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A subexponential bound on the cardinality of abelian quotients in finite transitive groups

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Abstract

We show that, for every transitive permutation group G of degree $n \geqslant 2$, the largest abelian quotient of G has cardinality at most $4^{n/\sqrt{\log_2 n}}$. This gives a positive answer to a 1989 outstanding question of László Kovács and Cheryl Praeger.

1. *Introduction*

László Kovács and Cheryl Praeger [5] have investigated large abelian quotients in arbitrary permutation groups of finite degree. Their work was motivated by recent (at that time) investigations on minimal permutation representations of a finite group [2]. One of the main results in [5] (which is independently proved in [1]) shows that, for every permutation group of degree n, the largest abelian quotient has order at most $3^{n/3}$. Clearly, this bound is attained, whenever n is a multiple of 3, by an elementary abelian 3-group of order $3^{n/3}$ having all of its orbits of cardinality 3. Furthermore, the authors conjecture that, for *transitive* groups of degree n, a subexponential bound in $n(\log_2 n)^{-1/2}$ holds. More history on this conjecture and more details can be found in the survey paper [8].

The first substantial evidence towards the conjecture goes back to the work of Aschbacher and Guralnick [1]; they proved the striking result that the largest abelian quotient of a *primitive* group of degree n has order at most n. In the concluding remarks, the authors also independently ask whether one can obtain a subexponential bound on the order of abelian quotients of transitive groups in terms of their degrees. We refer to [1, 8] for an infinite family of transitive groups G of degree n with $|G/G'|$ asymptotic to $\exp(bn/\sqrt{\log_2 n})$, for some constant b.

The second substantial evidence towards the conjecture is in [4], where many of the results in Section 7 get very close to the desired upper bound. In particular, [4, Theorem 7.6] says that if G is a transitive permutation group of degree $n \geqslant 2$ and $N \lhd G$ is a still transitive normal subgroup of G, then the product of the orders of the abelian composition factors of G/N is at most $4^{n/\sqrt{\log_2 n}}$.

In this paper, we settle in the affirmative the conjecture of Kovács and Praeger.

Theorem 1. *For every positive integer $n \geqslant 2$ and for every transitive permutation group G of degree n, we have*

$$|G/G'| \leqslant 4^{n/\sqrt{\log_2 n}}.$$

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The constant 4 in Theorem 1 should not be taken too seriously, but it seems remarkably hard to pin down the exact constant. The choice of the constant 4 in our work is a compromise: it makes the statement of Theorem 1 explicit and valid for every $n \geqslant 2$.

2. *Preliminaries*

Unless otherwise explicitly stated, all the logarithms are to base 2. Given a field $\mathbb{F}$, a group G, a subgroup H of G and an $\mathbb{F}H$-module W (or simply H-module), we denote by $W\uparrow_H^G$ the induced G-module of W from H to G, that is, $W\uparrow_H^G := W \otimes_{\mathbb{F}H} \mathbb{F}G$. Moreover, given a G-module M, we denote by $d_G(M)$ the minimal number of generators of M as a G-module. We are ready to report a fundamental result from [7].

Lemma 2.1 (See [7, Lemma 4]). *There is a universal constant b' such that whenever H is a subgroup of index $n \geqslant 2$ in a finite group G, $\mathbb{F}$ is a field, V is an H-module of dimension a over $\mathbb{F}$ and M is a G-submodule of the induced module $V\uparrow_H^G$, then*

$$d_G(M) \leqslant \frac{ab'n}{\sqrt{\log n}}.$$

Remark 2.2. Gareth Tracey, in his monumental work [10] on minimal sets of generators of transitive groups, has refined Lemma 2.1 in various directions. For instance, [10, Section 4.3] gives a more quantitative form of Lemma 2.1. Indeed, using the notation in Lemma 2.1, from [10, Corollary 4.27 (*iii*)], we deduce

$$d_G(M) \leqslant aE(n,p) \leqslant \begin{cases} an\frac{2}{c'\log n} & \text{when } 2 \leqslant n \leqslant 1260,\\ an\frac{2}{\sqrt{\pi \log n}} & \text{when } n > 1261,\end{cases}$$

where $c' := 0.552282$, p is the characteristic of M and $E(n,p)$ is explicitly defined in [10, Section 4]. In particular, we immediately see that in Lemma 2.1 we may take $b' := 2/\sqrt{\pi}$ whenever $n > 1261$. With the help of a computer, we have implemented the function $E(n,p)$ and we have checked that $E(n,p) \leqslant 2n/\sqrt{\pi \log n}$ also when $n \leqslant 1260$. Therefore, in Lemma 2.1 we may take $b' := 2/\sqrt{\pi}$.

Let R be a finite group. For each prime number p, let $a_p(R)$ be the number of abelian composition factors of R of order p, and let

$$a(R) := \sum_{p\,\text{prime}} a_p(R)\log p.$$

We now report a useful result of Pyber.

Lemma 2.3 (See [9, Theorem 2.10]). *Let $c_0 := \log_9(48 \cdot 24^{1/3})$. The product of the orders of the abelian composition factors of a primitive permutation group of degree r is at most $24^{-1/3}r^{1+c_0}$.*

From Lemma 2.3, we deduce the following.

Lemma 2.4. *Let R be a primitive group of degree r and let c_0 be the constant in Lemma* 2.3. *Then*

$$a(R) \leqslant (1+c_0)\log r - \log(24)/3.$$

Proof. By definition, the product of the orders of the abelian composition factors of R is

$$\prod_{p \text{ prime}} p^{a_p(R)} = \prod_{p \text{ prime}} 2^{a_p(R)\log p} = 2^{a(R)}.$$

From Lemma 2.3, this number is at most $24^{-1/3}r^{1+c_0}$. The proof follows by taking logarithms. □

Notice that Lemma 2.3 is often used in order to bound the composition length of a primitive permutation groups. A more precise bound on this composition length has been recently proved by Glasby, Praeger, Rosa and Verret [**3**, Theorem 1.3]. However, this stronger bound is not sufficient for our application, which requires information not only on the number of the composition factors but also on their order.

Finally, given a finite group G, we denote by G_{ab} the quotient group G/G'.

3. *Proof of Theorem* 1

Let R be a finite group, let Δ be a finite set and let $W := R\,\mathrm{wr}_\Delta\,\mathrm{Sym}(\Delta)$ be the wreath product of R via $\mathrm{Sym}(\Delta)$. We denote by

$$\pi : W \to \mathrm{Sym}(\Delta)$$

the projection of W over the top group $\mathrm{Sym}(\Delta)$. Let $\prod_{\delta\in\Delta} R_\delta$ be the base subgroup of W and, for each $\delta \in \Delta$, consider $W_\delta := \mathbf{N}_W(R_\delta)$. As

$$W_\delta = R_\delta \times R\,\mathrm{wr}\,\mathrm{Sym}(\Delta \setminus \{\delta\}),$$

we may consider the projection $\rho_\delta : W_\delta \to R_\delta$. Using this notation, we adapt the proof of [**6**, Lemma 2.5] to prove the following.

Lemma 3.1. *Let R be a finite group, let Δ be a set of cardinality at least* 2 *and let G be a subgroup of the wreath product $R\,\mathrm{wr}_\Delta\,\mathrm{Sym}(\Delta)$ with the properties*

(1) *$\pi(G)$ is transitive on Δ,*
(2) *$\rho_\delta(\mathbf{N}_G(R_\delta)) = R_\delta$, for every $\delta \in \Delta$.*

Then

$$\log|G_{\mathrm{ab}}| \leqslant \frac{a(R)b'|\Delta|}{\sqrt{\log|\Delta|}} + \log|(\pi(G))_{\mathrm{ab}}|,$$

where b' is the absolute constant appearing in Lemma 2.1, *and $a(R)$ is defined in Section* 2.

Proof. We argue by induction on the order of R. When $|R| = 1$, there is nothing to prove because $\pi(G) \cong G$ and hence $\log|G_{\mathrm{ab}}| = \log|(\pi(G))_{\mathrm{ab}}|$. Suppose then $R \neq 1$. We write

$$|G_{\mathrm{ab}}| = |G : G'M||G'M : G'| = |(G/M)_{\mathrm{ab}}||M : M \cap G'|. \tag{3.1}$$

Let L be a minimal normal subgroup of R. Fix $\delta_0 \in \Delta$. We identify L with a normal subgroup L_{δ_0} of the direct factor R_{δ_0} of the base group $\prod_{\delta\in\Delta} R_\delta$ of W. Let B_L be the direct product of the distinct G-conjugates of L_{δ_0} and consider $M := B_L \cap G$. We have $M \trianglelefteq G$ and

$$\frac{G}{M} = \frac{G}{B_L \cap G} \cong \frac{GB_L}{B_L}.$$

Now, from (1), we deduce that GB_L/B_L is isomorphic to a subgroup of the wreath product

$$(R/L)\mathrm{wr}_\Delta\,\mathrm{Sym}(\Delta).$$

Therefore, by induction,

$$\log|(G/M)_{\mathrm{ab}}| \leqslant \frac{a(R/L)b'|\Delta|}{\sqrt{\log|\Delta|}} + \log|(\pi(G))_{\mathrm{ab}}|. \tag{3.2}$$

We now distinguish two cases.

L IS NON-ABELIAN:
Since $M \trianglelefteq W_{\delta_0} \cap G$, we deduce $\rho_{\delta_0}(M) \trianglelefteq \rho_{\delta_0}(W_{\delta_0} \cap G)$. From (2), we have $\rho_{\delta_0}(W_{\delta_0} \cap G) = \rho_{\delta_0}(\mathbf{N}_G(R_{\delta_0})) = R_{\delta_0}$ and hence $\rho_{\delta_0}(M) \trianglelefteq R_{\delta_0}$. Observe that $\rho_{\delta_0}(M)$ is contained in L_{δ_0}. As L_{δ_0} is a minimal normal subgroup of R_{δ_0}, we get either $\rho_{\delta_0}(M) = 1$ or $\rho_{\delta_0}(M) = L_{\delta_0}$. From (1), $\pi(G)$ is transitive on Δ and hence either $\rho_\delta(M) = 1$ for each $\delta \in \Delta$, or $\rho_\delta(M) = L_\delta$ for each $\delta \in \Delta$.

Suppose $\rho_{\delta_0}(M) = 1$. As $\rho_\delta(M) = 1$ for each $\delta \in \Delta$, we get $M = 1$. Now the proof immediately follows from (3.2) because $G/M \cong G$.

Suppose $\rho_{\delta_0}(M) = L_{\delta_0}$. Then M is a subdirect product of $L^\Delta = \prod_{\delta\in\Delta} L_\delta$. As L is a non-abelian minimal normal subgroup of R, we deduce that M is a direct product of non-abelian simple groups. Thus M has no abelian composition factor and hence (3.1) gives $|G_{\mathrm{ab}}| = |(G/M)_{\mathrm{ab}}|$. Moreover, $a(R/L) = a(R)$, and hence, once again, the proof immediately follows from (3.2).

L IS ABELIAN:
As L is a minimal normal subgroup of R, it is an elementary abelian p_0-group, for some prime number p_0. Let a_{p_0} be the composition length of L. In particular,

$$a(R) = a(R/L) + a_{p_0}\log p_0.$$

The group B_L is abelian and the action of G by conjugation on B_L endows B_L with a natural structure of G-module. From its definition, as G-module, B_L is isomorphic to the induced module

$$L_{\delta_0}\uparrow_K^G,$$

where $K := \mathbf{N}_G(L_{\delta_0})$. From (1), G acts transitively on Δ and hence $|\Delta| = |G : \mathbf{N}_G(L_{\delta_0})| = |G : K|$. From Lemma 2.1, we deduce

$$d_G(M/(M\cap G')) \leqslant d_G(M) \leqslant \frac{a_{p_0}b'|\Delta|}{\sqrt{\log|\Delta|}}.$$

However, as G acts trivially by conjugation on $M/(M \cap G')$, we get that $d_G(M/(M \cap G'))$ is just the dimension of $M/(M \cap G')$ as a vector space over the prime field $\mathbb{Z}/p_0\mathbb{Z}$. Therefore,

$$|M : M \cap G'| \leqslant p_0^{(a_{p_0}b'|\Delta|/\sqrt{\log|\Delta|})}. \tag{3.3}$$

From (3.1), (3.2) and (3.3), we get

$$\begin{aligned}
\log|G_{\mathrm{ab}}| \leqslant & \log|(G/M)_{\mathrm{ab}}| + \log|M : M\cap G'| \\
\leqslant & \frac{a(R/L)b'|\Delta|}{\sqrt{\log|\Delta|}} + \log|(\pi(G))_{\mathrm{ab}}| + \log(p_0)\frac{a_{p_0}b'|\Delta|}{\sqrt{\log|\Delta|}} \\
= & (a(R/L) + a_{p_0}\log p_0)\frac{b'|\Delta|}{\sqrt{\log|\Delta|}} + \log|(\pi(G))_{\mathrm{ab}}| \\
= & a(R)\frac{b'|\Delta|}{\sqrt{\log|\Delta|}} + \log|(\pi(G))_{\mathrm{ab}}|.
\end{aligned}$$

□

With Lemma 3.1 in hand, we prove Theorem 1 by induction on n.

Let G be a transitive permutation group of degree $n \geqslant 2$. From the main result of [5], we have $|G_{\mathrm{ab}}| \leqslant 3^{n/3}$. Now the inequality $3^{n/3} \leqslant 4^{n/\sqrt{\log n}}$ is satisfied for each $n \leqslant 20\,603$. In particular, for the rest of the proof, we may suppose that $n \geqslant 20\,604$.

Suppose first that G is primitive. In this case, from [1], we have $|G_{\mathrm{ab}}| \leqslant n$ and the inequality $n \leqslant 4^{n/\sqrt{\log n}}$ follows with an easy computation.

Suppose now that G is imprimitive and let Ω be the domain of G. Among all non-trivial blocks of imprimitivity of G, choose one (say Λ) minimal with respect to the inclusion. Let $G_{\{\Lambda\}} := \{g \in G \mid \Lambda^g = \Lambda\}$ be the setwise stabilizer of Λ in G and let $R \leqslant \mathrm{Sym}(\Lambda)$ be the permutation group induced by $G_{\{\Lambda\}}$ in its action on Λ. The minimality of Λ yields that R acts primitively on Λ.

Let $\Delta := \{\Lambda^g \mid g \in G\}$ be the system of imprimitivity determined by the block Λ. Then G is a subgroup of the wreath product

$$R \,\mathrm{wr}_\Delta\, \mathrm{Sym}(\Delta).$$

We now use the notation of Lemma 3.1 for wreath products. In particular, let $\pi : R \,\mathrm{wr}_\Delta\, \mathrm{Sym}(\Delta) \to \mathrm{Sym}(\Delta)$ be the projection onto the top group $\mathrm{Sym}(\Delta)$ and for each $\delta \in \Delta$, let R_δ be the direct factor of the base group $\prod_{\delta\in\Delta} R_\delta$ corresponding to δ. From the fact that G acts transitively on Ω and from the definition of R, we get that the two hypotheses (1) and (2) are satisfied. Therefore, from Lemma 3.1 itself, we deduce

$$\log|G_{\mathrm{ab}}| \leqslant \frac{a(R)b'|\Delta|}{\sqrt{\log|\Delta|}} + \log|(\pi(G))_{\mathrm{ab}}|.$$

Set $r := |\Lambda|$. Thus $|\Delta| = n/r$. From Lemma 2.4 and from induction (as $n/r < n$), we get

$$\log|G_{\mathrm{ab}}| \leqslant \frac{b'(n/r)}{\sqrt{\log(n/r)}}\left((1+c_0)\log r - \frac{\log(24)}{3}\right) + 2\frac{(n/r)}{\sqrt{\log(n/r)}}. \tag{3.4}$$

From Remark 2.2, we see that we may take $b' = 2/\sqrt{\pi}$. Now, for $n \geqslant 20\,604$, a careful calculation shows that the right-hand side of (3.4) is at most $2n/\sqrt{\log n}$ for every divisor r of n with $4 < r < n$.

We now discuss the cases $r \in \{2,3,4\}$ separately. When $r = 2$, we have $a(R) = 1$ and hence

$$\log|G_{\mathrm{ab}}| \leqslant \frac{b'(n/2)}{\sqrt{\log(n/2)}} + 2\frac{(n/2)}{\sqrt{\log(n/2)}}. \tag{3.5}$$

Now, the right-hand side of (3.5) is less than $2n/\sqrt{\log n}$ for each $n \geqslant 20\,604$. The computation when $r \in \{3,4\}$ is analogous using $a(R) \leqslant 1 + \log(3)$ when $r = 3$, and $a(R) \leqslant 3 + \log(3)$ when $r = 4$.

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