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Slope inequalities for fibred surfaces and fibred threefolds

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0.1. Introduction

We call a fibred variety an complex projective algebraic variety X of dimension n with a surjective map $f\colon X \to B$ with connected fibres, over a smooth projective curve B. In this setting we define the *relative canonical sheaf* $\omega_f := \omega_X \otimes (f^*\omega_B)^\vee$ where ω_Y is the dualizing sheaf of the variety Y, and call K_f an associated divisor. A *slope inequality* is an inequality of the form:

$$K_f^n \geq A \deg f_*\omega_f, \tag{0.1.1}$$

where A is a positive number, ideally depending on the geometry of the fibration and $f_*\omega_f$. The rank $h^0(F, \omega_F)$ vector bundle is called the Hodge bundle of f. This inequality 0.1.1 is particular meaningful in case K_f has some positivity property, say for instance K_f is relatively ample, or nef.

In the surface case ($n = 2$), we have that

$$\deg f_*\omega_f = \chi_f = \chi(\mathcal{O}_X) - \chi(\mathcal{O}_F)\chi(\mathcal{O}_B)$$

(see 2.2.1). The general fibres F are smooth curves of genus say g.

The first slope inequality was proved independently by Xiao and Cornalba-Harris ([59],[21]) and it is the following:

Let $f\colon S \to B$ be a non-isotrivial fibred surface, over a smooth base curve B of genus $g(B) := b$. Called g the genus of the general smooth fibre F. Suppose that $g \geq 2$ then holds the inequality:

$$K_f^2 \geq \frac{4(g-1)}{g} \deg f_*\omega_f.$$

So, in this case A is an increasing function of the genus of the general fibres. The slope inequality is sharp: it is reached by certain hyperelliptic fibration. After the seminal papers of Xiao and Cornalba-Harris, many authors among all we have to cite Konno [38] [39] [40] have studied the problem of finding other slope inequalities for fibred surfaces, proving the influence of other invariants of the fibration.

In particular, in view of the result mentionted above, it is normal to expect a better bound increasing with the gonality of the fibres. There have been contributions in this direction due to Barja-Stoppino [11] (for trigonal fibres), Zucconi-Beorchia [15] (for 4-gonal fibres) and Cornalba-Stoppino [22] (for double covers). Also the Clifford index of the general fibres has proved to have an effect on the slope ([40]). On the other hand, another invariant has an influence on the slope: the relative irregularity of the fibration

$$q_f := q(S) - q(B) = q(S) - b.$$

Indeed it is proved alredy in Xiao's paper [59] that if $q_f > 1$ then the better bound
$$K_f^2 \geq 4\chi_f$$
holds.

This invariant is also related to the Hodge bundle $f_*\omega_f$. Let us recall that the Hodge bundle satisfies two decompositions called Fujita's decompositions [30], [31]:

$$f_*\omega_f = \mathcal{E} \oplus \mathcal{O}_B^{\oplus q_f} \quad \text{first Fujita's decomposition;} \tag{0.1.2}$$

$$f_*\omega_f = \mathcal{A} \oplus \mathcal{U} \quad \text{second Fujita's decomposition;} \tag{0.1.3}$$

where $\mathcal{A}$ is an ample vector bundle, $\mathcal{U}$ an unitary flat bundle of rank $u_f := \mathrm{rk}\,\mathcal{U}$ and $\mathcal{E}$ is nef and satisfies $H^1(B, \mathcal{E}) = 0$. Comparing the first and second Fujita's decomposition, since $\mathcal{U}$ is a flat bundle, we see that:
$$\mathcal{U} \supseteq \mathcal{O}_B^{q_f},$$
and so $u_f \geq q_f$.

The first slope inequality which involved the relative irregularity q_f, is due to Barja-Stoppino [10]:

$$K_f^2 \geq 4\frac{g-1}{g-\lfloor m/2 \rfloor} \deg f_*\omega_f \tag{0.1.4}$$

where $m := \min\{q_f, c_f\}$ and c_f is the maximun clifford index amoung the smoth fiber of f (see Section 2.2.3). This last inequality was latter improved by the result of Lu and Zuo [41]:

$$K_f^2 \geq 4\frac{g-1}{g-q_f/2} \deg f_*\omega_f. \tag{0.1.5}$$

The main results of this thesis are some new slope inequalities for fibred surfaces. In particular we obtain:

- a new bound increasing with $\min\{q_f, c_f\}$ sharper than (0.1.4) and in some cases better than the one of (0.1.5);
- a bound increasing with $\min\{c_f, u_f\}$ which is completely new.

In particular the results can be summarized as follows:

Theorem (Theorems 2.7 and 2.8). Let $f\colon S \to B$ be a relatively minimal fibred surface of genus $g \geq 2$; let $m := \min\{q_f, c_f\}$. The following inequalities hold:

$$K_f^2 \geq 2\frac{2g-2-m}{g-m}\chi_f, \tag{0.1.6}$$

$$K_f^2 \geq \begin{cases} 2\frac{(2g-2-u_f)}{(g-u_f)}\chi_f & \text{if } u_f \leq c_f; \\ 2\frac{(2g-2-c_f)(g-1-u_f)}{(g-1-c_f)(g-u_f)}\chi_f & \text{if } u_f \geq c_f. \end{cases} \tag{0.1.7}$$

REMARK 0.1.8. Note that all our bounds are asymptotically close to 4 for $g \gg 0$, and this is natural in view of all the known examples and conjectures. But when m is big with respect to g, the slope gets bigger, going asymptotically to 6. Let us observe that for odd genus, if the Clifford index is maximal $c_f = \lfloor \frac{g-1}{2} \rfloor$ and if $q_f \geq \frac{g-1}{2}$ (0.1.6) becomes Konno's bound ([40], thm 4.1). For Clifford index (hence gonality) close to $\frac{g-1}{2}$, yet not maximal, these bounds are new.

Our arguments make use of Xiao's method (Section 2.2.4). Basically, the idea of Xiao's tecnique can be described as follows: given a subsheaf of the Hodge bundle $\mathcal{G} \subseteq f_*\omega_f$, consider the linear sub-canonical system $\mathcal{G} \otimes \mathbb{C}(t) \subseteq H^0(F, K_F)$ induced on a general fibre $F = f^{-1}(t)$. If one has a lower estimate on the ratio of degree over projective dimension of the linear subsystems of $\mathcal{G} \otimes \mathbb{C}(t)$, then the method produces an inequality of the form $K_f^2 \geq b\deg(\mathcal{G})$, where b is a positive number depending on the lower estimate above. See Section 2.2.4 and Theorem 2.5 for precise statements. Taking as $\mathcal{G}$ the whole Hodge bundle, Clifford's Theorem says that for any subsystem $|V| \subseteq |\mathcal{G} \otimes \mathbb{C}(t)| = |K_{F_t}|$, the degree over dimension of $|V|$ is greater or equal to 2.

This info implemented in the Xiao's machinary gives the slope inequality 2.2.15 [59].

It is thus very natural to try and apply Xiao's method to the ample summand $\mathcal{A}$ of the second Fujita decomposition of the Hodge bundle (2.2), as $deg\mathcal{A} = \chi_f$. In [12], [10] the analog approach is discussed with the positive summand of the first Fujita decomposition (2.2.8). One of the difficulties with these approaches is that there seems to be no control on the base locus of the linear sub-canonical systems induced by $\mathcal{A}$ on the general fibre of f, neither on the linear stability (ref. Section 1.1) of this system.

However, one can still look to a lower bound for the ratio of degree over projective dimension of the linear subcanonical systems that improves Clifford's bound 2.

This is what it is done in this thesis, obtaining a new Clifford-type inequality for subcanonical systems over a non-hyperelliptic curve C, only depending on the codimension and on the Clifford index of C. This gives also the desired control on the base locus of the subcanonical systems.

THEOREM (Theorem 1.5). Let $C \subseteq \mathbb{P}^{g-1}$ be a canonical non-hyperelliptic curve. Let $V \subseteq H^0(C, \omega_C)$ be a linear subspace of codimension $k \leq g-2$. Then for any $W \subseteq V$ subspace of dimension $\dim W \geq 2$, we have:

$$\frac{\deg |W|}{\dim |W|} \geq \frac{2g-2-m}{g-m-1},$$

where $m := \min\{k, \mathrm{Cliff}(C)\}$.

Although the motivation is to apply Xiao's technique, we believe that this result is interesting on its own. The arguments are of genuine classical flavour.

The above result implies a stability result, as follows (see Section 1.1 for the definitions).

COROLLARY 0.1.9 (Corollary 1.5.3). Given $V \subseteq H^0(C, \omega_C)$ a vector subspace of codimension k and dimension ≥ 2, with $k \leq \mathrm{Cliff}(C)$. Then $\deg |V| \geq 2g-2-k$, i.e. the base locus of $|V|$ has degree smaller or equal to k. If $\deg |V| = 2g-2-k$, then $|V|$ is linearly semistable and in particular it is Chow semistable.

REMARK 0.1.10. This result should be compared also to [46], where linear stability of linear systems on curves is discussed in relation to the Clifford index.

REMARK 0.1.11. The slope inequalities have applications both to the geography of surfaces of general type (see for instance [52]) and to the ample cone of the moduli space of curves (see for instance [47] and [32]). These perspectives were the original point of view of Xiao and of Cornalba and Harris respectively.

We also find an example of a fibred surface where the inequality (0.1.6) is strictly greater than (0.1.7), following a construction due to Catanese and Dettweiler [16]. This fibration is a family of semistable curves and its general fibres are cyclic covers over $\mathbb{P}^1$ ramified over four points .The computation of u_f is obtained through a study of the Hodge bundle monodromy (see Chapter 3). In particular we find the invariants of the fibrations have the following values: $g = 6$, $c_f = 2$, $u_f = 2$ and $q_f = 0$. It would be interesting to obtain via the same method other examples with both g and u_f high, but this task doesn't seem to be easily obtained.

In the last part of the thesis we address the problem of slope inequalities for fibred threefolds.

In the last years, many authors have treated the case of slope inequalities of fibrations over curves with total space of higher dimension (see for instance [8]).

First of all let us recall that the Fujita's decompositions cited above (0.1.2,) hold for any fibration, as long as the base locus is a curve.

We focus on understanding the influence of u_f for the case of fibred threefolds. The main result cam be stated as follows

Theorem (Theorem 4.2). Let $f\colon \Sigma \to B$ be a fibred threefold, $f_*\omega_f = \mathcal{A}\oplus\mathcal{U}$ its Hodge bundle, and suppose that $\mathcal{A}$ is semistable. Then there exists a set of slope inequalities which depends on u_f and p_g, of the form

$$\lambda_f \geq A + \frac{Bu_f + C}{p_g - u_f}$$

where A, B are positive constants and C depends of the "embedding properties" of $\mathcal{A}$.

Comparing this inequalities with a result of Ohno [51], we give also:

Theorem (Theorem 4.3.2). Let $f\colon \Sigma \to B$ be a fibred threefold as above. Then if the genus of the base is zero or one, there exist a set of upper bounds for u_f of the form:

$$u_f \leq Ap_g + B$$

where A is a positive constant, and B depends of the "embedding properties" of $\mathcal{A}$.

This is a first upper bound for the unitary rank which depends on p_g.

This thesis is organized as follows. In Chapter 1, after some preliminaries on canonical curves and linear stability, we prove the main Clifford-type result for non-complete sub-canonical systems on non-hyperelliptic curves. We then discuss some consequences and give some natural examples.

In Chapter 2 we start by reviewing in Section 2.1 some basic results on fibred surfaces and their relative invariants. Then in Section 2.2.4 and Section 2.2.5 we give a review of the main theorems of Xiao's technique, in the form needed for our arguments. We state Xiao's method for fibred surfaces in full generality, following Konno's and Barja's papers, for any locally free subsheaf $\mathcal{G}$ of $f_*\mathcal{O}_S(D)$, where D is a nef divisor on S.

The proof of the main inequalities is carried on in Section 2.3.

In Chapter 3, following Catanese and Dettweiler's examples, we provide a first example of a fibred surface such that the inequality in the main theorem involving u_f is new.

In the last Chapter 4 we recall some basic definition of fibred threefold and state the Xiao–Konno–Ohno formula for fibred threefold, which extend the Xiao's tecnique. Using this latter we derived new slope inequality for threefold under the hypothesis of semistability of the ample summand of the Hodge bundle. Finally using a result of Ohno, we are eventually able to rephrase our slope inequalities as upper buonds for the unitary rank u_f in term of p_g if the genus of the base b is zero or one.

Assuptions and Notations. We work over $\mathbb{C}$. All varieties, unless otherwise stated, are assumed to be smooth and projective. Given a variety X and a divisor D on X, $H^0(X, D)$ means as usual $H^0(X, \mathcal{O}_X(D))$, $|D|$ its linear system and $\dim |D| = h^0(X, D) - 1$ its projective dimension.

CHAPTER 1

Curves

1.1. Dualizing sheaf and canonical morphism

Let C be a smooth curve. Over C is naturally defined a sheaf of holomorphic 1-forms ω_C called the *canonical sheaf*. It is involved in the Serre duality:

$$H^0(C, \mathcal{O}(D)) \simeq H^1(C, \omega_C(-D))^\vee,$$

where D is any divisor of C. For this last property ω_C is called *dualizing sheaf*. Since it is particularly useful for calculating cohomological dimensions we want to define a dualizing sheaf even over a singular curve.

Now let C be a reduced and not necessarily irreducible curve, we follow ([13] chapter II sec 6) to define the dualizing sheaf of C. If $\nu \colon \hat{C} \to C$ is the normalization of C, then $\hat{C}$ is a disjoint union of (non-singular) curves. For $x \in C$ singular, $\nu^{-l}(x)$ consists of finitely many points corresponding to the different branches of C through x. On C there is the normalization sequence:

$$0 \to \nu^* \mathcal{O}_C \to \mathcal{O}_{\hat{C}} \to Z \to 0$$

where Z is a skyscraper sheaf supported at the singular points $p_1, ..., p_k$ of C. Now we consider the sections ψ of $\omega_{\hat{C}}$ such that:

- over every set of points $q_1, ..., q_k$ inverse images of a singular point of C, ψ has at most $k-1$ poles counted with multiplicity;
- the sum of the residues of ψ over $q_1, ..., q_k$ is zero.

The $\mathcal{O}_C$-module given by the pushforward of these sections is the dualizing sheaf $\hat{\omega_C}$ of C. This sheaf has the following properties:

- if C is smooth $\hat{\omega_C} = \omega_C$;
- the dualizing sheaf satisfies the Serre's properties, hence for every divisor D of C:

$$H^1(C, \mathcal{O}(D)) \backsimeq H^0(C, \hat{\omega_C} \otimes \mathcal{O}(-D))^\vee;$$

- $\hat{\omega_C}$ has degree $\deg(\hat{\omega_C}) = 2(g_{ar}(C) - 1)$, where $g_{ar} := H^1(C, \mathcal{O}_C)$ is the arithmetic genus of C;
- the linear system $|\hat{K_C}|$ has base points over the singular points of C.

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Let C be a smooth projective curve of genus $g(C) = g \geq 2$, and let K_C (resp. ω_C) a canonical divisor (resp. line bundle). We fix a basis $\psi_1, ..., \psi_g$ of $H^0(C, \omega_C)$ and define the map:

$$\phi_K : C \longrightarrow \mathbb{P}(H^0(C, \omega_C)^\vee) \cong \mathbb{P}^{g-1}$$
$$p \mapsto (\psi_1(p) : ... : \psi_g(p))$$

which is called the *canonical morphism.*

Theorem 1.1 (Canonical Embedding ([1] pag 12-13)). *Let C be a smooth curve of genus $g \geq 2$. If C is not hyperelliptic then the canonical map is an embedding. Otherwise the canonical map factorizes as follows:*

$$\begin{array}{ccc} C & \xrightarrow{\phi_K} & \mathbb{P}^{g-1} \\ & {\scriptstyle\pi}\searrow \quad \nearrow{\scriptstyle\nu_{g-1}} & \\ & \mathbb{P}^1 & \end{array}$$

where π is the quotient map induced by the involution of C and ν_{g-1} is the Veronese embedding of degree $g-1$.

Definition 1.1.1 (Span of divisor). Let C be a smooth curve and $\phi \colon C \to \mathbb{P}^r$ be a holomorphic map. For any effective divisor D on C we denote by $\operatorname{Span}\phi(D)$ the intersection of all hyperplanes $H \subset \mathbb{P}^r$ such that:

- $\phi(C) \subseteq H$ or,
- $\phi^*(H) \geq D$.

In particular if $D = p_1 + .. + p_k$ and $\dim(\operatorname{Span}(D)) = k - 1$ the points $p_1, .., p_k$ are said to be *in general position.*

Remark 1.1.2. We consider now the canonical map

$$\phi_K \colon C \to \mathbb{P}^{g-1} \cong \mathbb{P}(H^0(C, K_C)^\vee)$$

. Every hyperplane of $\mathbb{P}^{g-1}$ has an equation given by a section of the canonical bundle K_C, or equivalently an element of $|K_C|$. $\operatorname{Span}(\phi_K(D))$ is the intersection of all hyperplanes that cut $\phi_K(C)$ in a divisor containing $\phi_K(D)$. This space has codimension $h^0(C, K_C - D)$.

This remark allows us to reformulate the Riemann-Roch theorem in geometric terms:

THEOREM 1.2 (Geometric version of Riemann-Roch [1]). *Given an effective divisor D of degree d on a smooth curve C of genus $g \geq 2$, we have:*

$$\dim(\operatorname{Span}\phi_K(D)) = d - 1 - \dim|D| = d - h^0(C, D).$$

PROOF. From Remark 1.1.2 we know that:

$$\dim(\operatorname{Span}(\phi_K(D))) = g - 1 - h^0(C, K_C - D).$$

Now recall Riemann-Roch therem we can rewrite:

$$\dim(\operatorname{Span}(\phi_K(D))) = g-1-h^0(C, K_C-D) = \deg(D)-h^0(C, D) = d-1-\dim|D|.$$

□

For a hyperelliptic curve, Theorem 1.2 allows us to give a complete characterization of its linear systems:

PROPOSITION 1.1.3. Let C be a hyperelliptic curve of genus g with involution i. Then every linear system $|D|$ of degree $d \leq g$ and dimension $\dim|D| = r$, is equivalent to:

$$rg^1_2 + p_1 + ... + p_{d-2r}$$

where $i(p_j) \neq p_k$ for every $j, k = 1, ..., d - 2r$.

PROOF. Let $D \in |D|$ a divisor. Since C is hyperelliptic it has an involution $i : C \to C$ that exchanges the two sheets over $\mathbb{P}^1$. Suppose that D has k couples of points which are orbits of i, or in other terms

$$D = (q_1 + i(q_1)) + ...(q_k + i(q_k)) + p_1 + ... + p_{d-2k}.$$

Now from Theorems 1.2 and 1.1:

$$r = \dim|D| = d - 1 \dim\operatorname{Span}(\nu^{g-1}(\pi(D))),$$

but since $\pi(D)$ is a set of $k + d - 2k = d - k$ distinct points of $\mathbb{P}^1$, under the Veronese's map they became $d - k$ points in general position in $\mathbb{P}^{g-1}$, i.e. $\dim\operatorname{Span}(\nu^{g-1}(\pi(D))) = d - k - 1$. Then we conclude that:

$$r = d - 1 - (d - 1 - k) = k$$

as we wanted. □

Now assume that C is non-hyperelliptic, i.e., that ϕ_K is an embedding. Often, with abuse of notation, we identify C and its points with the corresponding canonical image.

Given a linear subspace $V \subseteq H^0(C, \omega_C)$, let us consider:

$$Ann(V) := \{\theta \in H^0(C, \omega_C)^\vee \mid \theta(v) = 0 \quad \forall v \in V\} \subseteq H^0(C, \omega_C)^\vee.$$

We call this subspace *annihilator* of V. Let $\mathbb{A}\mathrm{nn}(V) := \mathbb{P}(Ann(V)) \subseteq \mathbb{P}(H^0(C, \omega_C)^\vee)$ be its projectivisation. Observe that the dimension of $\mathbb{A}\mathrm{nn}(V)$ is the codimension of V minus one. Thanks to Remark 1.1.2 we can re-define $\mathrm{Span}(D)$, in $\mathbb{P}^{g-1}$, as:

DEFINITION 1.1.4. Given an effective divisor D on C, its projective span is

$$\mathrm{Span}(D) = \mathrm{Span}\,\phi_K(D) := \mathbb{A}\mathrm{nn}(H^0(C, \omega_C(-D)) \subseteq \mathbb{P}(H^0(C, \omega_C)^\vee)$$

EXAMPLE 1.1.5. Given a point $p \in C$, $\mathrm{Span}\,\phi_K(p) = \{p\}$, while $\mathrm{Span}\,\phi_K(2p)$ is the line tangent to C in $\mathbb{P}^{g-1}$, $\mathrm{Span}\,\phi_K(3p)$ is the osculating plane to C, and so on. For n distinct points $p_1, \ldots, p_n$ on C if we call $D = p_1 + \ldots + p_n$, we have that $\mathrm{Span}\,\phi_K(D)$ coincides with the linear projective span of the points in $\mathbb{P}^{g-1}$.

Given a linear subspace $V \subseteq H^0(C, \omega_C)$ we call the *base locus* D_V of the linear system $|V|$ the scheme-theoretic intersection $D_V := \mathbb{A}\mathrm{nn}(V) \cap C$. Observe that the evaluation map of V is surjective onto $\omega_C(-D_V)$.

1.2. Gonality and Clifford index

DEFINITION 1.2.1. (Gonality) The *gonality* $\mathrm{gon}(C)$ of C is the following integer:

$$\mathrm{gon}(C) := \min\{\deg(\pi)|\ \pi : C \to \mathbb{P}^1 \text{ is a surjective morphism}\} = \min\{m|\ \exists\ g^1_m \text{ over } C\}.$$

REMARK 1.2.2. Every hyperelliptic curve has one g^1_2 and so is 2- gonal, and for every curve C of genus $g \geq 1$ we have that $\mathrm{gon}(C) \geq 2$.

DEFINITION 1.2.3. (Clifford index) Given a curve C of genus $g \geq 4$, we define its Clifford index $\mathrm{Cliff}(C)$ as:

$$\mathrm{Cliff}(C) := \min\{\deg(D) - 2(\dim|D|)\ \ |\ h^0(C, D) \geq 2,\ h^1(C, D) \geq 2\}.$$

In case $g = 2, 3$ we define the Clifford index as follows:

- if $g = 2$, $\mathrm{Cliff}(C) := 0$;
- if $g = 3$, $\mathrm{Cliff}(C) := 0$ (resp. 1) if C is hyperelliptic (resp. trigonal).

For every divisor D such that $h^0(C, D) \geq 2$ and $h^1(C, D) \geq 2$, we say that D *c*ontributes to the Clifford index.

We have the following upper bounds:

$$\mathrm{gon}(C) \leq \lfloor \frac{g+3}{2} \rfloor, \qquad \mathrm{Cliff}(C) \leq \lfloor \frac{g-1}{2} \rfloor,$$

with equality holding for a general curve in $\mathcal{M}_g$.

REMARK 1.2.4. In [44] Coppens and Martens proved the following inequalities

$$\text{gon}(C) - 3 \leq \text{Cliff}(C) \leq \text{gon}(C) - 2. \tag{1.2.5}$$

Ballico [5] showed that for a general curve C in the locally closed subset of curves in $\mathcal{M}_g$ of gonality $\text{gon}(C)$, it holds the equality on the right.

Let us discuss in some detail the case of plane curves

LEMMA 1.1. *Let $C \subset \mathbb{P}^2$ a smooth curve of degree $d \geq 2$, then there is no g^1_{d-2} but it exist a g^1_{d-1} so the gonality of C is $d-1$. The Clifford index of C is $d-4$.*

PROOF. Suppose that it exists a divisor D with degree $\deg(D) = e \leq d-2$ and $h^0(C, D) \geq 2$. From Bertini's Lemma [1] we can find, if necessary, another divisor $D' \in |D|$ such that $D' = p_1 + ... + p_e$ where the p_i's are all distinct points. Now the Riemann-Roch theorem tells us:

$$h^0(C, K_C - D) = g - 1 - e + h^0(C, D) \geq g - e + 1,$$

so D' does not impose independent conditions on the canonical system. Nonetheless the canonical system $|K_C|$ is cut out by hypersurfaces of degree $d-3$ of $\mathbb{P}^2$ so every $d-2$ points of C impose an independent condition, then an absurd. We note as for any $p \in C$ the lines through p define a g^1_{d-1}. Moreover the the linear system induced by the hyperplanes of $\mathbb{P}^2$, say $|H|$, defines a g^2_d of C, and its Clifford index is:

$$\text{Cliff}(H) = d - 4 = \text{gon}(C) - 3$$

then recalling Remark 1.2.4,we conclude that $\text{Cliff}(H) = \text{Cliff}(C)$. □

Gonality also has a very natural geometric interpretation via the Geometric Riemann-Roch Theorem:

PROPOSITION 1.2.6. For every effective divisor D over C,

$$\dim(\text{Span}(D)) \leq \deg(D) - 1.$$

If $\dim \text{Span}(D) < \deg(D) - 1$, then $\deg D \geq \text{gon}(C)$. If on the other hand k is an integer greater than or equal to $\text{gon}(C)$, then there exists a divisor D of degree $\deg D = k$ with $\dim \text{Span} D < \deg D - 1$.

PROOF. The first inequality is straightforward from Geometric Riemann Roch 1.2. Suppose now that $\dim \text{Span}(D) < \deg(D) - 1$; by Riemann Roch again $h^0(C, D) \geq 2$. So there exist a linear subspace $V \subseteq H^0(C, D)$ of dimension 2 and degree $\leq \deg D$. Thus $\text{gon}(C) \leq \deg D$. The other implication is immediate. □

REMARK 1.2.7. For example, a non-hyperelliptic curve C is trigonal if and only if there exists three collinear points on C, a curve C is 4-gonal (i.e. $\mathrm{gon}(C) = 4$) if and only if every three points p_1, p_2, p_3 of C are not collinear, but there exist a 4-uple of points of C that spans a plane.

REMARK 1.2.8. In general, if C has gonality $\mathrm{gon}(C) = k$ then holds:

$$\mathrm{Cliff}(C) \leq k - 2.$$

In fact, considering the g^1_k given by $|D|$, then from the geometric version of Riemann-Roch theorem we have:

$$h^1(C, D) = g - 1 - \dim \mathrm{Span}(D) = g - 2.$$

So if $g \geq 4$ the divisor $|D|$ contributes to the clifford index and we conclude that

$$\mathrm{Cliff}(C) \leq \deg D - 2\dim |D| = k - 2.$$

1.3. Clifford's theorem

Now we see one big player in this thesis, but first let we introduce a preliminar lemma:

THEOREM 1.3 (General position Lemma ([1] pag 107)). *Let $C \subset \mathbb{P}^r$ be an irreducible non-degenerate, possibly singular curve of degree d. Then a general hyperplane H intersects C in d points any r of which are linearly independent.*

REMARK 1.3.1. This lemma can be read as: "every subset of r points in $C \cap H$ span H, for almost every hyperplan H of $\mathbb{P}^r$".

THEOREM 1.4 (Clifford's Theorem). *Let C be a smooth curve of genus $g \geq 2$ and $|D|$ a (complete) linear system of degree d, then it holds:*

$$d \geq 2\dim |D|. \tag{1.3.2}$$

Moreover (1.3.2) is an equality if and if one of these conditions is satisfied:

- *D is the trivial or the canonical divisor;*
- *C is hyperelliptic and $|D|$ is equivalent to a multiple of the g^1_2.*

PROOF. We first show the inequality.
We claim that $\dim |D| \geq r$ if and only if the linear system $|D|$ contains every subset of r points of C, more precisely for every $p_1, .., p_r \in C$ (not necessarily distinct) there exists a divisor $P \in |D|$ of the form $P = p_1 + ... + p_r + P'$.

Proof of Claim. We proof the claim inductively. If $\dim|D| \geq 1$ then there exist a surjective map

$$\phi_{|D|} : C \to \mathbb{P}^1,$$

so every point of C is contained in a fiber of $\phi_{|D|}$, but every fiber is a divisor in $|D|$.

Now suppose the claim holds for $\dim|D| \geq r-1$ and consider a linear system $|D|$ of dimension not less than r. Choose an arbitrary point p of C and consider the subsystem $|D-p|$, which has dimension $\dim|D-p| \geq r-1$. For hypothesis $|D-p|$ contains every subset of $r-1$ points of C, then since p is choosen arbitrarely $|D|$ contains every subset of r points.

On the other hand if $|D|$ contains every r points of C, the subsystem $|D-p_1|$ must have codimension 1 in $|D|$. Similarly $|D-p_1-p_2|$ has codimension 1 in $|D-p_1|$, and so on until we arrive at $|D-p_1-...-p_k|$ which must have non negative dimension. So in $|D|$ we can span a linear subspace of dimension r then $\dim|D| \geq r$. □

Suppose now that $|D|$ and $|D'|$ are two linear systems of dimension r and r'. So for the previous lemma they contain respectively any r and r' points of C; the system $|D+D'|$, must contains any $r+r'$ points of C, so invoking again the lemma:

$$\dim|D+D'| \geq r+r = \dim|D| + \dim|D'|.$$

In particular every element of the canonical system $|K_C|$ can be written as the sum $D+(K_C-D)$, where $K_C \in |K_C|$ and $D \in |D|$, then set $D' := K_C - D$ we see that

$$g-1 = |K_C| = |D+D'| \geq |D| + |D'|.$$

Now from the Riemann-Roch theorem we have:

$$|D| - |D'| = d+1-g.$$

So summing up the two above inequalities we conclude

$$2\dim|D| \leq d.$$

Suppose now that (1.3.2) is an equality and D is neither trivial or canonical. Exchanging D and D' if necessary, we can set $\deg D \leq g-1$ and $\dim|D| > 0$. If C is not hyperelliptic, there exists an hyperplan of $\mathbb{P}^{g-1}$ that cuts the image $\phi(C)$ on $D+D'$. But D contains $g-1$ dependents points, against Lemma 1.3.

If otherwise C is hyperelliptic, from Remark 1.1.3, we know that every linear system $|D|$ is equivalent to

$$rg^1_2 + p_1 + ... + p_{d-2r}.$$

So the equality is satisfied if and only if $d = 2r$, i.e. $D = rg^1_2$, as wanted. □

Remark 1.3.3. This last statement can be restated as follows. We have seen that the Clifford index of a curve C, Cliff(C) is defined as:

$$\mathrm{Cliff}(C) := \min\{\deg(D) - 2(\dim|D|) \ \mid h^0(C, D) \geq 2,\ h^1(C, D) \geq 2\}.$$

Then Clifford's theorem tells us that $\mathrm{Cliff}(C) = 0$ if and only if it exists a special divisor D such that $\deg D = 2\dim|D|$, and this happens if and only if C is hyperelliptic.

1.4. Linear stability

Definition 1.4.1. (Linear (semi)stability) A linear system $|V|$ over C is *linearly stable* (resp. semistable) if for every linear subsystem $|W| \subseteq |V|$ we have:

$$\frac{\deg|W - D_W|}{\dim|W|} > \frac{\deg|V|}{\dim|V|} \quad \text{(resp. } \geq\text{)}$$

Remark 1.4.2. Let us make some remarks.

- Again, Clifford's theorem can be rephrased saying that the canonical system on a curve is linearly semistable, and it is stable if and only if the curve is non-hyperelliptic.
- The linear system $|V|$ and its linear subsystems $|W|$ are not necessarily complete;
- If $|V| \subseteq |L|$ has a non zero base locus D_V, then the linear subsystem:

  $$V(-D_V) := V \cap H^0(C, L - D_V)$$

  destabilizes it, because $\deg|V(-D_V)| < \deg|V|$ but $\dim|V(-D_V)| = \dim|V|$. So, systems with base points are always linearly unstable.
- Linear stability was introduced by Mumford in order to develop a simple method to prove GIT stability results, indeed, it is proven in [48] that linear semistability implies Chow stability and in [2] that linear stability implies Hilbert stability.

Remark 1.4.3. We want to recall the notion of *Chow's stability* for algebraic varieties, focusing in particular on the case of (smooth) curve. Let C be a non degenerate irreducible curve in $\mathbb{P}^n$, of degree d. We call $\mathbb{G}(n-1,n)$ the space of hyperplanes of $\mathbb{P}^n$, and $Z_C \subseteq \mathbb{G}(n-1,n)$ the locus defined as follows:

$$Z_C := \{H \in \mathbb{G}(n-1,n) : \quad H \cap C \neq \emptyset\}.$$

Z_C is an irreducible divisor of $\mathbb{G}(n-1,n)$ of degree d (ref [48]), so there exists a form $F_C \in H^0(\mathbb{G}(n-1,n), \mathcal{O}_{\mathbb{G}(n-1,n)}(d))$, called *Chow's form* whose zero locus is exactly Z_C. Now the action of the group $\mathrm{Sl}(n,\mathbb{C})$ on $\mathbb{P}^n$ induces an action over $\mathbb{G}(n-1,n)$ and similarly on $H^0(\mathbb{G}(n-1,n), \mathcal{O}_{\mathbb{G}(n-1,n)}(d))$, we can apply the notion of G.I.T. stability [48].

DEFINITION 1.4.4. Let $C \subseteq \mathbb{P}^n$ be a non degenerate irreducible curve, and F_C its chow form. We say that C is *Chow stable* if the form $F_C \in H^0(\mathbb{G}(n-1,n), \mathcal{O}_{\mathbb{G}(n-1,n)}(d))$ is G.I.T. stable under the action of the group $\mathrm{Sl}(n,\mathbb{C})$.

1.5. Clifford-type inequalities for sub-canonical systems

A linear system $|V|$ is linearly stable if its ratio $\deg|V|/\dim|V|$ bounds from below the ratio d/r for any $g^r_d \subseteq |V|$. Changing point of view, given a linear system on a curve, one can ask for a lower bound for this ratio d/r possibly lower than the original ratio $\deg|V|/\dim|V|$. This is what we do for canonical subsystems of non-hyperelliptic curves, obtaining a bound depending on the codimension and on the Clifford index of the curve. This new prospective is dictated by Xiao's method which we will use in the next chapter.

THEOREM 1.5. *Let $C \subseteq \mathbb{P}^{g-1}$ be a canonical non-hyperelliptic curve. Let $V \subseteq H^0(C,\omega_C)$ a linear subspace of codimension $k \leq g-2$. Then for any $W \subseteq V$ subspace of dimension $\dim W \geq 2$, we have:*

$$\frac{\deg|W - D_W|}{\dim|W|} \geq \frac{2g-2-m}{g-m-1}.$$

where $m := \min\{k, \mathrm{Cliff}(C)\}$.

PROOF. For any $W \subseteq V$ we have the evaluation morphism:

$$W \otimes \mathcal{O}_C \twoheadrightarrow \omega_C(-D_W),$$

where $D_W := \mathbb{A}\mathrm{nn}(W) \cap C$ is the base locus of $|W|$.

We begin by considering the case $m = k$.

LEMMA 1.2. *If $k \leq \mathrm{Cliff}(C)$, then $\deg|V| = \deg(\omega_C(-D_V)) \geq 2g-2-k$, i.e. $\deg D_V \leq k$.*

PROOF. We split the proof of the lemma in two cases:

- If $h^0(C, D_V) \geq 2$, since $h^0(C, \omega_C(-D_V)) \geq \dim V \geq 2$, then both D_V and $\omega_C(-D_V)$ contributes to the Clifford index of C, so we have:

$$\deg(\omega_C(-D_V)) \geq 2(h^0(C, \omega_C(-D_V)) - 1) + \mathrm{Cliff}(C) \geq 2(g-k-1) + k = 2g-2-k,$$

  as wanted.
- If $h^0(C, D_V) = 1$, by the geometric version of Riemann-Roch (Theorem 1.2), we have that:

$$\dim(\mathrm{Span}(D_V)) = \deg D_V - h^0(C, D_V) = \deg D_V - 1.$$

  Now, $\mathrm{Span}(D_V) \subseteq \mathbb{A}\mathrm{nn}\, V$ by construction, and

$$\dim \mathbb{A}\mathrm{nn}(V) = g - 1 - \dim(V) = g - 1 - (g - k) = k - 1.$$

  Therefore, we can conclude that $\deg D_V \leq k$, and the claim is proven.

□

Let's go back to the proof of Theorem 1.5. Let $W \subsetneq V$, with $\dim W \geq 2$. As done for Lemma 1.2, we analyze the two following cases:

(i) If $h^0(C, D_W) \geq 2$, hence D_W contributes to $\mathrm{Cliff}(C)$ since $h^1(C, D_W) = h^0(C, \omega_C(-D_W)) \geq \dim W \geq 2$, then:

$$\deg \omega_C(-D_W) \geq 2(h^0(C, \omega_C(-D_W)) - 1) + \mathrm{Cliff}(C) \geq 2(\dim W - 1) + k.$$

Hence:

$$\frac{\deg |W - D_W|}{\dim |W|} = \frac{\deg \omega_C(-D_W)}{\dim |W|} \geq 2 + \frac{k}{\dim |W|} \geq 2 + \frac{k}{\dim |V|} = \frac{2g-2-k}{g-k-1},$$

as wanted.

(ii) If $h^0(C, D_W) = 1$ we can conclude $\deg D_W \leq \dim(\mathbb{A}\mathrm{nn}\, W) + 1$ as in the proof of lemma 1.2. Setting $k_W := \dim(\mathbb{A}\mathrm{nn}\, W) + 1 = \mathrm{codim}(W)$, we have:

$$\frac{\deg |W - D_W|}{\dim |W|} = \frac{2g - 2 - \deg D_W}{g - k_W - 1} \geq \frac{2g - 2 - k_W}{g - k_W - 1}.$$

Since $W \subseteq V$ we can conclude that $k_W \geq k$.

Now, consider the function:

$$f\colon [0, g-1] \to \mathbb{R} \quad f(t) := \frac{2g-2-t}{g-t-1}. \tag{1.5.1}$$

As

$$f'(t) = \frac{g-1}{(g-t-1)^2} > 0 \qquad \forall t \in [0, g-1],$$

we have that f is monotone strictly increasing. So, since $k_W \geq k$, we obtain:

$$\frac{\deg|W - D_W|}{\dim|W|} \geq f(k_W) \geq f(k) = \frac{2g-2-k}{g-k-1},$$

as wanted.

Let us now treat the case $k \geq \mathrm{Cliff}(C) =: c$. We prove that for any $W \subseteq V$, with $\dim W \geq 2$:

$$\frac{\deg|W|}{\dim|W|} \geq \frac{2g-2-c}{g-c-1}.$$

Like we did above, we focus on two cases:

(i) if $h^0(C, D_W) \geq 2$, then D_W contributes to the Clifford index since

$$h^1(C, D_W) = h^0(C, \omega_C(-D_W)) \geq \dim W \geq 2.$$

So we have that

$$\deg(\omega_C(-D_W)) \geq 2(h^0(C, \omega_C(-D_W)) - 1) + c \geq 2\dim|W| + c.$$

Then it follows that:

$$\frac{\deg|W - D_W|}{\dim|W|} \geq 2 + \frac{c}{\dim|W|} \geq 2 + \frac{c}{g-c-1} = \frac{2g-2-c}{g-c-1}.$$

(ii) If otherwise $h^0(C, D_W) = 1$, then as in the previous case we can conclude:

$$\deg D_W \leq k_W$$

and since $k_W \geq k \geq c$, exploiting the monotonicity of the function f:

$$\frac{\deg|W|}{\dim|W|} = \frac{2g-2-\deg D_W}{g-1-k_W} \geq \frac{2g-2-k_W}{g-1-k_W} = f(k_W) \geq f(c) = \frac{2g-2-c}{g-c-1}.$$

□

Remark 1.5.2. Theorem 1.5 above is not a linear stability result for the system $|V|$ unless $k \leq \mathrm{Cliff}(C)$ and $\deg|V| = 2g-2-k$, i.e. $|V|$ has the biggest possible base locus, according to Lemma 1.2.

The above result implies a stability result, as follows.

Corollary 1.5.3. Let $V \subseteq H^0(C, \omega_C)$ be a vector subspace of codimension k, with $k \leq \mathrm{Cliff}(C)$. If

$$\deg|V| = 2g-2-k$$

then $|V|$ is linearly semistable. In particular the morphism induced on C is Chow semistable.

Proof. Let $W \subseteq V$. Let $h \geq k$ be the codimension of W in $H^0(C, \omega_C)$. By Lemma 1.5 we have that, for $\overline{m} = \min\{\text{Cliff}(C), h\}$,

$$\frac{\deg|W - D_W|}{\dim|W|} \geq \frac{2g - 2 - \overline{m}}{g - h - \overline{m}}.$$

Now, $\overline{m} \geq k$, and we are done by the monotonicity of the function f defined in (1.5.1):

$$\frac{\deg|W - D_W|}{\dim|W|} \geq \frac{2g - 2 - \overline{m}}{g - \overline{m} - 1} = f(\overline{m}) \geq f(k) = \frac{2g - 2 - k}{g - k - 1} = \frac{\deg|V|}{\dim|V|}.$$

□

Example 1.5.4. Given $k \leq \text{Cliff}\, C$ points $p_1, \ldots, p_k$ on C in general position, clearly the system $|\omega_C(-p_1 \ldots - p_k)|$ satisfies the assumptions of Corollary 1.5.3, as

$$\deg(\omega_C(-p_1 \ldots - p_k)) = 2g - 2 - k \quad \text{and} \quad h^0(C, \omega_C(-p_1 \ldots - p_k)) = g - k.$$

Example 1.5.5. We see here that indeed for *any* set of $k \leq \text{Cliff}(C)$ points on C, the system $|\omega_C(-p_1 \ldots - p_k)|$ satisfies the assumptions of Corollary 1.5.3. Indeed, we claim that

$$h^0(C, p_1 + \ldots + p_k) = 1.$$

Assume by contradiction that $h^0(C, p_1 + \ldots + p_k) \geq 2$: we would have a g^1_d on C with $d \leq k$ hence

$$\text{gon}(C) \leq d,$$

but from Ballico's result (4.2) we obtain:

$$k + 2 \leq \text{gon}(C) \leq d \leq k,$$

which gives a contradiction. From Riemann-Roch theorem

$$h^0(C, \omega_C(-p_1 \ldots - p_k)) = 2g - 2 - k + 1 - g + h^0(C, p_1 + \ldots + p_k) = g - k.$$

Hence the linear series $|\omega_C(-p_1 \ldots - p_k)|$ satisfies the hypothesis of Corollary 1.5.3, so it is linearly semistable.

1.6. Chevalley-Weil formula

The Riemann existence theorem [45] allows us to construct every smooth curve C as a finite cover over $\mathbb{P}^1$ with a given monodromy group G (that satisfies certain conditions). From this recipe it is interesting and will be useful later, to see how the cohomology groups ($H^1(C)$ in particular) of C behave under the action of G. For smooth curves this question is answered by the Chevalley-Weil theorem which we now introduce.

Let $f\colon C \to C'$ be a separated finite morphism of curves of degree d [45]. We call it a *Galois covering* if the field extension of the functions fields $K(C)/K(C')$ is Galois. Geometrically this means that the group of covering transformations:

$$\mathrm{Cov}(C/C') := \{\sigma : C \to C \ \mid f(\sigma(p)) = f(p) \quad \forall p \in C\}$$

acts trasitively on any fiber of f.

A Galois covering has the following properties:

- any fiber $f^{-1}(Q)$ with $Q \in C'$, has the same number of points (counted with multiplicity) equal to d;
- called $Q_1, ..., Q_r \in C'$ the branching points of f, and $P_i \in f^{-1}(Q_j)$ ramification points over Q_j, every P_i has the same multiplicity e_i called *ramification index*;
- for every point $P \in f^{-1}(Q)$ there exists a stabilizier subgroup of the covering group $G_P := \{g \in G \ \mid g(P) = P\}$, which is cyclic of cardinality $|G_P| = e_i$, and given two ramification points $P, P' \in f^{-1}Q$ their stabilizier subgroups are conjugated each other;

  An element of G_P is called a *local monodromy* if its primitive character $\chi_{G_p} \to \mathbb{C}^*$ is $\exp 2i\pi/e_i$.
- Given a sheaf $\mathcal{F}$ of $\mathcal{O}_C$, G acts on $f_*\mathcal{F}$, where the action on a local section ψ is:

$$g * \psi := \psi g^{-1}.$$

Following this last property, if we consider the sheaf of differential forms ω_C, then the action of G induces a representation over the vector space of global section $H^0(C, \omega_C)$. Given a character χ and called ρ_χ the irreducible representation associated, in general we can write:

$$d_\chi \nu_\chi = \dim H^0(C, \omega_C)^\chi,$$

where ν_χ labels the multiplicity of the representation and d_χ is the dimension of the irreducible one.

We recall that given an element $g \in G$ of order n, its matrix representation $\rho(g)$ has eigenvalues $\lambda_1, ..., \lambda_d$ which are of the form $\exp(2\pi i\alpha/n)$ with $\alpha = 0, 1, ..., n-1$.

In particular, this holds for the monodromies g_i of the branching points Q_i. We indicate with $N_{\alpha,i}$ the numbers of eigenvalues equal to $\exp(2\pi i\alpha/n)$.

We define $< q >:= q - \rfloor q \lfloor$ the fractional part of $q \in \mathbb{R}$, with $\rfloor q \lfloor$ integer part of q.

The Chevalley-Weil formula gives the multiplicities ν_χ as function of e_i and $N_{\alpha i}$:

THEOREM 1.6 (Chevalley-Weil formula). *Let $f : C \to C'$ be a Galois covering of curves with Galois group G. Let e_μ and $N_{\alpha\mu}$ as above. Then the multiplicity ν_χ of a given irreducible character χ in $H^0(C, \omega_C)$ is given by:*

$$\nu_\chi = d_\chi(g_Y - 1) + \sum_{i=1}^{n} \sum_{\alpha=1}^{\mu_i - 1} N_{i\alpha} < -\frac{\alpha}{\mu_i} > + \sigma$$

where $\sigma = 1$ if χ is the trivial character and zero otherwise.

In particular if $C' = \mathbb{P}^1$, we have $H^0(\mathbb{P}, \omega_{\mathbb{P}^1}) = H^0(C, \omega_C)_{\chi_{trivial}} = 0$. If G is abelian, then every irreducible representation has degree one. We can put together these remarks as follows:

COROLLARY 1.6.1. Let $f : C \to \mathbb{P}^1$ be a Galois covering with an abelian covering group G. Then the multiplicity ν_χ of a given non trivial character χ in $H^0(C, \omega_C)$ is given by:

$$\nu_\chi = -1 + \sum_{i=1}^{n} \sum_{\alpha=1}^{\mu_i - 1} N_{i\alpha} < -\frac{\alpha}{\mu_i} >$$

1.6.1. An easy application. Consider the smooth curve C defined by the affine equation in $\mathbb{A}^2$:

$$y^n = x(x-1)(x-\lambda),$$

with $n \geq 4$. This equation describes a cyclic covering of $\mathbb{P}^1$, branched over the four points $\{0, 1, \lambda, \infty\}$. Its Galois group is obviously $\mathbb{Z}/n\mathbb{Z}$. Call $\sigma \in \mathbb{Z}/n\mathbb{Z}$ a generator of G that acts on C as:

$$(x, y) \to (x, \eta(\sigma)y)$$

with $\eta(\sigma)$ primitive n-th root of unity. The characters group $\{\chi_i\}_{i=0}^{n-1}$ send $\chi_i : \eta(\sigma) \to \eta^i(\sigma)$.

Therefore we find that:

$$\text{at } Q_1, Q_2, Q_3 \qquad N_{\alpha,i} = 1 \text{ if } \alpha = i \qquad N_{\alpha,i} = 0 \text{ otherwise;}$$

$$\text{at } Q_4 \qquad N_{\alpha,i} = 1 \text{ if } \alpha = (n-3)i \qquad N_{\alpha,i} = 0 \text{ otherwise;}$$

Then according to the Chevalley-Weil formula we find that the multiplicity of the representation of η^i is:

$$\dim H^0(C,\omega_C)_i = v_i = -1+ < -\frac{i}{n} > + < -\frac{(n-3)i}{n} > +\sigma$$
$$= \begin{cases} 0 & \text{if } i=0; \\ 2-\rfloor \frac{3i}{n} \lfloor & \text{otherwise} \end{cases}$$

In Chapter 3 we will use a generalization of this example.

1.7. Some results on gonality of curves

Let C be a smooth curve. In general it can be a hard task to find the gonality $\gamma(C)$ (and the Clifford index) of C and only few general results are known. For the applications to fibred surfaces we will need some result, focusing in particular on the case of irreducible curves which are cyclic covering of $\mathbb{P}^1$ of degree d, with affine equation:

$$y^d = (x-\lambda_1)^{\alpha_1}(x-\lambda_2)^{\alpha_2}...(x-\lambda_n)^{\alpha_n} \tag{1.7.1}$$

where $\alpha_1+\alpha_2+...+\alpha_n \equiv 0 \mod n$, and $\lambda_1,...,\lambda_n \in \mathbb{C}$.

Remark. The condition $\alpha_1+\alpha_2+...+\alpha_n \equiv 0 \mod n$ forces that there is no branching point at infinity, i.e. that λ_i are all the branching points of C.

Remark. We notice that each permutation of the exponents does not affect the gonality, the same if we multiply each α_i by a factor k such that $gcd(k,d)=1$. So the result stated below applies precisely to the *class of equivalence* of equations linked by the trasformation

$$(\alpha_1,\alpha_2,...,\alpha_n) \mapsto (k\alpha_{\pi(1)},k\alpha_{\pi(2)},...,k\alpha_{\pi(n)})$$

with π permutation of $\{1,2,...,n\}$ and $gcd(k,d)=1$.

Theorem 1.7 (Hyperelliptic/elliptic criteria [58]). *Let C be the smooth curve associated to the affine equation 1.7.1. Then C is hyperelliptic or elliptic (in other word $\gamma(C)=2$) if and only if one of these conditions is fulfilled:*

(i) $d=2$;
(ii) $d\geq 4$ *is even and all but 2 of the* a_i *are equal to* $d/2$
(iii) $n=3$, $d\geq 3$
(iv) $n=4$, $d\geq 3$ *and* (a_1,a_2,a_3,a_4) *is equivalent to* $(a,a,d-a,d-a)$

Theorem 1.8 (Trigonality criteria [57]). *The curve C given by the equation 1.7.1 is trigonal if and only if one of the following conditions is satisfied:*

- $n = 3$ *and* $d \geq 7$, *and* $(\alpha_1, \alpha_2, \alpha_3) \sim (1, 2, d - 3)$;
- $n = 4$ *and* $d \geq 4$, *and* $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \sim (1, 1, 1, d - 3)$;
- $n = 4$ *and* $d \geq 5$, *and* $(\alpha_1, \alpha_2, \alpha_3, \alpha_3) \sim (1, 2, d - 2, d - 1)$;
- $n = 5$ *and* $d \geq 7$, *and* $(\alpha_1, ..., \alpha_5) \sim (1, 1, 1, d - 2, d - 1)$;
- $n = 6$ *and* $d \geq 7$, *and* $(\alpha_1, ..., \alpha_6) \sim (1, 1, 1, d - 1, d - 1, d - 1)$;
- $n \geq 5$ *and* $d = 3$;
- $n \geq 3$ *and* $d \geq 6$, d *is divisible by* 3, *and all but two of* α_i *are divisible by* $d/3$.

1.8. Vector bundles and their stability on curves

Definition 1.8.1 (Vector bundles on curves). A *vector bundle* $\mathcal{E}$ of rank n over a smooth curve B, is a complex manifold of dimension $n + \dim B = n + 1$ with the additional data:

(i) A smooth map $\pi \colon \mathcal{E} \to B$, such that every fiber $\mathcal{E}_x$ is a complex vector bundle of dimension n.
(ii) There exists an open cover $\mathfrak{U}$ (in the analytic topology) of X such that on every $U \in \mathfrak{U}$, it holds:

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\psi} & U \times \mathbb{C}^n \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle P} & \\ & U & \end{array}$$

where P is the projection on the first factor, and ψ is a biholomorphism.

Remark. If $U_i; U_j \in \mathfrak{U}$ are such that $U_i \cap U_j \neq \emptyset$, then there exists a function:

$$g_{ij} : U_i \cap U_j \to \mathrm{Gl}(n, \mathbb{C})$$

such that:

$$(\pi_{U_i}^{-1})\pi_{U_j}(x, v) = (x, g_{ij}(x)v) \quad \text{for } x \in U_i \cap U_j \text{ and } v \in \mathbb{C}^n.$$

Such functions are called *transition functions* and $\mathfrak{U}$ is called trivializing cover. A vector bundle whose transition functions are the identity matrix, is called a *trivial bundle.*

A vector bundle such that there exists a trivialing cover such that the transition functions are constant is called *flat bundle.*

Definition 1.8.2 (Degree and Slope of Vector Bundles). Given a vector bundle $\mathcal{E}$ over a curve B the *degree* of $\mathcal{E}$, $\deg(\mathcal{E})$, is the degree of the determinant line bundle $\det \mathcal{E} := \wedge^{\mathrm{rk}\,\mathcal{E}} \mathcal{E}$. Moreover we define its *slope* $\mu(\mathcal{E})$ as:

$$\mu(\mathcal{E}) := \frac{\deg(\mathcal{E})}{\mathrm{rk}(\mathcal{E})}$$

Definition 1.8.3 (Unitary flat vector bundle). A flat vector bundle of rank n is called *unitary* if its transition functions have values in the unitary group $\mathrm{U}(n, \mathbb{C}) \subseteq \mathrm{Gl}(n, \mathbb{C})$.

Remark 1.8.4. We remind that every unitary flat vector bundle $\mathcal{E}$ has degree zero.

Definition 1.8.5 (Stable/semistable vector bundle). A vector bundle $\mathcal{E}$ over a curve B is called *stable* (resp. *semistable*) if for every proper sub vector bundle $0 \neq \mathcal{F} \subset \mathcal{E}$ it holds:

$$\mu(\mathcal{F}) < \mu(\mathcal{E}) \quad \text{resp.} \quad \mu(\mathcal{F}) \leq \mu(\mathcal{E}).$$

Remark 1.8.6. Many authors define the condition of stability (resp. semistability) as a property that must holds for every *subsheaf* $\mathcal{F}$ of $\mathcal{E}$. Moreover it can be proven that ours definition is equivalent to this one, in fact for every subsheaf $\mathcal{F}$ its *saturation* $\bar{\mathcal{F}}$ is the smallest vector bundle that contains $\mathcal{F}$, and they satisfy the relation:

$$\mu(\mathcal{F}) \leq \mu(\bar{\mathcal{F}}).$$

Definition 1.8.7 (Harder-Narasimhan filtration). [36] Let $\mathcal{F}$ be a vector bundle over a smooth projective curve B. There exists a unique sequence of vector sub-bundles of $\mathcal{F}$:

$$0 = \mathcal{F}_0 \subsetneq \mathcal{F}_1 \subsetneq \ldots \subsetneq \mathcal{F}_{k-1} \subsetneq \mathcal{F}_k = \mathcal{F} \tag{1.8.8}$$

satisfying the conditions:

- for $i = 1, \ldots, k$ $\mathcal{F}_i/\mathcal{F}_{i-1}$ is a semistable vector bundle;
- For any $i = 1, \ldots, k$ setting $\mu_i := \mu(\mathcal{F}_i/\mathcal{F}_{i-1})$, we have:

$$\mu_1 > \mu_2 > \ldots > \mu_k.$$

The filtration (1.8.8) is called *Harder-Narasimhan filtration* of $\mathcal{F}$.

We set $\mu_-(\mathcal{F}) := \mu_k$, and call it the *final slope* of the sheaf.

Remark 1.8.9. Note that it holds the formula:

$$\deg \mathcal{F} = \sum_{i=1}^{k} r_i(\mu_i - \mu_{i+1}).$$

Indeed, considering the exact sequence of vector bundles:

$$0 \to \mathcal{F}_{k-1} \to \mathcal{F}_k \to \mathcal{F}_k/\mathcal{F}_{k-1} \to 0,$$

from the additivity property of degree, we can say $\deg \mathcal{F}_k = \deg \mathcal{F}_{k-1} + \deg \mathcal{F}_k/\mathcal{F}_{k-1}$. Similarly, we have that: $\deg \mathcal{F}_{k-1} = \deg \mathcal{F}_{k-2} + \deg \mathcal{F}_{k-1}/\mathcal{F}_{k-2}$, and so on. By induction we can conclude that:

$$\begin{aligned}\deg \mathcal{F}_k &= \deg(\mathcal{F}_k/\mathcal{F}_{k-1}) + \deg(\mathcal{F}_{k-1}/\mathcal{F}_{k-2}) + \ldots + \deg(\mathcal{F}_2/\mathcal{F}_1) + \deg(\mathcal{F}_1)\\ &= \sum_{i=1}^{k} \deg(\mathcal{F}_i/\mathcal{F}_{i-1}).\end{aligned}$$

Now, from the definition of slope, for every $i = 1, .., k$ we have $\deg \mathcal{F}_i/\mathcal{F}_{i-1} = \mu_i(r_i - r_{i-1})$, So, setting $\mu_{k+1} = 0$ and $r_{k+1} = r_k$, we obtain the desired formula

$$\deg \mathcal{F} = \deg \mathcal{F}_k = \sum_{i=1}^{k} \mu_i(r_i - r_{i-1}) = \sum_{i=1}^{k} r_i(\mu_i - \mu_{i+1}).$$

Let $\pi\colon \mathbb{P}_B(\mathcal{E}) \to B$ be the projective bundle of one dimensional quotients of $\mathcal{E}$ (Grothendieck's notations); and let $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ be the associated tautological line bundle.

Definition 1.8.10. We say that $\mathcal{E}$ is a *nef (resp. ample, semi ample) vector bundle* if $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is nef (resp. ample, semiample) over $\mathbb{P}_B(\mathcal{E})$.

Equivalently we can say:

Proposition 1.8.11. Let $\mathcal{E}$ be a vector bundles. Then $\mathcal{E}$ is nef if every quotient bundle $\mathcal{Q}$:

$$\mathcal{E}' \to \mathcal{E} \to \mathcal{Q}$$

has non-negative degree ([13]).

In particular we recall that:

Definition 1.8.12 (ample and semi-ample line bundle). The line bundle L is *semi-ample* if it there exists an integer m such that $L^{\otimes m}$ is base point free (or equivalently is globally generated)

The line bundle L is *ample* if exists an integer n such that $L^{\otimes n}$ is very ample, or in other term, given an embedding $\phi : B \to \mathbb{P}^r$, $L = \phi^*\mathcal{O}_{\mathbb{P}^r}(1)$.

CHAPTER 2

Surfaces

2.1. Preliminaries on fibred surfaces

DEFINITION 2.1.1. We call *fibred surface* or sometimes simply *fibration* the data of a surjective morphism with connected fibers $f\colon S \to B$ from a smooth projective surface S to a smooth projective curve B.

We denote with $b = g(B)$ the genus of the base curve. A general fibre F is a smooth curve and its genus $g = g(F)$ is by definition the genus of the fibration. From now on, we consider fibrations of genus $g \geq 2$.

Let $K_f := K_S - f^*K_B$ (resp. $\omega_f := \omega_S \otimes (f^*\omega_B)^\vee$) the relative canonical divisor (resp. line bundle). Recall that given a surface S a (-1)-curve is a non-singular rational curve $C \subseteq S$ such that $C^2 = -1$.

DEFINITION 2.1.2. We say that f is *relatively minimal* if it does not contain any (-1)-curves in its fibres.

DEFINITION 2.1.3. A divisor D on S is *relatively nef* with respect to the fibration f if for every irreducible component C of a fiber F, $D.C \geq 0$.

PROPOSITION 2.1.4. A fibration is relatively nef if and only if K_f is a relatively nef divisor.

PROOF. By duality theory [13] ω_f restricted to every fiber is the dualizing sheaf of the fiber. So for every smooth fiber F since $g = g(F) \geq 2$

$$K_f.F = \deg(\omega_f|_F) = \deg \omega_F = 2(g-1) > 0.$$

Every singular fiber $F = C_1 + ... + C_k$ can be decomposed in sum of irreducible components $C_1, ..., C_k$. For any $j = 1, ..., k$ we have

$$K_f.C_j = \deg(\omega_f|_{C_j}) = \deg \omega_{C_j} - C_j^2 = 2(g_{ar}(C_j) - 1) - C_j^2.$$

Thanks to the Zariski Lemma ([13], lemma 8.2, pag. 111) $C_j^2 \leq 0$ and in particular if the fiber is connected $C_j^2 < 0$ for every $j = 1, ..., k$. So the only components where $\deg \omega_f|_{C_j}$ is negative are the rational components E with $E^2 = -1$, but these do not exist since f is relatively minimal.

□

REMARK 2.1.5. We stress that every fibration $f\colon S \to B$ with genus $g \geq 2$ can be reduced to a relatively minimal fibration conctracting any (-1) curve contained in the fibers.

DEFINITION 2.1.6. We say that a fibred surface is:

- *smooth* if every fiber is smooth;
- *isotrivial* if all smooth fibres are mutually isomorphic;
- a *Kodaira fibration* if it is smooth but not isotrivial;
- *locally trivial* if f is smooth and isotrivial (equivalently if f is a fibre bundle ([13])):
- *trivial* if S is birationally equivalent to $F \times B$ and f corresponds to the projection on B. If $b > 0$ and f is relatively minimal this is equivalent to $S = F \times B$.
- *Semistable* if every fiber is nodal and f is relatively minimal.

We have the following relative numerical invariants for fibred surfaces:

- $K_f^2 = K_S^2 - 8(g-1)(b-1)$ the self-intersection of the relative canonical divisor;
- $\chi_f := \chi(\mathcal{O}_S) - (g-1)(b-1)$ the relative Euler characteristic (the equality follows from Leray's spectral sequence);
- $e_f := e(S) - e(B)e(F) = e(S) - 4(g-1)(b-1)$ the relative topological characteristic (with $e(X)$ topological characteristic of X);
- $q_f := q - b$ the relative irregularity, with $q = h^1(S, \mathcal{O}_S)$ irregularity of S.

These invariants are not all independent: they indeed are linked by the relation:

PROPOSITION 2.1.7 (Noether relation). With the notation above we have:

$$12\chi_f = K_f^2 + e_f$$

In section (2.2.1) we give a proof of the Noether relation.
For those invariants are also known the following inequalities:

- $K_f^2 \geq 0$ and $K_f^2 = 0$ if and only if f is locally trivial [3] (see Remark 2.2.16);
- $\chi_f \geq 0$ and $\chi_f = 0$ if and only if f is locally trivial [2];
- $e_f \geq 0$ and $e_f = 0$ if and only if f is smooth [13];
- $q_f \leq g$ and equality holds if and only if f is trivial [14].

REMARK 2.1.8. We note as since $e_f \geq 0$, from the Noether relation we see

$$12\chi_f \geq K_f^2$$

which is an equality if and only every fiber is smooth (i.e. f is smooth)

2.2. The Hodge bundle

DEFINITION 2.2.1. The rank g vector bundle $f_*\omega_f$ is called the *Hodge bundle* of the fibred surface.

For this vector bundle it holds:

PROPOSITION 2.2.2. $\chi_f = \deg f_*\omega_f$.

This result follows from the the Groethendieck-Riemann-Roch formula, which we now recall in its full generality.
Consider the following setting:

$$f\colon X \to Y \tag{2.2.3}$$

where f is a proper map between two smooth algebraic varieties X and Y. We recall that the *Todd class* $\mathrm{todd}(X)$ of a smooth algebraic variety X is an element of its rational cohomology ring $H^*(X, \mathbb{Q})$ which at the first degrees reads as:

$$\mathrm{todd}(X) = 1 + \frac{1}{2}c_1(X) + \frac{1}{12}(c_1^2(X) + c_2(X)) + ... \tag{2.2.4}$$

where $c_i(X)$ are che Chern classes of X. Let $\mathcal{E}$ be a vector bundle over X, then we define its *lower shriek* through f, $f_!(\mathcal{E}) \in K(Y)$ as

$$f_!(\mathcal{E}) = \sum_{i \geq 0} R^i f_* \mathcal{E} \tag{2.2.5}$$

and its chern character $\mathrm{ch}(\mathcal{E}) \in H^*(X, \mathbb{Q})$:

$$\mathrm{ch}(\mathcal{E}) = \sum_{i \geq 0} \mathrm{ch}(\mathcal{E})_i \in H^*(X, \mathbb{Q}) \tag{2.2.6}$$

where the first components are:

$$\mathrm{ch}_0(\mathcal{E}) = \mathrm{rk}(\mathcal{E});$$
$$\mathrm{ch}_1(\mathcal{E}) = \deg(\mathcal{E});$$
$$\mathrm{ch}_2(\mathcal{E}) = \frac{1}{2}(c_1(\mathcal{E})^2 - c_2(\mathcal{E}));$$

Now we are ready to state the Groetendieck-Riemann-Rock Formula([1] pag 333)

$$\mathrm{ch}(f_!(\mathcal{E})).\,\mathrm{todd}(Y) = f_*(\mathrm{ch}(\mathcal{E}).\,\mathrm{todd}(X)) \tag{2.2.7}$$

We remark that this formula says in particular that the upper degree term of $f_!(\mathcal{E}).\,\mathrm{todd}(Y) \in H^*(Y, \mathbb{Q})$ is equal to the upper degree term of $\mathrm{ch}(\mathcal{E}).\,\mathrm{todd}(X) \in$

$H^*(X, \mathbb{Q})$.

PROOF OF PROPOSITION (2.2.2). Firstly we recall the *Leray's spectral sequence* ([13] pag 14), which is an exact sequence of vector spaces, which in our case has this form:

$$0 \to H^1(B, f_*\omega_f) \to H^1(S, \omega_f) \to H^0(B, R^1 f_*\omega_f) \to 0.$$

For duality theory, since every fiber is connected $R^1 f_*\omega_f = \mathcal{O}_B$, and so

$$h^1(S, \omega_f) = h^1(B, f_*\omega_f) + 1.$$

From Hirzebruch-Riemann-Roch theorem we have:

$$\chi(K_f) = \frac{K_f.(K_f - K_S)}{2} + \chi_S = -2\chi_B\chi_F + \chi_S,$$
$$\chi(f_*\omega_f) = \deg f_*\omega_f + \operatorname{rk} f_*\omega_f(1-b) = \deg f_*\omega_f + g(1-b).$$

Applying Serre's duality we have:

$$h^2(K_f) = h^0(K_S - K_f) = h^0(f^*K_B) = h^0(K_B) = b.$$

So, combining the previous equations:

$$\chi(K_f) = h^0(K_f) - h^1(K_f) + h^2(K_f) = h^0(f_*K_f) - h^0(f_*K_f) + b - 1 = \chi(f_*\omega_f) + (b-1),$$

and recalling that:

$$\deg f_*\omega_f + (g-1)(1-b) = \chi(f_*\omega_f) + (b-1) = -2\chi_B\chi_F + \chi_S;$$

we finally obtain:

$$\deg f_*\omega_f = \chi_S - \chi_B\chi_F =: \chi_f.$$

as wanted. □

We have the following decompositions of the Hodge bundle as a direct summand of vector sub-bundles:

- (First Fujita decomposition [30])

(2.2.8) $$f_*\omega_f = \mathcal{O}_B^{\oplus q_f} \oplus \mathcal{E},$$

  where $\mathcal{E}$ is nef (see Section 1.8) and $H^0(B, \mathcal{E}^\vee) = 0$;
- (Second Fujita decomposition [31] [17])

(2.2.9) $$f_*\omega_f = \mathcal{A} \oplus \mathcal{U},$$

  with $\mathcal{A}$ ample and $\mathcal{U}$ unitary flat (See section 1.8).

DEFINITION 2.2.10. Following [34], we define the *unitary rank* u_f of the fibred surface to be the following integer

$$u_f := \mathrm{rk}\mathcal{U}.$$

REMARK 2.2.11. Comparing the two decompositions, since every trivial bundle is unitary flat, we have:

$$\mathcal{O}_B^{\oplus q_f} \subseteq \mathcal{U},$$

and then it holds that $q_f \leq u_f$. Moreover, $\deg \mathcal{U} = 0$ and $\deg \mathcal{A} > 0$, hence

$$\chi_f = \deg f_* \omega_f = \deg \mathcal{A},$$

and $u_f = g$ if and only if $\chi_f = 0$ (equivalently f is locally trivial). Catanese and Dettweiler first gave examples [16] [17] of fibred surfaces for which the unitary summand is not semiample, thus disproving a long standing conjecture of Fujita. They proved that semi-ampleness of the Hodge bundle is indeed equivalent to $\mathcal{U}$ having finite monodromy (see Chapter 3.1 for definition). In all the examples in loc. cit. $q_f = 0$, hence in particular the strict inequality $q_f < u_f$ holds. Note moreover that for any fibred surface such that the monodromy of $\mathcal{U}$ is infinite, the inequality $q_f < u_f$ also holds "up to base change", i.e. for any fibration $\tilde{f}$ obtained from f via base change, we still have $q_{\tilde{f}} < u_{\tilde{f}}$. On the other hand, if the monodromy is finite, then there exist a base change $a\colon \tilde{B} \to B$ such that the induced fibration $\tilde{f}$ has $q_{\tilde{f}} = u_{\tilde{f}}$. See [34].

2.2.1. Noether's relation.

We are now ready to prove Proposition 2.1.7

PROOF OF PROPOSITION 2.1.7. Now the proper morphism $f : S \to B$ given by a fibred surface, and let $\mathcal{E} = \omega_f$ the relative dualizing sheaf. Then the formula 2.2.7 in this case is:

$$\mathrm{ch}(f_!(\omega_f)).\,\mathrm{todd}(B) = f_*(\mathrm{ch}(\omega_f).\,\mathrm{todd}(S))$$

On the left side the highest degree component is:

$$\begin{aligned}(\mathrm{ch}(f_!(\omega_f)).\,\mathrm{todd}(B))_1 &= \mathrm{ch}(f_!(\omega_f)_0\,\mathrm{todd}(B)_1 + \mathrm{ch}(f_!(\omega_f)_1.\,\mathrm{todd}(B)_0 \\ &= (g-1)(b-1) + \deg f_*\omega_f,\end{aligned}$$

and on the right side is:

$$\begin{aligned}(\mathrm{ch}(\omega_f).\,\mathrm{todd}(S))_2 &= \mathrm{ch}(\omega_f)_0\,\mathrm{todd}(S)_2 + \mathrm{ch}(\omega_f)_1\,\mathrm{todd}(S)_1 + \mathrm{ch}(\omega_f)_2\,\mathrm{todd}(S)_0 \\ &= \frac{1}{12}(K_S^2 + e(S)) + \frac{K_f f^* K_B}{2}.\end{aligned}$$

So comparing the two sides:

$$(g-1)(b-1)+\deg f_*\omega_f = \frac{1}{12}(K_S^2+e(S))+\frac{K_f.f^*K_B}{2}. \tag{2.2.12}$$

If we now rewrite 2.2.12 using only e_f and ω_f, we have:

$$(g-1)(b-1)+\deg f_*\omega_f = \frac{1}{12}(K_f^2+e_f)+(g-1)(b-1), \tag{2.2.13}$$

and so the result is obtained. □

2.2.2. Classic slope inequality. A *slope inequality* for the fibred surface is an inequality of the form:

$$K_f^2 \geq a\chi_f, \tag{2.2.14}$$

where $a > 0$ is a positive rational number depending on the geometry of the fibration. The first of this kind of results is the celebrated slope inequality proved by Xiao in [59] and by Cornalba and Harris in [21] (see also [54]).

THEOREM 2.1 (Xiao-Cornalba-Harris Slope inequality). *Let $f\colon S \to B$ be a relatively minimal not isotrivial fibration, and let $g = g(F) \geq 2$ the genus of the general (smooth) fiber. Then holds*

$$K_f^2 \geq 4\frac{g-1}{g}\chi_f, \tag{2.2.15}$$

A third proof was given later by Moriwaki in [47]. So, here a is an increasing function of g.

This statement will be proven in section 2.2.5 via Xiao's method.

REMARK 2.2.16. Suppose that $K_f^2 = 0$. Then by the slope inequality we have $\chi_f = 0$ so f is locally trivial. If, on the other hand, f is locally trivial; then by (ii) we have $\chi_f = 0$, then by Noether's relation and the non-negativity of e_f we have $K_f^2 = 0$.

2.2.3. Clifford index and Gonality for families of curves. We have seen in Section 1.2 that gonality and Clifford index are some important properties of a smooth curve C. In particular gonality give the minimum degree of any cover $\pi\colon C \to \mathbb{P}^1$ over $\mathbb{P}^1$, and Clifford index says how much "long" is the free resolution of the ring of regular functions $R(C)$ ([35]).

Over the moduli space $\mathcal{M}_g$ of smooth curves of genus g, the function:

$$[C] \mapsto \mathrm{Cliff}(C)$$

is a well defined lower semicontinuous function [26] [29]. This allows us to give the following:

DEFINITION 2.2.17. Given $f\colon S \to B$ a relatively minimal fibred surface. We define

$$c_f := \max_{t\in B}\{\text{Cliff}(F_t)\,|F_t \text{ is a smooth fibre of } f\} = \text{Cliff}(F) \text{ for } F \text{ general fibre of } f$$

and call it the *Clifford index of f.*

2.2.4. Xiao's method. In this section we recall the main results of Xiao's method, introduced by Xiao in his seminar paper [59], and further developed by Konno and Barja. It is worth to develop in full generality and detail the construction, as the precise statement we need is not immediate to find in the literature.

Let $f\colon S \to B$ be a relatively minimal fibration and fix a divisor D on S. For every non zero vector subbundle $\mathcal{F} \subseteq f_*\mathcal{O}_S(D)$, the natural homomorphism

$$f^*\mathcal{F} \hookrightarrow f^*f_*\mathcal{O}_S(D) \longrightarrow \mathcal{O}_S(D)$$

yields a rational map

$$\begin{array}{ccc} S & \overset{\psi}{\dashrightarrow} & \mathbb{P}_B(\mathcal{F}) \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & B & \end{array}$$

such that $\pi \circ \psi = f$. The indeterminacy locus of the map ψ is described by the following result, whose proof is immediate.

THEOREM 2.2. *[Ohno [51]] In the above situation, there exists a blow up* $\epsilon\colon \hat{S} \to S$ *and a morphism* $\lambda := \psi\circ\epsilon\colon \hat{S} \to \mathbb{P}_B(\mathcal{F})$ *such that* $\lambda^*L_{\mathcal{F}} \sim \epsilon^*(D-Z)-E$ *where*

- Z *is an effective divisor on* S*;*
- E *is a* $\epsilon-$*exceptional effective divisor of* $\hat{S}$*;*
- $L_{\mathcal{F}}$ *a hyperplane section of* $\mathbb{P}_B(\mathcal{F})$ *i.e. a divisor associated to* $\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$.

DEFINITION 2.2.18. In this setting we define:

- $M(D,\mathcal{F}) := \lambda^*L_{\mathcal{F}}$ the *moving part* of the vector subbundle $\mathcal{F}$;
- $Z(D,\mathcal{F}) := \epsilon^*Z + E$ the *fixed part* of the vector subbundle $\mathcal{F}$;
- $N(D,\mathcal{F}) := M(D,\mathcal{F}) - \lambda^*\mu(\mathcal{F})F$ where we note that ϵ does not change the general fibre of f; then we can rewrite: $N(D,\mathcal{F}) = M(D,\mathcal{F}) - \mu(\mathcal{F})F$ with F a general fibre of f.

The Xiao's method makes a crucial use of the Harder-Narasimhan filtration and a following result of Miyaoka-Nakayama.

THEOREM 2.3 (Miyaoka-Nakayama). *Let $\mathcal{F}$ be a locally free sheaf on a projective curve B. Let Σ be the general fibre of $\pi : \mathbb{P}_C(\mathcal{F}) \to C$. The $\mathbb{Q}$-divisor $L_{\mathcal{F}} - x\Sigma$ is nef if and only if $x \leq \mu_-(\mathcal{F})$.*

REMARK 2.2.19. From Miyaoka-Nakayama's result we see straightforwardly that $\mu_-(\mathcal{F}) \geq 0$ if and only if $\mathcal{F}$ is a nef vector bundle on B.

REMARK 2.2.20. In the case $\mathcal{G} = f_*\omega_f$, it is important to notice that the second to last subsheaf is precisely the ample part in the second Fujita's decomposition: $\mathcal{F}_{l-1} = \mathcal{A}$. Indeed, $f_*\omega_f$ is nef, and the subsheaf $\mathcal{U} = f_*\omega_f/\mathcal{A}$ is a subsheaf of maximal rank in $f_*\omega_f$ with (minimal) degree 0.

Note that for the Hodge bundle the last slope μ_l is greater or equal to 0 and $\mu_l = 0$ if and only if $\mathcal{U} \neq 0$.

We are now ready to expose the heart of Xiao's method:

THEOREM 2.4. *(Xiao's key Lemma [59]) Let $f\colon S \to B$ be a fibred surface. Let D be a divisor on S and suppose that there exist a sequence of effective divisors:*

$$Z_1 \geq Z_2 \geq \ldots \geq Z_s \geq Z_{s+1} := 0,$$

and a sequence of rational numbers

$$\mu_1 > \mu_2 > \ldots\ldots > \mu_s \geq \mu_{s+1} := 0,$$

such that for every $i = 1, \ldots, s$ $N_i := D - Z_i - \mu_i F$ is a nef $\mathbb{Q}-$divisor. Then for any set of indices $\{j_1, \ldots, j_s\} \subseteq \{1, \ldots, l\}$ we have

$$D^2 \geq \sum_{i=1}^{s} (d_{j_i} + d_{j_{i+1}})(\mu_{j_i} - \mu_{j_{i+1}})$$

where $d_j := N_j F$.

PROOF. Just observe that the assumptions imply the following:

$$\begin{aligned} N^2_{j_{i+1}} - N^2_{j_i} &= (N_{j_{i+1}} + N_{j_i})(N_{j_{i+1}} - N_{j_i}) = (N_{j_{i+1}} + N_{j_i})(Z_{j_i} - Z_{j_{i+1}} - (\mu_i - \mu_{i+1})F) \\ &\geq (d_{j_i} + d_{j_{i+1}})(\mu_i - \mu_{i+1}), \end{aligned}$$

and that

$$\sum_{i=1}^{s}(N^2_{j_{i+1}} - N^2_{j_i}) = -N^2_{j_1} + N^2_{j_s} \leq N^2_{j_s} \leq D^2.$$

□

2.2.5. Main inequality. We are now ready to state the version of Xiao's basic result in the form needed. Note that this is an expanded version of the inequality stated in [12, Remark 24].

THEOREM 2.5. *Let $f\colon S \to B$ be a fibred surface. Let D be a nef divisor on S and $\mathcal{G} \subseteq f_*\mathcal{O}_S(D)$ be a rank r subsheaf. Let $d' = MF$ where $M = M(D, \mathcal{G})$.*

Suppose that there exists a real number $\alpha > 0$ such that for every linear subsystem $|P|$ of $|M_{|F}|$

$$\frac{\deg |P|}{\dim |P|} \geq \alpha. \tag{2.2.21}$$

(i) The following inequality holds:

$$D^2 \geq \frac{2\alpha(r-1)}{r} \deg \mathcal{G} = 2\alpha(r-1)\mu(\mathcal{G}). \tag{2.2.22}$$

(ii) If moreover $\mathcal{G}$ is nef, then, for every non negative integer $d \leq d'$, the following inequality holds:

$$D^2 \geq \frac{2\alpha d}{d+\alpha} \deg \mathcal{G}. \tag{2.2.23}$$

PROOF. Let

$$0 \subsetneq \mathcal{G}_1 \subsetneq \ldots \subsetneq \mathcal{G}_{k-1} \subsetneq \mathcal{G}_k = \mathcal{G} \tag{2.2.24}$$

be the Harder-Narasimhan filtration of $\mathcal{G}$. We note that in general this filtration need not necessarily be related to the Harder-Narasimhan filtration of $f_*\mathcal{O}_S(D)$ (although this will happen in the application: see Remark 2.3.2).

Following Ohno's construction in Theorem 2.2, we consider a suitable blow up $\nu\colon \hat{S} \to S$ and over $\hat{S}$ for every index i we consider the divisors $M_i := M(D, \mathcal{G}_i)$ and $Z_i := Z(D, \mathcal{G}_i)$, which are respectively nef and effective. Call $r_i = \operatorname{rk} \mathcal{G}_i$ and $d_i := M_i F$. We also set $\mathcal{G}_{k+1} := \mathcal{G}_k = \mathcal{G}$.

Let us first assume that $\mathcal{G}$ s nef and prove inequality (2.5). The final slope of $\mathcal{G}$ is $\mu_k \geq 0$ by Remark 2.2.19 and we can choose $\mu_{k+1} = 0$ and $Z_k = Z_{k+1}$. The sequence (Z_i, μ_i) clearly satisfies by construction:

$$Z_1 \geq Z_2 \geq \ldots \geq Z_k = Z_{k+1},$$

and

$$\mu_1 > \mu_2 > \ldots > \mu_k \geq \mu_{k+1} := 0.$$

Observing that μ_i coincides with $\mu_-(\mathcal{G}_i)$ we have by Miyaoka's Theorem 2.3 that the divisors

$$N_i := M(D, \mathcal{G}_i) - \mu_i F$$

are all nef $\mathbb{Q}-$divisors over $\hat{S}$. Since the intersection product is invariant under birational morphism we have $(\nu^*D)^2 = D^2$. So, we can apply Theorem 2.4 to estimate $(\nu^*D - Z_k)^2$. We make a wise use of the choice of the indexes in the theorem.

Firstly we use the set of indexes $\{1, \ldots, k\}$, obtaining the inequality

$$(\nu^*D - Z_k)^2 \geq \sum_{i=1}^{k} (d_i + d_{i+1})(\mu_i - \mu_{i+1}),$$

which in its extensive form reads as follows

$$(\nu^*D - Z_k)^2 \geq (d_1 + d_2)(\mu_1 - \mu_2) + \ldots + (d_{k-1} + d_k)(\mu_{k-1} - \mu_k) + (d_k + d_{k+1})(\mu_k).$$

Observe that assumption (2.2.21) implies that for any i, $d_i \geq \alpha(r_i - 1)$, because in case $r_1 = 1$, the inequality holds trivially. Using this inequality and the fact that $r_i \geq r_{i-1} + 1$ for $i = 1, \ldots, k-1$ and that $r_{k+1} = r_k$, we have:

$$\begin{aligned}(\nu^*D - Z_k)^2 &\geq \sum_{i=1}^{k} (d_i + d_{i+1})(\mu_i - \mu_{i+1}) \geq \\ &\geq 2\alpha(\sum_{i=1}^{k-1} r_i(\mu_i - \mu_{i+1}) + r_k\mu_k) - \alpha(\mu_1 + \mu_k) = \\ &= 2\alpha \deg \mathcal{G} - \alpha(\mu_1 + \mu_k).\end{aligned}$$

Consider now the list of indexes $\{1, k\}$: we have

$$(\nu^*D - Z_k)^2 \geq (d_1 + d_k)(\mu_1 - \mu_k) + (d_k + d_{k+1})(\mu_k) \geq d_k(\mu_1 + \mu_k).$$

Eventually, combining the last two inequalities we obtain:

$$(\nu^*D - Z_k)^2 \geq \frac{2\alpha d_k}{d_k + \alpha} \deg \mathcal{G}.$$

Now observe that

$$(\nu^*D - Z_k)^2 = D^2 - 2\nu^*DZ_k + Z_k^2 \leq D^2,$$

where the last inequality follows form the fact that ν^*D is nef and Z_k effective and from $Z_k^2 \leq 0$ by Hodge index theorem. Now, consider the following function:

$$g(t) := \frac{2\alpha t}{\alpha + t},$$

which is monotone increasing for $t \geq 0$. From the hypothesis we have $d_k \geq d$ so we can deduce that

$$D^2 \geq \frac{2\alpha d_k}{d_k + \alpha} \deg \mathcal{G} = g(d_k) \deg \mathcal{G} \geq g(d) \deg \mathcal{G} = \frac{2\alpha d}{d + \alpha} \deg \mathcal{G},$$

and the proof of inequality (2.5) is concluded under the assumption that $\mathcal{G}$ is nef.

In the non-nef case, just consider as in [12, Prop.8] the last nef subbundle in the Harder-Narasimhan sequence: $\mathcal{G}_s$, where $s = \max\{i \mid \mu_i \geq 0\}$. Applying the very same construction to $\mathcal{G}_s$ we can obtain

$$D^2 \geq \frac{2\alpha d_s}{d_s + \alpha} \deg \mathcal{G}_s \geq 2\frac{\alpha(r_s - 1)}{r_s} \deg \mathcal{G}_s \geq 2\frac{\alpha(r-1)}{r} \deg \mathcal{G},$$

where the second inequality is obtained by choosing $d = \alpha(r_s - 1)$, and the last inequality follows from the monotonicity of the function g above and from the fact that clearly $\deg \mathcal{G}_s \geq \deg \mathcal{G}$. So, also inequality (2.2.22) is proved. □

Remark 2.2.25. As proved in [8], the vector subbundle $\mathcal{G}_s$ in the proof of the above theorem is a maximal element in the set of nef sub-bundles of $\mathcal{G}$: for any nef sub-bundle of $\mathcal{G}$ it holds $\mathcal{F} \subseteq \mathcal{G}_s$.

In particular, if $|\mathcal{G} \otimes \mathbb{C}(t)|$ is linearly semistable for general $t \in B$, we can take:

$$\alpha = \frac{\deg |\mathcal{G} \otimes \mathbb{C}(t)|}{\dim |\mathcal{G} \otimes \mathbb{C}(t)|}.$$

and obtain the following well known result (see [12]).

Corollary 2.2.26. Let $f \colon S \to B$ be a fibred surface. Given D a nef divisor on S and $\mathcal{G} \subseteq f_*\mathcal{O}_S(D)$ a rank r subsheaf. Let $d = \deg |\mathcal{G} \otimes \mathbb{C}(t)|$ the degree of the linear system $|\mathcal{G} \otimes \mathbb{C}(t)|$, over a general fibre F_t. If $|\mathcal{G} \otimes \mathbb{C}(t)|$ is linearly semistable, then

$$D^2 \geq \frac{2d}{r} \deg \mathcal{G} = 2d\mu(\mathcal{G}).$$

2.3. Slope inequalities

As promised in this section we prove, via Xiao's method, the classical slope inequality of Xiao-Cornalba-Harris, which we recall:

Theorem 2.6 (Classic Slope inequality). *Let $f : S \to B$ be a relatively minimal not locally trivial fibration, and $g = g(F) \geq 2$ the genus of a general smooth fiber F. Then it holds:*

$$K_f^2 \geq \frac{4(g-1)}{g} \chi_f.$$

Proof. Consider the relative canonical divisor K_f and the sheaf $f_*\mathcal{O}_S(K_f)$. From duality theory we know that $K_{f_{|F}} = K_F$ so the adjuction formula tells us that:

$$K_f.F = \deg K_f = \deg K_F = 2(g-1).$$

We know from Clifford's theorem 1.4 that every special linear system $|L|$ over a smooth curve satisfies the bound:

$$\frac{\deg |L|}{\dim |L|} \geq 2$$

Then applying Theorem 2.2.14 setting $D = K_f$, $\mathcal{G} = f_*\mathcal{O}_S(K_f)$, $d = K_f.F$ and $\alpha = 2$:

$$K_f^2 \geq \frac{2\alpha d}{d+\alpha} \deg f_*\mathcal{O}_S(K_f) = \frac{4(g-1)}{g}\chi_f.$$

where we used Proposition 2.2.2. □

Now let $f\colon S \to B$ be a relative minimal fibration of genus $g \geq 2$. We are now ready to prove our main estimates on the slope of fibred surfaces.

Firstly, using the first Fujita decomposition (2.2.8) we give a bound that improves the main bound of Barja and Stoppino in [10]. Note that the proof is much simpler than the proof of [10], where the authors needed to lift a general projection on the fibre to obtain the desired subsheaf of the Hodge bundle.

THEOREM 2.7. *Let $m := \min\{q_f, c_f\}$. The following inequality holds:*

$$K_f^2 \geq 2\frac{2g-2-m}{g-m}\chi_f.$$

PROOF. First observe that in the hyperelliptic case $m = 0$ and the inequality is just the classical slope inequality. Assume that the general fibre is not hyperelliptic.

Let us consider the first Fujita decomposition (2.2.8).

$$f_*\omega_f = \mathcal{E} \oplus \mathcal{O}^{\oplus q_f}.$$

If $q_f \leq c_f$ consider the vector bundle $\mathcal{G} := \mathcal{E}$. If $q_f \geq c_f$ consider the vector bundle $\mathcal{G} := \mathcal{E} \oplus \mathcal{O}_B^{q_f - c_f}$ In both cases the fibre over a general $t \in B$ $\mathcal{G} \otimes \mathbb{C}(t) \subseteq H^0(F_t, K_{F_t})$ defines a linear subsystem of $H^0(F_t, K_{F_t})$ of codimension m.

Let us start by observing that in case that the first vector subbundle in the Harder-Narasimhan filtration of the Hodge bundle is of rank one (a line bundle), we have $d_1 = 0 = r_1 - 1$. By the remark above and Theorem 1.5, we can apply Theorem 2.5 to $D = K_f$ and $\mathcal{G}$ as defined above, with $\alpha = \frac{2g-2-m}{g-m-1}$. We thus obtain

$$K_f^2 \geq \frac{2\alpha d}{\alpha + d} \deg \mathcal{G} = 2\frac{2g-2-m}{g-m}\chi_f,$$

as desired. □

We shall now turn our attention on the influence of the unitary rank u_f on the slope.

THEOREM 2.8. *The following inequalities holds:*

$$K_f^2 \geq \begin{cases} 2\frac{(2g-2-u_f)}{(g-u_f)}\chi_f & u_f \leq c_f \\ 2\frac{(2g-2-c_f)(g-u_f-1)}{(g-c_f-1)(g-u_f)}\chi_f & u_f \geq c_f \end{cases}$$

PROOF. As above, we can assume that F is non-hyperelliptic. Consider the second Fujita decomposition (2.2.9) $f_*\omega_f = \mathcal{A} \oplus \mathcal{U}$. As already observed, we have that $\deg \mathcal{A} = \deg f_*\omega_f$. We distinguish the two following cases:

- If $u_f \leq c_f$, then consider $\mathcal{G} = \mathcal{A}$. From Theorem 1.8.8 we can estimate the degree of that linear subsystem as follows:

  $$\deg |\mathcal{A} \otimes \mathbb{C}(t)| \geq \frac{2g-2-m}{g-m-1}(g-u_f-1) = 2g-2-u_f =: d.$$

  Then, applying Theorem 2.5 with $D = K_f$ and $\mathcal{G} = \mathcal{A}$, we have:

  $$K_f^2 \geq \frac{2\alpha d}{\alpha + d} \deg \mathcal{A} = 2\frac{2g-2-u_f}{g-u_f}\chi_f,$$

  as wanted.
- If $u_f \geq c_f$, using Theorem 1.5 we estimate the degree of the linear system $|\mathcal{A} \otimes \mathbb{C}(t)|$ as:

  $$\deg |\mathcal{A} \otimes \mathbb{C}(t)| \geq \frac{2g-2-c_f}{g-c_f-1}(g-u_f-1) =: d.$$

  Then applying Theorem 2.5 with $D = K_f$ and $\mathcal{G} = \mathcal{A}$ we have:

  $$K_f^2 \geq \frac{2\alpha d}{\alpha + d} \deg \mathcal{A} = 2\frac{(2g-2-c_f)(g-u_f-1)}{(g-u_f)(g-c_f-1)}\chi_f,$$

  and the proof is concluded.

□

REMARK 2.3.1. Observe that these last inequalities are not symmetric in $\min\{u_f, c_f\}$ as the one of Theorem 2.7. In case there exists a unitary flat subsheaf $\mathcal{U}'$ of $\mathcal{U}$, with $\mathrm{rk}\,\mathcal{U}' \geq u_f - c_f$, one can improve the last inequality of Theorem 2.8. However, such a subsheaf $\mathcal{U}'$ need not to exist.

REMARK 2.3.2. It is worth making the following remark. In Xiao's method as shown in Section 2.2.4, we use the Harder-Narasimhan sequence of the subsheaf $\mathcal{G}$ of $f_*\mathcal{O}_S(D)$. This in general is not related to the Harder-Narasimhan sequence of $f_*\mathcal{O}_S(D)$ itself. But in case $\mathcal{G}$ is a nef subsheaf of the Hodge bundle that contains the ample summand $\mathcal{A}$, then the Harder-Narasimhan filtration of $\mathcal{G}$ clearly is the truncation of the filtration of $f_*\omega_f$.

2.4. Comparison with the known results

We recall now the main results for slope inequalities which depend on q_f and c_f; the Barja-Stoppino inequality [10]

$$K_f^2 \geq 4\frac{g-1}{g-\rfloor m/2\lfloor}\chi_f$$

where $m := \min\{q_f, c_f\}$, and the Lu-Zuo inequality [42]

$$K_f^2 \geq 4\frac{g-1}{g-q_f/2}\chi_f.$$

Let us now compare our results with the ones cited above:

- The first inequality (0.1.6) improves ([10] thm 1.3) and, more importantly, is stronger than ([41] thm 1.3) in case $q_f \leq c_f$. On the other hand in case $q_f \geq c_f$, inequality (0.1.6) gives a bound increasing with the Clifford index. The inequality in ([42], thm 1.6) can be better, but inequality (0.1.6) holds also when ([42] thm 1.6) is not applicable: no genericity assumptions is needed, nor assumptions on $g \gg m$. Moreover, for m big, or u_f and c_f close to $\frac{g-1}{2}$, the bound of inequalities (0.1.6) and (0.1.7) becomes close to 6 (see Remark 0.1.8 below).
- Inequalities (0.1.7) are the first known slope inequalities where u_f plays a role.
- Inequalities (0.1.7) are of particular interest in view of the fact cited above that u_f can be strictly bigger than q_f. In Section 3, following [18], we give a first example of a fibred surface where the second inequality is new. This fibred surface has invariants $g = 6$, $q_f = 0$, $c_f = u_f = 2$, and is not bielliptic. The bound of (0.1.7) is $K_S^2 \geq 4\chi_f$, while the other previously known bounds are strictly smaller or not applicable.

CHAPTER 3

An example

Now we want to expose a first example of a fibred surface in which the bound of Theorem 2.8 is better than the bound of Theorem 2.7 and of any other previous bound. The known examples of fibred surfaces with high unitary rank ([**16**, **17**, **18**, **43**]) all satisfy $c_f \leq 1$. It is therefore interesting to find examples with Clifford index close to the unitary rank. This is a first example in this direction.

We begin giving some preliminary notions:

3.1. Local systems and monodromy

DEFINITION 3.1.1 (Local system). Let X be topological space. A *local system* $\mathbb{V}$ over X is a locally constant sheaf of $\mathbb{C}$ vector spaces.

REMARK 3.1.2. We recall that a sheaf $\mathbb{V}$ (of $\mathbb{C}$ linear spaces) over a topological space X is locally constant, if for every point $p \in X$ exists an open neighbourhood U_p such that the restriction map:

$$\Gamma(\mathbb{V}, U_p) \to \mathbb{V}_p$$

is an isomorphism of $\mathbb{C}$-vector spaces.

REMARK 3.1.3. In particular if X is an algebraic variety, flat vector bundles over X correspond to local system: given a local system $\mathbb{V}$, $\mathbb{V} \otimes \mathcal{O}_X$ is a flat vector bundle.

Now suppose that X is an arc connected space.

Let $\mathbb{V}$ be a local system and fix a point $p \in X$. Then to every loop $\gamma : I \to X$ such that $\gamma(0) = \gamma(1) = p$, we can consider the local system $\gamma^*\mathbb{V}$ over the unit interval I.

Since I is contractable ([**23**]) for every couple of point $q, q' \in I$ there exists an isomorphism between the fibers $\mathbb{V}_{\gamma(q)}$ and $\mathbb{V}_{\gamma(q')}$, so in particular choosing $q = 0, q' = 1$ we can define an isomorphism F_γ:

$$F_\gamma : \mathbb{V}_p \to \mathbb{V}_p,$$

which is an automorphism of the vector space $\mathbb{V}_p$.

It may be shown (see [23]) that two homotopy equivalent loops γ, γ' give the same automorphism, and moreover it holds

$$F_{\gamma*\gamma'} = F_\gamma * F_{\gamma'}; \qquad F_{\gamma^{-1}} = (F_\gamma)^{-1}.$$

where $*$ is the composition law in the corrisponding set.

This allows us to define:

DEFINITION 3.1.4 (Monodromy of Local system). Let X an arc connected topological space and $\mathbb{V}$ a local system of rank n over it. Fixing a point $p \in X$ we can define a representation of $\pi_1(X,p)$:

$$\rho : \pi_1(X,p) \to \mathrm{Aut}(\mathbb{V}_p) \cong \mathrm{Gl}(n,\mathbb{C}).$$

We call $\rho(\pi_1(X,p)) \subseteq \mathrm{Gl}(n,\mathbb{C})$ the *monodromy* of $\mathbb{V}$.

We say that the monodromy is irreducible (resp. reducible) if ρ is irreducible (resp reducible) as representation.

3.2. Local systems over $\mathbb{P}^1$

Let $\mathcal{S} := \{s_1, ..., s_n\}$ be a set of n points of $\mathbb{P}^1$ and $\mathbb{L}$ a local system on rank one over $\mathbb{P}^1 \setminus \mathcal{S}$. Fix a point $q \in \mathbb{P}^1 \setminus \mathcal{S}$, then we can take n loops $\gamma_1, ..., \gamma_n$, where each γ_i surrounds only one the point s_i of $\mathcal{S}$, such that:

$$\pi_1(\mathbb{P}^1 \setminus \mathcal{S}, q) = \langle \gamma_1, ..., \gamma_n \;\; | \;\; \gamma_1 * \gamma_2 * ... * \gamma_N = I \rangle.$$

where I is the constant loop.

Each loop γ_i surrounds only one point of $\mathcal{S}$, s_i and we call $\alpha_i := \rho(\gamma_i)$. Without loss of generality [23] we can take the α_i such that $|\alpha_i| = 1$ and in particular we can write

$$\alpha_i = \exp(2i\pi\mu_i)$$

where $\mu_i \in [0,1)$ for $i = 1, ..., n$ is call the *local monodromy* of $\mathbb{L}$ around s_i.

These local systems support a cohomology theory, which for our purposes we collect in this statement (see [23] for details):

THEOREM 3.1 (Local system cohomology over $\mathbb{P}^1$). *Let $\mathbb{P}^1 \setminus \mathcal{S}$ as above, and $\mathbb{L}$ a local system of rank one over it. Suppose there exist an least one $\mu_i \neq 0$, then it holds:*

$$H^1(\mathbb{P}^1 \setminus \mathcal{S}, \mathbb{L}) = H^{(1,0)}(\mathbb{P}^1 \setminus \mathcal{S}, \mathbb{L}) \oplus H^{(0,1)}(\mathbb{P}^1 \setminus \mathcal{S}, \mathbb{L})$$
$$\dim H^1(\mathbb{P}^1 \setminus \mathcal{S}, \mathbb{L}) = n - 2$$
$$\dim H^{(1,0)}(\mathbb{P}^1 \setminus \mathcal{S}, \mathbb{L}) = -1 + \sum_{i=1}^{n} \mu_i$$

where μ_i are the local monodromies of $\mathbb{L}$.

3.3. A family of curves with $(g, u_f, c_f) = (6, 2, 2)$

We construct our fibred surface following the article of Catanese and Dettweiler [17]. Our fibration will have smooth fibers which are the normalizations of the family of plane curves parametrized by $t \in \mathbb{C} \setminus \{0, 1\}$ described by the equation:

$$z^7 = y_0 y_1 (y_0 - y_1)^2 (y_1 - t y_0)^3, \tag{3.3.1}$$

with $[y_0, y_1, z] \in \mathbb{P}^2$. We give a sketch of the construction. Starting from equation 3.3.1 we homogenize it as follows:

$$z^7 = y_0 y_1 (y_0 - y_1)^2 (y_1 t_0 - t_1 y_0)^3 t_0^4, \tag{3.3.2}$$

where $([y_0, y_1], [x_0, x_1]) \in \mathbb{P}^1 \times \mathbb{P}^1$ and z is a section of $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)$. This equation (3.3.2) defines a (singular) cyclic covering of $\mathbb{P}^1 \times \mathbb{P}^1$ with group $\mathbb{Z}/7\mathbb{Z}$.

$$\pi \colon \Sigma \to \mathbb{P}^1 \times \mathbb{P}^1$$

The composition of π with the first projection $p_1 \colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ is a singular family of singular curves over $\mathbb{P}^1$. We now perform as follows:

Let B be the normalization of D

$$B \xrightarrow{\alpha} D \xrightarrow{\beta} \mathbb{P}^1$$

$\tilde{\rho} := \alpha \circ \beta$ is a degree 7 morphism.

We now consider the fibred product $\Sigma \times_{\mathbb{P}^1} B$ induced by $\tilde{\rho}$ and π and let S be its normalization as described in the diagram below.

$$\begin{array}{ccccc} S & \longrightarrow & \Sigma \times_{\mathbb{P}^1} B & \longrightarrow & \Sigma \\ & \overset{f}{\searrow} & \downarrow & & \downarrow \\ & & B & \xrightarrow[\tilde{\rho}]{} & \mathbb{P}^1 \end{array}$$

Catanese and Dettweiler have proven the following results:

PROPOSITION 3.3.3 ([17] prop 4.1). The surface S is smooth.

THEOREM 3.2 ([17] prop 4.2). *Given the morphism $f \colon S \to B$ it holds:*

(i) for every fiber $g = g(F) = 6$;

(ii) the genus of the base curve $b = g(B) = 3$;

(iii) every fiber is smooth except for the three over the points of B corresponding to $t = 0, 1, \infty$; the singular fibers consist of two smooth curves of genus 3 intersecting trasversely;

(iv) f is an Albanese map, i.e. $q_f = 0$.

3.4. Monodromy and local systems

Let $f\colon S \to B$ our family of curves given above and consider the local system $\mathbb{V} := R^1 f_*\mathbb{C}$. This sheaf is also a local system over B and the vector bundle $\mathcal{V} := \mathbb{V} \otimes \mathcal{O}_B$ is the direct sum of the Hodge bundle and its complex coniugate:

$$\mathcal{V} = f_*\omega_f \oplus f_*\bar{\omega}_f.$$

Since every fiber (smooth or not) defines a cyclic cover of $\mathbb{P}^1$, the Galois covering group $G = \mathbb{Z}/7\mathbb{Z}$ acts of $\mathbb{V}$ (hence also on the subbundle $f_*\omega_f$), which decomposes as direct sum of eigenspaces

$$\mathbb{V} = \oplus_{j=1}^{6} \mathbb{V}_j \qquad \mathcal{H} = \oplus_{j=1}^{6} \mathcal{H}_j$$

where in each eigenspace $\mathbb{V}_j$ (respectively $\mathcal{H}_j$). G acts by multiply by the character χ_j.

We note that the stalks of any local subsystem $\mathbb{V}_j$, are the eigenspaces of the first cohomology group of the fibers with character χ_j. We can thus calculate its rank as follows ([17])

$$\operatorname{rk} \mathbb{V}_j = \dim H^1(C_\lambda, \mathbb{C})_j = \dim H^1(\mathbb{P}^1 \setminus \mathcal{S}, \mathbb{L}_j) = |\mathcal{S}| - 2 = 2.$$

We also have this result about the monodromy of these local subsystems:

Lemma 3.1 (irreducibility and non-finiteness of monodromy ([17])). *With the notation given above, each local subsystem $\mathbb{V}_j$ has a monodromy representation of $\pi_1(B, b_0)$ which is:*

(a) irreducible;
(b) not finite, i.e. the image of $\rho_j : \pi_1(B, b_0) \to \operatorname{Aut}(\mathbb{V}_{j b_0})$ has infinite cardinality.

3.5. Clifford index and unitary rank of the family

Now we want to calculate the invariants of our fibration:

Proposition 3.5.1 (Clifford index of the family). Let $f\colon S \to B$ be the fibration constructed in 3.3. the fibration is tetragonal and $c_f = 2$.

Proof. The smooth fibers C_λ of $f\colon S \to B$ have an affine plane model with equation:

$$y^7 = x(x-1)^2(x-\lambda)^3, \qquad \lambda \in \mathbb{C} \setminus \{0, 1\}.$$

Since these curves have genus 6, their gonality $\gamma(C_\lambda)$ is at most $\lfloor \frac{g+3}{2} \rfloor = 4$. Using the criterions of 1.7, 1.8 we exclude $\gamma(C_\lambda) = 2, 3$, so C_λ is necessarily tetragonal.

The Clifford index $c_f = \mathrm{Cliff}(C_\lambda)$ for λ general is therefore either 2 or 1 in case C_λ has a g^2_5. We want to exclude this last case: observe that C_λ has an automorphism of order 7 while for a plane curve of degree 5 the order of any cyclic subgroup of the automorphism has to divide one of the following integers

$$4, 5, 10, 16, 15, 20$$

by [4], corollary 8.

□

PROPOSITION 3.5.2 (Unitary rank of the family). Let $f\colon S \to B$ be the fibration constructed in 3.3. Then unitary rank u_f is 2.

PROOF. Let us consider again the decomposition of the Hodge bundle in eigenspaces of $G = \mathbb{Z}/7\mathbb{Z}$:

$$f_*\omega_f = \bigoplus_{j\in\mathbb{Z}/7\mathbb{Z}} \mathcal{H}_j.$$

Thanks to the Chevalley-Weil formula (1.6), we can compute the rank of every subbundle:

$$\mathrm{rk}\,\mathcal{H}_j = \dim H^0(C_x, \omega_{C_x})_j = \begin{cases} 0 & \text{if } i = 0, 6; \\ 1 & \text{if } i = 2, 3, 4, 5; \\ 2 & \text{if } i = 1 \end{cases} \tag{3.5.3}$$

Any of the four line bundles $\mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4, \mathcal{H}_5$ must be a summund of $\mathcal{A}$ in the second Fujita decomposition $f_*\omega_f = \mathcal{A} \oplus \mathcal{U}$. Indeed, if this was not the case each $\mathcal{H}_j$ with $j = 2, 3, 4, 5$ would be a flat line bundle, and also $\bar{\mathcal{H}}_{-j}$ would be flat, since $\mathbb{V}_j \otimes \mathcal{O}_B = \mathcal{H}_j \oplus \bar{\mathcal{H}}_{-j}$ is flat too. So the monodromy representation of $\mathbb{V}_j$ would be reducible, against Theorem 3.1. Now the summand $\mathcal{H}_1$ has rank 2 and it is a subbundle of $\mathcal{V}_1 \otimes \mathcal{O}_B$ so it coincides with this latter one and its therefore unitary flat.

□

REMARK 3.5.4. To complete our analysis we calculate the slope $\lambda_f := K_f^2/\chi_f$ of this family of curves.

By construction S is a cyclic cover of degree 7 of a ruled surfaces $\mathbb{P}^1 \times \mathbb{P}^1$ blown up at three points, i.e. S is a cover of the del pezzo surface $Z := Blow_{p_1,p_2,p_3}(\mathbb{P}^1 \times \mathbb{P}^1)$. This cover is ramified over $\mathcal{B}$, with monodromy group $\mathbb{Z}/7\mathbb{Z}$ for any component of $\mathcal{B}$, so:

$$K_S^2 = (7K_P + 6\mathcal{B})^2.$$

Called L_1, L_1 the total trasform of the hyperplane divisors of $\mathbb{P}^1 \times \mathbb{P}^1$, on Z, and E_1, E_2, E_3 the exceptional divisors,

$$K_Z = -2L_1 - 2L_2 + E_1 + E_2 + E_3;$$
$$\mathcal{B} = 4L_1 + 4L_2 - 2(E_1 + E_2 + E_3).$$

After a bit of calculation we conclude that $K_S^2 = 125$. Recalling the formula 2.1:

$$K_f^2 = K_S^2 - 8(b-1)(g-1) = 45$$

Since the family as only three singular fibers, and anyone of these have a single nodal point, we conclude that $e_f = 3$, and from Noether relation 2.1.7:

$$\chi_f = \frac{K_f^2 + e_f}{12} = \frac{48}{12} = 4.$$

Then the slope of $f : S \to B$ is

$$\lambda_f := K_f^2/\chi_f = \frac{45}{4} = 11,25.$$

CHAPTER 4

Fibred threefolds

4.1. Preliminaries on fibred threefolds

In this chapter we follow what has been done in [6].

Preliminarly we recall some taxonomy of algebraic varieties.

Every algebraic variety in what follows is supposed to be normal.

Let X be an algebraic variety of dimension n, we call $\mathrm{W}(X)$ and $\mathrm{Div}(X)$ respectively the Weil group and the Cartier group of divisors of X [37].

REMARK 4.1.1. For a non smooth variety X of dimension n, we call $X_0 \subset X$ the smooth locus, i.e. the complementary subset in X of the singular locus X_{sing}, which has codimension at least 2.

The Weil canonical divisor K_X is defined as the closure in X of the divisor associated to the canonical bundle $\Omega^n_{X_0}$ of X_0.

The canonical divisor has an important role in classification of algebraic varieties, indeed we define:

DEFINITION 4.1.2 ($\mathbb{Q}$-factorial, $\mathbb{Q}$-Gorenstein variety). An algebraic variety X is called $\mathbb{Q}$-*factorial*, if $\mathrm{W}(X) \otimes \mathbb{Q} = \mathrm{Div}(X) \otimes \mathbb{Q}$, i.e. for every Weil divisor D, it exists a couple of intergers m, n such that $nD = mZ$, with Z a Cartier divisor.
In particular X is called $\mathbb{Q}$-*Gorestein* if the canonical Weil divisor K_X has a multiple rK_V which is a Cartier divisor.
A $\mathbb{Q}$-Gorestein variety X has terminal (canonical) singularities if there exists a resolution of singularities $f: Y \to X$ such that

$$rK_Y = f^*(rK_X) + \sum_i a_i E_i$$

with $a_i > 0$ ($a_i \geq 0$) and E_i f-exceptional divisors.

Now we want focus on varieties of dimension three i.e. threefolds, which we want see as families of surfaces, so we introduce the notion:

DEFINITION 4.1.3 (Fibred threefold). A fibred threefold $f\colon \Sigma \to B$ is given by a triple (Σ, B, f) where:

- Σ is a $\mathbb{Q}-$factorial threefold with only terminal singularities;
- B a smooth projective curve of genus $g(B) := b$ called *base curve*;
- $f\colon \Sigma \to B$ a proper surjective morphism with connected fibers, such that the general fiber is a minimal surface of general type.

REMARK 4.1.4. Suppose that a fibred threefold Σ is $\mathbb{Q}$-factorial, and any general fiber F is a surface of general type. Let $p_g := h^2(F, \mathcal{O}_F)$ the geometric genus of F, then:

- the relative canonical sheaf $\omega_f := \omega_\Sigma \otimes f^*\omega_B$ is a nef $\mathbb{Q}-$line bundle;
- the Hodge bundle $f_*\omega_f$ is a nef vector bundle over B, of rank p_g.

We remind that ([30],[31]) for every fibration $f\colon X \to B$ over a smooth curve, the Hodge bundle satisfies the two Fujita's decompositions. So for a fibred threefold we define:

DEFINITION 4.1.5. Let $f : \Sigma \to B$ be a fibred threefold, and $f_*\omega_f = \mathcal{A} \oplus \mathcal{U}$ its Hodge bundle. We call $u_f := \operatorname{rk}\mathcal{U}$, the unitary rank of f.

Every fibred threefold, posseses a set of relative numerical invariants:

- $K_f^3 = K_T^3 - 2(b-1)K_F^2$
- $\Delta_f := \deg f_*\mathcal{O}(K_f)$
- $\chi_f := \chi_F\chi_B - \chi_\Sigma$

We stress that unlike what happens in the case of fibred surfaces χ_f is not equal to Δ_f, instead it holds:

PROPOSITION 4.1.6 ([6] Lemma 5.6). Let $f : \Sigma \to B$ be a fibreed threefold, then if $\chi_f \geq 0$ it holds:

$$\chi_f \leq \Delta_f.$$

PROOF. Using the Leray spectral sequence we can state:

$$\chi_f = \deg f_!\omega_f.$$

Now by definition of $f_!\omega_f$, since the fibers are surfaces:

$$\deg f_!\omega_f = \sum_{i\geq 0}(-1)^i \deg R^i f_*\omega_f = \deg f_*\omega_f - \deg R^1 f_*\omega_f + \deg R^2 f_*\omega_f,$$

and from relative duality

$$R^2 f_*\omega_f \simeq (f_*\mathcal{O}_\Sigma)^\vee \simeq (\mathcal{O}_B)^\vee \simeq \mathcal{O}_B.$$

Then

$$\chi_f = \deg f_*\omega_f - \deg R^1 f_*\omega_f. \tag{4.1.7}$$

Now we recall that ([6], prop 3.1) $\mathbb{R}^1 f_* \omega_f$ is a nef vector bundle, hence has non-negative degree, so

$$\chi_f \leq \Delta_f \tag{4.1.8}$$

as we wanted. □

REMARK 4.1.9. Since $\chi_f \neq \Delta_f$ we can define two distincted slopes λ_f^1, λ_f^2:

$$\lambda_f^1 := \frac{K_f^3}{\chi_f} \qquad \lambda_f^2 := \frac{K_f^3}{\Delta_f}$$

The first one λ_f^1, gives us more information regarding the fibers than λ_f^2, nonetheless λ_f^2 is the natural quantity estimated by the Xiao-Konno-Ohno Formula (4.1). Although we are lucky and proposition 4.1.6, tell us that:

$$\lambda_f^1 \geq \lambda_f^2$$

so every inequality of the form

$$\lambda_f^2 \geq A \tag{4.1.10}$$

is also an inequality of the same form for λ_f^1. In particular an inequality in the form (4.1.10) is called a *slope inequality*.

REMARK 4.1.11. Unlike the case of fibred surfaces, there is not an analogous Noether relation for fibred threefolds since the Groethendieck-Riemann-Roch formula 2.2.7 gives only a trivial relation for the relative invariants. So we must study separately the sign of every invariant.

REMARK 4.1.12. We remark that in general χ_f is not non-negative, so it is useful have some criteria for the non negativity of this invariant.

PROPOSITION 4.1.13 (Non-negativity criteria ([6])). Let $f : \Sigma \to B$ be a fibred threefold, call $\hat{\text{Alb}}$ the Albanese map of Σ

$$\hat{\text{Alb}} : \Sigma \to \text{Alb}\,.$$

Set $a := \dim im(\text{Alb}(\Sigma))$, then $\chi_f \geq 0$ if holds one on these:

- $b \leq 1$ and $\chi(\mathcal{O}_\Sigma) \leq 0$;
- $a = 2$ and $h^0(S, \omega_{\Sigma|B}) \neq 0$;
- $a = 3$ and all the special fibers are reduced;
- $a = 3$ and the general fiber is a surface of general type.

REMARK 4.1.14. Differently from χ_f, Δ_f has a behaviour similar to the case of fibred surfaces, indeed we can state:

PROPOSITION 4.1.15 (isotrivial criteria [6]). Let $f : \Sigma \to B$ be a fibred threefold. If $\Delta_f = 0$ and the canonical system $|K_F|$ of the general fiber F, defines a birational map, then f is isotrivial.

4.2. Xiao-Ohno-Konno formula for fibred threefolds

We can exibit an analogous formula for the (top) auto-intersection of divisors. In particular we can give a lower estimate of K_f^3 which depends only on the Hodge bundle and linear systems over the general fiber F.

We recall briefly the main characters of this section. Let $f : \Sigma \to B$ be a fibred threefold and f_*K_f be its Hodge bundle.

Given any subbundle $\mathcal{G} \subset f_*K_f$ we define:

- $Z_{\mathcal{G}}$ as the only effetive divisor on Σ such that the second map in the sequence
$$f^*\mathcal{G} \to \mathcal{O}_\Sigma(K_f) \to \mathcal{O}_\Sigma(K_f - Z_{\mathcal{G}})$$
is generically surjective;
- $\hat{M}_{\mathcal{G}} := K_f - Z_{\mathcal{G}}$ called the moving part of $\mathcal{G}$ and $M_{\mathcal{G}} := \hat{\mathcal{M}}_{\mathcal{G}}.F$;
- $N_{\mathcal{G}} := M_{\mathcal{G}} - \mu_{\mathcal{G}} F$, where $\mu_{\mathcal{G}}$ is the last slope of $\mathcal{G}$.

Moreover from Miyaoka-Nakayama Lemma 2.3, as long as $\mu_{\mathcal{G}} \geq 0$, the divisor $N_{\mathcal{G}}$ is nef.

THEOREM 4.1 (Xiao-Ohno-Konno Formula). *Given a fibred threefold $f : \Sigma \to B$, let K_f its relative canonical divisor and f_*K_f its Hodge bundle. If the Hodge bundle is nef and its Harder-Narashiman filtration is:*

$$0 \subsetneq \mathcal{E}_1 \subsetneq \mathcal{E}_2 \subsetneq ... \subsetneq \mathcal{E}_{l-1} \subsetneq \mathcal{E}_l = f_*K_f$$

with slope $\mu_1 > \mu_2 > ... > \mu_l \geq \mu_{l+1} := 0$, call $M_i := M_{\mathcal{E}_i}$ and $m \in \{1, 2, ..., l\}$ the smaller index such that the map induced by the linear system $|M_m|$ over the general fiber F has dimension 2.

Let $J := \{i_1, ..., i_p, ..., i_k\} \subseteq \{1, ..., k\}$ be any subset of indexes, with i_p the lowest index, such that $i_p \geq m$.

Then it holds the inequality:

$$K_f^3 \geq \sum_{i=p}^{k} (\mu_i - \mu_{i+1})((M_{i+1}^2 + M_{i+1}M_k + M_i^2) + \sum_{j=1}^{p-1} ((\mu_j - \mu_{j+1})M_p(M_j + M_{j+1}).$$

where the summation is performed over the set of indexes J.

Proof. Let $N_j := N_{\mathcal{E}_{i_j}}$ for any $j = 1, ..., k$ and fix $N_{k+1} := K_f$.
Then we can write:

$$\begin{aligned}
K_f^3 &= N_{k+1}^3 - N_k^3 + N_k^3 \\
&= (N_{k+1} - N_k)(N_{k+1}^2 + N_{k+1}N_k + N_k^2) + N_k^3 \\
&\geq (\mu_k - \mu_{k+1})(N_{k+1}^2 + N_{k+1}N_l + N_k^2).F + N_k^3 \\
&\geq (\mu_k - \mu_{k+1})(M_{k+1}^2 + M_{k+1}M_k + M_k^2) + N_k^3 - N_{k-1}^3 + N_{k-1}^3 \\
&\geq \sum_{i=m}^{k} (\mu_i - \mu_{i+1})((M_{i+1}^2 + M_{i+1}M_i + M_i^2) + N_m^3 \\
&\geq \sum_{i=m}^{k} (\mu_i - \mu_{i+1})((M_{i+1}^2 + M_{i+1}M_i + M_i^2) + N_m(N_m^2 - N_{m-1}^2 + N_{m-1}^2) \\
&\geq \sum_{i=m}^{k} (\mu_i - \mu_{i+1})((M_{i+1}^2 + M_{i+1}M_i + M_i^2) + N_m((N_m - N_{m-1})(N_m + N_{m-1}) + N_{m-1}^2) \\
&\geq \sum_{i=m}^{k} (\mu_i - \mu_{i+1})((M_{i+1}^2 + M_{i+1}M_i + M_i^2) + N_m.F.((\mu_{m-1} - \mu_m)(\hat{M}_m + \hat{M_{m-1}}) + ..) \\
&\geq \sum_{i=m}^{k} (\mu_i - \mu_{i+1})((M_{i+1}^2 + M_{i+1}M_i + M_i^2) + \sum_{j=1}^{m-1} M_m.((\mu_j - \mu_{j+1})(M_j + M_{j+1})
\end{aligned}$$

as wanted.

□

4.3. Slope inequalities

Now given the the Xiao-Konno-Ohno method we can state new slope inequalities for fibred threefolds, which holds under some (restrictive) conditions for the Hodge bundle.

We remark that in this section we put $\lambda_f^2 := \lambda_f$.

Remark 4.3.1. In what follows we use the notation:
Given a linear system $|M|$ on a surface F, of dimension k, we say that:

- $|M|$ is g.f.n.d. if induces a generically finite map $\phi_{|M|} : F \to \mathbb{P}^k$ which is not a double cover on the image $\phi_{|M|}(F)$ where $\phi_{|M|}(F)$ is a ruled surface;
- $|M|$ is g.f.d. if induces a generically finite map $\phi_{|M|} : F \to \mathbb{P}^k$ which is a double cover on the image $\phi_{|M|}(F)$ and $\phi_{|M|}(F)$ is a ruled surface and call $q := h^1(\phi_{|M|}(F))$.

- $|M|$ induces a fibration of gonality γ, if the map $\phi_{|M|} : F \to \mathbb{P}^k$ is a fibration on F which general (smooth) fiber C has gonality γ.

THEOREM 4.2 (Slope threefold ample part semistable). *Let $f : \Sigma \to B$ be a fibred threefold, $f_*\omega_f = \mathcal{A} \oplus \mathcal{U}$ its Hodge bundle where the ample summand $\mathcal{A}$ is semistable. Let $p_g := H^0(F, \omega_F) \geq 3$.*

If $\operatorname{rk}\mathcal{A} \geq 2$ and $|K_F|$ is g.f.n.d. it holds:

Ia $\lambda_f \geq 9 + \frac{3u_f - 20}{p_g - u_f}$ *if $|M_{\mathcal{A}}|$ is g.f.n.d.;*
Ib $\lambda_f \geq 7 + \frac{3u_f + 4q - 14}{p_g - u_f}$ *if $|M_{\mathcal{A}}|$ is f.g.d.*
Ic $\lambda_f \geq 8 + \frac{3u_f - 12}{p_g - u_f}$ *if $|M_{\mathcal{A}}|$ induces a fibration of gonality $\gamma \geq 5$;*
Id $\lambda_f \geq 7 + \frac{3u_f - 11}{p_g - u_f}$ *if $|M_{\mathcal{A}}|$ induces a fibration of gonality $\gamma \geq 4$;*
Ie $\lambda_f \geq 5 + \frac{3u_f - 10}{p_g - u_f}$ *if $|M_{\mathcal{A}}|$ induces a fibration of gonality $\gamma \geq 3$;*
If $\lambda_f \geq 5 + \frac{3u_f - 9}{p_g - u_f}$ *if $|M_{\mathcal{A}}|$ induces a fibration.*

If $\operatorname{rk}\mathcal{A} \geq 2$ and $|K_F|$ is g.f.d. it holds:

IIa $\lambda_f \geq 6 + \frac{2u_f + 6q - 11}{p_g - u_f}$ *if $|M_{\mathcal{A}}|$ is g.f.d.*
IIb $\lambda_f \geq 7 + \frac{2u_f + 2q - 9}{p_g - u_f}$ *if $|M_{\mathcal{A}}|$ induces a fibration of gonality $\gamma \geq 5$*
IIc $\lambda_f \geq 6 + \frac{2u_f + 2q - 8}{p_g - u_f}$ *if $|M_{\mathcal{A}}|$ induces a fibration of gonality $\gamma \geq 4$*
IId $\lambda_f \geq 5 + \frac{2u_f + 2q - 8}{p_g - u_f}$ *if $|M_{\mathcal{A}}|$ induces a fibration of gonality $\gamma \geq 3$*
IIe $\lambda_f \geq 4 + \frac{2u_f + 2q - 7}{p_g - u_f}$ *if $|M_{\mathcal{A}}|$ induces a fibration.*

If $\operatorname{rk}\mathcal{A} \geq 2$ and $|K_F|$ induces a fibration of gonality γ, it holds:

IIIa $\lambda_f \geq 10 + 5\frac{u_f - 2}{p_g - u_f}$ *if $\gamma \geq 5$;*
IIIb $\lambda_f \geq 8 + 4\frac{u_f - 2}{p_g - u_f}$ *if $\gamma \geq 4$;*
IIIc $\lambda_f \geq 6 + 3\frac{u_f - 2}{p_g - u_f}$ *if $\gamma \geq 3$;*
IIId $\lambda_f \geq 4 + 2\frac{u_f - 2}{p_g - u_f}$ *if $\gamma \geq 2$;*

Otherwise if $\mathcal{A}$ is a line bundle it holds:

$$\lambda_f \geq K_F^2.$$

PROOF. Since $\mathcal{A}$ is semistable, the Harder-Narashiman filtration of $f_*\omega_f$ is:

$$0 \subsetneq \mathcal{A} \subsetneq f_*\omega_f,$$

where $\mu_1 = \mu_{\mathcal{A}} = \deg f_*\omega_f / \operatorname{rk}\mathcal{A}$ and $\mu_2 = \mu_{f_*\omega_f} = \mu(\mathcal{U}) = 0$.

Now suppose that the map $\phi_{|\mathcal{A}|} : F \to \mathbb{P}^{\operatorname{rk}\mathcal{A}-1}$ induced by $|M_{\mathcal{A}}|$ is generically finite.

Appling the Xiao-Ohno-Konno formula 4.1, for the set of indexes $\{1, 2, 3\}$ we get:

$$\begin{aligned}K_f^3 &\geq (\mu_1 - \mu_2)(M_1^2 + M_1M_2 + M_2^2) + (\mu_2 - \mu_3)(M_2^2 + M_2M_3 + M_3^2) \\ &= \mu_1(M_1^2 + M_1M_2 + M_2^2).\end{aligned}$$

Since $|M_{\mathcal{A}}|$ is generically finite also $|K_F|$ is. Furthemore suppose that this latter is g.f.n.d.

If we suppose that $\phi_{|\mathcal{A}|}$ is g.f.n.d., then quoting ([6], Lemma 5.9) :

$$\begin{aligned}K_f^3 &\geq \mu_1(3(p_g - u_f) - 7 + 3(p_g - u_f) - 6 + 3p_g - 7) \\ &= \deg f_*K_f \frac{(9(p_g - u_f) - 20 + 3u_f)}{p_g - u_f} \\ &= \deg f_*K_f(9 + \frac{3u_f - 20}{p_g - u_f}).\end{aligned}$$

Otherwise if $\phi_{|\mathcal{A}|}$ is g.f.d., from the same lemma we have:

$$\begin{aligned}K_f^3 &\geq \mu_1(2(p_g - u_f) - 4 + 2q + 2(p_g - u_f) - 3 + 2q + 3p_g - 7) \\ &= \deg f_*\mathcal{O}(K_f)\frac{(7(p_g - u_f) - 14 + 3u_f + 4q)}{p_g - u_f} \\ &= \deg f_*\mathcal{O}_\Sigma(K_f)(7 + \frac{3u_f + 4q - 14}{p_g - u_f}).\end{aligned}$$

Now supposing that the map $\phi_A : F \to \mathbb{P}^{\operatorname{rk}\mathcal{A}-1}$ is a fibration, then the Xiao-Ohno-Konno formula (4.1) tell us that:

$$\begin{aligned}K_f^3 &\geq (\mu_1 - \mu_2)K_F(M_1 + M_2) + (\mu_2 - \mu_3)K_F(M_2 + K_F) \\ &= \mu_1 K_F(M_1 + M_2)\end{aligned}$$

Now if $|K_F|$ induces a generic finite map also $|M_2|$, which is the mobile part of $|K_F|$, does the same. So thanks to ([6] Lemma 5.9) we write:

$$\begin{aligned}&K_F M_2 \geq M_2^2 \geq 3p_g - 7. \quad \text{if } |K_F| \text{ is g.f.n.d.;} \\ &K_F M_2 \geq M_2^2 \geq 2p_g - 4 + 2q(S) \quad \text{if } |K_F| \text{ is g.f.d.}\end{aligned}$$

If we suppose that the general fiber C of $\phi_{|\mathcal{A}|}$ has gonality $\operatorname{gon}(C) \geq 5$, then by ([6] Lemma 5.9)

$$
\begin{aligned}
K_f^3 &\geq \deg f_*\mathcal{O}_\Sigma(K_f)\frac{(5(p_g-u_f-1)+3p_g-7)}{p_g-u_f}\\
&= \deg f_*\mathcal{O}_\Sigma(K_f)(8+\frac{3u_f-12}{p_g-u_f}).\\
K_f^3 &\geq \deg f_*\mathcal{O}_\Sigma(K_f)\frac{(5(p_g-u_f-1)+2p_g-4+2q)}{p_g-u_f}\\
&= \deg f_*\mathcal{O}_\Sigma(K_f)(7+\frac{2u_f+2q-9}{p_g-u_f}).
\end{aligned}
$$

Similarly for the other gonalities we get:

$$
\begin{aligned}
&K_f^3 \geq \deg f_*\mathcal{O}_\Sigma(K_f)(7+\frac{3u_f-11}{p_g-u_f}) && \text{if } \mathrm{gon}(C)\geq 4 \text{ and } |K_F| \text{ is a g.f.n.d.;}\\
&K_f^3 \geq \deg f_*\mathcal{O}_\Sigma(K_f)(6+\frac{2u_f+2q-8}{p_g-u_f}) && \text{if } \mathrm{gon}(C)\geq 4 \text{ and } |K_F| \text{ is a g.f.d.;}\\
&K_f^3 \geq \deg f_*\mathcal{O}_\Sigma(K_f)(6+\frac{3u_f-10}{p_g-u_f}) && \text{if } \mathrm{gon}(C)\geq 3 \text{ and } |K_F| \text{ is a g.f.n.d.;}\\
&K_f^3 \geq \deg f_*\mathcal{O}_\Sigma(K_f)(5+\frac{2u_f+2q-7}{p_g-u_f}) && \text{if } \mathrm{gon}(C)\geq 4 \text{ and } |K_F| \text{ is a g.f.d.;}\\
&K_f^3 \geq \deg f_*\mathcal{O}_\Sigma(K_f)(5+\frac{3u_f-9}{p_g-u_f}) && \text{for every } C \text{ and } |K_F| \text{ is a g.f.n.d.;}\\
&K_f^3 \geq \deg f_*\mathcal{O}_\Sigma(K_f)(4+\frac{2u_f+2q-6}{p_g-u_f}) && \text{if } |K_F| \text{ is a g.f.n.d.;}
\end{aligned}
$$

Now suppose that $|K_f|$ is g.f.d., then similarly at what see above it holds:

IIa $\lambda_f \geq 6+\frac{2u_f+6q-11}{p_g-u_f}$ if $|M_{\mathcal{A}}|$ is g.f.d.
IIb $\lambda_f \geq 7+\frac{2u_f+2q-9}{p_g-u_f}$ if $|M_{\mathcal{A}}|$ induces a fibration of gonality $\gamma \geq 5$
IIc $\lambda_f \geq 6+\frac{2u_f+2q-8}{p_g-u_f}$ if $|M_{\mathcal{A}}|$ induces a fibration of gonality $\gamma \geq 4$
IId $\lambda_f \geq 5+\frac{2u_f+2q-8}{p_g-u_f}$ if $|M_{\mathcal{A}}|$ induces a fibration of gonality $\gamma \geq 3$
IIe $\lambda_f \geq 4+\frac{2u_f+2q-7}{p_g-u_f}$ if $|M_{\mathcal{A}}|$ induces a fibration.

If $|K_f|$ (hence also $|M_2|$) induces a fibration of gonality $\gamma \geq 2$, then the same does $\phi_{|\mathcal{A}|}$ (see [6] thm 5.9 proof). So similarly to what does above we have:

$$K_f^3 \geq \deg f_*\mathcal{O}_\Sigma(K_f)(10 + 5\frac{u_f - 2}{p_g - u_f}) \qquad \text{if } \gamma \geq 4;$$

$$K_f^3 \geq \deg f_*\mathcal{O}_\Sigma(K_f)(8 + 4\frac{u_f - 2}{p_g - u_f}) \qquad \text{if } \gamma \geq 3;$$

$$K_f^3 \geq \deg f_*\mathcal{O}_\Sigma(K_f)(6 + 4\frac{u_f - 2}{p_g - u_f}) \qquad \text{if } \gamma \geq 2.$$

Now suppose that $\operatorname{rk}\mathcal{A} = 1$. Thanks to theorem 2.3 we can say that $K_f - \mu_{\mathcal{A}}F$ is nef and $K_f - \mu_2 F$ is pseudo-effective, so the intersection is non negative:

$$K_f^2(K_f - \mu_1 F) \geq 0.$$

Rearranging the terms, we get

$$K_f^3 \geq \mu_1 K_F^2 = \deg f_*\mathcal{O}_\Sigma(K_f)K_F^2,$$

as wanted. □

We now use the above inequalities to estimate u_f. In particular we give an upper bound for u_f which depends only on p_g. We recall this result of Ohno, which in our case can be stated as:

Theorem 4.3 ([51]). *Let $f : \Sigma \to B$ be a fibred threefold, then it holds:*

$$K_f^3 - 2(b-1)K_F^2 \leq 72\chi_f,$$

In particular if the genus of the base curve $b = 0, 1$ the above theorem tell us that:

$$\lambda_f = K_f^3/\chi_f \leq 72.$$

So substituing the variuos expressions for λ_f in theorem 4.2 we get:

Proposition 4.3.2 (Upper bound for u_f). Let $f : \Sigma \to B$ be a fibred threefold with $b = 0, 1$ and p_g the geometric genus of the general fiber F. Suppose that the ample summand $\mathcal{A}$ of the Hodge bundle is semistable. Then we have the following bounds for u_f:

(Ia) If $|K_F|$ and $|M_{\mathcal{A}}|$ are g.f.n.d.:

$$u_f \leq \frac{63p_g + 20}{66}.$$

(Ib) If $|K_F|$ is g.f.n.d. and $|M_{\mathcal{A}}|$ is g.f.d:

$$u_f \leq \frac{65p_g - 4q + 14}{68}.$$

(Ic) If $|K_F|$ is g.f.n.d. and $|M_{\mathcal{A}}|$ defines a fibration of gonality $\gamma \geq 5$:

$$u_f \leq \frac{64p_g + 12}{67}.$$

(Id) If $|K_F|$ is g.f.n.d. and $|M_{\mathcal{A}}|$ defines a fibration of gonality $\gamma \geq 4$:

$$u_f \leq \frac{65p_g + 11}{68}.$$

(Ie) If $|K_F|$ is g.f.n.d. and $|M_{\mathcal{A}}|$ defines a fibration of gonality $\gamma \geq 3$:

$$u_f \leq \frac{67p_g + 10}{70}.$$

(If) If $|K_F|$ is g.f.n.d. and $|M_{\mathcal{A}}|$ defines a fibration of gonality $\gamma \geq 2$:

$$u_f \leq \frac{67p_g + 9}{70}.$$

(IIa) If $|K_F|$ and $|M_{\mathcal{A}}|$ are g.f.n.d.:

$$u_f \leq \frac{66p_g - 6q + 11}{68}.$$

(IIb) If $|K_F|$ is g.f.d. and $|M_{\mathcal{A}}|$ defines a fibration of gonality $\gamma \geq 5$:

$$u_f \leq \frac{65p_g - 2q + 9}{67}.$$

(IIc) If $|K_F|$ is g.f.d. and $|M_{\mathcal{A}}|$ defines a fibration of gonality $\gamma \geq 4$:

$$u_f \leq \frac{66p_g - 2q + 8}{68}.$$

(IId) If $|K_F|$ is g.f.n. and $|M_{\mathcal{A}}|$ defines a fibration of gonality $\gamma \geq 3$:

$$u_f \leq \frac{67p_g - 2q + 8}{69}.$$

(IIe) If $|K_F|$ is g.f.d. and $|M_{\mathcal{A}}|$ defines a fibration of gonality $\gamma \geq 2$:

$$u_f \leq \frac{68p_g - 2q + 7}{70}.$$

(IIIa) If $|K_F|$ defines a fibration of gonality $\gamma \geq 5$:

$$u_f \leq \frac{62p_g + 124}{67}.$$

(IIIb) If $|K_F|$ defines a fibration of gonality $\gamma \geq 4$:

$$u_f \leq \frac{16p_g + 2}{17}.$$

(IIIc) If $|K_F|$ defines a fibration of gonality $\gamma \geq 3$:

$$u_f \leq \frac{22p_g + 2}{23}.$$

(IIId) If $|K_F|$ defines a fibration of gonality $\gamma \geq 2$:

$$u_f \leq \frac{34p_g + 2}{35}.$$

Bibliography

[1] E. Arbarello, M. Cornalba, P.A. Griffiths, J. Harris, Geometry of algebraic curves. Vol. I, GMW, **267**, Springer-Verlag, New York, 1985.

[2] E. Arbarello, M. Cornalba, P.A. Griffiths, Geometry of algebraic curves. Vol. 2, GMW, **267**, Springer-Verlag, New York, 2010.

[3] S.J. Arakelov, *Families of algebraic curves with fixed degeneracies,* Math. USSR Izvestija, **5** no. 6 (1971),1269–1293.

[4] E. Badr, F. Bars *Plane non-singular curves with an element of "large" order in its automorphism group* International Journal of Algebra and Computation Vol. 26, No. 02, pp. 399-433 (2016).

[5] E. Ballico, *On the Clifford index of algebraic curves*, Proc. Amer. Math. Soc. **97** (1986), 217–218.

[6] M.A. Barja, *On the slope and geography of fibred surfaces and threefolds*, PhD Thesis, 2000.

[7] M.A. Barja, *On the slope of bielliptic fibrations.* Proc. Amer. Math. Soc. **129** (2001), no. 7, 1899–1906.

[8] M.A. Barja, *Higher dimensional slope inequalities for irregular fibrations*, arXiv:2012.06889 [math.AG].

[9] M.A. Barja, V. Gonzàlez-Alonso, J.C. Naranjo, *Xiao's conjecture for general fibred surfaces*, J. reine angew. Math. **739** (2018), 297–308.

[10] M.A. Barja, L. Stoppino, *Linear stability of projected canonical curves with applications to the slope of fibred surfaces*, J. Math. Soc. Japan, **60**, No. 1 (2008), 171–192.

[11] M.A. Barja, L. Stoppino, *Slopes of trigonal fibred surfaces and of higher dimensional fibrations*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), **8**, (2009), No. 4, 647–658.

[12] M.A. Barja, L. Stoppino, *Stability conditions and positivity of invariants of fibrations*, Algebraic and complex geometry, Springer Proc. Math. Stat., **71**, 1–40, (2014).

[13] W.P. Barth, K. Hulek, C.A.M. Peters, A. Van de Ven, Compact complex surface, Second edition, Springer-Verlag, Berlin Heidelberg, 2004.

[14] A. Beauville, Appendix to *Inégalités numériques pour les surfaces de type général.*, by O. Debarre, Bull. Soc. Math. France **110** (1982), no. 3, 319–346.

[15] V. Beorchia, F. Zucconi, *On the slope of fourgonal semistable fibrations*, Math. Res. Lett. **25**, no. 3 (2018), 723–757.

[16] F. Catanese, M. Dettweiler, *The direct image of the relative dualizing sheaf needs not be semiample*, C. R. Math. Acad. Sci. Paris **352** (2014), no. 3, 241–244.

[17] F. Catanese, M. Dettweiler, *Vector bundles on curves coming from variation of Hodge structures*, Internat. J. Math. **27** (2016), no. 7, 1640001, 25 pp.

[18] F. Catanese, M. Dettweiler, *Answer to a question by Fujita on variation of Hodge structures*, Higher Dimensional Algebraic Geometry (Tokyo: Mathematical Society of Japan, 2017), 73–102.

[19] K. Chen, X. Lu, K. Zuo, *On the Oort conjecture for Shimura varieties of unitary and orthogonal types.* Compos. Math. **152** (2016), no. 5, 889–917.

[20] M. Coppens, G. Martens, *Secant spaces and Clifford's theorem*, Compositio Math. **78** (1991), 193–212.

[21] M. Cornalba, J. Harris, *Divisor classes associated to families of special varieties, with applications to the moduli space of curves* Ann. Sc. Ec. Norm. Sup., **21** (1988), no. 4, 455–475.

[22] M. Cornalba, L. Stoppino, *A sharp bound for the slope of double cover fibrations*, Michigan Math. J. **56** (2008), no. 3, 551–561.

[23] P. Deligne, G.D. Mostow, *Monodromy of hypergeometric functions and non-lattice integral monodromy* Publications Mathématiques de l'IHÉS, Tome 63 (1986) , pp. 5-89.

[24] P. Deligne, *Equations differentielles a points singuliers reguliers.* Lecture Notes in Mathematics,Vol. 163. Springer-Verlag, Berlin, 1970.

[25] P. Deligne, *Local behavior of Hodge structures at infinity* Studies in advanced mathematic vol 1, 1997.

[26] D. Eisenbud, M. Green, *Clifford indices of ribbons* Transaction of the american mathematical society Volume 347, Number 3, March 1995.

[27] D. Eisenbud, H. Lange, G. Martens, F.O. Scheryer *The Clifford dimension of a projective curve* Compositio Mathematica 72: 173-204, (1989).

[28] M. Enokizono, *Slopes of fibered surfaces with a finite cyclic automorphism*, Michigan Math. J. **66** (2017), no. 1, 125–154.

[29] L.Y. Fong, *L.-Y. Fong, Rational ribbons and deformations of hyperelliptic curves*, J. Algebraic Geom. 2 (1993), 295-307

[30] T. Fujita, *On Kähler fiber spaces over curves*, J. Math. Soc. Japan **30** (1978), no. 4, 779–794.

[31] T. Fujita, *The sheaf of relative canonical form of a Kähler fiber space over a curve*, Proc. Japan Acad. Ser. A Math. Sci. **54** (1978), no. 7, 183–184.

[32] A. Gibney, S. Keel, I. Morrison *Towards the ample cone of* $\overline{\mathcal{M}}_{g,n}$, J. Amer. Math. Soc. **15** (2002), no. 2, 273–294.

[33] V. Gonzàlez-Alonso, S. Torelli, *Families of curves with Higgs field of arbitrarily large kernel*, to appear in Bullettin of the L.M.S., 2020. DOI: 10.1112/blms.12437

[34] V. Gonzàlez-Alonso, L. Stoppino, S. Torelli, *On the rank of the flat unitary summand of the Hodge bundle*, Trans. Amer. Math. Soc.**372** (2019), 8663–8677.

[35] M. Green *Koszul cohomology and the geometry of projective varieties* J. Differential Geom. 19(1): 125-171 (1984).

[36] G. Harder, M. Narasimhan, *On the cohomology group of moduli spaces of vector bundles on curves*, Math. Ann., **212** (1975), 215–248.

[37] R. Hartshorne *Algebraic geometry* New York: Springer-Verlag, 1977.

[38] K. Konno, *Nonhyperelliptic fibrations of small genus and certain irregular canonical surfaces.* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 20 (1993), no. 4, 575–595.

[39] K. Konno, *A lower bound of the slope of trigonal fibrations,* Internat. J. Math. **7** (1996), no.1, 19–27.

[40] K. Konno, *Clifford index and the slope of fibered surfaces,* J. Algebraic Geom. **8** (1999), no. 2, 207–220.

[41] X. Lu, K. Zuo, *On the slope conjecture of Barja and Stoppino for fibred surfaces,* Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **19** (2019), no. 3, 1025–1064.

[42] X. Lu, K. Zuo, *On the gonality and the slope of a fibered surface,* Adv. Math. **324** (2018), 336–354.

[43] X. Lu, *Family of curves with large unitary summand in the Hodge bundle,* Math. Z. **291** (2019), no. 3-4, 1381–1387.

[44] G. Martens *Über den Clifford Index algebraischer Kurven*, J. Reine Angew. Math. 336 (1982), 83-90.

[45] R. Miranda *Algebraic Curves and Riemann Surfaces* American Mathematical Soc., 1995

[46] E.C. Mistretta, L. Stoppino, *Linear series on curves: stability and Clifford index*, Internat. J. Math., **23 (12)** (2012),1250121, 25 pp.

[47] A. Moriwaki, *A sharp slope inequality for general stable fibrations of curves,* J. Reine Angew. Math. **480** (1996), 177–195.

[48] D. Mumford, *Stability of projective varieties*, Enseign. Math. (2) **23** (1977), no. 1-2, 39–110.

[49] N. Nakayama, *Zariski-decomposition and abundance.* MSJ Memoirs, 14. Mathematical Society of Japan, Tokyo, 2004. xiv+277 pp.

[50] O. Ohkouchi, F. Sakai, *The gonality of singular planar curve.* Tokyo J. Math. Vol. 27, no. 1, 2004.

[51] K. Ohno, *Some inequality for minimal fibration of surface of general type over curves*, J. Math. Soc. Japan **44** (1992), no.4, 643–666.

[52] R. Pardini, *The Severi inequality* $K^2 \geq 4\chi$ *for surfaces of maximal Albanese dimension.* Invent. Math. **159** (2005), no.3, 669–672.

[53] G.P. Pirola, *On a conjecture of Xiao.* J. Reine Angew. Math. **431** (1992), 75–89.

[54] L. Stoppino, *Slope inequalities for fibred surfaces via GIT*, Osaka J. Math., **45** (2008), no. 4, 1027–1041.

[55] A. Schweizer, *Some remarks on bielliptic and trigonal curves*, arXiv:1512.07963 [math.AG]

[56] C. Voisin *Hodge Theory and Complex Algebraic Geometry I: Volume 1* Cambridge Studies in Advanced Mathematics vol 76

[57] N. Wangyu, *Cyclic covering of the projective line with prime gonality*, J.of Pure and Applied Algebra **219** (2015), 1704–1710.

[58] N. Wangyu, F. Sakai, *Hyperelliptic curves among cyclic coverings of the projective line, II*, Arch. Math. **102** (2) (2014) 113–116.

[59] G. Xiao, *Fibered algebraic surface with low slope*, Math. Ann. **276**, (1987), 449–466.

[60] G. Xiao, *Irregularity of surfaces with a linear pencil*, Duke Math. J., **55** (1987), no. 3, 597–602.

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