilibria in pure strategies only. We leave the characterization of equilibria in mixed strategies for future research. By requiring that players in both groups expect Pareto efficient equilibria to be played and the weaker group not to win in the second period, we determine the existence of a continuum of subgame perfect equilibria in which the advantaged group wins with certainty.

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2.A Proofs

2.A.1 Proof of Proposition 2.1

Let consider the different cases.

1. Suppose

$$(\gamma_1, \gamma_2) = (0, 0) \ .$$

Then,

$$x_{ij} = min\left\{\mathbf{x}_j\right\} = min\left\{\mathbf{x}_{-j}\right\} \text{ and } \sum_{i=1}^{n_j} x_{ij} = \sum_{i=1}^{n_{-j}} x_{i-j} = 0$$

so that

$$\pi_{ij}\left(\gamma_j, \gamma_{-j} | \boldsymbol{\alpha}\right) = \frac{1}{2}\left[\frac{1}{n_j}\right] v \ .$$

If agent ij deviates to $x_{ij} = e' > 0$, then

$$x_{ij} = min\left\{\mathbf{x}_j\right\} = min\left\{\mathbf{x}_{-j}\right\} \text{ and } \sum_{i=1}^{n_j} x_{ij} = e', \sum_{i=1}^{n_{-j}} x_{i-j} = 0$$

so that the deviation payoff is

$$\pi_{ij}^D\left(\gamma_j, \gamma_{-j} | \boldsymbol{\alpha}\right) = \frac{1}{2}\left[\left(1-\alpha_j\right)\frac{e'}{e'} + \frac{\alpha_j}{n_j}\right] v - e' \ .$$

Hence, for any player ij there is no incentive to deviate if and only if

$$\frac{v}{2n_j} \geq \frac{1}{2}\left[\left(1-\alpha_j\right)\frac{e'}{e'} + \frac{\alpha_j}{n_j}\right] v - e' \Leftrightarrow \alpha_j \geq \lim_{e' \to 0} 1 - \frac{2n_j e'}{\left(n_j - 1\right) v} \Leftrightarrow \alpha_j \geq 1 \ .$$

2. Suppose

$$(\gamma_1, \gamma_2) = (e, e) \text{ with } e \in \mathbb{R}_{++} \text{ and } \mathbf{x}_j = \mathbf{e} \ .$$

Then,

$$x_{ij} = min\left\{\mathbf{x}_j\right\} = min\left\{\mathbf{x}_{-j}\right\}, \sum_{i=1}^{n_j} x_{ij} = n_j\gamma_j \text{ and } \sum_{i=1}^{n_{-j}} x_{i-j} = n_{-j}\gamma_{-j}$$

so that

$$\pi_{ij}\left(\gamma_j, \gamma_{-j} | \boldsymbol{\alpha}\right) = \frac{1}{2}\left[\frac{1}{n_j}\right] v - e \ .$$

If agent ij deviates to $x_{ij} = e' = 0$, then

$$x_{ij} = min\left\{\mathbf{x}_j\right\} < min\left\{\mathbf{x}_{-j}\right\} and \sum_{i=1}^{n_j} x_{ij} = \left(n_j - 1\right)\gamma_j \text{ and}$$

$$\sum_{i=1}^{n_{-j}} x_{-j}\left(i\right) = n_{-j}\gamma_{-j}$$

so that the deviation payoff is

$$\pi_{ij}^{D}\left(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}\right) = 0 \leq \frac{v}{2n_j} - e \Leftrightarrow v \geq 2n_j e \ .$$

Note that any deviation $0 < e' < e$ is strictly payoff-dominated by $e' = 0$, so that we can now take into account upward deviations only. If agent ij deviates to $x_{ij} = e' > e$, then

$$x_{ij} = min\left\{\mathbf{x}_j\right\} = min\left\{\mathbf{x}_{-j}\right\}, \sum_{i=1}^{n_j} x_{ij} = \left(n_j - 1\right)\gamma_j + e' \text{ and}$$

$$\sum_{i=1}^{n_{-j}} x_{-j}\left(i\right) = n_{-j}\gamma_{-j}$$

so that the deviation payoff is

$$\pi_{ij}^{D}\left(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}\right) = \frac{1}{2}\left[\left(1-\alpha_j\right)\frac{e'}{\left(n_j-1\right)e + e'} + \frac{\alpha_j}{n_j}\right]v - e' < \frac{v}{2n_j} - e \Leftrightarrow$$

$$\Leftrightarrow \alpha_j \geq \lim_{\epsilon \to 0} 1 - \frac{2n_j e}{v} - \frac{2n_j\left(e+\epsilon\right)}{\left(n_j-1\right)v} \Leftrightarrow \alpha_j \geq 1 - \frac{2n_j e}{v} - \frac{2n_j e}{\left(n_j-1\right)v} \ .$$

Therefore,

$$\left(\gamma_1, \gamma_2\right) = \left(e, e\right) \text{ with } e \in \mathbb{R}_{++} \text{ and } \mathbf{x}_j = \mathbf{e}$$

is a Nash equilibrium for any

$$\alpha_j \in \left[1 - \frac{2n_j\gamma_j}{v} - \frac{2n_j\gamma_j}{(n_j-1)v}, 1\right] \text{ and } v \geq 2n_j\gamma_j \ .$$

3 and 4. Suppose

$$\left(\gamma_j, \gamma_{-j}\right) \in \left(0, e\right) \quad \forall e \in \mathbb{R}_{+} \text{ and } \mathbf{x}_j = \mathbf{e} \ .$$

Then

$$x_{ij} = \min\left\{\mathbf{x}_j\right\} = 0 < \min\left\{\mathbf{x}_{-j}\right\} = e \text{ and}$$

$$\sum_{i=1}^{n_j} x_{ij} = 0, \sum_{i=1}^{n_{-j}} x_{i-j} = n_{-j}e \ ,$$

so that

$$\pi_{ij}\left(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}\right) = 0.$$

If agent ij deviates to $x_{ij} = e'$, $\forall e' \in \mathbb{R}_{++}$, then

$$\min\left\{\mathbf{x}_j\right\} = 0 < \min\left\{\mathbf{x}_{-j}\right\} = e \text{ and } \sum_{i=1}^{n_j} x_{ij} = e', \sum_{i=1}^{n_{-j}} x_{ij} = n_{-j}e$$

so that the deviation payoff is

$$\pi_{ij}^{D}\left(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}\right) = -e' \ .$$

Hence, for any player ij there is no incentive to deviate. On the other hand,

$$\pi_{i-j}\left(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}\right) = \frac{1}{n_{-j}}v - e \ .$$

If agent $i-j$ deviates to any $e' \in \mathbb{R}_{++} \setminus \{e\}$, then

$$min\left\{\mathbf{x}_j\right\} = 0 < min\left\{\mathbf{x}_{-j}\right\} \in \mathbb{R}_{++} \text{ and}$$

$$\sum_{i=1}^{n_j} x_{ij} = 0, \sum_{i=1}^{n_{-j}} x_{i-j} = \left(n_{-j}-1\right)e + e'$$

so that the deviation payoff is

$$\pi_{i-j}^{D}\left(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}\right) =$$

$$\left[\left(1-\alpha_{-j}\right)\frac{e'}{\left(n_{-j}-1\right)e+e'} + \frac{\alpha_{-j}}{n_{-j}}\right]v - e' \leq \frac{1}{n_{-j}}v - e \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \alpha_{-j} \geq 1 - \frac{n_{-j}e}{v} - \frac{n_{-j}e'}{(n_{-j}-1)v} & \text{if } e' > e \\ \alpha_{-j} \leq 1 - \frac{n_{-j}e}{v} - \frac{n_{-j}e'}{(n_{-j}-1)v} & \text{if } e' < e \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \alpha_{-j} \geq \lim_{\epsilon\to 0} 1 - \frac{n_{-j}e}{v} - \frac{n_{-j}(e+\epsilon)}{(n_{-j}-1)v} \\ \alpha_{-j} \leq \lim_{\epsilon\to 0} 1 - \frac{n_{-j}e}{v} - \frac{n_{-j}(e-\epsilon)}{(n_{-j}-1)v} \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \alpha_{-j} \geq 1 - \frac{n_{-j}e}{v} - \frac{n_{-j}e}{(n_{-j}-1)v} \\ \alpha_{-j} \leq 1 - \frac{n_{-j}e}{v} - \frac{n_{-j}e}{(n_{-j}-1)v} \end{cases} \Leftrightarrow \alpha_{-j} = 1 - \frac{n_{-j}e}{v} - \frac{n_{-j}e}{(n_{-j}-1)\,v} \ .$$

Conversely, if agent $i-j$ deviates to $e' = 0$, then

$$min\left\{\mathbf{x}_j\right\} = min\left\{\mathbf{x}_{-j}\right\} = 0 \text{ and } \sum_{i=1}^{n_j} x_{ij} = 0, \sum_{i=1}^{n_{-j}} x_{i-j} = (n_{-j}-1)\,e$$

so that

$$\pi^D_{i-j}\left(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}\right) = \frac{1}{2}\frac{\alpha_{-j}}{n_{-j}}v \leq \frac{1}{n_{-j}}v - e \Leftrightarrow \alpha_{-j} \leq 2\left(1 - \frac{n_{-j}e}{v}\right) \ .$$

Note that

$$2\left(1 - \frac{n_{-j}e}{v}\right) \geq 1 - \frac{n_{-j}e}{v} - \frac{n_{-j}e}{(n_{-j}-1)\,v} \Leftrightarrow v \geq n_{-j}e - \frac{n_{-j}e}{n_{-j}-1} \ .$$

For $\alpha_j \in [0,1]$, the equilibrium

$$(\gamma_j, \gamma_{-j}) = (0, e) \quad \text{such that } \mathbf{x}_{-j} = \mathbf{e}$$

requires the following conditions

i.
$$\left\{1 - \frac{n^2_{-j}e}{(n_{-j}-1)\,v}\right\} \in [0,1] \Leftrightarrow \begin{cases} 1 - \frac{n^2_{-j}e}{(n_{-j}-1)v} \geq 0 \\ 1 - \frac{n^2_{-j}e}{(n_{-j}-1)v} \leq 1 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \frac{n^2_{-j}e}{(n_{-j}-1)\,v} \leq 1 \Leftrightarrow e \leq \frac{(n_{-j}-1)\,v}{n^2_{-j}}$$

ii.
$$v \geq \frac{n_{-j}\,(n_{-j}-2)}{n_{-j}-1}e \Leftrightarrow e \leq \frac{(n_{-j}-1)\,v}{n_{-j}\,(n_{-j}-2)}.$$

However, since

$$\frac{(n_{-j}-1)\,v}{n^2_{-j}} \leq \frac{(n_{-j}-1)\,v}{n_{-j}\,(n_{-j}-2)},$$

only the first inequality is binding.

Hence,

$$(\gamma_j, \gamma_{-j}) \in (0, e) \quad \forall e \in \mathbb{R}_{++} \text{ such that } \mathbf{x}_j = \mathbf{e}$$

is a Nash equilibrium if and only if

$$(\alpha_j, \alpha_{-j}) \in [0,1] \times \left\{1 - \frac{n_{-j}\gamma_{-j}}{v} - \frac{n_{-j}\gamma_{-j}}{(n_{-j}-1)\,v}\right\} \quad \forall v \geq \frac{n_{-j}^2\gamma_{-j}}{n_{-j}-1}.$$

5. Finally, note that similar arguments prove that the following strategy profiles are not second-period Nash equilibria in pure strategies:

(a)

$$(\gamma_1, \gamma_2) \text{ such that } \gamma_j \in \mathbb{R}_{++},\ min\left\{\mathbf{x}_1\right\} = min\left\{\mathbf{x}_2\right\} > 0,$$

$$2 \leq \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=min\{\mathbf{x}_j\}} \leq n_j - 1\ .^{2.A8}$$

Suppose

$$(\gamma_1, \gamma_2) \text{ such that } \gamma_j \in \mathbb{R}_{++}, min\left\{\mathbf{x}_1\right\} = min\left\{\mathbf{x}_2\right\} > 0 \text{ and}$$

$$2 \leq \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=min\{\mathbf{x}_j\}} \leq n_j - 1\ .$$

Then, if $x_{ij} = e \geq min\left\{\mathbf{x}_1\right\} = min\left\{\mathbf{x}_2\right\} > 0$,

$$x_{ij} \geq \min\left\{\mathbf{x}_j\right\} = \min\left\{\mathbf{x}_{-j}\right\} > 0 \text{ and}$$

$$\sum_{i=1}^{n_j} x_j\left(i\right) = n_j\gamma_j, \sum_{i=1}^{n_{-j}} x_{i-j} = n_{-j}\gamma_{-j}$$

so that

$$\pi_{ij}\left(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}\right) = \frac{1}{2}\left[\left(1-\alpha_j\right)\frac{e}{n_j\gamma_j} + \frac{\alpha_j}{n_j}\right]v - e.$$

If agent ij deviates to $e' \geq min\left\{\mathbf{x}_j\right\}$ *s.t.* $e' \neq e$, then

$$x_{ij} \geq \min\left\{\mathbf{x}_j\right\} = \min\left\{\mathbf{x}_{-j}\right\} > 0 \text{ and}$$

$$\sum_{i=1}^{n_j} x_{ij} = n_j\gamma_j - e + e', \sum_{i=1}^{n_{-j}} x_{i-j} = n_{-j}\gamma_{-j}$$

so that the deviation payoff is

$$\pi_{ij}^D\left(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}\right) = \frac{1}{2}\left[\left(1-\alpha_j\right)\frac{e'}{n_j\gamma_j - e + e'} + \frac{\alpha_j}{n_j}\right]v - e'\ .$$

[2.A8] Note that $\mathbb{1}_{x_{ij}=min\{\mathbf{x}_j\}}$ stands for the Indicator function taking value 1 when $x_{ij} = min\{\mathbf{x}_j\}$ for any ij.

In contrast, if agent ij deviates to $e' < min\left\{\mathbf{x}_j\right\}$, then

$$x_{ij} = \min\left\{\mathbf{x}_j\right\} < \min\left\{\mathbf{x}_{-j}\right\} \text{ and}$$

$$\sum_{i=1}^{n_j} x_{ij} = n_j\gamma_j - e + e', \sum_{i=1}^{n_{-j}} x_{i-j} = n_{-j}\gamma_{-j}$$

so that the deviation payoff is

$$\pi_{ij}^{D}\left(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}\right) = -e' \ .$$

We select the payoff-dominant downward deviation, that is $e' = 0$. Hence, for any player ij there is no incentive to deviate if and only if

i.

$$\frac{1}{2}\left[\left(1-\alpha_j\right)\frac{e}{n_j\gamma_j} + \frac{\alpha_j}{n_j}\right]v - e \geq$$

$$\frac{1}{2}\left[\left(1-\alpha_j\right)\frac{e'}{n_j\gamma_j - e + e'} + \frac{\alpha_j}{n_j}\right]v - e' \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \alpha_j \geq 1 - \frac{2n_j\gamma_j\left(n_j\gamma_j - e + e'\right)}{\left(n_j\gamma_j - e\right)v} & \text{if } e' > e \\ \alpha_j \leq 1 - \frac{2n_j\gamma_j\left(n_j\gamma_j - e + e'\right)}{\left(n_j\gamma_j - e\right)v} & \text{if } e' < e \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \alpha_j \geq \lim_{\epsilon \to 0} 1 - \frac{2n_j\gamma_j(n_j\gamma_j - min\{\mathbf{x}_j\} + min\{\mathbf{x}_j\} + \epsilon)}{(n_j\gamma_j - min\{\mathbf{x}_j\})v} & \text{if } e' > e \\ \alpha_j \leq \lim_{\epsilon \to 0} 1 - \frac{2n_j\gamma_j(n_j\gamma_j - max\{\mathbf{x}_j\} + max\{\mathbf{x}_j\} - \epsilon)}{(n_j\gamma_j - max\{\mathbf{x}_j\})v} & \text{if } e' < e \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \alpha_j \geq 1 - \frac{2(n_j\gamma_j)^2}{(n_j\gamma_j - min\{\mathbf{x}_j\})v} & \text{if } e' > e \\ \alpha_j \leq 1 - \frac{2(n_j\gamma_j)^2}{(n_j\gamma_j - max\{\mathbf{x}_j\})v} & \text{if } e' < e \end{cases}$$

and

ii.

$$\frac{1}{2}\left[\left(1-\alpha_j\right)\frac{e}{n_j\gamma_j} + \frac{\alpha_j}{n_j}\right]v - e \geq 0 \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \alpha_j \geq \frac{e}{\gamma_j - e}\left(\frac{2n_j\gamma_j}{v} - 1\right) & \gamma_j > e \\ v \geq 2n_j\gamma_j & \gamma_j = e \\ \alpha_j \leq \frac{e}{\gamma_j - e}\left(\frac{2n_j\gamma_j}{v} - 1\right) & \gamma_j < e \ . \end{cases}$$

Note that an α_j preventing both upward and downward deviations cannot exist, since the lower and upper bounds at point $a.$ do cross, as $max\left\{\mathbf{x}_j\right\} > min\left\{\mathbf{x}_j\right\}$. Hence,

$$(\gamma_1, \gamma_2) \text{ such that } \gamma_j \in \mathbb{R}_{++},\ min\left\{\mathbf{x}_1\right\} = min\left\{\mathbf{x}_2\right\} > 0 \text{ and}$$

$$2 \leq \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=min\{\mathbf{x}_j\}} \leq n_j - 1$$

is not a Nash equilibrium for any $\alpha_j \in \mathbb{R}$.

(b)

$$(\gamma_j, \gamma_{-j}) \text{ such that } \gamma_j = e \in \mathbb{R}_{++} \text{ and } \mathbf{x}_j = \mathbf{e}, \gamma_{-j} \in \mathbb{R}_{++},$$

$$min\left\{\mathbf{x}_j\right\} = min\left\{\mathbf{x}_{-j}\right\} > 0, 2 \leq \sum_{i=1}^{n_{-j}} \mathbb{1}_{x_{i-j}=min\{\mathbf{x}_{-j}\}} \leq n_{-j} - 1 \ .$$

The proof follows the corresponding arguments shown at point (a).

(c)

$$(\gamma_j, \gamma_{-j}) \text{ such that } \gamma_j = 0, \gamma_{-j} \in \mathbb{R}_{++}, min\left\{\mathbf{x}_1\right\} = min\left\{\mathbf{x}_2\right\} = 0 \text{ and}$$

$$2 \leq \sum_{i=1}^{n_{-j}} \mathbb{1}_{x_{i-j}=0} \leq n_{-j} - 2 \ .$$[2.A9]

Suppose

$$(\gamma_1, \gamma_2) \text{ s.t. } \gamma_j = 0, \gamma_{-j} \in \mathbb{R}_{++}, min\left\{\mathbf{x}_1\right\} = min\left\{\mathbf{x}_2\right\} \text{ and}$$

$$2 \leq \sum_{i=1}^{n_{-j}} \mathbb{1}_{x_{i-j}=0} \leq n_{-j} - 2 \ .$$

Then,

$$x_{ij} = \min\left\{\mathbf{x}_j\right\} = \min\left\{\mathbf{x}_{-j}\right\} = 0 \text{ and}$$

$$\sum_{i=1}^{n_j} x_j\left(i\right) = n_j \gamma_j, \sum_{i=1}^{n_{-j}} x_{i-j} = n_{-j} \gamma_{-j}$$

so that

$$\pi_{ij}\left(\gamma_j, \gamma_{-j} | \boldsymbol{\alpha}\right) = \frac{1}{2} \frac{v}{n_j} \ .$$

If agent ij deviates to $x_{ij} = e' > 0$, then

$$\min\left\{\mathbf{x}_j\right\} = \min\left\{\mathbf{x}_{-j}\right\} = 0 \text{ and}$$

[2.A9] Note that $\mathbb{1}_{x_{ij}=0}$ stands for the Indicator function taking value 1 when $x_{ij} = 0$ for any ij.

$$\sum_{i=1}^{n_j} x_{ij} = n_j\gamma_j - e + e', \sum_{i=1}^{n_{-j}} x_{i-j} = n_{-j}\gamma_{-j}$$

so that the deviation payoff is

$$\pi_{ij}^{D}\left(\gamma_j,\gamma_{-j}|\boldsymbol{\alpha}\right) = \frac{1}{2}\left[\left(1-\alpha_j\right)\frac{e'}{e'}+\frac{\alpha_j}{n_j}\right]v - e' \ .$$

Hence for any player ij there is no incentive to deviate if and only if

$$\frac{1}{2}\frac{v}{n_j} > \frac{1}{2}\left[\left(1-\alpha_j\right)+\frac{\alpha_j}{n_j}\right]v - 1 \Leftrightarrow \alpha_j \geq \lim_{e'\to 0} 1 - \frac{2n_je'}{(n_j-1)v} \Leftrightarrow \alpha_j \geq 1 \ .$$

On the other hand,

$$x_{i-j} \geq \min\left\{\mathbf{x}_j\right\} = \min\left\{\mathbf{x}_{-j}\right\} = 0 \text{ and}$$

$$\sum_{i=1}^{n_j} x_{ij} = n_j\gamma_j, \sum_{i=1}^{n_{-j}} x_{i-j} = n_{-j}\gamma_{-j}$$

so that

$$\pi_{i-j}\left(\gamma_j,\gamma_{-j}|\boldsymbol{\alpha}\right) = \frac{1}{2}\left[\left(1-\alpha_{-j}\right)\frac{e}{n_{-j}\gamma_{-j}}+\frac{\alpha_{-j}}{n_{-j}}\right]v - e \ .$$

If agent $i-j$ deviates to $x_{i-j} = e' \neq e$, then

$$\min\left\{\mathbf{x}_j\right\} = \min\left\{\mathbf{x}_{-j}\right\} = 0 \text{ and } \sum_{i=1}^{n_j} x_{ij} = n_j\gamma_j, \sum_{i=1}^{n_{-j}} x_{i-j} = n_{-j}\gamma_{-j} - e + e'$$

so that the deviation payoff is

$$\pi_{i-j}^{D}\left(\gamma_j,\gamma_{-j}|\boldsymbol{\alpha}\right) = \frac{1}{2}\left[\left(1-\alpha_{-j}\right)\frac{e'}{n_{-j}\gamma_{-j}-e+e'}+\frac{\alpha_{-j}}{n_{-j}}\right]v - e' \ .$$

Hence for any player ij there is no incentive to deviate if and only if

$$\frac{1}{2}\left[\left(1-\alpha_{-j}\right)\frac{e}{n_{-j}\gamma_{-j}}+\frac{\alpha_{-j}}{n_{-j}}\right]v - e \geq$$

$$\frac{1}{2}\left[\left(1-\alpha_j\right)\frac{e'}{n_j\gamma_j-e+e'}+\frac{\alpha_j}{n_j}\right]v - e' \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \alpha_{-j} \geq 1 - \frac{2n_{-j}\gamma_{-j}\left(n_{-j}\gamma_{-j}-e+e'\right)}{(n_{-j}\gamma_{-j}-e)v} & \forall\ e' > e \\ \alpha_{-j} \leq 1 - \frac{2n_{-j}\gamma_{-j}\left(n_{-j}\gamma_{-j}-e+e'\right)}{(n_{-j}\gamma_{-j}-e)v} & \forall\ e' < e \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \alpha_{-j} \geq \lim_{\epsilon \to 0} 1 - \frac{2n_{-j}\gamma_{-j}(n_{-j}\gamma_{-j} - min\{\mathbf{x}_{-j}\} + min\{\mathbf{x}_{-j}\} + \epsilon)}{(n_{-j}\gamma_{-j} - min\{\mathbf{x}_{-j}\})v} & \forall\ e' > e \\ \alpha_{-j} \leq \lim_{\epsilon \to 0} 1 - \frac{2n_{-j}\gamma_{-j}(n_{-j}\gamma_{-j} - max\{\mathbf{x}_{-j}\} + max\{\mathbf{x}_{-j}\} - \epsilon)}{(n_{-j}\gamma_{-j} - max\{\mathbf{x}_{-j}\})v} & \forall\ e' < e \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \alpha_{-j} \geq 1 - \frac{2(n_{-j}\gamma_{-j})^2}{(n_{-j}\gamma_{-j} - min\{\mathbf{x}_{-j}\})v} & \forall\ e' > e \\ \alpha_{-j} \leq 1 - \frac{2(n_{-j}\gamma_{-j})^2}{(n_{-j}\gamma_{-j} - max\{\mathbf{x}_{-j}\})v} & \forall\ e' < e \end{cases}$$

Note that the upper and lower bounds obtained above do cross, since $max\{\mathbf{x}_{-j}\} > min\{\mathbf{x}_{-j}\}$. Therefore,

$$(\gamma_j, \gamma_{-j}) \text{ such that } \gamma_j = 0, \gamma_{-j} \in \mathbb{R}_{++}, min\{\mathbf{x}_1\} = min\{\mathbf{x}_2\} = 0 \text{ and}$$

$$2 \leq \sum_{i=1}^{n_{-j}} \mathbb{1}_{x_{i-j}=0} \leq n_{-j} - 2$$

is not a Nash equilibrium for any $(\alpha_j, \alpha_{-j}) \in \mathbb{R} \times \mathbb{R}$.

(d)

$$(\gamma_j, \gamma_{-j}) \text{ such that } \gamma_j \in \mathbb{R}_{++}, min\{\mathbf{x}_1\} = min\{\mathbf{x}_2\} = 0 \text{ and}$$

$$2 \leq \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=0} \leq n_j - 2\ .$$

The proof follows the corresponding arguments shown at point (c).

(e)

$$(\gamma_1, \gamma_2) \quad \text{such that} \quad \gamma_j = 0, \gamma_{-j} \in \mathbb{R}_{++}, min\{\mathbf{x}_1\} = min\{\mathbf{x}_2\} \text{ and}$$

$$\sum_{i=1}^{n_{-j}} \mathbb{1}_{x_{i-j}=0} = n_{-j} - 1\ .$$

Suppose

$$(\gamma_1, \gamma_2) \text{ such that } \gamma_j = 0, \gamma_{-j} \in \mathbb{R}_{++}, min\{\mathbf{x}_1\} = min\{\mathbf{x}_2\} \text{ and}$$

$$\sum_{i=1}^{n_{-j}} \mathbb{1}_{x_{i-j}=0} = n_{-j} - 1\ .$$

Then,

$$x_{ij} = \min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} = 0 \text{ and}$$

$$\sum_{i=1}^{n_j} x_{ij} = n_j\gamma_j, \sum_{i=1}^{n_{-j}} x_{i-j} = n_{-j}\gamma_{-j}$$

so that

$$\pi_{ij}\left(\gamma_j,\gamma_{-j}|\boldsymbol{\alpha}\right)=\frac{1}{2}\frac{v}{n_j}\ .$$

If agent ij deviates to any $x_{ij}=e'>0$, then

$$\min\left\{\mathbf{x}_j\right\}=\min\left\{\mathbf{x}_{-j}\right\}=0 \text{ and } \sum_{i=1}^{n_j}x_{ij}=e',\sum_{i=1}^{n_{-j}}x_{i-j}=n_{-j}\gamma_{-j}$$

so that the deviation payoff is

$$\pi_{ij}^{D}\left(\gamma_j,\gamma_{-j}|\boldsymbol{\alpha}\right)=\frac{1}{2}\left[\left(1-\alpha_j\right)\frac{e'}{e'}+\frac{\alpha_j}{n_j}\right]v-e'\ .$$

Hence, for any player ij there is no incentive to deviate if and only if

$$\frac{1}{2}\frac{v}{n_j}\geq\frac{1}{2}\left[\left(1-\alpha_j\right)+\frac{\alpha_j}{n_j}\right]v-e'\Leftrightarrow\alpha_j\geq\lim_{e'\to 0}1-\frac{2n_je'}{(n_j-1)v}\Leftrightarrow\alpha_j\geq 1\ .$$

On the other hand, consider $x_{i-j}=0$, that is

$$x_{i-j}=\min\left\{\mathbf{x}_j\right\}=\min\left\{\mathbf{x}_{-j}\right\}=0 \text{ and}$$

$$\sum_{i=1}^{n_j}x_{ij}=n_j\gamma_j,\sum_{i=1}^{n_{-j}}x_{i-j}=n_{-j}\gamma_{-j}$$

so that

$$\pi_{i-j}\left(\gamma_j,\gamma_{-j}|\boldsymbol{\alpha}\right)=\frac{1}{2}\frac{\alpha_{-j}}{n_{-j}}v\ .$$

If agent $i-j$ deviates to $x_{i-j}=e'\in\mathbb{R}_{++}$, then

$$\min\left\{\mathbf{x}_{-j}\right\}=\min\left\{\mathbf{x}_j\right\}=0 \text{ and } \sum_{i=1}^{n_j}x_{ij}=n_j\gamma_j,\sum_{i=1}^{n_{-j}}x_{i-j}=n_{-j}\gamma_{-j}+e'$$

so that the deviation payoff is

$$\pi_{i-j}^{D}\left(\gamma_j,\gamma_{-j}|\boldsymbol{\alpha}\right)=\frac{1}{2}\left[\left(1-\alpha_{-j}\right)\frac{e'}{n_{-j}\gamma_{-j}+e'}+\frac{\alpha_{-j}}{n_{-j}}\right]v-e'\ .$$

Hence, for any player $i-j$ such that $x_{i-j}=0$ there is no incentive to deviate if and only if

$$\frac{1}{2}\frac{\alpha_{-j}}{n_{-j}}v\geq\frac{1}{2}\left[\left(1-\alpha_{-j}\right)\frac{e'}{n_{-j}\gamma_{-j}+e'}+\frac{\alpha_{-j}}{n_{-j}}\right]v-e'\Leftrightarrow$$

$$\Leftrightarrow \alpha_{-j} \geq \lim_{e' \to 0} 1 - \frac{2\left(n_{-j}\gamma_{-j} + e'\right)}{v} \Leftrightarrow$$

$$\Leftrightarrow \alpha_{-j} \geq 1 - \frac{2n_{-j}\gamma_{-j}}{v} \ .$$

Consider the unique player $i-j$ such that $x_{i-j} = e \in \mathbb{R}_{++}$, that is

$$x_{i-j} \geq min\left\{\mathbf{x}_j\right\} = min\left\{\mathbf{x}_{-j}\right\} = 0 \text{ and}$$

$$\sum_{i=1}^{n_j} x_{ij} = n_j\gamma_j, \sum_{i=1}^{n_{-j}} x_{i-j} = n_{-j}\gamma_{-j}$$

so that

$$\pi_{i-j}\left(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}\right) = \frac{1}{2}\left[\left(1-\alpha_{-j}\right)\frac{e}{e} + \frac{\alpha_{-j}}{n_{-j}}\right]v - e \ .$$

It is straightforward to see that deviating to any small $e' > 0$ strictly payoff-dominates any positive effort level $e > e'$ for player $i-j$, so that an equilibrium cannot exist. On the other hand, for completeness sake, if player $i-j$ deviates to $x_{i-j} = 0$, then

$$x_{i-j} = min\left\{\mathbf{x}_j\right\} = min\left\{\mathbf{x}_{-j}\right\} = 0 \text{ and } \sum_{i=1}^{n_j} x_{ij} = n_j\gamma_j, \sum_{i=1}^{n_{-j}} x_{i-j} = 0$$

so that the deviation payoff is

$$\pi^D_{i-j}\left(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}\right) = \frac{1}{2}\frac{v}{n_{-j}} \ .$$

Hence, for any player $i-j$ there is no incentive to deviate if and only if

$$\frac{1}{2}\left[\left(1-\alpha_{-j}\right)\frac{e'}{e'} + \frac{\alpha_{-j}}{n_{-j}}\right]v - e' \geq \frac{1}{2}\frac{v}{n_{-j}} \Leftrightarrow$$

$$\alpha_{-j} \leq \lim_{e' \to 0} 1 - \frac{2n_{-j}e'}{\left(n_{-j}-1\right)v} \Leftrightarrow \alpha_{-j} \leq 1 \ .$$

Therefore,

$$\left(0, \gamma_{-j}\right) \text{ such that } \gamma_{-j} \in \mathbb{R}_{++} \text{ and } \sum_{i=1}^{n_{-j}} \mathbb{1}_{x_{i-j}=0} = n_{-j} - 1$$

is not a Nash equilibrium for any

$$\left(\alpha_j, \alpha_{-j}\right) \in \mathbb{R} \times \mathbb{R} \ .$$

(f)

$$(\gamma_1, \gamma_2) \text{ such that } \gamma_j \in \mathbb{R}_{++} \text{ and } \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=0} = n_j - 1 \ .$$

Suppose

$$(\gamma_1, \gamma_2) \text{ such that } \gamma_j \in \mathbb{R}_{++} \text{ and } \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=0} = n_j - 1 \ .$$

Then, the proof follows the corresponding arguments shown at point (e), so that

$$(\gamma_1, \gamma_2) \text{ such that } \gamma_j \in \mathbb{R}_{++} \text{ and } \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=0} = n_j - 1$$

is a not a Nash equilibrium for any

$$(\alpha_1, \alpha_2) \in \mathbb{R} \times \mathbb{R} \ .$$

(g)

$$(\gamma_1, \gamma_2) \text{ such that } \gamma_j \in \mathbb{R}_{++} \ , \ min \left\{\mathbf{x}_1\right\} = min \left\{\mathbf{x}_2\right\} > 0 \text{ and}$$

$$\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=min\{\mathbf{x}_j\}} = 1 \ .$$

Suppose

$$(\gamma_1, \gamma_2) \text{ such that } \gamma_j \in \mathbb{R}_{++}, min \left\{\mathbf{x}_1\right\} = min \left\{\mathbf{x}_2\right\} > 0 \text{ and}$$

$$\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=min\{\mathbf{x}_j\}} = 1 \ .$$

Then, if $x_{ij} = e > min \left\{\mathbf{x}_1\right\} = min \left\{\mathbf{x}_2\right\} > 0$,

$$x_{ij} > \min \left\{\mathbf{x}_j\right\} = \min \left\{\mathbf{x}_{-j}\right\} > 0 \text{ and}$$

$$\sum_{i=1}^{n_j} x_{ij} = n_j \gamma_j, \sum_{i=1}^{n_{-j}} x_{i-j} = n_{-j} \gamma_{-j}$$

so that

$$\pi_{ij} \left(\gamma_j, \gamma_{-j} | \boldsymbol{\alpha}\right) = \frac{1}{2} \left[\left(1 - \alpha_j\right) \frac{e}{n_j \gamma_j} + \frac{\alpha_j}{n_j} \right] v - e \ .$$

If agent ij deviates to $e' \geq min \left\{\mathbf{x}_j\right\}$ s.t. $e' \neq e$, then

$$x_{ij} \geq \min \left\{\mathbf{x}_j\right\} = \min \left\{\mathbf{x}_{-j}\right\} > 0 \text{ and}$$

$$\sum_{i=1}^{n_j} x_{ij} = n_j\gamma_j - e + e', \sum_{i=1}^{n_{-j}} x_{i-j} = n_{-j}\gamma_{-j}$$

so that the deviation payoff is

$$\pi_{ij}^{D}\left(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}\right) = \frac{1}{2}\left[\left(1-\alpha_j\right)\frac{e'}{n_j\gamma_j - e + e'} + \frac{\alpha_j}{n_j}\right]v - e' \ .$$

In contrast, if $x_{ij} = e \geq min\left\{\mathbf{x}_j\right\} = min\left\{\mathbf{x}_1\right\} = min\left\{\mathbf{x}_2\right\} > 0$, and agent ij deviates to $e' < min\left\{\mathbf{x}_j\right\}$, then

$$x_{ij} = \min\left\{\mathbf{x}_j\right\} < \min\left\{\mathbf{x}_{-j}\right\} \text{ and}$$

$$\sum_{i=1}^{n_j} x_{ij} = n_j\gamma_j - e + e', \sum_{i=1}^{n_{-j}} x_{i-j} = n_{-j}\gamma_{-j}$$

so that the deviation payoff is

$$\pi_{ij}^{D}\left(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}\right) = -e' \ .$$

We select the payoff-dominant downward deviation, that is $e' = 0$. On the other hand, if the unique agent ij exerting effort $e = min\left\{\mathbf{x}_j\right\}$ deviates to $e' > e$, then

$$x_{ij} \geq \min\left\{\mathbf{x}_j\right\} > \min\left\{\mathbf{x}_{-j}\right\} \text{ and}$$

$$\sum_{i=1}^{n_j} x_{ij} = n_j\gamma_j - e + e', \sum_{i=1}^{n_{-j}} x_{i-j} = n_{-j}\gamma_{-j}$$

so that the deviation payoff is

$$\pi_{ij}^{D}\left(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}\right) = \left[\left(1-\alpha_j\right)\frac{e'}{n_j\gamma_j - e + e'} + \frac{\alpha_j}{n_j}\right]v - e' \ .$$

Hence, for any player ij there is no incentive to deviate if and only if

i.

$$\frac{1}{2}\left[\left(1-\alpha_j\right)\frac{e}{n_j\gamma_j} + \frac{\alpha_j}{n_j}\right]v - e \geq$$

$$\frac{1}{2}\left[\left(1-\alpha_j\right)\frac{e'}{n_j\gamma_j - e + e'} + \frac{\alpha_j}{n_j}\right]v - e' \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \alpha_j \geq 1 - \frac{2n_j\gamma_j\left(n_j\gamma_j - e + e'\right)}{\left(n_j\gamma_j - e\right)v} & \forall\ e' > e \\ \alpha_j \leq 1 - \frac{2n_j\gamma_j\left(n_j\gamma_j - e + e'\right)}{\left(n_j\gamma_j - e\right)v} & \forall\ e' < e \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \alpha_j \geq \lim_{\epsilon \to 0} 1 - \frac{2n_j\gamma_j(n_j\gamma_j - min\{\mathbf{x}_j\} + min\{\mathbf{x}_j\} + \epsilon)}{(n_j\gamma_j - min\{\mathbf{x}_j\})v} & \forall\ e' > e \\ \alpha_j \leq \lim_{\epsilon \to 0} 1 - \frac{2n_j\gamma_j(n_j\gamma_j - max\{\mathbf{x}_j\} + max\{\mathbf{x}_j\} - \epsilon)}{(n_j\gamma_j - max\{\mathbf{x}_j\})v} & \forall\ e' < e \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \alpha_j \geq 1 - \frac{2(n_j\gamma_j)^2}{(n_j\gamma_j - min\{\mathbf{x}_j\})v} & \forall\ e' > e \\ \alpha_j \leq 1 - \frac{2(n_j\gamma_j)^2}{(n_j\gamma_j - max\{\mathbf{x}_j\})v} & \forall\ e' < e \end{cases}$$

ii.

$$\frac{1}{2}\left[(1-\alpha_j)\frac{e}{n_j\gamma_j} + \frac{\alpha_j}{n_j}\right]v - e \geq 0 \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \alpha_j \geq \frac{e}{\gamma_j - e}\left(\frac{2n_j\gamma_j}{v} - 1\right) & \gamma_j > e \\ v \geq 2n_j\gamma_j & \gamma_j = e \\ \alpha_j \leq \frac{e}{\gamma_j - e}\left(\frac{2n_j\gamma_j}{v} - 1\right) & \gamma_j < e \end{cases}$$

iii.

$$\frac{1}{2}\left[(1-\alpha_j)\frac{e}{n_j\gamma_j} + \frac{\alpha_j}{n_j}\right]v - e \geq \left[(1-\alpha_j)\frac{e'}{n_j\gamma_j - e + e'} + \frac{\alpha_j}{n_j}\right]v - e' \Leftrightarrow$$

$$\Leftrightarrow \alpha_j \lesseqgtr \frac{(2n_j\gamma_j e' - e\,(n_j\gamma_j - e + e'))\,v + (e - e')\,2n_j\gamma_j\,(n_j\gamma_j - e + e')}{(2n_j\gamma_j e' - (\gamma_j + e)\,(n_j\gamma_j - e + e'))\,v}$$

Note that the lower and upper bounds at point i. do cross, since $max\,\{\mathbf{x}_j\} > min\,\{\mathbf{x}_j\}$. Therefore,

$$(\gamma_1, \gamma_2) \text{ such that } \gamma_j \in \mathbb{R}_{++}, min\,\{\mathbf{x}_1\} = min\,\{\mathbf{x}_2\} > 0 \text{ and}$$

$$\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij} = min\{\mathbf{x}_j\}} = 1$$

is not a Nash equilibrium for any

$$(\alpha_1, \alpha_2) \in \mathbb{R} \times \mathbb{R}\ .$$

(h)

$$(\gamma_j, \gamma_{-j}) \text{ such that } \gamma_j = e \in \mathbb{R}_{++} \text{ and } \mathbf{x}_j = \mathbf{e}, \gamma_{-j} \in \mathbb{R}_{++}\ ,$$

$$min\,\{\mathbf{x}_j\} = min\,\{\mathbf{x}_{-j}\} > 0 \text{ and } \sum_{i=1}^{n_{-j}} \mathbb{1}_{x_{i-j} = min\{\mathbf{x}_{-j}\}} = 1\ .$$

The proof follows the corresponding arguments shown at point (g).

(i)

$$(\gamma_j, \gamma_{-j}) \text{ such that } \gamma_j = 0 \text{ , } \gamma_{-j} \in \mathbb{R}_{++} \text{ and } \sum_{i=1}^{n_{-j}} \mathbb{1}_{x_{i-j}=0} = 1 \text{ .}$$

Suppose

$$(\gamma_1, \gamma_2) \text{ such that } \gamma_j = 0, \gamma_{-j} \in \mathbb{R}_{++}, min\left\{\mathbf{x}_1\right\} = min\left\{\mathbf{x}_2\right\} \text{ and}$$

$$\mathbb{1}_{x_{i-j}=0} = 1 \text{ .}$$

Then,

$$x_{ij} = \min\left\{\mathbf{x}_j\right\} = \min\left\{\mathbf{x}_{-j}\right\} = 0 \text{ and}$$

$$\sum_{i=1}^{n_j} x_{ij} = n_j \gamma_j, \sum_{i=1}^{n_{-j}} x_{i-j} = n_{-j}\gamma_{-j}$$

so that

$$\pi_{ij}\left(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}\right) = \frac{1}{2}\frac{v}{n_j} \text{ .}$$

If agent ij deviates to $x_{ij} = e'$, then

$$\min\left\{\mathbf{x}_j\right\} = \min\left\{\mathbf{x}_{-j}\right\} = 0 \text{ and}$$

$$\sum_{i=1}^{n_j} x_{ij} = n_j\gamma_j - e + e', \sum_{i=1}^{n_{-j}} x_{i-j} = n_{-j}\gamma_{-j}$$

so that the deviation payoff is

$$\pi_{ij}^{D}\left(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}\right) = \frac{1}{2}\left[(1-\alpha_j)\frac{e'}{e'} + \frac{\alpha_j}{n_j}\right]v - e' \text{ .}$$

Hence for any player ij there is no incentive to deviate if and only if

$$\frac{1}{2}\frac{v}{n_j} > \frac{1}{2}\left[(1-\alpha_j) + \frac{\alpha_j}{n_j}\right]v - e' \Leftrightarrow$$

$$\alpha_j \geq \lim_{e' \to 0} 1 - \frac{2n_j}{(n_j-1)v} \Leftrightarrow \alpha_j \geq 1 \text{ .}$$

On the other hand, consider $x_{i-j} = 0$, that is

$$x_{i-j} = \min\left\{\mathbf{x}_j\right\} = \min\left\{\mathbf{x}_{-j}\right\} = 0 \text{ and}$$

$$\sum_{i=1}^{n_j} x_{ij} = n_j\gamma_j, \sum_{i=1}^{n_{-j}} x_{i-j} = n_{-j}\gamma_{-j}$$

so that

$$\pi_{i-j}\left(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}\right) = \frac{1}{2}\frac{\alpha_{-j}}{n_{-j}}v \ .$$

If agent $i-j$ deviates to $x_{i-j} = e' \in \mathbb{R}_{++}$, then

$$\min\left\{\mathbf{x}_{-j}\right\} > \min\left\{\mathbf{x}_j\right\} = 0 \text{ and}$$

$$\sum_{i=1}^{n_j} x_{ij} = n_j\gamma_j, \sum_{i=1}^{n_{-j}} x_{i-j} = n_{-j}\gamma_{-j} + e'$$

so that the deviation payoff is

$$\pi^D_{i-j}\left(\alpha, \gamma_j, \gamma'_{-j}\right) = \left[\left(1-\alpha_{-j}\right)\frac{e'}{n_{-j}\gamma_{-j}+e'} + \frac{\alpha_{-j}}{n_{-j}}\right]v - e' \ .$$

Hence, for the unique player $i-j$ such that $x_{i-j} = 0$ there is no incentive to deviate if and only if

$$\frac{1}{2}\frac{\alpha_{-j}}{n_{-j}}v \geq \left[\left(1-\alpha_{-j}\right)\frac{e'}{n_{-j}\gamma_{-j}+e'} + \frac{\alpha_{-j}}{n_{-j}}\right]v - e' \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \alpha_{-j} \geq \frac{2n_{-j}e'\left(v-n_{-j}\gamma_{-j}-e'\right)}{\left(2n_{-j}e'-n_{-j}\gamma_{-j}-e'\right)v} & \text{if } 2n_{-j}e' - n_{-j}\gamma_{-j} - e' > 0 \\ \alpha_{-j} \leq \frac{2n_{-j}e'\left(v-n_{-j}\gamma_{-j}-e'\right)}{\left(2n_{-j}e'-n_{-j}\gamma_{-j}-e'\right)v} & \text{if } 2n_{-j}e' - n_{-j}\gamma_{-j} - e' < 0 \end{cases}$$

Consider any player $i-j$ such that $x_{i-j} > 0$, that is

$$x_{i-j} > \min\left\{\mathbf{x}_j\right\} = \min\left\{\mathbf{x}_{-j}\right\} = 0 \text{ and}$$

$$\sum_{i=1}^{n_j} x_j\left(i\right) = n_j\gamma_j, \sum_{i=1}^{n_{-j}} x_{i-j} = n_{-j}\gamma_{-j}$$

so that

$$\pi_{i-j}\left(\alpha, \gamma_j, \gamma_{-j}\right) = \frac{1}{2}\left[\left(1-\alpha_{-j}\right)\frac{e}{n_{-j}\gamma_{-j}} + \frac{\alpha_{-j}}{n_{-j}}\right] - e \ .$$

If agent $i-j$ deviates to $x_{i-j} = e' \in \mathbb{R}_+$, then

$$\min\left\{\mathbf{x}_j\right\} = \min\left\{\mathbf{x}_{-j}\right\} = 0 \text{ and}$$

$$\sum_{i=1}^{n_j} x_{ij} = n_j\gamma_j, \sum_{i=1}^{n_{-j}} x_{i-j} = n_{-j}\gamma_{-j} - e + e' \ ,$$

so that the deviation payoff is

$$\pi^D_{i-j}\left(\alpha,\gamma_j,\gamma'_{-j}\right)=\frac{1}{2}\left[\left(1-\alpha_{-j}\right)\frac{e'}{n_{-j}\gamma_{-j}-e+e'}+\frac{\alpha_{-j}}{n_{-j}}\right]v-e'\ .$$

Hence, for any player $i-j$ there is no incentive to deviate if and only if

$$\frac{1}{2}\left[\left(1-\alpha_{-j}\right)\frac{e}{n_{-j}\gamma_{-j}}+\frac{\alpha_{-j}}{n_{-j}}\right]v-e\geq$$

$$\frac{1}{2}\left[\left(1-\alpha_{-j}\right)\frac{e'}{n_{-j}\gamma_{-j}-e+e'}+\frac{\alpha_{-j}}{n_{-j}}\right]v-e'\Leftrightarrow$$

$$\Leftrightarrow\begin{cases}\alpha_{-j}\geq 1-\frac{2n_{-j}\gamma_{-j}\left(n_{-j}\gamma_{-j}-e+e'\right)}{(n_{-j}\gamma_{-j}-e)v} & \forall\ e'>e\\ \alpha_{-j}\leq 1-\frac{2n_{-j}\gamma_{-j}\left(n_{-j}\gamma_{-j}-e+e'\right)}{(n_{-j}\gamma_{-j}-e)v} & \forall\ e'<e\end{cases}\Leftrightarrow$$

$$\Leftrightarrow\begin{cases}\alpha_{-j}\geq \lim_{\epsilon\to 0}1-\frac{2n_{-j}\gamma_{-j}(n_{-j}\gamma_{-j}-min\{\mathbf{x}_{-j}\}+min\{\mathbf{x}_{-j}\}+\epsilon)}{(n_{-j}\gamma_{-j}-min\{\mathbf{x}_{-j}\})v} & \forall\ e'>e\\ \alpha_{-j}\leq \lim_{\epsilon\to 0}1-\frac{2n_{-j}\gamma_{-j}(n_{-j}\gamma_{-j}-max\{\mathbf{x}_{-j}\}+max\{\mathbf{x}_{-j}\}-\epsilon)}{(n_{-j}\gamma_{-j}-max\{\mathbf{x}_{-j}\})v} & \forall\ e'<e\end{cases}\Leftrightarrow$$

$$\Leftrightarrow\begin{cases}\alpha_{-j}\geq 1-\frac{2(n_{-j}\gamma_{-j})^2}{(n_{-j}\gamma_{-j}-min\{\mathbf{x}_{-j}\})v} & \forall\ e'>e\\ \alpha_{-j}\leq 1-\frac{2(n_{-j}\gamma_{-j})^2}{(n_{-j}\gamma_{-j}-max\{\mathbf{x}_{-j}\})v} & \forall\ e'<e\end{cases}$$

Note that the upper and lower bounds above do cross, since $max\left\{\mathbf{x}_{-j}\right\}>min\left\{\mathbf{x}_{-j}\right\}$. Therefore,

$$\left(\gamma_j,\gamma_{-j}\right)\ \text{such that}\ \gamma_j=0,\gamma_{-j}\in\mathbb{R}_{++},min\left\{\mathbf{x}_1\right\}=min\left\{\mathbf{x}_2\right\}=0\ \text{and}$$

$$\mathbb{1}_{x_{i-j}=0}=1$$

is not a Nash equilibrium for any

$$\left(\alpha_j,\alpha_{-j}\right)\in\mathbb{R}\times\mathbb{R}\ .$$

(j)

$$\left(\gamma_1,\gamma_2\right)\ \text{such that}\ \gamma_j\in\mathbb{R}_{++}\ \text{and}\ \sum_{i=1}^{n_j}\mathbb{1}_{x_{ij}=0}=1\ .$$

The proof follows the corresponding arguments shown at point (i).

■

2.A.2 Proof of Corollary 2.2

In the restricted case, the equilibrium

$$(\gamma_j, \gamma_{-j}) = (e, e) \quad \text{such that } \mathbf{x}_{-j} = \mathbf{e}$$

to be sustained by any $(\alpha_1, \alpha_2) \in [0,1] \times [0,1]$ and not by a subset of $[0,1]$ requires the following conditions

(a)

$$\begin{cases} 1 - \frac{2n_1^2 e}{(n_1-1)v} \leq 0 \\ 1 - \frac{2n_2^2 e}{(n_2-1)v} \leq 0 \end{cases} \Leftrightarrow \begin{cases} e \geq \frac{(n_1-1)}{2n_1^2} v \\ e \geq \frac{(n_2-1)}{2n_2^2} v \end{cases} \Leftrightarrow e \geq \frac{(n_1-1)}{2n_1^2} v \ ;$$

(b)

$$e \leq \frac{v}{2n_2}.$$

Hence

$$e \in \left[\frac{(n_1-1)}{2n_1^2} v, \frac{v}{2n_2}\right]$$

and

$$\left[\frac{(n_1-1)}{2n_1^2} v, \frac{v}{2n_2}\right] \neq \varnothing \Leftrightarrow \frac{1}{n_2} \geq \frac{(n_1-1)}{n_1^2} \Leftrightarrow n_1^2 - n_2 n_1 + n_2 \geq 0 \Leftrightarrow$$

$$n_1 \leq \frac{n_2 - \sqrt{n_2^2 - 4n_2}}{2} \quad \vee \quad n_1 \geq \frac{n_2 + \sqrt{n_2^2 - 4n_2}}{2}.$$

Thus

$$\left[\frac{(n_1-1)}{2n_1^2} v, \frac{v}{2n_2}\right] \neq \varnothing \Leftrightarrow n_1 \in \left[\frac{n_2 + \sqrt{n_2^2 - 4n_2}}{2}, n_2\right] \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} n_1 = n_2 \\ n_1 = n_2 - 1 \\ n_1 = 2, n_2 = 4. \end{cases}$$

■

2.B The Preference Ordering according to a Group Utilitarian Welfare Function

Let

$$\succ_j$$

denote the preference ordering according to a utilitarian welfare function of group j, as induced by the payoff function described above.

Then,

LEMMA 1.1. *The preference relations according to the group utilitarian welfare functions are such that:*

- *if* $n_1 = n_2 - 1$,

$$\begin{aligned}&(e_l, 0) \succ_1 (e_m, 0) \sim_1 (0,0) \succ_1 (e_h, 0) \succ_1 (e, e) \succ_1 (e_{max}, e_{max}) \succ_1 \\ &\succ_1 (0, e_l) \sim_1 (0, e_m) \sim_1 (0, e_h),\end{aligned}$$

$$\begin{aligned}&(0, e_l) \succ_2 (0, e_m) \sim_2 (0,0) \succ_2 (0, e_h) \succ_2 (e, e) \succ_2 (e_{max}, e_{max}) \sim_2 \\ &\sim_2 (e_l, 0) \sim_2 (e_m, 0) \sim_2 (e_h, 0);\end{aligned}$$

- *if* $n_1 = n_2$, *the preference relations according to the group utilitarian welfare functions are:*

$$\begin{aligned}&(e_l, 0) \succ_1 (e_m, 0) \sim_1 (0,0) \succ_1 (e_h, 0) \succ_1 (e, e) \succ_1 (e_{max}, e_{max}) \sim_1 \\ &\sim_1 (0, e_l) \sim_1 (0, e_m) \sim_1 (0, e_h),\end{aligned}$$

$$\begin{aligned}&(0, e_l) \succ_2 (0, e_m) \sim_2 (0,0) \succ_2 (0, e_h) \succ_2 (e, e) \succ_2 (e_{max}, e_{max}) \sim_2 \\ &\sim_2 (e_l, 0) \sim_2 (e_m, 0) \sim_2 (e_h, 0).\end{aligned}$$

Proof. It is immediate to derive the following preference relations according to group utilitarian welfare functions over the different second-period equilibria:

- $$(e',0) \succ_1 (0,e^{''})\ \forall\ e' \in \left(0, \frac{(n_1-1)\,v}{n_1^2}\right] \ \text{ and } \ e^{''} \in \left(0, \frac{(n_2-1)\,v}{n_2^2}\right],$$
  $$(0,e^{''}) \succ_2 (e',0)\ \forall\ e' \in \left(0, \frac{(n_1-1)\,v}{n_1^2}\right] \ \text{ and } \ e^{''} \in \left(0, \frac{(n_2-1)\,v}{n_2^2}\right];$$

- if $n_1 = n_2 - 1$
  $$(e,e) \succ_1 (0,e^{''})\ \forall\ e \in \left[\frac{(n_1-1)\,v}{2n_1^2}, \frac{v}{2n_2}\right] \text{ and } e^{''} \in \left(0, \frac{(n_2-1)\,v}{n_2^2}\right],$$
  $$(e,e) \succ_2 (e',0)\ \forall\ e \in \left[\frac{(n_1-1)\,v}{2n_1^2}, \frac{v}{2n_2}\right) \text{ and } e' \in \left(0, \frac{(n_1-1)\,v}{n_1^2}\right],$$
  $$(e,e) \sim_2 (e',0) \text{ if } e = \frac{v}{2n_2} \text{ and } \forall\ e' \in \left(0, \frac{(n_1-1)\,v}{n_1^2}\right];$$

- if $n_1 = n_2$
  $$(e,e) \succ_1 (0,e^{''})\ \forall\ e \in \left[\frac{(n_1-1)\,v}{2n_1^2}, \frac{v}{2n_2}\right) \text{ and } e^{''} \in \left(0, \frac{(n_2-1)\,v}{n_2^2}\right],$$
  $$(e,e) \sim_1 (0,e^{''}) \text{ if } e = \frac{v}{2n_2} \text{ and } \forall\ e^{''} \in \left(0, \frac{(n_2-1)\,v}{n_2^2}\right];$$
  $$(e,e) \succ_2 (e',0)\ \forall\ e \in \left[\frac{(n_1-1)\,v}{2n_1^2}, \frac{v}{2n_2}\right) \text{ and } e' \in \left(0, \frac{(n_1-1)\,v}{n_1^2}\right],$$
  $$(e,e) \sim_2 (e',0) \text{ if } e = \frac{v}{2n_2} \text{ and } \forall\ e' \in \left(0, \frac{(n_1-1)\,v}{n_1^2}\right];$$

- $$(0,0) \succ_1 (e,e)\ \forall\ e \in \left[\frac{(n_1-1)\,v}{2n_1^2}, \frac{v}{2n_2}\right],$$
  $$(0,0) \succ_2 (e,e)\ \forall\ e \in \left[\frac{(n_1-1)\,v}{2n_1^2}, \frac{v}{2n_2}\right];$$

- $$(e',0) \succ_1 (e,e)\ \forall\ e' \in \left(0, \frac{(n_1-1)\,v}{n_1^2}\right] \text{ and } e \in \left[\frac{(n_1-1)\,v}{2n_1^2}, \frac{v}{2n_2}\right],$$
  $$(0,e^{''}) \succ_2 (e,e)\ \forall\ e^{''} \in \left(0, \frac{(n_2-1)\,v}{n_2^2}\right] \text{ and } e \in \left[\frac{(n_2-1)\,v}{2n_2^2}, \frac{v}{2n_2}\right];$$

- $$(e',0) \succ_1 (0,0)\ \forall\ e' \in \left(0, \frac{v}{2n_1}\right)\ ,$$

$$(e',0) \sim_1 (0,0) \text{ if } e' = \frac{v}{2n_1}\ ,$$

$$(0,0) \succ_1 (e',0)\ \forall\ e' \in \left(\frac{v}{2n_1}, \frac{(n_1-1)\,v}{n_1^2}\right]\ ,$$

$$(0,e^{''}) \succ_2 (0,0)\ \forall\ e^{''} \in \left(0, \frac{v}{2n_2}\right),$$

$$(0,e^{''}) \sim_2 (0,0) \text{ if } e^{''} = \frac{v}{2n_2}\ ,$$

$$(0,0) \succ_2 (e^{''},0)\ \forall\ e^{''} \in \left(\frac{v}{2n_2}, \frac{(n_2-1)\,v}{n_2^2}\right]\ .$$

QED

Thus, we may conclude with the following result.

COROLLARY 1.1. *According to group utilitarian welfare functions, the Pareto efficient effort equilibria are*

$$(e_l, 0), \quad (0, e_l) \quad \textit{and} \quad (0,0).$$

2.C Analytical Examples

2.C.1 Beliefs About Second-Period Equilibria

We present some instances of possible beliefs about the second-period equilibria:

1. suppose that the groups expect the effort stage equilibrium

$$\forall\,(\alpha_1,\alpha_2)\in[0,1]\times[0,1]\,,\quad \left(\gamma_1^I,\gamma_2^I\right)=(e,e)\ \text{ with } \mathbf{x}_j=\mathbf{e}$$

   where

$$e\in\left[\frac{(n_1-1)\,v}{2n_1^2},\frac{v}{2n_2}\right)$$

   so that the players' continuation payoffs are

$$\pi_{ij}\left(\alpha_j,\alpha_{-j}|\gamma_1^I,\gamma_2^I\right)=\frac{1}{2}\left[(1-\alpha_j)\,\frac{e}{n_je}+\alpha_j\frac{1}{n_j}\right]v-e=\frac{1}{2n_j}v-e\ .$$

2. Suppose that the groups expect the effort stage equilibrium

$$\left(\gamma_1^{II},\gamma_2^{II}\right)=\begin{cases}(e,e)\ \text{ with } \mathbf{x}_j=\mathbf{e} & if\quad(\alpha_1,\alpha_2)\in[0,1)\times[0,1)\\ (0,0) & if\quad(\alpha_1,\alpha_2)\in\{1\}\times\{1\}\end{cases}$$

   where

$$e\in\left[\frac{(n_1-1)\,v}{2n_1^2},\frac{v}{2n_2}\right)$$

   so that the players' continuation payoffs are

$$\pi_{ij}\left(\alpha_j,\alpha_{-j}|\gamma_1^{II},\gamma_2^{II}\right)=\begin{cases}\frac{1}{2n_j}v-e & if\quad(\alpha_1,\alpha_2)\in[0,1)\times[0,1)\\ \frac{1}{2n_j}v & if\quad(\alpha_1,\alpha_2)\in\{1\}\times\{1\}\end{cases}$$

3. Suppose that the groups expect the effort stage equilibrium

$$\left(\gamma_1^{III},\gamma_2^{III}\right)=\begin{cases}(e,e)\ \text{ with } \mathbf{x}_j=\mathbf{e} & otherwise\\ (0,e) & if\quad(\alpha_1,\alpha_2)\in[0,1]\times\left\{1-\frac{n_2^2e}{(n_2-1)v}\right\}\end{cases}$$

where

$$e \in \begin{cases} \left[\frac{(n_1-1)v}{2n_1^2}, \frac{v}{2n_2}\right) & \text{if } \left(\gamma_1^{III}, \gamma_2^{III}\right) = (e,e) \text{ with } \mathbf{x}_j = \mathbf{e} \\ \left(0, \frac{v}{2n_2}\right) & \text{if } \left(\gamma_1^{III}, \gamma_2^{III}\right) = (0,e) \end{cases}$$

so that the players' continuation payoffs are

$$\pi_{i1}\left(\alpha_1, \alpha_2 | \gamma_1^{III}, \gamma_2^{III}\right) = \begin{cases} \frac{1}{2n_1}v - e & \mathit{otherwise} \\ 0 & \mathit{if} \quad (\alpha_1, \alpha_2) \in [0,1] \times \left\{1 - \frac{n_2^2 e}{(n_2-1)v}\right\} \end{cases}$$

and

$$\pi_{i2}\left(\alpha_1, \alpha_2 | \gamma_1^{III}, \gamma_2^{III}\right) = \begin{cases} \frac{1}{2n_2}v - e & \mathit{otherwise} \\ \frac{1}{n_2}v - e & \mathit{if} \quad (\alpha_1, \alpha_2) \in [0,1] \times \left\{1 - \frac{n_2^2 e}{(n_2-1)v}\right\} \end{cases}$$

4. Suppose that the groups expect the effort stage equilibrium

$$\left(\gamma_1^{IV}, \gamma_2^{IV}\right) = \begin{cases} (e,e) \text{ with } \mathbf{x}_j = \mathbf{e} & \mathit{otherwise} \\ (e,0) & \mathit{if} \quad (\alpha_1, \alpha_2) \in \left\{1 - \frac{n_1^2 e}{(n_1-1)v}\right\} \times [0,1] \end{cases}$$

where

$$e \in \begin{cases} \left[\frac{(n_1-1)v}{2n_1^2}, \frac{v}{2n_2}\right) & \text{if } \left(\gamma_1^{IV}, \gamma_2^{IV}\right) = (e,e) \text{ with } \mathbf{x}_j = \mathbf{e} \\ \left(0, \frac{v}{2n_1}\right) & \text{if } \left(\gamma_1^{IV}, \gamma_2^{IV}\right) = (e,0) \end{cases}$$

so that the players' continuation payoffs are

$$\pi_{i1}\left(\alpha_1, \alpha_2 | \gamma_1^{IV}, \gamma_2^{IV}\right) = \begin{cases} \frac{1}{2n_1}v - e & \mathit{otherwise} \\ \frac{1}{n_1}v - e & \mathit{if} \quad (\alpha_1, \alpha_2) \in \left\{1 - \frac{n_1^2 e}{(n_1-1)v}\right\} \times [0,1] \end{cases}$$

and

$$\pi_{i2}\left(\alpha_1, \alpha_2 | \gamma_1^{IV}, \gamma_2^{IV}\right) = \begin{cases} \frac{1}{2n_2}v - e & \mathit{otherwise} \\ 0 & \mathit{if} \quad (\alpha_1, \alpha_2) \in \left\{1 - \frac{n_1^2 e}{(n_1-1)v}\right\} \times [0,1] \end{cases}$$

5. Suppose that the groups expect the effort stage equilibrium

$$\left(\gamma_1^{V}, \gamma_2^{V}\right) = \begin{cases} (e,e) \text{ with } \mathbf{x}_j = \mathbf{e} & \mathit{otherwise} \\ (0,0) & \mathit{if} \quad (\alpha_1, \alpha_2) \in \{1\} \times \{1\} \\ (0,e) & \mathit{if} \quad (\alpha_1, \alpha_2) \in [0,1) \times \left\{1 - \frac{n_2^2 e}{(n_2-1)v}\right\} \end{cases}$$

where

$$e \in \begin{cases} \left[\frac{(n_1-1)v}{2n_1^2}, \frac{v}{2n_2}\right) & \text{if } \left(\gamma_1^V, \gamma_2^V\right) = (e,e) \text{ with } \mathbf{x}_j = \mathbf{e} \\ \left(0, \frac{v}{2n_2}\right) & \text{if } \left(\gamma_1^V, \gamma_2^V\right) = (0,e) \end{cases}$$

so that the players' continuation payoffs are

$$\pi_{i1}\left(\alpha_1, \alpha_2 | \gamma_1^V, \gamma_2^V\right) = \begin{cases} \frac{1}{2n_1}v - e & otherwise \\ \frac{1}{2n_1}v & if \quad (\alpha_1, \alpha_2) \in \{1\} \times \{1\} \\ 0 & if \quad (\alpha_1, \alpha_2) \in [0,1] \times \left\{1 - \frac{n_2^2 e}{(n_2-1)v}\right\} \end{cases}$$

and

$$\pi_{i2}\left(\alpha_1, \alpha_2 | \gamma_1^V, \gamma_2^V\right) = \begin{cases} \frac{1}{2n_2}v - e & otherwise \\ \frac{1}{2n_2}v & if \quad (\alpha_1, \alpha_2) \in \{1\} \times \{1\} \\ \frac{1}{n_2}v - e & if \quad (\alpha_1, \alpha_2) \in [0,1] \times \left\{1 - \frac{n_2^2 e}{(n_2-1)v}\right\} \end{cases}$$

6. Suppose that the groups expect the effort stage equilibrium

$$\left(\gamma_1^{VI}, \gamma_2^{VI}\right) = \begin{cases} (e,e) \text{ with } \mathbf{x}_j = \mathbf{e} & otherwise \\ (0,0) & if \quad (\alpha_1, \alpha_2) \in \{1\} \times \{1\} \\ (e,0) & if \quad (\alpha_1, \alpha_2) \in \left\{1 - \frac{n_1^2 e}{(n_1-1)v}\right\} \times [0,1) \end{cases}$$

where

$$e \in \begin{cases} \left[\frac{(n_1-1)v}{2n_1^2}, \frac{v}{2n_2}\right) & \text{if } \left(\gamma_1^{VI}, \gamma_2^{VI}\right) = (e,e) \text{ with } \mathbf{x}_j = \mathbf{e} \\ \left(0, \frac{v}{2n_1}\right) & \text{if } \left(\gamma_1^{VI}, \gamma_2^{VI}\right) = (e,0) \end{cases}$$

so that the players' continuation payoffs are

$$\pi_{i1}\left(\alpha_1, \alpha_2 | \gamma_1^{VI}, \gamma_2^{VI}\right) = \begin{cases} \frac{1}{2n_1}v - e & otherwise \\ \frac{1}{2n_1}v & if \quad (\alpha_1, \alpha_2) \in \{1\} \times \{1\} \\ \frac{1}{n_1}v - e & if \quad (\alpha_1, \alpha_2) \in \left\{1 - \frac{n_1^2 e}{(n_1-1)v}\right\} \times [0,1] \end{cases}$$

and

$$\pi_{i2}\left(\alpha_1, \alpha_2 | \gamma_1^{VI}, \gamma_2^{VI}\right) = \begin{cases} \frac{1}{2n_2}v - e & otherwise \\ \frac{1}{2n_2}v & if \quad (\alpha_1, \alpha_2) \in \{1\} \times \{1\} \\ 0 & if \quad (\alpha_1, \alpha_2) \in \left\{1 - \frac{n_1^2 e}{(n_1-1)v}\right\} \times [0,1] \end{cases}$$

7.a Suppose that the groups expect the effort stage equilibrium

$$\left(\gamma_1^{VIIa}, \gamma_2^{VIIa}\right) =$$

$$\begin{cases} (e,e) \ \text{ with } \mathbf{x}_j = \mathbf{e} & otherwise \\ (e,0) & if \quad (\alpha_1,\alpha_2) \in \left\{1 - \frac{n_1^2 e}{(n_1-1)v}\right\} \times [0,1] \\ (0,e) & if \quad (\alpha_1,\alpha_2) \in \left\{\left[0, 1 - \frac{n_1^2 e}{(n_1-1)v}\right) \cup \left(1 - \frac{n_1^2 e}{(n_1-1)v}, 1\right]\right\} \times \left\{1 - \frac{n_2^2 e}{(n_2-1)v}\right\} \end{cases}$$

where

$$e \in \begin{cases} \left[\frac{(n_1-1)v}{2n_1^2}, \frac{v}{2n_2}\right) & \text{if } \left(\gamma_1^{VIIa}, \gamma_2^{VIIa}\right) = (e,e) \ \text{ with } \mathbf{x}_j = \mathbf{e} \\ \left(0, \frac{v}{2n_1}\right) & \text{if } \left(\gamma_1^{VIIa}, \gamma_2^{VIIa}\right) = (e,0) \\ \left(0, \frac{v}{2n_2}\right) & \text{if } \left(\gamma_1^{VIIa}, \gamma_2^{VIIa}\right) = (0,e) \end{cases}$$

so that the players' continuation payoffs are

$$\pi_{i1}\left(\alpha_1, \alpha_2 | \gamma_1^{VIIa}, \gamma_2^{VIIa}\right) =$$

$$\begin{cases} \frac{1}{2n_1} v - e & otherwise \\ \frac{1}{n_1} v - e & if \quad (\alpha_1,\alpha_2) \in \left\{1 - \frac{n_1^2 e}{(n_1-1)v}\right\} \times [0,1] \\ 0 & if \quad (\alpha_1,\alpha_2) \in \left\{\left[0, 1 - \frac{n_1^2 e}{(n_1-1)v}\right) \cup \left(1 - \frac{n_1^2 e}{(n_1-1)v}, 1\right]\right\} \times \left\{1 - \frac{n_2^2 e}{(n_2-1)v}\right\} \end{cases}$$

and

$$\pi_{i2}\left(\alpha_1, \alpha_2 | \gamma_1^{VIIa}, \gamma_2^{VIIa}\right) =$$

$$\begin{cases} \frac{1}{2n_2} v - e & otherwise \\ 0 & if \quad (\alpha_1,\alpha_2) \in \left\{1 - \frac{n_1^2 e}{(n_1-1)v}\right\} \times [0,1] \\ \frac{1}{n_2} v - e & if \quad (\alpha_1,\alpha_2) \in \left\{\left[0, 1 - \frac{n_1^2 e}{(n_1-1)v}\right) \cup \left(1 - \frac{n_1^2 e}{(n_1-1)v}, 1\right]\right\} \times \left\{1 - \frac{n_2^2 e}{(n_2-1)v}\right\} \end{cases}$$

7.b Suppose that the groups expect the effort stage equilibrium

$$\left(\gamma_1^{VIIb}, \gamma_2^{VIIb}\right) =$$

$$\begin{cases} (e,e) \ \text{ with } \mathbf{x}_j = \mathbf{e} & otherwise \\ (e,0) & if \quad (\alpha_1,\alpha_2) \in \left\{1 - \frac{n_1^2 e}{(n_1-1)v}\right\} \times \left\{\left[0, 1 - \frac{n_2^2 e}{(n_2-1)v}\right) \cup \left(1 - \frac{n_2^2 e}{(n_2-1)v}, 1\right]\right\} \\ (0,e) & if \quad (\alpha_1,\alpha_2) \in [0,1] \times \left\{1 - \frac{n_2^2 e}{(n_2-1)v}\right\} \end{cases}$$

where

$$e \in \begin{cases} \left[\frac{(n_1-1)v}{2n_1^2}, \frac{v}{2n_2}\right) & \text{if } \left(\gamma_1^{VIIb}, \gamma_2^{VIIb}\right) = (e, e) \text{ with } \mathbf{x}_j = \mathbf{e} \\ \left(0, \frac{v}{2n_1}\right) & \text{if } \left(\gamma_1^{VIIb}, \gamma_2^{VIIb}\right) = (e, 0) \\ \left(0, \frac{v}{2n_2}\right) & \text{if } \left(\gamma_1^{VIIb}, \gamma_2^{VIIb}\right) = (0, e) \end{cases}$$

so that the players' continuation payoffs are

$$\pi_{i1}\left(\alpha_1, \alpha_2 | \gamma_1^{VIIb}, \gamma_2^{VIIb}\right) =$$

$$\begin{cases} \frac{1}{2n_1}v - e & otherwise \\ \frac{1}{n_1}v - e \quad if & (\alpha_1, \alpha_2) \in \left\{1 - \frac{n_1^2 e}{(n_1-1)v}\right\} \times \left\{\left[0, 1 - \frac{n_2^2 e}{(n_2-1)v}\right) \cup \left(1 - \frac{n_2^2 e}{(n_2-1)v}, 1\right]\right\} \\ 0 & if \quad (\alpha_1, \alpha_2) \in [0,1] \times \left\{1 - \frac{n_2^2 e}{(n_2-1)v}\right\} \end{cases}$$

and

$$\pi_{i2}\left(\alpha_1, \alpha_2 | \gamma_1^{VIIb}, \gamma_2^{VIIb}\right) =$$

$$\begin{cases} \frac{1}{2n_2}v - e & otherwise \\ 0 \quad if & (\alpha_1, \alpha_2) \in \left\{1 - \frac{n_1^2 e}{(n_1-1)v}\right\} \times \left\{\left[0, 1 - \frac{n_2^2 e}{(n_2-1)v}\right) \cup \left(1 - \frac{n_2^2 e}{(n_2-1)v}, 1\right]\right\} \\ \frac{1}{n_2}v - e & if \quad (\alpha_1, \alpha_2) \in [0,1] \times \left\{1 - \frac{n_2^2 e}{(n_2-1)v}\right\} \end{cases}$$

8.a Suppose that the groups expect the effort stage equilibrium

$$\left(\gamma_1^{VIIIa}, \gamma_2^{VIIIa}\right) =$$

$$\begin{cases} (e, e) \text{ with } \mathbf{x}_j = \mathbf{e} & otherwise \\ (0, 0) & if \quad (\alpha_1, \alpha_2) \in \{1\} \times \{1\} \\ (e, 0) & if \quad (\alpha_1, \alpha_2) \in \left\{1 - \frac{n_1^2 e}{(n_1-1)v}\right\} \times [0, 1) \\ (0, e) & \left\{\left[0, 1 - \frac{n_1^2 e}{(n_1-1)v}\right) \cup \left(1 - \frac{n_1^2 e}{(n_1-1)v}, 1\right)\right\} \times \left\{1 - \frac{n_2^2 e}{(n_2-1)v}\right\} \end{cases}$$

where

$$e \in \begin{cases} \left[\frac{(n_1-1)v}{2n_1^2}, \frac{v}{2n_2}\right) & \text{if } \left(\gamma_1^{VIIIa}, \gamma_2^{VIIIa}\right) = (e, e) \text{ with } \mathbf{x}_j = \mathbf{e} \\ \left(0, \frac{v}{2n_1}\right) & \text{if } \left(\gamma_1^{VIIIa}, \gamma_2^{VIIIa}\right) = (e, 0) \\ \left(0, \frac{v}{2n_2}\right) & \text{if } \left(\gamma_1^{VIIIa}, \gamma_2^{VIIIa}\right) = (0, e) \end{cases}$$

so that the players' continuation payoffs are

$$\pi_{i1}\left(\alpha_1, \alpha_2 | \gamma_1^{VIIIa}, \gamma_2^{VIIIa}\right) =$$

$$\begin{cases} \frac{1}{2n_1}v-e & \mathit{otherwise} \\ \frac{1}{2n_1}v & \mathit{if} \quad (\alpha_1,\alpha_2)\in\{1\}\times\{1\} \\ \frac{1}{n_1}v-e & \mathit{if} \quad (\alpha_1,\alpha_2)\in\left\{1-\frac{n_1^2e}{(n_1-1)v}\right\}\times[0,1) \\ 0 & \mathit{if} \quad (\alpha_1,\alpha_2)\in\left\{\left[0,1-\frac{n_1^2e}{(n_1-1)v}\right)\cup\left(1-\frac{n_1^2e}{(n_1-1)v},1\right)\right\}\times\left\{1-\frac{n_2^2e}{(n_2-1)v}\right\} \end{cases}$$

and

$$\pi_{i2}\left(\alpha_1,\alpha_2|\gamma_1^{VIIIa},\gamma_2^{VIIIa}\right)=$$

$$\begin{cases} \frac{1}{2n_2}v-e & \mathit{otherwise} \\ \frac{1}{2n_1}v & \mathit{if} \quad (\alpha_1,\alpha_2)\in\{1\}\times\{1\} \\ 0 & \mathit{if} \quad (\alpha_1,\alpha_2)\in\left\{1-\frac{n_1^2e}{(n_1-1)v}\right\}\times[0,1) \\ \frac{1}{n_2}v-e & \mathit{if} \quad (\alpha_1,\alpha_2)\in\left\{\left[0,1-\frac{n_1^2e}{(n_1-1)v}\right)\cup\left(1-\frac{n_1^2e}{(n_1-1)v},1\right)\right\}\times\left\{1-\frac{n_2^2e}{(n_2-1)v}\right\} \end{cases}$$

8.b Suppose that the groups expect the effort stage equilibrium

$$\left(\gamma_1^{VIIIb},\gamma_2^{VIIIb}\right)=$$

$$\begin{cases} (e,e)\ \text{ with } \mathbf{x}_j=\mathbf{e} & \mathit{otherwise} \\ (0,0) & \mathit{if} \quad (\alpha_1,\alpha_2)\in\{1\}\times\{1\} \\ (e,0) & \mathit{if} \quad (\alpha_1,\alpha_2)\in\left\{1-\frac{n_1^2e}{(n_1-1)v}\right\}\times\left\{\left[0,1-\frac{n_2^2e}{(n_2-1)v}\right)\cup\left(1-\frac{n_2^2e}{(n_2-1)v},1\right)\right\} \\ (0,e) & \mathit{if} \quad (\alpha_1,\alpha_2)\in[0,1)\times\left\{1-\frac{n_2^2e}{(n_2-1)v}\right\} \end{cases}$$

where

$$e\in\begin{cases} \left[\frac{(n_1-1)v}{2n_1^2},\frac{v}{2n_2}\right) & \text{if } \left(\gamma_1^{VIIIb},\gamma_2^{VIIIb}\right)=(e,e)\ \text{ with } \mathbf{x}_j=\mathbf{e} \\ \left(0,\frac{v}{2n_1}\right) & \text{if } \left(\gamma_1^{VIIIb},\gamma_2^{VIIIb}\right)=(e,0) \\ \left(0,\frac{v}{2n_2}\right) & \text{if } \left(\gamma_1^{VIIIb},\gamma_2^{VIIIb}\right)=(0,e) \end{cases}$$

so that the players' continuation payoffs are

$$\pi_{i1}\left(\alpha_1,\alpha_2|\gamma_1^{VIIIb},\gamma_2^{VIIIb}\right)=$$

$$\begin{cases} \frac{1}{2n_1}v-e & \mathit{otherwise} \\ \frac{1}{2n_1}v & \mathit{if} \quad (\alpha_1,\alpha_2)\in\{1\}\times\{1\} \\ \frac{1}{n_1}v-e & \mathit{if} \quad (\alpha_1,\alpha_2)\in\left\{1-\frac{n_1^2e}{(n_1-1)v}\right\}\times\left\{\left[0,1-\frac{n_2^2e}{(n_2-1)v}\right)\cup\left(1-\frac{n_2^2e}{(n_2-1)v},1\right)\right\} \\ 0 & \mathit{if} \quad (\alpha_1,\alpha_2)\in[0,1)\times\left\{1-\frac{n_2^2e}{(n_2-1)v}\right\} \end{cases}$$

and

$$\pi_{i2}\left(\alpha_1,\alpha_2|\gamma_1^{VIIIb},\gamma_2^{VIIIb}\right)=$$

$$\begin{cases} \frac{1}{2n_2}v-e & otherwise \\ \frac{1}{2n_1}v & if \quad (\alpha_1,\alpha_2)\in\{1\}\times\{1\} \\ 0 & if \quad (\alpha_1,\alpha_2)\in\left\{1-\frac{n_1^2e}{(n_1-1)v}\right\}\times\left\{\left[0,1-\frac{n_2^2e}{(n_2-1)v}\right)\cup\left(1-\frac{n_2^2e}{(n_2-1)v},1\right)\right\} \\ \frac{1}{n_2}v-e & if \quad (\alpha_1,\alpha_2)\in[0,1)\times\left\{1-\frac{n_2^2e}{(n_2-1)v}\right\} \end{cases}$$

2.C.2 Subgame Perfect Equilibria

As an illustration, we provide a few examples of subgame perfect equilibria in pure strategies:

1. if

   $$n_1\geq 4, n_1\in\{n_2-1,n_2\} \text{ and } e\in\left[\frac{(n_1-1)\,v}{2n_1^2},\frac{v}{2n_2}\right) \;,$$

   $$\left(\alpha_1^{SGP},\alpha_2^{SGP}\right)=[0,1]\times[0,1]$$

   $$\left(\gamma_1^{SGP},\gamma_2^{SGP}\right)=(e,e)\,\forall\,(\alpha_1,\alpha_2)\in[0,1]\times[0,1]$$

   with

   $$\left(\alpha_1^{SGP},\alpha_2^{SGP}\right)=[0,1]\times[0,1] \text{ and } \left(\gamma_1^{SGP},\gamma_2^{SGP}\right)=(e,e)$$

   as equilibrium outcomes.

   Comment: if players of both groups expect the equilibrium in which groups tie on any admissible positive effort level, then, clearly, any $\alpha_j\in[0,1]$ is optimal in the first period, that is any incentivization scheme is optimal when expecting such a second-period equilibrium. Such a tie is particularly plausible for $n_1=n_2\geq 4$, in which case no group is advantaged;

2. if

   $$n_1\geq 4, n_1\in\{n_2-1,n_2\} \text{ and } e\in\left[\frac{(n_1-1)\,v}{2n_1^2},\frac{v}{2n_2}\right) \;,$$

   $$\left(\alpha_1^{SGP},\alpha_2^{SGP}\right)=\{1\}\times\{1\}$$

   $$\left(\gamma_1^{SGP},\gamma_2^{SGP}\right)=\begin{cases} (e,e) \text{ with } \mathbf{x}_j=\mathbf{e} & if \quad (\alpha_1,\alpha_2)\in[0,1)\times[0,1) \\ (0,0) & if \quad (\alpha_1,\alpha_2)\in\{1\}\times\{1\} \end{cases}$$

   with

   $$\left(\alpha_1^{SGP},\alpha_2^{SGP}\right)=\{1,1\} \text{ and } \left(\gamma_1^{SGP},\gamma_2^{SGP}\right)=(0,0)$$

as equilibrium outcomes.

Comment: if players in both groups expect the zero-effort equilibrium and the positive-effort tying equilibrium are played in the $\alpha_1 \times \alpha_2$ space, then a fully equalizing sharing rule enables to select the Pareto efficient second-period equilibrium $(\gamma_1, \gamma_2) = (0, 0)$ as the equilibrium outcome;

3. if

$$n_1 \geq 4, n_1 \in \{n_2 - 1, n_2\} \text{ and}$$

$$e \in \begin{cases} \left[\frac{(n_1-1)v}{2n_1^2}, \frac{v}{2n_2}\right) & \text{if } (\gamma_1, \gamma_2) = (e, e) \text{ with } \mathbf{x}_j = \mathbf{e} \\ \left(0, \frac{v}{2n_2}\right) & \text{if } (\gamma_1, \gamma_2) = (0, e) \end{cases},$$

$$\left(\alpha_1^{SGP}, \alpha_2^{SGP}\right) = [0, 1] \times \left\{1 - \frac{n_2^2 e}{(n_2 - 1)\, v}\right\}$$

$$\left(\gamma_1^{SGP}, \gamma_2^{SGP}\right) = \begin{cases} (e, e) \text{ with } \mathbf{x}_j = \mathbf{e} & otherwise \\ (0, e) & if \quad (\alpha_1, \alpha_2) \in [0, 1] \times \left\{1 - \frac{n_2^2 e}{(n_2-1)v}\right\} \end{cases}$$

with

$$\left(\alpha_1^{SGP}, \alpha_2^{SGP}\right) = [0, 1] \times \left\{1 - \frac{n_2^2 e}{(n_2 - 1)\, v}\right\} \text{ and } \left(\gamma_1^{SGP}, \gamma_2^{SGP}\right) = (0, e)$$

as equilibrium outcomes.

Comment: if players in both groups expect the second-period equilibrium in which group 2 wins and the equilibrium in which groups tie on a positive level of effort in the $\alpha_1 \times \alpha_2$ space, then the best-response for group 2 is selecting α_2 ensuring $(\gamma_1, \gamma_2) = (0, e)$, making the first-period choice of group 1 irrelevant and ensuring group-2 victory. Such expectations are easily conceivable in the case $n_1 = n_2$, namely when no group is advantaged. However, they are quite unreasonable if $n_1 < n_2$, for they imply the advantaged group expecting to lose in the second-period.

4. if

$$n_1 \geq 4, n_1 \in \{n_2 - 1, n_2\} \text{ and}$$

$$e \in \begin{cases} \left[\frac{(n_1-1)v}{2n_1^2}, \frac{v}{2n_2}\right) & \text{if } (\gamma_1, \gamma_2) = (e, e) \text{ with } \mathbf{x}_j = \mathbf{e} \\ \left(0, \frac{v}{2n_1}\right] & \text{if } (\gamma_1, \gamma_2) = (e, 0) \end{cases},$$

$$\left(\alpha_1^{SGP}, \alpha_2^{SGP}\right) = \left\{1 - \frac{n_1^2 e}{(n_1 - 1)\, v}\right\} \times [0, 1]$$

$$\left(\gamma_1^{SGP}, \gamma_2^{SGP}\right) = \begin{cases} (e, e) \text{ with } \mathbf{x}_j = \mathbf{e} & otherwise \\ (e, 0) & if \quad (\alpha_1, \alpha_2) \in \left\{1 - \frac{n_1^2 e}{(n_1-1)v}\right\} \times [0, 1] \end{cases}$$

with

$$\left(\alpha_1^{SGP},\alpha_2^{SGP}\right)=\left\{1-\frac{n_1^2e}{\left(n_1-1\right)v}\right\}\times[0,1] \text{ and } \left(\gamma_1^{SGP},\gamma_2^{SGP}\right)=(e,0)$$

as equilibrium outcomes.

Comment: if players in both groups expect the second-period equilibrium in which group 1 wins and the equilibrium in which groups tie on a positive level of effort in the $\alpha_1\times\alpha_2$ space, then the best-response for group 1 is selecting α_1 ensuring $(\gamma_1,\gamma_2)=(e,0)$, making the first-period choice of group 2 irrelevant and ensuring group-1 victory. Such expectations are easily conceivable for both $n_1=n_2$, namely when no group is advantaged, and $n_1<n_2$, for they imply the advantaged group expecting to win in the second-period;

5. if

$$n_1\geq 4, n_1\in\{n_2-1,n_2\} \text{ and}$$

$$e\in\begin{cases}\left[\frac{(n_1-1)v}{2n_1^2},\frac{v}{2n_2}\right) & \text{if } (\gamma_1,\gamma_2)=(e,e) \text{ with } \mathbf{x}_j=\mathbf{e}\\ \left(0,\frac{v}{2n_2}\right) & \text{if } (\gamma_1,\gamma_2)=(0,e)\end{cases},$$

$$\left(\alpha_1^{SGP},\alpha_2^{SGP}\right)\in\{1\}\times\{1\}$$

$$\left(\gamma_1^{SGP},\gamma_2^{SGP}\right)=\begin{cases}(e,e) \text{ with } \mathbf{x}_j=\mathbf{e} & otherwise\\ (0,0) & if \quad (\alpha_1,\alpha_2)\in\{1\}\times\{1\}\\ (0,e) & if \quad (\alpha_1,\alpha_2)\in[0,1)\times\left\{1-\frac{n_2^2e}{(n_2-1)v}\right\}\end{cases}$$

with

$$\left(\alpha_1^{SGP},\alpha_2^{SGP}\right)\in\{1\}\times\{1\} \text{ and } \left(\gamma_1^{SGP},\gamma_2^{SGP}\right)=(0,0)$$

as equilibrium outcomes.

Comment: if players in both groups expect the second-period equilibrium in which group 2 wins, the equilibrium in which groups tie on a positive level of effort, and the equilibrium in which groups tie on the zero-effort equilibrium in the $\alpha_1\times\alpha_2$ space, then best-responses of both groups select (α_1,α_2) ensuring the zero-effort equilibrium in the second-period. However, note that such expectations about the second-period equilibria sustain the first-period equilibrium ensuring the victory of group 2 as well, which is the disadvantaged one for $n_1<n_2$;

6. if

$$n_1\geq 4, n_1\in\{n_2-1,n_2\} \text{ and}$$

$$e \in \begin{cases} \left[\frac{(n_1-1)v}{2n_1^2}, \frac{v}{2n_2}\right) & \text{if } (\gamma_1,\gamma_2)=(e,e) \text{ with } \mathbf{x}_j=\mathbf{e} \\ \left(0, \frac{v}{2n_1}\right) & \text{if } (\gamma_1,\gamma_2)=(e,0) \end{cases},$$

$$\left(\alpha_1^{SGP},\alpha_2^{SGP}\right) \in \{1\}\times\{1\}$$

$$\left(\gamma_1^{SGP},\gamma_2^{SGP}\right) = \begin{cases} (e,e) \text{ with } \mathbf{x}_j=\mathbf{e} & otherwise \\ (0,0) & if \quad (\alpha_1,\alpha_2)\in\{1\}\times\{1\} \\ (e,0) & if \quad (\alpha_1,\alpha_2)\in\left\{1-\frac{n_1^2 e}{(n_1-1)v}\right\}\times[0,1) \end{cases}$$

with

$$\left(\alpha_1^{SGP},\alpha_2^{SGP}\right) \in \{1\}\times\{1\} \text{ and } \left(\gamma_1^{SGP},\gamma_2^{SGP}\right)=(0,0)$$

as equilibrium outcomes.

Comment: if players in both groups expect the second-period equilibrium in which group 1 wins, the equilibrium in which groups tie on a positive level of effort, and the equilibrium in which groups tie on the zero-effort equilibrium in the $\alpha_1 \times \alpha_2$ space, then best-responses of both groups select (α_1, α_2) ensuring the zero-effort equilibrium in the second-period. However, note that such expectations about the second-period equilibria sustain the first-period equilibrium ensuring the victory of group 1 as well, which is the advantaged one for $n_1 < n_2$;

7.a if

$$n_1 \geq 4, n_1 \in \{n_2-1, n_2\} \text{ and}$$

$$e \in \begin{cases} \left[\frac{(n_1-1)v}{2n_1^2}, \frac{v}{2n_2}\right) & \text{if } (\gamma_1,\gamma_2)=(e,e) \text{ with } \mathbf{x}_j=\mathbf{e} \\ \left(0, \frac{v}{2n_1}\right] & \text{if } (\gamma_1,\gamma_2)=(e,0) \\ \left(0, \frac{v}{2n_2}\right) & \text{if } (\gamma_1,\gamma_2)=(0,e) \end{cases},$$

$$\left(\alpha_1^{SGP},\alpha_2^{SGP}\right) = \left\{1-\frac{n_1^2 e}{(n_1-1)\,v}\right\}\times[0,1]$$

$$\left(\gamma_1^{SGP},\gamma_2^{SGP}\right) =$$

$$\begin{cases} (e,e) \text{ with } \mathbf{x}_j=\mathbf{e} & otherwise \\ (e,0) & if \quad (\alpha_1,\alpha_2)\in\left\{1-\frac{n_1^2 e}{(n_1-1)v}\right\}\times[0,1] \\ (0,e) & if \quad (\alpha_1,\alpha_2)\in\left\{\left[0,1-\frac{n_1^2 e}{(n_1-1)v}\right)\cup\left(1-\frac{n_1^2 e}{(n_1-1)v},1\right]\right\}\times\left\{1-\frac{n_2^2 e}{(n_2-1)v}\right\} \end{cases}$$

with

$$\left(\alpha_1^{SGP},\alpha_2^{SGP}\right) = \left\{1-\frac{n_1^2 e}{(n_1-1)\,v}\right\}\times[0,1] \text{ and } \left(\gamma_1^{SGP},\gamma_2^{SGP}\right)=(e,0)$$

as equilibrium outcomes.

Comment: if players in both groups expect the second-period equilibrium in which group 1 wins, the equilibrium in which groups tie on a positive level of effort, and the equilibrium in which group 2 wins in the $\alpha_1 \times \alpha_2$ space, then best-responses of both groups select (α_1, α_2) ensuring the group-1 victory. However, note that such expectations about the second-period equilibria sustain the first-period equilibrium ensuring the victory of group 2 as well, which is the disadvantaged one for $n_1 < n_2$, as shown in the next example;

7.b if

$$n_1 \geq 4, n_1 \in \{n_2 - 1, n_2\} \text{ and}$$

$$e \in \begin{cases} \left[\frac{(n_1-1)v}{2n_1^2}, \frac{v}{2n_2}\right) & \text{if } (\gamma_1, \gamma_2) = (e, e) \text{ with } \mathbf{x}_j = \mathbf{e} \\ \left(0, \frac{v}{2n_1}\right) & \text{if } (\gamma_1, \gamma_2) = (e, 0) \\ \left(0, \frac{v}{2n_2}\right) & \text{if } (\gamma_1, \gamma_2) = (0, e) \end{cases},$$

$$\left(\alpha_1^{SGP}, \alpha_2^{SGP}\right) = [0, 1] \times \left\{1 - \frac{n_2^2 e}{(n_2 - 1)\, v}\right\}$$

$$\left(\gamma_1^{SGP}, \gamma_2^{SGP}\right) =$$

$$\begin{cases} (e, e) \text{ with } \mathbf{x}_j = \mathbf{e} & \textit{otherwise} \\ (e, 0) & \textit{if} \quad (\alpha_1, \alpha_2) \in \left\{1 - \frac{n_1^2 e}{(n_1-1)v}\right\} \times \left\{\left[0, 1 - \frac{n_2^2 e}{(n_2-1)v}\right) \cup \left(1 - \frac{n_2^2 e}{(n_2-1)v}, 1\right]\right\} \\ (0, e) & \textit{if} \quad (\alpha_1, \alpha_2) \in [0, 1] \times \left\{1 - \frac{n_2^2 e}{(n_2-1)v}\right\} \end{cases}$$

with

$$\left(\alpha_1^{SGP}, \alpha_2^{SGP}\right) = [0, 1] \times \left\{1 - \frac{n_2^2 e}{(n_2 - 1)\, v}\right\} \text{ and } \left(\gamma_1^{SGP}, \gamma_2^{SGP}\right) = (0, e)$$

as equilibrium outcomes.

Comment: if players in both groups expect the second-period equilibrium in which group 1 wins, the equilibrium in which groups tie on a positive level of effort, and the equilibrium in which group 2 wins in the $\alpha_1 \times \alpha_2$ space, then best-responses of both groups select (α_1, α_2) ensuring the group-2 victory, despite being the disadvantaged group for $n_1 < n_2$;

8.a if

$$n_1 \geq 4, n_1 \in \{n_2 - 1, n_2\} \text{ and}$$

$$e \in \begin{cases} \left[\frac{(n_1-1)v}{2n_1^2}, \frac{v}{2n_2}\right) & \text{if } (\gamma_1, \gamma_2) = (e, e) \text{ with } \mathbf{x}_j = \mathbf{e} \\ \left(0, \frac{v}{2n_1}\right) & \text{if } (\gamma_1, \gamma_2) = (e, 0) \\ \left(0, \frac{v}{2n_2}\right) & \text{if } (\gamma_1, \gamma_2) = (0, e) \end{cases},$$

$$\left(\alpha_1^{SGP}, \alpha_2^{SGP}\right) = \left\{1 - \frac{n_1^2 e}{(n_1-1)v}\right\} \times [0,1]$$

$$\left(\gamma_1^{SGP}, \gamma_2^{SGP}\right) =$$

$$\begin{cases} (e,e) \text{ with } \mathbf{x}_j = \mathbf{e} & \mathit{otherwise} \\ (0,0) & \mathit{if} \quad (\alpha_1, \alpha_2) \in \{1\} \times \{1\} \\ (e,0) & \mathit{if} \quad (\alpha_1, \alpha_2) \in \left\{1 - \frac{n_1^2 e}{(n_1-1)v}\right\} \times [0,1) \\ (0,e) & \left\{\left[0, 1 - \frac{n_1^2 e}{(n_1-1)v}\right) \cup \left(1 - \frac{n_1^2 e}{(n_1-1)v}, 1\right)\right\} \times \left\{1 - \frac{n_2^2 e}{(n_2-1)v}\right\} \end{cases}$$

with

$$\left(\alpha_1^{SGP}, \alpha_2^{SGP}\right) = \left\{1 - \frac{n_1^2 e}{(n_1-1)v}\right\} \times [0,1] \text{ and } \left(\gamma_1^{SGP}, \gamma_2^{SGP}\right) = (e, 0)$$

as equilibrium outcomes.

Comment: if players in both groups expect the second-period equilibrium in which group 1 wins, the equilibrium in which groups tie on a positive level of effort, the equilibrium in which groups tie on a zero level of effort, and the equilibrium in which group 2 wins in the $\alpha_1 \times \alpha_2$ space, then best-responses of both groups select (α_1, α_2) ensuring the group-1 victory. However, note that such expectations about the second-period equilibria sustain the first-period equilibrium ensuring the victory of group 2 as well, which is the disadvantaged one for $n_1 < n_2$, as shown in the next example;

8.b if

$$n_1 \geq 4, n_1 \in \{n_2 - 1, n_2\} \text{ and}$$

$$e \in \begin{cases} \left[\frac{(n_1-1)v}{2n_1^2}, \frac{v}{2n_2}\right) & \text{if } (\gamma_1, \gamma_2) = (e, e) \text{ with } \mathbf{x}_j = \mathbf{e} \\ \left(0, \frac{v}{2n_1}\right) & \text{if } (\gamma_1, \gamma_2) = (e, 0) \\ \left(0, \frac{v}{2n_2}\right) & \text{if } (\gamma_1, \gamma_2) = (0, e) \end{cases},$$

$$\left(\alpha_1^{SGP}, \alpha_2^{SGP}\right) = [0,1] \times \left\{1 - \frac{n_2^2 e}{(n_2-1)v}\right\}$$

$$\left(\gamma_1^{SGP}, \gamma_2^{SGP}\right) =$$

$$\begin{cases} (e,e) \text{ with } \mathbf{x}_j = \mathbf{e} & \textit{otherwise} \\ (0,0) & \textit{if} \quad (\alpha_1,\alpha_2) \in \{1\} \times \{1\} \\ (e,0) & \textit{if} \quad (\alpha_1,\alpha_2) \in \left\{1-\frac{n_1^2 e}{(n_1-1)v}\right\} \times \left\{\left[0, 1-\frac{n_2^2 e}{(n_2-1)v}\right) \cup \left(1-\frac{n_2^2 e}{(n_2-1)v}, 1\right)\right\} \\ (0,e) & \textit{if} \quad (\alpha_1,\alpha_2) \in [0,1) \times \left\{1-\frac{n_2^2 e}{(n_2-1)v}\right\} \end{cases}$$

with

$$\left(\alpha_1^{SGP}, \alpha_2^{SGP}\right) = [0,1] \times \left\{1 - \frac{n_2^2 e}{(n_2-1)\,v}\right\} \text{ and } \left(\gamma_1^{SGP}, \gamma_2^{SGP}\right) = (0,e)$$

as equilibrium outcomes;

Comment: if players in both groups expect the second-period equilibrium in which group 1 wins, the equilibrium in which groups tie on a positive level of effort, the equilibrium in which groups tie on a zero level of effort, and the equilibrium in which group 2 wins in the $\alpha_1 \times \alpha_2$ space, then best-responses of both groups select (α_1, α_2) ensuring the victory of group 2, despite being the disadvantaged one for $n_1 < n_2$.

3 Max-Min Group Contests with Incomplete Information à la Global Games

Nam in omni actione principaliter intenditur ab agente, sive necessitate naturae sive voluntarie agat, propriam similitudinem explicare; unde fit quod omne agens, in quantum huiusmodi, delectatur, quia, cum omne quod est appetat suum esse, ac in agendo agentis esse modammodo amplietur, sequitur de necessitate delectatio... Nihil igitur agit nisi tale existens quale patiens fieri debet.

(Dante)

Max-Min Group Contests with Incomplete Information à la Global Games [*]

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Abstract

The main novelty of this paper is the introduction of incomplete information à la global games into max-min group contests with binary actions. Depending on whether the complete information assumption is relaxed on the value of the prize or on the cost of providing effort, I obtain different results in terms of equilibrium selection: in the first case, there exist both an equilibrium in (monotonic) switching strategies and an equilibrium robust to incomplete information in the sense of Kajii and Morris (1997), in which no player exerts effort in both groups, whereas in the second one there exists a unique equilibrium in (monotonic) switching-strategies. Then, I discuss the presence of the group-size paradox for both classes of games. The results are thus extended to the case of M groups, and the properties of Bayes-Nash equilibria for these classes of games are investigated. Finally, I show a limit-uniqueness and a noise independent selection result.

JEL classification: D74, D71, C72

Keywords: Group Contests; Incomplete Information; Global Games

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3.1 Introduction

Complete information games with strategic complementarities do often display multiple Nash equilibria. This is the case for group contests, as well. When within-group complementarities in group contests are modelled by the weakest-link impact function, this multiplicity translates into the continuum of pure strategy Nash equilibria result found by Chowdhury et al. (2016) for the case of a public good prize; whereas Gilli and Sorrentino (2024) and Gilli and Sorrentino (2025) analyze the private good prize setting with binary and continuous effort provision choices, respectively, and confirm the strong multiplicity result. These works share the complete information assumption. Departing from this assumption, Barbieri, Kovenock, et al. (2019) confirm the multiplicity result for group contests with the weakest-link impact function, a public good prize and incomplete information about the cost of exerting effort. The authors focus on the interplay between dispersed private information about the cost of effort and the weakest-link impact function and characterize the set of Bayes-Nash equilibria in pure strategies. Despite allowing for very general distributions of the cost of effort, with common support being an unnecessary assumption, nevertheless, the uniqueness result they get is limited to nondegenerate equilibria without mass-points at the top. Moreover, they find a continuum of nondegenerate equilibria with mass-points at the top and degenerate equilibria. Group-public randomization has been recently adopted as a way to address the issue of multiplicity of equilibria in group contests with the weakest-link and the best-shot impact functions by Barbieri and Topolyan (2024). The authors find that group-public randomization exacerbates the multiplicity in weakest-link group contests, whereas it selects a unique equilibrium in best-shot counterparts. However, they are able to find an equilibrium refinement which delivers equilibrium uniqueness, that is stability against coalitional deviations, or rather a strong Bayes-Nash equilibrium. Following Cass and Shell (1983), they do claim that group-public randomization represents a source of extrinsic incomplete information, rather than an intrinsic one, as it does not affect payoff-relevant parameters.

The main novelty of this paper is the introduction of incomplete information à la global games into max-min group contests with binary actions. As well known, global games introduced by Carlsson and Damme (1993a) relax the complete information assumption in 2×2 games in such a way that a unique equilibrium in switching strategies is selected as the noise vanishes, independently from its distribution, as the result of iterated deletion of (interim) strictly-dominated strategies. In this paper, I perturb complete information about the value of the prize contested and about the cost of providing effort, separately. In the first case, I find both a unique equilibrium in (monotonic) switching-strategies and an equilibrium robust to incomplete information à la Kajii and Morris (1997), in which no player exerts effort; in the second one, I obtain a unique equilibrium in (monotonic) switching-strategies. The properties of the payoff-structure of the underlying complete information games are the key to understand this difference. My results are closely related to the generalization of payoffs perturbation of 2×2 games to the n-player case of stag hunt games due to Carlsson and Damme (1993b), where it is apparent that risk-dominance fails to be the appropriate equilibrium selection criterion, when going beyond the two-player case. Then, I identify the presence of the so-called group-size paradox driven by the weakest-link impact function in our two-group contest, in terms of both probability of winning and expected payoff. Moreover, I provide an M-group generalization of the two-group contest model under incomplete information à la global games, proving that my results do not crucially hinge on the two-group assumption. On the other hand, somehow mirroring Carlsson and Damme (1993a), I show a limit-uniqueness and a noise independent selection result, but in a less general fashion than what done by the two authors. Finally, I provide the intuition as to why our results are not generally communication-proof, by considering the impact of cheap talk within-groups on players' second order beliefs.

As stressed in Gilli and Sorrentino (2024), many economic applications can be modelled as competition between groups with agents choosing whether to exert effort or not. Contests with binary decisions have also been the object of a wide theoretical and experimental literature, spanning from corporate science, to sabotage activities and contests for status, as reviewed by Sheremeta (2018). Entry decision in a market where consortia of enterprises could possibly compete is an example of a max-min group contest with binary actions. [3.1] Technology adoption by groups of companies can be modeled similarly. In these last two examples, incomplete information about the value of the market share, or about the efficiency of a new technology to be employed, is a particularly sensible modeling choice. On the other hand, sports, music performances and research activities are some settings presented by Barbieri, Kovenock, et al. (2019) as possible applications of their perfectly-discriminating group contest with perfect within-group complementarities. Back to Chowdhury et al. (2016), R&D competition, negative campaigning on multiple dimensions for products or elections, cyber-security conflict are additional examples presented as being suitable examples of max-min group contests.

The paper is structured as follows. In Section 3.2 the formal model with both complete information and incomplete information is presented under two different specifications. Section 3.3 presents two examples to clarify the parallelism between group contests and the supermodular payoff structure perturbed in the global games à la Carlsson and Damme (1993a) and how equilibrium selection naturally arises when modelling incomplete information à la global games. Section 3.4 derives the set of Nash equilibria of the complete information game, while Section 3.5 is the core of the paper deriving the set of Bayes-Nash equilibria for the two classes of incomplete information games. Section 3.6 addresses whether there is the so-called group-size paradox in the two model specifications delivered in Section 3.5. Section 3.7 extends the two-group contest model under incomplete information to an M-group contest model, while Section 3.8 delivers results regarding limit-uniqueness and noise independent selection, and provides the intuition as to why our results are not generally communication-proof. Finally, Section 3.9 concludes.

3.2 The Model

Let us consider a group contest defined by the following elements:

1. two **groups,** denoted by $j \in \{1, 2\}$;

[3.1] I thank Stefano Barbieri for suggesting this example.

2. each group has $n_j \geq 2$ members, where $n_1 \geq n_2$ without loss of generality. The total number of agents is $n_1 + n_2 = N$. As notation device, let us write ij or $j(i)$ for **agents** $i \in \{1, \ldots, n_j\}$ of group j;

3. the **choice** of member i in group j, to increase the possibility of getting the prize, is denoted by $x_j(i) \in \{0, 1\}$. Let $\mathbf{x}_j$ be the vector of all agents' efforts of group j, and $\mathbf{x}$ the vector of all agents' efforts. Moreover, let $x_j(i) = 1$ be denoted by a and $x_j(i) = 0$ by $\overline{a}$; let us define the average exerted effort in group j, or rather the participation rate in group j as

$$\gamma_j = \frac{1}{n_j} \sum_{i=1}^{n_j} x_{ij} \in [0, 1].$$

Moreover, when $\gamma_j \in (0, 1)$, denote by

$$\gamma_j^+ = \frac{1}{n_j} \left(\sum_{i=1}^{n_j} x_{ij} + 1 \right) \in [0, 1]$$

the share of active agents at a marginal increase and by

$$\gamma_j^- = \frac{1}{n_j} \left(\sum_{i=1}^{n_j} x_{ij} - 1 \right) \in [0, 1]$$

the share of active agents at a marginal decrease.

4. a club good **prize** worth $v \in \mathbb{R}$ to be allocated to one of the two groups: thus, the prize v can be a bad. Note that the value of the prize contested $v \in \mathbb{R}$ could be conceived as the continuation payoff from a possible second-stage of the game in which players compete by exerting continuous effort, after a binary decision about market entry or new technology adoption has been made, for instance. [3.2]

5. the **impact function** of group j is given by the weakest-link technology

$$X_j = \min \{x_j(i) \in \{0, 1\}, i \in \{1, ..., n_j\}\};$$

[3.2] I do thank Stefano Barbieri for this suggestion.

6. the **contest success function** is given by the *all-pay auction:*

$$p_j(X_1, X_2) = \begin{cases} 1 & if \quad X_j > X_{-j} \\ \frac{1}{2} & if \quad X_j = X_{-j} \\ 0 & if \quad X_j < X_{-j}; \end{cases}$$

7. the individual **cost of effort** $C(x_j(i)) = x_j(i)$.

As a consequence of these modelling characteristics, player ij has the expected **payoff**

$$\pi_{ij}(\mathbf{x}_1, \mathbf{x}_2) = p_j v - x_{ij} = \begin{cases} v - x_j(i) & if \quad \min\{\mathbf{x}_j\} > \min\{\mathbf{x}_{-j}\} \\ \frac{1}{2}v - x_j(i) & if \quad \min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} \\ -x_j(i) & if \quad \min\{\mathbf{x}_j\} < \min\{\mathbf{x}_{-j}\}. \end{cases}$$

Now I am able to provide a formal definition of a binary max-min group contest with a public good prize.

DEFINITION 3.1. *A* Binary Max-Min Group Contest $BMMGC^*$ *is a one-stage game* $BMMGC^* = \langle\{1, 2\}, N, B_{ij}, \pi_{ij}\rangle$ *defined by*

1. *the set of groups* $\{1, 2\}$;
2. *the set of players* $N = \{1, ..., n_1 + n_2\}$;
3. *the set of actions* $B_{ij} = \{0, 1\}$: *for each player* ij*, the choice of the effort* $x_j(i)$;
4. *the payoff functions for each player* $ij \in N$

$$\pi_{ij}(\mathbf{x}_1, \mathbf{x}_2) = p_j v - x_{ij} = \begin{cases} v - x_j(i) & if \quad \min\{\mathbf{x}_j\} > \min\{\mathbf{x}_{-j}\} \\ \frac{1}{2}v - x_j(i) & if \quad \min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} \\ -x_j(i) & if \quad \min\{\mathbf{x}_j\} < \min\{\mathbf{x}_{-j}\} \end{cases}.$$

The notation used in this paper is summed up in table 3.1.

Variable	Meaning
ij or $j\,(i)$	agent i of group j
$\{1, ..., n_j\}$	set of agents in group j
$x_j\,(i)$ or x_{ij}	effort of agent i in group j
$X_j = \min\{x_j\,(i) \in \{0,1\}\,, i \in \{1, ..., n_j\}\}$	impact of effort of all agents in group j
$\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2)$	vector of efforts of all agents
$C\,(x_j\,(i)) = x_j\,(i)$	cost of effort for agent i of group j
$p_j\,(X_1, X_2)$	probability of group j of winning the contest
$\pi_{ij}\,(\mathbf{x}_1, \mathbf{x}_2)$	payoff function of agent i of group j
$\gamma_j = \frac{1}{n_j}\sum_{i=1}^{n_j} x_{ij} \in [0,1]$	share of active agents in group j

Table 3.1

3.3 An Example

Let us consider a $BMMGC^*$ with two members for each group. [3.3] W.l.g., let players $1, 2$ belong to group 1 and players $3, 4$ to group 2. Consider the following geometric representation of the game, where player 3 "moves horizontally", while player 4 "moves vertically":

3 a $\overline{a}$

1/2	a	$\overline{a}$
a	$\frac{v}{2}-1; \frac{v}{2}-1; \frac{v}{2}-1; \frac{v}{2}-1$	$-1; 0; v-1; v-1$
$\overline{a}$	$0; -1; v-1; v-1$	$0; 0; v-1; v-1$

1/2	a	$\overline{a}$
a	$v-1; v-1; 0; -1$	$\frac{v}{2}-1; \frac{v}{2}; \frac{v}{2}; \frac{v}{2}-1$
$\overline{a}$	$\frac{v}{2}; \frac{v}{2}-1; \frac{v}{2}; \frac{v}{2}-1$	$\frac{v}{2}; \frac{v}{2}; \frac{v}{2}; \frac{v}{2}-1$

a

1/2	a	$\overline{a}$
a	$v-1; v-1; -1; 0$	$\frac{v}{2}-1; \frac{v}{2}; \frac{v}{2}-1; \frac{v}{2}$
$\overline{a}$	$\frac{v}{2}; \frac{v}{2}-1; \frac{v}{2}-1; \frac{v}{2}$	$\frac{v}{2}; \frac{v}{2}; \frac{v}{2}-1; \frac{v}{2}$

1/2	a	$\overline{a}$
a	$v-1; v-1; 0; 0$	$\frac{v}{2}-1; \frac{v}{2}; \frac{v}{2}; \frac{v}{2}$
$\overline{a}$	$\frac{v}{2}; \frac{v}{2}-1; \frac{v}{2}; \frac{v}{2}$	$\frac{v}{2}; \frac{v}{2}; \frac{v}{2}; \frac{v}{2}$

$\overline{a}$

4

It is straightforward to derive the following properties:

[3.3] This section is a direct application of the example carried out by Carlsson and Damme (1993a) in their introduction.

- if $v > 2$, there are four strict Nash equilibria in pure strategies

  $$NE \equiv \{(a,a,a,a)\,;(\overline{a},\overline{a},a,a)\,;(a,a,\overline{a},\overline{a})\,;(\overline{a};\overline{a};\overline{a};\overline{a})\}$$

  and a Nash equilibrium in symmetric strictly-mixed strategies $\sigma_i^*(a) = \frac{2}{v}\ \forall i \in \{1,2,3,4\}$;

- if $v = 2$, there are four Nash equilibria in pure strategies

  $$NE \equiv \{(a,a,a,a)\,;(\overline{a},\overline{a},a,a)\,;(a,a,\overline{a},\overline{a})\,;(\overline{a};\overline{a};\overline{a};\overline{a})\}\,;$$

- if $v < 2$, the unique Nash equilibrium derived by strict-dominance is $(\overline{a},\overline{a},\overline{a},\overline{a})$;

- if $v > 2$, $(a,a,\overline{a},\overline{a})$ payoff-dominates $(\overline{a},\overline{a},\overline{a},\overline{a})$ for group 1 and $(\overline{a},\overline{a},a,a)$ payoff-dominates $(\overline{a},\overline{a},\overline{a},\overline{a})$ for group 2;[3.4]

- if $v > 4$, (a,a,a,a) and $(a,a,\overline{a},\overline{a})$ are the risk-dominant equilibrium strategy profiles for group 1 and, symmetrically, (a,a,a,a) and $(\overline{a},\overline{a},a,a)$ are the risk-dominant equilibrium strategy profiles for group 2. As a matter of fact, let us compare the Nash products of (a,a,a,a) and $(\overline{a},\overline{a},a,a)$. Then, for group 1:

  $$\left(\frac{v}{2}-1\right)\left(\frac{v}{2}-1\right) > (0-(-1))\,(0-(-1)) \Leftrightarrow \left(\frac{v}{2}-1\right)^2 > 1 \Leftrightarrow v > 4,$$

  that is, for $v > 4$, (a,a) is associated with the largest Nash product.

  Moreover, let us compare the Nash products of $(a,a,\overline{a},\overline{a})$ and $(\overline{a},\overline{a},\overline{a},\overline{a})$. Then, for group 1:

  $$\left(v-1-\frac{v}{2}\right)\left(v-1-\frac{v}{2}\right) > \left(\frac{v}{2}-\left(\frac{v}{2}-1\right)\right)\left(\frac{v}{2}-\left(\frac{v}{2}-1\right)\right) \Leftrightarrow \left(\frac{v}{2}-1\right)^2 > 1 \Leftrightarrow v > 4\,,$$

  that is, for $v > 4$, (a,a) is associated with the largest Nash product.

  The same inequalities hold symmetrically for group 2 as well;

[3.4] For the formulation of payoff-dominace and risk-dominance concepts see Harsanyi and Selten (1988).

- if $2 < v < 4$, $(\overline{a}, \overline{a}, a, a)$ and $(\overline{a}, \overline{a}, \overline{a}, \overline{a})$ are the risk-dominant equilibrium strategy profiles for group 1 and, symmetrically, $(a, a, \overline{a}, \overline{a})$ and $(\overline{a}, \overline{a}, \overline{a}, \overline{a})$ are the risk-dominant equilibrium strategy profiles for group 2 . Clearly this follows from what shown at the previous point for both groups;

- overall, there is a one-sided dominance region: for $v < 2$, a is a strictly dominated action.

Finally, note that, for $2 < v < 4$, $(a, a, \overline{a}, \overline{a})$ is the payoff-dominant equilibrium strategy profile for group 1, whereas $(\overline{a}, \overline{a}, a, a)$ and $(\overline{a}, \overline{a}, \overline{a}, \overline{a})$ are the risk-dominant equilibrium strategy profiles for group 1, and, symmetrically, $(\overline{a}, \overline{a}, a, a)$ is the payoff-dominant equilibrium strategy profile for group 2, whereas $(a, a, \overline{a}, \overline{a})$ and $(\overline{a}, \overline{a}, \overline{a}, \overline{a})$ are the risk-dominant equilibrium strategy profiles for group 2. Hence, there is a tension between payoff-dominance and risk-dominance. Now let us consider a slight variation of the game above and let:

- the individual **cost of effort** $C(x_j(i)) = c$ with $c \in \mathbb{R}$ and the club good **prize** worth $v > 0$. Thus, cost of effort may be negative, which means that agents could enjoy effort per se, while the prize v is always worth positive utils, so that it is a good.[3.5]

Then, I have the following representation of the game, where player 3 "moves horizontally" and player 4 "moves vertically":

3 a $\overline{a}$

1/2	a	$\overline{a}$
a	$\frac{v}{2} - c; \frac{v}{2} - c; \frac{v}{2} - c; \frac{v}{2} - c$	$-c; 0; v - c; v - c$
$\overline{a}$	$0; -c; v - c; v - c$	$0; 0; v - c; v - c$

1/2	a	$\overline{a}$
a	$v - c; v - c; 0; -c$	$\frac{v}{2} - c; \frac{v}{2}; \frac{v}{2}; \frac{v}{2} - c$
$\overline{a}$	$\frac{v}{2}; \frac{v}{2} - c; \frac{v}{2}; \frac{v}{2} - c$	$\frac{v}{2}; \frac{v}{2}; \frac{v}{2}; \frac{v}{2} - c$

a

1/2	a	$\overline{a}$
a	$v - c; v - c; -c; 0$	$\frac{v}{2} - c; \frac{v}{2}; \frac{v}{2} - c; \frac{v}{2}$
$\overline{a}$	$\frac{v}{2}; \frac{v}{2} - c; \frac{v}{2} - c; \frac{v}{2}$	$\frac{v}{2}; \frac{v}{2}; \frac{v}{2} - c; \frac{v}{2}$

1/2	a	$\overline{a}$
a	$v - c; v - c; 0; 0$	$\frac{v}{2} - c; \frac{v}{2}; \frac{v}{2}; \frac{v}{2}$
$\overline{a}$	$\frac{v}{2}; \frac{v}{2} - c; \frac{v}{2}; \frac{v}{2}$	$\frac{v}{2}; \frac{v}{2}; \frac{v}{2}; \frac{v}{2}$

$\overline{a}$

4

[3.5] Clearly, under complete information, for $c \in \mathbb{R}_{++}$ and $v \in \mathbb{R}_{++}$, the cost of effort $C_{ij}(x_j(i))$ can always be normalized to one via a simple change of variables.

It is straightforward to derive the following properties:

- if $c < 0$, the unique Nash equilibrium derived by strict dominance is (a, a, a, a);

- if $c > \frac{v}{2}$, the unique Nash equilibrium derived by strict dominance is $(\overline{a}, \overline{a}, \overline{a}, \overline{a})$;

- if $0 \leq c < \frac{v}{2}$, there are four strict Nash equilibria in pure strategies

$$NE = \{(a, a, a, a), (a, a, \overline{a}, \overline{a}), (\overline{a}, \overline{a}, a, a), (\overline{a}, \overline{a}, \overline{a}, \overline{a})\}$$

  and an equilibrium in symmetric strictly mixed strategies $\sigma_i^*(a) = \frac{2c}{v} \quad \forall i \in \{1, 2, 3, 4\}$;

- if $c = \frac{v}{2}$, there are four Nash equilibria in pure strategies

$$NE = \{(a, a, a, a), (a, a, \overline{a}, \overline{a}), (\overline{a}, \overline{a}, a, a), (\overline{a}, \overline{a}, \overline{a}, \overline{a})\};$$

- if $c < \frac{v}{2}$, $(a, a, \overline{a}, \overline{a})$ payoff-dominates $(\overline{a}, \overline{a}, \overline{a}, \overline{a})$ for group 1 and $(\overline{a}, \overline{a}, a, a)$ payoff-dominates $(\overline{a}, \overline{a}, \overline{a}, \overline{a})$ for group 2;

- if $0 \leq c < \frac{v}{4}$, (a, a, a, a) and $(a, a, \overline{a}, \overline{a})$ are the risk-dominant equilibrium strategy profiles for group 1 and, symmetrically, (a, a, a, a) and $(\overline{a}, \overline{a}, a, a)$ are the risk-dominant equilibrium strategy profiles for group 2 . As a matter of fact, let us compare the Nash products of (a, a, a, a) and $(\overline{a}, \overline{a}, a, a)$. Then, for group 1:

$$\left(\frac{v}{2} - c\right)\left(\frac{v}{2} - c\right) > (0 - (-c))(0 - (-c)) \Leftrightarrow \left(\frac{v}{2} - c\right)^2 > c^2 \Leftrightarrow v > 4c \Leftrightarrow c < \frac{v}{4},$$

  that is, for $c < \frac{v}{4}$, (a, a) is associated with the largest Nash product.

  Moreover, let us compare the Nash products of $(a, a, \overline{a}, \overline{a})$ and $(\overline{a}, \overline{a}, \overline{a}, \overline{a})$. Then, for group 1:

$$\left(v - c - \frac{v}{2}\right)\left(v - c - \frac{v}{2}\right) > \left(\frac{v}{2} - \left(\frac{v}{2} - c\right)\right)\left(\frac{v}{2} - \left(\frac{v}{2} - c\right)\right) \Leftrightarrow \left(\frac{v}{2} - c\right)^2 > c^2 \Leftrightarrow c < \frac{v}{4},$$

  that is, for $c < \frac{v}{4}$, (a, a) is associated with the largest Nash product.

The same inequalities hold symmetrically for group 2 as well;

- if $\frac{v}{4} < c < \frac{v}{2}$, $(\overline{a}, \overline{a}, a, a)$ and $(\overline{a}, \overline{a}, \overline{a}, \overline{a})$ are the risk-dominant equilibrium strategy profiles for group 1 and, symmetrically, $(a, a, \overline{a}, \overline{a})$ and $(\overline{a}, \overline{a}, \overline{a}, \overline{a})$ are the risk-dominant equilibrium strategy profiles for group 2 . Clearly this follows from what shown at the previous point for both groups;
- overall, there are two dominance regions: for $c < 0$, $\overline{a}$ is a strictly dominated action; for $c > \frac{v}{2}$, a is a strictly dominated action.

Finally, note that, for $\frac{v}{4} < c < \frac{v}{2}$, $(a, a, \overline{a}, \overline{a})$ is the payoff-dominant equilibrium strategy profile for group 1, whereas $(\overline{a}, \overline{a}, a, a)$ and $(\overline{a}, \overline{a}, \overline{a}, \overline{a})$ are the risk-dominant equilibrium strategy profiles for group 1, and, symmetrically, $(\overline{a}, \overline{a}, a, a)$ is the payoff-dominant equilibrium strategy profile for group 2, whereas $(a, a, \overline{a}, \overline{a})$ and $(\overline{a}, \overline{a}, \overline{a}, \overline{a})$ are the risk-dominant equilibrium strategy profiles for group 2. Hence, there is a tension between payoff-dominance and risk-dominance.

In turn, I would like to draw a possible comparison with the classical example due to Carlsson and Damme (1993a) about a 2 × 2 game under complete information, reported in table 4.1.

	α_2	β_2
α_1	x, x	$x, 0$
β_1	$0, x$	$4, 4$

Table 3.1: Game $g(x)$ by Carlsson and Damme (1993a) .

Carlsson and Damme (1993a) highlight the following properties of this game under complete information:

- if $x > 4$, the unique Nash equilibrium derived by strict dominance is (α_1, α_2);
- if $x < 0$, the unique Nash equilibrium derived by strict dominance is (β_1, β_2);
- if $0 < x < 4$, there are two strict Nash equilibria, that is (α_1, α_2) and (β_1, β_2);

- if $x \in (2,4)$, (α_1, α_2) is the risk-dominant equilibrium;
- if $x \in (0,2)$, (β_1, β_2) is the risk-dominant equilibrium;
- overall, there are two dominance regions.

Finally, note that, for $2 < x < 4$, (β_1, β_2) is the payoff-dominant equilibrium, whereas (α_1, α_2) is the risk-dominant equilibrium: there is a tension between payoff-dominance and risk-dominance.

3.3.1 Incomplete Information about the Prize

Let us consider the case where the individual **cost of effort** $C(x_j(i)) = x_j(i)$ is commonly known. Henceforth, I refer to this game as $g(v)$. I closely follow Carlsson and Damme (1993a) introducing incomplete information about the prize v as follows:

- let V be a random variable which is uniform on some interval $[\underline{v}, \overline{v}]$ including the dominance region and the threshold for the risk-dominance, e.g. $[1,5]$;
- given the unknown realization v, each player $i \in \{1,2,3,4\}$ idiosyncratically observes the realization of a random variable V_i, uniform on $[v-\varepsilon, v+\varepsilon]$ for some $\varepsilon > 0$, so that the players' observation errors $V_1 - v$, $V_2 - v$, $V_3 - v$ and $V_4 - v$ are independent;
- after these idiosyncratic observations, each player $i \in \{1,2,3,4\}$ simultaneously and independently decides whether to exert effort or not and gets a payoff as described by the strategic form game of $g(v)$;
- note that $E(V|v_i) = v_i$, if i observes $v_i \in [\underline{v}+\varepsilon, \overline{v}-\varepsilon]$ so that $V|v_i \sim U(v_i - \varepsilon, v_i + \varepsilon)$;
- furthermore, for $v_i \in [\underline{v}+\varepsilon, \overline{v}-\varepsilon]$, the conditional distribution of the teammate's or opponents' observation will be centered around v_i with support $[v_i - 2\varepsilon, v_i + 2\varepsilon]$. Hence, $Prob\,[V_{-i} < v_i|v_i] = Prob[V_{-i} > v_i|v_i] = \frac{1}{2}$.

Now, let us further assume $\varepsilon < \left|\frac{v}{2} - 1\right|$ and suppose player $i \in \{1,2,3,4\}$ observes $v_i < 2$. Then, i's conditionally expected payoff from exerting effort, that is choosing a, is smaller than the one from exerting no effort, that is choosing $\overline{a}$. Accordingly, $\overline{a}$ is a conditionally strictly dominant action for player $i \in \{1, 2, 3, 4\}$ whenever she observes $v_i < 2$. Suppose $i = 1$ without loss of generality. Iterating this dominance argument, if players $-i \in \{2, 3, 4\}$ are forced to play $\overline{a}$ whenever they observe $v_{-i} < 2$, then player i, observing $v_i = 2$ has to assign at least probability $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$ to $(\overline{a}_2, \overline{a}_3, \overline{a}_4)$. Thus, i's conditionally expected payoff difference from not exerting effort versus exerting effort, that is choosing $\overline{a}_i$ over a_i, will be at least $\frac{1}{2}$, so that a_i can be discarded by iterated dominance for $v_i = 2$. Let v_i^* be the smallest observation such that a_i cannot be excluded by iterated dominance. Then, it is possible to show that $v_i^* = 4$. Note that $v_i = 4$ is the threshold for the risk-dominance regions as well. As a matter of fact, when $v_i = 4$, the conditionally expected payoff from exerting effort equals

$$\frac{1}{8}\left(\frac{4}{2}-1\right)+\frac{1}{8}(-1)+\frac{1}{8}(4-1)+\frac{1}{8}\left(\frac{4}{2}-1\right)+\frac{1}{8}(4-1)+\frac{1}{8}\left(\frac{4}{2}-1\right)+\frac{1}{8}(4-1)+\frac{1}{8}\left(\frac{4}{2}-1\right)=\frac{3}{2},$$

while the conditionally expected payoff from not exerting effort equals

$$\frac{1}{8}*0+\frac{1}{8}*0+\frac{1}{8}\left(\frac{4}{2}\right)+\frac{1}{8}\left(\frac{4}{2}\right)+\frac{1}{8}\left(\frac{4}{2}\right)+\frac{1}{8}\left(\frac{4}{2}\right)+\frac{1}{8}\left(\frac{4}{2}\right)+\frac{1}{8}\left(\frac{4}{2}\right)=\frac{3}{2}.$$

The cutoff $v_i^* = 4$ is the unique threshold that can be established from the lower dominance regions by iterated deletion of strictly dominated strategies, since it is the unique value for $v_i \in [v - \varepsilon, v + \varepsilon]$ solving

$$\frac{1}{8}\left(\frac{v_i}{2}-1\right)+\frac{1}{8}(-1)+\frac{1}{8}(v_i-1)+\frac{1}{8}\left(\frac{v_i}{2}-1\right)+\frac{1}{8}(v_i-1)+\frac{1}{8}\left(\frac{v_i}{2}-1\right)+\frac{1}{8}(v_i-1)+\frac{1}{8}\left(\frac{v_i}{2}-1\right)$$

$$=\frac{1}{8}*0+\frac{1}{8}*0+\frac{1}{8}\left(\frac{v_i}{2}\right)+\frac{1}{8}\left(\frac{v_i}{2}\right)+\frac{1}{8}\left(\frac{v_i}{2}\right)+\frac{1}{8}\left(\frac{v_i}{2}\right)+\frac{1}{8}\left(\frac{v_i}{2}\right)+\frac{1}{8}\left(\frac{v_i}{2}\right).$$

The same kind of reasoning cannot be carried out for large observations of v, since it does not exist an upper dominance region. Conversely, this is possible in our second setting in which there is incomplete information about the cost of effort itself. As a matter of fact, in the latter there are both a lower dominance region and an upper dominance region. As an intuition, observe that in the first setting, ∀ $v \geq 2$, whether bearing a sunk cost of effort or not depends on considerations about the teammate and the rivals' behavior, for the probability of getting the positive payoff component, i.e. the prize, depends on the entire profile of players' strategies. Conversely, in the second setting for $c < 0$, exerting effort brings a positive payoff component stemming from effort itself for certainty, even in the case in which the teammate does not exert effort. Finally, incentive compatibility above the cutoff is ensured as $Prob\,(V_{-i} < 4|v_i) \leq 1/2 \;\forall\; v_i \geq 4$ holds.

Hence, in $g\,(v)$ under incomplete information à la global games, for sufficiently small ε, there is a unique equilibrium in (monotonic) cutoff strategies, such that $\forall i \in \{1,2,3,4\}$:

$$x_i^*\,(v_i) = \begin{cases} 1 & \text{if } v_i > 4 \\ 0 & \text{if } v_i \leq 4\ . \end{cases}$$

Nonetheless, given the absence of an upward dominance region, the following equilibrium $\forall i \in \{1,2,3,4\}$ exists as in De Mesquita (2011):

$$x_i^{**}\,(v_i) = 0\ \forall v_i \in [\underline{v} + \varepsilon, \overline{v} - \varepsilon]\,.$$

As a matter of fact, at $(\overline{a}_1, \overline{a}_2, \overline{a}_3, \overline{a}_4)$ any deviation is strictly dominated for any $v_i \in [\underline{v} + \varepsilon,$ $\overline{v} - \varepsilon]$, so that $(\overline{a}_1, \overline{a}_2, \overline{a}_3, \overline{a}_4)$ is robust to incomplete information in the sense of Kajii and Morris (1997).

3.3.2 Incomplete Information about the Cost of Effort

Let us consider the case where the individual **cost of effort** is $C\,(x_j\,(i)) = c$ with $c \in \mathbb{R}$ and the club good **prize** worth $v > 0$ is commonly known. Henceforth, I refer to this game as $g\,(c)$. I closely follow Carlsson and Damme (1993a) introducing incomplete information about the cost of effort c as follows:

- let C be a random variable which is uniform on some interval $[\underline{c}, \overline{c}]$ including both dominance regions, e.g. $[-v, +v]$;
- given the unknown realization c, each player $i \in \{1,2,3,4\}$ idiosyncratically observes the realization c_i of a signal C_i, uniform on $[c - \varepsilon, c + \varepsilon]$ for some $\varepsilon > 0$, so that the players' observation errors $C_1 - c$, $C_2 - c$, $C_3 - c$ and $C_4 - c$ are independent;

- after these idiosyncratic observations, each player $i \in \{1, 2, 3, 4\}$ simultaneously and independently decides whether to exert effort or not and gets a payoff as described by the strategic form game $g(c)$;

- note that $E(C|c_i) = c_i$, if i observes $c_i \in [\underline{c} + \varepsilon, \overline{c} - \varepsilon]$ so that $C|c_i \sim U(c_i - \varepsilon, c_i + \varepsilon)$;

- furthermore, for $c_i \in [\underline{c} + \varepsilon, \overline{c} - \varepsilon]$, the conditional distribution of the teammate's or opponents' observation will be centered around c_i with support $[c_i - 2\varepsilon, c_i + 2\varepsilon]$. Hence, $Prob\,[C_{-i} < c_i|c_i] = Prob[C_{-i} > c_i|c_i] = \frac{1}{2}$.

Now, let us further assume $\varepsilon< \left|\frac{2\overline{c}-v}{4}\right|$ and suppose player $i \in \{1, 2, 3, 4\}$ observes $c_i > \frac{v}{2}$. Then, i's conditionally expected payoff from exerting effort, that is choosing a, is smaller than the one from exerting no effort, that is choosing $\overline{a}$. Accordingly, $\overline{a}$ is a conditionally strictly dominant action for player $i \in \{1, 2, 3, 4\}$ whenever she observes $c_i > \frac{v}{2}$. Suppose $i = 1$ without loss of generality. Iterating this dominance argument, if players $-i \in \{2, 3, 4\}$ are forced to play $\overline{a}$ whenever they observe $c_{-i} > \frac{v}{2}$, then player i, observing $c_i = \frac{v}{2}$ has to assign at least probability $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$ to $(\overline{a}_2, \overline{a}_3, \overline{a}_4)$. Thus, i's conditionally expected payoff difference from not exerting effort versus exerting effort, that is choosing $\overline{a}_i$ over a_i will be at least $\frac{1}{4}v$, so that a_i can be discarded by iterated dominance for $c_i = \frac{v}{2}$. Let c_i^* be the smallest observation such that a_i cannot be excluded by iterated dominance. Then, it is possible to show that $c_i^* = \frac{v}{4}$. Note that $c_i = \frac{v}{4}$ is the threshold for the risk-dominance regions as well. As a matter of fact, when $c_i = \frac{v}{4}$, the conditionally expected payoff from exerting effort equals

$$\frac{1}{8}\left(\frac{v}{2} - \frac{v}{4}\right) + \frac{1}{8}\left(-\frac{v}{4}\right) + \frac{1}{8}\left(v - \frac{v}{4}\right) + \frac{1}{8}\left(\frac{v}{2} - \frac{v}{4}\right) + \frac{1}{8}\left(v - \frac{v}{4}\right) + \frac{1}{8}\left(\frac{v}{2} - \frac{v}{4}\right) +$$
$$+\frac{1}{8}\left(v - \frac{v}{4}\right) + \frac{1}{8}\left(\frac{v}{2} - \frac{v}{4}\right) = \frac{3}{8}v$$

and the conditionally expected payoff from not exerting effort equals

$$\frac{1}{8}(0) + \frac{1}{8}(0) + \frac{1}{8}\left(\frac{v}{2}\right) + \frac{1}{8}\left(\frac{v}{2}\right) + \frac{1}{8}\left(\frac{v}{2}\right) + \frac{1}{8}\left(\frac{v}{2}\right) + \frac{1}{8}\left(\frac{v}{2}\right) + \frac{1}{8}\left(\frac{v}{2}\right) = \frac{3}{8}v.$$

The cutoff $c_i^* = \frac{v}{4}$ is the unique threshold that can be established from the upper dominance region by iterated deletion of strictly dominated strategies, since it is the unique value for $c_i \in [c-\epsilon, c+\epsilon]$ solving

$$\frac{1}{8}\left(\frac{v}{2}-c_i\right)+\frac{1}{8}\left(-c_i\right)+\frac{1}{8}\left(v-c_i\right)+\frac{1}{8}\left(\frac{v}{2}-c_i\right)+\frac{1}{8}\left(v-c_i\right)+\frac{1}{8}\left(\frac{v}{2}-c_i\right)+\frac{1}{8}\left(v-c_i\right)+\frac{1}{8}\left(\frac{v}{2}-c_i\right)$$

$$=\frac{1}{8}\left(0\right)+\frac{1}{8}\left(0\right)+\frac{1}{8}\left(\frac{v}{2}\right)+\frac{1}{8}\left(\frac{v}{2}\right)+\frac{1}{8}\left(\frac{v}{2}\right)+\frac{1}{8}\left(\frac{v}{2}\right)+\frac{1}{8}\left(\frac{v}{2}\right)+\frac{1}{8}\left(\frac{v}{2}\right).$$

The same kind of reasoning can be carried out for small observations of c, since it does exist a lower dominance region.

Again, let us assume $\varepsilon < \left|-\frac{c}{2}\right|$ and suppose player $i \in \{1,2,3,4\}$ observes $c_i < 0$. Then, i's conditionally expected payoff from exerting effort, that is choosing a, is positive and greater than the one from exerting no effort, that is choosing $\overline{a}$. Accordingly, $\overline{a}$ is a conditionally strictly dominant action for player $i \in \{1,2,3,4\}$ whenever she observes $c_i < 0$. Iterating this dominance argument, suppose $i = 1$ without loss of generality. Then, if players $-i \in \{2,3,4\}$ are forced to play a whenever they observe $c_{-i} < 0$, player i, observing $c_i = 0$ has to assign at least probability $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$ to (a_2, a_3, a_4). Thus, i's conditionally expected payoff difference from exerting effort versus not exerting effort, that is choosing a_i over $\overline{a}_i$, will be at least $\frac{v}{4}$, so that $\overline{a}_i$ can be discarded by iterated dominance for $c_i = 0$. Let c_i^{**} be the smallest observation such that $\overline{a}_i$ cannot be excluded by iterated dominance. Then, it is possible to show that $c_i^{**} = \frac{v}{4}$. Note that $c_i = \frac{v}{4}$ is the threshold for the risk-dominance regions as well. As a matter of fact, when $c_i = \frac{v}{4}$, the conditionally expected payoff from exerting effort equals

$$\frac{1}{8}\left(\frac{v}{2}-\frac{v}{4}\right)+\frac{1}{8}\left(-\frac{v}{4}\right)+\frac{1}{8}\left(v-\frac{v}{4}\right)+\frac{1}{8}\left(\frac{v}{2}-\frac{v}{4}\right)+\frac{1}{8}\left(v-\frac{v}{4}\right)+\frac{1}{8}\left(\frac{v}{2}-\frac{v}{4}\right)+$$

$$+\frac{1}{8}\left(v-\frac{v}{4}\right)+\frac{1}{8}\left(\frac{v}{2}-\frac{v}{4}\right)=\frac{3}{8}v$$

and the conditionally expected payoff from not exerting effort equals

$$\frac{1}{8}\left(0\right)+\frac{1}{8}\left(0\right)+\frac{1}{8}\left(\frac{v}{2}\right)+\frac{1}{8}\left(\frac{v}{2}\right)+\frac{1}{8}\left(\frac{v}{2}\right)+\frac{1}{8}\left(\frac{v}{2}\right)+\frac{1}{8}\left(\frac{v}{2}\right)+\frac{1}{8}\left(\frac{v}{2}\right)=\frac{3}{8}v.$$

The cutoff $c_i^{**} = \frac{v}{4}$ is the unique threshold that can be established from the lower dominance region by iterated deletion of strictly dominated strategies, since it is the unique value for $c_i \in [c-\varepsilon, c+\varepsilon]$ solving

$$\frac{1}{8}\left(\frac{v}{2}-c_i\right)+\frac{1}{8}\left(-c_i\right)+\frac{1}{8}\left(v-c_i\right)+\frac{1}{8}\left(\frac{v}{2}-c_i\right)+\frac{1}{8}\left(v-c_i\right)+\frac{1}{8}\left(\frac{v}{2}-c_i\right)+\frac{1}{8}\left(v-c_i\right)+\frac{1}{8}\left(\frac{v}{2}-c_i\right)$$

$$=\frac{1}{8}\left(0\right)+\frac{1}{8}\left(0\right)+\frac{1}{8}\left(\frac{v}{2}\right)+\frac{1}{8}\left(\frac{v}{2}\right)+\frac{1}{8}\left(\frac{v}{2}\right)+\frac{1}{8}\left(\frac{v}{2}\right)+\frac{1}{8}\left(\frac{v}{2}\right)+\frac{1}{8}\left(\frac{v}{2}\right) \ .$$

Hence, $c_i^* = c_i^{**}$ and in $g\left(c\right)$ under incomplete information à la global games, for sufficiently small ε, there exists a unique equilibrium in switching strategies such that $\forall i \in \{1,2,3,4\}$

$$x_i^*\left(c_i\right) = \begin{cases} 1 & \text{if } c_i < \frac{v}{4} \\ 0 & \text{if } c_i \geq \frac{v}{4} \ . \end{cases}$$

3.3.3 Observations

Overall, I can state some general points from the example above:

- under complete information, there are multiple Nash equilibria in pure strategies in a max-min two-group four-player contest with binary actions and a public good prize, independently from whether the cost of effort equates effort itself or a parameter belonging to the set of real numbers.

- Focusing on the geometric representation of each two-player group it is apparent the similarity with the symmetric 2x2 game by Carlsson and Damme (1993a): in both of them there is a supermodular payoff-structure and the cardinality of the actions set is equal to two, as in classical stag hunt games. Accordingly, symmetric Nash equilibria in pure strategies naturally emerge.

- In both examples I highlight a tension between payoff-dominance and risk-dominance, as in the example due to Carlsson and Damme (1993a).

- Relaxing complete information à la global games induces the existence of an equilibrium in (monotonic) switching strategies, whose cutoff coincides with the one of the risk-dominance region.

- Equilibrium selection happens even for “a pinch of uncertainty”, no matter how small ε is.
- Whether the selection induced delivers uniqueness or not crucially depends on the properties of the payoffs structure under complete information: in particular, the presence of both an upward and a downward dominance region is conducive to a unique equilibrium in (monotonic) switching strategies by deletion of interim-strictly dominated strategies when departing from the complete information assumption in the sense of Carlsson and Damme (1993a).
- Focusing on mixed-strategies would not affect the equilibria in (monotonic) switching strategies, since they are derived by iterated deletion of conditionally strictly dominated strategies and pure dominated strategies cannot be part of the support of any equilibrium mixed strategies.
- Finally, note that whether the risk-dominant equilibrium in $g\left(v_i\right)$ and $g\left(c_i\right)$ coincides with the risk-dominant equilibrium in the actual game selected by Nature, i.e. $g\left(v\right)$ and $g\left(c\right)$ respectively, or not, depends on whether ε is sufficiently small, that is for $\varepsilon < \left|v-2\right|$ and $\varepsilon < \left|c-\frac{v}{4}\right|$, respectively.

Once highlighted the main properties of our example, the general model and the mechanisms guiding to the related results should be more transparent in the next section.

3.4 The Nash Equilibria of the Complete Information Binary Max-Min Group Contest

To simplify notation and presentation, the NE of the $BMMGC^{*}$ will be presented in terms of share of active agents, i.e., $\left(\gamma_1,\gamma_2\right) \in [0,1] \times [0,1]$.

PROPOSITION 3.1. *In the $BMMGC^*$,*

- *if $v > 2$, there are four strict Nash equilibria in pure strategies*

$$NE \equiv \{(\gamma_1, \gamma_2) = (1,1)\,; (\gamma_1, \gamma_2) = (1,0)\,; (\gamma_1, \gamma_2) = (0,1)\,; (\gamma_1, \gamma_2) = (0,0)\}$$

  and a Nash equilibrium in within-group symmetric strictly-mixed strategies

$$\sigma^*_{ij}\left(x_{ij} = 1\right) = \left(\frac{2}{v}\right)^{1/(n_j - 1)} \quad \forall i \in \{1, \ldots, n_j\} \;\; \textit{and } j \in \{1,2\}\,;$$

- *if $v = 2$, there are four Nash equilibria in pure strategies*

$$NE \equiv \{(\gamma_1, \gamma_2) = (1,1)\,; (\gamma_1, \gamma_2) = (1,0)\,; (\gamma_1, \gamma_2) = (0,1)\,; (\gamma_1, \gamma_2) = (0,0)\}\,;$$

- *if $v < 2$, there is a unique Nash equilibrium in pure strategies derived by strict-dominance*

$$(\gamma_1, \gamma_2) = (0,0) \;\; .$$

Overall, there is a one-sided dominance region: for $v < 2$, $x_{ij} = 1$ is a strictly dominated action for any $ij \in \{1, \ldots, N\}$.

Proof. See Appendix 3.A.1. □

The following result is immediate from proposition 1.

COROLLARY 3.1. *In the $BMMGC^*$, there are no within-group asymmetric Nash equilibria in pure strategies.*

Moreover, by taking a generalization of risk-dominance close to the unilateral deviation stability due to Güth (1992), it is easy to prove the following result. [3.6]

[3.6] In the class of 2×2-games, risk-dominance à la Harsanyi and Selten (1988) and unilateral deviation stability à la Güth (1992) coincide. However, for more than two players the tracing procedure does not generally produce the same equilibrium selection result as according to the largest Nash product.

PROPOSITION 3.2. *In the $BMMGC^{*}$,*

- *for $v > 2$, $(\gamma_j, \gamma_{-j}) = (1, 0)$ is the payoff-dominant equilibrium for group $j \in \{1, 2\}$;*
- *for $v > 4$, $(\gamma_j, \gamma_{-j}) = (1, 1)$ and $(\gamma_j, \gamma_{-j}) = (1, 0)$ are the risk-dominant equilibrium strategy profiles for group $j \in \{1, 2\}$;*
- *for $2 < v < 4$, $(\gamma_j, \gamma_{-j}) = (0, 1)$ and $(\gamma_j, \gamma_{-j}) = (0, 0)$ are the risk-dominant equilibrium strategy profiles for group $j \in \{1, 2\}$.*

Proof. See Appendix 3.A.2 □

REMARK 3.1. *Note that, for $2 < v < 4$ and any group $j \in \{1, 2\}$, $(\gamma_j, \gamma_{-j}) = (1, 0)$ is the payoff-dominant equilibrium strategy profile for group $j \in \{1, 2\}$, whereas $(\gamma_j, \gamma_{-j}) = (0, 1)$ and $(\gamma_j, \gamma_{-j}) = (0, 0)$ are the risk-dominant equilibrium strategy profiles for group $j \in \{1, 2\}$: there is a tension between payoff-dominance and risk-dominance.*

Now consider a slight variation of the game above and let:

- the individual cost of effort $C\left(x_j\left(i\right)\right) = c$ with $c \in \mathbb{R}$. Thus, cost of effort may be negative, which means that agents could enjoy effort per se;
- the club good prize $v > 0$, so that it is a good.[3.7]

Henceforth, I term this variation as $BMMGC^{*b}$. Then, it is straightforward to derive the following results.

PROPOSITION 3.3. *In the $BMMGC^{*b}$,*

- *if $c < 0$, there is a unique Nash equilibrium in pure strategies derived by strict-dominance*

$$(\gamma_1, \gamma_2) = (1, 1) \ ;$$

[3.7] As stressed previously in our example, under complete information, for $c \in \mathbb{R}_{++}$ and $v \in \mathbb{R}_{++}$, the cost of effort $C_{ij}\left(x_j\left(i\right)\right)$ can always be normalized to one via a simple change of variables.

- *if* $c > \frac{v}{2}$, *there is a unique Nash equilibrium in pure strategies derived by strict-dominance*

$$(\gamma_1, \gamma_2) = (0, 0) \ ;$$

- *if* $0 < c < \frac{v}{2}$, *there are four strict Nash equilibria in pure strategies*

$$NE = \{(\gamma_1, \gamma_2) = (1, 1), (\gamma_1, \gamma_2) = (1, 0), (\gamma_1, \gamma_2) = (0, 1), (\gamma_1, \gamma_2) = (0, 0)\}$$

  and an equilibrium in within-group symmetric strictly mixed strategies $\sigma_i^* (x_{ij} = 1) = \left(\frac{2c}{v}\right)^{1/(n_j - 1)} \ \forall i \in \{1, \dots, n_j\}$ *and* $j \in \{1, 2\}$*;*

- *if* $c = 0$, *the set of Nash equilibria in pure strategies is*

$$\begin{aligned} NE = & \left\{ (\gamma_1, \gamma_2) = (1, 1), (\gamma_1, \gamma_2) = (1, 0), (\gamma_1, \gamma_2) = (0, 1), (\gamma_1, \gamma_2) = (0, 0) \right\} \cup \\ & \cup \left\{ (\gamma_1, \gamma_2) \ \textit{such that} \ \gamma_j \in (0, 1) \ \textit{and} \ 1 \leq \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} \leq n_j - 2 \ \forall j \in \{1, 2\} \right\} \cup \\ & \cup \left\{ (\gamma_j, \gamma_{-j}) = (\gamma_j, 0) \ \textit{such that} \ \gamma_j \in (0, 1) \ \textit{and} \ 1 \leq \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} \leq n_j - 2 \ \forall j \in \{1, 2\} \right\} \ ; \end{aligned}$$

- *if* $c = \frac{v}{2}$, *there are four Nash equilibria in pure strategies*

$$NE = \{(\gamma_1, \gamma_2) = (1, 1), (\gamma_1, \gamma_2) = (1, 0), (\gamma_1, \gamma_2) = (0, 1), (\gamma_1, \gamma_2) = (0, 0)\} \ .$$

Overall, there are two dominance regions: for $c < 0$, $x_{ij} = 0$ *is a strictly dominated action for any* $ij \in \{1, \dots, N\}$*: for* $c > \frac{v}{2}$, $x_{ij} = 1$ *is a strictly dominated action for any* $ij \in \{1, \dots, N\}$.

Proof. See Appendix 3.A.3. □

As before, by taking a generalization of risk-dominance close to the unilateral deviation stability due to Güth (1992), it is easy prove the following result.

PROPOSITION 3.4. *In the* $BMMGC^{*b}$,

- *for* $0 \leq c < \frac{v}{2}$, $(\gamma_j, \gamma_{-j}) = (1, 0)$ *is the payoff-dominant equilibrium for group* $j \in \{1, 2\}$*;*
- *for* $0 \leq c < \frac{v}{4}$, $(\gamma_j, \gamma_{-j}) = (1, 1)$ *and* $(\gamma_j, \gamma_{-j}) = (1, 0)$ *are the risk-dominant equilibrium strategy profiles for group* $j \in \{1, 2\}$ *;*
- *for* $\frac{v}{4} < c < \frac{v}{2}$, $(\gamma_j, \gamma_{-j}) = (0, 1)$ *and* $(\gamma_j, \gamma_{-j}) = (0, 0)$ *are the risk-dominant equilibrium strategy profiles for group* $j \in \{1, 2\}$ *.*

Proof. See Appendix 3.A.4. ☐

REMARK 3.2. *Note that, for* $\frac{v}{4} < c < \frac{v}{2}$ *and any group* $j \in \{1, 2\}$*,* $(\gamma_j, \gamma_{-j}) = (1, 0)$ *is the payoff-dominant equilibrium strategy profile for group* $j \in \{1, 2\}$*, whereas* $(\gamma_j, \gamma_{-j}) = (0, 1)$ *and* $(\gamma_j, \gamma_{-j}) = (0, 0)$ *are the risk-dominant equilibrium strategy profiles for group* $j \in \{1, 2\}$*: there is a tension between payoff-dominance and risk-dominance.*

3.5 Bayes-Nash Equilibria with Incomplete Information à la Global games

3.5.1 Incomplete Information about the Prize

Let us consider the case where the individual **cost of effort** is $C(x_j(i)) = x_j(i)$, the $BMMGC^*$ model. I closely follow Carlsson and Damme (1993a) introducing incomplete information about the prize v as follows:

- let V be a random variable which is uniform on some interval $[\underline{v}, \overline{v}]$ including the dominance region and the threshold for the risk-dominance, e.g. $[1, 5]$;
- given the unknown realization v, each player $ij \in \{1, \ldots, n_j\}$ $\forall j \in \{1, 2\}$ idiosyncratically observes the realization v_{ij} of a signal V_{ij}, uniform on $[v - \varepsilon, v + \varepsilon]$ for some $0 < \varepsilon < \left|\frac{v}{2} - 1\right|$, so that the players' observation errors $V_{ij} - v$ $\forall ij \in \{1, \ldots, n_j\}$ and $\forall j \in \{1, 2\}$ are independent;

- after these idiosyncratic observations, each player $ij \in \{1, \dots, n_j\}$ $\forall j \in \{1, 2\}$ simultaneously and independently decides whether to exert effort or not and gets a payoff as described above.

Henceforth, I refer to this game as $g_1(v)$. Then, I am able to obtain the following result.

PROPOSITION 3.5. *In the $g_1(v)$, there is a unique equilibrium in (monotonic) switching strategies, such that $\forall ij \in \{1, \dots, n_j\}$ and $\forall j \in \{1, 2\}$:*

$$x^*_{ij}(v_{ij}) = \begin{cases} 1 & \textit{if } v_{ij} > 2^{n_j} \\ 0 & \textit{if } v_{ij} \leq 2^{n_j} \end{cases}$$

other than the equilibrium

$$x^{**}_{ij}(v_{ij}) = 0 \ \forall v_{ij} \in [\underline{v} + \varepsilon, \overline{v} - \varepsilon] .$$

Proof. See Appendix 3.A.5. ☐

REMARK 3.3. *The absence of an upward dominance region is conducive to the existence of the equilibrium $x^{**}_{ij}(v_{ij}) = 0 \ \forall v_{ij} \in [\underline{v} + \varepsilon, \overline{v} - \varepsilon]$, so that the equilibrium $(\gamma_1, \gamma_2) = (0, 0)$ in the BMMGC* is robust to incomplete information in the sense of Kajii and Morris (1997).*

REMARK 3.4. *The existence of an equilibrium in (monotonic) switching strategies in the $g_1(v)$ is ensured as long as $0 < \varepsilon < \left|\frac{v}{2} - 1\right|$. However, equilibrium selection happens even for "a pinch of uncertainty", no matter how small ε is.*

REMARK 3.5. *The cutoff of the equilibrium in (monotonic) switching strategies, $v_{ij} = 2^{n_j}$, does not coincide with the one of the risk-dominance region, that is $v_{ij} = 4$ for any $j \in \{1, 2\}$, differently from what happens in the two-group four-player example. This is very close to the point made by Carlsson and Damme (1993b) for n-player stag hunt games, where the authors stress that risk-dominance fails as an equilibrium selection criterion when I depart from the 2×2 case.*

3.5.2 Incomplete Information about the Cost of Effort

Let us consider the case where the individual **cost of effort** is $C\left(x_j\left(i\right)\right) = c$ with $c \in \mathbb{R}$ and the club good **prize** worth $v > 0$, that is the $BMMGC^{*b}$ model. I closely follow Carlsson and Damme (1993a) introducing incomplete information about the cost of effort c as follows:

- let C be a random variable which is uniform on some interval $[\underline{c}, \overline{c}]$ including both dominance regions, e.g. $[-v, +v]$;
- given the unknown realization c, each player $ij \in \{1, \ldots, n_j\}$ $\forall j \in \{1, 2\}$ idiosyncratically observes the realization c_{ij} of a signal C_{ij}, uniform on $[c - \varepsilon, c + \varepsilon]$ for some $0 < \varepsilon < min\left\{\left|\frac{2\overline{c}-v}{4}\right|, \left|\frac{\underline{c}}{2}\right|\right\}$, so that the players' observation errors $C_{ij} - c$ $\forall ij \in \{1, \ldots, n_j\}$ and $\forall j \in \{1, 2\}$ are independent;
- after these idiosyncratic observations, each player $ij \in \{1, \ldots, n_j\}$ $\forall j \in \{1, 2\}$ simultaneously and independently decides whether to exert effort or not and gets a payoff as described above;

Henceforth, I refer to this game as $g_2\left(c\right)$. Then I am able to obtain the following result.

PROPOSITION 3.6. *In the* $g_2\left(c\right)$*, there is a unique equilibrium in (monotonic) switching strategies, such that* $\forall ij \in \{1, \ldots, n_j\}$ *and* $\forall j \in \{1, 2\}$*:*

$$x^*_{ij}\left(c_{ij}\right) = \begin{cases} 1 & \textit{if } c_{ij} < 2^{-n_j} v \\ 0 & \textit{if } c_{ij} \geq 2^{-n_j} v \ . \end{cases}$$

REMARK 3.6. *The presence of both an upward dominance region and a downward dominance region is conducive to the selection of a unique equilibrium in (monotonic) switching strategies.*

Proof. See Appendix 3.A.6. □

REMARK 3.7. *The existence of a unique equilibrium in (monotonic) switching strategies in the* $g_2\left(c\right)$ *is ensured as long as* $0 < \varepsilon < min\left\{\left|\frac{2\overline{c}-v}{4}\right|, \left|\frac{\underline{c}}{2}\right|\right\}$*. However, equilibrium selection happens even for "a pinch of uncertainty", no matter how small* ε *is.*

REMARK 3.8. *Note that the cutoff of the equilibrium in (monotonic) switching strategies, i.e.* $c_{ij} = 2^{-n_j} v$*, does not coincide with the one of the risk-dominance region, that is* $c_{ij} = \frac{v}{4}$ *for any* $j \in \{1, 2\}$*, differently from what happens in the two-group four-player example. This is very close to the point made by Carlsson and Damme (1993b) for n-player stag hunt games, where the authors stress that risk-dominance fails as an equilibrium selection criterion when I depart from the 2×2 case.*

Differently from what is obtained by Barbieri, Kovenock, et al. (2019) for a deterministic two-group contest with the weakest-link impact function, continuous effort, a public good prize commonly valued across groups and incomplete information about the cost of effort, in the $g_2(c)$ there are no multiple equilibria in pure strategies but a unique equilibrium in (monotonic) switching strategies. The uniqueness result achieved by Barbieri, Kovenock, et al. (2019) regards only the class of nondegenerate Bayes-Nash equilibria without mass at the top. As a matter of fact, the authors obtain a continuum of non-degenerate Bayes-Nash equilibria with mass at the top, other than degenerate Bayes-Nash equilibria, where players perfectly align effort choices on the highest cost type: a degeneracy result consinstent with the complete information model due to Chowdhury et al. (2016). Semidegenerate equilibria, that is equilibria in which effort levels are dispersed just in one group, are found only for the setting with asymmetric prize valuations between the two groups, i.e. $v_1 \neq v_2$.

3.6 The Group-Size Paradox

In this section I calculate the probabilities of winning the prize v and the expected payoffs for both groups at the unique equilibrium in (monotonic) switching strategies in $g_1(v)$ and $g_2(c)$, to assess the presence of the so-called group-size paradox for both classes of incomplete information games.

3.6.1 The Group-Size Paradox in $g_1(v)$

Let us consider the $g_1(v)$. Then it is posssible to derive the probability of winning the prize v for group $j \in \{1, 2\}$, as the following proposition shows.

PROPOSITION 3.7. *In the $g_1(v)$, the probability of winning the prize v for group $j \in \{1,2\}$ at the cutoff equilibrium equals:*

- *if $\underline{v}+\varepsilon \leq 2^{n_j} \leq \overline{v}-\varepsilon$ and $\underline{v}+\varepsilon \leq 2^{n_{-j}} \leq \overline{v}-\varepsilon$,*

$$\begin{aligned} Prob\,(j\ \ wins\ v) = & \left(1-\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\underline{v}-2\varepsilon}\right)^{n_j}\cdot\left[1-\left(1-\frac{2^{n_{-j}}-\underline{v}-\varepsilon}{\overline{v}-\underline{v}-2\varepsilon}\right)^{n_{-j}}\right]+ \\ & +\frac{1}{2}\left[1-\left(1-\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\underline{v}-2\varepsilon}\right)^{n_j}\right]\cdot\left[1-\left(1-\frac{2^{n_{-j}}-\underline{v}-\varepsilon}{\overline{v}-\underline{v}-2\varepsilon}\right)^{n_{-j}}\right]+ \\ & +\frac{1}{2}\left(1-\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n_j}\cdot\left(1-\frac{2^{n_{-j}}-\underline{v}-\varepsilon}{\overline{v}-\underline{v}-2\varepsilon}\right)^{n_{-j}} \quad ; \end{aligned}$$

- *if $\underline{v}+\varepsilon \leq 2^{n_j} \leq \overline{v}-\varepsilon$ and $2^{n_{-j}} > \overline{v}-\varepsilon$,*

$$\begin{aligned} Prob\,(j\ \ wins\ v) = & \left(1-\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\underline{v}-2\varepsilon}\right)^{n_j}+ \\ & +\frac{1}{2}\left[1-\left(1-\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\underline{v}-2\varepsilon}\right)^{n_j}\right] \quad ; \end{aligned}$$

- *if $2^{n_j} > \overline{v}-\varepsilon$ and $\underline{v}+\varepsilon \leq 2^{n_{-j}} \leq \overline{v}-\varepsilon$,*

$$Prob\,(j\ \ wins\ v) = \frac{1}{2}\left[1-\left(1-\frac{2^{n_{-j}}-\underline{v}-\varepsilon}{\overline{v}-\underline{v}-2\varepsilon}\right)^{n_{-j}}\right] \quad ;$$

- *if $2^{n_j} > \overline{v}-\varepsilon$ and $2^{n_{-j}} > \overline{v}-\varepsilon$,*

$$Prob\,(j\ \ wins\ v) = \frac{1}{2} \ .$$

Proof. See Appendix 3.A.7 □

COROLLARY 3.2. *In the $g_1(v)$, the probability of winning the prize v for group j at the equilibrium in (monotonic) switching strategies is:*

- *decreasing in n_j at the threshold $n_j^* = log(\overline{v}-\varepsilon)/log(2)\ \forall\ 0<\varepsilon<|\frac{v}{2}-1|$;*
- *increasing in n_{-j} at the threshold $n_{-j}^* = log(\overline{v}-\varepsilon)/log(2)\ \forall\ 0<\varepsilon<|\frac{v}{2}-1|$.*

Proof. It follows immediately from Proposition 3.7. ☐

COROLLARY 3.3. *In the* $g_1(v)$, *if* $log\left(\underline{v}+\varepsilon\right)/log\left(2\right) \leq n_j \leq log(\overline{v}-\varepsilon)/log(2)-1$ *and* $log\left(\underline{v}+\varepsilon\right)/log\left(2\right) \leq n_{-j} \leq log(\overline{v}-\varepsilon)/log(2)-1 \ \forall \ 0<\varepsilon<|\frac{\underline{v}}{2}-1|$, *the probability of winning the prize* v *for group* j *at the equilibrium in (monotonic) switching strategies is:*

- *decreasing in* n_j*;*
- *increasing in* n_{-j}*.*

Proof. See Appendix 3.A.8. ☐

Moreover, once computed the probabilities of winning, it is immediate to retrieve the expected payoffs at the equilibrium in (monotonic) switching strategies.

PROPOSITION 3.8. *In the* $g_1(v)$, *the expected payoff at the cutoff equilibrium* $x^*_{ij}(v_{ij})$ *for any* $ij \in \{1,\ldots,n_j\}\ \forall j \in \{1,2\}$ *equals:*[3.8]

- *for* $\underline{v}+\varepsilon \leq 2^{n_j} \leq \overline{v}-\varepsilon$ *and* $\underline{v}+\varepsilon \leq 2^{n_{-j}} \leq \overline{v}-\varepsilon$,

$$\begin{aligned}
\mathbb{E}\left[\pi_{ij}\left(\mathbf{x}^*_j,\mathbf{x}^*_{-j}\right)\right] = {} & \left(1-\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n_j} \cdot \left[1-\left(1-\frac{2^{n_{-j}}-\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n_{-j}}\right] \cdot \left(\frac{\underline{v}+\overline{v}}{2}-1\right)+ \\
& + \left(\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right) \cdot \left[1-\left(1-\frac{2^{n_{-j}}-\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n_{-j}}\right] \cdot \frac{\underline{v}+\overline{v}}{4} + \\
& + \left(1-\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n_j} \cdot \left(1-\frac{2^{n_{-j}}-\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n_{-j}} \cdot \left(\frac{\underline{v}+\overline{v}}{4}-1\right) + \\
& + \left(1-\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right) \cdot \left[1-\left(1-\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n_j-1}\right] \cdot \\
& \cdot \left[1-\left(1-\frac{2^{n_{-j}}-\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n_{-j}}\right] \cdot \left(\frac{\underline{v}+\overline{v}}{4}-1\right)+ \\
& + \left(1-\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right) \cdot \left[1-\left(1-\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n_j-1}\right] \cdot \\
& \cdot \left(1-\frac{2^{n_{-j}}-\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n_{-j}} \cdot (-1) \ ;
\end{aligned}$$

[3.8] By $\mathbb{E}\left[\pi_{ij}\left(\mathbf{x}^*_j,\mathbf{x}^*_{-j}\right)\right]$ I mean the expected payoff at the cutoff equilibrium $x^*_{ij}(v_{ij})$ for any $ij \in \{1,\ldots,n_j\}$ $\forall j \in \{1,2\}$, where $\mathbf{x}^*_j, \mathbf{x}^*_{-j}$ are the vectors of equilibrium strategies profiles for the two groups.

- *for* $\underline{v} + \varepsilon \leq 2^{n_j} \leq \overline{v} - \varepsilon$ *and* $2^{n_{-j}} > \overline{v} - \varepsilon$*,*

$$\mathbb{E}\left[\pi_{ij}\left(\mathbf{x}_j^*, \mathbf{x}_{-j}^*\right)\right] = \left(1 - \frac{2^{n_j} - \underline{v} - \varepsilon}{\overline{v} - \varepsilon - \underline{v} - \varepsilon}\right)^{n_j} \cdot \left(\frac{\underline{v} + \overline{v}}{2} - 1\right) + \left(\frac{2^{n_j} - \underline{v} - \varepsilon}{\overline{v} - \varepsilon - \underline{v} - \varepsilon}\right) \cdot \frac{\underline{v} + \overline{v}}{4} + \\ + \left(1 - \frac{2^{n_j} - \underline{v} - \varepsilon}{\overline{v} - \varepsilon - \underline{v} - \varepsilon}\right) \cdot \left[1 - \left(1 - \frac{2^{n_j} - \underline{v} - \varepsilon}{\overline{v} - \varepsilon - \underline{v} - \varepsilon}\right)^{n_j - 1}\right] \cdot \left(\frac{\underline{v} + \overline{v}}{4} - 1\right) \;;$$

- *for* $2^{n_j} > \overline{v} - \varepsilon$ *and* $\underline{v} + \varepsilon \leq 2^{n_{-j}} \leq \overline{v} - \varepsilon$*,*

$$\mathbb{E}\left[\pi_{ij}\left(\mathbf{x}_j^*, \mathbf{x}_{-j}^*\right)\right] = \left[1 - \left(1 - \frac{2^{n_{-j}} - \underline{v} - \varepsilon}{\overline{v} - \varepsilon - \underline{v} - \varepsilon}\right)^{n_{-j}}\right] \cdot \frac{\underline{v} + \overline{v}}{4} \;;$$

- *for* $2^{n_j} > \overline{v} - \varepsilon$ *and* $2^{n_{-j}} > \overline{v} - \varepsilon$*,*

$$\mathbb{E}\left[\pi_{ij}\left(\mathbf{x}_j^*, \mathbf{x}_{-j}^*\right)\right] = \frac{\underline{v} + \overline{v}}{4} \;.$$

Proof. See Appendix 3.A.9. □

COROLLARY 3.4. *In the* $g_1(v)$*, the expected payoff for any* $ij \in \{1, \ldots, n_j\}$ $\forall j \in \{1, 2\}$ *at the equilibrium in (monotonic) switching strategies* $x_{ij}^*(v_{ij})$ *is:*

- *decreasing in* n_j *at the threshold* $n_j^* = log(\overline{v} - \varepsilon)/log(2)$ $\forall\ 0 < \varepsilon < |\frac{\underline{v}}{2} - 1|$;
- *increasing in* n_{-j} *at the threshold* $n_{-j}^* = log(\overline{v} - \varepsilon)/log(2)$ $\forall\ 0 < \varepsilon < |\frac{\underline{v}}{2} - 1|$.

Proof. It follows immediately from Proposition 3.8. □

COROLLARY 3.5. *In the* $g_1(v)$*, if* $log\left(\underline{v} + \varepsilon\right)/log\left(2\right) \leq n_j \leq log\left(\overline{v} - \varepsilon\right)/log\left(2\right) - 1$ *and* $log\left(\underline{v} + \varepsilon\right)/log\left(2\right) \leq n_{-j} \leq log\left(\overline{v} - \varepsilon\right)/log(2) - 1$ $\forall\ 0 < \varepsilon < \left|\frac{\underline{v}}{2} - 1\right|$*, the expected payoff for any* $ij \in \{1, \ldots, n_j\}$ *and* $j \in \{1, 2\}$ *at the equilibrium in (monotonic) switching strategies* $x_{ij}^*(v_{ij})$ *is:*

- *decreasing in* n_j *if* $\mathbb{E}(V) > \frac{2^{n_j+1}}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\left[\left(\frac{\overline{v}-\varepsilon-2^x}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n_j} - \left(\frac{\overline{v}-\varepsilon-2^{n_j+1}}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n_j+1}\right]^{-1}$*;*
- *increasing in* n_{-j} *if* $\mathbb{E}(V) > 0$.

Proof. See Appendix 3.A.10. ☐

REMARK 3.9. *Overall, in the $g_1(v)$ a smaller group is more likely to win, as shown by corollaries 3.2 and 3.3, and it attains a higher expected payoff if the expected value of the prize is sufficiently high, as shown by corollary 3.5. Clearly, a higher probability of winning is beneficial for a player if the value of the prize contested is sufficiently large. Therefore, in the $g_1(v)$ I find conditions for the so-called group-size paradox.*

3.6.2 The Group-Size Paradox in $g_2(c)$

Let us now turn to the $g_2(c)$ model. Then it is possible to derive the probability of winning the prize v for group $j \in \{1, 2\}$, as the following proposition shows.

PROPOSITION 3.9. *In the $g_2(c)$, the probability of winning the prize v for group $j \in \{1, 2\}$ at the cutoff equilibrium equals:*

$$
\begin{aligned}
Prob\,(j\ \ wins\ v) = & \left(\frac{2^{-n_j}v - \underline{c} - \varepsilon}{\overline{c} - \underline{c} - 2\varepsilon}\right)^{n_j} \cdot \left[1 - \left(\frac{2^{-n_{-j}}v - \underline{c} - \varepsilon}{\overline{c} - \underline{c} - 2\varepsilon}\right)^{n_{-j}}\right] + \\
& + \frac{1}{2}\left[1 - \left(\frac{2^{-n_j}v - \underline{c} - \varepsilon}{\overline{c} - \underline{c} - 2\varepsilon}\right)^{n_j}\right] \cdot \left[1 - \left(\frac{2^{-n_{-j}}v - \underline{c} - \varepsilon}{\overline{c} - \underline{c} - 2\varepsilon}\right)^{n_{-j}}\right] + \\
& + \frac{1}{2}\left(\frac{2^{-n_j}v - \underline{c} - \varepsilon}{\overline{c} - \underline{c} - 2\varepsilon}\right)^{n_j} \cdot \left(\frac{2^{-n_{-j}}v - \underline{c} - \varepsilon}{\overline{c} - \underline{c} - 2\varepsilon}\right)^{n_{-j}} \quad .
\end{aligned}
$$

Proof. See Appendix 3.A.11. ☐

REMARK 3.10. *Contrary to $g_1(v)$, in the $g_2(c)$ the probability of winning the prize v for group j at the equilibrium in (monotonic) switching strategies is not increasing in n_j at the threshold $n_j^* = log(\frac{v}{\overline{c}-\varepsilon})/log(2)$ $\forall 0 < \varepsilon < min\left\{|\frac{2\overline{c}-v}{4}|, |\frac{\underline{c}}{2}|\right\}$ and increasing in n_{-j} at the threshold $n_{-j}^* = log(\frac{v}{\overline{c}-\varepsilon})/log(2)$ $\forall\ 0 < \varepsilon < min\left\{|\frac{2\overline{c}-v}{4}|, |\frac{\underline{c}}{2}|\right\}$, since $2^{-n_j}v > \underline{c} + \varepsilon\ \forall\ \underline{c} < 0$, $0 < \varepsilon < min\left\{|\frac{2\overline{c}-v}{4}|, |\frac{\underline{c}}{2}|\right\}$ and $\forall n_j \geq 2$ and $2^{-n_j}v < \overline{c} - \varepsilon\ \forall\ \overline{c} > \frac{v}{2}$, $0 < \varepsilon < min\left\{|\frac{2\overline{c}-v}{4}|, |\frac{\underline{c}}{2}|\right\}$ and $\forall\ n_j \geq 2$.*

COROLLARY 3.6. *In the $g_2(c)$, for any $ij \in \{1, \ldots, n_j\}$ the probability of winning at the equilibrium in (monotonic) switching strategies $x_{ij}^*(c_{ij})$ is*

- *decreasing in* n_j *;*
- *increasing in* n_{-j} *.*

Proof. See Appendix 3.A.12. $\square$

Moreover, once computed the probabilities of winning, it is immediate to retrieve the expected payoffs at the equilibrium in (monotonic) switching strategies.

PROPOSITION 3.10. *In the* $g_2(c)$*, the expected payoff at the cutoff equilibrium* $x^*_{ij}(c_{ij})$ *for any* $ij \in \{1, \ldots, n_j\}$ $\forall j \in \{1, 2\}$ *equals:*[3.9]

$$\begin{aligned}
\mathbb{E}\left[\pi_{ij}\left(\mathbf{x}^*_j, \mathbf{x}^*_{-j}\right)\right] = & \left(\frac{2^{-n_j}v - \underline{c} - \varepsilon}{\overline{c} - \varepsilon - \underline{c} - \varepsilon}\right)^{n_j} \cdot \left[1 - \left(\frac{2^{-n_{-j}}v - \underline{c} - \varepsilon}{\overline{c} - \varepsilon - \underline{c} - \varepsilon}\right)^{n_{-j}}\right] \cdot \left(v - \frac{\underline{c} + \overline{c}}{2}\right) + \\
& + \left(1 - \frac{2^{-n_j}v - \underline{c} - \varepsilon}{\overline{c} - \varepsilon - \underline{c} - \varepsilon}\right) \cdot \left[1 - \left(\frac{2^{-n_{-j}}v - \underline{c} - \varepsilon}{\overline{c} - \varepsilon - \underline{c} - \varepsilon}\right)^{n_{-j}}\right] \cdot \frac{v}{2} + \\
& + \left(\frac{2^{-n_j}v - \underline{c} - \varepsilon}{\overline{c} - \varepsilon - \underline{c} - \varepsilon}\right)^{n_j} \cdot \left(\frac{2^{-n_{-j}}v - \underline{c} - \varepsilon}{\overline{c} - \varepsilon - \underline{c} - \varepsilon}\right)^{n_{-j}} \cdot \left(\frac{v}{2} - \frac{\underline{c} + \overline{c}}{2}\right) + \\
& + \left(\frac{2^{-n_j}v - \underline{c} - \varepsilon}{\overline{c} - \varepsilon - \underline{c} - \varepsilon}\right) \cdot \left[1 - \left(\frac{2^{-n_j}v - \underline{c} - \varepsilon}{\overline{c} - \varepsilon - \underline{c} - \varepsilon}\right)^{n_j - 1}\right] \cdot \\
& \cdot \left[1 - \left(\frac{2^{-n_{-j}}v - \underline{c} - \varepsilon}{\overline{c} - \varepsilon - \underline{c} - \varepsilon}\right)^{n_{-j}}\right] \cdot \left(\frac{v}{2} - \frac{\underline{c} + \overline{c}}{2}\right) + \\
& + \left(\frac{2^{-n_j}v - \underline{c} - \varepsilon}{\overline{c} - \varepsilon - \underline{c} - \varepsilon}\right) \cdot \left[1 - \left(\frac{2^{-n_j}v - \underline{c} - \varepsilon}{\overline{c} - \varepsilon - \underline{c} - \varepsilon}\right)^{n_j - 1}\right] \cdot \\
& \cdot \left(\frac{2^{-n_{-j}}v - \underline{c} - \varepsilon}{\overline{c} - \varepsilon - \underline{c} - \varepsilon}\right)^{n_{-j}} \cdot \left(-\frac{\underline{c} + \overline{c}}{2}\right) .
\end{aligned}$$

Proof. See Appendix 3.A.13. $\square$

REMARK 3.11. *Contrary to* $g_1(v)$*, in the* $g_2(c)$ *the expected payoff for any* $ij \in \{1, \ldots, n_j\}$ *and* $j \in \{1, 2\}$ *at the equilibrium in (monotonic) switching strategies is not increasing in* n_j *at the threshold* $n^*_j = log(\frac{v}{\overline{c}-\varepsilon})/log(2)$ $\forall\ 0 < \varepsilon < min\left\{|\frac{2\overline{c}-v}{4}|, |\frac{\underline{c}}{2}|\right\}$ *and increasing in* n_{-j} *at the threshold* $n^*_{-j} = log(\frac{v}{\overline{c}-\varepsilon})/log(2)$ $\forall\ 0 < \varepsilon < min\left\{|\frac{2\overline{c}-v}{4}|, |\frac{\underline{c}}{2}|\right\}$ *, since* $2^{-n_j}v > \underline{c} + \varepsilon$ $\forall$ $\underline{c} < 0$*,*

[3.9] By $\mathbb{E}\left[\pi_{ij}\left(\mathbf{x}^*_j, \mathbf{x}^*_{-j}\right)\right]$ I mean the expected payoff at the cutoff equilibrium $x^*_{ij}(c_{ij})$ for any $ij \in \{1, \ldots, n_j\}$ $\forall j \in \{1, 2\}$, where $\mathbf{x}^*_j, \mathbf{x}^*_{-j}$ are the vectors of equilibrium strategies profiles for the two groups.

$0 < \varepsilon < min\left\{|\frac{2\overline{c}-v}{4}|, |\frac{\underline{c}}{2}|\right\}$ *and* $\forall\ n_j \geq 2$ *and* $2^{-n_j}v < \overline{c} - \varepsilon\ \forall\ \overline{c} > \frac{v}{2}$, $0 < \varepsilon < min\left\{|\frac{2\overline{c}-v}{4}|, |\frac{\underline{c}}{2}|\right\}$ *and* $\forall\ n_{-j} \geq 2$.

COROLLARY 3.7. *In the* $g_2(c)$, *the expected payoff for any* $ij \in \{1, \ldots, n_j\}$ *and* $j \in \{1, 2\}$ *at the equilibrium in (monotonic) switching strategies* $x^*_{ij}(c_{ij})$ *is:*

- *decreasing in* n_j *if*

$$\mathbb{E}(C) < \frac{2^{n_j+1}(\overline{c}-\varepsilon)-v}{2}\left(\frac{2^{-1-n_j}-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)^{n_j} + 2^{n_j}(\overline{c}-\varepsilon-\underline{c}-\varepsilon)\left(\frac{2^{-n_j}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)^{n_j};$$

- *increasing in* n_{-j} .

Proof. See Appendix 3.A.14. □

REMARK 3.12. *Overall, in the* $g_2(c)$ *a smaller group is more likely to win and attains a higher expected payoff if the expected value of the cost of effort is sufficiently small. Therefore, in the* $g_2(c)$ *I find conditions for the so-called group-size paradox.*

3.7 Extension to an M-Group Model

In this section I assess the robustness of the results obtained under the two-group assumption by extending our model to the M-group case with $M \geq 2$. I will directly inspect the two incomplete information cases separately as done for the two-group setting, that is about the prize contested and the cost of exerting effort.

3.7.1 Incomplete Information à la Global Games about the Prize and M Groups

Let us define the $BMMGC^*$ as the $BMMGC^*$ with M groups, where $M \geq 2$, and $n_1 \geq \ldots \geq n_M \geq 2$ without loss of generality.

I closely follow Carlsson and Damme (1993a) introducing incomplete information about the prize v as follows:

- let V be a random variable which is uniform on some interval $[\underline{v}, \overline{v}]$ including the dominance region and the threshold for the risk-dominance, e.g. $[1, 5]$;
- given the unknown realization v, each player $ij \in \{1, \ldots, n_j\}$ $\forall j \in \{1, \ldots, M\}$ idiosyncratically observes the realization v_{ij} of a signal V_{ij}, uniform on $[v - \varepsilon, v + \varepsilon]$ for some $0 < \varepsilon < \left| \frac{\underline{v}}{2} - \frac{M}{2(M-1)} \right|$, so that the players' observation errors $V_{ij} - v$ $\forall ij \in \{1, \ldots, n_j\}$ and $\forall j \in \{1, 2\}$ are independent;
- after these idiosyncratic observations, each player $ij \in \{1, \ldots, n_j\}$ $\forall j \in \{1, \ldots, M\}$ simultaneously and independently decides whether to exert effort or not and gets a payoff as described above.

Henceforth, I refer to this game as $g_3(v)$. Then, I am able to obtain the following result.

PROPOSITION 3.11. *In the $g_3(v)$, there is an equilibrium in (monotonic) switching strategies such that $\forall ij \in \{1, \ldots, n\}$ and $\forall j \in \{1, \ldots, M\}$:*

$$x^*_{ij}(v_{ij}) = \begin{cases} 1 & \text{if } v_{ij} > v^*_j \\ 0 & \text{if } v_{ij} \leq v^*_j \end{cases},$$

where

$$v^*_j = \left(2^{1-n} \left(1 - 2^{-n}\right)^{M-1} \left(1 - \frac{1}{M}\right) + 2^{1-n} \sum_{k=1}^{M-1} \frac{\binom{M-1}{k} \left(2^{-n}\right)^k \left(1 - 2^{-n}\right)^{M-1-k}}{k+1} \right)^{-1},$$

other than the equilibrium $\forall\ ij \in \{1, \ldots, n_j\}$ and $\forall j \in \{1, \ldots, M\}$

$$x^{**}_{ij}(v_{ij}) = 0 \ \forall v_{ij} \in [\underline{v} + \varepsilon, \overline{v} - \varepsilon].$$

Proof. See Appendix 3.A.15. □

REMARK 3.13. *The absence of an upward dominance region is conducive to the existence of the equilibrium $x^{**}_{ij}(v_{ij}) = 0$ $\forall v_{ij} \in [\underline{v} + \varepsilon, \overline{v} - \varepsilon]$, so that the equilibrium $(\gamma_1, \ldots, \gamma_M) = (0, 0)$ in the $BMMGC^*$ is robust to incomplete information in the sense of Kajii and Morris (1997).*

REMARK 3.14. *The existence of an equilibrium in (monotonic) switching strategies in the* $g_3(v)$ *is ensured for groups of symmetric sizes as long as* $0 < \varepsilon < \left|\frac{v}{2} - \frac{M}{2(M-1)}\right|$. *However, equilibrium selection happens even for "a pinch of uncertainty", no matter how small* ε *is. Iterated dominance is conducive to a threshold for* v_{ij} *equal to* $M/(M-1) \cdot 2^{n_j-1}$ *which is lower than* v_j^* *for* $M > 2$*, due to the fact that for* $M > 2$ *expected payoff differences do depend on the probability of members of the other groups not exerting effort as well. In the case of asymmetric group sizes, equilibrium existence is likely to be linked to the size of the noise. The analysis of this setting is left for future research.*

3.7.2 Incomplete Information à la Global Games about the Cost of Effort and M Groups

Let us define the $BMMAMGC^{*b}$ as the $BMMAGC^{*b}$ with M groups, where $M \geq 2$, and $n_1 \geq \ldots \geq n_M \geq 2$ without loss of generality.

I closely follow Carlsson and Damme (1993a) introducing incomplete information about the cost of effort c as follows:

- let C be a random variable which is uniform on some interval $[\underline{c}, \overline{c}]$ including both dominance regions, e.g. $[-v, +v]$;
- given the unknown realization c, each player $ij \in \{1, \ldots, n_j\}$ $\forall j \in \{1, \ldots, M\}$ idiosyncratically observes the realization c_{ij} of a signal C_{ij}, uniform on $[c - \varepsilon, c + \varepsilon]$ for some $0 < \varepsilon < min\{|\overline{c}/2 - v(M-1)/2M|, |\underline{c}/2|\}$, so that the players' observation errors $C_{ij} - c$ $\forall ij \in \{1, \ldots, n_j\}$ and $\forall j \in \{1, 2\}$ are independent;
- after these idiosyncratic observations, each player $ij \in \{1, \ldots, n_j\}$ $\forall j \in \{1, \ldots, M\}$ simultaneously and independently decides whether to exert effort or not and gets a payoff as described above;

Henceforth, I refer to this game as $g_4(c)$. Then I am able to obtain the following result.

PROPOSITION 3.12. *In* $g_4(c)$ *with symmetric group sizes there is an equilibrium in (monotonic) cutoff strategies, such that* $\forall ij \in \{1, \ldots, n\}$ *and* $\forall j \in \{1, \ldots, M\}$*:*

$$x_{ij}^{*}\left(c_{ij}\right)=\begin{cases}1 & \textit{if } c_{ij}<c_{j}^{*}\\ 0 & \textit{if } c_{ij}\geq c_{j}^{*}\end{cases},$$

where

$$c_{j}^{*}=v\left(2^{1-n}\left(1-2^{-n}\right)^{M-1}\left(1-\frac{1}{M}\right)+2^{1-n}\sum_{k=1}^{M-1}\frac{\binom{M-1}{k}\left(2^{-n}\right)^{k}\left(1-2^{-n}\right)^{M-1-k}}{k+1}\right).$$

Proof. See Appendix 3.A.16. □

REMARK 3.15. *In the $g_4(c)$ I prove the existence, not the uniqueness of an equilibrium in (monotonic) switching strategies for groups symmetric in size for $M > 2$. This equilibrium is shown to be unique for $M = 2$ and . Starting from the two dominance regions, iterated deletion of strictly dominated strategies is conducive to two different thresholds, i.e. $(2^{1-n_j}/M)v$ and $vM/(M-1)\cdot 2^{-n_j-1}$. For $M > 2$ the expected payoff differences do depend on the probability of the rivals choosing not to exert effort as well, for there are more opportunities to tie, so that the iterated dominance argument does not deliver uniqueness. Moreover, the threshold c_j^{ast} is found by guess and solve. The formal analysis of uniqueness even just in the class of equilibria in (monotonic) switching strategies would require to study more formally the set of thresholds equating the expected payoff differences to zero and it is left for future research. Groups with asymmetric sizes would likely require conditions on the size of the noise ε. The study of this setting is left for future research as well.*

REMARK 3.16. *The existence of an equilibrium in (monotonic) switching strategies for symmetric group sizes in $g_4(c)$ is guaranteed no matter how small ε is, i.e. $0 < \varepsilon < min\{|\overline{c}/2-v(M-1)/2M|, |\frac{\underline{c}}{2}|\}$*

3.8 Further Results

3.8.1 Limit-Uniqueness Result

One of the main results due to Carlsson and Damme (1993a) is about the robustness of the equilibrium in (monotonic) switching strategies as the noise tends to zero, i.e. the so-called limit-uniqueness result. I can easily show that, under our uniform information structure, the main results obtained so far hold as the noise fades away, as stated by the following lemma.

LEMMA 3.1. *As the size of noise tends to zero, i.e.* $\varepsilon \to 0$*, propositions 4.5, 4.6, 3.11, 4.9 hold.*

Proof. See Appendix 3.A.17. □

3.8.2 Noise Independent Selection

The strength of the equilibrium selection phenomenon in global games highlighted by Carlsson and Damme (1993a) is also due to its invariance with respect to both the prior distribution and the distribution of the noise, as long as the the support of the latter is sufficiently small and all other assumptions about independence and continuity in the information structure are preserved. Moreover, if it is employed a uniform improper prior, then exact results hold, as shown by Carlsson and Damme (1993a), even without assuming a sufficiently small support for the noise. I will focus on the latter case to prove the invariance of our results with respect to the choice of the noise distribution.

PROPOSITION 3.13. *Under the uniform improper prior distribution for* V *and* C*, the results in propositions 4.5-3.11 and 4.6-4.9, respectively, are invariant to the exact distribution of the noise, provided that it is symmetric, with mean zero and unit variance.*

Proof. See Appendix 3.A.18. □

3.8.3 Cheap Talk Within Groups

When there are within-group strategic complementarities, allowing group members to communicate could enhance coordination and prevent wasteful effort provision. However, differently from Barbieri, Kovenock, et al. (2019) where private information is about the dispersed individual cost of effort, our information structure is such that the private signals are about the common prize valuation or about the common cost of effort. Accordingly, by means of a simple example, I can provide the underlying intuition as to how our equilibrium selection results are not communication-proof by showing how symmetricity of second-order beliefs – a key driver of the equilibrium selection mechanism I explored – fails when teammates communicate.

- Let players 1 and 2 belong to group 1 and players 3 and 4 to group 2 in $g(v)$.
- Let teammates communicate between them about the realization of their private signals v_i.
- Then, $\mathbb{E}(V|v_1, v_2) = \dfrac{v_1 + v_2}{2}$ and $\mathbb{E}(V|v_3, v_4) = \dfrac{v_3 + v_4}{2}$.
- Moreover, let us denote $\mathbb{E}(V|v_1, v_2) = \dfrac{v_1 + v_2}{2}$ by $\overline{V}_{12}$, and $\mathbb{E}(V|v_3, v_4) = \dfrac{v_3 + v_4}{2}$ by $\overline{V}_{34}$.
- Then, $\mathit{Prob}\left(\overline{V}_{34} < \overline{v}_{12}|v_1, v_2\right) \neq 1/2$, in general.
- Hence, in general iterated deletion of strictly dominated strategies does not deliver equilibrium selection according to risk-dominance as in our standard example.

Note that a similar argument applies for $g(c)$.

3.9 Conclusions

I introduced incomplete information à la global games in a two-group max-min group contest with binary actions, relaxing the complete information assumption about the value of the prize contested and the cost of providing effort, separately. In the first case, there are both an equilibrium in (monotonic) switching strategies and an equilibrium robust to incomplete information in the sense of Kajii and Morris (1997); in the second one, a unique equilibrium in (monotonic) switching-strategies emerges. The existence result in both settings is extended to the case of M groups with symmetric sizes, while the analysis of equilibrium uniqueness for both symmetric and asymmetric groups is left for future research. On the other hand, in the two-group setting, given the uniform information structure, it is straightforward to calculate the probability of winning for each group and the expected payoffs at the equilibrium in switching strategies in both incomplete information cases, so that conditions ensuring the emergence of the co-called group-size paradox are highlighted. Finally I somehow replicate the limit-uniqueness and noise-independent selection results due to Carlsson and Damme (1993a), but in a more restrictive fashion, and I provide some intuition as to why our results are generally not communication-proof. Therefore, introducing incomplete information à la global games in max-min group contests with binary actions does not only deliver informational realism, but it also reduces significantly the burden of equilibrium multiplicity, or rather indeterminacy, which affects deterministic group contests with continuous efforts and a public good prize under both complete information and under incomplete information, as in Chowdhury et al. (2016) and Barbieri, Kovenock, et al. (2019), respectively. Equilibrium selection is delivered abstracting from group-public randomization devices as in turn done in Barbieri and Topolyan (2024). I would like to stress that this selection result could be relevant for applications of deterministic two-group contests with binary actions, among which I emphasized consortia of enterprises entering a market or adopting a new technology, research groups, international alliances, group strikes and military conflict in the introduction.

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3.A PROOFS

3.A.1 Proof of Proposition 3.1

Suppose

$$(\gamma_1, \gamma_2) \text{ such that } \gamma_j \in (0,1) \text{ and } 1 \leq \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} \leq n_j - 1 \; \forall j \in \{1,2\} \; .$$[3.A1]

Then,

$$min\,\{\mathbf{x}_1\} = min\,\{\mathbf{x}_2\} \; .$$

Suppose $x_j\,(i) = 1$, then

$$\pi_{ij}\,(\gamma_j, \gamma_{-j}) = \frac{v}{2} - 1 \; .$$

If agent ij deviates to $x_j\,(i) = 0$, then

$$min\,\{\mathbf{x}_1\} = min\,\{\mathbf{x}_2\}$$

so that the deviation payoff is

$$\pi_{ij}^{D}\left(\gamma_j^{-}, \gamma_{-j}\right) = \frac{v}{2}.$$

Hence, for player ij there is an incentive to deviate $\forall v \in \mathbb{R}$, since

$$\frac{v}{2} - 1 < \frac{v}{2} \; \forall v \in \mathbb{R} \; .$$

Thus,

$$(\gamma_1, \gamma_2) \text{ such that } \gamma_j \in (0,1) \text{ and } 1 \leq \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} \leq n_j - 1 \; \forall j \in \{1,2\}$$

is not a Nash equilibrium $\forall v \in \mathbb{R}$.

Suppose

$$(\gamma_1, \gamma_2) = (1,1) \; .$$

[3.A1] $\mathbb{1}_{x_{ij}=1}$ stands for the Indicator random variable taking value 1 when player ij chooses $x_{ij} = 1$, that is she exerts effort.

Then,

$$min\left\{\mathbf{x}_1\right\} = min\left\{\mathbf{x}_2\right\}$$

so that

$$\pi_{ij}\left(\gamma_j, \gamma_{-j}\right) = \frac{v}{2} - 1 \ .$$

If agent ij deviates to $x_j\left(i\right) = 0$, then

$$min\left\{\mathbf{x}_j\right\} = 0 < min\left\{\mathbf{x}_{-j}\right\}$$

so that the deviation payoff is

$$\pi_{ij}^{D}\left(\gamma_j^{-}, \gamma_{-j}\right) = 0 \ .$$

Hence, for player ij there is no incentive to deviate if and only if

$$\frac{v}{2} - 1 \geq 0 \Leftrightarrow v \geq 2 \ .$$

Thus,

$$\left(\gamma_1, \gamma_2\right) = \left(1, 1\right)$$

is a Nash equilibrium in pure strategies $\forall v \geq 2$.

Suppose

$$\left(\gamma_j, \gamma_{-j}\right) = \left(1, 0\right) \forall j \in \left\{1, 2\right\} \ .$$

Then,

$$min\left\{\mathbf{x}_j\right\} = 1 > min\left\{\mathbf{x}_{-j}\right\} = 0$$

so that

$$\pi_{ij}\left(\gamma_j, \gamma_{-j}\right) = v - 1 \ .$$

If agent ij deviates to $x_j\left(i\right) = 0$, then

$$min\left\{\mathbf{x}_j\right\} = 0 = min\left\{\mathbf{x}_{-j}\right\}$$

so that the deviation payoff is

$$\pi_{ij}^{D}\left(\gamma_j^{-},\gamma_{-j}\right)=\frac{v}{2}\ .$$

Hence, for agent ij there is no incentive to deviate if and only if

$$v-1\geq\frac{v}{2}\Leftrightarrow v\geq 2\ .$$

On the other hand, if agent $i-j$ deviates to $x_{-j}\left(i\right)=1$, then

$$min\left\{\mathbf{x}_j\right\}=1>min\left\{\mathbf{x}_{-j}\right\}=0$$

so that the deviation payoff is

$$\pi_{i-j}^{D}\left(\gamma_j,\gamma_{-j}^{+}\right)=-1\ .$$

Hence, for player $i-j$ deviating is a strictly dominated action as

$$\pi_{i-j}\left(\gamma_j,\gamma_{-j}\right)=0>\pi_{i-j}^{D}\left(\gamma_j,\gamma_{-j}^{+}\right)=-1\ .$$

Thus,

$$\left(\gamma_j,\gamma_{-j}\right)=\left(1,0\right)\ \ \forall j\in\left\{1,2\right\}$$

is a Nash equilibrium in pure strategies for any $v\geq 2$.

Suppose

$$\left(\gamma_1,\gamma_2\right)=\left(0,0\right)\ \ .$$

Then

$$min\left\{\mathbf{x}_1\right\}=min\left\{\mathbf{x}_2\right\}$$

so that

$$\pi_{ij}\left(\gamma_1,\gamma_2\right)=\frac{v}{2}\ \forall j\in\left\{1,2\right\}\ \ .$$

If agent ij deviates to $x_j(i) = 1$, then

$$min\{\mathbf{x}_1\} = min\{\mathbf{x}_2\}$$

so that

$$\pi_{ij}^D\left(\gamma_j^+, \gamma_{-j}\right) = \frac{v}{2} - 1 \ .$$

Hence, for player ij deviating is a strictly dominated action as

$$\pi_{ij}(\gamma_j, \gamma_{-j}) = \frac{v}{2} > \pi_{ij}^D\left(\gamma_j^D, \gamma_{-j}\right) = \frac{v}{2} - 1 \ \forall j \in \{1, 2\} \ .$$

Thus,

$$(\gamma_1, \gamma_2) = (0, 0)$$

is a Nash equilibrium in pure strategies for any $v \in \mathbb{R}$.

Let $\sigma_{ij}(x_{ij} = 1)$ be the within-group symmetric randomization over pure strategy $x_{ij} = 1$ for player ij, then

$$EU_{ij}(x_{ij} = 1) = EU_{ij}(x_{ij} = 0) \Leftrightarrow$$

$$\Leftrightarrow Prob\left((n_j - 1)\gamma_j = n_j - 1\right) \cdot Prob\left(n_{-j}\gamma_{-j} < n_{-j}\right) \cdot (v - 1) + \Big(Prob\left((n_j - 1)\gamma_j < n_j - 1\right) \cdot$$

$$Prob\left(n_{-j}\gamma_{-j} < n_{-j}\right) + Prob\left((n_j - 1)\gamma_j = n_j - 1\right) \cdot Prob\left(n_{-j}\gamma_{-j} = n_{-j}\right)\Big)\left(\frac{v}{2} - 1\right) +$$

$$+ Prob\left((n_j - 1)\gamma_j < n_j - 1\right) \cdot Prob\left(n_{-j}\gamma_{-j} = n_{-j}\right) \cdot (-1) = Prob\left(n_{-j}\gamma_{-j} < n_{-j}\right) \cdot \frac{v}{2} \Leftrightarrow$$

$$\Leftrightarrow \left(\sigma_{ij}(x_{ij} = 1)\right)^{n_j - 1} \cdot \left(1 - \left(\sigma_{i-j}(x_{i-j} = 1)\right)^{n_{-j}}\right) \cdot (v - 1) + \Big(1 - \left(\sigma_{ij}(x_{ij} = 1)\right)^{n_j - 1} \cdot$$

$$\left(1 - \left(\sigma_{i-j}(x_{i-j} = 1)\right)^{n_{-j}}\right) + \left(\sigma_{ij}(x_{ij} = 1)\right)^{n_j - 1} \cdot \left(\sigma_{i-j}(x_{i-j} = 1)\right)^{n_{-j}}\Big) \cdot \left(\frac{v}{2} - 1\right) +$$

$$+ \left(1 - \left(\sigma_{ij}(x_{ij} = 1)\right)^{n_j - 1}\right) \cdot \left(\sigma_{i-j}(x_{i-j} = 1)\right)^{n_{-j}} \cdot (-1) = \left(1 - \left(\sigma_{i-j}(x_{i-j} = 1)\right)^{n_{-j}}\right) \cdot \frac{v}{2} \Leftrightarrow$$

$$\Leftrightarrow \sigma_{ij}^*(x_{ij} = 1) = \left(\frac{2}{v}\right)^{1/(n_j - 1)} \ \forall i \in \{1, \ldots, n_j\} \text{ and } j \in \{1, 2\} \ .$$

Thus,

$$\sigma^*_{ij}\left(x_{ij}=1\right)=\left(\frac{2}{v}\right)^{1/(n_j-1)} \quad \forall i\in\{1,\ldots,n_j\} \text{ and } j\in\{1,2\} \ .$$

is a Nash equilibrium in within-group symmetric strictly-mixed strategies $\forall\ v>2$. QED

3.A.2 Proof of Proposition 3.2

Following the formulation of payoff-dominance and risk-dominance concepts by Harsanyi and Selten (1988), it is straightforward to state that:

- for $v>2$, $(\gamma_1,\gamma_2)=(1,0)$ payoff-dominates $(\gamma_1,\gamma_2)=(0,0)$ for group 1 and $(\gamma_1,\gamma_2)=(0,1)$ payoff-dominates $(\gamma_1,\gamma_2)=(0,0)$ for group 2, since

  $$\pi_{ij}\left(\gamma_j=1,\gamma_{-j}=0\right)=v-1>\pi_{ij}\left(\gamma_j=0,\gamma_{-j}=0\right)=\frac{v}{2}\Leftrightarrow v>2 \ .$$

- for $v>4$, $\gamma_j=1$ is the risk-dominant equilibrium strategy profile for group $j\in\{1,2\}$. As a matter of fact, let us compare the Nash products of $(\gamma_j,\gamma_{-j})=(1,1)$ and $(\gamma_j,\gamma_{-j})=(0,1)$. Then, for group j:

  $$\left(\frac{v}{2}-1\right)^{n_j}>(0-(-1))^{n_j}\Leftrightarrow\left(\frac{v}{2}-1\right)^{n_j}>1\Leftrightarrow v>4,$$

  that is, for $v>4$, $(\gamma_1,\gamma_2)=(1,1)$ is associated with the largest Nash product.

  Moreover, let us compare the Nash products of $(\gamma_j,\gamma_{-j})=(1,0)$ and $(\gamma_j,\gamma_{-j})=(0,0)$. Then, for group j:

  $$\left(v-1-\frac{v}{2}\right)^{n_j}>\left(\frac{v}{2}-\left(\frac{v}{2}-1\right)\right)^{n_j}\Leftrightarrow\left(\frac{v}{2}-1\right)^{n_j}>1\Leftrightarrow v>4 \ ,$$

  that is, for $v>4$, $(\gamma_j,\gamma_{-j})=(1,0)$ is associated with the largest Nash product;

- for $2<v<4$, $(\gamma_j,\gamma_{-j})=(0,1)$ and $(\gamma_j,\gamma_{-j})=(0,0)$ are the risk-dominant equilibrium strategy profiles for group $j\in\{1,2\}$. Clearly this follows from what shown at the previous point for both groups. QED

3.A.3 Proof of Proposition 3.3

Suppose

$$(\gamma_1, \gamma_2) \text{ such that } \gamma_j \in (0,1) \text{ and } 1 \leq \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} \leq n_j - 1 \; \forall j \in \{1,2\} \; .$$

Then,

$$min\left\{\mathbf{x}_1\right\} = min\left\{\mathbf{x}_2\right\} \; .$$

Suppose $x_j\left(i\right) = 1$, then

$$\pi_{ij}\left(\gamma_j, \gamma_{-j}\right) = \frac{v}{2} - c \; .$$

If agent ij deviates to $x_j\left(i\right) = 0$, then

$$min\left\{\mathbf{x}_1\right\} = min\left\{\mathbf{x}_2\right\}$$

so that the deviation payoff is

$$\pi_{ij}^{D}\left(\gamma_j^{-}, \gamma_{-j}\right) = \frac{v}{2}.$$

Hence, for player ij there is no incentive to deviate $\forall v \in \mathbb{R}_{++}$ if and only if

$$\frac{v}{2} - c \geq \frac{v}{2} \Leftrightarrow c \leq 0 \; .$$

On the other hand, suppose $x_j\left(i\right) = 0$ and $c \leq 0$, then

$$\pi_{ij}\left(\gamma_j, \gamma_{-j}\right) = \frac{v}{2}$$

If agent ij deviates to $x_j\left(i\right) = 1$, then

$$min\left\{\mathbf{x}_1\right\} = min\left\{\mathbf{x}_2\right\}$$

so that the deviation payoff is

$$\pi_{ij}^{D}\left(\gamma_j^{+},\gamma_{-j}\right)=\begin{cases} v-c & \text{if } n_j\gamma_j=n_j-1 \\ \frac{v}{2}-c & \text{otherwise} \end{cases} .$$

Hence, for player ij there is no incentive to deviate if and only if

$$c=0 \text{ and } n_j\gamma_j<n_j-1 \ .$$

Thus,

$$(\gamma_1,\gamma_2) \ \text{ such that } \gamma_j\in(0,1) \ \text{ and } 1\leq\sum_{i=1}^{n_j}\mathbb{1}_{x_{ij}=1}\leq n_j-2 \ \forall j\in\{1,2\}$$

is a Nash equilibrium if and only if $c=0$.

Suppose

$$(\gamma_j,\gamma_{-j})=(\gamma_j,0) \ \text{ such that } \gamma_j\in(0,1) \ \text{ and } 1\leq\sum_{i=1}^{n_j}\mathbb{1}_{x_{ij}=1}\leq n_j-1 \ \forall j\in\{1,2\} \ .$$

Then,

$$min\left\{\mathbf{x}_1\right\}=min\left\{\mathbf{x}_2\right\}.$$

Suppose $x_j\left(i\right)=1$, then

$$\pi_{ij}\left(\gamma_j,\gamma_{-j}\right)=\frac{v}{2}-c \text{ and } \pi_{i-j}\left(\gamma_j,\gamma_{-j}\right)=\frac{v}{2} \ .$$

If agent ij deviates to $x_j\left(i\right)=0$, then

$$min\left\{\mathbf{x}_1\right\}=min\left\{\mathbf{x}_2\right\}$$

so that the deviation payoff is

$$\pi_{ij}^{D}\left(\gamma_j^{-},\gamma_{-j}\right)=\frac{v}{2}.$$

Hence, for player ij there is no incentive to deviate $\forall\ v \in \mathbb{R}_{++}$ if and only if

$$\frac{v}{2} - c \geq \frac{v}{2} \Leftrightarrow c \leq 0 \ .$$

On the other hand, suppose $x_j\left(i\right) = 0$ and $c \leq 0$, then

$$\pi_{ij}\left(\gamma_j, \gamma_{-j}\right) = \frac{v}{2} \ .$$

If agent ij deviates to $x_j\left(i\right) = 1$, then

$$min\left\{\mathbf{x}_1\right\} = min\left\{\mathbf{x}_2\right\}$$

so that the deviation payoff is

$$\pi^D_{ij}\left(\gamma^+_j, \gamma_{-j}\right) = \begin{cases} v - c & \text{if } n_j\gamma_j = n_j - 1 \\ \frac{v}{2} - c & \text{otherwise} \ . \end{cases}$$

Hence, for player ij there is no incentive to deviate if and only if

$$c = 0 \text{ and } n_j\gamma_j < n_j - 1 \ .$$

If agent $i-j$ deviates to $x_{-j}\left(i\right) = 1$, then

$$min\left\{\mathbf{x}_j\right\} = min\left\{\mathbf{x}_{-j}\right\}$$

so that the deviation payoff is

$$\pi^D_{i-j}\left(\gamma_j, \gamma^+_{-j}\right) = \frac{v}{2} - c \ .$$

Hence, for player $i-j$ there is no incentive to deviate if and only if

$$\frac{v}{2} \geq \frac{v}{2} - c \Leftrightarrow c \geq 0 \ .$$

Thus,

$$(\gamma_j, \gamma_{-j}) = (\gamma_j, 0) \text{ such that } \gamma_j \in (0,1) \text{ and } 1 \leq \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} \leq n_j - 2 \; \forall j \in \{1,2\} \ .$$

is a Nash equilibrium if and only if $c = 0$.

Suppose

$$(\gamma_1, \gamma_2) = (1,1) \ .$$

Then,

$$min\left\{\mathbf{x}_1\right\} = min\left\{\mathbf{x}_2\right\}$$

so that

$$\pi_{ij}\left(\gamma_j, \gamma_{-j}\right) = \frac{v}{2} - c \ .$$

If agent ij deviates to $x_j\left(i\right) = 0$, then

$$min\left\{\mathbf{x}_j\right\} = 0 < min\left\{\mathbf{x}_{-j}\right\}$$

so that the deviation payoff is

$$\pi_{ij}^D\left(\gamma_j^-, \gamma_{-j}\right) = 0 \ .$$

Hence, for player ij there is no incentive to deviate if and only if

$$\frac{v}{2} - c \geq 0 \Leftrightarrow c \leq \frac{v}{2} \ .$$

Thus,

$$(\gamma_1, \gamma_2) = (1,1)$$

is a Nash equilibrium in pure strategies $\forall c \leq \frac{v}{2}$.

Suppose

$$(\gamma_j, \gamma_{-j}) = (1,0) \, \forall j \in \{1,2\} \ .$$

Then,

$$min\left\{\mathbf{x}_j\right\} = 1 > min\left\{\mathbf{x}_{-j}\right\} = 0$$

so that

$$\pi_{ij}\left(\gamma_j, \gamma_{-j}\right) = v - c \ .$$

If agent ij deviates to $x_j\left(i\right) = 0$, then

$$min\left\{\mathbf{x}_j\right\} = 0 = min\left\{\mathbf{x}_{-j}\right\}$$

so that the deviation payoff is

$$\pi_{ij}^D\left(\gamma_j^-, \gamma_{-j}\right) = \frac{v}{2} \ .$$

Hence, for agent ij there is no incentive to deviate if and only if

$$v - c \geq \frac{v}{2} \Leftrightarrow c \leq \frac{v}{2} \ .$$

On the other hand, if agent $i-j$ deviates to $x_{-j}\left(i\right) = 1$, then

$$min\left\{\mathbf{x}_j\right\} = 1 > min\left\{\mathbf{x}_{-j}\right\} = 0$$

so that the deviation payoff is

$$\pi_{i-j}^D\left(\gamma_j, \gamma_{-j}^+\right) = -c \ .$$

Hence, for player $i-j$ there is no incentive to deviate if and only if

$$\pi_{i-j}\left(\gamma_j, \gamma_{-j}\right) = 0 \geq \pi_{i-j}^D\left(\gamma_j, \gamma_{-j}^+\right) = -c \Leftrightarrow c \geq 0 \ .$$

Thus,

$$\left(\gamma_j, \gamma_{-j}\right) = \left(1, 0\right) \ \forall j \in \left\{1, 2\right\}$$

is a Nash equilibrium in pure strategies for any $0 \leq c \leq \frac{v}{2}$.

Suppose

$$(\gamma_1, \gamma_2) = (0,0) \ .$$

Then

$$min\left\{\mathbf{x}_1\right\} = min\left\{\mathbf{x}_2\right\}$$

so that

$$\pi_{ij}\left(\gamma_1, \gamma_2\right) = \frac{v}{2} \ \forall j \in \{1,2\} \ .$$

If agent ij deviates to $x_j\left(i\right) = 1$, then

$$min\left\{\mathbf{x}_1\right\} = min\left\{\mathbf{x}_2\right\}$$

so that

$$\pi_{ij}^{D}\left(\gamma_j^{+}, \gamma_{-j}\right) = \frac{v}{2} - c \ .$$

Hence, for player ij there is no incentive to deviate if and only if

$$\pi_{ij}\left(\gamma_j, \gamma_{-j}\right) = \frac{v}{2} \geq \pi_{ij}^{D}\left(\gamma_j^{D}, \gamma_{-j}\right) = \frac{v}{2} - c \Leftrightarrow c \geq 0 \ \forall j \in \{1,2\} \ .$$

Thus,

$$(\gamma_1, \gamma_2) = (0,0)$$

is a Nash equilibrium in pure strategies for any $c \geq 0$.

Let $\sigma_{ij}\left(x_{ij} = 1\right)$ be the within-group symmetric randomization over pure strategy $x_{ij} = 1$ for player ij $\forall j \in \{1,2\}$, then

$$EU_{ij}\left(x_{ij} = 1\right) = EU\left(x_{ij} = 0\right) \Leftrightarrow$$

$$\Leftrightarrow Prob\left(\left(n_j - 1\right)\gamma_j = n_j - 1\right) \cdot Prob\left(n_{-j}\gamma_{-j} < n_{-j}\right) \cdot \left(v - c\right) + \Big(Prob\left(\left(n_j - 1\right)\gamma_j < n_j - 1\right) \cdot$$

$$Prob\left(n_{-j}\gamma_{-j} < n_{-j}\right) + Prob\left(\left(n_j - 1\right)\gamma_j = n_j - 1\right) \cdot Prob\left(n_{-j}\gamma_{-j} = n_{-j}\right)\Big)\left(\frac{v}{2} - c\right) +$$

$$+Prob\left(\left(n_j - 1\right)\gamma_j < n_j - 1\right) \cdot Prob\left(n_{-j}\gamma_{-j} = n_{-j}\right) \cdot \left(-c\right) = Prob\left(n_{-j}\gamma_{-j} < n_{-j}\right) \cdot \frac{v}{2} \Leftrightarrow$$

$$\Leftrightarrow \left(\sigma_{ij}\left(x_{ij}=1\right)\right)^{n_j-1}\cdot\left(1-\left(\sigma_{i-j}\left(x_{i-j}=1\right)\right)^{n-j}\right)\cdot(v-c)+\Big(1-\left(\sigma_{ij}\left(x_{ij}=1\right)\right)^{n_j-1}\cdot$$

$$\left(1-\left(\sigma_{i-j}\left(x_{i-j}=1\right)\right)^{n-j}\right)+\left(\sigma_{ij}\left(x_{ij}=1\right)\right)^{n_j-1}\cdot\left(\sigma_{i-j}\left(x_{i-j}=1\right)\right)^{n-j}\Big)\cdot\left(\frac{v}{2}-c\right)+$$

$$+\left(1-\left(\sigma_{ij}\left(x_{ij}=1\right)\right)^{n_j-1}\right)\cdot\left(\sigma_{i-j}\left(x_{i-j}=1\right)\right)^{n-j}\cdot(-c)=\left(1-\left(\sigma_{i-j}\left(x_{i-j}=1\right)\right)^{n-j}\right)\cdot\frac{v}{2}\Leftrightarrow$$

$$\Leftrightarrow \sigma^*_{ij}\left(x_{ij}=1\right)=\left(\frac{2c}{v}\right)^{1/(n_j-1)}\quad\forall i\in\{1,\ldots,n_j\}\ \text{ and } j\in\{1,2\};$$

Thus,

$$\sigma^*_{ij}\left(x_{ij}=1\right)=\left(\frac{2c}{v}\right)^{1/(n_j-1)}\quad\forall i\in\{1,\ldots,n_j\}\ \text{ and } j\in\{1,2\}$$

is a Nash equilibrium in within-group symmetric strictly-mixed strategies $\forall\ 0<c<\frac{v}{2}$.

QED

3.A.4 Proof of Proposition 3.4

Following the formulation of payoff-dominance and risk-dominance concepts by Harsanyi and Selten (1988), it is straightforward to state that in the $BMMGC^{*b}$:

- if $0<c\leq\frac{v}{2}$, $(\gamma_1,\gamma_2)=(1,0)$ payoff-dominates $(\gamma_1,\gamma_2)=(0,0)$ for group 1 and $(\gamma_1,\gamma_2)=(0,1)$ payoff-dominates $(\gamma_1,\gamma_2)=(0,0)$ for group 2, since

$$\pi_{ij}\left(\gamma_j=1,\gamma_{-j}=0\right)=v-c>\pi_{ij}\left(\gamma_j=0,\gamma_{-j}=0\right)=\frac{v}{2}\Leftrightarrow c<\frac{v}{2};$$

- if $c=0$, for group j, $(\gamma_j,\gamma_{-j})=(1,0)$ payoff-dominates $(\gamma_1,\gamma_2)=(0,0)\,,(\gamma_j,\gamma_{-j})=(\gamma_j,0)$ such that $\gamma_j\in(0,1)$ and $1\leq\sum_{i=1}^{n_j}\mathbb{1}_{x_{ij}=1}\leq n_j-2,(\gamma_1,\gamma_2)$ such that $\gamma_j\in(0,1)$ and $1\leq\sum_{i=1}^{n_j}\mathbb{1}_{x_{ij}=1}\leq n_j-2\ \forall j\in\{1,2\}$ since

$$\begin{cases} \pi_{ij}\left(\gamma_j = 1, \gamma_{-j} = 0\right) = v \\ \pi_{ij}\left(\gamma_j = 0, \gamma_{-j} = 0\right) = \frac{v}{2} \\ \pi_{ij}\left(\gamma_j, \gamma_{-j}\right) = \frac{v}{2} \\ \pi_{ij}\left(\gamma_j, 0\right) = \frac{v}{2} \\ \pi_{ij}\left(\gamma_j = 0, \gamma_{-j} = 1\right) = 0 \ ; \end{cases}$$

- if $0 < c < \frac{v}{4}$, $\gamma_1 = 1$ is the risk-dominant equilibrium strategy profile for group 1 and, symmetrically, $\gamma_2 = 1$ for group 2 . As a matter of fact, let us compare the Nash products of $(\gamma_j, \gamma_{-j}) = (1,1)$ and $(\gamma_j, \gamma_{-j}) = (0,1)\ \forall j \in \{1,2\}$. Then, for group j:

  $$\left(\frac{v}{2} - c\right)^{n_j} > (0 - (-c))^{n_j} \Leftrightarrow \left(\frac{v}{2} - c\right)^{n_j} > c^{n_j} \Leftrightarrow v > 4c \Leftrightarrow c < \frac{v}{4},$$

  that is, for $c < \frac{v}{4}$, $(\gamma_j, \gamma_{-j}) = (1,1)$ is associated with the largest Nash product.

  Moreover, let us compare the Nash products of $(\gamma_j, \gamma_{-j}) = (1,0)$ and $(\gamma_j, \gamma_{-j}) = (0,0)$. Then, for group j:

  $$\left(v - c - \frac{v}{2}\right)^{n_j} > \left(\frac{v}{2} - \left(\frac{v}{2} - c\right)\right)^{n_j} \Leftrightarrow \left(\frac{v}{2} - c\right)^{n_j} > c^{n_j} \Leftrightarrow c < \frac{v}{4}\ ,$$

  that is, for $c < \frac{v}{4}$, $(\gamma_j, \gamma_{-j}) = (1,0)$ is associated with the largest Nash product;

- if $c = 0$, $\gamma_1 = 1$ is the risk-dominant equilibrium strategy profile for group 1 and, symmetrically, $\gamma_2 = 1$ for group 2 . As a matter of fact, let us compare the Nash products of $(\gamma_j, \gamma_{-j}) = (1,1)$ and $(\gamma_j, \gamma_{-j}) = (0,1)\ \forall j \in \{1,2\}$. Then, for group j:

  $$\left(\frac{v}{2} - c\right)^{n_j} > (0 - (-c))^{n_j} \Leftrightarrow \left(\frac{v}{2}\right)^{n_j} > 0^{n_j} \Leftrightarrow v > 0\ ,$$

  that is, for $c = 0$, $(\gamma_j, \gamma_{-j}) = (1,1)$ is associated with the largest Nash product.

  Moreover, let us compare the Nash products of $(\gamma_j, \gamma_{-j}) = (1,0)$ and $(\gamma_j, \gamma_{-j}) = (0,0)$. Then, for group 1:

$$\left(v-c-\frac{v}{2}\right)^{n_j} > \left(\frac{v}{2}-\left(\frac{v}{2}-c\right)\right)^{n_j} \Leftrightarrow \left(\frac{v}{2}\right)^{n_j} > 0^{n_j} \Leftrightarrow v > 0 \ ,$$

that is, for $v > 0$, $(\gamma_j, \gamma_{-j}) = (1, 0)$ is associated with the largest Nash product.

Furthermore, let us compare the Nash products of $(\gamma_j, \gamma_{-j}) = (1, 0)$ and $(\gamma_j, \gamma_{-j}) = (\gamma_j, 0)$ such that $\gamma_j \in (0, 1)$ and $1 \leq \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}} \leq n_j - 2 \ \forall j \in \{1, 2\}$. Then, for group 1:

$$\left(v-c-\frac{v}{2}\right)^{n_j} > \left(\frac{v}{2}-c-\frac{v}{2}\right)^{n_j\gamma_j} \cdot \left(\frac{v}{2}-\left(\frac{v}{2}-c\right)\right)^{n_j(1-\gamma_j)} \Leftrightarrow \left(\frac{v}{2}\right)^{n_j} > 0 \Leftrightarrow v > 0 \ ,$$

that is, for $v > 0$, $(\gamma_j, \gamma_{-j}) = (1, 0)$ is associated with the largest Nash product.

Finally, let us compare the Nash products of $(\gamma_j, \gamma_{-j}) = (1, 0)$ and (γ_1, γ_2) such that $\gamma_j \in (0, 1)$ and $1 \leq \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}} \leq n_j - 2 \ \forall j \in \{1, 2\}$. Then, for group j:

$$\left(v-c-\frac{v}{2}\right)^{n_j} > \left(\frac{v}{2}-c-\frac{v}{2}\right)^{n_j\gamma_j} \cdot \left(\frac{v}{2}-\left(\frac{v}{2}-c\right)\right)^{n_j(1-\gamma_j)} \Leftrightarrow \left(\frac{v}{2}\right)^{n_j} > 0 \Leftrightarrow v > 0 \ ,$$

that is, for $v > 0$, $(\gamma_j, \gamma_{-j}) = (1, 0)$ is associated with the largest Nash product;

- if $\frac{v}{4} < c < \frac{v}{2}$, $(\gamma_j, \gamma_{-j}) = (0, 1)$ and $(\gamma_j, \gamma_{-j}) = (0, 0)$ are the risk-dominant equilibrium strategy profiles for group $j \in \{1, 2\}$. Clearly this follows from what shown at the previous point for both groups.

QED

3.A.5 Proof of Proposition 3.5

In the $g_1(v)$, note that $E(V|v_{ij}) = v_{ij}$, if i observes $v_{ij} \in [\underline{v} + \varepsilon, \overline{v} - \varepsilon]$ so that $V|v_{ij} \sim U(v_{ij} - \varepsilon, v_{ij} + \varepsilon)$. Furthermore, for $v_{ij} \in [\underline{v} + \varepsilon, \overline{v} - \varepsilon]$, the conditional distribution of the teammates' or opponents' observation will be centered around v_{ij} with support $[v_{ij} - 2\varepsilon, v_i + 2\varepsilon]$. Hence, $Prob\,[V_{-ij} < v_{ij}|v_{ij}] = Prob[V_{-ij} > v_{ij}|v_{ij}] = \frac{1}{2} \ \forall ij \in \{1, \ldots, n_j\}$ and $j \in \{1, 2\}$.

Now, suppose player $ij \in \{1, \ldots, n_j\} \ \forall j \in \{1, 2\}$ observes $v_{ij} < 2$. Then, ij's conditionally expected payoff from exerting effort, that is choosing $x_{ij} = 1$, is smaller than the one from exerting no effort, that is choosing $x_{ij} = 0$. Accordingly, $x_{ij} = 0$ is a conditionally strictly dominant action for player $ij \in \{1, \dots, n_j\}$ $\forall j \in \{1, 2\}$ whenever she observes $v_{ij} < 2$. Let us denote p_{-ij} the conditional expected probability attached by player ij to any other player $-ij$ within the same group choosing not to exert effort. Likewise, p_{i-j} is the conditional expected probability attached by ij to any other member of the rival group $-j$ not exerting effort. Then, for player ij the conditionally expected payoff from exerting effort equals

$$\begin{aligned}
&Prob\left(\left(n_j-1\right)\gamma_j = n_j-1\right) \cdot Prob\left(n_{-j}\gamma_{-j} < n_{-j}\right) \cdot \left(v_{ij}-1\right) + \Big(Prob\left(\left(n_j-1\right)\gamma_j < n_j-1\right) \\
&\cdot Prob\left(n_{-j}\gamma_{-j} < n_{-j}\right) + Prob\left(\left(n_j-1\right)\gamma_j = n_j-1\right) \cdot Prob\left(n_{-j}\gamma_{-j} = n_{-j}\right)\Big)\left(\frac{v_{ij}}{2}-1\right) + \\
&+ Prob\left(\left(n_j-1\right)\gamma_j < n_j-1\right) \cdot Prob\left(n_{-j}\gamma_{-j} = n_{-j}\right) \cdot (-1) \\
&= \prod_{-i}\left(1-p_{-ij}\right) \cdot \left(1-\prod_{i}\left(1-p_{i-j}\right)\right) \cdot \left(v_{ij}-1\right) + \left(\left(1-\prod_{-i}\left(1-p_{-ij}\right)\right) \cdot \left(1-\prod_{i}\left(1-p_{i-j}\right)\right) + \right. \\
&\left. + \prod_{-i}\left(1-p_{-ij}\right) \cdot \prod_{i}\left(1-p_{i-j}\right)\right)\left(\frac{v_{ij}}{2}-1\right) + \left(1-\prod_{-i}\left(1-p_{-ij}\right)\right) \cdot \prod_{i}\left(1-p_{i-j}\right) \cdot (-1) \ ,
\end{aligned}$$

while the conditionally expected payoff from not exerting effort equals

$$Prob\left(n_{-j}\gamma_{-j} < n_{-j}\right) \cdot \frac{v_{ij}}{2} = \left(1-\prod_{i}\left(1-p_{i-j}\right)\right) \cdot \frac{v_{ij}}{2} \ .$$

Hence, the expected payoff difference from exerting effort versus not exerting effort equals

$$\prod_{-i}\left(1-p_{-ij}\right)\left(\frac{v_{ij}}{2}\right) - 1 \ ,$$

and the expected payoff difference from not exerting effort versus exerting effort equals

$$1-\prod_{-i}\left(1-p_{-ij}\right)\left(\frac{v_{ij}}{2}\right) \ .$$

Note that, for $v_{ij} = 2$, $p_{-ij} \geq 1/2$, so that exerting effort can be discarded by strict dominance, since the expected payoff difference from not exerting effort versus exerting effort equals at least $(1/2)^{n_j-1}$. Note that I imposed by assumption that $0 < \varepsilon < \left|\frac{\underline{v}}{2} - 1\right|$, so that $v_{ij} - 2\varepsilon > \underline{v}$ for $v_{ij} = 2$. Let v_{ij}^* be the smallest observation such that $x_{ij} = 1$ cannot be excluded by iterated dominance. Then, it is possible to show that $v_{ij}^* = 2^{n_j}$. Note that $v_{ij} = 4$ is the threshold for the risk-dominance regions. As a matter of fact, at $v_{ij} = 2^{n_j}$ the conditional expected payoff difference from not exerting effort versus exerting effort equals at least zero, so that exerting effort cannot be discarded by strict dominance.

The cutoff $v_{ij}^{*} = 2^{n_j}$ is the unique threshold that can be established from the lower dominance regions by iterated deletion of strictly dominated strategies, since it is the unique value for v_{ij} solving

$$\left(\frac{1}{2}\right)^{n_j-1} \cdot \left(1-\left(\frac{1}{2}\right)^{n_{-j}}\right) \cdot (v_{ij}-1) + \left(\left(1-\left(\frac{1}{2}\right)^{n_{-j}}\right) \cdot \left(1-\left(\frac{1}{2}\right)^{n_j-1}\right) + \right.$$

$$\left. + \left(\frac{1}{2}\right)^{n_{-j}} \cdot \left(\frac{1}{2}\right)^{n_j-1}\right) \cdot \left(\frac{v_{ij}}{2}-1\right) + \left(1-\left(\frac{1}{2}\right)^{n_j-1}\right) \cdot \left(\frac{1}{2}\right)^{n-j} \cdot (-1) = \left(1-\left(\frac{1}{2}\right)^{n_{-j}}\right) \cdot \frac{v_{ij}}{2} \ .$$

The same kind of reasoning cannot be carried out for large observations of v, since it does not exist an upper dominance region. Conversely, this is possible in our second setting in which there is incomplete information about the cost of effort itself. As a matter of fact, in the latter there are both a lower and an upper dominance region. Note further that incentive compatibility above the cutoff v_j^{*} is ensured as $p_{-ij} = p_{i-j} \leq 1/2 \ \forall \ v_{ij} \geq 2^{n_j}$ holds.

Hence, in $g_1(v)$ under incomplete information à la global games there is a unique equilibrium in (monotonic) cutoff strategies, such that $\forall ij \in \{1, \ldots, n_j\}$ and $\forall j \in \{1, 2\}$:

$$x_{ij}^{*}(v_{ij}) = \begin{cases} 1 & \text{if } v_{ij} > 2^{n_j} \\ 0 & \text{if } v_{ij} \leq 2^{n_j} \end{cases}$$

Nonetheless, given the absence of an upward dominance region, the following equilibrium $\forall ij \in \{1, \ldots, n_j\}$ and $\forall j \in \{1, 2\}$ exists as in De Mesquita (2011):

$$x_{ij}^{**}(v_{ij}) = 0 \ \forall v_{ij} \in [\underline{v}+\varepsilon, \overline{v}-\varepsilon] \,.$$

Note that at $(\gamma_1, \gamma_2) = (0, 0)$ any deviation is strictly dominated for any $v_{ij} \in [\underline{v}+\varepsilon, \overline{v}-\varepsilon]$, so that $(\gamma_1, \gamma_2) = (0, 0)$ in the $BMMGC^{*}$ is robust to incomplete information in the sense of Kajii and Morris (1997).

QED

3.A.6 Proof of Proposition 3.6

In the $g_2(c)$, note that $E(C|c_{ij}) = c_{ij}$, if ij observes $c_{ij} \in [\underline{c}+\varepsilon, \overline{c}-\varepsilon]$ so that $C|c_{ij} \sim U(c_{ij}-\varepsilon \,, c_{ij}+\varepsilon)$. Furthermore, for $c_{ij} \in [\underline{c}-\varepsilon, \overline{c}+\varepsilon]$, the conditional distribution of the teammates' or opponents' observation will be centered around c_{ij} with support $[c_{ij} - 2\varepsilon, c_{ij} + 2\varepsilon]$. Hence, $Prob\left[C_{-ij} < c_{ij}|c_{ij}\right] = Prob[C_{-ij} > c_{ij}|c_{ij}] = \frac{1}{2}$.

Now, suppose player $ij \in \{1, \ldots, n_j\}$ $\forall j \in \{1, 2\}$ observes $c_{ij} > \frac{v}{2}$. Then, ij's conditionally expected payoff from exerting effort, that is choosing $x_{ij} = 1$, is smaller than the one from exerting no effort, that is choosing $x_{ij} = 0$. Accordingly, $x_{ij} = 0$ is a conditionally strictly dominant action for player $ij \in \{1, \ldots, n_j\}$ $\forall j \in \{1, 2\}$ whenever she observes $c_{ij} > \frac{v}{2}$. Again, let us assume $\varepsilon < |\frac{c}{2}|$ and suppose player $ij \in \{1, \ldots, n_j\}$ $\forall j \in \{1, 2\}$ observes $c_{ij} < 0$. Then, ij's conditionally expected payoff from exerting effort, that is choosing $x_{ij} = 1$, is positive and greater than the one from exerting no effort, that is choosing $x_{ij} = 0$. Accordingly, $x_{ij} = 1$ is a conditionally strictly dominant action for player $ij \in \{1, \ldots, n_j\}$ $\forall j \in \{1, 2\}$ whenever she observes $c_{ij} < 0$. Then for player ij the conditionally expected payoff from exerting effort equals

$$\begin{aligned}
& Prob\left(\left(n_j-1\right)\gamma_j = n_j - 1\right) \cdot Prob\left(n_{-j}\gamma_{-j} < n_{-j}\right) \cdot \left(v - c_{ij}\right) + \Big(Prob\left(\left(n_j-1\right)\gamma_j < n_j - 1\right) \\
& \cdot Prob\left(n_{-j}\gamma_{-j} < n_{-j}\right) + Prob\left(\left(n_j-1\right)\gamma_j = n_j - 1\right) \cdot Prob\left(n_{-j}\gamma_{-j} = n_{-j}\right)\Big)\left(\frac{v}{2} - c_{ij}\right) + \\
& + Prob\left(\left(n_j-1\right)\gamma_j < n_j - 1\right) \cdot Prob\left(n_{-j}\gamma_{-j} = n_{-j}\right) \cdot \left(-c_{ij}\right) \\
& = \prod_{-i}\left(1 - p_{-ij}\right) \cdot \left(1 - \prod_{i}\left(1 - p_{i-j}\right)\right) \cdot \left(v - c_{ij}\right) + \left(\left(1 - \prod_{-i}\left(1 - p_{-ij}\right)\right) \cdot \left(1 - \prod_{i}\left(1 - p_{i-j}\right)\right) + \right. \\
& \left. + \prod_{-i}\left(1 - p_{-ij}\right) \cdot \prod_{i}\left(1 - p_{i-j}\right)\right)\left(\frac{v}{2} - c_{ij}\right) + \left(1 - \prod_{-i}\left(1 - p_{-ij}\right)\right) \cdot \prod_{i}\left(1 - p_{i-j}\right) \cdot \left(-c_{ij}\right) \ ,
\end{aligned}$$

while the conditionally expected payoff from not exerting effort equals

$$Prob\left(n_{-j}\gamma_{-j} < n_{-j}\right) \cdot \frac{v}{2} = \left(1 - \prod_{i}\left(1 - p_{i-j}\right)\right) \cdot \frac{v}{2} \ .$$

Hence, the expected payoff difference from exerting effort versus not exerting effort equals

$$\prod_{-i}\left(1 - p_{-ij}\right)\left(\frac{v}{2}\right) - c_{ij} \ ,$$

and the expected payoff difference from not exerting effort versus exerting effort equals

$$c_{ij} - \prod_{-i}\left(1 - p_{-ij}\right)\left(\frac{v}{2}\right) \ .$$

Note that, for $c_{ij} = v/2$, $p_{-ij} \geq 1/2$, so that exerting effort can be discarded by strict dominance, since the expected payoff difference from not exerting effort versus exerting effort equals at least $v/2 \cdot \left(1-(1/2)^{n_j-1}\right)$. Note that I imposed by assumption that $\varepsilon < \left|\frac{2\overline{c}-v}{4}\right|$, so that $c_{ij}+2\varepsilon < \overline{c}$ for $c_{ij}=\frac{v}{2}$. Let c_{ij}^{*} be the smallest observation such that $x_{ij}=1$ cannot be excluded by iterated dominance. Then, it is possible to show that $c_{ij}^{*}=2^{-n_j}v$. Note that $v_{ij}=v/4$ is the threshold for the risk-dominance regions. As a matter of fact, at $c_{ij}=2^{-n_j}v$ the conditional expected payoff difference from not exerting effort versus exerting effort equals at least zero, so that exerting effort cannot be discarded by strict dominance.

The cutoff $c_{ij}^{*}=2^{-n_j}v$ is the unique threshold that can be established from the upper dominance region by iterated deletion of strictly dominated strategies, since it is the unique value for c_{ij} solving

$$\left(\frac{1}{2}\right)^{n_j-1}\cdot\left(1-\left(\frac{1}{2}\right)^{n_{-j}}\right)\cdot\left(v-c_{ij}\right)+\left(\left(1-\left(\frac{1}{2}\right)^{n_{-j}}\right)\cdot\left(1-\left(\frac{1}{2}\right)^{n_j-1}\right)+\left(\frac{1}{2}\right)^{n_{-j}}\cdot\right.$$

$$\left.\cdot\left(\frac{1}{2}\right)^{n_j-1}\right)\cdot\left(\frac{v}{2}-c_{ij}\right)+\left(1-\left(\frac{1}{2}\right)^{n_j-1}\right)\cdot\left(\frac{1}{2}\right)^{n-j}\cdot\left(-c_{ij}\right)=\left(1-\left(\frac{1}{2}\right)^{n_{-j}}\right)\cdot v$$

On the other hand, observe that , for $c_{ij}=0$, $p_{-ij}\leq 1/2$, so that not exerting effort can be discarded by strict dominance, since the expected payoff difference from exerting effort versus not exerting effort equals at least $2^{-n_j}v$. Note that I imposed by assumption that $\varepsilon < |\underline{c}/2|$, so that $c_{ij}-2\varepsilon > \underline{c}$ for $c_{ij}=0$. Let c_{ij}^{**} be the smallest observation such that $x_{ij}=0$ cannot be excluded by iterated dominance. Then, it is possible to show that $c_{ij}^{**}=2^{-n_j}v$. Note that $v_{ij}=v/4$ is the threshold for the risk-dominance regions. As a matter of fact, at $c_{ij}=2^{-n_j}v$ the conditional expected payoff difference from exerting effort versus not exerting effort equals at least zero, so that not exerting effort cannot be discarded by strict dominance.

The cutoff $c_{ij}^{**}=2^{-n_j}v$ is the unique threshold that can be established from the lower dominance region by iterated deletion of strictly dominated strategies, since it is the unique value for c_{ij} solving

$$\left(\frac{1}{2}\right)^{n_j-1}\cdot\left(1-\left(\frac{1}{2}\right)^{n_{-j}}\right)\cdot\left(v-c_{ij}\right)+\left(\left(1-\left(\frac{1}{2}\right)^{n_{-j}}\right)\cdot\left(1-\left(\frac{1}{2}\right)^{n_j-1}\right)+\left(\frac{1}{2}\right)^{n_{-j}}\cdot\right.$$

$$\left.\cdot\left(\frac{1}{2}\right)^{n_j-1}\right)\cdot\left(\frac{v}{2}-c_{ij}\right)+\left(1-\left(\frac{1}{2}\right)^{n_j-1}\right)\cdot\left(\frac{1}{2}\right)^{n-j}\cdot\left(-c_{ij}\right)=\left(1-\left(\frac{1}{2}\right)^{n_{-j}}\right)\cdot\frac{v}{2}\ .$$

Hence, $c_{ij}^{*}=c_{ij}^{**}$ and there exists a unique equilibrium in switching strategies in $g_2\left(c\right)$ such that

$\forall ij \in \{1, \ldots, n_j\}$ and $\forall j \in \{1, 2\}$

$$x_{ij}^{*}\left(c_{ij}\right) = \begin{cases} 1 & \text{if } c_{ij} < 2^{-n_j} v \\ 0 & \text{if } c_{ij} \geq 2^{-n_j} v \ . \end{cases}$$

QED

3.A.7 Proof of Proposition 3.7

In the $g_1\left(v\right)$, given the contest success function $P_j\left(X_j, X_{-j}\right) \forall j \in \{1, 2\}$, the probability of winning the prize v for group $j \in \{1, 2\}$ is:

$$\begin{aligned} Prob\left(j \text{ wins } v\right) = & Prob\left[\left(\gamma_j^{*}, \gamma_{-j}^{*}\right) = (1, 0)\right] + \frac{1}{2} Prob\left[\left(\gamma_j^{*}, \gamma_{-j}^{*}\right) = (0, 0)\right] + \\ & + \frac{1}{2} Prob\left[\left(\gamma_j^{*}, \gamma_{-j}^{*}\right) = (1, 1)\right] \ . \end{aligned}$$

On the other hand, the probability of winning the prize v for group $j \in \{1, 2\}$ at the cutoff equilibrium $x_{ij}^{*}\left(v_{ij}\right)$ depends on whether or not 2^{n_j} belongs to $[\underline{v} + \varepsilon, \overline{v} - \varepsilon]$, where v_{ij} is uniformly distributed. Hence, I will consider all possible cases:

- if $\underline{v} + \varepsilon \leq 2^{n_j} \leq \overline{v} - \varepsilon$ and $\underline{v} + \varepsilon \leq 2^{n_{-j}} \leq \overline{v} - \varepsilon$, [3.A2]

$$\begin{aligned} Prob\left[\left(\gamma_j, \gamma_{-j}\right) = (1, 0)\right] &= \left(1 - \frac{2^{n_j} - \underline{v} - \varepsilon}{\overline{v} - \varepsilon - \underline{v} - \varepsilon}\right)^{n_j} \cdot \left[1 - \left(1 - \frac{2^{n_{-j}} - \underline{v} - \varepsilon}{\overline{v} - \varepsilon - \underline{v} - \varepsilon}\right)^{n_{-j}}\right] \ ; \\ Prob\left[\left(\gamma_j, \gamma_{-j}\right) = (0, 0)\right] &= \left[1 - \left(1 - \frac{2^{n_j} - \underline{v} - \varepsilon}{\overline{v} - \varepsilon - \underline{v} - \varepsilon}\right)^{n_j}\right] \cdot \left[1 - \left(1 - \frac{2^{n_{-j}} - \underline{v} - \varepsilon}{\overline{v} - \varepsilon - \underline{v} - \varepsilon}\right)^{n_{-j}}\right] \ ; \\ Prob\left[\left(\gamma_j, \gamma_{-j}\right) = (1, 1)\right] &= \left(1 - \frac{2^{n_j} - \underline{v} - \varepsilon}{\overline{v} - \varepsilon - \underline{v} - \varepsilon}\right)^{n_j} \cdot \left(1 - \frac{2^{n_{-j}} - \underline{v} - \varepsilon}{\overline{v} - \varepsilon - \underline{v} - \varepsilon}\right)^{n_{-j}} \ . \end{aligned}$$

  Hence,

$$\begin{aligned} Prob\left(j \text{ wins } v\right) = & Prob\left[\left(\gamma_j, \gamma_{-j}\right) = (1, 0)\right] + \frac{1}{2} Prob\left[\left(\gamma_j, \gamma_{-j}\right) = (0, 0)\right] + \\ & + \frac{1}{2} Prob\left[\left(\gamma_j, \gamma_{-j}\right) = (1, 1)\right] \end{aligned}$$

[3.A2] Note that for $\gamma_{-j} = 0$, it suffices that just one $i - j$ chooses $x_{i-j}\left(v_{i-j}\right) = 0$, due to the weakest-link impact function.

$$
\begin{aligned}
&=\left(1-\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\underline{v}-2\varepsilon}\right)^{n_j}\cdot\left[1-\left(1-\frac{2^{n_{-j}}-\underline{v}-\varepsilon}{\overline{v}-\underline{v}-2\varepsilon}\right)^{n_{-j}}\right]+\\
&+\frac{1}{2}\left[1-\left(1-\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\underline{v}-2\varepsilon}\right)^{n_j}\right]\cdot\left[1-\left(1-\frac{2^{n_{-j}}-\underline{v}-\varepsilon}{\overline{v}-\underline{v}-2\varepsilon}\right)^{n_{-j}}\right]+\\
&+\frac{1}{2}\left(1-\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\underline{v}-2\varepsilon}\right)^{n_j}\cdot\left(1-\frac{2^{n_{-j}}-\underline{v}-\varepsilon}{\overline{v}-\underline{v}-2\varepsilon}\right)^{n_{-j}}\ .
\end{aligned}
$$

- If $\underline{v}+\varepsilon\leq 2^{n_j}\leq\overline{v}-\varepsilon$ and $2^{n_{-j}}<\underline{v}+\varepsilon$,

$$
\begin{aligned}
&Prob\left[(\gamma_j,\gamma_{-j})=(1,0)\right]=0\ ;\\
&Prob\left[(\gamma_j,\gamma_{-j})=(0,0)\right]=0\ ;\\
&Prob\left[(\gamma_j,\gamma_{-j})=(1,1)\right]=\left(1-\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\underline{v}-2\varepsilon}\right)^{n_j}\ .
\end{aligned}
$$

Hence,

$$
\begin{aligned}
Prob\,(j\text{ wins }v)=&Prob\left[(\gamma_j,\gamma_{-j})=(1,0)\right]+\frac{1}{2}Prob\left[(\gamma_j,\gamma_{-j})=(0,0)\right]+\\
&+\frac{1}{2}Prob\left[(\gamma_j,\gamma_{-j})=(1,1)\right]\\
=&\frac{1}{2}\left(1-\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\underline{v}-2\varepsilon}\right)^{n_j}\ .
\end{aligned}
$$

- If $\underline{v}+\varepsilon\leq 2^{n_j}\leq\overline{v}-\varepsilon$ and $2^{n_{-j}}>\overline{v}-\varepsilon$,

$$
\begin{aligned}
&Prob\left[(\gamma_j,\gamma_{-j})=(1,0)\right]=\left(1-\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\underline{v}-2\varepsilon}\right)^{n_j}\ ;\\
&Prob\left[(\gamma_j,\gamma_{-j})=(0,0)\right]=1-\left(1-\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\underline{v}-2\varepsilon}\right)^{n_j}\ ;\\
&Prob\left[(\gamma_j,\gamma_{-j})=(1,1)\right]=0\ .
\end{aligned}
$$

Hence,

$$
\begin{aligned}
Prob\,(j\text{ wins }v)=&Prob\left[(\gamma_j,\gamma_{-j})=(1,0)\right]+\frac{1}{2}Prob\left[(\gamma_j,\gamma_{-j})=(0,0)\right]+\\
&+\frac{1}{2}Prob\left[(\gamma_j,\gamma_{-j})=(1,1)\right]\\
=&\left(1-\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\underline{v}-2\varepsilon}\right)^{n_j}+\frac{1}{2}\left[1-\left(1-\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\underline{v}-2\varepsilon}\right)^{n_j}\right]\ .
\end{aligned}
$$

- If $2^{n_j} > \overline{v} - \varepsilon$ and $\underline{v} + \varepsilon \leq 2^{n_{-j}} \leq \overline{v} - \varepsilon$,

$$\begin{aligned} &Prob\left[(\gamma_j, \gamma_{-j}) = (1,0)\right] = 0 \ ; \\ &Prob\left[(\gamma_j, \gamma_{-j}) = (0,0)\right] = 1 - \left(1 - \frac{2^{n_{-j}} - \underline{v} - \varepsilon}{\overline{v} - \underline{v} - 2\varepsilon}\right)^{n_{-j}} \ ; \\ &Prob\left[(\gamma_j, \gamma_{-j}) = (1,1)\right] = 0 \ . \end{aligned}$$

Hence,

$$\begin{aligned} Prob\,(j \text{ wins } v) =& Prob\left[(\gamma_j, \gamma_{-j}) = (1,0)\right] + \frac{1}{2} Prob\left[(\gamma_j, \gamma_{-j}) = (0,0)\right] + \\ &+ \frac{1}{2} Prob\left[(\gamma_j, \gamma_{-j}) = (1,1)\right] \\ =& \frac{1}{2}\left[1 - \left(1 - \frac{2^{n_{-j}} - \underline{v} - \varepsilon}{\overline{v} - \underline{v} - 2\varepsilon}\right)^{n_{-j}}\right] \ . \end{aligned}$$

- If $2^{n_j} > \overline{v} - \varepsilon$ and $2^{n_{-j}} > \overline{v} - \varepsilon$,

$$\begin{aligned} &Prob\left[(\gamma_j, \gamma_{-j}) = (1,0)\right] = 0 \ ; \\ &Prob\left[(\gamma_j, \gamma_{-j}) = (0,0)\right] = 1 \ ; \\ &Prob\left[(\gamma_j, \gamma_{-j}) = (1,1)\right] = 0 \ . \end{aligned}$$

Hence,

$$\begin{aligned} Prob\,(j \text{ wins } v) =& Prob\left[(\gamma_j, \gamma_{-j}) = (1,0)\right] + \frac{1}{2} Prob\left[(\gamma_j, \gamma_{-j}) = (0,0)\right] + \\ &+ \frac{1}{2} Prob\left[(\gamma_j, \gamma_{-j}) = (1,1)\right] = \frac{1}{2} \ . \end{aligned}$$

Finally, note that $2^{n_j} \geq \underline{v} + \varepsilon \ \forall \ \underline{v} < 2$ and $0 < \varepsilon < |\frac{\underline{v}}{2} - 1|$.

QED

3.A.8 Proof of Corollary 3.3

Let us consider the setting in which $log\,(\underline{v} + \varepsilon)\,/log\,(2) \leq n_j \leq log(\overline{v} - \varepsilon)/log(2) - 1$ and $log\,(\underline{v} + \varepsilon)\,/log\,(2) \leq n_{-j} \leq log(\overline{v} - \varepsilon)/log(2) - 1 \ \forall \ 0 < \varepsilon < |\frac{\underline{v}}{2} - 1|$, so that a unitary increase in group sizes does not cross the threshold $n_j^* = n_{-j}^* = log(\overline{v} - \varepsilon)/log(2) \ \forall \ 0 < \varepsilon < |\frac{\underline{v}}{2} - 1|$. Then,

$$\begin{aligned}\Delta_j Prob\,(\text{ j wins v }) &= Prob\,(\text{ j wins v ; } n_j+1) - Prob\,(\text{ j wins v ; } n_j)\\ &= \frac{1}{2}\left(-\left(\frac{\overline{v}-\varepsilon-2^{n_j}}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n_j}+\left(\frac{\overline{v}-\varepsilon-2^{n_j+1}}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n_j+1}\right)<0\ ,\end{aligned}$$

and

$$\begin{aligned}\Delta_{-j} Prob\,(\text{ j wins v }) &= Prob\,(\text{ j wins v ; } n_{-j}+1) - Prob\,(\text{ j wins v ; } n_{-j})\\ &= \frac{1}{2}\left(\left(\frac{\overline{v}-\varepsilon-2^{n_{-j}}}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n_{-j}}-\left(\frac{\overline{v}-\varepsilon-2^{n_{-j}+1}}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n_{-j}+1}\right)>0\ .\end{aligned}$$

QED

3.A.9 Proof of Proposition 3.8

First of all, in the $g_1(v)$ the expected value of the prize according to the uniform prior distribution is $E[V] = \frac{\underline{v}+\overline{v}}{2}$ .Then, let us consider all the distinct cases:

- for $\underline{v}+\varepsilon \leq 2^{n_j} \leq \overline{v}-\varepsilon$ and $\underline{v}+\varepsilon \leq 2^{n_{-j}} \leq \overline{v}-\varepsilon$,

$$\begin{aligned}\mathbb{E}\left[\pi_{ij}\left(\mathbf{x}_j^*,\mathbf{x}_{-j}^*\right)\right] = &\ Prob\,(\text{ all ij receive a signal higher than } 2^{n_j}\ )\ \cdot\\ &\cdot Prob\,(\text{ at least one i-j receives a signal not higher than } 2^{n_{-j}}\ )\cdot\left(\frac{\underline{v}+\overline{v}}{2}-1\right)+\\ &+Prob\,(\text{ ij receives a signal not higher than } 2^{n_j}\ )\ \cdot\\ &\cdot Prob\,(\text{ at least one i-j receives a signal not higher than } 2^{n_{-j}}\ )\cdot\frac{\underline{v}+\overline{v}}{4}+\\ &+Prob\,(\text{ all ij receive a signal higher than } 2^{n_j}\ )\cdot\\ &\cdot Prob\,(\text{ all i-j receive a signal higher than } 2^{n_{-j}}\ )\cdot\left(\frac{\underline{v}+\overline{v}}{4}-1\right)+\\ &+Prob\,(\text{ ij receives a signal higher than } 2^{n_j}\ )\cdot\\ &\cdot Prob\,(\text{ at least one -ij receives a signal not higher than } 2^{n_j}\ )\cdot\\ &\cdot Prob\left(\text{ at least one i-j receives a signal not higher than } 2^{n-j}\ \right)\cdot\left(\frac{\underline{v}+\overline{v}}{4}-1\right)+\\ &+Prob\,(\text{ ij receives a signal higher than } 2^{n_j}\ )\cdot\\ &\cdot Prob\,(\text{ at least one -ij receives a signal not higher than } 2^{n_j})\cdot\\ &\cdot Prob\,(\text{ all i-j receive a signal higher than } 2^{n_{-j}}\ )\cdot(-1)\ ,\end{aligned}$$

where

$$
\begin{aligned}
&\mathit{Prob}\,(\text{ all ij receive a signal higher than } 2^{n_j}\ ) = \left(1-\frac{2^{n_j}\ -\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n_j} \ ,\\
&\mathit{Prob}\,(\text{ at least one i-j receives a signal not higher than } 2^{n-j}\ ) = 1-\left(1-\frac{2^{n-j}\ -\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n-j} \ ,\\
&\mathit{Prob}\,(\text{ ij receives a signal not higher than } 2^{n_j}) = \frac{2^{n_j}\ -\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon} \ ,\\
&\mathit{Prob}\,(\text{ all i-j receive a signal higher than } 2^{n-j}\ ) = \left(1-\frac{2^{n-j}\ -\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n-j} \ ,\\
&\mathit{Prob}\,(\text{ at least one agent -ij receives a signal not higher than } 2^{n_j}) = 1-\left(1-\frac{2^{n_j}\ -\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n_j-1} \ .
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\mathbb{E}\left[\pi_{ij}\left(\mathbf{x}_j^*,\mathbf{x}_{-j}^*\right)\right] = &\left(1-\frac{2^{n_j}\ -\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n_j}\cdot\left[1-\left(1-\frac{2^{n-j}\ -\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n-j}\right]\cdot\left(\frac{\underline{v}+\overline{v}}{2}-1\right)+\\
&+\left(\frac{2^{n_j}\ -\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)\cdot\left[1-\left(1-\frac{2^{n-j}\ -\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n-j}\right]\cdot\frac{\underline{v}+\overline{v}}{4}\ +\\
&+\left(1-\frac{2^{n_j}\ -\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n_j}\cdot\left(1-\frac{2^{n-j}\ -\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n-j}\cdot\left(\frac{\underline{v}+\overline{v}}{4}-1\right)\ +\\
&+\left(1-\frac{2^{n_j}\ -\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)\cdot\left[1-\left(1-\frac{2^{n_j}\ -\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n_j-1}\right]\cdot\\
&\cdot\left[1-\left(1-\frac{2^{n-j}\ -\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n-j}\right]\cdot\left(\frac{\underline{v}+\overline{v}}{4}-1\right)+\\
&+\left(1-\frac{2^{n_j}\ -\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)\cdot\left[1-\left(1-\frac{2^{n_j}\ -\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n_j-1}\right]\cdot\\
&\cdot\left(1-\frac{2^{n-j}\ -\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n-j}\cdot(-1)\ ;
\end{aligned}
$$

- for $\underline{v}+\varepsilon\leq 2^{n_j}\ \leq\overline{v}-\varepsilon$ and $2^{n-j}\ \geq\overline{v}-\varepsilon$,

$$
\begin{aligned}
\mathbb{E}\left[\pi_{ij}\left(\mathbf{x}_j^*,\mathbf{x}_{-j}^*\right)\right] = &\ \mathit{Prob}\,(\text{ all ij receive a signal higher than } 2^{n_j}\ )\ \cdot\\
&\cdot\mathit{Prob}\,(\text{ at least one i-j receives a signal not higher than } 2^{n-j}\ )\cdot\left(\frac{\underline{v}+\overline{v}}{2}-1\right)+\\
&+\mathit{Prob}\,(\text{ ij receives a signal not higher than } 2^{n_j}\ )\ \cdot\\
&\cdot\mathit{Prob}\,(\text{ at least one i-j receives a signal not higher than } 2^{n-j}\ )\cdot\frac{\underline{v}+\overline{v}}{4}\ +\\
&+\mathit{Prob}\,(\text{ all ij receive a signal higher than } 2^{n_j}\ )\cdot\\
&\cdot\mathit{Prob}\,(\text{ all i-j receive a signal higher than } 2^{n-j}\ )\cdot\left(\frac{\underline{v}+\overline{v}}{4}-1\right)+
\end{aligned}
$$

$$
\begin{aligned}
&+ Prob\,(\text{ ij receives a signal higher than } 2^{n_j}\ )\cdot\\
&\cdot Prob\,(\text{ at least one -ij receives a signal not higher than } 2^{n_j}\ )\cdot\\
&\cdot Prob\left(\text{ at least one i-j receives a signal not higher than } 2^{n-j}\ \right)\cdot\left(\frac{\underline{v}+\overline{v}}{4}-1\right)+\\
&+ Prob\,(\text{ ij receives a signal higher than } 2^{n_j}\ )\cdot\\
&\cdot Prob\,(\text{ at least one -ij receives a signal not higher than } 2^{n_j})\cdot\\
&\cdot Prob\,(\text{ all i-j receive a signal higher than } 2^{n-j}\ )\cdot(-1)\ ,
\end{aligned}
$$

where

$$
\begin{aligned}
&Prob\,(\text{ all ij receive a signal higher than } 2^{n_j}\ ) = \left(1-\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n_j}\ ,\\
&Prob\,(\text{ at least one i-j receives a signal not higher than } 2^{n-j}\ ) = 1\ ,\\
&Prob\,(\text{ ij receives a signal not higher than } 2^{n_j}) = \frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\ ,\\
&Prob\,(\text{ all i-j receive a signal higher than } 2^{n-j}\ ) = 0\ ,\\
&Prob\,(\text{ at least one agent -ij receives a signal not higher than } 2^{n_j}) = 1-\left(1-\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n_j-1}\ .
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\mathbb{E}\left[\pi_{ij}\left(\mathbf{x}_j^*,\mathbf{x}_{-j}^*\right)\right] &= \left(1-\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n_j}\cdot\left(\frac{\underline{v}+\overline{v}}{2}-1\right)+\left(\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)\cdot\frac{\underline{v}+\overline{v}}{4}+\\
&+\left(1-\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)\cdot\left[1-\left(1-\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n_j-1}\right]\cdot\left(\frac{\underline{v}+\overline{v}}{4}-1\right)\ ;
\end{aligned}
$$

- for $2^{n_j} > \overline{v}-\varepsilon$ and $\underline{v}+\varepsilon \leq 2^{n-j} \leq \overline{v}-\varepsilon$,

$$
\begin{aligned}
\mathbb{E}\left[\pi_{ij}\left(\mathbf{x}_j^*,\mathbf{x}_{-j}^*\right)\right] = &\ Prob\,(\text{ all ij receive a signal higher than } 2^{n_j}\ )\ \cdot\\
&\cdot Prob\,(\text{ at least one i-j receives a signal not higher than } 2^{n-j}\ )\cdot\left(\frac{\underline{v}+\overline{v}}{2}-1\right)+\\
&+ Prob\,(\text{ ij receives a signal not higher than } 2^{n_j}\ )\ \cdot\\
&\cdot Prob\,(\text{ at least one i-j receives a signal not higher than } 2^{n-j}\ )\cdot\frac{\underline{v}+\overline{v}}{4}\ +\\
&+ Prob\,(\text{ all ij receive a signal higher than } 2^{n_j}\ )\cdot\\
&\cdot Prob\,(\text{ all i-j receive a signal higher than } 2^{n-j}\ )\cdot\left(\frac{\underline{v}+\overline{v}}{4}-1\right)+
\end{aligned}
$$

$$+ Prob\,(\text{ ij receives a signal higher than } 2^{n_j}\ )\cdot$$

$$\cdot\, Prob\,(\text{ at least one -ij receives a signal not higher than } 2^{n_j}\ )\cdot$$

$$\cdot\, Prob\left(\text{ at least one i-j receives a signal not higher than } 2^{n-j}\ \right)\cdot\left(\frac{\underline{v}+\overline{v}}{4}-1\right)+$$

$$+ Prob\,(\text{ ij receives a signal higher than } 2^{n_j}\ )\cdot$$

$$\cdot\, Prob\,(\text{ at least one -ij receives a signal not higher than } 2^{n_j})\cdot$$

$$\cdot\, Prob\,(\text{ all i-j receive a signal higher than } 2^{n-j}\ )\cdot(-1)\ \ ,$$

where

$$Prob\,(\text{ all ij receive a signal higher than } 2^{n_j}\ ) = 0\ ,$$

$$Prob\,(\text{ at least one i-j receives a signal not higher than } 2^{n-j}\ ) = 1-\left(1-\frac{2^{n-j}-\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n-j}\ ,$$

$$Prob\,(\text{ ij receives a signal not higher than } 2^{n_j}) = 1\ ,$$

$$Prob\,(\text{ all i-j receive a signal higher than } 2^{n-j}\ ) = \left(1-\frac{2^{n-j}-\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n-j}\ ,$$

$$Prob\,(\text{ at least one agent -ij receives a signal not higher than } 2^{n_j}) = 1\ .$$

Hence,

$$\mathbb{E}\left[\pi_{ij}\left(\mathbf{x}_j^*,\mathbf{x}_{-j}^*\right)\right] = \left[1-\left(1-\frac{2^{n-j}-\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n-j}\right]\cdot\frac{\underline{v}+\overline{v}}{4}\ ;$$

- for $2^{n_j} > \overline{v}-\varepsilon$ and $2^{n-j} > \overline{v}-\varepsilon$,

$$\mathbb{E}\left[\pi_{ij}\left(\mathbf{x}_j^*,\mathbf{x}_{-j}^*\right)\right] = Prob\,(\text{ all ij receive a signal higher than } 2^{n_j}\ )\ \cdot$$

$$\cdot\, Prob\,(\text{ at least one i-j receives a signal not higher than } 2^{n-j}\ )\cdot\left(\frac{\underline{v}+\overline{v}}{2}-1\right)+$$

$$+ Prob\,(\text{ ij receives a signal not higher than } 2^{n_j}\ )\ \cdot$$

$$\cdot\, Prob\,(\text{ at least one i-j receives a signal not higher than } 2^{n-j}\ )\cdot\frac{\underline{v}+\overline{v}}{4}\ +$$

$$+ Prob\,(\text{ all ij receive a signal higher than } 2^{n_j}\ )\cdot$$

$$\cdot\, Prob\,(\text{ all i-j receive a signal higher than } 2^{n-j}\ )\cdot\left(\frac{\underline{v}+\overline{v}}{4}-1\right)+$$

$+ \, Prob$ (ij receives a signal higher than 2^{n_j}) $\cdot$

$\cdot \, Prob$ (at least one -ij receives a signal not higher than 2^{n_j}) $\cdot$

$\cdot \, Prob$ (at least one i-j receives a signal not higher than 2^{n-j}) $\cdot \left(\frac{\underline{v} + \overline{v}}{4} - 1 \right) +$

$+ \, Prob$ (ij receives a signal higher than 2^{n_j}) $\cdot$

$\cdot \, Prob$ (at least one -ij receives a signal not higher than 2^{n_j}) $\cdot$

$\cdot \, Prob$ (all i-j receive a signal higher than 2^{n-j}) $\cdot (-1)$,

where

$Prob$ (all ij receive a signal higher than 2^{n_j}) $= 0$,

$Prob$ (at least one i-j receives a signal not higher than 2^{n-j}) $= 1$,

$Prob$ (ij receives a signal not higher than 2^{n_j}) $= 1$,

$Prob$ (all i-j receive a signal higher than 2^{n-j}) $= 0$,

$Prob$ (at least one agent -ij receives a signal not higher than 2^{n_j}) $= 1$.

Hence,

$$\mathbb{E}\left[\pi_{ij}\left(\mathbf{x}_j^*, \mathbf{x}_{-j}^*\right)\right] = \frac{\underline{v} + \overline{v}}{4} \ .$$

QED

3.A.10 Proof of Corollary 3.5

In the $g_1(v)$, for any $log\left(\underline{v} + \varepsilon\right)/log\left(2\right) \leq n_j \leq log\left(\overline{v} - \varepsilon\right)/log\left(2\right) - 1$ and $log\left(\underline{v} + \varepsilon\right)/log\left(2\right) \leq n_{-j} \leq log\left(\overline{v} - \varepsilon\right)/log(2) - 1 \ \forall \ 0 < \varepsilon < \left|\frac{\underline{v}}{2} - 1\right|$:

$$\begin{aligned} \Delta_j \mathbb{E}\left[\pi_{ij}\left(\mathbf{x}_j^*, \mathbf{x}_{-j}^*\right)\right] &= \mathbb{E}\left[\pi_{ij}\left(\mathbf{x}_j^*, \mathbf{x}_{-j}^*; n_j + 1\right)\right] - \mathbb{E}\left[\pi_{ij}\left(\mathbf{x}_j^*, \mathbf{x}_{-j}^*; n_j\right)\right] \\ &= \frac{2^{n_j}}{\overline{v} - \varepsilon - \underline{v} - \varepsilon} + \frac{\left(\underline{v} + \overline{v}\right)\left(\left(\frac{\overline{v} - \varepsilon - 2^{n_j + 1}}{\overline{v} - \varepsilon - \underline{v} - \varepsilon}\right)^{n_j + 1} - \left(\frac{\overline{v} - \varepsilon - 2^{n_j}}{\overline{v} - \varepsilon - \underline{v} - \varepsilon}\right)^{n_j}\right)}{4} \ . \end{aligned}$$

Then, it follows

$$\begin{aligned}
&\Delta_j \mathbb{E}\left[\pi_{ij}\left(\mathbf{x}_j^*, \mathbf{x}_{-j}^*\right)\right] < 0 \Leftrightarrow \\
&\Leftrightarrow \mathbb{E}\left(V\right) > \frac{2^{n_j+1}}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\left[\left(\frac{\overline{v}-\varepsilon-2^{x}}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n_j} - \left(\frac{\overline{v}-\varepsilon-2^{n_j+1}}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n_j+1}\right]^{-1} \ .
\end{aligned}$$

Moreover,

$$\begin{aligned}
\Delta_{-j}\mathbb{E}\left[\pi_{ij}\left(\mathbf{x}_j^*, \mathbf{x}_{-j}^*\right)\right] &= \mathbb{E}\left[\pi_{ij}\left(\mathbf{x}_j^*, \mathbf{x}_{-j}^*; n_{-j}+1\right)\right] - \mathbb{E}\left[\pi_{ij}\left(\mathbf{x}_j^*, \mathbf{x}_{-j}^*; n_{-j}\right)\right] \\
&= \frac{\left(\underline{v}+\overline{v}\right)\left[\left(\frac{\overline{v}-\varepsilon-2^{n_j}}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n_{-j}} - \left(\frac{\overline{v}-\varepsilon-2^{n_j+1}}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n_{-j}+1}\right]}{4} \ .
\end{aligned}$$

Then, it follows

$$\Delta_{-j}\mathbb{E}\left[\pi_{ij}\left(\mathbf{x}_j^*, \mathbf{x}_{-j}^*\right)\right] > 0 \Leftrightarrow \mathbb{E}\left(V\right) > 0 \ .$$

QED

3.A.11 Proof of Proposition 3.9

In the $g_2\left(c\right)$, given the contest success function $P_j\left(X_j, X_{-j}\right) \forall j \in \{1,2\}$, the probability of winning the prize v for group $j \in \{1,2\}$ is:

$$\begin{aligned}
Prob\left(j \text{ wins } v\right) =& Prob\left[\left(\gamma_j^*, \gamma_{-j}^*\right) = (1,0)\right] + \frac{1}{2}Prob\left[\left(\gamma_j^*, \gamma_{-j}^*\right) = (0,0)\right] + \\
&+\frac{1}{2}Prob\left[\left(\gamma_j^*, \gamma_{-j}^*\right) = (1,1)\right] \ .
\end{aligned}$$

On the other hand, the probability of winning the prize v for group $j \in \{1,2\}$ at the cutoff equilibrium $x_{ij}^*\left(c_{ij}\right)$ depends on whether or not $2^{-n_j}v$ belongs to $\left[\underline{c}+\varepsilon, \overline{c}-\varepsilon\right]$, where c_{ij} is uniformly distributed. However, note that $2^{-n_j}v > \underline{c}+\varepsilon \ \forall \ \underline{c} < 0$, $0 < \varepsilon < min\left\{|\frac{2\overline{c}-v}{4}|, |\frac{\underline{c}}{2}|\right\}$ and $\forall \ n_j \geq 2$.

Moreover, $2^{-n_j}v < \overline{c}-\varepsilon \ \forall \ \overline{c} > \frac{v}{2}$, $0 < \varepsilon < min\left\{|\frac{2\overline{c}-v}{4}|, |\frac{\underline{c}}{2}|\right\}$ and $\forall \ n_{-j} \geq 2$. Therefore, I will consider restrict our attention to the unique possible case, i.e. $\underline{c}+\varepsilon < 2^{-n_j}v < \overline{c}-\varepsilon$ and

$\underline{c} + \varepsilon < 2^{-n_{-j}} v < \overline{c} - \varepsilon$. Accordingly, [3.A3]

$$\begin{aligned}
Prob\left[(\gamma_j, \gamma_{-j}) = (1,0)\right] &= \left(\frac{2^{-n_j} v - \underline{c} - \varepsilon}{\overline{c} - \underline{c} - 2\varepsilon}\right)^{n_j} \cdot \left[1 - \left(\frac{2^{n_{-j}} v - \underline{c} - \varepsilon}{\overline{c} - \underline{c} - 2\varepsilon}\right)^{n_{-j}}\right] \ ; \\
Prob\left[(\gamma_j, \gamma_{-j}) = (0,0)\right] &= \left[1 - \left(\frac{2^{-n_j} v - \underline{c} - \varepsilon}{\overline{c} - \underline{c} - 2\varepsilon}\right)^{n_j}\right] \cdot \left[1 - \left(\frac{2^{-n_{-j}} v - \underline{c} - \varepsilon}{\overline{c} - \underline{c} - 2\varepsilon}\right)^{n_{-j}}\right] \ ; \\
Prob\left[(\gamma_j, \gamma_{-j}) = (1,1)\right] &= \left(\frac{2^{-n_j} v - \underline{c} - \varepsilon}{\overline{c} - \underline{c} - 2\varepsilon}\right)^{n_j} \cdot \left(\frac{2^{n_{-j}} v - \underline{c} - \varepsilon}{\overline{c} - \underline{c} - 2\varepsilon}\right)^{n_{-j}} \ .
\end{aligned}$$

Hence,

$$\begin{aligned}
Prob\,(j \text{ wins } v) =& Prob\left[(\gamma_j, \gamma_{-j}) = (1,0)\right] + \frac{1}{2} Prob\left[(\gamma_j, \gamma_{-j}) = (0,0)\right] + \\
&+ \frac{1}{2} Prob\left[(\gamma_j, \gamma_{-j}) = (1,1)\right] \\
&= \left(\frac{2^{-n_j} v - \underline{c} - \varepsilon}{\overline{c} - \underline{c} - 2\varepsilon}\right)^{n_j} \cdot \left[1 - \left(\frac{2^{-n_{-j}} v - \underline{c} - \varepsilon}{\overline{c} - \underline{c} - 2\varepsilon}\right)^{n_{-j}}\right] + \\
&+ \frac{1}{2}\left[1 - \left(\frac{2^{-n_j} v - \underline{c} - \varepsilon}{\overline{c} - \underline{c} - 2\varepsilon}\right)^{n_j}\right] \cdot \left[1 - \left(\frac{2^{-n_{-j}} v - \underline{c} - \varepsilon}{\overline{c} - \underline{c} - 2\varepsilon}\right)^{n_{-j}}\right] + \\
&+ \frac{1}{2}\left(\frac{2^{-n_j} v - \underline{c} - \varepsilon}{\overline{c} - \underline{c} - 2\varepsilon}\right)^{n_j} \cdot \left(\frac{2^{-n_{-j}} v - \underline{c} - \varepsilon}{\overline{c} - \underline{c} - 2\varepsilon}\right)^{n_{-j}} \ .
\end{aligned}$$

QED

3.A.12 Proof of Corollary 3.6

In the $g_2(c)$, for any $n_j \geq n_{-j} \geq 2$,

$$\begin{aligned}
\Delta_j Prob\,(\text{ j wins v }) &= Prob\,(\text{ j wins v }; n_j + 1) - Prob\,(\text{ j wins v }; n_j) \\
&= \frac{1}{4}\Bigg(-2\left(\frac{2^{-n_j} v - \underline{c} - \varepsilon}{\overline{c} - \varepsilon - \underline{c} - \varepsilon}\right)^{n_j} + \\
&+ \frac{4^{-n_j}\left(v - 2^{n_j+1}\left(\underline{c} + \varepsilon\right)\right)\left(\frac{2^{-n_j} v - 2(\underline{c} + \varepsilon)}{\overline{c} - \varepsilon - \underline{c} - \varepsilon}\right)^{n_j}}{\overline{c} - \varepsilon - \underline{c} - \varepsilon}\Bigg) < 0 \ .
\end{aligned}$$

[3.A3] Note that for $\gamma_{-j} = 0$, it suffices that just one $i-j$ chooses $x_{i-j}\left(c_{i-j}\right) = 0$, due to the weakest-link impact function.

$$\begin{aligned}\Delta_{-j} Prob\,(\text{ j wins v }) &= Prob\,(\text{ j wins v };n_{-j}+1) - Prob\,(\text{ j wins v };n_{-j})\\ &= \frac{1}{4}\left(2\left(\frac{2^{-n_{-j}}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)^{n_{-j}}+\right.\\ &\left.-\frac{4^{-n_{-j}}\left(v-2^{n_{-j}+1}\left(\underline{c}+\varepsilon\right)\right)\left(\frac{2^{-n_{-j}}v-2(\underline{c}+\varepsilon)}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)^{n_{-j}}}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)>0\ .\end{aligned}$$

QED

3.A.13 Proof of Proposition 3.10

First of all, in the $g_2(c)$ the expected value of the cost of effort according to the uniform prior distribution is $E\left[C\right]=\frac{\underline{c}+\overline{c}}{2}$. Moreover, $2^{-n_j}v>\underline{c}+\varepsilon\ \forall\ \underline{c}<0$, $0<\varepsilon<min\left\{|\frac{2\overline{c}-v}{4}|,|\frac{\underline{c}}{2}|\right\}$ and $\forall n_j\geq 2$ and $2^{-n_j}v<\overline{c}-\varepsilon\ \forall\ \overline{c}>\frac{v}{2}$, $0<\varepsilon<min\left\{|\frac{2\overline{c}-v}{4}|,|\frac{\underline{c}}{2}|\right\}$ and $\forall\ n_{-j}\geq 2$. Then,

$$\begin{aligned}\mathbb{E}\left[\pi_{ij}\left(\mathbf{x}_j^*,\mathbf{x}_{-j}^*\right)\right] = &\ Prob\left(\text{ all ij receive a signal smaller than } 2^{-n_j}v\ \right)\cdot\\ &\cdot Prob\left(\text{ at least one i-j receives a signal not smaller than } 2^{-n_{-j}}v\ \right)\cdot\left(v-\frac{\underline{c}+\overline{c}}{2}\right)+\\ &+Prob\left(\text{ ij receives a signal not smaller than } 2^{-n_j}v\ \right)\cdot\\ &\cdot Prob\left(\text{ at least one i-j receives a signal not smaller than } 2^{-n_{-j}}v\ \right)\cdot\frac{v}{2}+\\ &+Prob\left(\text{ all ij receive a signal smaller than } 2^{-n_j}v\ \right)\cdot\\ &\cdot Prob\left(\text{ all i-j receive a signal smaller than } 2^{-n_{-j}}v\ \right)\cdot\left(\frac{v}{2}-\frac{\underline{c}+\overline{c}}{2}\right)+\\ &+Prob\left(\text{ ij receives a signal smaller than } 2^{-n_j}v\ \right)\cdot\\ &\cdot Prob\left(\text{ at least one -ij receives a signal not smaller than } 2^{-n_j}v\ \right)\cdot\\ &\cdot Prob\left(\text{ at least one i-j receives a signal not smaller than } 2^{-n_{-j}}v\ \right)\cdot\left(\frac{v}{2}-\frac{\underline{c}+\overline{c}}{2}\right)+\\ &+Prob\left(\text{ ij receives a signal smaller than } 2^{-n_j}v\ \right)\cdot\\ &\cdot Prob\left(\text{ at least one -ij receives a signal not smaller than } 2^{-n_j}v\right)\cdot\\ &\cdot Prob\left(\text{ all i-j receive a signal smaller than } 2^{-n_{-j}}v\ \right)\cdot\left(-\frac{\underline{c}+\overline{c}}{2}\right)\ ,\end{aligned}$$

where

$$Prob\left(\text{ all ij receive a signal smaller than } 2^{-n_j}v\ \right) = \left(\frac{2^{-n_j}v - \underline{c} - \varepsilon}{\overline{c} - \varepsilon - \underline{c} - \varepsilon}\right)^{n_j} \quad ,$$

$$Prob\left(\text{ at least one i-j receives a signal not smaller than } 2^{-n_{-j}}v\ \right) = 1 - \left(\frac{2^{-n_{-j}}v - \underline{c} - \varepsilon}{\overline{c} - \varepsilon - \underline{c} - \varepsilon}\right)^{n_{-j}} \quad ,$$

$$Prob\left(\text{ ij receives a signal not smaller than } 2^{-n_j}v\right) = 1 - \frac{2^{-n_j}v - \underline{c} - \varepsilon}{\overline{c} - \varepsilon - \underline{c} - \varepsilon} \ ,$$

$$Prob\left(\text{ all i-j receive a signal smaller than } 2^{-n_{-j}}v\ \right) = \left(\frac{2^{-n_{-j}}v - \underline{c} - \varepsilon}{\overline{c} - \varepsilon - \underline{c} - \varepsilon}\right)^{n_{-j}} \quad ,$$

$$Prob\left(\text{ at least one agent -ij receives a signal not smaller than } 2^{-n_j}v\right) = 1 - \left(\frac{2^{-n_j}v - \underline{c} - \varepsilon}{\overline{c} - \varepsilon - \underline{c} - \varepsilon}\right)^{n_j - 1} \quad .$$

Hence,

$$\begin{aligned}
\mathbb{E}\left[\pi_{ij}\left(\mathbf{x}_j^*, \mathbf{x}_{-j}^*\right)\right] = & \left(\frac{2^{-n_j}v - \underline{c} - \varepsilon}{\overline{c} - \varepsilon - \underline{c} - \varepsilon}\right)^{n_j} \cdot \left[1 - \left(\frac{2^{-n_{-j}}v - \underline{c} - \varepsilon}{\overline{c} - \varepsilon - \underline{c} - \varepsilon}\right)^{n_{-j}}\right] \cdot \left(v - \frac{\underline{c} + \overline{c}}{2}\right) + \\
& + \left(1 - \frac{2^{-n_j}v - \underline{c} - \varepsilon}{\overline{c} - \varepsilon - \underline{c} - \varepsilon}\right) \cdot \left[1 - \left(\frac{2^{-n_{-j}}v - \underline{c} - \varepsilon}{\overline{c} - \varepsilon - \underline{c} - \varepsilon}\right)^{n_{-j}}\right] \cdot \frac{v}{2} + \\
& + \left(\frac{2^{-n_j}v - \underline{c} - \varepsilon}{\overline{c} - \varepsilon - \underline{c} - \varepsilon}\right)^{n_j} \cdot \left(\frac{2^{-n_{-j}}v - \underline{c} - \varepsilon}{\overline{c} - \varepsilon - \underline{c} - \varepsilon}\right)^{n_{-j}} \cdot \left(\frac{v}{2} - \frac{\underline{c} + \overline{c}}{2}\right) \ + \\
& + \left(\frac{2^{-n_j}v - \underline{c} - \varepsilon}{\overline{c} - \varepsilon - \underline{c} - \varepsilon}\right) \cdot \left[1 - \left(\frac{2^{-n_j}v - \underline{c} - \varepsilon}{\overline{c} - \varepsilon - \underline{c} - \varepsilon}\right)^{n_j - 1}\right] \cdot \\
& \cdot \left[1 - \left(\frac{2^{-n_{-j}}v - \underline{c} - \varepsilon}{\overline{c} - \varepsilon - \underline{c} - \varepsilon}\right)^{n_{-j}}\right] \cdot \left(\frac{v}{2} - \frac{\underline{c} + \overline{c}}{2}\right) + \\
& + \left(\frac{2^{-n_j}v - \underline{c} - \varepsilon}{\overline{c} - \varepsilon - \underline{c} - \varepsilon}\right) \cdot \left[1 - \left(\frac{2^{-n_j}v - \underline{c} - \varepsilon}{\overline{c} - \varepsilon - \underline{c} - \varepsilon}\right)^{n_j - 1}\right] \cdot \\
& \cdot \left(\frac{2^{-n_{-j}}v - \underline{c} - \varepsilon}{\overline{c} - \varepsilon - \underline{c} - \varepsilon}\right)^{n_{-j}} \cdot \left(-\frac{\underline{c} + \overline{c}}{2}\right) \ .
\end{aligned}$$

QED

3.A.14 Proof of Corollary 3.7

In the $g_2(c)$, for any $n_j \leq log\left(\overline{v} - \varepsilon\right)/log\left(2\right) - 1$ and $n_{-j} \leq log\left(\overline{v} - \varepsilon\right)/log(2) - 1$ and $\forall\ 0 < \varepsilon < \left|\frac{v}{2} - 1\right|$

:

$$\begin{aligned}\Delta_j \mathbb{E}\left[\pi_{ij}\left(\mathbf{x}_j^*, \mathbf{x}_{-j}^*\right)\right] &= \mathbb{E}\left[\pi_{ij}\left(\mathbf{x}_j^*, \mathbf{x}_{-j}^*; n_j+1\right)\right] - \mathbb{E}\left[\pi_{ij}\left(\mathbf{x}_j^*, \mathbf{x}_{-j}^*; n_j\right)\right] \\ &= \frac{2^{-1-n_j}}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\left[\frac{\left(\underline{c}+\overline{c}\right)v}{2} - \frac{\left(2^{n_j+1}\left(\underline{c}+\varepsilon\right)-v\right)v}{2}\left(\frac{2^{-n_j-1}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)^{n_j} + \right. \\ &\left. - 2^{n_j}v\left(\overline{c}-\varepsilon-\underline{c}-\varepsilon\right)\left(\frac{2^{-n_j}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)^{n_j}\right] .\end{aligned}$$

Then, it follows

$$\Delta_j \mathbb{E}\left[\pi_{ij}\left(\mathbf{x}_j^*, \mathbf{x}_{-j}^*\right)\right] < 0 \Leftrightarrow$$

$$\Leftrightarrow \mathbb{E}\left(C\right) < \frac{2^{n_j+1}\left(\underline{c}+\varepsilon\right)-v}{2}\left(\frac{2^{-n_j-1}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)^{n_j} + 2^{n_j}\left(\overline{c}-\varepsilon-\underline{c}-\varepsilon\right)\left(\frac{2^{-n_j}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)^{n_j} .$$

Moreover,

$$\begin{aligned}\Delta_{-j} \mathbb{E}\left[\pi_{ij}\left(\mathbf{x}_j^*, \mathbf{x}_{-j}^*\right)\right] &= \mathbb{E}\left[\pi_{ij}\left(\mathbf{x}_j^*, \mathbf{x}_{-j}^*; n_{-j}+1\right)\right] - \mathbb{E}\left[\pi_{ij}\left(\mathbf{x}_j^*, \mathbf{x}_{-j}^*; n_{-j}\right)\right] \\ &= \frac{v}{4}\left[\left(\frac{2^{-n_{-j}}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)^{n_{-j}} - \left(\frac{2^{-n_{-j}}v-2\left(\underline{c}-\varepsilon\right)}{2\left(\overline{c}-\varepsilon-\underline{c}-\varepsilon\right)}\right)^{n_{-j}+1}\right] > 0 .\end{aligned}$$

QED

3.A.15 Proof of Proposition 3.11

In the $g_3(v)$, note that $E\left(V|v_{ij}\right) = v_{ij}$, if ij observes $v_{ij} \in [\underline{v}+\varepsilon, \overline{v}-\varepsilon]$ so that $V|v_{ij} \sim U(v_{ij}-\varepsilon,\ v_{ij}+\varepsilon)$. Furthermore, for $v_{ij} \in [\underline{v}+\varepsilon, \overline{v}-\varepsilon]$, the conditional distribution of the teammates' or opponents' observation will be centered around v_{ij} with a triangular distribution over the support $[v_{ij}-2\varepsilon, v_{ij}+2\varepsilon]$.

Hence, $Prob\left[V_{-ij} < v_{ij}|v_{ij}\right] = Prob[V_{-ij} > v_{ij}|v_{ij}] = \frac{1}{2}\ \forall ij \in \{1, \ldots, n_j\}$ and $j \in \{1, \ldots, M\}$. Moreover, let us define Q_k as the set of all subsets of cardinality k formed by groups different from j, that is $Q_k = \{-J \in \{\{1, \ldots, M\} \setminus \{j\}\} \mid |-J| = k\}$, where $|\cdot|$ denotes the cardinality of a set.

Let us denote p_{-ij} the conditional expected probability attached by player ij to any other player $-ij$ within the same group choosing not to exert effort. Likewise, p_{i-j} is the conditional expected probability attached by ij to any other member of the rival group $-j$ not exerting effort. Then, for player ij the conditionally expected payoff from exerting effort equals [3.A4]

[3.A4] Due the auction-type contest success function and the presence of $M \geq 2$ groups, I have to consider the possibility of a tie with up to $M-1$ groups.

$$
\begin{aligned}
&Prob\,(n_j\gamma_j = n_j)\cdot\prod_{-j} Prob\,(n_{-j}\gamma_{-j} < n_{-j})\,(v_{ij}-1)+\\
&+Prob\,(n_j\gamma_j = n_j)\cdot\sum_{k=1}^{M-1}\sum_{-J\in Q_k}\prod_{-j\in -J} Prob\,(n_{-j}\gamma_{-j} = n_{-j})\cdot\prod_{-j'\in Q_k\setminus -J} Prob\,(n_{j'}\gamma_{j'} < n_{j'})\cdot\left(\frac{v_{ij}}{k+1}-1\right)+\\
&+Prob\,(n_j\gamma_j < n_j)\cdot\sum_{k=1}^{M-1}\sum_{-J\in Q_k}\prod_{-j\in -J} Prob\,(n_{-j}\gamma_{-j} = n_{-j})\cdot\prod_{-j'\in Q_k\setminus -J} Prob\,(n_{-j'}\gamma_{-j'} < n_{-j'})\cdot(-1)+\\
&+Prob\,(n_j\gamma_j < n_j)\cdot\prod_{-j} Prob\,(n_{-j}\gamma_{-j} < n_{-j})\left(\frac{v_{ij}}{M}-1\right)\\
&=\prod_{-i}(1-p_{-ij})\cdot\prod_{-j}\left(1-\prod_{i}(1-p_{i-j})\right)(v_{ij}-1)+\\
&+\prod_{-i}(1-p_{-ij})\cdot\sum_{k=1}^{M-1}\sum_{-J\in Q_k}\prod_{-j\in -J}\prod_{i}(1-p_{i-j})\cdot\prod_{-j'\in Q_k\setminus J}\left(1-\prod_{i}(1-p_{i-j'})\right)\cdot\left(\frac{v_{ij}}{k+1}-1\right)+\\
&+\left(1-\prod_{-i}(1-p_{-ij})\right)\cdot\prod_{-j}\prod_{i}(1-p_{i-j})\,(-1)+\\
&+\left(1-\prod_{-i}(1-p_{-ij})\right)\prod_{-j}\left(1-\prod_{i}(1-p_{i-j})\right)\left(\frac{v_{ij}}{M}-1\right)\ ,
\end{aligned}
$$

while the conditionally expected payoff from not exerting effort equals

$$
\prod_{-j} Prob\,(n_{-j}\gamma_{-j} < n_{-j})\,\frac{v_{ij}}{M}=\prod_{-j}\left(1-\prod_{i}(1-p_{i-j})\right)\frac{v}{M}\ .
$$

Hence, for player ij the conditional expected payoff difference from exerting effort versus not exerting effort equals

$$
\begin{aligned}
&\prod_{-i}(1-p_{-ij})\cdot\Bigg(\prod_{-j}\left(1-\prod_{i}(1-p_{i-j})\right)\cdot\left(\frac{M-1}{M}\right)+\\
&+\sum_{k=1}^{M-1}\sum_{-J\in Q_k}\prod_{-j\in -J}\prod_{i}(1-p_{i-j})\cdot\prod_{-j'\in Q_k\setminus -J}\left(1-\prod_{i}(1-p_{i-j'})\right)\cdot\left(\frac{1}{k+1}\right)\Bigg)\cdot v_{ij}+\\
&-\left(\prod_{-j}\left(1-\prod_{i}(1-p_{i-j})\right)+\sum_{k=1}^{M-1}\sum_{-J\in Q_k}\prod_{-j\in -J}\prod_{i}(1-p_{i-j})\cdot\prod_{-j'\in Q_k\setminus -J}\left(1-\prod_{i}(1-p_{i-j'})\right)\right)\\
&=\prod_{-i}(1-p_{-ij})\cdot\Bigg(\prod_{-j}\left(1-\prod_{i}(1-p_{i-j})\right)\cdot\left(\frac{M-1}{M}\right)+\\
&+\sum_{k=1}^{M-1}\sum_{-J\in Q_k}\prod_{-j\in -J}\prod_{i}(1-p_{i-j})\cdot\prod_{-j'\in Q_k\setminus -J}\left(1-\prod_{i}(1-p_{i-j'})\right)\cdot\left(\frac{1}{k+1}\right)\Bigg)\cdot v_{ij}-1\ ,
\end{aligned}
$$

and the conditional expected payoff difference from not exerting effort versus exerting effort equals

$$\begin{aligned}
&\left(\prod_{-j}\left(1-\prod_{i}(1-p_{i-j})\right)+\sum_{k=1}^{M-1}\sum_{-J\in Q_k}\prod_{-j\in -J}\prod_{i}(1-p_{i-j})\cdot\prod_{-j'\in Q_k\setminus -J}\left(1-\prod_{i}(1-p_{i-j'})\right)\right)+\\
&-\prod_{-i}(1-p_{-ij})\cdot\left(\prod_{-j}\left(1-\prod_{i}(1-p_{i-j})\right)\cdot\left(\frac{M-1}{M}\right)+\right.\\
&\left.+\sum_{k=1}^{M-1}\sum_{-J\in Q_k}\prod_{-j\in -J}\prod_{i}(1-p_{i-j})\cdot\prod_{-j'\in Q_k\setminus -J}\left(1-\prod_{i}(1-p_{i-j'})\right)\cdot\left(\frac{1}{k+1}\right)\right)\cdot v_{ij}\\
&=1-\prod_{-i}(1-p_{-ij})\cdot\left(\prod_{-j}\left(1-\prod_{i}(1-p_{i-j})\right)\cdot\left(\frac{M-1}{M}\right)+\right.\\
&\left.+\sum_{k=1}^{M-1}\sum_{-J\in Q_k}\prod_{-j\in -J}\prod_{i}(1-p_{i-j})\cdot\prod_{-j'\in Q_k\setminus -J}\left(1-\prod_{i}(1-p_{i-j'})\right)\cdot\left(\frac{1}{k+1}\right)\right)\cdot v_{ij}\ .
\end{aligned}$$

Observe that for player ij the expected payoff difference from exerting effort versus not exerting effort is decreasing in the probability of the teammates not exerting effort, while it is increasing in the probability of the rival groups not exerting effort. Hence, by setting $p_{-ij}=1$, $p_{i-j}=0$, $p_{i-j'}=0$, the expected payoff difference reduces to -1. Thus, for any player ij $x_{ij}=1$ is never a dominant action. On the other hand, for player ij the expected payoff difference from not exerting effort versus exerting effort is increasing in the probability of the teammates not exerting effort, while it is decreasing in the probability of the rival groups not exerting effort. Hence, by setting $p_{-ij}=0$, $p_{i-j}=1$, $p_{i-j'}=1$, the expected payoff difference reduces to $1-v_{ij}(M-1)/M$. From the latter we derive threshold of one dominance region: for $v_{ij}<M/(M-1)$, $x_{ij}=0$ is a strictly dominant action for any player ij. Let us suppose player ij observes $v_{ij}=M/(M-1)$ and $\varepsilon<|\underline{v}/2-M/2(M-1)|$ so that $M/(M-1)-2\varepsilon\geq\underline{v}$. Then she will attach probability at least 1/2 to any other player not exerting effort. Accordingly, if we set $p_{-ij}=1/2$, $p_{i-j}=1$, $p_{i-j'}=1$, then we get that the first value of v_{ij} such that exerting effort cannot be discarded by iterated dominance is $v_{ij}=M/(M-1)\cdot 2^{n_j-1}$. Note that for $M=2$, we get $v^*_{ij}=2^{n_j}$ as in proposition 3.5. However, observe that for more than two groups, the expected payoff difference from not exerting effort and exerting effort depends not only on p_{-ij} but also on p_{i-j}, so that a cutoff strategy around the threshold of the iterated-dominance region cannot be sustained in equilibrium for more than two groups. Let us assume that players follow the following cutoff strategy

$$x_{ij}\left(v_{ij}\right)=\begin{cases}1 & \text{if } v_{ij}>k\\ 0 & \text{if } v_{ij}\leq k\ ,\end{cases}$$

where k is obtained by equating the expected payoff differences to zero with $p_{-ij}=1/2$, $p_{i-j}=1/2$,

$p_{i-j'} = 1/2$, that is

$$k = \left(2^{1-n_j} \prod_{-j} \left(1 - 2^{-n_{-j}}\right) \left(1 - \frac{1}{M}\right) + 2^{1-n_j} \sum_{k=1}^{M-1} \frac{\sum_{-J \in Q_k} \prod_{-j \in -J} \left(2^{-n_{-j}}\right) \prod_{-j' \in Q_k \setminus J} \left(1 - 2^{-n_{-j'}}\right)}{k+1}\right)^{-1} .$$

Then, for the strategy above to be an equilibrium it must satisfy incentive compatibility constraints from below and and above the cutoff, that is $\forall\ v_{ij} \in [M/(M-1) \cdot 2^{n_j - 1}, k]$

$$\begin{aligned} &1 - \prod_{-i} (1 - p_{-ij}) \cdot \left(\prod_{-j} \left(1 - \prod_i (1 - p_{i-j})\right) \cdot \left(\frac{M-1}{M}\right) + \right. \\ &\left. + \sum_{k=1}^{M-1} \sum_{-J \in Q_k} \prod_{-j \in -J} \prod_i (1 - p_{i-j}) \cdot \prod_{-j' \in Q_k \setminus -J} \left(1 - \prod_i (1 - p_{i-j'})\right) \cdot \left(\frac{1}{k+1}\right) \right) \cdot v_{ij} \geq 0 \end{aligned}$$

with

$$p_{-ij} = p_{i-j} = p_{i-j'} = \begin{cases} 0 & \text{if } k \leq v_{ij} - 2\varepsilon \\ \dfrac{(k - v_{ij} + 2\varepsilon)^2}{8\varepsilon^2} & \text{if } v_{ij} - 2\varepsilon < k \leq v_{ij} \end{cases} ,$$

and $\forall v_{ij} > k$

$$\begin{aligned} &\prod_{-i} (1 - p_{-ij}) \cdot \left(\prod_{-j} \left(1 - \prod_i (1 - p_{i-j})\right) \cdot \left(\frac{M-1}{M}\right) + \right. \\ &\left. + \sum_{k=1}^{M-1} \sum_{-J \in Q_k} \prod_{-j \in -J} \prod_i (1 - p_{i-j}) \cdot \prod_{-j' \in Q_k \setminus -J} \left(1 - \prod_i (1 - p_{i-j'})\right) \cdot \left(\frac{1}{k+1}\right) \right) \cdot v_{ij} - 1 \geq 0 \ . \end{aligned}$$

with

$$p_{-ij} = p_{i-j} = p_{i-j'} = \begin{cases} 1 - \dfrac{(v_{ij} - k + 2\varepsilon)^2}{8\varepsilon^2} & \text{if } v_{ij} < k \leq v_{ij} + 2\varepsilon \\ 1 & \text{if } k > v_{ij} + 2\varepsilon \ . \end{cases}$$

Remind that the expected payoff difference is increasing in p_{-ij} and decreasing in p_{i-j} from below the cutoff, while it is decreasing in p_{-ij} and increasing in p_{i-j} from above. However, with groups symmetric in size $p_{-ij}^{n-1} > p_{i-j}^{n}$ holds. Hence, a sufficient condition for the two inequalities to hold is that groups are symmetric in size, i.e. $n_j = n \geq 2\ \forall j \in \{1, \ldots, M\}$, no matter how small ε is. Further note that for $n_j = n \geq\ \forall j \in \{1, \ldots, M\}$

$$k = \left(2^{1-n} \left(1 - 2^{-n}\right)^{M-1} \left(1 - \frac{1}{M}\right) + 2^{1-n} \sum_{k=1}^{M-1} \frac{\binom{M-1}{k} \left(2^{-n}\right)^k \left(1 - 2^{-n}\right)^{M-1-k}}{k+1}\right)^{-1} .$$

Hence, in $g_3(v)$ with symmetric group sizes there is an equilibrium in (monotonic) cutoff strategies, such that $\forall ij \in \{1, \ldots, n\}$ and $\forall j \in \{1, \ldots, M\}$:

$$x_{ij}^{*}\left(v_{ij}\right) = \begin{cases} 1 & \text{if } v_{ij} > v_j^{*} \\ 0 & \text{if } v_{ij} \leq v_j^{*} \ , \end{cases}$$

where

$$v_j^{*} = \left(2^{1-n}\left(1-2^{-n}\right)^{M-1}\left(1-\frac{1}{M}\right) + 2^{1-n}\sum_{k=1}^{M-1}\frac{\binom{M-1}{k}\left(2^{-n}\right)^{k}\left(1-2^{-n}\right)^{M-1-k}}{k+1}\right)^{-1} .$$

Nonetheless, given the absence of an upward dominance region, the following equilibrium $\forall ij \in \{1, \ldots, n_j\}$ and $\forall j \in \{1, 2\}$ exists as in De Mesquita (2011):

$$x_{ij}^{**}\left(v_{ij}\right) = 0 \ \forall v_{ij} \in \left[\underline{v} + \varepsilon, \overline{v} - \varepsilon\right].$$

Note that at $(\gamma_1, \ldots, \gamma_M) = (0, \ldots, 0)$ any deviation is strictly dominated for any $v_{ij} \in [\underline{v} + \varepsilon, \overline{v} - \varepsilon]$, so that $(\gamma_1, \ldots, \gamma_M) = (0, \ldots, 0)$ in the $BMMGC^{*}$ is robust to incomplete information in the sense of Kajii and Morris (1997) even with asymmetric group sizes.

QED

3.A.16 Proof of Proposition 3.12

In the $g_4\left(c\right)$, note that $E\left(C|c_{ij}\right) = c_{ij}$, if ij observes $c_{ij} \in [\underline{c} + \varepsilon, \overline{c} - \varepsilon]$ so that $C|c_{ij} \sim U(c_{ij} - \varepsilon,\ c_{ij} + \varepsilon)$. Furthermore, for $c_{ij} \in [\underline{c} - \varepsilon, \overline{c} + \varepsilon]$, the conditional distribution of the teammates' or opponents' observation will be centered around c_{ij} with a triangular distribution over the support $[c_{ij} - 2\varepsilon, c_{ij} + 2\varepsilon]$.

Hence, $Prob\left[C_{-ij} < c_{ij}|c_{ij}\right] = Prob[C_{-ij} > c_{ij}|c_{ij}] = \frac{1}{2}$. Moreover, let us define Q_k as the set of all subsets of cardinality k formed by groups different from j, that is $Q_k = \{-J \in \{\{1, \ldots, M\} \setminus \{j\}\} \mid |-J| = k\}$, where $|\cdot|$ denotes the cardinality of a set. Let us denote p_{-ij} the conditional expected probability attached by player ij to any other player $-ij$ within the same group choosing not to exert effort. Likewise, p_{i-j} is the conditional expected probability attached by ij to any other member of the rival group $-j$ not exerting effort. Then, for player ij the conditionally expected payoff from exerting effort equals [3.A5]

[3.A5] Due the auction-type contest success function and the presence of $M \geq 2$ groups, I have to consider the possibility of a tie with up to $M - 1$ groups.

$$\begin{aligned}
&Prob\left(n_j\gamma_j = n_j\right) \cdot \prod_{-j} Prob\left(n_{-j}\gamma_{-j} < n_{-j}\right)\left(v - c_{ij}\right) + \\
&+ Prob\left(n_j\gamma_j = n_j\right) \cdot \sum_{k=1}^{M-1} \sum_{-J\in Q_k} \prod_{-j\in -J} Prob\left(n_{-j}\gamma_{-j} = n_{-j}\right) \cdot \prod_{-j'\in Q_k\setminus -J} Prob\left(n_{j'}\gamma_{j'} < n_{j'}\right) \cdot \left(\frac{v}{k+1} - c_{ij}\right) + \\
&+ Prob\left(n_j\gamma_j < n_j\right) \cdot \sum_{k=1}^{M-1} \sum_{-J\in Q_k} \prod_{-j\in -J} Prob\left(n_{-j}\gamma_{-j} = n_{-j}\right) \cdot \prod_{-j'\in Q_k\setminus -J} Prob\left(n_{-j'}\gamma_{-j'} < n_{-j'}\right) \cdot \left(-c_{ij}\right) + \\
&+ Prob\left(n_j\gamma_j < n_j\right) \cdot \prod_{-j} Prob\left(n_{-j}\gamma_{-j} < n_{-j}\right)\left(\frac{v}{M} - c_{ij}\right)
\end{aligned}$$

$$\begin{aligned}
&= \prod_{-i}\left(1 - p_{-ij}\right) \cdot \prod_{-j}\left(1 - \prod_{i}\left(1 - p_{i-j}\right)\right)\left(v - c_{ij}\right) + \\
&+ \prod_{-i}\left(1 - p_{-ij}\right) \cdot \sum_{k=1}^{M-1} \sum_{-J\in Q_k} \prod_{-j\in -J} \prod_{i}\left(1 - p_{i-j}\right) \cdot \prod_{-j'\in Q_k\setminus J}\left(1 - \prod_{i}\left(1 - p_{i-j'}\right)\right) \cdot \left(\frac{v}{k+1} - c_{ij}\right) + \\
&+ \left(1 - \prod_{-i}\left(1 - p_{-ij}\right)\right) \cdot \prod_{-j}\prod_{i}\left(1 - p_{i-j}\right)\left(-c_{ij}\right) + \\
&+ \left(1 - \prod_{-i}\left(1 - p_{-ij}\right)\right)\prod_{-j}\left(1 - \prod_{i}\left(1 - p_{i-j}\right)\right)\left(\frac{v}{M} - c_{ij}\right) \quad ,
\end{aligned}$$

while the conditionally expected payoff from not exerting effort equals

$$\prod_{-j} Prob\left(n_{-j}\gamma_{-j} < n_{-j}\right)\frac{v}{M} = \prod_{-j}\left(1 - \prod_{i}\left(1 - p_{i-j}\right)\right)\frac{v}{M} \ .$$

Hence, for player ij the conditional expected payoff difference from exerting effort versus not exerting effort equals

$$\begin{aligned}
&\prod_{-i}\left(1 - p_{-ij}\right) \cdot \left(\prod_{-j}\left(1 - \prod_{i}\left(1 - p_{i-j}\right)\right) \cdot \left(\frac{M-1}{M}\right) + \right. \\
&\left. + \sum_{k=1}^{M-1} \sum_{-J\in Q_k} \prod_{-j\in -J} \prod_{i}\left(1 - p_{i-j}\right) \cdot \prod_{-j'\in Q_k\setminus -J}\left(1 - \prod_{i}\left(1 - p_{i-j'}\right)\right) \cdot \left(\frac{1}{k+1}\right)\right) \cdot v + \\
&- \left(\prod_{-j}\left(1 - \prod_{i}\left(1 - p_{i-j}\right)\right) + \sum_{k=1}^{M-1} \sum_{-J\in Q_k} \prod_{-j\in -J} \prod_{i}\left(1 - p_{i-j}\right) \cdot \prod_{-j'\in Q_k\setminus -J}\left(1 - \prod_{i}\left(1 - p_{i-j'}\right)\right)\right) c_{ij} \\
&= \prod_{-i}\left(1 - p_{-ij}\right) \cdot \left(\prod_{-j}\left(1 - \prod_{i}\left(1 - p_{i-j}\right)\right) \cdot \left(\frac{M-1}{M}\right) + \right. \\
&\left. + \sum_{k=1}^{M-1} \sum_{-J\in Q_k} \prod_{-j\in -J} \prod_{i}\left(1 - p_{i-j}\right) \cdot \prod_{-j'\in Q_k\setminus -J}\left(1 - \prod_{i}\left(1 - p_{i-j'}\right)\right) \cdot \left(\frac{1}{k+1}\right)\right) \cdot v - c_{ij} \ ,
\end{aligned}$$

and the conditional expected payoff difference from not exerting effort versus exerting effort equals

$$\begin{aligned}
&\left(\prod_{-j}\left(1-\prod_{i}(1-p_{i-j})\right)+\sum_{k=1}^{M-1}\sum_{-J\in Q_k}\prod_{-j\in -J}\prod_{i}(1-p_{i-j})\cdot\prod_{-j'\in Q_k\setminus -J}\left(1-\prod_{i}(1-p_{i-j'})\right)\right)c_{ij}+\\
&-\prod_{-i}(1-p_{-ij})\cdot\left(\prod_{-j}\left(1-\prod_{i}(1-p_{i-j})\right)\cdot\left(\frac{M-1}{M}\right)+\right.\\
&\left.+\sum_{k=1}^{M-1}\sum_{-J\in Q_k}\prod_{-j\in -J}\prod_{i}(1-p_{i-j})\cdot\prod_{-j'\in Q_k\setminus -J}\left(1-\prod_{i}(1-p_{i-j'})\right)\cdot\left(\frac{1}{k+1}\right)\right)\cdot v\\
&=c_{ij}-\prod_{-i}(1-p_{-ij})\cdot\left(\prod_{-j}\left(1-\prod_{i}(1-p_{i-j})\right)\cdot\left(\frac{M-1}{M}\right)+\right.\\
&\left.+\sum_{k=1}^{M-1}\sum_{-J\in Q_k}\prod_{-j\in -J}\prod_{i}(1-p_{i-j})\cdot\prod_{-j'\in Q_k\setminus -J}\left(1-\prod_{i}(1-p_{i-j'})\right)\cdot\left(\frac{1}{k+1}\right)\right)\cdot v\ .
\end{aligned}$$

Observe that for player ij the expected payoff difference from exerting effort versus not exerting effort is decreasing in the probability of the teammates not exerting effort, while it is increasing in the probability of the rival groups not exerting effort. Hence, by setting $p_{-ij}=1$, $p_{i-j}=0$, $p_{i-j'}=0$, the expected payoff difference reduces to $-c_{ij}$. Thus, for any player ij $x_{ij}=1$ is a dominant action for $c_{ij}<0$. On the other hand, for player ij the expected payoff difference from not exerting effort versus exerting effort is increasing in the probability of the teammates not exerting effort, while it is decreasing in the probability of the rival groups not exerting effort. Hence, by setting $p_{-ij}=0$, $p_{i-j}=1$, $p_{i-j'}=1$, the expected payoff difference reduces to $c_{ij}-v(M-1)/M$. From the latter we derive threshold of one dominance region: for $c_{ij}>v(M-1)/(M)$, $x_{ij}=0$ is a strictly dominant action for any player ij. Let us suppose player ij observes $c_{ij}=0$ and $\varepsilon<|\underline{c}/2|$ so that $0-2\varepsilon\geq\underline{c}$. Then she will attach probability at least $1/2$ to any other player exerting effort. Accordingly, if we set $p_{-ij}=1/2$, $p_{i-j}=0$, $p_{i-j'}=0$, then we get that the first value of c_{ij} such that not exerting effort cannot be discarded by iterated dominance is $c^{d}_{ij}=(2^{1-n_j}/M)v$.

Let us suppose player ij observes $c_{ij}=v(M-1)/M$ and $\varepsilon<|\overline{c}/2-v(M-1)/2M|$ so that $vM/(M-1)+2\varepsilon\leq\overline{c}$. Then she will attach probability at least $1/2$ to any other player not exerting effort. Accordingly, if we set $p_{-ij}=1/2$, $p_{i-j}=1$, $p_{i-j'}=1$, then we get that the first value of c_{ij} such that exerting effort cannot be discarded by iterated dominance is $c^{dd}_{ij}=vM/(M-1)\cdot 2^{-n_j-1}$. Note that for $M=2$, I get a unique equilibrium by iterated dominance, given that $c^{d}_{ij}=c^{dd}_{ij}=c^{*}_{ij}=2^{-n_j}v$ as in proposition 3.6. However, observe that for more than two groups, the expected payoff difference from not exerting effort and exerting effort depends not only on p_{-ij} but also on p_{i-j}, so that the two thresholds of the dominance regions do not coincide in general. Let us assume that players adopt the following cutoff strategy

$$x_{ij}\left(c_{ij}\right)=\begin{cases} 1 & \text{if } c_{ij}<k \\ 0 & \text{if } c_{ij}\geq k\ , \end{cases}$$

where k is obtained by equating the expected payoff differences to zero with $p_{-ij}=1/2$, $p_{i-j}=1/2$, $p_{i-j'}=1/2$, that is k is equal to

$$k=v\cdot\left(2^{1-n_j}\prod_{-j}\left(1-2^{-n_{-j}}\right)\left(1-\frac{1}{M}\right)+2^{1-n_j}\sum_{k=1}^{M-1}\frac{\sum_{-J\in Q_k}\prod_{-j\in -J}\left(2^{-n_{-j}}\right)\prod_{-j'\in Q_k\setminus J}\left(1-2^{-n_{-j'}}\right)}{k+1}\right)\ .$$

Then, for the strategy above to be an equilibrium it must satisfy incentive compatibility constraints from above and below the cutoff, that is $\forall\ v_{ij}\in k,[k,vM/(M-1)\cdot 2^{-n_j-1}]$

$$\begin{aligned} &c_{ij}-\prod_{-i}\left(1-p_{-ij}\right)\cdot\left(\prod_{-j}\left(1-\prod_{i}\left(1-p_{i-j}\right)\right)\cdot\left(\frac{M-1}{M}\right)+\right.\\ &\left.+\sum_{k=1}^{M-1}\sum_{-J\in Q_k}\prod_{-j\in -J}\prod_{i}\left(1-p_{i-j}\right)\cdot\prod_{-j'\in Q_k\setminus -J}\left(1-\prod_{i}\left(1-p_{i-j'}\right)\right)\cdot\left(\frac{1}{k+1}\right)\right)\cdot v\geq 0 \end{aligned}$$

with

$$p_{-ij}=p_{i-j}=p_{i-j'}=\begin{cases}1 & \text{if } v/4\leq c_i-2\varepsilon \\ 1-\dfrac{(v/4-c_i+2\varepsilon)^2}{8\varepsilon^2} & \text{if } c_i-2\varepsilon<v/4\leq c_i\end{cases}$$

and $\forall c_{ij}\in\left[\left(2^{1-n_j}/M\right)v,k\right)$

$$\begin{aligned} &\prod_{-i}\left(1-p_{-ij}\right)\cdot\left(\prod_{-j}\left(1-\prod_{i}\left(1-p_{i-j}\right)\right)\cdot\left(\frac{M-1}{M}\right)+\right.\\ &\left.+\sum_{k=1}^{M-1}\sum_{-J\in Q_k}\prod_{-j\in -J}\prod_{i}\left(1-p_{i-j}\right)\cdot\prod_{-j'\in Q_k\setminus -J}\left(1-\prod_{i}\left(1-p_{i-j'}\right)\right)\cdot\left(\frac{1}{k+1}\right)\right)\cdot v-c_{ij}\geq 0\ . \end{aligned}$$

with

$$p_{i-j}=p_{-ij}=p_{i-j'}=\begin{cases}\dfrac{(c_i-v/4+2\varepsilon)^2}{8\varepsilon^2} & \text{if } c_i<v/4<c_i+2\varepsilon \\ 0 & \text{if } v/4>c_i+2\varepsilon\ .\end{cases}$$

Remind that the expected payoff difference is increasing in p_{-ij} and decreasing in p_{i-j} from above the cutoff, while it is decreasing in p_{-ij} and increasing in p_{i-j} from below. However, with groups symmetric in size $p_{-ij}^{n-1}>p_{i-j}^{n}$ holds. Hence, a sufficient condition for the two inequalities to hold is that groups are symmetric in size, i.e. $n_j=n\geq 2\ \forall j\in\{1,\ldots,M\}$, no matter how small ε is. Further note that for $n_j = n \geq \ \forall j \in \{1, \ldots, M\}$

$$k = v \cdot \left(2^{1-n} \left(1 - 2^{-n}\right)^{M-1} \left(1 - \frac{1}{M}\right) + 2^{1-n} \sum_{k=1}^{M-1} \frac{\binom{M-1}{k} \left(2^{-n}\right)^k \left(1 - 2^{-n}\right)^{M-1-k}}{k+1} \right) .$$

Hence, in $g_4(c)$ with symmetric group sizes there is an equilibrium in (monotonic) cutoff strategies, such that $\forall ij \in \{1, \ldots, n\}$ and $\forall j \in \{1, \ldots, M\}$:

$$x^*_{ij}(v_{ij}) = \begin{cases} 1 & \text{if } c_{ij} > c^*_j \\ 0 & \text{if } c_{ij} \leq c^*_j \ , \end{cases}$$

where

$$c^*_j = v \left(2^{1-n} \left(1 - 2^{-n}\right)^{M-1} \left(1 - \frac{1}{M}\right) + 2^{1-n} \sum_{k=1}^{M-1} \frac{\binom{M-1}{k} \left(2^{-n}\right)^k \left(1 - 2^{-n}\right)^{M-1-k}}{k+1} \right) .$$

QED

3.A.17 Proof of Lemma 3.1

Under the uniform information structure of $g_1(v)$, as the noise ε tends to zero, it holds for any ij:

$$\lim_{\varepsilon \to 0} Prob\left[V_{-ij} < v_{ij} | v_{ij}\right] = \lim_{\varepsilon \to 0} Prob\left[V_{-ij} > v_{ij} | v_{ij}\right] = \lim_{\varepsilon \to 0} \frac{v_{ij} - v_{ij} + 2\varepsilon}{v_{ij} + 2\varepsilon - v_{ij} + 2\varepsilon} = \frac{1}{2} \ .$$

Then, iterated deletion of conditionally/interim strictly-dominated strategies follows until the threshold obtained in $g_1(v)$ and $g_3(v)$, as proved for proposition 4.5. Moreover, incentive compatibility around the equilibrium cutoff in proposition 3.11 for $g_3(v)$ holds for $Prob[V_{-ij} < v^*_j | v_{ij}] = Prob[V_{i-j} < v^*_j | v_{ij}] \leq 1/2$ with $v_{ij} \geq v^*_j$ and $Prob[V_{-ij} < v^*_j | v_{ij}] = Prob[V_{i-j} < v^*_j | v_{ij}] \geq 1/2$ with $v_{ij} \leq v^*_j$. These inequalities hold for $\varepsilon \to 0$ as well:

$$p_{-ij} = p_{i-j} = p_{i-j'} = \begin{cases} 0 & \text{if } k \leq v_{ij} - 2\varepsilon \\ \dfrac{(k - v_{ij} + 2\varepsilon)^2}{8\varepsilon^2} & \text{if } v_{ij} - 2\varepsilon < k \leq v_{ij} \end{cases} \Leftrightarrow \begin{cases} 0 & \text{if } k < v_{ij} \\ \dfrac{1}{2} & \text{if } v_{ij} = k \ . \end{cases}$$

and

$$p_{-ij} = p_{i-j} = p_{i-j'} = \begin{cases} 1 - \dfrac{(v_{ij} - k + 2\varepsilon)^2}{8\varepsilon^2} & \text{if } v_{ij} < k \leq v_{ij} + 2\varepsilon \\ 1 & \text{if } k > v_{ij} + 2\varepsilon \end{cases} \Leftrightarrow \begin{cases} \dfrac{1}{2} & \text{if } v_{ij} = k \\ 1 & \text{if } k > v_{ij} \ . \end{cases}$$

On the other hand, under the uniform information structure of $g_2(c)$ and $g_4(c)$, as the noise ε tends to zero, it holds for any ij:

$$\lim_{\varepsilon \to 0} Prob\left[C_{-ij} < c_{ij} | c_{ij}\right] = \lim_{\varepsilon \to 0} Prob\left[C_{-ij} > c_{ij} | c_{ij}\right] = \lim_{\varepsilon \to 0} \frac{c_{ij} - c_{ij} + 2\varepsilon}{c_{ij} + 2\varepsilon - c_{ij} + 2\varepsilon} = \frac{1}{2} \ .$$

Then, iterated deletion of conditionally/interim strictly-dominated strategies follows until the threshold obtained in $g_2(c)$, as proved for propositions 4.6. Moreover, incentive compatibility around the equilibrium cutoff in proposition 4.9 regarding $g_4(c)$ holds for $Prob[C_{-ij} > c^*_j | c_{ij}] = Prob[C_{i-j} > c^*_j | c_{ij}] \leq 1/2$ with $c_{ij} \leq c^*_j$ and $Prob[C_{-ij} > c^*_j | c_{ij}] = Prob[C_{i-j} > c^*_j | c_{ij}] \geq 1/2$ with $c_{ij} \geq c^*_j$. These inequalities hold for $\varepsilon \to 0$ as well:

$$p_{-ij} = p_{i-j} = p_{i-j'} = \begin{cases} \dfrac{(c_{ij} - c^*_j + 2\varepsilon)^2}{8\varepsilon^2} & \text{if } c_{ij} < c^*_j < c_{ij} + 2\varepsilon \\ 0 & \text{if } c^*_j > c_{ij} + 2\varepsilon \ . \end{cases} \Leftrightarrow \begin{cases} \dfrac{1}{2} & \text{if } c_{ij} = c^*_j \\ 0 & \text{if } c^*_j > c_{ij} \ . \end{cases}$$

and

$$p_{-ij} = p_{i-j} = p_{i-j'} = \begin{cases} 1 & \text{if } c^*_j \leq c_{ij} - 2\varepsilon \\ 1 - \dfrac{(c^*_j - c_{ij} + 2\varepsilon)^2}{8\varepsilon^2} & \text{if } c_{ij} - 2\varepsilon < c^*_j \leq c_{ij} \end{cases} \Leftrightarrow \begin{cases} 1 & \text{if } c^*_j < c_{ij} \\ \dfrac{1}{2} & \text{if } c_{ij} = c^*_j \ . \end{cases}$$

QED

3.A.18 Proof of Proposition 3.13

I first consider incomplete information about the prize, as follows:

- let V be a random variable which is uniformly distributed on $\mathbb{R}$;
- each player $ij \in \{1, \ldots, n_j\} \, \forall j \in \{1, \ldots, M\}$, with $M \geq 2$, idiosyncratically observes the realization of a random variable $V_{ij} = V + \varepsilon E_{ij}$, where E_{ij} are independent and symmetric random variables with mean zero and variance one and $\varepsilon > 0$ is a scale parameter, so that players' observation errors are independent.

- Then, the prior distribution for V is the uniform improper, so that $f(V) \propto 1$, that is it is constant over $\mathbb{R}$.
- Moreover, the likelihood distribution is

  $$f\left(V_{ij}=v_{ij}|V\right)=f\left(E_{ij}=\frac{v_{ij}-V}{\varepsilon}\right)\ .$$

  Note that the likelihood function is maximized for $V=v_{ij}$, since E_{ij} is symmetric around 0.
- Using Bayes' rule, the posterior for V given v_{ij}, is proportional to the likelihood

  $$f\left(V|v_{ij}\right)\propto f\left(V_{ij}=v_{ij}|V\right)\cdot f\left(V\right)\ .$$

  Since $f\left(V\right)\propto 1$, this simplifies to

  $$f\left(V|v_{ij}\right)\propto f\left(E_{ij}=\frac{v_{ij}-V}{\varepsilon}\right)\ .$$

  The posterior is symetric around $V=v_{ij}$, so that

  $$\mathbb{E}\left[V|v_{ij}\right]=v_{ij}\ .$$

  Accordingly, to evaluate the conditional distribution of $V_{-ij}|v_{ij}$, I can write $V_{-ij}=V+\varepsilon E_{-ij}$ as

  $$V_{-ij}=\left(v_{ij}-\varepsilon E_{ij}\right)+\varepsilon E_{-ij}=v_{ij}+\varepsilon\left(E_{-ij}-E_{ij}\right)\ .$$

  Conditional on v_{ij}, the term $E_{-ij}-E_{ij}$ is symmetric around 0, since E_{-ij} and E_{ij} are symmetric independent random variables, so that $f\left(V_{-ij}|v_{ij}\right)$ is symmetric around v_{ij}.
- It follows,

  $$\begin{aligned}&Prob\left(V_{-ij}<v_{ij}|v_{ij}\right)=Prob\left(V_{-ij}>v_{ij}|v_{ij}\right)=\frac{1}{2}\ ,\\&Prob\left(V_{-ij}<v_j^*|v_{ij}\right)=Prob(V_{i-j}<v_j^*|v_{ij})\geq\frac{1}{2}\quad\forall\ v_{ij}\leq v_j^*\ ,\\&Prob\left(V_{-ij}<v_j^*|v_{ij}\right)=Prob(V_{i-j}<v_j^*|v_{ij})\leq\frac{1}{2}\quad\forall\ v_{ij}\geq v_j^*\ .\end{aligned}$$

- Clearly, $Prob\left[V_{i-j}<v_{ij}|v_{ij}\right]=Prob\left[V_{i-j}>v_{ij}|v_{ij}\right]=\frac{1}{2}\ \forall i\in\{1,\dots,n_{-j}\}$ and $\forall-j\neq j$.
- Iterated deletion of conditionally/interim strictly-dominated strategies follows until the threshold

obtained in $g_1(v)$with symmetric group sizes, as proved for proposition 4.5, and incentive compatibility around the cutoff v_j^* holds in $g_3(v)$, as proved for proposition 3.11.

I then consider incomplete information about the cost of effort, as follows:

- let C be a random variable which is uniformly distributed on $\mathbb{R}$;
- each player $ij \in \{1, \ldots, n_j\} \, \forall j \in \{1, \ldots, M\}$, with $M \geq 2$, idiosyncratically observes the realization of a random variable $C_{ij} = C + \varepsilon E_{ij}$, where E_{ij} are independent and symmetric random variables with mean zero and variance one and $\varepsilon > 0$ is a scale parameter, so that players' observation errors are independent.
- Then, the prior distribution for C is the uniform improper, so that $f(C) \propto 1$, that is it is constant over $\mathbb{R}$.
- Moreover, the likelihood distribution is

  $$f\left(C_{ij} = c_{ij} | C\right) = f\left(E_{ij} = \frac{c_{ij} - C}{\varepsilon}\right) \ .$$

  Note that the likelihood function is maximized for $C = c_{ij}$, since E_{ij} is symmetric around 0.
- Using Bayes' rule, the posterior for C given c_{ij}, is proportional to the likelihood

  $$f\left(C | c_{ij}\right) \propto f\left(C_{ij} = c_{ij} | C\right) \cdot f\left(C\right) \ .$$

  Since $f(C) \propto 1$, this simplifies to

  $$f\left(C | c_{ij}\right) \propto f\left(E_{ij} = \frac{c_{ij} - C}{\varepsilon}\right) \ .$$

  The posterior is symetric around $C = c_{ij}$, so that

  $$\mathbb{E}\left[C | c_{ij}\right] = c_{ij} \ .$$
- Accordingly, to evaluate the conditional distribution of $C_{-ij} | c_{ij}$, I can write $C_{-ij} = C + \varepsilon E_{-ij}$ as

  $$C_{-ij} = \left(c_{ij} - \varepsilon E_{ij}\right) + \varepsilon E_{-ij} = c_{ij} + \varepsilon\left(E_{-ij} - E_{ij}\right) \ .$$

Conditional on c_{ij}, the term $E_{-ij} - E_{ij}$ is symmetric around 0, since E_{-ij} and E_{ij} are symmetric independent random variables, so that $f\left(C_{-ij}|c_{ij}\right)$ is symmetric around c_{ij}.

- It follows,

$$\begin{aligned}
&Prob\left(C_{-ij} < c_{ij}|c_{ij}\right) = Prob\left(C_{-ij} > c_{ij}|c_{ij}\right) = \frac{1}{2}\\
&Prob\left(C_{-ij} > c_j^*|c_{ij}\right) = Prob(C_{i-j} > c_j^*|c_{ij}) \geq \frac{1}{2} \quad \forall\ c_{ij} \geq c_j^*\ ,\\
&Prob\left(C_{-ij} > c_j^*|c_{ij}\right) = Prob(C_{i-j} > c_j^*|c_{ij}) \leq \frac{1}{2} \quad \forall\ c_{ij} \leq c_j^*\ .
\end{aligned}$$

- Clearly, $Prob\left[C_{i-j} < c_{ij}|c_{ij}\right] = Prob\left[C_{i-j} > c_{ij}|c_{ij}\right] = \frac{1}{2}\ \forall i \in \{1, \ldots, n_{-j}\}$ and $\forall -j \neq j$.

- Iterated deletion of conditionally/interim strictly-dominated strategies follows until the threshold obtained in $g_2\left(c\right)$, as proved for proposition 4.6, and incentive compatibility around the cutoff c_j^* holds in $g_4\left(c\right)$with symmetric group sizes, as proved for propositions 4.9.

QED

4 Max-Max Group Contests with Incomplete Information à la Global Games

Aus unbeschreiblicher Verwandlung stammen
solche Gebilde – Fühl! und glaub!
Wir leidens oft: zu Asche werden Flammen;
doch: in der Kunst: zur Flamme wird der Staub.

Hier ist Magie. In das Bereich des Zaubers
scheint das gemeine Wort hinaufgestuft...
und ist doch wirklich wie der Ruf des Taubers,
der nach der unsichtbaren Taube ruft.

(Rainer Maria Rilke)

Max-Max Group Contests with Incomplete Information à la Global Games [*]

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Abstract

In this paper we introduce incomplete information à la global games into a deterministic two-group contest with the best-shot impact function and binary actions and we characterize the set of equilibria. Depending on whether the complete information assumption is relaxed on the value of the prize or on the cost of providing effort, we obtain different results in terms of equilibrium existence: in the first case, equilibria in (monotonic) switching strategies do not exist, whereas in the second one there exists an equilibrium in (monotonic) switching-strategies for a sufficiently high level of noise. Then, we discuss the presence of the group-size paradox for the case in which incomplete information is modeled on the cost of effort. The equilibrium existence result for this setting is thus extended to the case of M groups.

JEL classification: D74, D71, C72

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4.1 Introduction

Mirroring the terminology introduced by Chowdhury, Lee, and Topolyan (2016) for deterministic group contests with the weakest-link impact function, we call “max-max” group contests with the auction-type contest success function and the best-shot impact function, so that the aggregated effort within a group equals the maximum of the efforts provided. Group contests with the best-shot impact function can describe competition by groups for the generation of innovative ideas in $R\&D$, military conflict and provision of a discrete public good, as stressed by Chowdhury, Lee, and Sheremeta (2013) and Barbieri, Malueg, and Topolyan (2014). In these settings we bolster the idea that effort provision choice can be conceived and modeled as a binary variable for generation of innovative projects, delivery of exceptional skill-based output, military attack and duels. Despite being non-standard, the assumption of a binary action set is not new in contest theory, rather it has been adopted by a wide theoretical and experimental literature, spanning from corporate science, to sabotage activities and contests for status, as reviewed by Sheremeta (2018).

Deterministic group contests with the best-shot impact function display multiplicity of equilibria under both complete and payoff-relevant incomplete information, as shown by Barbieri, Malueg, and Topolyan (2014) and Barbieri and Malueg (2016), respectively. However, a recent contribution by Barbieri and Topolyan (2024) shows that group-public randomization delivers equilibrium uniqueness in best-shot group contests. Group-public randomization is conceived by the authors as a source of extrinsic incomplete information, that is not payoff-relevant. On the other hand, in Chapter 3, we show that introducing incomplete information à la global games about the value of the prize or the cost of effort in deterministic group contests with the weakest-link impact function delivers interesting equilibrium selection properties. In this paper, as done for deterministic group contests with the weakest-link impact function in Chapter 3, we follow Carlsson and Damme (1993a) by introducing payoff-relevant incomplete information à la global games about the value of the prize contested and the cost of effort into a max-max two-group contest with binary actions. In the first case, we prove the non-existence of an equilibrium in (monotonic) switching-strategies, whereas in the second case we are able to show the existence of an equilibrium in (monotonic) switching strategies for a sufficiently high level of noise, not resulting by iterated deletion of (interim) strictly dominated strategies, as in Carlsson and Damme (1993a) for 2×2 games, but by a guess-and-solve approach. Risk-dominance is shown to be a valid equilibrium selection criterion for two-group contests with two players per group, but fails to be pivotal as the number of team members exceeds two, confirming what shown by Carlsson and Damme (1993b) for the generalization of stag-hunt games to the n-player case. These results are different to what obtained for max-min group contests with incomplete information à la global games, where within-group complementarities enable the application of iterated deletion of strictly dominated strategies and trigger equilibrium existence and uniqueness, depending on the two settings considered for incomplete information modeling, i.e. prize valuation or cost of effort. Moreover, we show the non-existence of an equilibrium robust to incomplete information, in the sense of Kajii and Morris (1997), in which no player exerts effort for any possible private value of the prize. Other dimensions along which max-max group contests differ from the weakest-link counterpart are the presence of the so-called group-size paradox and the generalization to the M-group case. As a matter of fact, whether an increase in group size translates into a lower probability of winning and a lower expected payoff cannot be established generally for this class of games, so that we provide numerical examples, only. Moreover, the generalization of our analysis to the M-group case, highlights that the equilibrium in (monotonic) switching strategies in the cost-of-effort setting is rooted in a different threshold with respect to the weakest-link M-group counterpart, as shown in Chapter 3.

The paper is structured as follows. In Section 4.2 the formal model with both complete information and incomplete information is presented under two different specifications. Section 4.3 presents two examples which should clarify the parallelism between group contests and the supermodular payoff structure perturbed in the global games à la Carlsson and Damme (1993a) and how an equilibrium in (monotonic) switching strategies naturally arises or not when modeling incomplete information à la global games, stressing the difference incurring with the weakest-link case address in Chapter 3. Section 4.4 derives the set of Nash equilibria of the complete information game, while Section 4.5 is the core of the paper establishing non existence and existence of Bayes-Nash equilibria for the two classes of incomplete information games. Section 4.6 addresses whether there is the so-called group-size paradox in just one of the two model specifications delivered in Section 4.5, providing numerical examples. Section 4.7 extends the two-group contest model under incomplete information to an M-group contest model. Finally, Section 4.8 concludes.

4.2 The Model

Let us consider a deterministic group contest defined by the following elements:

1. two **groups,** denoted by $j \in \{1, 2\}$;
2. each group has $n_j \geq 2$ members, where $n_1 \geq n_2$ without loss of generality. The total number of

agents is $n_1 + n_2 = N$. As notation device, let us write ij or $j(i)$ for **agents** $i \in \{1, \ldots, n_j\}$ of group j;

3. the **choice** of member $i \in \{1, \ldots, n_j\}$ in group $j \in \{1, 2\}$, to increase the possibility of getting the prize, is denoted by $x_j(i) \in \{0, 1\}$. Let $\mathbf{x}_j$ be the vector of all agents' efforts of group j, and $\mathbf{x}$ the vector of all agents' efforts. Moreover, let $x_j(i) = 1$ be denoted by a and $x_j(i) = 0$ by $\overline{a}$; let us define the average exerted effort in group j, or rather the participation rate in group j as

$$\gamma_j = \frac{1}{n_j} \sum_{i=1}^{n_j} x_{ij} \in [0, 1];$$

   Moreover, when $\gamma_j \in (0, 1)$, denote by

$$\gamma_j^+ = \frac{1}{n_j} \left(\sum_{i=1}^{n_j} x_{ij} + 1 \right) \in [0, 1]$$

   the share of active agents at a marginal increase and by

$$\gamma_j^- = \frac{1}{n_j} \left(\sum_{i=1}^{n_j} x_{ij} - 1 \right) \in [0, 1]$$

   the share of active agents at a marginal decrease.

4. a club good **prize** worth $v \in \mathbb{R}$ is allocated to one of the two groups: thus, the prize v can be worth negative utils, which means that it can be a bad;

5. the **impact function** of group j is given by the best-shot technology

$$X_j = \max \{x_j(i) \in \{0, 1\}, i \in \{1, ..., n_j\}\};$$

6. the **contest success function** is given by the *all-pay auction:*

$$p_j(X_1, X_2) = \begin{cases} 1 & if \quad X_j > X_{-j} \\ \frac{1}{2} & if \quad X_j = X_{-j} \\ 0 & if \quad X_j < X_{-j}; \end{cases}$$

7. the individual **costs of effort** $C_{ij}(x_j(i)) = x_j(i)$.

As a consequence of these modelling characteristics, player ij has the expected **payoff**

$$\pi_{ij}\left(\mathbf{x}_1,\mathbf{x}_2\right)=p_j v-x_{ij}=$$

$$=\begin{cases} v-x_j\left(i\right) & if \quad \max\left\{\mathbf{x}_j\right\}>\max\left\{\mathbf{x}_{-j}\right\} \\ \frac{1}{2}v-x_j\left(i\right) & if \quad \max\left\{\mathbf{x}_j\right\}=\max\left\{\mathbf{x}_{-j}\right\} \\ -x_j\left(i\right) & if \quad \max\left\{\mathbf{x}_j\right\}<\max\left\{\mathbf{x}_{-j}\right\}. \end{cases}$$

Now we are able to provide a formal definition of a binary max-min group contest with a public good prize.

DEFINITION 4.1. *A* Binary Max-Max Group Contest $BMMAGC^*$ *is a one-stage game* $BMMAGC^*$ $=\left\langle\left\{1,2\right\},N,B_{ij},\pi_{ij}\right\rangle$ *defined by*

1. *the set of groups* $\left\{1,2\right\};$
2. *the set of players* $N=\left\{1,...,n_1+n_2\right\};$
3. *the set of actions* $B_{ij}=\left\{0,1\right\}:$ *for each player* ij*, the choice of the effort* $x_j\left(i\right);$
4. *the payoff functions for each player* $ij\in N$

$$\pi_{ij}\left(\mathbf{x}_1,\mathbf{x}_2\right)=p_j v-x_{ij}=$$

$$=\begin{cases} v-x_j\left(i\right) & if \quad \max\left\{\mathbf{x}_j\right\}>\max\left\{\mathbf{x}_{-j}\right\} \\ \frac{1}{2}v-x_j\left(i\right) & if \quad \max\left\{\mathbf{x}_j\right\}=\max\left\{\mathbf{x}_{-j}\right\} \\ -x_j\left(i\right) & if \quad \max\left\{\mathbf{x}_j\right\}<\max\left\{\mathbf{x}_{-j}\right\} \end{cases}.$$

The notation used in this paper is summed up in table 4.1.

Variable	Meaning
ij or $j(i)$	agent i of group j
$\{1, ..., n_j\}$	set of agents in group j
$x_j(i)$ or x_{ij}	effort of agent i in group j
$X_j = \max\{x_j(i) \in \{0,1\}, i \in \{1, ..., n_j\}\}$	impact of effort of all agents in group j
$\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2)$	vector of efforts of all agents
$C(x_j(i)) = x_j(i)$	cost of effort for agent i of group j
$p_j(X_1, X_2)$	probability of group j of winning the contest
$\pi_{ij}(\mathbf{x}_1, \mathbf{x}_2)$	payoff function of agent i of group j
$\gamma_j = \frac{1}{n_j}\sum_{i=1}^{n_j} x_{ij} \in [0,1]$	share of active agents in group j

Table 4.1

4.3 An Example

Let us consider a $BMMAGC^*$ with two members for each group.[4.1] W.l.g., let players $1, 2$ belong to group 1 and players $3, 4$ to group 2. Consider the following geometric representation of the game, where player 3 "moves horizontally", while player 4 "moves vertically":

3

a

$1/2$	a	$\overline{a}$
a	$\frac{v}{2}-1; \frac{v}{2}-1; \frac{v}{2}-1; \frac{v}{2}-1$	$\frac{v}{2}-1; \frac{v}{2}; \frac{v}{2}-1; \frac{v}{2}-1$
$\overline{a}$	$\frac{v}{2}; \frac{v}{2}-1; \frac{v}{2}-1; \frac{v}{2}-1$	$0; 0; v-1; v-1$

$\overline{a}$

$1/2$	a	$\overline{a}$
a	$\frac{v}{2}-1; \frac{v}{2}-1; \frac{v}{2}; \frac{v}{2}-1$	$\frac{v}{2}-1; \frac{v}{2}; \frac{v}{2}; \frac{v}{2}-1$
$\overline{a}$	$\frac{v}{2}; \frac{v}{2}-1; \frac{v}{2}; \frac{v}{2}-1$	$0; 0; v; v-1$

a

$1/2$	a	$\overline{a}$
a	$\frac{v}{2}-1; \frac{v}{2}-1; \frac{v}{2}-1; \frac{v}{2}$	$\frac{v}{2}-1; \frac{v}{2}; \frac{v}{2}-1; \frac{v}{2}$
$\overline{a}$	$\frac{v}{2}; \frac{v}{2}-1; \frac{v}{2}-1; \frac{v}{2}$	$0; 0; v-1; v$

$1/2$	a	$\overline{a}$
a	$v-1; v-1; 0; 0$	$v-1; v; 0; 0$
$\overline{a}$	$v; v-1; 0; 0$	$\frac{v}{2}; \frac{v}{2}; \frac{v}{2}; \frac{v}{2}$

$\overline{a}$

4

4.3.1 The Set of Nash Equilibria in the Two-player Two-group Example

It is straightforward to derive the following properties:

[4.1] This section is a direct application of the example carried out by Carlsson and Damme (1993a) in their introduction.

- if $v > 2$, there are four strict Nash equilibria in pure strategies

  $$NE = \{(a, \overline{a}, \overline{a}, a)\,; (\overline{a}, a, \overline{a}, a)\,; (a, \overline{a}, a, \overline{a})\,; (\overline{a}, a, a, \overline{a})\}$$

  and a Nash equilibrium in symmetric strictly-mixed strategies $\sigma_i^*(a) = 1 - \frac{2}{v} \quad \forall i \in \{1, 2, 3, 4\}$;

- if $v = 2$, there are four Nash equilibria in pure strategies

  $$\begin{aligned} NE = &\{(a, \overline{a}, \overline{a}, a)\,; (\overline{a}, a, \overline{a}, a)\,; (a, \overline{a}, a, \overline{a})\,; (\overline{a}, a, a, \overline{a})\,; (\overline{a}, \overline{a}, \overline{a}, a)\} \cup \\ &\cup \{(a, \overline{a}, \overline{a}, \overline{a})\,; (\overline{a}, a, \overline{a}, \overline{a})\,; (\overline{a}, \overline{a}, a, \overline{a})\,; (\overline{a}, \overline{a}, \overline{a}, \overline{a})\}\,; \end{aligned}$$

- if $v < 2$, the unique Nash equilibrium derived by strict-dominance is $(\overline{a}, \overline{a}, \overline{a}, \overline{a})$;
- if $v > 2$,
  - i. $(\overline{a}, a, \overline{a}, a)$ and $(\overline{a}, a, a, \overline{a})$ are the payoff-dominant equilibria for player 1;[4.2]
  - ii. $(a, \overline{a}, \overline{a}, a)$ and $(a, \overline{a}, a, \overline{a})$ are the payoff-dominant equilibria for player 2;
  - iii. $(a, \overline{a}, \overline{a}, a)$ and $(\overline{a}, a, \overline{a}, a)$ are the payoff-dominant equilibria for player 3;
  - iv. $(a, \overline{a}, a, \overline{a})$ and $(\overline{a}, a, a, \overline{a})$ are the payoff-dominant equilibria for player 4;
- if $v = 2$,
  - i. $(\overline{a}, a, \overline{a}, \overline{a})$ is the payoff-dominant equilibrium for player 1;
  - ii. $(a, \overline{a}, \overline{a}, \overline{a})$ is the payoff-dominant equilibrium for player 2;
  - iii. $(\overline{a}, \overline{a}, \overline{a}, a)$ is the payoff-dominant equilibrium for player 3;
  - iv. $(\overline{a}, \overline{a}, a, \overline{a})$ is the payoff-dominant equilibrium for player 4;
- if $v > 4$,
  - i. $(a, \overline{a}, \overline{a}, a)$ and $(a, \overline{a}, a, \overline{a})$ are the risk-dominant equilibrium strategy profiles for player 1. As a matter of fact, let us compute the deviation losses of $(a, \overline{a}, \overline{a}, a)$, $(\overline{a}, a, \overline{a}, a)$, $(a, \overline{a}, a, \overline{a})$, $(\overline{a}, a, a, \overline{a})$ for player 1. Then,

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[4.2] For the formulation of payoff-dominace and risk-dominance concepts see Harsanyi and Selten (1988).

- $(a,\overline{a},\overline{a},a) \rightarrow \frac{v}{2}-1-0=\frac{v}{2}-1$;
- $(\overline{a},a,\overline{a},a) \rightarrow \frac{v}{2}-\frac{v}{2}+1=1$;
- $(a,\overline{a},a,\overline{a}) \rightarrow \frac{v}{2}-1-0=\frac{v}{2}-1$;
- $(\overline{a},a,a,\overline{a}) \rightarrow \frac{v}{2}-\frac{v}{2}+1=1$.

Hence, for $v>4$, $(a,\overline{a},\overline{a},a)$ and $(a,\overline{a},a,\overline{a})$ are associated with the largest deviation losses for player 1;

ii. $(\overline{a},a,\overline{a},a)$ and $(\overline{a},a,a,\overline{a})$ are the risk-dominant equilibrium strategy profiles for player 2. As a matter of fact, let us compute the deviation losses of $(a,\overline{a},\overline{a},a)$, $(\overline{a},a,\overline{a},a)$, $(a,\overline{a},a,\overline{a})$ and $(\overline{a},a,a,\overline{a})$ for player 2. Then,

- $(a,\overline{a},\overline{a},a) \rightarrow \frac{v}{2}-\frac{v}{2}+1=1$;
- $(\overline{a},a,\overline{a},a) \rightarrow \frac{v}{2}-1-0=\frac{v}{2}-1$;
- $(a,\overline{a},a,\overline{a}) \rightarrow \frac{v}{2}-\frac{v}{2}+1=1$;
- $(\overline{a},a,a,\overline{a}) \rightarrow \frac{v}{2}-1-0=\frac{v}{2}-1$.

Hence, for $v>4$, $(\overline{a},a,\overline{a},a)$ and $(\overline{a},a,a,\overline{a})$ are associated with the largest deviation losses for player 2;

iii. $(a,\overline{a},a,\overline{a})$ and $(a,\overline{a},a,\overline{a})$ are the risk-dominant equilibrium strategy profiles for player 3. As a matter of fact, let us compute the deviation losses of $(a,\overline{a},\overline{a},a)$, $(\overline{a},a,\overline{a},a)$, $(a,\overline{a},a,\overline{a})$ and $(\overline{a},a,a,\overline{a})$ for player 3. Then,

- $(a,\overline{a},\overline{a},a) \rightarrow \frac{v}{2}-\frac{v}{2}+1=1$;
- $(\overline{a},a,\overline{a},a) \rightarrow \frac{v}{2}-\frac{v}{2}+1=1$;
- $(a,\overline{a},a,\overline{a}) \rightarrow \frac{v}{2}-1-0=\frac{v}{2}-1$;
- $(\overline{a},a,a,\overline{a}) \rightarrow \frac{v}{2}-1-0=\frac{v}{2}-1$.

Hence, for $v>4$, $(a,\overline{a},a,\overline{a})$ and $(a,\overline{a},a,\overline{a})$ are associated with the largest deviation losses for player 3;

iv. $(a,\overline{a},\overline{a},a)$ and $(\overline{a},a,\overline{a},a)$ are the risk-dominant equilibrium strategy profiles for player 4. As a matter of fact, let us compute the deviation losses of $(a,\overline{a},\overline{a},a)$, $(\overline{a},a,\overline{a},a)$, $(a,\overline{a},a,\overline{a})$ and $(\overline{a},a,a,\overline{a})$ for player 4. Then,

- $(a,\overline{a},\overline{a},a) \rightarrow \frac{v}{2}-1-0=\frac{v}{2}-1$;
- $(\overline{a},a,\overline{a},a) \rightarrow \frac{v}{2}-1-0=\frac{v}{2}-1$;

- $(a,\overline{a},a,\overline{a}) \to \frac{v}{2} - \frac{v}{2} + 1 = 1$;
- $(\overline{a},a,a,\overline{a}) \to \frac{v}{2} - \frac{v}{2} + 1 = 1$.

Hence, for $v > 4$, $(a,\overline{a},\overline{a},a)$ and $(\overline{a},a,\overline{a},a)$ are associated with the largest deviation losses for player 4;

- if $2 < v < 4$,

  i. $(\overline{a},a,\overline{a},a)$ and $(\overline{a},a,a,\overline{a})$ are the risk-dominant equilibrium strategy profiles for player 1;

  ii. $(a,\overline{a},\overline{a},a)$ and $(a,\overline{a},a,\overline{a})$ are the risk-dominant equilibrium strategy profiles for player 2;

  iii. $(a,\overline{a},\overline{a},a)$ and $(\overline{a},a,\overline{a},a)$ are the risk-dominant equilibrium strategy profiles for player 3;

  iv. $(a,\overline{a},a,\overline{a})$ and $(\overline{a},a,a,\overline{a})$ are the risk-dominant equilibrium strategy profiles for player 4.

  Clearly this follows from what shown at the previous point for both groups;

- if $v = 2$,

  i. $(\overline{a},a,\overline{a},a)\,,(\overline{a},a,a,\overline{a})$ and $(\overline{a},a,\overline{a},\overline{a})$ are the risk-dominant equilibrium strategy profiles for player 1. As a matter of fact, let us compute the deviation losses of $(a,\overline{a},\overline{a},a)\,,\ (\overline{a},a,\overline{a},a)\,,\ (a,\overline{a},a,\overline{a})\,,\ (\overline{a},a,a,\overline{a})\,,(\overline{a},\overline{a},\overline{a},a)\,,(a,\overline{a},\overline{a},\overline{a})\,,(\overline{a},a,\overline{a},\overline{a})\,,$ $(\overline{a},\overline{a},a,\overline{a})\,,(\overline{a},\overline{a},\overline{a},\overline{a})$ for player 1.

  Then,

  - $(a,\overline{a},\overline{a},a) \to \frac{v}{2} - 1 - 0 = \frac{v}{2} - 1 = 0$;
  - $(\overline{a},a,\overline{a},a) \to \frac{v}{2} - \frac{v}{2} + 1 = 1$;
  - $(a,\overline{a},a,\overline{a}) \to \frac{v}{2} - 1 - 0 = \frac{v}{2} - 1 = 0$;
  - $(\overline{a},a,a,\overline{a}) \to \frac{v}{2} - \frac{v}{2} + 1 = 1$;
  - $(\overline{a},\overline{a},\overline{a},a) \to 0 - \frac{v}{2} + 1 = 0$;
  - $(a,\overline{a},\overline{a},\overline{a}) \to v - 1 - \frac{v}{2} = 0$;
  - $(\overline{a},a,\overline{a},\overline{a}) \to v - v + 1 = 1$;
  - $(\overline{a},\overline{a},a,\overline{a}) \to 0 - \frac{v}{2} + 1 = 0$;

- $(\overline{a},\overline{a},\overline{a},\overline{a}) \rightarrow \frac{v}{2} - v + 1 = 0$.

Hence, for $v = 2$, $(\overline{a},a,\overline{a},a),(\overline{a},a,a,\overline{a}),(\overline{a},a,\overline{a},\overline{a})$ are associated with the largest deviation losses for player 1;

ii. $(a,\overline{a},\overline{a},a),(a,\overline{a},a,\overline{a})$ and $(a,\overline{a},\overline{a},\overline{a})$ are the risk-dominant equilibrium strategy profiles for player 2. As a matter of fact, let us compute the deviation losses of $(a,\overline{a},\overline{a},a)$, $(\overline{a},a,\overline{a},a)$, $(a,\overline{a},a,\overline{a}),(\overline{a},a,a,\overline{a}),(\overline{a},\overline{a},\overline{a},a)$, $(a,\overline{a},\overline{a},\overline{a}),(\overline{a},a,\overline{a},\overline{a})$, $(\overline{a},\overline{a},a,\overline{a}),(\overline{a},\overline{a},\overline{a},\overline{a})$ for player 2. Then,

- $(a,\overline{a},\overline{a},a) \rightarrow \frac{v}{2} - \frac{v}{2} + 1 = 1$;
- $(\overline{a},a,\overline{a},a) \rightarrow \frac{v}{2} - 1 - 0 = \frac{v}{2} - 1 = 0$;
- $(a,\overline{a},a,\overline{a}) \rightarrow \frac{v}{2} - \frac{v}{2} + 1 = 1$;
- $(\overline{a},a,a,\overline{a}) \rightarrow \frac{v}{2} - 1 - 0 = \frac{v}{2} - 1 = 0$;
- $(\overline{a},\overline{a},\overline{a},a) \rightarrow 0 - \frac{v}{2} + 1 = 0$;
- $(a,\overline{a},\overline{a},\overline{a}) \rightarrow v - v + 1 = 1$;
- $(\overline{a},a,\overline{a},\overline{a}) \rightarrow v - 1 - \frac{v}{2} = 0$;
- $(\overline{a},\overline{a},a,\overline{a}) \rightarrow 0 - \frac{v}{2} + 1 = 0$;
- $(\overline{a},\overline{a},\overline{a},\overline{a}) \rightarrow \frac{v}{2} - v + 1 = 0$.

Hence, for $v = 2$, $(a,\overline{a},\overline{a},a),(a,\overline{a},a,\overline{a})$ and $(a,\overline{a},\overline{a},\overline{a})$ are associated with the largest deviation losses for player 2;

iii. $(a,\overline{a},\overline{a},a)$, $(\overline{a},a,\overline{a},a)$ and $(\overline{a},\overline{a},\overline{a},a)$ are the risk-dominant equilibrium strategy profiles for player 3. As a matter of fact, let us compute the deviation losses of $(a,\overline{a},\overline{a},a)$, $(\overline{a},a,\overline{a},a)$, $(a,\overline{a},a,\overline{a}),(\overline{a},a,a,\overline{a}),(\overline{a},\overline{a},\overline{a},a),(a,\overline{a},\overline{a},\overline{a}),(\overline{a},a,\overline{a},\overline{a})$, $(\overline{a},\overline{a},a,\overline{a})$, $(\overline{a},\overline{a},\overline{a},\overline{a})$ for player 3.

Then,

- $(a,\overline{a},\overline{a},a) \rightarrow \frac{v}{2} - \frac{v}{2} + 1 = 1$;
- $(\overline{a},a,\overline{a},a) \rightarrow \frac{v}{2} - \frac{v}{2} + 1 = 1$;
- $(a,\overline{a},a,\overline{a}) \rightarrow \frac{v}{2} - 1 - 0 = \frac{v}{2} - 1 = 0$;
- $(\overline{a},a,a,\overline{a}) \rightarrow \frac{v}{2} - 1 - 0 = \frac{v}{2} - 1 = 0$;
- $(\overline{a},\overline{a},\overline{a},a) \rightarrow v - v + 1 = 1$;
- $(a,\overline{a},\overline{a},\overline{a}) \rightarrow 0 - \frac{v}{2} + 1 = 0$;
- $(\overline{a},a,\overline{a},\overline{a}) \rightarrow 0 - \frac{v}{2} + 1 = 0$;

- $(\overline{a},\overline{a},a,\overline{a}) \to v-1-\frac{v}{2}=0$;
- $(\overline{a},\overline{a},\overline{a},\overline{a}) \to \frac{v}{2}-v+1=0$.

Hence, for $v=2$, $(a,\overline{a},\overline{a},a)$, $(\overline{a},a,\overline{a},a)$ and $(\overline{a},\overline{a},\overline{a},a)$ are associated with the largest deviation losses for player 3;

iv. $(a,\overline{a},a,\overline{a})$, $(\overline{a},a,a,\overline{a})$ and $(\overline{a},\overline{a},a,\overline{a})$ are the risk-dominant equilibrium strategy profiles for player 4. As a matter of fact, let us compute the deviation losses of $(a,\overline{a},\overline{a},a)$, $(\overline{a},a,\overline{a},a)$, $(a,\overline{a},a,\overline{a})$, $(\overline{a},a,a,\overline{a})$, $(\overline{a},\overline{a},\overline{a},a)$, $(a,\overline{a},\overline{a},\overline{a})$, $(\overline{a},a,\overline{a},\overline{a})$, $(\overline{a},\overline{a},a,\overline{a})$, $(\overline{a},\overline{a},\overline{a},\overline{a})$ for player 4.

Then,

- $(a,\overline{a},\overline{a},a) \to \frac{v}{2}-1-0=\frac{v}{2}-1=0$;
- $(\overline{a},a,\overline{a},a) \to \frac{v}{2}-1-0=\frac{v}{2}-1=0$;
- $(a,\overline{a},a,\overline{a}) \to \frac{v}{2}-\frac{v}{2}+1=1$;
- $(\overline{a},a,a,\overline{a}) \to \frac{v}{2}-\frac{v}{2}+1=1$;
- $(\overline{a},\overline{a},\overline{a},a) \to v-1-\frac{v}{2}=0$;
- $(a,\overline{a},\overline{a},\overline{a}) \to 0-\frac{v}{2}+1=0$;
- $(\overline{a},a,\overline{a},\overline{a}) \to 0-\frac{v}{2}+1=0$;
- $(\overline{a},\overline{a},a,\overline{a}) \to v-v+1=1$;
- $(\overline{a},\overline{a},\overline{a},\overline{a}) \to \frac{v}{2}-v+1=0$.

Hence, for $v=2$, $(a,\overline{a},a,\overline{a})$, $(\overline{a},a,a,\overline{a})$ and $(\overline{a},\overline{a},a,\overline{a})$ are associated with the largest deviation losses for player 4;

- overall, there is a one-sided dominance region: for $v<2$, a is a strictly dominated action.

Finally, note that, for $v>4$,

i. $(\overline{a},a,\overline{a},a)$ and $(\overline{a},a,a,\overline{a})$ are the payoff-dominant equilibria for player 1, whereas $(a,\overline{a},\overline{a},a)$ and $(a,\overline{a},a,\overline{a})$ are the risk-dominant equilibrium strategy profiles for player 1;

ii. $(a,\overline{a},\overline{a},a)$ and $(a,\overline{a},a,\overline{a})$ are the payoff-dominant equilibria for player 2, whereas $(\overline{a},a,\overline{a},a)$ and $(\overline{a},a,a,\overline{a})$ are the risk-dominant equilibrium strategy profiles for player 2;

iii. $(a,\overline{a},\overline{a},a)$ and $(\overline{a},a,\overline{a},a)$ are the payoff-dominant equilibria for player 3, whereas $(a,\overline{a},a,\overline{a})$ and $(a,\overline{a},a,\overline{a})$ are the risk-dominant equilibrium strategy profiles for player 3;

iv. $(a, \overline{a}, a, \overline{a})$ and $(\overline{a}, a, a, \overline{a})$ are the payoff-dominant equilibria for player 4, whereas $(a, \overline{a}, \overline{a}, a)$ and $(\overline{a}, a, \overline{a}, a)$ are the risk-dominant equilibrium strategy profiles for player 4.

Hence, there is a tension between payoff-dominance and risk-dominance.

Now let us consider a slight variation of the game above and let:

- the individual **costs of effort** $C(x_j(i)) = c$ with $c \in \mathbb{R}$ and the club good **prize** worth $v > 0$. Thus, costs of effort may be negative, which means that agents could enjoy effort per se, while the prize v is always worth positive utils, so that it is a good. [4.3]

Then, we have the following representation of the game, where player 3 "moves horizontally" and player 4 "moves vertically":

3 a $\overline{a}$

1/2	a	$\overline{a}$
a	$\frac{v}{2}-c; \frac{v}{2}-c; \frac{v}{2}-c; \frac{v}{2}-c$	$\frac{v}{2}-c; \frac{v}{2}; \frac{v}{2}-c; \frac{v}{2}-c$
$\overline{a}$	$\frac{v}{2}; \frac{v}{2}-c; \frac{v}{2}-c; \frac{v}{2}-c$	$0; 0; v-c; v-c$

1/2	a	$\overline{a}$
a	$\frac{v}{2}-c; \frac{v}{2}-c; \frac{v}{2}; \frac{v}{2}-c$	$\frac{v}{2}-c; \frac{v}{2}; \frac{v}{2}; \frac{v}{2}-c$
$\overline{a}$	$\frac{v}{2}; \frac{v}{2}-c; \frac{v}{2}; \frac{v}{2}-c$	$0; 0; v; v-c$

a

1/2	a	$\overline{a}$
a	$\frac{v}{2}-c; \frac{v}{2}-c; \frac{v}{2}-c; \frac{v}{2}$	$\frac{v}{2}-c; \frac{v}{2}; \frac{v}{2}-c; \frac{v}{2}$
$\overline{a}$	$\frac{v}{2}; \frac{v}{2}-c; \frac{v}{2}-c; \frac{v}{2}$	$0; 0; v-c; v$

1/2	a	$\overline{a}$
a	$v-c; v-c; 0; 0$	$v-c; v; 0; 0$
$\overline{a}$	$v; v-c; 0; 0$	$\frac{v}{2}; \frac{v}{2}; \frac{v}{2}; \frac{v}{2}$

$\overline{a}$

4

It is straightforward to derive the following properties:

- if $c < 0$, the unique Nash equilibrium derived by strict dominance is (a, a, a, a);
- if $c > \frac{v}{2}$, the unique Nash equilibrium derived by strict dominance is $(\overline{a}, \overline{a}, \overline{a}, \overline{a})$;
- if $0 < c < \frac{v}{2}$, there are four strict Nash equilibria in pure strategies

$$NE = \{(a, \overline{a}, \overline{a}, a); (\overline{a}, a, \overline{a}, a); (a, \overline{a}, a, \overline{a}); (\overline{a}, a, a, \overline{a})\}$$

and an equilibrium in symmetric strictly mixed strategies $\sigma_i^*(a) = 1 - \frac{2c}{v}$ $\forall i \in \{1, 2, 3, 4\}$;

[4.3] Clearly, under complete information, for $c \in \mathbb{R}_{++}$ and $v \in \mathbb{R}_{++}$, the cost of effort $C_{ij}(x_j(i))$ can always be normalized to one via a simple change of variables.

- if $c = 0$, there are nine Nash equilibria in pure strategies

$$NE = \{(a,\overline{a},\overline{a},a)\,;(\overline{a},a,\overline{a},a)\,;(a,\overline{a},a,\overline{a})\,;(\overline{a},a,a,\overline{a})\,;(a,a,a,a)\} \cup$$
$$\cup \{(a,\overline{a},a,a)\,;(\overline{a},a,a,a)\,;(a,a,a,\overline{a})\,;(a,a,\overline{a},a)\}\,;$$

- if $c = \frac{v}{2}$, there are nine Nash equilibria in pure strategies

$$NE = \{(a,\overline{a},\overline{a},a)\,;(\overline{a},a,\overline{a},a)\,;(a,\overline{a},a,\overline{a})\,;(\overline{a},a,a,\overline{a})\,;(\overline{a},\overline{a},\overline{a},a)\} \cup$$
$$\cup \{(a,\overline{a},\overline{a},\overline{a})\,;(\overline{a},a,\overline{a},\overline{a})\,;(\overline{a},\overline{a},a,\overline{a})\,;(\overline{a},\overline{a},\overline{a},\overline{a})\}$$

- if $\frac{v}{4} < c < \frac{v}{2}$,
  - i. $(\overline{a},a,\overline{a},a)$ and $(\overline{a},a,a,\overline{a})$ are the payoff-dominant equilibria for player 1;
  - ii. $(a,\overline{a},\overline{a},a)$ and $(a,\overline{a},a,\overline{a})$ are the payoff-dominant equilibria for player 2;
  - iii. $(a,\overline{a},\overline{a},a)$ and $(\overline{a},a,\overline{a},a)$ are the payoff-dominant equilibria for player 3;
  - iv. $(a,\overline{a},a,\overline{a})$ and $(\overline{a},a,a,\overline{a})$ are the payoff-dominant equilibria for player 4;
- if $c = 0$, there is no payoff-dominant equilibrium for any player $i \in \{1,2,3,4\}$;
- if $c = \frac{v}{2}$,
  - i. $(\overline{a},a,\overline{a},\overline{a})$ is the payoff-dominant equilibrium for player 1;
  - ii. $(a,\overline{a},\overline{a},\overline{a})$ is the payoff-dominant equilibrium for player 2;
  - iii. $(\overline{a},\overline{a},\overline{a},a)$ is the payoff-dominant equilibrium for player 3;
  - iv. $(\overline{a},\overline{a},a,\overline{a})$ is the payoff-dominant equilibrium for player 4;
- if $0 < c < \frac{v}{4}$,
  - i. $(a,\overline{a},\overline{a},a)$ and $(a,\overline{a},a,\overline{a})$ are the risk-dominant equilibrium strategy profiles for player 1. As a matter of fact, let us compute the deviation losses of $(a,\overline{a},\overline{a},a)$, $(\overline{a},a,\overline{a},a)$, $(a,\overline{a},a,\overline{a})$, $(\overline{a},a,a,\overline{a})$ for player 1. Then,
    - $(a,\overline{a},\overline{a},a) \to \frac{v}{2} - c - 0 = \frac{v}{2} - c$;
    - $(\overline{a},a,\overline{a},a) \to \frac{v}{2} - \frac{v}{2} + c = c$;

- $(a, \overline{a}, a, \overline{a}) \rightarrow \frac{v}{2} - c - 0 = \frac{v}{2} - c$;
- $(\overline{a}, a, a, \overline{a}) \rightarrow \frac{v}{2} - \frac{v}{2} + c = c$.

Hence, for $0 < c < \frac{v}{4}$, $(a, \overline{a}, \overline{a}, a)$ and $(a, \overline{a}, a, \overline{a})$ are associated with the largest deviation losses for player 1;

ii. $(\overline{a}, a, \overline{a}, a)$ and $(\overline{a}, a, a, \overline{a})$ are the risk-dominant equilibrium strategy profiles for player 2. As a matter of fact, let us compute the deviation losses of $(a, \overline{a}, \overline{a}, a)$, $(\overline{a}, a, \overline{a}, a)$, $(a, \overline{a}, a, \overline{a})$ and $(\overline{a}, a, a, \overline{a})$ for player 2. Then,

- $(a, \overline{a}, \overline{a}, a) \rightarrow \frac{v}{2} - \frac{v}{2} + c = c$;
- $(\overline{a}, a, \overline{a}, a) \rightarrow \frac{v}{2} - c - 0 = \frac{v}{2} - c$;
- $(a, \overline{a}, a, \overline{a}) \rightarrow \frac{v}{2} - \frac{v}{2} + c = c$;
- $(\overline{a}, a, a, \overline{a}) \rightarrow \frac{v}{2} - c - 0 = \frac{v}{2} - c.$

Hence, for $0 < c < \frac{v}{4}$, $(\overline{a}, a, \overline{a}, a)$ and $(\overline{a}, a, a, \overline{a})$ are associated with the largest deviation losses for player 2;

iii. $(a, \overline{a}, a, \overline{a})$ and $(a, \overline{a}, a, \overline{a})$ are the risk-dominant equilibrium strategy profiles for player 3. As a matter of fact, let us compute the deviation losses of $(a, \overline{a}, \overline{a}, a)$, $(\overline{a}, a, \overline{a}, a)$, $(a, \overline{a}, a, \overline{a})$ and $(\overline{a}, a, a, \overline{a})$ for player 3. Then,

- $(a, \overline{a}, \overline{a}, a) \rightarrow \frac{v}{2} - \frac{v}{2} + c = c$;
- $(\overline{a}, a, \overline{a}, a) \rightarrow \frac{v}{2} - \frac{v}{2} + c = c$;
- $(a, \overline{a}, a, \overline{a}) \rightarrow \frac{v}{2} - c - 0 = \frac{v}{2} - c;$
- $(\overline{a}, a, a, \overline{a}) \rightarrow \frac{v}{2} - c - 0 = \frac{v}{2} - c$.

Hence, for $0 < c < \frac{v}{4}$, $(a, \overline{a}, a, \overline{a})$ and $(a, \overline{a}, a, \overline{a})$ are associated with the largest deviation losses for player 3;

iv. $(a, \overline{a}, \overline{a}, a)$ and $(\overline{a}, a, \overline{a}, a)$ are the risk-dominant equilibrium strategy profiles for player 4. As a matter of fact, let us compute the deviation losses of $(a, \overline{a}, \overline{a}, a)$, $(\overline{a}, a, \overline{a}, a)$, $(a, \overline{a}, a, \overline{a})$ and $(\overline{a}, a, a, \overline{a})$ for player 4. Then,

- $(a, \overline{a}, \overline{a}, a) \rightarrow \frac{v}{2} - c - 0 = \frac{v}{2} - c$;
- $(\overline{a}, a, \overline{a}, a) \rightarrow \frac{v}{2} - c - 0 = \frac{v}{2} - c$;
- $(a, \overline{a}, a, \overline{a}) \rightarrow \frac{v}{2} - \frac{v}{2} + c = c$;
- $(\overline{a}, a, a, \overline{a}) \rightarrow \frac{v}{2} - \frac{v}{2} + c = c$.

Hence, for $0 < c < \frac{v}{4}$, $(a,\overline{a},\overline{a},a)$ and $(\overline{a},a,\overline{a},a)$ are associated with the largest deviation losses for player 4;

- if $\frac{v}{4} < c < \frac{v}{2}$,

  i. $(\overline{a},a,\overline{a},a)$ and $(\overline{a},a,a,\overline{a})$ are the risk-dominant equilibrium strategy profiles for player 1;

  ii. $(a,\overline{a},\overline{a},a)$ and $(a,\overline{a},a,\overline{a})$ are the risk-dominant equilibrium strategy profiles for player 2;

  iii. $(a,\overline{a},\overline{a},a)$ and $(\overline{a},a,\overline{a},a)$ are the risk-dominant equilibrium strategy profiles for player 3;

  iv. $(a,\overline{a},a,\overline{a})$ and $(\overline{a},a,a,\overline{a})$ are the risk-dominant equilibrium strategy profiles for player 4.

  Clearly this follows from what shown at the previous point for both groups;

- if $c = 0$,

  i. $(a,\overline{a},\overline{a},a)\,,(a,\overline{a},a,\overline{a})$ and $(a,\overline{a},a,a)$ are the risk-dominant equilibrium strategy profiles for player 1. As a matter of fact, let us compute the deviation losses of $(a,\overline{a},\overline{a},a)\,,\,(\overline{a},a,\overline{a},a)\,,\,(a,\overline{a},a,\overline{a})\,,(\overline{a},a,a,\overline{a})\,,(a,a,a,a)\,,$ $(a,\overline{a},a,a)\,,(\overline{a},a,a,a)\,,(a,a,a,\overline{a})\,,(a,a,\overline{a},a)$ for player 1.

     Then,

     - $(a,\overline{a},\overline{a},a) \rightarrow \frac{v}{2} - c - 0 = \frac{v}{2}$;
     - $(\overline{a},a,\overline{a},a) \rightarrow \frac{v}{2} - \frac{v}{2} + c = 0$;
     - $(a,\overline{a},a,\overline{a}) \rightarrow \frac{v}{2} - c - 0 = \frac{v}{2}$;
     - $(\overline{a},a,a,\overline{a}) \rightarrow \frac{v}{2} - \frac{v}{2} + c = 0$;
     - $(a,a,a,a) \rightarrow \frac{v}{2} - c - \frac{v}{2} = 0$;
     - $(a,\overline{a},a,a) \rightarrow \frac{v}{2} - c - 0 = \frac{v}{2}$;
     - $(\overline{a},a,a,a) \rightarrow \frac{v}{2} - \frac{v}{2} + c = 0$;
     - $(a,a,a,\overline{a}) \rightarrow \frac{v}{2} - c - \frac{v}{2} = 0$;
     - $(a,a,\overline{a},a) \rightarrow \frac{v}{2} - c - \frac{v}{2} = 0$.

     Hence, for $c = 0$, $(a,\overline{a},\overline{a},a)\,,(a,\overline{a},a,\overline{a})$ and $(a,\overline{a},a,a)$ are associated with the largest deviation losses for player 1;

ii. $(\overline{a}, a, \overline{a}, a)$, $(\overline{a}, a, a, \overline{a})$ and $(\overline{a}, a, a, a)$ are the risk-dominant equilibrium strategy profiles for player 2. As a matter of fact, let us compute the deviation losses of $(a, \overline{a}, \overline{a}, a)$, $(\overline{a}, a, \overline{a}, a)$, $(a, \overline{a}, a, \overline{a})$, $(\overline{a}, a, a, \overline{a})$, (a, a, a, a), $(a, \overline{a}, a, a)$, $(\overline{a}, a, a, a)$, $(a, a, a, \overline{a})$, $(a, a, \overline{a}, a)$ for player 2.

Then,

- $(a, \overline{a}, \overline{a}, a) \rightarrow \frac{v}{2} - \frac{v}{2} + c = 0$;
- $(\overline{a}, a, \overline{a}, a) \rightarrow \frac{v}{2} - c - 0 = \frac{v}{2}$;
- $(a, \overline{a}, a, \overline{a}) \rightarrow \frac{v}{2} - \frac{v}{2} + c = 0$;
- $(\overline{a}, a, a, \overline{a}) \rightarrow \frac{v}{2} - c - 0 = \frac{v}{2}$;
- $(a, a, a, a) \rightarrow \frac{v}{2} - c - \frac{v}{2} = 0$;
- $(a, \overline{a}, a, a) \rightarrow \frac{v}{2} - \frac{v}{2} + c = 0$;
- $(\overline{a}, a, a, a) \rightarrow \frac{v}{2} - c - 0 = \frac{v}{2}$;
- $(a, a, a, \overline{a}) \rightarrow \frac{v}{2} - c - \frac{v}{2} = 0$;
- $(a, a, \overline{a}, a) \rightarrow \frac{v}{2} - c - \frac{v}{2} = 0$.

Hence, for $c = 0$, $(\overline{a}, a, \overline{a}, a)$, $(\overline{a}, a, a, \overline{a})$ and $(\overline{a}, a, a, a)$ are associated with the largest deviation losses for player 2;

iii. $(a, \overline{a}, a, \overline{a})$, $(\overline{a}, a, a, \overline{a})$ and $(a, a, a, \overline{a})$ are the risk-dominant equilibrium strategy profiles for player 3. As a matter of fact, let us compute the deviation losses of $(a, \overline{a}, \overline{a}, a)$, $(\overline{a}, a, \overline{a}, a)$, $(a, \overline{a}, a, \overline{a})$, $(\overline{a}, a, a, \overline{a})$, (a, a, a, a), $(a, \overline{a}, a, a)$, $(\overline{a}, a, a, a)$, $(a, a, a, \overline{a})$, $(a, a, \overline{a}, a)$ for player 3.

Then,

- $(a, \overline{a}, \overline{a}, a) \rightarrow \frac{v}{2} - \frac{v}{2} + c = 0$;
- $(\overline{a}, a, \overline{a}, a) \rightarrow \frac{v}{2} - \frac{v}{2} + c = 0$;
- $(a, \overline{a}, a, \overline{a}) \rightarrow \frac{v}{2} - c - 0 = \frac{v}{2}$;
- $(\overline{a}, a, a, \overline{a}) \rightarrow \frac{v}{2} - c - 0 = \frac{v}{2}$;
- $(a, a, a, a) \rightarrow \frac{v}{2} - c - \frac{v}{2} = 0$;
- $(a, \overline{a}, a, a) \rightarrow \frac{v}{2} - c - \frac{v}{2} = 0$;
- $(\overline{a}, a, a, a) \rightarrow \frac{v}{2} - c - \frac{v}{2} = 0$;
- $(a, a, a, \overline{a}) \rightarrow \frac{v}{2} - c - 0 = \frac{v}{2}$;
- $(a, a, \overline{a}, a) \rightarrow \frac{v}{2} - \frac{v}{2} + c = 0$.

Hence, for $c = 0$, $(a, \overline{a}, a, \overline{a})$, $(\overline{a}, a, a, \overline{a})$ and $(a, a, a, \overline{a})$ are associated with the largest deviation losses for player 3;

iv. $(a, \overline{a}, \overline{a}, a)$, $(\overline{a}, a, \overline{a}, a)$ and $(a, a, \overline{a}, a)$ are the risk-dominant equilibrium strategy profiles for player 4. As a matter of fact, let us compute the deviation losses of $(a, \overline{a}, \overline{a}, a)$, $(\overline{a}, a, \overline{a}, a)$, $(a, \overline{a}, a, \overline{a})$, $(\overline{a}, a, a, \overline{a})$, (a, a, a, a) , $(a, \overline{a}, a, a)$, $(\overline{a}, a, a, a)$, $(a, a, a, \overline{a})$, $(a, a, \overline{a}, a)$ for player 4.

Then,

- $(a, \overline{a}, \overline{a}, a) \rightarrow \frac{v}{2} - c - 0 = \frac{v}{2}$;
- $(\overline{a}, a, \overline{a}, a) \rightarrow \frac{v}{2} - c - 0 = \frac{v}{2}$;
- $(a, \overline{a}, a, \overline{a}) \rightarrow \frac{v}{2} - \frac{v}{2} + c = 0$;
- $(\overline{a}, a, a, \overline{a}) \rightarrow \frac{v}{2} - \frac{v}{2} + c = 0$;
- $(a, a, a, a) \rightarrow \frac{v}{2} - c - \frac{v}{2} = 0$;
- $(a, \overline{a}, a, a) \rightarrow \frac{v}{2} - c - \frac{v}{2} = 0$;
- $(\overline{a}, a, a, a) \rightarrow \frac{v}{2} - c - \frac{v}{2} = 0$;
- $(a, a, a, \overline{a}) \rightarrow \frac{v}{2} - \frac{v}{2} + c = 0$;
- $(a, a, \overline{a}, a) \rightarrow \frac{v}{2} - c - 0 = \frac{v}{2}$.

Hence, for $c = 0$, $(a, \overline{a}, \overline{a}, a)$, $(\overline{a}, a, \overline{a}, a)$ and $(a, a, \overline{a}, a)$ are associated with the largest deviation losses for player 4;

- if $c = \frac{v}{2}$,

i. $(\overline{a}, a, \overline{a}, a)$, $(\overline{a}, a, a, \overline{a})$ and $(\overline{a}, a, \overline{a}, \overline{a})$ are the risk-dominant equilibrium strategy profiles for player 1. As a matter of fact, let us compute the deviation losses of $(a, \overline{a}, \overline{a}, a)$, $(\overline{a}, a, \overline{a}, a)$, $(a, \overline{a}, a, \overline{a})$, $(\overline{a}, a, a, \overline{a})$, $(\overline{a}, \overline{a}, \overline{a}, a)$, $(a, \overline{a}, \overline{a}, \overline{a})$, $(\overline{a}, a, \overline{a}, \overline{a})$, $(\overline{a}, \overline{a}, a, \overline{a})$, $(\overline{a}, \overline{a}, \overline{a}, \overline{a})$ for player 1.

Then,

- $(a, \overline{a}, \overline{a}, a) \rightarrow \frac{v}{2} - c - 0 = \frac{v}{2} - \frac{v}{2} = 0$;
- $(\overline{a}, a, \overline{a}, a) \rightarrow \frac{v}{2} - \frac{v}{2} + c = \frac{v}{2}$;
- $(a, \overline{a}, a, \overline{a}) \rightarrow \frac{v}{2} - c - 0 = \frac{v}{2} - \frac{v}{2} = 0$;
- $(\overline{a}, a, a, \overline{a}) \rightarrow \frac{v}{2} - \frac{v}{2} + c = \frac{v}{2}$;
- $(\overline{a}, \overline{a}, \overline{a}, a) \rightarrow 0 - \frac{v}{2} + c = 0$;

- $(a, \overline{a}, \overline{a}, \overline{a}) \rightarrow v - c - \frac{v}{2} = 0$;
- $(\overline{a}, a, \overline{a}, \overline{a}) \rightarrow v - v + c = \frac{v}{2}$;
- $(\overline{a}, \overline{a}, a, \overline{a}) \rightarrow 0 - \frac{v}{2} + c = 0$;
- $(\overline{a}, \overline{a}, \overline{a}, \overline{a}) \rightarrow \frac{v}{2} - v + c = 0$.

Hence, for $c = \frac{v}{2}$, $(\overline{a}, a, \overline{a}, a), (\overline{a}, a, a, \overline{a}), (\overline{a}, a, \overline{a}, \overline{a})$ are associated with the largest deviation losses for player 1;

ii. $(a, \overline{a}, \overline{a}, a)$, $(a, \overline{a}, a, \overline{a})$ and $(a, \overline{a}, \overline{a}, \overline{a})$ are the risk-dominant equilibrium strategy profiles for player 2. As a matter of fact, let us compute the deviation losses of $(a, \overline{a}, \overline{a}, a), (\overline{a}, a, \overline{a}, a)$, $(a, \overline{a}, a, \overline{a}), (\overline{a}, a, a, \overline{a}), (\overline{a}, \overline{a}, \overline{a}, a)$, $(a, \overline{a}, \overline{a}, \overline{a}), (\overline{a}, a, \overline{a}, \overline{a})$, $(\overline{a}, \overline{a}, a, \overline{a}), (\overline{a}, \overline{a}, \overline{a}, \overline{a})$ for player 2. Then,

- $(a, \overline{a}, \overline{a}, a) \rightarrow \frac{v}{2} - \frac{v}{2} + c = \frac{v}{2}$;
- $(\overline{a}, a, \overline{a}, a) \rightarrow \frac{v}{2} - c - 0 = \frac{v}{2} - c = 0$;
- $(a, \overline{a}, a, \overline{a}) \rightarrow \frac{v}{2} - \frac{v}{2} + c = \frac{v}{2}$;
- $(\overline{a}, a, a, \overline{a}) \rightarrow \frac{v}{2} - c - 0 = \frac{v}{2} - c = 0$;
- $(\overline{a}, \overline{a}, \overline{a}, a) \rightarrow 0 - \frac{v}{2} + c = 0$;
- $(a, \overline{a}, \overline{a}, \overline{a}) \rightarrow v - v + c = \frac{v}{2}$;
- $(\overline{a}, a, \overline{a}, \overline{a}) \rightarrow v - c - \frac{v}{2} = 0$;
- $(\overline{a}, \overline{a}, a, \overline{a}) \rightarrow 0 - \frac{v}{2} + c = 0$;
- $(\overline{a}, \overline{a}, \overline{a}, \overline{a}) \rightarrow \frac{v}{2} - v + c = 0$.

Hence, for $c = \frac{v}{2}$, $(a, \overline{a}, \overline{a}, a), (a, \overline{a}, a, \overline{a})$ and $(a, \overline{a}, \overline{a}, \overline{a})$ are associated with the largest deviation losses for player 2;

iii. $(a, \overline{a}, \overline{a}, a)$, $(\overline{a}, a, \overline{a}, a)$ and $(\overline{a}, \overline{a}, \overline{a}, a)$ are the risk-dominant equilibrium strategy profiles for player 3. As a matter of fact, let us compute the deviation losses of $(a, \overline{a}, \overline{a}, a)$, $(\overline{a}, a, \overline{a}, a)$, $(a, \overline{a}, a, \overline{a}), (\overline{a}, a, a, \overline{a}), (\overline{a}, \overline{a}, \overline{a}, a), (a, \overline{a}, \overline{a}, \overline{a}), (\overline{a}, a, \overline{a}, \overline{a})$, $(\overline{a}, \overline{a}, a, \overline{a}), (\overline{a}, \overline{a}, \overline{a}, \overline{a})$ for player 3.

Then,

- $(a, \overline{a}, \overline{a}, a) \rightarrow \frac{v}{2} - \frac{v}{2} + c = c$;
- $(\overline{a}, a, \overline{a}, a) \rightarrow \frac{v}{2} - \frac{v}{2} + c = c$;
- $(a, \overline{a}, a, \overline{a}) \rightarrow \frac{v}{2} - c - 0 = \frac{v}{2} - c = 0$;
- $(\overline{a}, a, a, \overline{a}) \rightarrow \frac{v}{2} - c - 0 = \frac{v}{2} - c = 0$;

- $(\overline{a}, \overline{a}, \overline{a}, a) \to v - v + c = c$;
- $(a, \overline{a}, \overline{a}, \overline{a}) \to 0 - \frac{v}{2} + c = 0$;
- $(\overline{a}, a, \overline{a}, \overline{a}) \to 0 - \frac{v}{2} + c = 0$;
- $(\overline{a}, \overline{a}, a, \overline{a}) \to v - c - \frac{v}{2} = 0$;
- $(\overline{a}, \overline{a}, \overline{a}, \overline{a}) \to \frac{v}{2} - v + c = 0$.

Hence, for $c = \frac{v}{2}$, $(a, \overline{a}, \overline{a}, a)$, $(\overline{a}, a, \overline{a}, a)$ and $(\overline{a}, \overline{a}, \overline{a}, a)$ are associated with the largest deviation losses for player 3;

iv. $(a, \overline{a}, a, \overline{a})$, $(\overline{a}, a, a, \overline{a})$ and $(\overline{a}, \overline{a}, a, \overline{a})$ are the risk-dominant equilibrium strategy profiles for player 4. As a matter of fact, let us compute the deviation losses of $(a, \overline{a}, \overline{a}, a)$, $(\overline{a}, a, \overline{a}, a)$, $(a, \overline{a}, a, \overline{a})$, $(\overline{a}, a, a, \overline{a})$, $(\overline{a}, \overline{a}, \overline{a}, a)$, $(a, \overline{a}, \overline{a}, \overline{a})$, $(\overline{a}, a, \overline{a}, \overline{a})$, $(\overline{a}, \overline{a}, a, \overline{a})$, $(\overline{a}, \overline{a}, \overline{a}, \overline{a})$ for player 4.

Then,

- $(a, \overline{a}, \overline{a}, a) \to \frac{v}{2} - c - 0 = \frac{v}{2} - c = 0$;
- $(\overline{a}, a, \overline{a}, a) \to \frac{v}{2} - c - 0 = \frac{v}{2} - c = 0$;
- $(a, \overline{a}, a, \overline{a}) \to \frac{v}{2} - \frac{v}{2} + c = \frac{v}{2}$;
- $(\overline{a}, a, a, \overline{a}) \to \frac{v}{2} - \frac{v}{2} + c = \frac{v}{2}$;
- $(\overline{a}, \overline{a}, \overline{a}, a) \to v - c - \frac{v}{2} = 0$;
- $(a, \overline{a}, \overline{a}, \overline{a}) \to 0 - \frac{v}{2} + c = 0$;
- $(\overline{a}, a, \overline{a}, \overline{a}) \to 0 - \frac{v}{2} + c = 0$;
- $(\overline{a}, \overline{a}, a, \overline{a}) \to v - v + c = \frac{v}{2}$;
- $(\overline{a}, \overline{a}, \overline{a}, \overline{a}) \to \frac{v}{2} - v + c = 0$.

Hence, for $c = \frac{v}{2}$, $(a, \overline{a}, a, \overline{a})$, $(\overline{a}, a, a, \overline{a})$ and $(\overline{a}, \overline{a}, a, \overline{a})$ are associated with the largest deviation losses for player 4;

- overall, there are two dominance regions: for $c > \frac{v}{2}$, a is a strictly dominated action; for $c < 0$, $\overline{a}$ ia a strictly dominated action.

Finally, note that, for $0 < c < \frac{v}{4}$,

i. $(\overline{a}, a, \overline{a}, a)$ and $(\overline{a}, a, a, \overline{a})$ are the payoff-dominant equilibria for player 1, whereas $(a, \overline{a}, \overline{a}, a)$ and $(a, \overline{a}, a, \overline{a})$ are the risk-dominant equilibrium strategy profiles for player 1;

ii. $(a, \overline{a}, \overline{a}, a)$ and $(a, \overline{a}, a, \overline{a})$ are the payoff-dominant equilibria for player 2, whereas $(\overline{a}, a, \overline{a}, a)$ and $(\overline{a}, a, a, \overline{a})$ are the risk-dominant equilibrium strategy profiles for player 2;

iii. $(a, \overline{a}, \overline{a}, a)$ and $(\overline{a}, a, \overline{a}, a)$ are the payoff-dominant equilibria for player 3, whereas $(a, \overline{a}, a, \overline{a})$ and $(a, \overline{a}, a, \overline{a})$ are the risk-dominant equilibrium strategy profiles for player 3;

iv. $(a, \overline{a}, a, \overline{a})$ and $(\overline{a}, a, a, \overline{a})$ are the payoff-dominant equilibria for player 4, whereas $(a, \overline{a}, \overline{a}, a)$ and $(\overline{a}, a, \overline{a}, a)$ are the risk-dominant equilibrium strategy profiles for player 4.

Hence, there is a tension between payoff-dominance and risk-dominance.

Introducing Incomplete Information à la Global Games

We would like to draw a possible comparison with the classical example due to Carlsson and Damme (1993a) about a 2×2 game under complete information, reported in table 4.1.

	α_2	β_2
α_1	x, x	$x, 0$
β_1	$0, x$	$4, 4$

Table 4.1: Game $g(x)$ by Carlsson and Damme (1993a) .

Carlsson and Damme (1993a) highlight the following properties of this game under complete information:

- if $x > 4$, the unique Nash equilibrium derived by strict dominance is (α_1, α_2);
- if $x < 0$, the unique Nash equilibrium derived by strict dominance is (β_1, β_2);
- if $0 < x < 4$, there are two strict Nash equilibria, that is (α_1, α_2) and (β_1, β_2);
- if $x \in (2, 4)$, (α_1, α_2) is the risk-dominant equilibrium;
- if $x \in (0, 2)$, (β_1, β_2) is the risk-dominant equilibrium;
- overall, there are two dominance regions.

Finally, note that, for $2 < x < 4$, (β_1, β_2) is the payoff-dominant equilibrium, whereas (α_1, α_2) is the risk-dominant equilibrium: there is a tension between payoff-dominance and risk-dominance.

4.3.2 Incomplete Information about the Prize

Let us consider the case where the individual **costs of effort** is $C_{ij}\left(x_j\left(i\right)\right) = x_j\left(i\right)$. Henceforth, we refer to this game as $g\left(v\right)$. We closely follow Carlsson and Damme (1993a) introducing incomplete information about the prize v as follows:

- let V be a random variable which is uniform on the real line, i.e. including the dominance region and the threshold for the risk-dominance;
- given the unknown realization v of V, each player $i \in \{1, 2, 3, 4\}$ idiosyncratically observes a the realization v_i of signal V_i, uniform on $[v - \varepsilon, v + \varepsilon]$ for some $\varepsilon > 0$, so that the players' observation errors $V_1 - v$, $V_2 - v$, $V_3 - v$ and $V_4 - v$ are independent;
- after these idiosyncratic observations, each player $i \in \{1, 2, 3, 4\}$ simultaneously and independently decides whether to exert effort or not and gets a payoff as described by the strategic form game $g\left(v\right)$;
- note that $E\left(V|v_i\right) = v_i$, if i observes $v_i \in [\underline{v} + \varepsilon, \overline{v} - \varepsilon]$ so that $V|v_i \sim U\left(v_i - \varepsilon, v_i + \varepsilon\right)$;
- furthermore, for $v_i \in [\underline{v} + \varepsilon, \overline{v} - \varepsilon]$, the conditional distribution of the teammate's or opponents' observation will be centered around v_i with support $[v_i - 2\varepsilon, v_i + 2\varepsilon]$. Hence, $Prob\left[V_{-i} < v_i|v_i\right] = Prob[V_{-i} > v_i|v_i] = \frac{1}{2}$. Note that $V_{-i}|V_i = v_i$ is distributed as a triangular random variable over $[v_i - 2\varepsilon, v_i + 2\varepsilon]$.

Now, let us further assume v_i in $[\underline{v} + 2\varepsilon, \overline{v} - 2\varepsilon]$ so that second-order beliefs are well-defined and suppose player $i \in \{1, 2, 3, 4\}$ observes $v_i < 2$. Then, i's conditionally expected payoff from exerting effort, that is choosing a, is smaller than the one from exerting no effort, that is choosing $\overline{a}$. Accordingly, $\overline{a}$ is a conditionally strictly dominant action for player $i \in \{1, 2, 3, 4\}$ whenever she observes $v_i < 2$. Note that for player 1, for instance, the expected payoff difference $\pi_1(\overline{a}|v_1) - \pi_1(a|v_1) = 1 - Prob(\overline{a}_2|v_1) \cdot v_1/2$. Then, we show that it is not possible to iterate this dominance argument. If players $-i \in \{2, 3, 4\}$ are forced to play $\overline{a}$ whenever they observe $v_{-i} < 2$, then player i, observing $v_i = 2$ has to assign at least probability $\frac{1}{2}$ to $\overline{a}_{-i}$. Thus, i's conditionally expected payoff difference from not exerting effort versus exerting effort, i.e. choosing $\overline{a}_i$ over a_i, will be between 0 and $\frac{1}{2}$, so that a_i cannot be discarded by iterated strict dominance for $v_i = 2$. Let us now assume players follow monotonic switching strategies around a cutoff $k = 4$ such that

$$x_i(v_i) = \begin{cases} 1 & \text{if } v_i > 4 \\ 0 & \text{if } v_i \leq 4\ . \end{cases}$$

Note that $v_i = 4$ is the threshold for the risk-dominance regions as well. When $v_i = k = 4$, the conditionally expected payoff from exerting effort equals

$$\frac{1}{8}\left(\frac{4}{2}-1\right)+\frac{1}{8}\left(\frac{4}{2}-1\right)+\frac{1}{8}\left(\frac{4}{2}-1\right)+\frac{1}{8}\left(\frac{4}{2}-1\right)+\frac{1}{8}\left(\frac{4}{2}-1\right)+\frac{1}{8}\left(\frac{4}{2}-1\right)+\frac{1}{8}(4-1)+\frac{1}{8}(4-1)=\frac{3}{2},$$

while the conditionally expected payoff from not exerting effort equals

$$\frac{1}{8}\left(\frac{4}{2}\right)+\frac{1}{8}*0+\frac{1}{8}\left(\frac{4}{2}\right)+\frac{1}{8}*0+\frac{1}{8}\left(\frac{4}{2}\right)+\frac{1}{8}*0+\frac{1}{8}(4)+\frac{1}{8}\left(\frac{4}{2}\right)=\frac{3}{2}.$$

Note that

$$Prob(V_{-i} \leq 4|V_i = v_i) = \begin{cases} 0 & \text{if } 4 \leq v_i - 2\varepsilon \\ \dfrac{(4-v_i+2\varepsilon)^2}{8\varepsilon^2} & \text{if } v_i - 2\varepsilon < 4 \leq v_i \\ 1 - \dfrac{(v_i-4+2\varepsilon)^2}{8\varepsilon^2} & \text{if } v_i < 4 \leq v_i + 2\varepsilon \\ 1 & \text{if } 4 \geq v_i + 2\varepsilon \end{cases}$$

Assume $i = 1$ without loss of generality. Then incentive-compatibility for a switching strategy around cutoff $k = 4$ requires

$$\pi_1(\overline{a}|v_1) - \pi_1(a|v_1) = 1 - Prob(V_2 \leq 4|V_1 = v_1)\frac{v_1}{2} \geq 0 \qquad \forall\ 2 \leq v_1 \leq 4$$
$$\pi_1(a|v_1) - \pi_1(\overline{a}|v_1) = Prob(V_2 \leq 4|V_1 = v_1)\frac{v_1}{2} - 1 \geq 0 \qquad \forall\ v_1 > 4$$

Accordingly $\forall v_1 \in [2,4]$,

$$1 - Prob(V_2 \leq 4|V_1 = v_1)\frac{v_1}{2} \geq 0 \Leftrightarrow 1 - \left(1 - \frac{(v_1-4+2\varepsilon)^2}{8\varepsilon^2}\right)\frac{v_1}{2} \geq 0 \Leftrightarrow$$

$$\varepsilon \geq \frac{(v_1-4)^2}{2\left(4-\sqrt{2}\sqrt{\frac{(v_1-4)^2(v_1-2)}{v_1}}-v_1\right)} \ ,$$

and $\forall\ v_1 \in (4, \overline{v}-2\varepsilon]$

$$Prob(V_2 \leq 4|V_1 = v_1)\frac{v_1}{2} - 1 \geq 0 \Leftrightarrow \frac{(4-v_1+2\varepsilon)^2}{8\varepsilon^2}\cdot\frac{v_1}{2} - 1 \Leftrightarrow$$

$$\Leftrightarrow \varepsilon \geq \frac{(v_1-4)^2}{2\left(v_1-2\left(2+\sqrt{\frac{(v_1-4)^2}{v_1}}\right)\right)} \ .$$

Both thresholds are increasing in v_1, so for $v_1 \to \infty$, $\varepsilon \to \infty$ and it does not exist a finite value of ε ensuring incentive compatibility for all possible realizable signals v_1. Note that the argument above is independent of the choice of k. The cutoff $v_i^* = 4$ solves

$$\frac{1}{8}\left(\frac{v_i}{2}-1\right)+\frac{1}{8}\left(\frac{v_i}{2}-1\right)+\frac{1}{8}\left(\frac{v_i}{2}-1\right)+\frac{1}{8}\left(\frac{v_i}{2}-1\right)+\frac{1}{8}\left(\frac{v_i}{2}-1\right)+\frac{1}{8}\left(\frac{v_i}{2}-1\right)+\frac{1}{8}\left(v_i-1\right)+\frac{1}{8}\left(v_i-1\right)=$$

$$=\frac{1}{8}\left(\frac{v_i}{2}\right)+\frac{1}{8}*0+\frac{1}{8}\left(\frac{v_i}{2}\right)+\frac{1}{8}*0+\frac{1}{8}\left(\frac{v_i}{2}\right)+\frac{1}{8}*0+\frac{1}{8}\left(v_i\right)+\frac{1}{8}\left(\frac{v_i}{2}\right).$$

In contrast with what we showed in Chapter 3 for the weakest-link impact function, it is not possible to establish the existence of an equilibrium robust to incomplete information in the sense of Kajii and Morris (1997) in which no player exerts effort, since $x_i\left(v_i\right) = 1$ is not a dominated action $\forall v_i \in [\underline{v}+\varepsilon, \overline{v}-\varepsilon]$ and $\forall i \in \{1,2,3,4\}$.

Hence, we conclude that in $g\left(v\right)$ under incomplete information à la global games, there are no equilibria in (monotonic) cutoff strategies, such that $\forall i \in \{1,2,3,4\}$:

$$x_i^*\left(v_i\right) = \begin{cases} 1 & \text{if } v_i > k \\ 0 & \text{if } v_i \leq k\ . \end{cases}$$

4.3.3 Incomplete Information about the Cost of Effort

Let us consider the case where the individual **costs of effort** is $C\left(x_j\left(i\right)\right) = c$ with $c \in \mathbb{R}$ and the club good **prize** worth $v > 0$. Henceforth, we refer to this game as $g\left(c\right)$. We closely follow Carlsson and Damme (1993a) introducing incomplete information about the cost of effort c

as follows:

- let C be a random variable which is uniform on the real line, clearly including both dominance regions;
- given the unknown realization c of C, each player $i \in \{1, 2, 3, 4\}$ idiosyncratically observes the realization c_i of a signal C_i, uniform on $[c - \varepsilon, c + \varepsilon]$ for some $\varepsilon > 0$, so that the players' observation errors $C_1 - c$, $C_2 - c$, $C_3 - c$ and $C_4 - c$ are independent;
- after these idiosyncratic observations, each player $i \in \{1, 2, 3, 4\}$ simultaneously and independently decides whether to exert effort or not and gets a payoff as described by the strategic form game $g(c)$;
- note that $E(C|c_i) = c_i$, if i observes $c_i \in [\underline{c} + \varepsilon, \overline{c} - \varepsilon]$ so that $C|c_i \sim U(c_i - \varepsilon, c_i + \varepsilon)$;
- furthermore, for $c_i \in [\underline{c} + \varepsilon, \overline{c} - \varepsilon]$, the conditional distribution of the teammate's or opponents' observation will be centered around c_i with support $[c_i - 2\varepsilon, c_i + 2\varepsilon]$. Hence, $Prob[C_{-i} < c_i|c_i] = Prob[C_{-i} > c_i|c_i] = \frac{1}{2}$Note that $C_{-i}|C_i = c_i$ is distributed as a triangular random variable over $[c_i - 2\varepsilon, c_i + 2\varepsilon]$.

Suppose player $i \in \{1, 2, 3, 4\}$ observes $c_i > \frac{v}{2}$. Then, i's conditionally expected payoff from exerting effort, that is choosing a, is smaller than the one from exerting no effort, that is choosing $\overline{a}$. Accordingly, $\overline{a}$ is a conditionally strictly dominant action for player $i \in \{1, 2, 3, 4\}$ whenever she observes $c_i > \frac{v}{2}$. Suppose $i = 1$ without loss of generality. Iterating this dominance argument, if players $-i \in \{2, 3, 4\}$ are forced to play $\overline{a}$ whenever they observe $c_{-i} > \frac{v}{2}$, then player i, observing $c_i = \frac{v}{2}$ has to assign at least probability $\frac{1}{2}$ to $\overline{a}_i$. Note that for player 1, for instance, the expected payoff difference from not exerting versus exerting effort equals $\pi_1(\overline{a}|c_1) - \pi_1(a|c_1) = c_1 - Prob(\overline{a}_2|c_1) \cdot v/2$. Thus, i's conditionally expected payoff difference from not exerting versus exerting effort will span from 0 to $v/4$, that is a_i cannot be discarded by iterated strict dominance at $c_i = v/2$. Similarly, for $c_i < 0$ it is strictly dominant to exert effort for any player. Nonetheless, iterated strict-dominance cannot be extended above $c_i = 0$: at $c_i = 0$ the expected payoff difference spans from 0 to $v/4$. Let us now assume players follow monotonic switching strategies around a cutoff $k = v/4$ such that

$$x_i\left(c_i\right) = \begin{cases} 1 & \text{if } c_i < \frac{v}{4} \\ 0 & \text{if } c_i \geq \frac{v}{4} \ . \end{cases}$$

Note that $c_i = v/4$ is the threshold for the risk-dominance regions as well. When $c_i = k = v/4$, the conditionally expected payoff from exerting effort equals

$$\frac{1}{8}\left(\frac{v}{2}-\frac{v}{4}\right)+\frac{1}{8}\left(\frac{v}{2}-\frac{v}{4}\right)+\frac{1}{8}\left(\frac{v}{2}-\frac{v}{4}\right)+\frac{1}{8}\left(\frac{v}{2}-\frac{v}{4}\right)+\frac{1}{8}\left(\frac{v}{2}-\frac{v}{4}\right)+\frac{1}{8}\left(\frac{v}{2}-\frac{v}{4}\right)+$$
$$+\frac{1}{8}\left(v-\frac{v}{4}\right)+\frac{1}{8}\left(v-\frac{v}{4}\right)=\frac{3}{8}v$$

and the conditionally expected payoff from not exerting effort equals

$$\frac{1}{8}\left(\frac{v}{2}\right)+\frac{1}{8}*0+\frac{1}{8}\left(\frac{v}{2}\right)+\frac{1}{8}*0+\frac{1}{8}\left(\frac{v}{2}\right)+\frac{1}{8}*0+\frac{1}{8}\left(v\right)+\frac{1}{8}\left(\frac{v}{2}\right)=\frac{3}{8}v \ .$$

Note that

$$Prob(C_{-i} \geq v/4 | C_i = c_i) = \begin{cases} 1 & \text{if } v/4 \leq c_i - 2\varepsilon \\ 1 - \dfrac{(v/4 - c_i + 2\varepsilon)^2}{8\varepsilon^2} & \text{if } c_i - 2\varepsilon < v/4 \leq c_i \\ \dfrac{(c_i - v/4 + 2\varepsilon)^2}{8\varepsilon^2} & \text{if } c_i < v/4 < c_i + 2\varepsilon \\ 0 & \text{if } v/4 > c_i + 2\varepsilon \end{cases}$$

Assume $i = 1$ without loss of generality. Then incentive-compatibility for a switching strategy around cutoff $k = v/4$ requires

$$\pi_1(\overline{a}|c_1) - \pi_1(a|c_1) = c_1 - Prob\left(C_2 \geq v/4 \mid C_1 = c_1\right)\frac{v}{2} \geq 0 \quad \forall\ v/4 \leq c_1 \leq v/2\ ,$$
$$\pi_1(a|v_1) - \pi_1(\overline{a}|v_1) = Prob(C_2 \geq v/4 \mid C_1 = c_1)\frac{v}{2} - c_1 \geq 0 \quad \forall\ 0 \leq c_1 < v/4\ .$$

Then, $\forall c_1 \in [v/4, v/2]$

$$c_1 - Prob(C_2 \geq 4 | C_1 = c_1)\frac{v}{2} \geq 0 \Leftrightarrow c_1 - \left(1 - \frac{\left(v/4 - c_1 + 2\varepsilon\right)^2}{8\varepsilon^2}\right)\frac{v}{2} \geq 0 \Leftrightarrow$$

$$\Leftrightarrow \varepsilon \geq \lim_{c_1 \to v/4^+} \frac{(v-4c_1)^2}{8\left(4c_1 - v - \sqrt{2}\sqrt{\frac{(v-4c_1)^2(v-2c_1)}{v}}\right)} = \frac{v}{4} \ .$$

and $\forall c_1 \in [0, v/4)$

$$Prob(C_2 \geq 4|C_1 = c_1)\frac{v}{2} - c_1 \geq 0 \Leftrightarrow \left(\frac{(c_i - v/4 + 2\varepsilon)^2}{8\varepsilon^2}\right)\frac{v}{2} - c_1 \geq 0$$

$$\Leftrightarrow \varepsilon \geq \lim_{c_1 \to v/4^-} \frac{(v-4c_1)^2}{8\left(v - 4c_1 - 2\sqrt{\frac{c_1(v-4c_1)^2}{v}}\right)} = \frac{v}{4} \ .$$

Effort contributions are perfect substitutes within groups so that, if player 1 were sure that player 2 would provide effort, then for player 1 it would be strictly dominant not to provide effort. A sufficiently large amount of noise, i.e., allows strategic uncertainty to hold even far away from the cutoff point. Establishing whether $k = v/4$ is the unique possible cutoff for an equilibrium in (monotonic) switching strategies would require to find the set of fixed-points of the expected payoff difference for any possible k. The cutoff $c_i^* = \frac{v}{4}$ solves

$$\frac{1}{8}\left(\frac{v}{2} - c_i\right) + \frac{1}{8}\left(\frac{v}{2} - c_i\right) + \frac{1}{8}\left(\frac{v}{2} - c_i\right) + \frac{1}{8}\left(\frac{v}{2} - c_i\right) + \frac{1}{8}\left(\frac{v}{2} - c_i\right) + \frac{1}{8}\left(\frac{v}{2} - c_i\right) + \frac{1}{8}(v - c_i) + \frac{1}{8}(v - c_i) =$$

$$= \frac{1}{8}\left(\frac{v}{2}\right) + \frac{1}{8} * 0 + \frac{1}{8}\left(\frac{v}{2}\right) + \frac{1}{8} * 0 + \frac{1}{8}\left(\frac{v}{2}\right) + \frac{1}{8} * 0 + \frac{1}{8}(v) + \frac{1}{8}\left(\frac{v}{2}\right).$$

Hence, we conclude that in $g(c)$ under incomplete information à la global games, for sufficiently large ε, i.e. $\varepsilon \geq v/4$, there exists an equilibrium in switching strategies such that $\forall i \in \{1, 2, 3, 4\}$

$$x_i^*(c_i) = \begin{cases} 1 & \text{if } c_i < \frac{v}{4} \\ 0 & \text{if } c_i \geq \frac{v}{4} \ . \end{cases}$$

4.3.4 Observations

Overall, we can state some general points from the example above:

- under complete information, there are multiple Nash equilibria in pure strategies in a max-max two-group four-player contest with binary actions and a public good prize.
- In both examples we highlight a tension between payoff-dominance and risk-dominance,

as in the example due to Carlsson and Damme (1993a).

- Assuming incomplete information à la global games induces the existence of an equilibrium in (monotonic) switching strategies, whose cutoff coincides with the one of the risk-dominance region, only if incomplete information is modeled about the cost of exerting effort. The argument is shown by a "guess-and-solve" approach.
- Equilibrium selection happens for a sufficiently large amount of noise, due to the presence of within-group substitutability in effort provision.
- Iterated-deletion of strictly dominated strategies does not select a unique equilibrium in monotonic switching strategies, no matter whether complete information is relaxed on the prize contested or on the cost of exerting effort. The combinations of dominance regions and within-group perfect complementarities are crucial for the equilibrium existence and uniqueness in max-min group contests as shown in Chapter 3.
- Equilibrium uniqueness even in the class of equilibria in monotonic switching strategies needs careful consideration, for the expected payoff difference from below and from above the cutoff is not monotonically increasing in the player's private signal, so that the existence of a unique cutoff setting the expected payoff difference to zero is not guaranteed.
- Differently from what shown in Chapter 3 for the weakest-link impact function, the presence of a one-sided dominance region is not conducive to the coexistence of an equilibrium in (monotonic) switching-strategies and an equilibrium robust to incomplete information à la Kajii and Morris (1997) in which no player exerts effort, since, at $(\overline{a}_1, \overline{a}_2, \overline{a}_3, \overline{a}_4)$, $x_i(v_i) = 1$ is not a strictly dominated action $\forall v_i \in [\underline{v} + \varepsilon, \overline{v} - \varepsilon]$ and $i \in \{1, 2, 3, 4\}$. Moreover, note that in the complete information game with $v \in \mathbb{R}$ and $C_{ij}(x_j(i)) = x_j(i)$ there does not exist any equilibrium strategy profile which is played for every possible value for v, that is $v \in \mathbb{R}$, in contrast with what happens under the weakest-link technology for $(\overline{a}_1, \overline{a}_2, \overline{a}_3, \overline{a}_4)$.

Once highlighted the main properties of our example, the general model and the mechanisms guiding to the related results should be more transparent.

4.4 The Set of Nash Equilibria of Binary Max-Max Group Contests

To simplify notation and presentation, the NE of the $BMMAGC^*$ will be presented in terms of share of active agents, i.e. $(\gamma_1, \gamma_2) \in [0,1] \times [0,1]$. The following results characterize the set of Nas equilibria of $BMMAGC^*$.

PROPOSITION 4.1. *In the $BMMAGC^*$,*

- *if $v > 2$, there are $n_1 \cdot n_2$ strict Nash equilibria in pure strategies* [4.4]

$$NE \equiv \left\{ (\gamma_1, \gamma_2) \;\; \textit{such that} \; \gamma_j \in (0,1) \;\; \textit{and} \; \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1 \; \forall j \in \{1,2\} \right\}$$

  and a Nash equilibrium in within-group symmetric strictly-mixed strategies

$$\sigma^*_{ij}\left(x_{ij} = 1\right) = \left(1 - \frac{2}{v}\right)^{1/(n_j - 1)} \quad \forall \; i \in \{1, \ldots, n_j\} \;\; \textit{and} \; j \in \{1,2\}\,;$$

- *if $v = 2$, there are $n_1 \cdot n_2 + n_1 + n_2 + 1$ Nash equilibria in pure strategies*

$$\begin{aligned} NE \equiv & \left\{ (\gamma_1, \gamma_2) \;\; \textit{such that} \; \gamma_j \in (0,1) \;\; \textit{and} \; \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1 \; \forall j \in \{1,2\} \right\} \cup \\ & \cup \left\{ (\gamma_j, \gamma_{-j}) = (\gamma_j, 0) \;\; \textit{such that} \; \gamma_j \in (0,1) \;\; \textit{and} \; \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1 \; \forall j \in \{1,2\} \right\} \cup \\ & \cup \left\{ (\gamma_1, \gamma_2) = (0,0) \right\} \end{aligned}$$

[4.4] $\mathbb{1}_{x_{ij}=1}$ stands for the Indicator random variable taking value 1 when player ij chooses $x_{ij} = 1$, that is she exerts effort.

- *if* $v < 2$*, there is a unique Nash equilibrium in pure strategies derived by strict-dominance*

$$(\gamma_1, \gamma_2) = (0, 0) \ .$$

Overall, there is a one-sided dominance region: for $v < 2$*,* $x_{ij} = 1$ *is a strictly dominated action* $\forall \ ij \in \{1, \ldots, N\}$*.*

Proof. See Appendix 4.A.1. □

Moreover, by taking a generalization of risk-dominance close to the unilateral deviation stability due to Güth (1992), it is easy to prove the following result. [4.5]

PROPOSITION 4.2. *In the BMMAGC*,*

- *for* $v > 2$*,* (γ_1, γ_2) *such that* $\gamma_j \in (0, 1)$ *and* $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1 \ \forall j \in \{1, 2\}$ *are the payoff-dominant equilibrium strategy profiles for any* ij *such that* $x_{ij} = 0$*;*
- *for* $v > 4$*,* (γ_1, γ_2) *such that* $\gamma_j \in (0, 1)$ *and* $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1 \ \forall j \in \{1, 2\}$ *are the risk-dominant equilibrium strategy profiles for any* ij *such that* $x_{ij} = 1$ *;*
- *for* $2 < v < 4$*,* (γ_1, γ_2) *such that* $\gamma_j \in (0, 1)$ *and* $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1 \ \forall j \in \{1, 2\}$ *are the risk-dominant equilibrium strategy profiles for any* ij *such that* $x_{ij} = 0$;
- *for* $v = 2$*,* $(\gamma_j, \gamma_{-j}) = (\gamma_j, 0)$ *such that* $\gamma_j \in (0, 1)$ *and* $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1 \ \forall j \in \{1, 2\}$ *are the payoff-dominant equilibrium strategy profiles for any player* ij *such that* $x_{ij} = 0$ *;*
- *for* $v = 2$*,* (γ_1, γ_2) *such that* $\gamma_j \in (0, 1)$ *and* $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1 \ \forall j \in \{1, 2\}$ *and* $(\gamma_j, \gamma_{-j}) = (\gamma_j, 0)$ *such that* $\gamma_j \in (0, 1)$ *and* $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 0 \ \forall j \in \{1, 2\}$ *are the risk-dominant equilibria for any* ij *such that* $x_{ij} = 0$.

Proof. See Appendix 4.A.2. □

[4.5] In the class of 2×2-games, risk-dominance à la Harsanyi and Selten (1988) and unilateral deviation stability à la Güth (1992) coincide. However, for more than two players the tracing procedure does not generally produce the same equilibrium selection result as according to the largest Nash product.

REMARK 4.1. *Note that, for* $v > 4$*,* (γ_1, γ_2) *such that* $\gamma_j \in (0,1)$ *and* $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1 \ \forall j \in \{1,2\}$ *are the payoff-dominant equilibria for any* ij *such that* $x_{ij} = 0$*, whereas* (γ_1, γ_2) *such that* $\gamma_j \in (0,1)$ *and* $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1 \ \forall j \in \{1,2\}$ *are the risk-dominant equilibria for any* ij *such that* $x_{ij} = 1$ *: there is a tension between payoff-dominance and risk-dominance.*

Now consider a slight variation of the game above and let:

- the individual cost of effort $C(x_j(i)) = c$ with $c \in \mathbb{R}$. Thus, costs of effort may be negative, which means that agents could enjoy effort per se;
- the club good prize $v > 0$, i.e. the prize v is always worth positive utils, so that it is a good.

Henceforth, we term this variation as $BMMAGC^{*b}$. Then, it is straightforward to derive the following results.

PROPOSITION 4.3. *In the* $BMMAGC^{*b}$*,*

- *if* $c < 0$*, there is a unique Nash equilibrium in pure strategies derived by strict-dominance*

$$(\gamma_1, \gamma_2) = (1,1) \ \ ;$$

- *if* $c > \frac{v}{2}$*, there is a unique Nash equilibrium in pure strategies derived by strict-dominance*

$$(\gamma_1, \gamma_2) = (0,0) \ \ ;$$

- *if* $0 < c < \frac{v}{2}$*, there are* $n_1 \cdot n_2$ *strict Nash equilibria in pure strategies*

$$NE \equiv \left\{ (\gamma_1, \gamma_2) \ \textit{such that} \ \gamma_j \in (0,1) \ \textit{and} \ \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1 \ \forall j \in \{1,2\} \right\}$$

and an equilibrium in within-group symmetric strictly mixed strategies

$$\sigma_i^* \left(x_{ij} = 1\right) = \left(1 - \frac{2c}{v}\right)^{1/(n_j-1)} \quad \forall\ i \in \{1, \ldots, n_j\}\ \ \textit{and}\ j \in \{1, 2\}\,;$$

- *if* $c = 0$*, the set of Nash equilibria in pure strategies is*

$$\begin{aligned} NE \equiv & \{(\gamma_1, \gamma_2)\ \ \textit{such that}\ \gamma_j \in (0,1)\ \ \textit{and}\ \ \forall j \in \{1,2\}\} \cup \\ & \cup \{(\gamma_j, \gamma_{-j}) = (\gamma_j, 1)\ \ \textit{such that}\ \gamma_j \in (0,1)\ \ \textit{and}\ \ \forall j \in \{1,2\}\} \cup \\ & \cup \{(\gamma_1, \gamma_2) = (1,1)\} \end{aligned}$$

- *if* $c = \frac{v}{2}$*, there are* $n_1 \cdot n_2 + n_1 + n_2 + 1$ *Nash equilibria in pure strategies*

$$\begin{aligned} NE \equiv & \left\{(\gamma_1, \gamma_2)\ \ \textit{such that}\ \gamma_j \in (0,1)\ \ \textit{and}\ \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1\ \forall j \in \{1,2\}\right\} \cup \\ & \cup \left\{(\gamma_j, \gamma_{-j}) = (\gamma_j, 0)\ \ \textit{such that}\ \gamma_j \in (0,1)\ \ \textit{and}\ \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1\ \forall j \in \{1,2\}\right\} \cup \\ & \cup \{(\gamma_1, \gamma_2) = (0,0)\} \end{aligned}$$

Overall, there are two dominance regions: for $c < 0$*,* $x_{ij} = 0$ *is a strictly dominated action for any* $ij \in \{1, \ldots, N\}$*: for* $c > \frac{v}{2}$*,* $x_{ij} = 1$ *is a strictly dominated action for any* $ij \in \{1, \ldots, N\}$*.*

Proof. See Appendix 4.A.3. □

As before, by taking a generalization of risk-dominance close to the unilateral deviation stability due to Güth (1992), it is easy prove the following result.

PROPOSITION 4.4. *In the* $BMMAGC^{*b}$*,*

- *for* $0 < c < \frac{v}{2}$*,* (γ_1, γ_2) *such that* $\gamma_j \in (0,1)$ *and* $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1\ \forall j \in \{1,2\}$ *are the payoff-dominant equilibrium strategy profiles for any* ij *such that* $x_{ij} = 0$*;*

- *for* $0 < c < \frac{v}{4}$, (γ_1, γ_2) *such that* $\gamma_j \in (0,1)$ *and* $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1$ $\forall j \in \{1,2\}$ *are the risk-dominant equilibrium strategy profiles for any* ij *such that* $x_{ij} = 1$ *;*
- *for* $\frac{v}{4} < c < \frac{v}{2}$, (γ_1, γ_2) *such that* $\gamma_j \in (0,1)$ *and* $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1$ $\forall j \in \{1,2\}$ *are the risk-dominant equilibrium strategy profiles for any* ij *such that* $x_{ij} = 0$;
- *for* $c = \frac{v}{2}$, $(\gamma_j, \gamma_{-j}) = (\gamma_j, 0)$ *such that* $\gamma_j \in (0,1)$ *and* $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1$ $\forall j \in \{1,2\}$ *are the payoff-dominant equilibrium strategy profiles for any player* ij *such that* $x_{ij} = 0$ *;*
- *for* $c = \frac{v}{2}$, (γ_1, γ_2) *such that* $\gamma_j \in (0,1)$ *and* $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1$ $\forall j \in \{1,2\}$ *and* $(\gamma_j, \gamma_{-j}) = (\gamma_j, 0)$ *such that* $\gamma_j \in (0,1)$ *and* $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 0$ $\forall j \in \{1,2\}$ *are the risk-dominant equilibria for any* ij *such that* $x_{ij} = 0$.
- *for* $c = 0$*, there is no payoff-dominant equilibrium strategy profile* $\forall ij \in \{1, \ldots, N\}$*;*
- *for* $c = 0$, (γ_1, γ_2) *such that* $\gamma_j \in (0,1)$ *and* $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1$ $\forall j \in \{1,2\}$ *are the risk-dominant equilibrium strategy profiles for any* ij *such that* $x_{ij} = 1$.

Proof. See Appendix 4.A.4. ☐

REMARK 4.2. *Note that, for* $\frac{v}{4} < c < \frac{v}{2}$ *and any group* $j \in \{1,2\}$, $(\gamma_j, \gamma_{-j}) = (1,0)$ *is the payoff-dominant equilibrium strategy profile for group* $j \in \{1,2\}$*, whereas* $(\gamma_j, \gamma_{-j}) = (0,1)$ *and* $(\gamma_j, \gamma_{-j}) = (0,0)$ *are the risk-dominant equilibrium strategy profiles for group* $j \in \{1,2\}$*: there is a tension between payoff-dominance and risk-dominance.*

4.5 The Set of Bayes-Nash Equilibria with Incomplete Information à la Global Games.

4.5.1 Incomplete Information à la Global Games about the Prize

Let us consider the case where the individual **cost of effort** is $C(x_j(i)) = x_j(i)$, that is the $BMMAGC^*$ model. We closely follow Carlsson and Damme (1993a) introducing incomplete information about the prize v as follows:

- let V be a random variable which is uniform on the real line, that is including the dominance region and the threshold for the risk-dominance;
- given the unknown realization v of V, each player $ij \in \{1, \ldots, n_j\}$ $\forall j \in \{1, 2\}$ idiosyncratically observes the realization v_{ij} of a signal V_{ij}, uniform on $[v - \varepsilon, v + \varepsilon]$ for some $\varepsilon > 0$, so that the players' observation errors $V_{ij} - v$ $\forall ij \in \{1, \ldots, n_j\}$ and $\forall j \in \{1, 2\}$ are independent;
- after these idiosyncratic observations, each player $ij \in \{1, \ldots, n_j\}$ $\forall j \in \{1, 2\}$ simultaneously and independently decides whether to exert effort or not and gets a payoff as described above.

Henceforth, we refer to this game as $g_1(v)$. Then, we are able to obtain the following result.

PROPOSITION 4.5. *In the $g_1(v)$, there are no equilibria in (monotonic) switching strategies, such that $\forall ij \in \{1, \ldots, n_j\}$, $\forall j \in \{1, 2\}$ and $\forall\ k \in \mathbb{R}$:*

$$x_{ij}^{*}(v_{ij}) = \begin{cases} 1 & \text{if } v_{ij} > k \\ 0 & \text{if } v_{ij} \leq k \end{cases},$$

Proof. See Appendix 4.A.5 ☐

REMARK 4.3. *In contrast with what we showed in Chapter 3 for the weakest-link impact function, it is not possible to establish the existence of an equilibrium robust to incomplete information in the sense of Kajii and Morris (1997) in which no player exerts effort, since $x_{ij}(v_{ij}) = 1$ is not a dominated action $\forall v_{ij} \in [\underline{v} + \varepsilon, \overline{v} - \varepsilon]$ and $\forall ij \in \{1, \ldots, N\}$.*

REMARK 4.4. *The proof of the non-existence result hinges on the prior distribution being unbounded and on the fact that there are no within-group complementarities. It can be shown that, with a proper uniform prior, checking incentive compatibility for the highest signal realization, i.e. $v_{ij} = \overline{v} - \varepsilon$, is non-trivial, for symmetry of both first-order and, more importantly, second-order beliefs breaks down.*

4.5.2 Incomplete Information à la global games about the Cost of Effort

Let us consider the case where the individual **costs of effort** is $C\left(x_j\left(i\right)\right) = c$ with $c \in \mathbb{R}$ and the club good **prize** worth $v > 0$, that is the $BMMAGC^{*b}$ model . We closely follow Carlsson and Damme (1993a) introducing incomplete information about the cost of effort c as follows:

- let C be a random variable which is uniform on the real line, clearly including both dominance regions;
- given the unknown realization c of C, each player $ij \in \{1, \ldots, n_j\}$ $\forall j \in \{1, 2\}$ idiosyncratically observes the realization c_{ij} of a signal C_{ij}, uniform on $[c - \varepsilon, c + \varepsilon]$ for some $\varepsilon > 0$, so that the players' observation errors $C_{ij} - c$ $\forall ij \in \{1, \ldots, n_j\}$ and $\forall j \in \{1, 2\}$ are independent;
- after these idiosyncratic observations, each player $ij \in \{1, \ldots, n_j\}$ $\forall j \in \{1, 2\}$ simultaneously and independently decides whether to exert effort or not and gets a payoff as described above;

Henceforth, we refer to this game as $g_2\left(c\right)$. Then we are able to obtain the following result.

PROPOSITION 4.6. *Let us define*

$$\underline{\varepsilon}(c_{ij}^{max}) = \frac{2^{-n_j-1}\left(v - 2^{n_j} \cdot c_{ij}^{max}\right)^2}{2^{n_j} \cdot c_{ij}^{max} - 2^{n_j+\frac{1}{2}}\sqrt{-4^{-n_j}\left(2^{\frac{1}{n_j-1}}\left(\frac{c_{ij}^{max}}{v}\right)^{\frac{1}{n_j-1}} - 1\right)\left(v - 2^{n_j} \cdot c_{ij}^{max}\right)^2} - v},$$

$$c_{ij}^{max} = \underset{c_{ij} \in (v/2^{n_j}, v/2]}{\arg\max} \frac{2^{-n_j-1}\left(v - 2^{n_j} \cdot c_{ij}\right)^2}{2^{n_j} \cdot c_{ij} - 2^{n_j+\frac{1}{2}}\sqrt{-4^{-n_j}\left(2^{\frac{1}{n_j-1}}\left(\frac{c_{ij}}{v}\right)^{\frac{1}{n_j-1}} - 1\right)\left(v - 2^{n_j} \cdot c_{ij}\right)^2} - v}.$$

Then, in the $g_2\left(c\right)$, for $\varepsilon \geq \underline{\varepsilon}(c_{ij}^{max})$ there is an equilibrium in (monotonic) switching strategies, *such that* $\forall ij \in \{1, \ldots, n_j\}$ *and* $\forall j \in \{1, 2\}$*:*

$$x^*_{ij}\left(c_{ij}\right) = \begin{cases} 1 & \text{if } c_{ij} < 2^{-n_j} v \\ 0 & \text{if } c_{ij} \geq 2^{-n_j} v \ . \end{cases}$$

Proof. See Appendix 4.A.6. ☐

REMARK 4.5. *The presence of both an upward dominance region and a downward dominance region is crucial to the existence of an equilibrium in (monotonic) switching strategies.*

REMARK 4.6. *The existence of an equilibrium in (monotonic) switching strategies in the* $g_2\left(c\right)$ *is ensured for a sufficiently high level of noise* ε *which preserves the incentive compatibility of the cutoff equilibrium despite within-group strategic substitutability. Accordingly, limit-uniqueness result obtained for the max-min case in Chapter 3 is not attainable with the best-shot impact function. Moreover, equilibrium uniqueness would require to find the set of thresholds* k *setting the expected payoff to zero. The latter is not strictly increasing in* c_{ij} *and thus the set is not guaranteed to be a singleton. The study of equilibrium uniqueness in the class of equilibria in (monotonic) switching strategies is left for future research.*

REMARK 4.7. *The size of the threshold* $\underline{\varepsilon}(c_{ij}^{max})$ *can be determined by numerical optimization for any* n_j *with (generally) an interior maximizer: simulations show that* $\underline{\varepsilon}(c_{ij}^{max}) < v/3$*.*

REMARK 4.8. *Note that the cutoff of the equilibrium in (monotonic) switching strategies, i.e.* $c_{ij} = 2^{-n_j} v$*, does not coincide with the one of the risk-dominance region, that is* $c_{ij} = \frac{v}{4}$ *for any* $j \in \{1, 2\}$*, differently from what happens in the two-group four-player example. This is very close to the point made by Carlsson and Damme (1993b) for n-player stag hunt games, where the authors stress that risk-dominance fails as an equilibrium selection criterion when we depart from the 2×2 case.*

REMARK 4.9. *The existence result holds with a proper uniform prior distributed on a sufficiently large interval* $[\underline{c}, \overline{c}]$ *comprising both dominance regions, e.g.* $[-v, v]$ *as well. On the other hand, observe that, under the uniform improper prior assumption, the existence* *result can be easily extended to the case in which private signals are distributed normally but still conditionally independent. The exact size of the threshold* $\underline{\varepsilon}(c_{ij}^{max})$ *ensuring existence will be different, for it hinges on to the distributional assumption. However, a formal study of noise independence mimicking the result obtained for the max-min case in Chapter 3 would require the study of convolution of conditionally independent random variables and it is left for future research.*

4.6 The Group-Size Paradox

In this section we calculate the probabilities of winning the prize v and the expected payoffs for both groups at the equilibrium in (monotonic) switching strategies in $g_2(c)$ with a bounded uniform prior over $[\underline{c}, \overline{c}]$, in order to assess the presence of the so-called group-size paradox.

4.6.1 The Group-Size Paradox in $g_2(c)$

Let us now turn to the $g_2(c)$ model. Let us assume C is distributed uniformly on some sufficiently large interval $[\underline{c}, \overline{c}]$. Then it is posssible to derive the probability of winning the prize v for group $j \in \{1, 2\}$, as the following proposition shows.

PROPOSITION 4.7. *In the* $g_2(c)$ *with a bounded uniform prior over* $[\underline{c}, \overline{c}]$*, the probability of winning the prize* v *for group* $j \in \{1, 2\}$ *at the cutoff equilibrium equals:*

$$\begin{aligned} Prob\,(j\ \ wins\ v) = & \left[1-\left(1-\frac{2^{-n_j}v-\underline{c}-\varepsilon}{\overline{c}-\underline{c}-2\varepsilon}\right)^{n_j}\right]\cdot\left(1-\frac{2^{n_{-j}}v-\underline{c}-\varepsilon}{\overline{c}-\underline{c}-2\varepsilon}\right)^{n_{-j}}+ \\ & +\frac{1}{2}\left(1-\frac{2^{-n_j}v-\underline{c}-\varepsilon}{\overline{c}-\underline{c}-2\varepsilon}\right)^{n_j}\cdot\left(1-\frac{2^{-n_{-j}}v-\underline{c}-\varepsilon}{\overline{c}-\underline{c}-2\varepsilon}\right)^{n_{-j}}+ \\ & +\frac{1}{2}\left[1-\left(1-\frac{2^{-n_j}v-\underline{c}-\varepsilon}{\overline{c}-\underline{c}-2\varepsilon}\right)^{n_j}\right]\cdot\left[1-\left(1-\frac{2^{n_{-j}}v-\underline{c}-\varepsilon}{\overline{c}-\underline{c}-2\varepsilon}\right)^{n_{-j}}\right]\ . \end{aligned}$$

Proof. See Appendix 4.A.7. ☐

REMARK 4.10. *In the* $g_2(c)$ *the probability of winning the prize* v *for group* j *at the equilibrium in (monotonic) switching strategies is not increasing in* n_j *at the threshold* $n_j^* = log(\frac{v}{\overline{c}-\varepsilon})/log(2)\ \forall\ \varepsilon \geq \underline{\varepsilon}(c_{ij}^{max})$ *and increasing in* n_{-j} *at the threshold* $n_{-j}^* = log(\frac{v}{\overline{c}-\varepsilon})/log(2)$

$\forall \varepsilon \geq \underline{\varepsilon}(c_{ij}^{max})$, *since* $2^{-n_j}v > \underline{c} + \varepsilon \; \forall \; \underline{c} < 0$, $\forall \varepsilon \geq \underline{\varepsilon}(c_{ij}^{max})$ *and* $\forall n_j \geq 2$, *and* $2^{-n_j}v < \overline{c} - \varepsilon$ $\forall \; \overline{c} > \frac{v}{2}$, $\forall \varepsilon \geq \underline{\varepsilon}(c_{ij}^{max})$ *and* $\forall \; n_j \geq 2$.

REMARK 4.11. *Note that whether the probability of winning the prize* v $\forall j \in \{1, 2\}$ *is increasing or decreasing with respect to group size* n_j *cannot be established analytically for any* $\overline{c}, \underline{c}, \varepsilon$ *and* v*, so that the presence of the so-called group-size paradox depends on the specific details of the uniform prior distribution, prize value and group sizes. Nonetheless, numerical solutions can be found as the following example clarifies.*

EXAMPLE 4.1. *Let us consider:*

- $[\underline{c} + \varepsilon, \overline{c} - \varepsilon] = [-1, 1000]$;
- $n_j = 2$ *and* $n_{-j} = 3$;
- $v = 800$ *utils ;*
- $\mathbb{E}\left[C\right] = \frac{\underline{c} + \overline{c}}{2}$.

Then,

$$\begin{aligned}\Delta_j Prob\,(\ j\ wins\ v\ ) &= Prob\,(\ j\ wins\ v\ ; n_j + 1) - Prob\,(\ j\ wins\ v\ ; n_j) \\ &= \frac{1}{4}\left(\frac{2^{-n_j}\left(v - 2^{n_j+1}\left(\overline{c} - \varepsilon\right)\right)\left(\frac{\overline{c} - \varepsilon - 2^{-n_j-1}v}{\overline{c} - \varepsilon - \underline{c} - \varepsilon}\right)^{n_j}}{\overline{c} - \varepsilon - \underline{c} - \varepsilon} + \right. \\ &\left. + 2\left(\frac{\overline{c} - \varepsilon - 2^{-n_j}v}{\overline{c} - \varepsilon - \underline{c} - \varepsilon}\right)^{n_j}\right) .\end{aligned}$$

so that we can find numerically:

- $\Delta_j Prob\,(\ j\ wins\ v\ ) < 0$ *if* $n_j \leq 12$,
- $\Delta_j Prob\,(\ j\ wins\ v\ ) > 0$ *if* $n_j > 12$.

Moreover,

$$\begin{aligned}\Delta_{-j} Prob\,(\ j\ wins\ v\ ) &= Prob\,(\ j\ wins\ v\ ; n_{-j}+1) - Prob\,(\ j\ wins\ v\ ; n_{-j}) \\ &= \frac{1}{4}\left(\frac{2^{-n_{-j}}\left(v - 2^{n_{-j}+1}\left(\overline{c}-\varepsilon\right)\right)\left(\frac{\overline{c}-\varepsilon-2^{-n_{-j}-1}v}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)^{n_{-j}}}{\overline{c}-\varepsilon-\underline{c}-\varepsilon} + \right. \\ &\left. - 2\left(\frac{\overline{c}-\varepsilon-2^{-n_{-j}}v}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)^{n_{-j}} \right) .\end{aligned}$$

so that we can find numerically:

- $\Delta_{-j}\,(\ j\ wins\ v\ ) > 0 \quad if \quad n_{-j} \leq 12,$
- $\Delta_{-j} Prob\,(\ j\ wins\ v\ ) < 0 \quad if \quad n_{-j} > 12\ .$

Therefore, in this specific example, the impact of an increase of group sizes on the probability of winning the prize is non-monotonic. Furthermore, the finite differences with respect to the two group sizes display different signs.

Moreover, once computed the probabilities of winning, it is immediate to retrieve the expected payoffs at the equilibrium in (monotonic) switching strategies.

PROPOSITION 4.8. *In the* $g_2(c)$ *with a bounded uniform prior over* $[\underline{c}, \overline{c}]$*, the expected payoff at the cutoff equilibrium* $x^*_{ij}(c_{ij})$ *for any* $ij \in \{1, \dots, n_j\}$ $\forall j \in \{1, 2\}$ *equals:* [4.6]

$$\begin{aligned}\mathbb{E}\left[\pi_{ij}\left(\mathbf{x}^*_j, \mathbf{x}^*_{-j}\right)\right] = &\left(\frac{2^{-n_j}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)\cdot\left(1-\frac{2^{-n_{-j}}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)^{n_{-j}}\cdot\left(v-\frac{\underline{c}+\overline{c}}{2}\right)+ \\ &+\left(1-\frac{2^{-n_j}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)\cdot\left[1-\left(1-\frac{2^{-n_j}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)^{n_j-1}\right]\cdot \\ &\cdot\left(1-\frac{2^{-n_{-j}}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)^{n_{-j}}\cdot v\ +\end{aligned}$$

[4.6] By $\mathbb{E}\left[\pi_{ij}\left(\mathbf{x}^*_j, \mathbf{x}^*_{-j}\right)\right]$ we mean the expected payoff at the cutoff equilibrium $x^*_{ij}(c_{ij})$ for any $ij \in \{1, \dots, n_j\}$ $\forall j \in \{1, 2\}$, where $\mathbf{x}^*_j, \mathbf{x}^*_{-j}$ are the vectors of equilibrium strategies profiles for the two groups.

$$
\begin{aligned}
&+\left(\frac{2^{-n_j}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)\cdot\left(1-\left(1-\frac{2^{-n_{-j}}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)^{-n_{-j}}\right)\cdot\left(\frac{v}{2}-\frac{\underline{c}+\overline{c}}{2}\right)\ +\\
&+\left(1-\frac{2^{-n_j}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)\cdot\left(1-\left(1-\frac{2^{-n_j}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)^{n_j-1}\right)\cdot\\
&\cdot\left(1-\left(1-\frac{2^{-n_{-j}}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)^{n_{-j}}\right)\cdot\frac{v}{2}\ +\\
&+\left(1-\frac{2^{-n_j}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)^{n_j}\cdot\left(1-\frac{2^{-n_{-j}}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)^{n_{-j}}\cdot\frac{v}{2}\ .
\end{aligned}
$$

Proof. See Appendix 4.A.8. ☐

REMARK 4.12. *In the* $g_2\left(c\right)$ *the expected payoff for any* $ij \in \{1,\ldots,n_j\}$ $\forall j \in \{1,2\}$ *at the equilibrium in (monotonic) switching strategies is not increasing in* n_j *at the threshold* $n_j^* = log(\frac{v}{\overline{c}-\varepsilon})/log(2)$ $\forall\ \varepsilon \geq \underline{\varepsilon}(c_{ij}^{max})$ *and increasing in* n_{-j} *at the threshold* $n_{-j}^* = log(\frac{v}{\overline{c}-\varepsilon})/log(2)$ $\forall\ \varepsilon \geq \underline{\varepsilon}(c_{ij}^{max})$ *, since* $2^{-n_j}v > \underline{c}+\varepsilon\ \forall\ \underline{c} < 0$, $\forall\ \varepsilon \geq \underline{\varepsilon}(c_{ij}^{max})$ *and* $\forall\ n_j \geq 2$ *and* $2^{-n_j}v < \overline{c}-\varepsilon$ $\forall\ \overline{c} > \frac{v}{2}$, $\forall\ \varepsilon \geq \underline{\varepsilon}(c_{ij}^{max})$ *and* $\forall\ n_{-j} \geq 2$ *.*

REMARK 4.13. *Note that whether the expected payoff* $\forall\ ij \in \{1,\ldots,n_j\}$ $\forall j \in \{1,2\}$ *is increasing or decreasing with respect to group size* n_j *cannot be established analytically for any* $\overline{c}, \underline{c}, \varepsilon$ *and* v*, so that the presence of the so-called group-size paradox depends on the specific details of the uniform prior distribution, prize value and group sizes. Nonetheless, numerical solutions can be found as the following example clarifies.*

EXAMPLE 4.2. *Let us consider:*

- $[\underline{c}+\varepsilon, \overline{c}-\varepsilon] = [-1, 1000]$ *;*
- $n_j = 2$ *and* $n_{-j} = 3$ *;*
- $v = 800$ *utils ;*
- $\mathbb{E}\left[C\right] = \frac{\underline{c}+\overline{c}}{2}$ *.*

Then,

$$\begin{aligned}\Delta_j \mathbb{E}\left[\pi_{ij}\left(\mathbf{x}_j^*,\mathbf{x}_{-j}^*\right)\right] &= \mathbb{E}\left[\pi_{ij}\left(\mathbf{x}_j^*,\mathbf{x}_{-j}^*;n_j+1\right)\right]-\mathbb{E}\left[\pi_{ij}\left(\mathbf{x}_j^*,\mathbf{x}_{-j}^*;n_j\right)\right]\\ &= \frac{2^{-n_j-2}\Big(\left(\underline{c}+\varepsilon+\overline{c}-\varepsilon\right)v-\left(2^{n_j+1}\left(\overline{c}-\varepsilon\right)-v\right)\left(\frac{\overline{c}-\varepsilon-2^{-n_j-1}v}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)^{n_j}\Big)}{\overline{c}-\varepsilon-\underline{c}+\varepsilon}+\\ &+\frac{2^{-n_j-2}\Big(2^{n_j+1}\left(\overline{c}-\varepsilon-\underline{c}-\varepsilon\right)v\left(\frac{\overline{c}-\varepsilon-2^{-n_j}v}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)^{n_j}\Big)}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\ ,\end{aligned}$$

so that we can find numerically:

- $\Delta_j \mathbb{E}\left[\pi_{ij}\left(\mathbf{x}_j^*,\mathbf{x}_{-j}^*\right)\right] > 0$ *if* $n_j = 2$ *or* $n_j \geq 12$,
- $\Delta_j \mathbb{E}\left[\pi_{ij}\left(\mathbf{x}_j^*,\mathbf{x}_{-j}^*\right)\right] < 0$ *if* $2 < n_j < 12$.

Moreover,

$$\begin{aligned}\Delta_{-j} \mathbb{E}\left[\pi_{ij}\left(\mathbf{x}_j^*,\mathbf{x}_{-j}^*\right)\right] &= \mathbb{E}\left[\pi_{ij}\left(\mathbf{x}_j^*,\mathbf{x}_{-j}^*;n_{-j}+1\right)\right]-\mathbb{E}\left[\pi_{ij}\left(\mathbf{x}_j^*,\mathbf{x}_{-j}^*;n_{-j}\right)\right]\\ &= \frac{v}{4}\Big(\frac{2^{-n_{-j}}\left(2^{n_{-j}+1}\left(\overline{c}-\varepsilon\right)-v\right)\left(\frac{\overline{c}-\varepsilon-2^{-n_{-j}-1}v}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)^{n_{-j}}}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\Big)+\\ &-\frac{v}{2}\left(\frac{\overline{c}-\varepsilon-2^{-n_{-j}}v}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)^{n_{-j}}\ ,\end{aligned}$$

so that we can find numerically:

- $\Delta_{-j} \mathbb{E}\left[\pi_{ij}\left(\mathbf{x}_j^*,\mathbf{x}_{-j}^*\right)\right] > 0$ *if* $n_{-j} \leq 12$,
- $\Delta_{-j} \mathbb{E}\left[\pi_{ij}\left(\mathbf{x}_j^*,\mathbf{x}_{-j}^*\right)\right] < 0$ *if* $n_{-j} > 12$.

REMARK 4.14. *The need of numerical examples to verify the presence of the so-called group-size paradox constitutes a sharp difference with respect to what obtained for the weakest-link case, as shown in Chapter 3, where in the general case , i.e.* $log\left(\underline{c}+\varepsilon\right)/log\left(2\right) \leq 2^{n_j} \leq log\left(\overline{c}-\varepsilon\right)/log\left(2\right)-1$ *, an increase in group size translates into a lower probability of winning and a lower expected payoff for sufficiently a sufficiently low cost effort.*

4.7 Extension to an M-Group Model

In this section we assess the robustness of the results obtained under the two-group assumption by extending our model to the M-group case with $M \geq 2$ for the case in which incomplete information is modeled about the cost of exerting effort.

4.7.1 Incomplete Information à la Global Games about the Cost of Effort and M Groups

Let us define the $BMMAMGC^{*b}$ as the $BMMAGC^{*b}$ with M groups, where $M \geq 2$, and $n_1 \geq \ldots \geq n_M \geq 2$ without loss of generality.

We closely follow Carlsson and Damme (1993a) introducing incomplete information about the cost of effort c as follows:

- let C be a random variable which is uniform on the real line, clearly including both dominance regions;
- given the unknown realization c, each player $ij \in \{1, \ldots, n_j\}$ $\forall j \in \{1, \ldots, M\}$ idiosyncratically observes the realization of a random variable C_{ij}, uniform on $[c - \varepsilon, c + \varepsilon]$ for some $\varepsilon > 0$, so that the players' observation errors $C_{ij} - c$ $\forall ij \in \{1, \ldots, n_j\}$ and $\forall j \in \{1, 2\}$ are independent;
- after these idiosyncratic observations, each player $ij \in \{1, \ldots, n_j\}$ $\forall j \in \{1, \ldots, M\}$ simultaneously and independently decides whether to exert effort or not and gets a payoff as described above;

Henceforth, we refer to this game as $g_3(c)$. Then we are able to obtain the following result.

PROPOSITION 4.9. *In $g_3(c)$ under incomplete information à la global games for $\varepsilon \geq \underline{\varepsilon}(c_{ij}^{max})$ there is an equilibrium in (monotonic) cutoff strategies, such that $\forall ij \in \{1, \ldots, n_j\}$ and $\forall j \in \{1, \ldots, M\}$:*

$$x^*_{ij}\left(c_{ij}\right) = \begin{cases} 1 & \mathit{if}\ c_{ij} < c^*_j \\ 0 & \mathit{if}\ c_{ij} \geq c^*_j, \end{cases}$$

where

$$c^*_j = 2^{1-n_j}\left(2^{-\sum_{-j} n_{-j}}\left(\frac{M-1}{M}\right) + \sum_{k=1}^{M-1}\frac{\sum_{-J\in Q_k}\prod_{-j\in -J}\left(1-2^{-n_{-j}}\right)\cdot\prod_{-j'\in Q_k\setminus -J}2^{-n_{-j'}}}{k+1}\right)v\ ,$$

$$Q_k = \{-J \in \{\{1,\ldots,M\}\setminus\{j\}\} \mid |-J| = k\}\ ,$$

and $\underline{\varepsilon}(c^{max}_{ij})$ *is the minimum value* ε *such that* $\forall\ c_{ij} \in [k, v\cdot(M-1)/(M)]$

$$\begin{aligned} c_{ij} - &\left(\prod_{-i}p_{-ij}\cdot\prod_{-j}\prod_{i}p_{i-j}\left(\frac{M-1}{M}\right)+\right.\\ &\left.+\prod_{-i}p_{-ij}\sum_{k=1}^{M-1}\frac{\sum_{-J\in Q_k}\prod_{-j\in -J}\left(1-\prod_i p_{i-j}\right)\cdot\prod_{-j'\in Q_k\setminus -J}\prod_i p_{i-j'}}{k+1}\right)v\ , \end{aligned}$$

where $p_{-ij} = p_{i-j} = p_{i-j'} = 1-(k-c_{ij}+2\varepsilon)^2/(8\varepsilon^2)$*, and* $\forall\ c_{ij} \in [0,k)$

$$\begin{aligned} &\left(\prod_{-i}p_{-ij}\cdot\prod_{-j}\prod_{i}p_{i-j}\left(\frac{M-1}{M}\right)+\right.\\ &\left.+\prod_{-i}p_{-ij}\sum_{k=1}^{M-1}\frac{\sum_{-J\in Q_k}\prod_{-j\in -J}\left(1-\prod_i p_{i-j}\right)\cdot\prod_{-j'\in Q_k\setminus -J}\prod_i p_{i-j'}}{k+1}\right)v - c_{ij}\ , \end{aligned}$$

where $p_{-ij} = p_{i-j} = p_{i-j'} = (c_{ij}-k+2\varepsilon)^2/(8\varepsilon^2)$.

Proof. See Appendix 4.A.9. □

REMARK 4.15. *Note that proposition 4.9 delivers an existence result, not a uniqueness one, even in the class of equilibria in (monotonic) switching strategies. Existence is intimately related to the presence of both an upper and a lower dominance region, as the the proof should clarify.*

REMARK 4.16. *Equilibrium existence is guaranteed for a sufficiently large level of noise* ε*,* *which prevents strategic substitutability from breaking incentive compatibility, as in the* $g_2(c)$*. Numerical simulations for three groups show that* $\underline{\varepsilon}(c_{ij}^{max}) < v/3$*.* [4.7]

4.8 Conclusions

We introduced incomplete information à la global games in a two-group max-max group contest with binary actions, relaxing the complete information assumption about the value of the prize contested and the cost of providing effort, separately. In the first case, we prove the non-existence of an equilibrium in (monotonic) switching strategies; in the second one, the existence of an equilibrium in (monotonic) switching-strategies emerges for a sufficiently high level of noise. The existence result is extended to the general M-group case. Moreover, in the two-group setting, with a bounded uniform prior, it is straightforward to calculate the probability of winning for each group and the expected payoffs at the equilibrium in switching strategies, but numerical examples are needed to assess the presence of the group-size paradox. We leave for future research the study of equilibrium uniqueness in the class of equilibrium in (monotonic) switching strategies and the exploration of possible independence from distributional assumptions. Therefore, it remains an open question whether introducing incomplete information à la global games in deterministic group contests with binary actions and the best-shot impact function reduces significantly the burden of equilibrium multiplicity or not. Equilibrium indeterminacy affects deterministic group contests with continuous efforts and a public good prize under both complete information and under incomplete information, as in Barbieri, Malueg, and Topolyan (2014) and Barbieri and Malueg (2016), respectively. Finding a setting in which payoff-relevant incomplete information, that is intrinsic private information, can lead to equilibrium uniqueness would complement the results by Barbieri and Topolyan (2024) obtained through the adoption of a group-public randomization device represents an exciting line of research. This work represents a first step in this direction and it helped clarifying the importance of within-group complementarities, i.e. the weakest-link case, with respect to substitutes, i.e. the best-shot setting, for the application of iterated-dominance

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[4.7] The *Mathematica* code is available upon request.

arguments as in Carlsson and Damme (1993a). We would like to stress that the study of the properties of such equilibria is relevant for applications of deterministic two-group contests with binary actions, among which we emphasized the classical examples stemming from the related literature in the introduction, such as *R&D* competition and military conflict.

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4.A PROOFS

4.A.1 Proof of Proposition 4.1

Suppose

$$(\gamma_1, \gamma_2) \text{ such that } \gamma_j \in (0,1) \text{ and } 1 \leq \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} \leq n_j - 1 \ \forall j \in \{1,2\} \ .$$

Then,

$$max\left\{\mathbf{x}_1\right\} = max\left\{\mathbf{x}_2\right\} \ .$$

Suppose $2 \leq \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} \leq n_j - 1$ and $x_j\left(i\right) = 1$, then

$$\pi_{ij}\left(\gamma_j, \gamma_{-j}\right) = \frac{v}{2} - 1 \ .$$

If agent ij deviates to $x_j\left(i\right) = 0$, then

$$max\left\{\mathbf{x}_1\right\} = max\left\{\mathbf{x}_2\right\}$$

so that the deviation payoff is

$$\pi_{ij}^D\left(\gamma_j^-, \gamma_{-j}\right) = \frac{v}{2}.$$

Hence, for player ij there is an incentive to deviate $\forall v \in R$, since

$$\frac{v}{2} - 1 < \frac{v}{2} \ \forall v \in \mathbb{R} \ .$$

On the other hand, suppose $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1$ and $x_j\left(i\right) = 1$, then

$$\pi_{ij}\left(\gamma_j, \gamma_{-j}\right) = \frac{v}{2} - 1 \ .$$

If agent ij deviates to $x_j(i) = 0$, then

$$max\{\mathbf{x}_1\} < max\{\mathbf{x}_2\}$$

so that the deviation payoff is

$$\pi_{ij}^D\left(\gamma_j^-, \gamma_{-j}\right) = 0.$$

Hence, for player ij there is no incentive to deviate if and only if $v \geq 2$, since

$$\frac{v}{2} - 1 \geq 0 \ \forall v \geq 2 \ .$$

Moreover, suppose $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 0$ and $x_j(i) = 1$, then

$$\pi_{ij}(\gamma_j, \gamma_{-j}) = \frac{v}{2} \ .$$

If agent ij deviates to $x_j(i) = 1$, then

$$max\{\mathbf{x}_1\} = max\{\mathbf{x}_2\}$$

so that the deviation payoff is

$$\pi_{ij}^D\left(\gamma_j^-, \gamma_{-j}\right) = \frac{v}{2} - 1 \ .$$

Hence, for player ij there is no incentive to deviate $\forall v \in R$, since

$$\frac{v}{2} - 1 < \frac{v}{2} \ \forall v \in \mathbb{R} \ .$$

Thus,

$$(\gamma_1, \gamma_2) \text{ such that } \gamma_j \in (0,1) \text{ and } \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1 \ \forall j \in \{1,2\}$$

is a Nash equilibrium in pure strategies $\forall v \geq 2$.

Suppose

$$(\gamma_1, \gamma_2) = (1,1) \ .$$

Then,

$$max\left\{\mathbf{x}_1\right\} = max\left\{\mathbf{x}_2\right\}$$

so that

$$\pi_{ij}\left(\gamma_j, \gamma_{-j}\right) = \frac{v}{2} - 1 \ .$$

If agent ij deviates to $x_j\left(i\right) = 0$, then

$$max\left\{\mathbf{x}_j\right\} = max\left\{\mathbf{x}_{-j}\right\}$$

so that the deviation payoff is

$$\pi_{ij}^{D}\left(\gamma_j^{-}, \gamma_{-j}\right) = \frac{v}{2} \ .$$

Hence, for player ij there is an incentive to deviate $\forall v \in R$ since

$$\frac{v}{2} - 1 < \frac{v}{2} \forall v \in \mathbb{R} \ .$$

Thus,

$$\left(\gamma_1, \gamma_2\right) = \left(1, 1\right)$$

is not a Nash equilibrium in pure strategies $\forall v \in \mathbb{R}$.

Suppose

$$\left(\gamma_j, \gamma_{-j}\right) = \left(1, 0\right) \forall j \in \left\{1, 2\right\} \ .$$

Then,

$$max\left\{\mathbf{x}_j\right\} = 1 > max\left\{\mathbf{x}_{-j}\right\} = 0$$

so that

$$\pi_{ij}\left(\gamma_j, \gamma_{-j}\right) = v - 1 \ .$$

If agent ij deviates to $x_j\left(i\right) = 0$, then

$$max\left\{\mathbf{x}_j\right\} > max\left\{\mathbf{x}_{-j}\right\}$$

so that the deviation payoff is

$$\pi_{ij}^{D}\left(\gamma_j^{-},\gamma_{-j}\right)=v\ .$$

Hence, for agent ij there is an incentive to deviate $\forall v \in R$, since

$$v-1<v\forall v\in\mathbb{R}\ .$$

On the other hand, for completeness sake, if agent $i-j$ deviates to $x_{-j}\left(i\right)=1$, then

$$max\left\{\mathbf{x}_j\right\}=max\left\{\mathbf{x}_{-j}\right\}=1$$

so that the deviation payoff is

$$\pi_{i-j}^{D}\left(\gamma_j,\gamma_{-j}^{+}\right)=\frac{v}{2}-1\ .$$

Hence, for player $i-j$ there are no incentives to deviate if and only if

$$\pi_{i-j}\left(\gamma_j,\gamma_{-j}\right)=0\geq\pi_{i-j}^{D}\left(\gamma_j,\gamma_{-j}^{+}\right)=\frac{v}{2}-1\Leftrightarrow v\leq 2\ .$$

Thus,

$$\left(\gamma_j,\gamma_{-j}\right)=\left(1,0\right)\ \ \forall j\in\left\{1,2\right\}$$

is not a Nash equilibrium in pure strategies for any $v\in\mathbb{R}$.

Suppose

$$\left(\gamma_1,\gamma_2\right)=\left(0,0\right)\ \ .$$

Then

$$max\left\{\mathbf{x}_1\right\}=max\left\{\mathbf{x}_2\right\}$$

so that

$$\pi_{ij}\left(\gamma_1,\gamma_2\right)=\frac{v}{2}\ \forall j\in\left\{1,2\right\}\ \ .$$

If agent ij deviates to $x_j\left(i\right)=1$, then

$$max\left\{\mathbf{x}_j\right\}>max\left\{\mathbf{x}_{-j}\right\}$$

so that

$$\pi_{ij}^{D}\left(\gamma_j^{+},\gamma_{-j}\right)=v-1\ .$$

Hence, for player ij there is no incentive to deviate if and only if

$$\pi_{ij}\left(\gamma_j,\gamma_{-j}\right)=\frac{v}{2}\geq\pi_{ij}^{D}\left(\gamma_j^{D},\gamma_{-j}\right)=v-1\Leftrightarrow v\leq 2\ \forall j\in\left\{1,2\right\}\ .$$

Thus,

$$\left(\gamma_1,\gamma_2\right)=\left(0,0\right)$$

is a Nash equilibrium in pure strategies for any $v\leq 2$.

Let $\sigma_{ij}\left(x_{ij}=1\right)$ be the within-group symmetric randomization over pure strategy $x_{ij}=1$ for player ij, then

$$EU_{ij}\left(x_{ij}=1\right)=EU_{ij}\left(x_{ij}=0\right)\Leftrightarrow$$

$$\Leftrightarrow Prob\left(n_{-j}\gamma_{-j}=0\right)\cdot\left(v-1\right)+Prob\left(n_{-j}\gamma_{-j}\geq 1\right)\cdot\left(\frac{v}{2}-1\right)=$$

$$=\Big(Prob\left(n_j\gamma_j\geq 1\right)\cdot Prob\left(n_{-j}\gamma_{-j}\geq 1\right)+Prob\left(n_j\gamma_j=0\right)\cdot Prob\left(n_{-j}\gamma_{-j}=0\right)\Big)\cdot\frac{v}{2}+$$

$$+Prob\left(n_j\gamma_j=0\right)\cdot Prob\left(n_{-j}\gamma_{-j}\geq 1\right)\cdot\left(0\right)+Prob\left(n_j\gamma_j\geq 1\right)\cdot Prob\left(n_{-j}\gamma_{-j}=0\right)\cdot v\Leftrightarrow$$

$$\Leftrightarrow\Big(1-\left(\sigma_{i-j}\left(x_{i-j}=1\right)\right)^{n_{-j}}\Big)\cdot\left(v-1\right)+\Big(1-\left(1-\sigma_{i-j}\left(x_{i-j}=1\right)\right)^{n_{-j}}\Big)\cdot\left(\frac{v}{2}-1\right)=$$

$$=\Big(\Big(1-\left(1-\sigma_{ij}\left(x_{ij}=1\right)\right)^{n_j-1}\Big)\cdot\left(1-\left(1-\sigma_{i-j}\left(x_{i-j}=1\right)\right)^{n_{-j}}\right)+$$

$$+\left(1-\sigma_{ij}\left(x_{ij}=1\right)\right)^{n_j-1}\cdot\left(1-\sigma_{i-j}\left(x_{i-j}=1\right)\right)^{n_{-j}}\Big)\cdot\frac{v}{2}+$$

$$+\Big(1-\left(1-\sigma_{ij}\left(x_{ij}=1\right)\right)^{n_j-1}\Big)\cdot\left(1-\sigma_{i-j}\left(x_{i-j}=1\right)\right)^{n_{-j}}\cdot v$$

$$\Leftrightarrow\sigma_{ij}^{*}\left(x_{ij}=1\right)=\left(1-\frac{2}{v}\right)^{1/\left(n_j-1\right)}\quad\forall i\in\left\{1,\ldots,n_j\right\}\text{ and }j\in\left\{1,2\right\}\ .$$

Thus,

$$\sigma^*_{ij}\left(x_{ij}=1\right)=\left(1-\frac{2}{v}\right)^{1/(n_j-1)}\quad\forall i\in\{1,\ldots,n_j\}\ \text{ and } j\in\{1,2\}\ .$$

is a Nash equilibrium in within-group symmetric strictly-mixed strategies $\forall v>2$.

QED

4.A.2 Proof of Proposition 4.2

Following the formulation of payoff-dominance and risk-dominance concepts by Harsanyi and Selten (1988), it is straightforward to state that:

- for $v>2$ and any ij such that $x_{ij}=0$, any (γ_1,γ_2) such that $\gamma_j\in(0,1)$ and $\sum_{i=1}^{n_j}\mathbb{1}_{x_{ij}=1}=1\ \forall j\in\{1,2\}$ payoff-dominate (γ_1,γ_2) such that $\gamma_j\in(0,1)$ and $\sum_{i=1}^{n_j}\mathbb{1}_{x_{ij}=1}=1\ \forall j\in\{1,2\}$ in which $x_{ij}=1, since$

  $$\pi_{ij}\left((\gamma_j,\gamma_{-j})\text{ s.t. }x_{ij}=0\right)=\frac{v}{2}>\pi_{ij}\left((\gamma_j,\gamma_{-j})\text{ s.t. }x_{ij}=1\right)=\frac{v}{2}-1\forall v>2\text{ and }\forall j\in\{1,2\}\ .$$

- for $v>4$, (γ_1,γ_2) such that $\gamma_j\in(0,1)$ and $\sum_{i=1}^{n_j}\mathbb{1}_{x_{ij}=1}=1\ \forall j\in\{1,2\}$ are the risk-dominant equilibrium strategy profiles for any ij such that $x_{ij}=1$. As a matter of fact, let us compare the deviation losses of (γ_1,γ_2) such that $\gamma_j\in(0,1)$ and $\sum_{i=1}^{n_j}\mathbb{1}_{x_{ij}=1}=1\ \forall j\in\{1,2\}$ for any ij such that $x_{ij}=1$ and (γ_1,γ_2) such that $\gamma_j\in(0,1)$ and $\sum_{i=1}^{n_j}\mathbb{1}_{x_{ij}=1}=1\ \forall j\in\{1,2\}$ for any ij such that $x_{ij}=0$. Then, for any ij:

  $$\left(\frac{v}{2}-1-0\right)>\left(\frac{v}{2}-\left(\frac{v}{2}-1\right)\right)\Leftrightarrow v>4,$$

  that is, for $v>4$, (γ_1,γ_2) such that $\gamma_j\in(0,1)$ and $\sum_{i=1}^{n_j}\mathbb{1}_{x_{ij}=1}=1\ \forall j\in\{1,2\}$ for any ij such that $x_{ij}=1$ are associated with the largest Nash difference; [4.A1]

- for $2<v<4$, (γ_1,γ_2) such that $\gamma_j\in(0,1)$ and $\sum_{i=1}^{n_j}\mathbb{1}_{x_{ij}=1}=1\ \forall j\in\{1,2\}$ are the risk-dominant equilibrium strategy profiles for any ij such that $x_{ij}=0$. Clearly this follows

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[4.A1] Here we do not employ deviation losses to determine the risk-dominant equilibrium, since we define it at a single-player level and not at group level. In the latter case, for $v>2$ all Nash equilibria in pure strategies would be clearly equivalent in terms of risk-dominance.

from what shown at the previous point for both groups;

- for $v = 2$, $(\gamma_j, \gamma_{-j}) = (\gamma_j, 0)$ such that $\gamma_j \in (0,1)$ and $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1 \ \forall j \in \{1,2\}$ are the payoff-dominant equilibria for any player ij such that $x_{ij} = 0$, since

$$\pi_{ij}\left((\gamma_j, 0) \text{ s.t. } x_{ij} = 0\right) = v \ ,$$

which is the highest attainable payoff;

- for $v = 2$, (γ_1, γ_2) such that $\gamma_j \in (0,1)$ and $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1 \ \forall j \in \{1,2\}$ and $(\gamma_j, \gamma_{-j}) = (\gamma_j, 0)$ such that $\gamma_j \in (0,1)$ and $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 0 \ \forall j \in \{1,2\}$ are the risk-dominant equilibria for any ij such that $x_{ij} = 0$. As a matter of fact, let us compare the deviation losses of the set of equilibria above with the ones of (γ_1, γ_2) such that $\gamma_j \in (0,1)$ and $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 0 \ \forall j \in \{1,2\}$, $(\gamma_j, \gamma_{-j}) = (\gamma_j, 0)$ such that $\gamma_j \in (0,1)$ and $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1 \ \forall j \in \{1,2\}$ for any ij such that $x_{ij} = 1$ and $(\gamma_1, \gamma_2) = (0,0)$. Then,

$$\left(\frac{v}{2} - \left(\frac{v}{2} - 1\right)\right) = (v - (v-1)) > \left(\frac{v}{2} - 1 - 0\right) = \left(v - 1 - \frac{v}{2}\right) = \left(0 - \left(\frac{v}{2} - 1\right)\right) = \left(\frac{v}{2} - (v-1)\right) \ .$$

QED

4.A.3 Proof of Proposition 4.3

- Suppose

$$(\gamma_1, \gamma_2) \text{ such that } \gamma_j \in (0,1) \text{ and } 1 \le \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} \le n_j - 1 \ \forall j \in \{1,2\} \ .$$

Then,

$$max\left\{\mathbf{x}_1\right\} = max\left\{\mathbf{x}_2\right\} \ .$$

Suppose $x_j(i) = 1$ and $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1$, then

$$\pi_{ij}\left(\gamma_j, \gamma_{-j}\right) = \frac{v}{2} - c \ .$$

If agent ij deviates to $x_j(i) = 0$, then

$$max\left\{\mathbf{x}_j\right\} < max\left\{\mathbf{x}_{-j}\right\}$$

so that the deviation payoff is

$$\pi_{ij}^{D}\left(\gamma_j^{-}, \gamma_{-j}\right) = 0 \ .$$

Hence, for player ij there is no incentive to deviate $\forall v \in \mathbb{R}_{++}$ if and only if

$$\frac{v}{2} - c \geq 0 \Leftrightarrow c \leq \frac{v}{2} \ .$$

Moreover, suppose $x_j(i) = 0$, then

$$\pi_{ij}\left(\gamma_j, \gamma_{-j}\right) = \frac{v}{2} \ .$$

If agent ij deviates to $x_j(i) = 1$, then

$$max\left\{\mathbf{x}_1\right\} = max\left\{\mathbf{x}_2\right\}$$

so that the deviation payoff is

$$\pi_{ij}^{D}\left(\gamma_j^{+}, \gamma_{-j}\right) = \frac{v}{2} - c \ .$$

Hence, for player ij there is no incentive to deviate if and only if

$$\frac{v}{2} \geq \frac{v}{2} - c \Leftrightarrow c \geq 0 \ .$$

Conversely, suppose $x_j(i) = 1$ and $1 < \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} \leq n_j - 1$, then

$$\pi_{ij}\left(\gamma_j, \gamma_{-j}\right) = \frac{v}{2} - c \ .$$

If agent ij deviates to $x_j(i) = 0$, then

$$max\{\mathbf{x}_j\} = max\{\mathbf{x}_{-j}\}$$

so that the deviation payoff is

$$\pi_{ij}^D\left(\gamma_j^-, \gamma_{-j}\right) = \frac{v}{2}.$$

Hence, for player ij there is no incentive to deviate $\forall v \in \mathbb{R}_{++}$ if and only if

$$\frac{v}{2} - c \geq \frac{v}{2} \Leftrightarrow c \leq 0 \ .$$

On the other hand, suppose $x_j(i) = 0$, then

$$\pi_{ij}(\gamma_j, \gamma_{-j}) = \frac{v}{2} \ .$$

If agent ij deviates to $x_j(i) = 1$, then

$$max\{\mathbf{x}_1\} = max\{\mathbf{x}_2\}$$

so that the deviation payoff is

$$\pi_{ij}^D\left(\gamma_j^+, \gamma_{-j}\right) = \frac{v}{2} - c \ .$$

Hence, for player ij there is no incentive to deviate if and only if

$$\frac{v}{2} \geq \frac{v}{2} - c \Leftrightarrow c \geq 0 \ .$$

Thus, for $0 < c \leq \frac{v}{2}$,

$$(\gamma_1, \gamma_2) \text{ such that } \gamma_j \in (0,1)$$

is a Nash equilibrium if and only if $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1$.

Moreover,

$$(\gamma_1, \gamma_2) \text{ such that } \gamma_j \in (0,1) \text{ and } 1 < \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} \leq n_j - 1 \; \forall j \in \{1,2\}$$

is a Nash equilibrium if and only if $c = 0$.

- Suppose

$$(\gamma_j, \gamma_{-j}) = (\gamma_j, 0) \text{ such that } \gamma_j \in (0,1) \text{ and } 1 \leq \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} \leq n_j - 1 \; \forall j \in \{1,2\} \; .$$

Then,

$$max\left\{\mathbf{x}_j\right\} > max\left\{\mathbf{x}_{-j}\right\} \; .$$

Suppose $x_j(i) = 1$ and $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1$, then

$$\pi_{ij}(\gamma_j, \gamma_{-j}) = v - c \text{ and } \pi_{i-j}(\gamma_j, \gamma_{-j}) = 0 \; .$$

If agent ij deviates to $x_j(i) = 0$, then

$$max\left\{\mathbf{x}_j\right\} = max\left\{\mathbf{x}_{-j}\right\}$$

so that the deviation payoff is

$$\pi_{ij}^{D}\left(\gamma_j^{-}, \gamma_{-j}\right) = \frac{v}{2}.$$

Hence, for player ij there is no incentive to deviate $\forall v \in \mathbb{R}_{++}$ if and only if

$$v - c \geq \frac{v}{2} \Leftrightarrow c \leq \frac{v}{2} \; .$$

Moreover, if agent $i-j$ deviates to $x_{-j}(i) = 1$, then

$$max\left\{\mathbf{x}_j\right\} = max\left\{\mathbf{x}_{-j}\right\}$$

so that the deviation payoff is

$$\pi_{ij}^{D}\left(\gamma_j, \gamma_{-j}^{-}\right) = \frac{v}{2} - c \ .$$

Hence, for player $i-j$ there is no incentive to deviate $\forall v \in \mathbb{R}_{++}$ if and only if

$$0 \geq \frac{v}{2} - c \Leftrightarrow c \geq \frac{v}{2} \ .$$

Conversely, suppose $x_j\left(i\right) = 1$ and $1 < \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} \leq n_j - 1$, then

$$\pi_{ij}\left(\gamma_j, \gamma_{-j}\right) = v - c \text{ and } \pi_{i-j}\left(\gamma_j, \gamma_{-j}\right) = 0 \ .$$

For player $i-j$ everything remains unchanged from the previous case, whereas if agent ij deviates to $x_j\left(i\right) = 0$, then

$$max\left\{\mathbf{x}_j\right\} > max\left\{\mathbf{x}_{-j}\right\}$$

so that the deviation payoff is

$$\pi_{ij}\left(\gamma_j^{-}, \gamma_{-j}\right) = v \ .$$

Hence, for player ij there is no incentive to deviate $\forall v \in \mathbb{R}_{++}$ if and only if

$$v - c \geq v \Leftrightarrow c \leq 0 \ .$$

Clearly, for players ij such that $x_{ij} = 0$, everything remains unchanged with respect to the previous case.

Thus,

$$\left(\gamma_j, \gamma_{-j}\right) = \left(\gamma_j, 0\right) \text{ such that } \gamma_j \in \left(0,1\right) \text{ and } \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1 \ \forall j \in \left\{1,2\right\} \ .$$

is a Nash equilibrium if and only if $c = \frac{v}{2}$.

- Suppose

$$\left(\gamma_j, \gamma_{-j}\right) = \left(\gamma_j, 1\right) \text{ such that } \gamma_j \in (0,1) \text{ and } 1 \leq \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} \leq n_j - 1 \; \forall j \in \{1,2\} \; .$$

Then,

$$max\left\{\mathbf{x}_j\right\} = max\left\{\mathbf{x}_{-j}\right\} \; .$$

Suppose $x_j\left(i\right) = 1$ and $1 < \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} \leq n_j - 1$, then

$$\pi_{ij}\left(\gamma_j, \gamma_{-j}\right) = \frac{v}{2} - c \text{ and } \pi_{i-j}\left(\gamma_j, \gamma_{-j}\right) = \frac{v}{2} - c \; .$$

If agent ij deviates to $x_j\left(i\right) = 0$, then

$$max\left\{\mathbf{x}_j\right\} = max\left\{\mathbf{x}_{-j}\right\}$$

so that the deviation payoff is

$$\pi_{ij}^D\left(\gamma_j^-, \gamma_{-j}\right) = \frac{v}{2} \; .$$

Hence, for player ij there is no incentive to deviate $\forall v \in \mathbb{R}_{++}$ if and only if

$$\frac{v}{2} - c \geq \frac{v}{2} \Leftrightarrow c \leq 0 \; .$$

Conversely, suppose $x_j\left(i\right) = 0$ and $1 < \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} \leq n_j - 1$, then

$$\pi_{ij}\left(\gamma_j, \gamma_{-j}\right) = \frac{v}{2} - c \text{ and } \pi_{i-j}\left(\gamma_j, \gamma_{-j}\right) = \frac{v}{2} \; .$$

If agent ij deviates to $x_j\left(i\right) = 1$, then

$$max\left\{\mathbf{x}_j\right\} = max\left\{\mathbf{x}_{-j}\right\}$$

so that the deviation payoff is

$$\pi_{ij}^{D}\left(\gamma_{j}^{-},\gamma_{-j}\right)=\frac{v}{2}-c\ .$$

Hence, for player ij there is no incentive to deviate $\forall v\in\mathbb{R}_{++}$ if and only if

$$\frac{v}{2}\geq\frac{v}{2}-c\Leftrightarrow c\geq 0\ .$$

The same arguments hold for player $i-j$, so that for player ij there is no incentive to deviate $\forall v\in\mathbb{R}_{++}$ if and only if

$$\frac{v}{2}-c\geq\frac{v}{2}\Leftrightarrow c\leq 0\ .$$

On the other hand, suppose $x_{j}\left(i\right)=1$ and $\sum_{i=1}^{n_{j}}\mathbb{1}_{x_{ij}=1}=1$, then

$$\pi_{ij}\left(\gamma_{j},\gamma_{-j}\right)=\frac{v}{2}-c\text{ and }\pi_{i-j}\left(\gamma_{j},\gamma_{-j}\right)=\frac{v}{2}-c\ .$$

If agent ij deviates to $x_{j}\left(i\right)=0$, then

$$max\left\{\mathbf{x}_{j}\right\}<max\left\{\mathbf{x}_{-j}\right\}$$

so that the deviation payoff is

$$\pi_{ij}^{D}\left(\gamma_{j}^{-},\gamma_{-j}\right)=0\ .$$

Hence, for player ij there is no incentive to deviate $\forall v\in\mathbb{R}_{++}$ if and only if

$$\frac{v}{2}-c\geq 0\Leftrightarrow c\leq\frac{v}{2}\ .$$

Conversely, suppose $x_{j}\left(i\right)=0$ and $\sum_{i=1}^{n_{j}}\mathbb{1}_{x_{ij}=1}=1$, then

$$\pi_{ij}\left(\gamma_{j},\gamma_{-j}\right)=\frac{v}{2}-c\text{ and }\pi_{i-j}\left(\gamma_{j},\gamma_{-j}\right)=\frac{v}{2}\ .$$

If agent ij deviates to $x_j\left(i\right)=1$, then

$$max\left\{\mathbf{x}_j\right\}=max\left\{\mathbf{x}_{-j}\right\}$$

so that the deviation payoff is

$$\pi^D_{ij}\left(\gamma^-_j,\gamma_{-j}\right)=\frac{v}{2}-c\ .$$

Hence, for player ij there is no incentive to deviate $\forall v\in\mathbb{R}_{++}$ if and only if

$$\frac{v}{2}\geq\frac{v}{2}-c\Leftrightarrow c\geq 0\ .$$

If agent $i-j$ deviates to $x_j\left(i\right)=0$, then

$$max\left\{\mathbf{x}_j\right\}=max\left\{\mathbf{x}_{-j}\right\}$$

so that the deviation payoff is

$$\pi^D_{ij}\left(\gamma^-_j,\gamma_{-j}\right)=\frac{v}{2}\ .$$

Hence, for player ij there is no incentive to deviate $\forall v\in\mathbb{R}_{++}$ if and only if

$$\frac{v}{2}-c\geq\frac{v}{2}\Leftrightarrow c\leq 0\ .$$

Thus,

$$\left(\gamma_j,\gamma_{-j}\right)=\left(\gamma_j,1\right)\ \text{such that}\ \gamma_j\in\left(0,1\right)\ \text{and}\ 1\leq\sum_{i=1}^{n_j}\mathbb{1}_{x_{ij}=1}\leq n_j-1\ \forall j\in\left\{1,2\right\}$$

is a Nash equilibrium in pure strategies if and only if $c=0$.

- Suppose

$$(\gamma_1, \gamma_2) = (1, 1) \ \ .$$

Then,

$$max\{\mathbf{x}_1\} = max\{\mathbf{x}_2\}$$

so that

$$\pi_{ij}(\gamma_j, \gamma_{-j}) = \frac{v}{2} - c \ .$$

If agent ij deviates to $x_j(i) = 0$, then

$$max\{\mathbf{x}_j\} = max\{\mathbf{x}_{-j}\}$$

so that the deviation payoff is

$$\pi_{ij}^{D}\left(\gamma_j^{-}, \gamma_{-j}\right) = \frac{v}{2} \ .$$

Hence, for player ij there is no incentive to deviate if and only if

$$\frac{v}{2} - c \geq \frac{v}{2} \Leftrightarrow c \leq 0 \ .$$

Thus,

$$(\gamma_1, \gamma_2) = (1, 1)$$

is a Nash equilibrium in pure strategies if and only if $c \leq 0$.

- Suppose

$$(\gamma_j, \gamma_{-j}) = (1, 0) \, \forall j \in \{1, 2\} \ \ .$$

Then,

$$max\{\mathbf{x}_j\} > max\{\mathbf{x}_{-j}\}$$

so that

$$\pi_{ij}(\gamma_j, \gamma_{-j}) = v - c \ .$$

If agent ij deviates to $x_j(i) = 0$, then

$$max\{\mathbf{x}_j\} > max\{\mathbf{x}_{-j}\}$$

so that the deviation payoff is

$$\pi_{ij}^{D}\left(\gamma_j^{-}, \gamma_{-j}\right) = v \ .$$

Hence, for agent ij there is no incentive to deviate if and only if

$$v - c \geq v \Leftrightarrow c \leq 0 \ .$$

On the other hand, if agent $i-j$ deviates to $x_{-j}(i) = 1$, then

$$max\{\mathbf{x}_j\} = max\{\mathbf{x}_{-j}\} = 0$$

so that the deviation payoff is

$$\pi_{i-j}^{D}\left(\gamma_j, \gamma_{-j}^{+}\right) = \frac{v}{2} - c \ .$$

Hence, for player $i-j$ there is no incentive to deviate if and only if

$$\pi_{i-j}(\gamma_j, \gamma_{-j}) = 0 \geq \pi_{i-j}^{D}\left(\gamma_j, \gamma_{-j}^{+}\right) = \frac{v}{2} - c \Leftrightarrow c \geq \frac{v}{2} \ .$$

Thus,

$$(\gamma_j, \gamma_{-j}) = (1, 0) \ \ \forall j \in \{1, 2\}$$

is a not a Nash equilibrium in pure strategies for any $c \in R$.

- Suppose

$$(\gamma_1, \gamma_2) = (0, 0) \ \ .$$

Then

$$max\{\mathbf{x}_1\} = max\{\mathbf{x}_2\}$$

so that

$$\pi_{ij}\left(\gamma_1,\gamma_2\right)=\frac{v}{2}\ \forall j\in\{1,2\}\ .$$

If agent ij deviates to $x_j\left(i\right)=1$, then

$$max\left\{\mathbf{x}_1\right\}>max\left\{\mathbf{x}_2\right\}$$

so that

$$\pi_{ij}^{D}\left(\gamma_j^{+},\gamma_{-j}\right)=v-c\ .$$

Hence, for player ij there is no incentive to deviate if and only if

$$\pi_{ij}\left(\gamma_j,\gamma_{-j}\right)=\frac{v}{2}\geq\pi_{ij}^{D}\left(\gamma_j^{D},\gamma_{-j}\right)=v-c\Leftrightarrow c\geq\frac{v}{2}\ \forall j\in\{1,2\}\ .$$

Thus,

$$\left(\gamma_1,\gamma_2\right)=(0,0)$$

is a Nash equilibrium in pure strategies for any $c\geq\frac{v}{2}$.

- Let $\sigma_{ij}\left(x_{ij}=1\right)$ be the within-group symmetric randomization over pure strategy $x_{ij}=1$ for player ij $\forall j\in\{1,2\}$, then

$$EU_{ij}\left(x_{ij}=1\right)=EU_{ij}\left(x_{ij}=0\right)\Leftrightarrow$$

$$\Leftrightarrow Prob\left(n_{-j}\gamma_{-j}=0\right)\cdot(v-c)+Prob\left(n_{-j}\gamma_{-j}\geq1\right)\cdot\left(\frac{v}{2}-c\right)=$$

$$=\Big(Prob\left(n_j\gamma_j\geq1\right)\cdot Prob\left(n_{-j}\gamma_{-j}\geq1\right)+Prob\left(n_j\gamma_j=0\right)\cdot Prob\left(n_{-j}\gamma_{-j}=0\right)\Big)\cdot\frac{v}{2}+$$

$$+Prob\left(n_j\gamma_j=0\right)\cdot Prob\left(n_{-j}\gamma_{-j}\geq1\right)\cdot(0)+Prob\left(n_j\gamma_j\geq1\right)\cdot Prob\left(n_{-j}\gamma_{-j}=0\right)\cdot v\Leftrightarrow$$

$$\Leftrightarrow\left(1-\left(\sigma_{i-j}\left(x_{i-j}=1\right)\right)^{n-j}\right)\cdot(v-c)+\left(1-\left(1-\sigma_{i-j}\left(x_{i-j}=1\right)\right)^{n-j}\right)\cdot\left(\frac{v}{2}-c\right)=$$

$$=\left(\left(1-\left(1-\sigma_{ij}\left(x_{ij}=1\right)\right)^{n_j-1}\right)\cdot\left(1-\left(1-\sigma_{i-j}\left(x_{i-j}=1\right)\right)^{n-j}\right)+\right.$$

$$+\left(1-\sigma_{ij}\left(x_{ij}=1\right)\right)^{n_j-1}\cdot\left(1-\sigma_{i-j}\left(x_{i-j}=1\right)\right)^{n-j}\Big)\cdot\frac{v}{2}+$$

$$+\Big(1-\left(1-\sigma_{ij}\left(x_{ij}=1\right)\right)^{n_j-1}\Big)\cdot\left(1-\sigma_{i-j}\left(x_{i-j}=1\right)\right)^{n-j}\cdot v$$

$$\Leftrightarrow \sigma^*_{ij}\left(x_{ij}=1\right)=\left(1-\frac{2c}{v}\right)^{1/(n_j-1)} \quad \forall i\in\{1,\ldots,n_j\} \text{ and } j\in\{1,2\}\ .$$

Thus,

$$\sigma^*_{ij}\left(x_{ij}=1\right)=\left(1-\frac{2c}{v}\right)^{1/(n_j-1)} \quad \forall i\in\{1,\ldots,n_j\} \text{ and } j\in\{1,2\}\ .$$

is a Nash equilibrium in within-group symmetric strictly-mixed strategies $\forall\ 0<c<\frac{v}{2}$.

QED

4.A.4 Proof of Proposition 4.4

Following the formulation of payoff-dominance and risk-dominance concepts by Harsanyi and Selten (1988), it is straightforward to state that in the $BMMAGC^{*b}$:

- for $0<c<\frac{v}{2}$ and any ij such that $x_{ij}=0$, any (γ_1,γ_2) such that $\gamma_j\in(0,1)$ and $\sum_{i=1}^{n_j}\mathbb{1}_{x_{ij}=1}=1\ \forall j\in\{1,2\}$ payoff-dominate (γ_1,γ_2) such that $\gamma_j\in(0,1)$ and $\sum_{i=1}^{n_j}\mathbb{1}_{x_{ij}=1}=1\ \forall j\in\{1,2\}$ in which $x_{ij}=1$,since $\forall\ 0<c<\frac{v}{2}$ and $\forall j\in\{1,2\}$

$$\pi_{ij}\left(\left(\gamma_j,\gamma_{-j}\right)\text{ s.t. } x_{ij}=0\right)=\frac{v}{2}>\pi_{ij}\left(\left(\gamma_j,\gamma_{-j}\right)\text{ s.t. } x_{ij}=1\right)=\frac{v}{2}-c\ .$$

- for $0<c<\frac{v}{4}$, (γ_1,γ_2) such that $\gamma_j\in(0,1)$ and $\sum_{i=1}^{n_j}\mathbb{1}_{x_{ij}=1}=1\ \forall j\in\{1,2\}$ are the risk-dominant equilibrium strategy profiles for any ij such that $x_{ij}=1$. As a matter of fact, let us compare the deviation losses of (γ_1,γ_2) such that $\gamma_j\in(0,1)$ and $\sum_{i=1}^{n_j}\mathbb{1}_{x_{ij}=1}=1\ \forall j\in\{1,2\}$ for any ij such that $x_{ij}=1$ and (γ_1,γ_2) such that $\gamma_j\in(0,1)$ and $\sum_{i=1}^{n_j}\mathbb{1}_{x_{ij}=1}=1\ \forall j\in\{1,2\}$ for any ij such that $x_{ij}=0$. Then, for any ij:

$$\left(\frac{v}{2}-c-0\right)>\left(\frac{v}{2}-\left(\frac{v}{2}-c\right)\right)\Leftrightarrow 0<c<\frac{v}{4},$$

that is, for $0<c<\frac{v}{4}$, (γ_1,γ_2) such that $\gamma_j\in(0,1)$ and $\sum_{i=1}^{n_j}\mathbb{1}_{x_{ij}=1}=1\ \forall j\in\{1,2\}$ for any ij such that $x_{ij} = 1$ are associated with the largest Nash difference; [4.A2]

- for $\frac{v}{4} < c < \frac{v}{2}$, (γ_1, γ_2) such that $\gamma_j \in (0,1)$ and $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1 \ \forall j \in \{1,2\}$ are the risk-dominant equilibrium strategy profiles for any ij such that $x_{ij} = 0$. Clearly this follows from what shown at the previous point for both groups.

- for $c = \frac{v}{2}$, $(\gamma_j, \gamma_{-j}) = (\gamma_j, 0)$ such that $\gamma_j \in (0,1)$ and $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1 \ \forall j \in \{1,2\}$ are the payoff-dominant equilibria for any player ij such that $x_{ij} = 0$, since

$$\pi_{ij}\left((\gamma_j, 0) \text{ s.t. } x_{ij} = 0\right) = v \ ,$$

  which is the highest attainable payoff;

- for $c = \frac{v}{2}$, (γ_1, γ_2) such that $\gamma_j \in (0,1)$ and $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1 \ \forall j \in \{1,2\}$ and $(\gamma_j, \gamma_{-j}) = (\gamma_j, 0)$ such that $\gamma_j \in (0,1)$ and $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 0 \ \forall j \in \{1,2\}$ are the risk-dominant equilibria for any ij such that $x_{ij} = 0$. As a matter of fact, let us compare the deviation losses of the set of equilibria above with the ones of (γ_1, γ_2) such that $\gamma_j \in (0,1)$ and $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 0 \ \forall j \in \{1,2\}$, $(\gamma_j, \gamma_{-j}) = (\gamma_j, 0)$ such that $\gamma_j \in (0,1)$ and $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1 \ \forall j \in \{1,2\}$ for any ij such that $x_{ij} = 1$ and $(\gamma_1, \gamma_2) = (0,0)$. Then,

$$\left(\frac{v}{2} - \left(\frac{v}{2} - c\right)\right) = (v - (v - c)) > \left(\frac{v}{2} - c - 0\right) = \left(v - c - \frac{v}{2}\right) = \left(0 - \left(\frac{v}{2} - c\right)\right) = \left(\frac{v}{2} - (v - c)\right) \ .$$

- for $c = 0$, there is no payoff-dominant equilibrium strategy profile $\forall ij \in \{1, \dots, N\}$. As a matter of fact,

$$\pi_{ij}\left(\forall \left(\gamma_1, \gamma_2\right) \in NE\right) = \frac{v}{2} \ .$$

- for $c = 0$, (γ_1, γ_2) such that $\gamma_j \in (0,1)$ and $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1 \ \forall j \in \{1,2\}$ are the risk-dominant equilibrium strategy profiles for any ij such that $x_{ij} = 1$. As a matter of fact, let us compare the deviation losses of the set of equilibria above with the ones of $\{(\gamma_1, \gamma_2)$ such that $\gamma_j \in (0,1)$ and $\forall j \in \{1,2\}\}$ in which $x_{ij} = 0$,$\{(\gamma_j, \gamma_{-j}) = (\gamma_j, 1)$ such that $\gamma_j \in (0,1)$ and $\forall j \in \{1,2\}\}$ and $\{(\gamma_1, \gamma_2) = (1,1)\}$. Then,

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[4.A2] Here we do not employ Nash products to determine the risk-dominant equilibrium, since we define it at a single-player level and not at group level. In the latter case, for $0 < c < \frac{v}{2}$ all Nash equilibria in pure strategies would be clearly equivalent in terms of risk-dominance.

$$\left(\frac{v}{2}-c-0\right)>\left(\frac{v}{2}-\left(\frac{v}{2}-c\right)\right)=\left(\frac{v}{2}-c-\frac{v}{2}\right)=\left(\frac{v}{2}-\left(\frac{v}{2}-c\right)\right)=\left(\frac{v}{2}-c-\frac{v}{2}\right)\ .$$

QED

4.A.5 Proof of Proposition 4.5

In the $g_1(v)$, note that $E(V|v_{ij})=v_{ij}$ so that $V|v_{ij}\sim U(v_{ij}-\varepsilon,v_{ij}+\varepsilon)$. Furthermore, for $v_{ij}\in[\underline{v}+\varepsilon,\overline{v}-\varepsilon]$, the conditional distribution of the teammates' or opponents' observation will be centered around v_{ij} with a triangular distribution over the support $[v_{ij}-2\varepsilon,v_{ij}+2\varepsilon]$. Hence, $Prob\,[V_{-ij}<v_{ij}|v_{ij}]=Prob[V_{-ij}>v_{ij}|v_{ij}]=\frac{1}{2}\ \forall ij\in\{1,\dots,n_j\}$ and $j\in\{1,2\}$.

Now, suppose player $ij\in\{1,\dots,n_j\}\ \forall j\in\{1,2\}$ observes $v_{ij}<2$. Then, ij's conditionally expected payoff from exerting effort, that is choosing $x_{ij}=1$, is smaller than the one from exerting no effort, that is choosing $x_{ij}=0$. Accordingly, $x_{ij}=0$ is a conditionally strictly dominant action for player $ij\in\{1,\dots,n_j\}\ \forall j\in\{1,2\}$ whenever she observes $v_{ij}<2$. Let us denote p_{-ij} the conditional expected probability attached by player ij to any other player $-ij$ within the same group choosing not to exert effort. Likewise, p_{i-j} is the conditional expected probability attached by ij to any other member of the rival group $-j$ not exerting effort. Then, for player ij the conditionally expected payoff from exerting effort equals

$$Prob\,(n_{-j}\gamma_{-j}=0)\cdot(v_{ij}-1)+Prob\,(n_{-j}\gamma_{-j}\geq 1)\cdot\left(\frac{v_{ij}}{2}-1\right)=$$

$$=\prod_{i}p_{i-j}\cdot(v_{ij}-1)+\left(1-\prod_{i}p_{i-j}\right)\cdot\left(\frac{v_{ij}}{2}-1\right)=\left(1+\prod_{i}p_{i-j}\right)\left(\frac{v_{ij}}{2}-1\right),$$

while the conditionally expected payoff from not exerting effort equals

$$\Big(Prob\,(n_j\gamma_j\geq 1)\cdot Prob\,(n_{-j}\gamma_{-j}\geq 1)+Prob\,(n_j\gamma_j=0)\cdot Prob\,(n_{-j}\gamma_{-j}=0)\Big)\cdot\frac{v_{ij}}{2}+$$

$$+Prob\,(n_j\gamma_j=0)\cdot Prob\,(n_{-j}\gamma_{-j}\geq 1)\cdot(0)+Prob\,(n_j\gamma_j\geq 1)\cdot Prob\,(n_{-j}\gamma_{-j}=0)\cdot v_{ij}\Leftrightarrow$$

$$\Leftrightarrow\left(\left(1-\prod_{-i}p_{-ij}\right)\cdot\left(1-\prod_{i}p_{i-j}\right)+\prod_{-i}p_{-ij}\cdot\prod_{i}p_{i-j}\right)\cdot\frac{v_{ij}}{2}+$$

$$+\prod_{-i} p_{-ij} \cdot \left(1 - \prod_{i} p_{i-j}\right) \cdot 0 + \left(1 - \prod_{-i} p_{-ij}\right) \cdot \prod_{i} p_{i-j} \cdot v_{ij} = \left(1 - \prod_{-i} p_{-ij} + \prod_{i} p_{i-j}\right) \frac{v_{ij}}{2} \ .$$

Hence, the expected payoff difference from exerting effort versus not exerting effort equals

$$\prod_{-i} p_{-ij} \left(\frac{v_{ij}}{2}\right) - 1 \ ,$$

and the expected payoff difference from not exerting effort versus exerting effort equals

$$1 - \prod_{-i} p_{-ij} \left(\frac{v_{ij}}{2}\right) \ .$$

Note that, for $v_{ij} = 2$, $p_{-ij} \geq 1/2$, so that exerting effort cannot be discarded by strict dominance, since the expected payoff difference from not exerting effort versus exerting effort equals zero for $p_{-ij} = 1$. Let us now suppose players follow monotonic switching strategies around a cutoff $k = 2^{n_j}$ such that

$$x_{ij}\left(v_{ij}\right) = \begin{cases} 1 & \text{if } v_{ij} > 2^{n_j} \\ 0 & \text{if } v_{ij} \leq 2^{n_j} \ . \end{cases}$$

Then, for the strategy above to be an equilibrium it must satisfy incentive compatibility constraints from below the cutoff and from above, that is

$$1 - \left(1 - \frac{\left(v_{ij} - 2^{n_j} + 2\varepsilon\right)^2}{8\varepsilon^2}\right)^{n_j - 1} \cdot \frac{v_{ij}}{2} \geq 0 \qquad \forall\ v_{ij} \in [2, 2^{n_j}],$$

and

$$\left(\frac{\left(2^{n_j} - v_{ij} + 2\varepsilon\right)^2}{8\varepsilon^2}\right)^{n_j - 1} \frac{v_{ij}}{2} - 1 \qquad \forall\ v_{ij} \in (2^{n_j}, \overline{v} - 2\varepsilon] \ .$$

From which we get the two acceptable solutions

$$\varepsilon \geq \frac{\left(2^{n_j} - v_{ij}\right)^2}{2\left(2^{n_j} - v_{ij} - \sqrt{2}\sqrt{-\left(\left(2^{\frac{1}{n_j - 1}} \left(\frac{1}{v_{ij}}\right)^{\frac{1}{n_j - 1}} - 1\right)\left(2^{n_j} - v_{ij}\right)^2\right)}\right)} \qquad \forall\ v_{ij} \in [2, 2^{n_j}],$$

and

$$\varepsilon \geq \frac{\left(2^{n_j} - v_{ij}\right)^2}{2\left(v_{ij} - 2^{n_j} - \sqrt{2^{\frac{n_j}{n_j - 1}} \left(2^{n_j} - v_{ij}\right)^2 \left(\frac{1}{v_{ij}}\right)^{\frac{1}{n_j - 1}}}\right)} \qquad \forall\ v_{ij} \in (2^{n_j}, \overline{v} - 2\varepsilon] \ .$$

Note that the two lower bounds are increasing in v_{ij} and that we have to check incentive compatibility for $v_{ij} \geq 2$. Moreover, second-order beliefs are defined up to for $v_{ij} \leq \overline{v} - 2\varepsilon$. Hence, the value of ε ensuring incentive-compatibility of the monotonic switching strategy for all possible private signals v_{ij} is the one computed for $\underline{\varepsilon}$ at v_{ij}^{max}, that is

$$\underline{\varepsilon}(v_{ij}^{max}) = \frac{\left(2^{n_j} - v_{ij}^{max}\right)^2}{2\left(v_{ij}^{max} - 2^{n_j} - \sqrt{2^{\frac{n_j}{n_j-1}}\left(2^{n_j} - v_{ij}^{max}\right)^2\left(\frac{1}{v_{ij}^{max}}\right)^{\frac{1}{n_j-1}}}\right)} \ ,$$

which tends to infinity for $v_{ij}^{max} \to \infty$.

Finally, observe that the argument above is independent of k being equal to 2^{n_j}, that is it holds for any k.

Hence, in $g_1(v)$ under incomplete information à la global games there are no equilibria in (monotonic) cutoff strategies, such that $\forall ij \in \{1, \ldots, n_j\}$ and $\forall j \in \{1, 2\}$:

$$x_{ij}^*(v_{ij}) = \begin{cases} 1 & \text{if } v_{ij} > k \\ 0 & \text{if } v_{ij} \leq k \ . \end{cases}$$

QED

4.A.6 Proof of Proposition 4.6

In the $g_2(c)$, note that $E(C|c_{ij}) = c_{ij}$ so that $C|c_{ij} \sim U(c_{ij} - \varepsilon, c_{ij} + \varepsilon)$. Furthermore, for $c_{ij} \in [\underline{c} - \varepsilon, \overline{c} + \varepsilon]$, the conditional distribution of the teammates' or opponents' observation will be centered around c_{ij} with a triangular distribution over the support $[c_{ij} - 2\varepsilon, c_{ij} + 2\varepsilon]$. Thus, $Prob\,[C_{-ij} < c_{ij}|c_{ij}]$ $= Prob[C_{-ij} > c_{ij}|c_{ij}] = \frac{1}{2}$.

Now, suppose player $ij \in \{1, \ldots, n_j\}$ $\forall j \in \{1, 2\}$ observes $c_{ij} > \frac{v}{2}$. Then, ij's conditionally expected payoff from exerting effort, that is choosing $x_{ij} = 1$, is smaller than the one from exerting no effort, that is choosing $x_{ij} = 0$. Accordingly, $x_{ij} = 0$ is a conditionally strictly dominant action for player $ij \in \{1, \ldots, n_j\}$ $\forall j \in \{1, 2\}$ whenever she observes $c_{ij} > \frac{v}{2}$. On the other hand, suppose player $ij \in \{1, \ldots, n_j\}$ $\forall j \in \{1, 2\}$ observes $c_{ij} < 0$. Then, ij's conditionally expected payoff from exerting effort, that is choosing $x_{ij} = 1$, is positive and greater than the one from exerting no effort, that is choosing $x_{ij} = 0$. Accordingly, $x_{ij} = 1$ is a conditionally strictly dominant action for player $ij \in \{1, \ldots, n_j\}$ $\forall j \in \{1, 2\}$ whenever she observes $c_{ij} < 0$. Let us denote p_{-ij} the conditional expected probability attached by player ij to any other player $-ij$ within the same group choosing not to exert effort. Likewise, p_{i-j} is the conditional expected probability attached by ij to any other member of the rival group $-j$ not exerting effort. Then, for player ij the conditionally expected payoff from exerting effort equals

$$Prob\left(n_{-j}\gamma_{-j}=0\right)\cdot\left(v-c_{ij}\right)+Prob\left(n_{-j}\gamma_{-j}\geq 1\right)\cdot\left(\frac{v}{2}-c_{ij}\right)=$$

$$=\prod_{i}p_{i-j}\cdot\left(v-c_{ij}\right)+\left(1-\prod_{i}p_{i-j}\right)\cdot\left(\frac{v}{2}-c_{ij}\right)=\left(1+\prod_{i}p_{i-j}\right)\left(\frac{v}{2}-c_{ij}\right),$$

while the conditionally expected payoff from not exerting effort equals

$$\Big(Prob\left(n_{j}\gamma_{j}\geq 1\right)\cdot Prob\left(n_{-j}\gamma_{-j}\geq 1\right)+Prob\left(n_{j}\gamma_{j}=0\right)\cdot Prob\left(n_{-j}\gamma_{-j}=0\right)\Big)\cdot\frac{v}{2}+$$

$$+Prob\left(n_{j}\gamma_{j}=0\right)\cdot Prob\left(n_{-j}\gamma_{-j}\geq 1\right)\cdot(0)+Prob\left(n_{j}\gamma_{j}\geq 1\right)\cdot Prob\left(n_{-j}\gamma_{-j}=0\right)\cdot v\Leftrightarrow$$

$$\Leftrightarrow\left(\left(1-\prod_{-i}p_{-ij}\right)\cdot\left(1-\prod_{i}p_{i-j}\right)+\prod_{-i}p_{-ij}\cdot\prod_{i}p_{i-j}\right)\cdot\frac{v}{2}+$$

$$+\prod_{-i}p_{-ij}\cdot\left(1-\prod_{i}p_{i-j}\right)\cdot 0+\left(1-\prod_{-i}p_{-ij}\right)\cdot\prod_{i}p_{i-j}\cdot v=\left(1-\prod_{-i}p_{-ij}+\prod_{i}p_{i-j}\right)\frac{v}{2}\ .$$

Hence, the expected payoff difference from exerting effort versus not exerting effort equals

$$\prod_{-i}p_{-ij}\left(\frac{v}{2}\right)-c_{ij}\ ,$$

and the expected payoff difference from not exerting effort versus exerting effort equals

$$c_{ij}-\prod_{-i}p_{-ij}\left(\frac{v}{2}\right)\ .$$

Note that, for $c_{ij}=0$, $p_{-ij}\leq 1/2$, so that not exerting effort cannot be discarded by strict dominance, since the expected payoff difference from exerting effort versus not exerting effort equals zero for $p_{-ij}=0$. Moreover, for $c_{ij}=v/2$, $p_{-ij}\geq 1/2$, so that exerting effort cannot be discarded by strict dominance, since the expected payoff difference from not exerting effort versus exerting effort equals zero for $p_{-ij}=1$. Let us now suppose players follow monotonic switching strategies around a cutoff $k=v/2^{n_j}$ such that

$$x_{ij}\left(v_{ij}\right)=\begin{cases}1 & \text{if } c_{ij}<v/2^{n_j}\\ 0 & \text{if } c_{ij}\geq v/2^{n_j}\ .\end{cases}$$

Then, for the strategy above to be an equilibrium it must satisfy incentive compatibility constraints from above the cutoff and from below, that is

$$c_{ij} - \left(1 - \frac{\left(v/2^{n_j} - c_{ij} + 2\varepsilon\right)^2}{8\varepsilon^2}\right)^{n_j-1} \cdot \frac{v}{2} \geq 0 \quad \forall\; c_{ij} \in [v/2^{n_j}, v/2]\ ,$$

and

$$\left(\frac{\left(c_{ij} - v/2^{n_j} + 2\varepsilon\right)^2}{8\varepsilon^2}\right)^{n_j-1} \cdot \frac{v}{2} - c_{ij} \geq 0 \quad \forall\; c_{ij} \in [0, v/2^{n_j})\ .$$

From which we get the two acceptable solutions

$$\varepsilon \geq \frac{2^{-n_j-1}\left(v - 2^{n_j} \cdot c_{ij}\right)^2}{2^{n_j} \cdot c_{ij} - v - 2^{n_j+\frac{1}{2}}\sqrt{-4^{-n_j}\left(2^{\frac{1}{n_j-1}}\left(\frac{c_{ij}}{v}\right)^{\frac{1}{n_j-1}} - 1\right)\left(v - 2^{n_j} \cdot c_{ij}\right)^2}} \qquad \forall c_{ij} \in [v/2^{n_j}, v/2]\ ,$$

and

$$\varepsilon \geq \frac{2^{-n_j-1}\left(v - 2^{n_j} \cdot c_{ij}\right)^2}{v - 2^{n_j} \cdot c_{ij} - 2^{n_j+\frac{1}{2}}\sqrt{2^{\frac{1}{n_j-1}-2n_j}\left(\frac{c_{ij}}{v}\right)^{\frac{1}{n-1}}\left(v - 2^{n_j} \cdot c_{ij}\right)^2}} \qquad \forall c_{ij} \in [0, v/2^{n_j})\ .$$

Note that the threshold is increasing in c_{ij} $\forall c_{ij} \in [0, v/2^{n_j})$ and tends to $v(n_j - 1)/2^{n_j}$. On the other hand, $\forall c_{ij} \in [v/2^{n_j}, v/2]$ the threshold is monotonically decreasing for $n_j = 2$, while it is in non-monotone for $n_j \geq 3$. Note further that $\forall c_{ij} \in [v/2^{n_j}, v/2]$ $\lim_{c_{ij} \to v/2^{n_j}+} \underline{\varepsilon} = (n_j - 1)v/2^{n_j}$, while $\lim_{c_{ij} \to v/2^-} \underline{\varepsilon} = 2^{-n_j-2}\left(2^{n_j} - 2\right)v$. Therefore, we can focus only on the incentive compatibility constraint for signals above the cutoff. Clearly, a general analytical solution for study of the maximum of $\underline{\varepsilon}(c_{ij})$ is not attainable $\forall n_j \geq 2$. However, c_{ij}^{max} exists for the objective function is continuous over the bounded interval $[v/2^{n_j}, v/2]$ by defining

$$\begin{aligned}\underline{\varepsilon}\left(c_{ij} = \frac{v}{2^{n_j}}\right) &= \lim_{c_{ij} \to v/2^{n_j}+} \frac{2^{-n_j-1}\left(v - 2^{n_j} \cdot c_{ij}\right)^2}{2^{n_j} \cdot c_{ij} - 2^{n_j+\frac{1}{2}}\sqrt{-4^{-n_j}\left(2^{\frac{1}{n_j-1}}\left(\frac{c_{ij}}{v}\right)^{\frac{1}{n_j-1}} - 1\right)\left(v - 2^{n_j} \cdot c_{ij}\right)^2} - v} = \\ &= \frac{\left(n_j - 1\right)v}{2^{n_j}}\ .\end{aligned}$$

and it solves

$$c_{ij}^{max} = \underset{c_{ij} \in [v/2^{n_j}, v/2]}{\arg\max} \frac{2^{-n_j-1}\left(v - 2^{n_j} \cdot c_{ij}\right)^2}{2^{n_j} \cdot c_{ij} - 2^{n_j+\frac{1}{2}}\sqrt{-4^{-n_j}\left(2^{\frac{1}{n_j-1}}\left(\frac{c_{ij}}{v}\right)^{\frac{1}{n_j-1}} - 1\right)\left(v - 2^{n_j} \cdot c_{ij}\right)^2} - v}\ .$$

Hence, for $\varepsilon \geq \underline{\varepsilon}(c_{ij}^{max})$ there exists an equilibrium in switching strategies in $g_2\left(c\right)$ such that

$\forall ij \in \{1, \ldots, n_j\}$ and $\forall j \in \{1, 2\}$

$$x_{ij}^{*}\left(c_{ij}\right) = \begin{cases} 1 & \text{if } c_{ij} < 2^{-n_j} v \\ 0 & \text{if } c_{ij} \geq 2^{-n_j} v \ . \end{cases}$$

QED

4.A.7 Proof of Proposition 4.7

In the $g_2(c)$ with a bounded uniform prior , given the contest success function $P_j\left(X_j, X_{-j}\right) \forall j \in \{1, 2\}$, the probability of winning the prize v for group $j \in \{1, 2\}$ is: [4.A3]

$$\begin{aligned} Prob\left(j \text{ wins } v\right) = & Prob\left[\left(X_j^{*}, X_{-j}^{*}\right) = (1, 0)\right] + \frac{1}{2} Prob\left[\left(X_j^{*}, X_{-j}^{*}\right) = (0, 0)\right] + \\ & + \frac{1}{2} Prob\left[\left(X_j^{*}, X_{-j}^{*}\right) = (1, 1)\right] \ . \end{aligned}$$

On the other hand, the probability of winning the prize v for group $j \in \{1, 2\}$ at the cutoff equilibrium $x_{ij}^{*}\left(c_{ij}\right)$ depends on whether or not $2^{-n_j} v$ belongs to $[\underline{c} + \varepsilon, \overline{c} - \varepsilon]$, where c_{ij} is uniformly distributed. However, we will restrict our attention to the unique interesting case, i.e. $\underline{c} + \varepsilon < 2^{-n_j} v < \overline{c} - \varepsilon$ and $\underline{c} + \varepsilon < 2^{-n_{-j}} v < \overline{c} - \varepsilon$.

Accordingly,

$$\begin{aligned} Prob\left[\left(X_j^{*}, X_{-j}^{*}\right) = (1, 0)\right] &= \left[1 - \left(1 - \frac{2^{-n_j} v - \underline{c} - \varepsilon}{\overline{c} - \underline{c} - 2\varepsilon}\right)^{n_j}\right] \cdot \left(1 - \frac{2^{n_{-j}} v - \underline{c} - \varepsilon}{\overline{c} - \underline{c} - 2\varepsilon}\right)^{n_{-j}} \quad ; \\ Prob\left[\left(X_j^{*}, X_{-j}^{*}\right) = (0, 0)\right] &= \left(1 - \frac{2^{-n_j} v - \underline{c} - \varepsilon}{\overline{c} - \underline{c} - 2\varepsilon}\right)^{n_j} \cdot \left(1 - \frac{2^{-n_{-j}} v - \underline{c} - \varepsilon}{\overline{c} - \underline{c} - 2\varepsilon}\right)^{n_{-j}} \quad ; \\ Prob\left[\left(X_j^{*}, X_{-j}^{*}\right) = (1, 1)\right] &= \left[1 - \left(1 - \frac{2^{-n_j} v - \underline{c} - \varepsilon}{\overline{c} - \underline{c} - 2\varepsilon}\right)^{n_j}\right] \cdot \left[1 - \left(1 - \frac{2^{n_{-j}} v - \underline{c} - \varepsilon}{\overline{c} - \underline{c} - 2\varepsilon}\right)^{n_{-j}}\right] \ . \end{aligned}$$

[4.A3] Note that for $X_j = 1$, it suffices that just one ij chooses $x_{ij}\left(c_{ij}\right) = 1$, due to the best-shot impact function.

Hence,

$$\begin{aligned}
Prob\,(j \text{ wins } v) =& Prob\left[\left(X_j^*, X_{-j}^*\right) = (1,0)\right] + \frac{1}{2} Prob\left[\left(X_j^*, X_{-j}^*\right) = (0,0)\right] + \\
&+ \frac{1}{2} Prob\left[\left(X_j^*, X_{-j}^*\right) = (1,1)\right] \\
=& \left[1 - \left(1 - \frac{2^{-n_j} v - \underline{c} - \varepsilon}{\overline{c} - \underline{c} - 2\varepsilon}\right)^{n_j}\right] \cdot \left(1 - \frac{2^{n_{-j}} v - \underline{c} - \varepsilon}{\overline{c} - \underline{c} - 2\varepsilon}\right)^{n_{-j}} + \\
&+ \frac{1}{2} \left(1 - \frac{2^{-n_j} v - \underline{c} - \varepsilon}{\overline{c} - \underline{c} - 2\varepsilon}\right)^{n_j} \cdot \left(1 - \frac{2^{-n_{-j}} v - \underline{c} - \varepsilon}{\overline{c} - \underline{c} - 2\varepsilon}\right)^{n_{-j}} + \\
&+ \frac{1}{2} \left[1 - \left(1 - \frac{2^{-n_j} v - \underline{c} - \varepsilon}{\overline{c} - \underline{c} - 2\varepsilon}\right)^{n_j}\right] \cdot \left[1 - \left(1 - \frac{2^{n_{-j}} v - \underline{c} - \varepsilon}{\overline{c} - \underline{c} - 2\varepsilon}\right)^{n_{-j}}\right] \ .
\end{aligned}$$

QED

4.A.8 Proof of Proposition 4.8

First of all, in the $g_2\,(c)$ with a bounded uniform prior, the expected value of the cost of effort according to the uniform prior distribution is $E\,[C] = \frac{\underline{c} + \overline{c}}{2}$. Moreover, we will restrict our attention to the unique interesting case, i.e. $\underline{c} + \varepsilon < 2^{-n_j} v < \overline{c} - \varepsilon$ and $\underline{c} + \varepsilon < 2^{-n_{-j}} v < \overline{c} - \varepsilon$.

Then,

$$\begin{aligned}
\mathbb{E}\left[\pi_{ij}\left(\mathbf{x}_j^*, \mathbf{x}_{-j}^*\right)\right] =& Prob\,(\text{ ij receives a signal smaller than } 2^v\ ) \ \cdot \\
&\cdot Prob\,\left(\text{ no i-j receives a signal smaller than } 2^{-n_{-j}} v\ \right) \cdot \left(\frac{v}{2} - \frac{\underline{c} + \overline{c}}{2}\right) + \\
&+ Prob\,\left(\text{ ij receives a signal higher than or equal to } 2^{-n_j} v\ \right) \ \cdot \\
&\cdot Prob\,\left(\text{ at least one -ij receives a signal lower than } 2^{-n_j} v\ \right) \cdot \\
&\cdot Prob\,\left(\text{ no i-j receives a signal lower than } 2^{-n_{-j}} v\ \right) \cdot \frac{v}{2} + \\
&+ Prob\,\left(\text{ ij receives a signal lower than } 2^{-n_j} v\ \right) \cdot \\
&\cdot Prob\,\left(\text{ at least one i-j receives a signal lower than } 2^{-n_{-j}} v\ \right) \cdot \left(\frac{v}{2} - \frac{\underline{c} + \overline{c}}{2}\right) + \\
&+ Prob\,\left(\text{ ij receives a signal higher than or equal to } 2^{-n_j} v\ \right) \cdot \\
&\cdot Prob\,\left(\text{ at least one -ij receives a signal smaller than } 2^{-n_j} v\right) \cdot \\
&\cdot Prob\,\left(\text{ at a least one i-j receives a signal smaller than } 2^{-n_{-j}} v\ \right) \cdot \frac{v}{2} + \\
&+ Prob\,\left(\text{ no ij receives a signal smaller than } 2^{-n_j} v\ \right) \cdot \\
&\cdot Prob\,\left(\text{ no i-j receives a signal smaller than } 2^{-n_{-j}} v\ \right) \cdot \frac{v}{2} \ ,
\end{aligned}$$

where

$$
\begin{aligned}
&\mathit{Prob}\left(\text{ ij receives a signal lower than } 2^{-n_j}v\ \right)=\frac{2^{-n_j}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\ ,\\
&\mathit{Prob}\left(\text{ no i-j receives a signal lower than } 2^{-n_{-j}}v\ \right)=\left(1-\frac{2^{-n_{-j}}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)^{n_{-j}}\ ,\\
&\mathit{Prob}\left(\text{ ij receives a signal higher than or equal to } 2^{-n_j}v\right)=1-\frac{2^{-n_j}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\ ,\\
&\mathit{Prob}\left(\text{ at least one agent -ij receives a signal lower than } 2^{-n_j}v\right)=1-\left(1-\frac{2^{-n_j}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)^{n_j-1}\ ,\\
&\mathit{Prob}\left(\text{ at least one i-j receives a signal lower than } 2^{-n_{-j}}v\ \right)=1-\left(1-\frac{2^{-n_{-j}}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)^{n_{-j}}\\
&\mathit{Prob}\left(\text{ no ij receives a signal smaller than } 2^{-n_j}v\ \right)=\left(1-\frac{2^{-n_j}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)^{n_j}\ .
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\mathbb{E}\left[\pi_{ij}\left(\mathbf{x}_j^*,\mathbf{x}_{-j}^*\right)\right]=&\left(\frac{2^{-n_j}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)\cdot\left(1-\frac{2^{-n_{-j}}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)^{n_{-j}}\cdot\left(v-\frac{\underline{c}+\overline{c}}{2}\right)+\\
&+\left(1-\frac{2^{-n_j}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)\cdot\left[1-\left(1-\frac{2^{-n_j}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)^{n_j-1}\right]\cdot\\
&\cdot\left(1-\frac{2^{-n_{-j}}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)^{n_{-j}}\cdot v+\\
&+\left(\frac{2^{-n_j}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)\cdot\left(1-\left(1-\frac{2^{-n_{-j}}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)^{-n_{-j}}\right)\cdot\left(\frac{v}{2}-\frac{\underline{c}+\overline{c}}{2}\right)\ +\\
&+\left(1-\frac{2^{-n_j}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)\cdot\left(1-\left(1-\frac{2^{-n_j}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)^{n_j-1}\right)\cdot\\
&\cdot\left(1-\left(1-\frac{2^{-n_{-j}}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)^{n_{-j}}\right)\cdot\frac{v}{2}+\\
&+\left(1-\frac{2^{-n_j}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)^{n_j}\cdot\left(1-\frac{2^{-n_{-j}}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)^{n_{-j}}\cdot\frac{v}{2}\ .
\end{aligned}
$$

QED

4.A.9 Proof of Proposition 4.9

In the $g_3(c)$, note that $E(C|c_{ij})=c_{ij}$ so that $C|c_{ij}\sim U(c_{ij}-\varepsilon,c_{ij}+\varepsilon)$. Furthermore, for $c_{ij}\in[\underline{c}-\varepsilon,\overline{c}+\varepsilon]$, the conditional distribution of the teammates' or opponents' observation will be centered around c_{ij} with a triangular distribution over the support $[c_{ij} - 2\varepsilon, c_{ij} + 2\varepsilon]$. Hence, $Prob\,[C_{-ij} < c_{ij}|c_{ij}] = Prob[C_{-ij} > c_{ij}|c_{ij}] = \frac{1}{2}$. Moreover, let us define Q_k as the set of all subsets of cardinality k formed by groups different from j, that is $Q_k = \{-J \in \{\{1, \ldots, M\} \setminus \{j\}\} \mid |-J| = k\}$, where $|\cdot|$ denotes the cardinality of a set.

Let us denote p_{-ij} the conditional expected probability attached by player ij to any other player $-ij$ within the same group choosing not to exert effort. Likewise, p_{i-j} is the conditional expected probability attached by ij to any other member of the rival group $-j$ not exerting effort. Then, for player ij the conditionally expected payoff from exerting effort equals [4.A4]

$$\begin{aligned}
&Prob\left(\sum_{-j\neq j} n_{-j}\gamma_{-j} = 0\right) \cdot (v - c_{ij}) + \\
&+ \sum_{k=1}^{M-1} \sum_{-J\in Q_k} \prod_{-j\in -J} Prob\,(n_{-j}\gamma_{-j} \geq 1) \cdot \prod_{-j'\in Q_k\setminus -J} Prob\,(n_{-j'}\gamma_{-j'} = 0) \cdot \left(\frac{v}{k+1} - c_{ij}\right) = \\
&= \prod_{-j}\prod_{i} p_{i-j} \cdot (v - c_{ij}) + \sum_{k=1}^{M-1} \sum_{-J\in Q_k} \prod_{-j\in -J} \left(1 - \prod_{i} p_{i-j}\right) \prod_{-j'\in Q_k\setminus -J} \prod_{i} p_{i-j'} \cdot \left(\frac{v}{k+1} - c_{ij}\right) \ ,
\end{aligned}$$

while the conditionally expected payoff from not exerting effort equals

$$\begin{aligned}
&Prob\left(\sum_{j=1}^{M} n_j\gamma_j = 0\right) \cdot \frac{v}{M} + Prob\,(n_j\gamma_j \geq 1) \cdot Prob\left(\sum_{-j\neq j} n_{-j}\gamma_{-j} = 0\right) \cdot v + \\
&+ Prob\,(n_j\gamma_j \geq 1) \cdot \sum_{k=1}^{M-1} \sum_{-J\in Q_k} \prod_{-j\in -J} Prob\,(n_{-j}\gamma_{-j} \geq 1) \prod_{-j'\in Q_k\setminus J} Prob\left(n_{-j'}\gamma_{-j'} = 0\right) \cdot \frac{v}{k+1} \\
&= \prod_{-i} p_{-ij} \prod_{-j}\prod_{i} p_{i-j} \cdot \frac{v}{M} + \prod_{-j}\prod_{i} p_{i-j} \left(1 - \prod_{-i} p_{-ij}\right) \cdot v + \\
&+ \left(1 - \prod_{-i} p_{-ij}\right) \sum_{k=1}^{M-1} \frac{v \sum_{-J\in Q_k} \prod_{-j\in -J} \left(1 - \prod_i p_{i-j}\right) \prod_{-j'\in Q_k\setminus -J} \prod_i p_{ij'}}{k+1} \ .
\end{aligned}$$

Hence, for player ij the conditional expected payoff difference from exerting effort versus not exerting

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[4.A4] Due to the auction-type contest success function and the presence of $M \geq 2$ groups, we have to consider the possibility of a tie with up to $M - 1$ groups.

effort equals

$$\begin{aligned}&\left(\prod_{-i} p_{-ij}\cdot\prod_{-j}\prod_{i} p_{i-j}\left(\frac{M-1}{M}\right)+\right.\\&\left.+\prod_{-i} p_{-ij}\sum_{k=1}^{M-1}\frac{\sum_{-J\in Q_k}\prod_{-j\in -J}\left(1-\prod_i p_{i-j}\right)\cdot\prod_{-j'\in Q_k\setminus -J}\prod_i p_{i-j'}}{k+1}\right)v-c_{ij}\end{aligned}$$

and the conditional expected payoff difference from not exerting effort versus exerting effort equals

$$\begin{aligned}&c_{ij}-\left(\prod_{-i} p_{-ij}\cdot\prod_{-j}\prod_{i} p_{i-j}\left(\frac{M-1}{M}\right)+\right.\\&\left.+\prod_{-i} p_{-ij}\sum_{k=1}^{M-1}\frac{\sum_{-J\in Q_k}\prod_{-j\in -J}\left(1-\prod_i p_{i-j}\right)\cdot\prod_{-j'\in Q_k\setminus -J}\prod_i p_{i-j'}}{k+1}\right)v\ .\end{aligned}$$

Observe that for player ij the expected payoff difference from exerting effort versus not exerting effort is increasing in the probability of both the teammates and the rival groups not exerting effort. Hence, by setting $p_{-ij}=0$, $p_{i-j}=0$, $p_{i-j'}=0$, the expected payoff difference reduces to c_{ij}. From the latter we derive threshold of one dominance region: for $c_{ij}<0$, $x_{ij}=1$ is a strictly dominant action for any player ij. On the other hand, for player ij the expected payoff difference from not exerting effort versus exerting effort is decreasing in the probability of both the teammates and the rival groups not exerting effort. Hence, by setting $p_{-ij}=1$, $p_{i-j}=1$, $p_{i-j'}=1$, the expected payoff difference reduces to $c_{ij}-v(M-1)/M+$. From the latter we derive the threshold of one dominance region: for $c_{ij}>v\cdot(M-1)/M$, $x_{ij}=0$ is a strictly dominant action for any player ij. Observe further that it is not possible to iterate the dominance argument at $c_{ij}=0$ and $c_{ij}=v\cdot(M-1)/M$.

Let us now suppose players follow the following cutoff strategy

$$x_{ij}\left(c_{ij}\right)=\begin{cases}1 & \text{if } c_{ij}<k\\ 0 & \text{if } c_{ij}\geq k\ ,\end{cases}$$

where k is obtained by equating the expected payoff differences to zero with $p_{-ij}=1/2$, $p_{i-j}=1/2$, $p_{i-j'}=1/2$, that is $k=(B/A)\cdot v$, with

$$k=2^{1-n_j}\left(2^{-\sum_{-j}n_{-j}}\left(\frac{M-1}{M}\right)+\sum_{k=1}^{M-1}\frac{\sum_{-J\in Q_k}\prod_{-j\in -J}\left(1-2^{-n_{-j}}\right)\cdot\prod_{-j'\in Q_k\setminus -J}2^{-n_{-j'}}}{k+1}\right)v\ .$$

Then, for the strategy above to be an equilibrium it must satisfy incentive compatibility constraints from above the cutoff and from below, that is $\forall\ c_{ij} \in [k, v \cdot (M-1)/(M)]$

$$c_{ij} - \left(\prod_{-i} p_{-ij} \cdot \prod_{-j} \prod_{i} p_{i-j} \left(\frac{M-1}{M} \right) + \right.$$
$$\left. + \prod_{-i} p_{-ij} \sum_{k=1}^{M-1} \frac{\sum_{-J \in Q_k} \prod_{-j \in -J} (1 - \prod_i p_{i-j}) \cdot \prod_{-j' \in Q_k \setminus -J} \prod_i p_{i-j'}}{k+1} \right) v \ .$$

where $p_{-ij} = p_{i-j} = p_{i-j'} = 1 - (k - c_{ij} + 2\varepsilon)^2/(8\varepsilon^2)$ and

$$\left(\prod_{-i} p_{-ij} \cdot \prod_{-j} \prod_{i} p_{i-j} \left(\frac{M-1}{M} \right) + \right.$$
$$\left. + \prod_{-i} p_{-ij} \sum_{k=1}^{M-1} \frac{\sum_{-J \in Q_k} \prod_{-j \in -J} (1 - \prod_i p_{i-j}) \cdot \prod_{-j' \in Q_k \setminus -J} \prod_i p_{i-j'}}{k+1} \right) v - c_{ij} \qquad \forall\ c_{ij} \in [0, k),$$

where $p_{-ij} = p_{i-j} = p_{i-j'} = (c_{ij} - k + 2\varepsilon)^2/(8\varepsilon^2)$. Observe that for a sufficiently high-value of the noise parameter ε the conditional probability each player ij will attach to any other player not exerting effort will be not too far away from 1/2 preserving incentive compatibility for signals away from the cutoff. Hence, for $\varepsilon \geq \underline{\varepsilon}(c_{ij}^{max})$, where c_{ij}^{max} is the private signal c_{ij} corresponding to which the minimum value of ε required to preserve incentive compatibility is the highest, the cutoff strategy above does not display profitable deviations. Such value always exists for the objective function is continuous over the bounded interval $[0, v \cdot (M-1)/M]$: it can be found by numerical optimization, depending on the number of groups M and their size n_j.

Hence, for $\varepsilon \geq \underline{\varepsilon}(c_{ij}^{max})$ there exists an equilibrium in switching strategies in $g_3(c)$ such that $\forall ij \in \{1, \ldots, n_j\}$ and $\forall j \in \{1, \ldots, M\}$:

$$x_{ij}^*(c_{ij}) = \begin{cases} 1 & \text{if } c_{ij} < c_j^* \\ 0 & \text{if } c_{ij} \geq c_j^* \ . \end{cases}$$

where

$$c_j^* = 2^{1-n_j} \left(2^{-\sum_{-j} n_{-j}} \left(\frac{M-1}{M} \right) + \sum_{k=1}^{M-1} \frac{\sum_{-J \in Q_k} \prod_{-j \in -J} (1 - 2^{-n_{-j}}) \cdot \prod_{-j' \in Q_k \setminus -J} 2^{-n_{-j'}}}{k+1} \right) v \ .$$

QED