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On stable outcomes of approval, plurality, and negative plurality games

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Abstract We prove two results on the generic determinacy of Nash equilibrium in voting games. The first one is for negative plurality games. The second one is for approval games under the condition that the number of candidates is equal to three. These results are combined with the analogous one obtained in De Sinopoli (Games Econ Behav 34:270–286, 2001) for plurality rule to show that, for generic utilities, three of the most well-known scoring rules, plurality, negative plurality and approval, induce finite sets of equilibrium outcomes in their corresponding derived games—at least when the number of candidates is equal to three. This is a necessary requirement for the development of a systematic comparison amongst these three voting rules and a useful aid to compute the stable sets of equilibria Mertens (Math Oper Res 14:575–625, 1989) of the induced voting games. To conclude, we provide some examples of voting environments with three candidates where we carry out this comparison.

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1 Introduction

The Gibbard–Satterthwaite Theorem teaches us that we must limit the number of desirable properties that we can ask of our voting systems. But collective decisions still need to be made whose outcome is legitimized by the participation in the decision process of all the individuals that will be affected by its outcome. This paper is a contribution to a positive research agenda that aims at understanding how the electoral system determines the political outcome.

An electoral system must be judged in terms of how it maps the constituency's preferences into the set of possible political outcomes. For this reason, it is important to describe the incentives created by the different voting games generated by the different electoral systems and to characterize, in as much detail as possible, their sets of equilibrium outcomes. The first step of this process is to qualify what we mean by *equilibrium*. Often voting games have unreasonable Nash equilibria that do not successfully capture plausible voting behavior. Farquharson (1969) suggested the *sophisticated voting principle*: reasonable equilibria must survive iterated deletion of dominated strategies. Within the more general framework of finite games, the literature on equilibrium refinements has proposed a number of other equilibrium concepts and rationality requirements.[1] It seems that *Mertens' stability* (Mertens 1989) is the equilibrium concept that satisfies the most comprehensive list of desirable game theoretical properties, including stability against iterated deletion of dominated strategies. Therefore, it appears to be the most suitable tool to make equilibrium analysis in voting games.

Of course, the task of comparing voting procedures would be much more feasible if there always was a unique equilibrium outcome. Unfortunately, it is often the case that uniqueness can only be obtained after imposing restrictive assumptions that are not necessarily compelling in every voting situation. Thus, it seems that we have to put up with multiplicity of equilibria if we want to deal with a broader realm of voting environments and that we should, at best, hope for finiteness in the set of equilibrium outcomes. However, this is again impossible if we do not restrict the set of possible preference profiles that the electorate can have. We have to, at least, restrict attention to generic preferences (i.e. generic points in the space of utility vectors) to obtain an appealing terrain where we can analyze voting systems and make comparisons amongst them.[2]

Indeed, De Sinopoli (2001) shows that, for generic plurality games, the set of Nash equilibrium outcomes is finite. (Under plurality, each voter votes for just one candidate, the candidate with the most votes wins the election and ties are broken randomly). In this paper, we first obtain the analogous result for negative plurality. (Under negative plurality, each voter casts a negative vote for just one candidate, the candidate with the least negative votes wins the election and ties are broken randomly.) Secondly, we prove that under approval voting and generic utilities the set of equilibrium distributions with at most three candidates in their support is finite. (Under approval voting, each voter

[1] See van Damme (1991) for an excellent review.

[2] Debreu (1970) makes an analogous argument but in the context of pure exchange economies. See also Harsanyi (1973), Park (1997), Govindan and McLennan (2001), Govindan and Wilson (2001).

casts a ballot that gives one and only one approval vote to as many candidates as she wants, the candidate with most approval votes wins the election and ties are broken randomly.) These results imply that, if utilities are generic, then each of the stable sets of the voting games generated by plurality, negative plurality and approval (with three candidates) maps into a unique outcome. Ideally, we would like to obtain the general result for approval for an arbitrary number of candidates. We hope that our detailed analysis of negative plurality and the partial result for approval help shed light on this general case.

These results can be used to compare plurality, negative plurality and approval in some voting environments with three candidates. As we have already mentioned, stable sets satisfy iterated deletion of dominated strategies (and the specific order of deletion does not matter), i.e. each stable set contains a stable set of the game obtained by deleting dominated strategies. Therefore, proving that, for generic preferences, each stable set maps into a unique outcome implies that, generically, a dominance solvable game has a unique stable outcome. Uniqueness of equilibrium outcomes is an important property in voting scenarios. It is argued by Myerson and Weber (1993) that the number of equilibrium outcomes has political significance because the larger the number of equilibria, the wider is the scope for focal manipulation by political leaders. Moreover, there already are results available that give sufficient conditions such that plurality, negative plurality and approval voting games are dominance solvable (e.g. Dhillon and Lockwood (2004) and Buenrostro et al. (2013)) which can therefore be read as sufficient conditions so that those voting games have a unique stable outcome. It follows that the set of voting games that are dominance solvable is quite relevant because they generate a unique stable outcome whose computation is very tractable. For instance, De Sinopoli et al. (2013) compute stable outcomes in a family of voting environments where plurality seems to do better than approval. In each of those examples, there is a unique stable outcome in the approval voting game and in such an outcome the Condorcet winner is never elected. On the other hand, plurality always generates a stable set where the Condorcet winner is elected with probability one. In turn, we illustrate the main results in this paper by computing the stable outcomes generated by plurality, negative plurality and approval voting in some voting environments. In particular, along the same lines as Myerson (2002), we consider a voting environment where plurality generates discriminatory equilibria in which a universally preferred candidate is not regarded as a serious contender (Example 1). We also present a voting environment where negative plurality generates too few discriminatory equilibria, further implying that in every equilibrium outcome, a universally disliked candidate is considered a serious contender (Example 2). As in Myerson (2002), approval voting gives a good balance between plurality and negative plurality in those environments. Furthermore, we present a robust voting environment where the unique stable outcome of the negative plurality selects the Condorcet loser with probability one (Example 3). To conclude, we also show that there exists an open set of utilities where negative plurality seems to outperform approval voting (Example 4).

In the next section we introduce the voting model in general terms. It can be easily specialized to approval, plurality and negative plurality voting. Section 3 contains results on the generic determinacy of Nash equilibria in negative plurality and approval games. It also introduces some basic properties of stable sets (Mertens 1989) and combines them with the previous results to derive some properties about the stable sets of equilibria in generic plurality, negative plurality and approval voting games. These are used in the last section to study several simple examples that show some of the ways in which the set of equilibrium outcomes varies as we change the voting system. Proofs for the generic determinacy of Nash equilibrium in negative plurality and approval voting are in the Appendix.

2 The voting model

We consider an election with electorate $N \equiv \{1, \ldots, n\}$ and set of candidates $K \equiv \{1, \ldots, k\}$. Each voter $i \in N$ casts a ballot $v_i \in V \subset \mathbb{Z}^k$, where V is the set of ballots allowed by the electoral system. A ballot $v_i = (v_i^1, \ldots, v_i^k)$ is a vector with as many entries as candidates where, for every candidate $c \in K$, v_i^c is the number of votes given by voter i to candidate c.

An electoral system must specify the set V of permissible ballots and an election rule that selects a winning candidate from K for each ballot profile $v = (v_1, \ldots, v_n) \in \mathcal{V} \equiv V^N$. For example, in an election with three candidates, the set of possible ballots V_p allowed by plurality rule consists of four elements, namely, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ and $(0, 0, 0)$—the zero vector corresponds to abstention. The set of ballots V_a allowed by approval voting is obtained by enlarging the set of ballots allowed by plurality rule with $(1, 1, 0)$, $(1, 0, 1)$, $(0, 1, 1)$ and $(1, 1, 1)$.[3]

Finally, the set of ballots available under negative plurality is $V_{np} \equiv \{(-1, 0, 0), (0, -1, 0), (0, 0, -1), (0, 0, 0)\}$.

Let $\Delta(K)$ denote the set of probability distributions on K. It is reasonable to choose an election rule $p : \mathcal{V} \to \Delta(K)$ that makes the candidates that obtain more support more likely to win. Given a ballot profile $v \in \mathcal{V}$, the set of winning candidates is

$$W(v) = \left\{ c \in K : \sum_{i=1}^{n} v_i^c \geq \sum_{i=1}^{n} v_i^d \text{ for all } d \in K \right\}. \tag{2.1}$$

And the probability $p(c \mid v)$ that candidate c wins the election if voters cast the ballot profile v is

$$p(c \mid v) = \begin{cases} 0 & \text{if } c \notin W(v), \\ 1/\#W(v) & \text{if } c \in W(v). \end{cases} \tag{2.2}$$

Henceforth we fix the election rule p to be as defined in (2.2).

Voter i's set of mixed strategies is $\Sigma_i \equiv \Delta(V)$. As usual, $\Sigma \equiv \prod_{i=1}^{n} \Sigma_i$ is the set of mixed strategy profiles. The probability attached to the ballot profile $v = (v_1, \ldots, v_n)$ by the mixed strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n)$ is $\sigma(v) \equiv \prod_{i=1}^{n} \sigma_i(v_i)$. Therefore, the probability that candidate c is elected when voters use the mixed strategy profile σ is $p(c \mid \sigma) = \sum_{v \in \mathcal{V}} \sigma(v)\, p(c \mid v)$.

Within this framework, a set of possible ballots V together with a utility vector u defines a voting game (V, u). The utility vector specifies for each voter i and each

3 Given the election rule below, (1,1,1) is equivalent to abstention $(0, 0, 0)$.

candidate c, the utility $u_i(c)$ to voter i if candidate c gets elected. Therefore, once the electoral system is fixed, a voting game is given by a point in $u \in U \equiv \mathbb{R}^{nk}$. The expected utility derived by voter i, if voters play according to the mixed strategy profile σ, is computed in the usual manner $U_i(\sigma) = \sum_{c \in K} p(c \mid \sigma) u_i(c)$.

Given a voting game (V, u), a Nash equilibrium is a strategy profile σ such that for every voter i and every ballot v_i,

$$U_i(\sigma) \geq U_i(\sigma_{-i}, v_i).$$

3 Strategic stability and sophisticated voting

We begin this section with a positive result on the generic determinacy of Nash equilibria in negative plurality and approval voting games.[4] De Sinopoli (2001) already shows that generic plurality games have finitely many equilibrium outcomes. Here we use the term "generic subset" meaning that its complement is a closed, lower-dimensional, semi-algebraic subset of utilities.[5] Having fixed a generic subset, we often refer to its elements as *generic points*. Hence we will talk of a generic game if its associated utility vector belongs to the generic space where the determinacy results holds.

In the Appendix we prove the following two propositions.

Proposition 1 *For generic negative plurality games, the set of Nash equilibrium outcomes is finite.*

Proposition 2 *For generic approval voting games, the set of Nash equilibrium outcomes that only place positive probability on three or fewer candidates is finite.*

Whether or not the result extends to the set of all Nash equilibrium outcomes in approval games remains an open question.

As an application, we use these genericity results to establish a link between *strategic stability* (Mertens 1989) and *sophisticated voting* (Farquharson 1969) in negative plurality and approval voting games.

It is well-known that the reduced game that is obtained after applying iterated deletion of dominated strategies may depend on the order of elimination.[6] Farquharson (1969) avoids this problem when defining sophisticated voting by deleting *every* dominated strategy present in each round. Following De Sinopoli (2000), we eliminate this restriction and define an outcome as *sophisticated* if it can be isolated by at least one order of deletion of dominated strategies. (That is, e.g., a voting game can have more than one sophisticated outcome).

On the other hand, we have *strategic stability* as defined by Mertens (1989, p. 585).

4 Govindan and McLennan (2001) offer a counterexample that shows that this result does not extend to general games.

5 A set is semi-algebraic if it is defined by a finite system of polynomial inequalities. A function or a correspondence is semi-algebraic if its graph is a semi-algebraic set. Every set and correspondence defined in this paper is semi-algebraic.

6 Typically, voting games do not have strictly dominated strategies. Here and throughout the paper, by *dominated strategy* we mean a *weakly dominated strategy*.

Mertens' stable sets satisfy a number of desirable properties that make them an appealing concept to analyze, among others, voting games. We do not need the precise definition of Mertens' stability, but only the following three properties (cf. Mertens (1989)):

(α) Every game has a stable set.
(β) Stable sets are connected sets of admissible (i.e. undominated) equilibria.
(γ) A stable set contains a stable set of every game that is obtained after deleting a dominated strategy.

Note that, in our voting games, property (γ) implies that, for each sophisticated outcome, every stable set contains a strategy profile that generates it. Combining these properties we obtain:

Proposition 3 *If a voting game has finitely many Nash equilibrium outcomes and it has a sophisticated outcome then this is the unique sophisticated outcome of the game. Furthermore, it is also the unique stable outcome of the game.*

Proof Property (β) ensures that every strategy profile in each stable set generates the same outcome. By assumption, there are finitely many Nash equilibrium outcomes, hence, finitely many stable outcomes. Property (γ) implies the desired result. □

The main result of this paper readily follows from combining Propositions 1, 2, and 3.

Theorem 1 *If a generic negative plurality game has a sophisticated outcome then it is unique and it coincides with the unique stable outcome of the game. Moreover, the same result holds for generic approval voting games with a sophisticated outcome such that three or fewer candidates win with positive probability.*

De Sinopoli (2001) shows that the analogous result holds for generic plurality games. In the next section, we use these results to compute the stable sets of equilibria in some simple examples and to show some ways in which the electoral system determines the political outcome.

4 Comparing voting systems

In what follows we take the viewpoint that the Condorcet winner, whenever it exists, is the most desirable alternative from a social perspective and that the Condorcet loser is the least desirable alternative. We begin with the formal definitions:

Definition 1 (Condorcet Winner). A candidate $c \in K$ is the *Condorcet winner* if

$$\#\{i \in N : u_i(c) > u_i(d)\} > \#\{i \in N : u_i(c) < u_i(d)\} \quad \text{for all } d \neq c.$$

Furthermore, we say that a candidate $c \in K$ is a *weak-Condorcet winner* if

$$\#\{i \in N : u_i(c) > u_i(d)\} \geq \#\{i \in N : u_i(c) < u_i(d)\} \quad \text{for all } d \in K, \text{ and}$$

$$\#\{i \in N : u_i(c) > u_i(d')\} > \#\{i \in N : u_i(c) < u_i(d')\} \quad \text{for some } d' \neq c.$$

If the Condorcet winner exists then it is unique. On the other hand, if there is a weak-Condorcet winner then it may not be the only one. The definitions of Condorcet loser and weak-Condorcet loser are the obvious ones.

We consider voting environments with three candidates that are always labelled a, b and c. Note that, with three candidates, each voter has only two undominated strategies: under approval, (1) approving her best candidate and (2) approving her two best candidates; under plurality, (1) voting for her best candidate and (2) voting for her second-best candidate; under negative plurality, (1) voting against her worst candidate and (2) voting against her second-best candidate. To economize on notation, we may say that a voter has preferences $u \equiv (u_a, u_b, u_c)$ if the utility value that she derives when candidate a, b or c wins the election is, respectively, u_a, u_b and u_c.

Before continuing we remark that every voting environment that we consider has a neighborhood where every property that we derive for the corresponding plurality, negative plurality and approval voting games holds. Therefore, there is an open set of utilities that is contained in the generic set of Theorem 1 for which the corresponding voting games have these same properties.

4.1 Discriminatory equilibria

Using Poisson games, Myerson (2002) studies voting rules in terms of their tendency to admit *discriminatory equilibria*, that is, equilibria in which voters disregard a candidate as not a serious contender. He finds that plurality rule tends to generate too many discriminatory equilibria while negative plurality tends to generate too few. Furthermore, approval seems to give a good balance between the two.

To illustrate the results in the previous section we consider two examples where we compute the stable outcomes of the plurality, negative plurality and approval games. In the first one, analogously to Myerson (2002), we show how a universally liked candidate can win with probability zero in a stable set of a plurality game. We borrow Myerson's terminology "above the fray" to indicate candidate a's privileged position in the election.

Example 1 (Above the fray). There are three candidates a, b and c and, for some integer $m > 1$, $2m$ voters grouped into two equally sized subsets. Voters in the first subset have preferences $a \succ b \succ c$ and voters in the second subset have preferences $a \succ c \succ b$. From a social perspective it is clear that candidate a is the most preferred alternative. Furthermore, $\{b, c\}$ is the set of weak-Condorcet losers. Nonetheless, the plurality game has a stable set where voters in the first subset vote for b and voters in the second subset vote for c. It is easy to see that this strategy combination is a strict Nash equilibrium and, therefore, a singleton stable set.[7] On the other hand, the approval game is dominance solvable. In every undominated strategy profile, every voter approves candidate a. Therefore, the approval games has a unique stable outcome which is the election of candidate a with probability one. It can also be proven that, once we eliminate dominated strategies, every Nash equilibrium of the negative plurality game leads to the election of candidate a too.[8] By properties (α), (β) and (γ), candidate a wins with probability one in the unique stable outcome of the negative plurality game.

We have just illustrated how negative plurality may eliminate discriminatory equilibria. Paralleling Myerson (2002) we now show that negative plurality may, in fact, generate too few discriminatory equilibria, which can also be harmful.

Again, we borrow the phrase "one bad apple" from Myerson to express the idea that the existence of one bad candidate can spoil the whole election.

Example 2 (One bad apple). There are three candidates a, b and c and, for some integer m, $3m$ voters that are grouped into three equally sized subsets. Voters in the first subset have utility function $(3, 1, 0)$, voters in the second subset have utility function $(1, 3, 0)$ and voters in the third subset have preferences that are either $a \succ b \succ c$ or $b \succ a \succ c$. It should be clear that from a social perspective, candidate c should not win the election. The negative plurality game has a stable set such that voters in the first subset vote against b, voters in the second subset vote against a and voters in the third subset vote against c. The strategy profile is a strict Nash equilibrium and, therefore, a singleton stable set.

In turn, both the approval voting game and the plurality game are dominance solvable.[9] The unique sophisticated equilibrium leads to the election of the Condorcet winner in both games (in this example this could be candidate a, b, or both could be weak-Condorcet winners). By Theorem 1 this is the unique stable outcome under both plurality and approval.

4.2 Electing Condorcet Losers

We now show a striking property of negative plurality. In an election where the Condorcet loser exists, negative plurality may select it with probability one.

Example 3 (Negative plurality selects the Condorcet loser with probability one in the unique stable outcome). Take five voters with preferences

[7] Every strict equilibrium is an *absorbing retract* (Kalai and Samet 1984) and every absorbing retract contains a stable set (Mertens 1992, p. 562).

[8] This comes from the fact that, in every undominated strategy profile, no voter casts a negative vote against a. Furthermore, in any undominated strategy profile such that some other candidate also receives zero negative votes some voter has an incentive to deviate.

[9] Note that in either system, no voter votes for candidate c, therefore, the voting game is reduced to a two-candidate contest between candidates a and b.

$$u_1 = u_2 = (3, 0, 1)$$
$$u_3 = u_4 = (0, 3, 1)$$
$$u_5 = (3, 2, 0).$$

Candidate a is the Condorcet winner and candidate c is the Condorcet loser. The negative plurality game is dominance solvable. First, eliminate every dominated strategy in the game (for each voter, the only undominated strategies are voting against her worst candidate and voting against her second best candidate). In the reduced game, voter 3 and 4's dominant strategy is to vote against a. After we eliminate voting against c from the strategy sets of voters 3 and 4, voter 5's dominant strategy is voting against c. In the last round of elimination, we find that voting against c is a dominated strategy for voters 1 and 2, hence, these voters vote against b. Under the resulting strategy profile, candidate a collects 2 negative votes, candidate b also collects 2 negative votes, and candidate c collects just one negative vote. Therefore, in the unique stable outcome of the negative plurality game, the Condorcet loser wins the election.

Consider now approval voting. In the first round of elimination, keep only the strategies where every voter approves her most preferred candidate and does not approve her least preferred one. In the next round, we find that voter 1 and 2's dominant strategy is to approve only candidate a. Given that, voters 2 and 3 approve only candidate b. Finally, voter 5 approves candidate a, which makes a the winner of the election.

Hence, there are robust voting environments where negative plurality selects the Condorcet loser with probability one. Nonetheless, plurality and approval suffer from a similar flaw too. In Example 1, we have seen how plurality rule can generate a stable outcome where the set of weak-Condorcet losers wins with probability one. Similarly, the generic approval game in De Sinopoli et al. (2013, Example 2) has a unique stable outcome where the set of weak-Condorcet losers also wins with probability one.

4.3 More on Approval Voting

Approval voting has received a lot of attention by the literature on political economy, see for instance Brams and Fishburn (1978), Fishburn and Brams (1981), or more recently, Brams and Sanver (2006). However, De Sinopoli et al. (2013) provide a family of examples where approval seems to be outperformed by plurality. Even if one decides to advocate for approval voting over other voting systems, it is important to understand its limitations and the kind of situations where it is not the ideal voting system. In any case, we think that these examples prove that none of the voting systems considered here is unambiguously superior to the rest.

In the next example, we compare approval voting with negative plurality. There is a unique weak-Condorcet winner and a unique weak-Condorcet loser. Moreover, every voter prefers the weak-Condorcet winner to the weak-Condorcet loser. In the unique stable outcome of the approval game both candidates are elected with the same probability. In the negative plurality game, the unique stable outcome has the weak-Condorcet winner being elected with probability one.

Example 4 Take six voters with preferences:

$$u_1 = u_2 = u_3 = (1, 3, 0)$$
$$u_4 = u_5 = u_6 = (3, 0, 2)$$

Candidate a is the unique weak-Condorcet winner: it ties with b and wins against c in pairwise contests. Meanwhile candidate b ties with both a and c in pairwise contests. It follows that c is the unique weak-Condorcet loser.

The approval game is dominance solvable. Let us identify strategies (ballots) with the collection of candidates approved by the voter. First eliminate dominated strategies so that voters 1, 2 and 3 are left with b and ab and voters 4, 5 and 6 are left with a and ac. Take voter 1, she will approve candidate b for sure. But when considering whether or not approving her second-ranked candidate a she knows that a will receive at least 3 approval votes, that b will receive exactly 3 approval votes (counting hers) and that c will receive at most 3 approval votes. Since the only case in which approving a pays off for voter 1 is when a and c are the only two candidates tied at the top and this case is impossible, approving both a and b is dominated by approving only b. The same is true for voters 2 and 3.

Voter 4 approves candidate a anyway. It follows that she knows that both a and b will receive exactly three approval votes and c at most 2. So the only case where approving c matters is when it takes exactly 2 votes. In such a case, voter 4 prefers a three-way lottery among the three candidates to a two-way lottery between candidates a and b. The analysis is symmetric for voters 5 and 6, hence, they all will approve both a and c.

This yields a unique stable outcome where the three candidates are elected with probability $1/3$. That is, both the weak-Condorcet winner and the weak-Condorcet loser are elected with the same probability *even though every voter strictly prefers a to c*. On the other hand, the negative plurality game is also dominance solvable (we leave the analysis to the reader) and leads to a unique stable outcome where candidate a is elected with probability one.

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Appendix 1: Generic determinacy of Nash equilibrium in negative plurality games

Henceforth, for every $i \in N$ we fix the set of pure strategies to be equal to

$$V_{np} \equiv \left\{ (v^1, \ldots, v^k) \in \{0, -1\}^k : \sum_{c \in K} v^c \in \{0, -1\} \right\}. \quad (4.1)$$

With slight abuse of notation we denote by c both candidate $c \in K$ and the ballot that gives a negative vote to candidate c. When that is the case, we say that a voter casts a negative vote against c or, simply, that she votes against c. The symbol 0 represents abstention. Therefore, we may write $V_{np} = \{0\} \cup K$.

Before proving Proposition 1 we point out one complication of the analysis of negative plurality. There are examples robust to slight utility perturbations such that, even if more than one candidate wins with positive probability, abstention can be a best response for some voters. Note that the same is not true with plurality or approval because voting for the most preferred candidate among those who win with positive probability always yields a strictly larger payoff than abstention (as long as the voter is not completely indifferent among all the winning candidates).

Example 5 Consider a negative plurality voting game with set of voters $N = \{1, 2, 3, 4\}$ and set of candidates $K = \{a, b, c, d\}$. Writing $u_i = (u_i(a), u_i(b), u_i(c), u_i(d))$ for voter i 's utility vector, voters' preferences are given by:

$$u_1 = (0, 0, -1, 0),\ u_2 = (4, 3, 0, 6),\ u_3 = (6, 3, 0, 4),\ u_4 = (0, -\varepsilon, -1, 0).$$

where $\varepsilon > 0$ is a suitable small number. A Nash equilibrium of this voting game is $\sigma = (c, \frac{1}{2}a + \frac{1}{2}b, \frac{1}{2}b + \frac{1}{2}d, 0)$. Under this Nash equilibrium, candidates a and d win with probability $3/8$, and candidate b wins with probability $1/4$. Note that for voter 4, voting against b, her least preferred candidate among those who win with positive probability, is not a best response. The reason is that if voter 4 votes against b then candidate c, her least preferred candidate overall, wins with positive probability. Moreover, every game in a neighborhood has a close by Nash equilibrium with the same characteristics.

Nevertheless, we must point out that both abstention and voting against candidate c are best responses for voter 4 and that abstaining is always a dominated strategy (by voting against the least preferred candidate overall).

Thus, in a Nash equilibrium of a negative plurality game, a voter may find it optimal to abstain even in close races. It should also be clear that voting against a candidate that wins with zero probability is "similar" to abstention in the sense that, once we fix the behavior of the rest of the voters, it does not affect the probability distribution over winning candidates.

We focus on Nash equilibria of negative plurality where more than one candidate wins with positive probability. We call such Nash equilibria *nondegenerate equilibria*.

Definition 2 The Nash equilibrium σ is *nondegenerate* if the probability distribution $p(\sigma) \equiv (p(c \mid \sigma))_{c \in K}$ that it induces on candidates satisfies $p(c \mid \sigma) < 1$ for every $c \in K$.

Given that there are exactly k probability distributions where only one candidate wins (the point masses on the elements of K), it is enough to prove that the set of equilibrium distributions induced by nondegenerate equilibria is finite.

Recall that $W(v)$ is the set of candidates that receive the least negative votes under the ballot profile v. Given some collection C of ballot profiles, let us write

$W(C) = \bigcup_{v\in C} W(v)$. For any strategy profile σ we let $C(\sigma) \equiv \{v : \sigma(v) > 0\}$ denote the *carrier* of σ. Note that $C(\sigma)$ has a product structure, i.e. $C(\sigma) = \prod_{i\in N} C_i(\sigma_i)$, where $C_i(\sigma_i) = \{v_i \in V : \sigma_i(v_i) > 0\}$. Any strategy profile σ with carrier C satisfies $p(c \mid \sigma) < 1$ for every $c \in K$ if and only if $\#W(C) > 1$. Therefore, if $\#W(C) > 1$ we say that the carrier C is nondegenerate. Given a nondegenerate carrier C, we can construct the set of candidates that cannot win if voter i abstains. Every ballot in C_i that consists of a negative vote against one of such candidates is equivalent to abstention. That set of ballots, together with abstention, is denoted $A_i(C)$. In symbols, $A_i(C) \equiv C_i \setminus W(C_{-i})$. Note that for any $v_i, v_i' \in A_i(C)$ and any $v_{-i} \in C_{-i}$ we have $W(v_{-i}, v_i) = W(v_{-i}, v_i')$ and, consequently, if $C(\sigma) = C$ we also have $p(c \mid \sigma_{-i}, v_i) = p(c \mid \sigma_{-i}, v_i')$ for every $c \in K$.

Insofar as we aim to establish a result that holds for generic utilities, we can restrict the analysis to utility vectors where no player is indifferent between two candidates. The set of all such utility vectors is denoted $\tilde{U}$. The set $\tilde{U}$ is obtained removing a finite number of lower-dimensional hyperplanes from U and its closure coincides with U.

Assumption 1 For every voter $i \in N$ and every pair of candidates $c, d \in K$ we have $u_i(c) \neq u_i(d)$.

We fix a point $u \in \tilde{U}$, a nondegenerate carrier C and a Nash equilibrium σ such that $C(\sigma) = C$. Take an arbitrary ballot profile $v^* \in C$ that satisfies $v_i^* \in A_i(C)$ whenever $A_i(C) \neq \emptyset$ (otherwise v_i^* is an arbitrary element of C_i). For each $i \in N$, let $\hat{K}_i \equiv C_i \setminus \left(A_i(C) \cup \{v_i^*\}\right)$.

Since σ is a Nash equilibrium, for each voter $i \in N$ and each pure strategy $c \in C_i$, the following equality holds:

$$\sum_{d\in K} p(d \mid \sigma_{-i}, c)u_i(d) = \sum_{d\in K} p(d \mid \sigma_{-i}, v_i^*)u_i(d).$$

Subtracting from both sides voter i's expected utility if she abstains and letting $\pi(d \mid \sigma_{-i}, c) \equiv p(d \mid \sigma_{-i}, c) - p(d \mid \sigma_{-i}, 0)$, we can rewrite the previous equality as:

$$\sum_{d\in K} \pi(d \mid \sigma_{-i}, c)u_i(d) = \sum_{d\in K} \pi(d \mid \sigma_{-i}, v_i^*)u_i(d). \tag{4.2}$$

For each voter i, let us select from (4.2) the equalities corresponding to ballots $c \in \hat{K}_i$. Rearranging those equalities, for all $c \in \hat{K}_i$, we obtain:

$$\begin{aligned} -\sum_{d\in \hat{K}_i} &\left(\pi(d \mid \sigma_{-i}, c) - \pi(d \mid \sigma_{-i}, v_i^*)\right) u_i(d) \\ &= \sum_{d\notin \hat{K}_i} \left(\pi(d \mid \sigma_{-i}, c) - \pi(d \mid \sigma_{-i}, v_i^*)\right) u_i(d). \end{aligned} \tag{4.3}$$

Therefore, for each voter $i \in N$ we have $\hat{k}_i \equiv \#\hat{K}_i$ equalities. Suppose that we know the values assumed by u_i over candidates in $K \setminus \hat{K}_i$. We call this vector u_i^*. We can interpret the $\hat{k}_i$ equalities as a system of $\hat{k}_i$ equations in $\hat{k}_i$ unknowns; the set of unknowns being the values assumed by u over candidates in $\hat{K}_i$. Let us call this vector of unknowns u_i° so that $u_i = (u_i^{\circ}, u_i^{*})$. We let X_i^C denote the $\hat{k}_i \times \hat{k}_i$ matrix of coefficients of this system of equations. Hence, the (c, d)-th entry of X_i^C is

$$X_i^C(c,d) = -\pi(d \mid \sigma_{-i}, c) + \pi(d \mid \sigma_{-i}, v_i^{*}). \tag{4.4}$$

We need to show that the matrix X_i^C is always invertible. To this end, we need the next Lemma. Recall that a square matrix is an *M-matrix* (Ostrowski 1955, p. 95) if all diagonal elements are strictly positive, all nondiagonal elements weakly negative, and all principal minors of all orders strictly positive.

Lemma 1 (Ostrowski 1955, p. 97). *Let Π be an $n \times n$ M-matrix and let $\pi = (\pi_1, \ldots, \pi_j, \ldots, \pi_n)$ be a weakly positive vector. The determinant of the $n \times n$ matrix X whose (i, j)-th element is given by $X_{ij} = \Pi_{ij} + \pi_j$ is strictly positive.*

Note that the matrix Π_i^C whose (c, d)-th element is $-\pi(d \mid \sigma_{-i}, c)$ and the vector $(\pi(d \mid \sigma_{-i}, v_i^{*}))_{d \in \hat{K}_i}$ decompose the matrix X_i^C in the way described in Lemma 1. Therefore, our task is to prove now that Π_i^C is an M-matrix and that $(\pi(d \mid \sigma_{-i}, v_i^{*}))_{d \in \hat{K}_i}$ is a weakly positive vector. We start with the latter result.

Lemma 2 *Every element of the vector $(\pi(d \mid \sigma_{-i}, v_i^{*}))_{d \in \hat{K}_i}$ is weakly positive.*

Proof Recall that we chose v_i^{*} so that $v_i^{*} \in A_i(C)$ if $A_i(C) \neq \emptyset$. If $v_i^{*} \in A_i(C)$ then $\pi(d \mid \sigma_{-i}, v_i^{*}) = 0$ for every candidate d because v_i^{*} is equivalent to abstention. If $v_i^{*} \notin A_i(C)$ then a negative vote against candidate v_i^{*} can never decrease the probability that some other candidate $d \neq v_i^{*}$ gets elected, therefore, $\pi(d \mid \sigma_{-i}, v_i^{*}) \geq 0$ for every $d \in \hat{K}_i$ (note that $v_i^{*} \notin \hat{K}_i$ by definition). □

To prove that Π_i^C is an M-matrix we proceed in several stages. The first one is to show some properties about its entries.

Lemma 3 *The following assertions hold:*

(i) Every nondiagonal element of Π_i^C is weakly negative.
(ii) Every diagonal element of Π_i^C is strictly positive.
(iii) Every row in Π_i^C adds up to some weakly positive number.

Proof Part (i) follows from the proof of Lemma 2.

To prove part (ii) we need to show that a negative vote against a candidate always decreases the probability that she wins the election. For any $c \in \hat{K}_i$ we have $c \notin A_i(C)$, so there exists a ballot profile $v_{-i} \in C_{-i}$ such that c is the candidate that collects the least number of negative votes under $(v_{-i}, 0)$. If some other candidate receives as many negative votes, or just one negative vote more than c, then candidate c wins with less probability under (v_{-i}, c) than under $(v_{-i}, 0)$. Given that every ballot profile in C_{-i} receives positive probability under σ_{-i} we obtain $p(c \mid \sigma_{-i}, c) < p(c \mid \sigma_{-i}, 0)$. Hence, suppose that candidate c receives at least two negative votes fewer than any other candidate under every ballot profile such that candidate c wins with positive probability. The carrier C is nondegenerate, so there must be another ballot profile v_i such that c is not the only candidate that receives the least negative votes under $(v_{-i}, 0)$. Since C_{-i} has a product structure, we can obtain the ballot profile v_{-i} from v_{-i} by changing one coordinate at a time. During this process we must find some ballot profile $v_{-i} \in C_{-i}$ such that, under $(v_{-i}, 0)$, candidate c either obtains as many negative votes as some other winning candidate or wins the election outright but collecting just one negative vote fewer than another candidate. Given that every ballot profile in C_{-i} receives positive probability under σ_{-i}, we conclude again that $p(c \mid \sigma_{-i}, c) < p(c \mid \sigma_{-i}, 0)$.

Part (iii) follows because the decrease $\pi(c \mid \sigma_{-i}, c)$ in the probability that candidate c gets elected when player i votes negatively for c is necessarily equal to the increase in the probability that some candidate $d \in K \setminus \{c\}$ is elected. That is,

$$-\pi(c \mid \sigma_{-i}, c) = \sum_{d \in K \setminus \{c\}} \pi(d \mid \sigma_{-i}, c) \geq \sum_{d \in \hat{K}_i \setminus \{c\}} \pi(d \mid \sigma_{-i}, c),$$

with strict inequality whenever $\pi(d \mid \sigma_{-i}, c) > 0$ for some candidate $d \notin \hat{K}_i$. ◻

In view of Lemma 3(i)–(ii), to prove that TI_i^C is an M-matrix we now just need to show that all principal minors of all orders are strictly positive. To establish this, we use result C_9 in Plemmons (1977) which says that it is enough to prove that all *real* eigenvalues of TI_i^C are strictly positive. The next Lemma is the first step in this direction.

Lemma 4 *The real part of every eigenvalue of TI_i^C is weakly positive.*

Proof The Gershgorin Circle Theorem (Gershgorin 1931) tells us that every eigenvalue of a square matrix $A = (a_{cd})$ can be found in one of the closed disks $D(a_{cc}, R_c)$ with center a_{cc} and radius $R_c = \sum_{d \neq c} |a_{cd}|$. Therefore, every eigenvalue of TI_i^C lies in some closed disk with center the strictly positive (by Lemma 3(ii)) real number $-\pi(c \mid \sigma_{-i}, c)$ and with radius $\sum_{d \in \hat{K}_i \setminus \{c\}} \pi(d \mid \sigma_{-i}, c)$. Lemma 3(iii) implies that the real part of every eigenvalue of TI_i^C is weakly positive. ◻

In order to prove that every real eigenvalue is indeed strictly positive we now show that TI_i^C is nonsingular. Lemma 3(i)–(iii) show that TI_i^C is a *dominant diagonal* matrix. Recall that a matrix $A = (a_{cd})$ is dominant diagonal if $|a_{cc}| > \sum_{d \neq c} |a_{cd}|$ for every row c.

Price (1951) gives the following bound on the determinant $|A|$ of a dominant diagonal matrix:

$$\prod_c \left(|a_{cc}| - \sum_{d > c} |a_{cd}| \right) \leq |A|. \tag{4.5}$$

Now we can prove:

Lemma 5 *The matrix TI_i^C in nonsingular and, therefore, the matrix X_i^C is also nonsingular.*

Proof Reorder the rows and columns of TI_i^C so that columns (rows) corresponding to voter i 's more preferred candidates appear before columns (rows) corresponding to voter i 's less preferred candidates. With this reordering of the matrix, if $-\pi(c \mid \sigma_{-i}, c) \leq \sum_{d \succ c} \pi(d \mid \sigma_{-i}, c)$ then the decrease in the probability that candidate c is elected is equal to the increase in the probability that candidates worse than c (according to voter i 's preferences) win the election. This provides a contradiction because, using Assumption 1, voter i's utility from voting against c would be strictly lower than under abstention, which contradicts the fact that σ is a Nash equilibrium. Consequently, $-\pi(c \mid \sigma_{-i}, c) > \sum_{d>c} \pi(d \mid \sigma_{-i}, c)$ for every candidate c.

In light of Lemma 3(i)–(ii) we can apply Eq. (4.5) to TI_i^C knowing that every term on the left-hand side of (4.5) is strictly positive. Therefore, TI_i^C is nonsingular and, given that we already established that every real eigenvalue of this matrix is weakly positive, TI_i^C is also an M-matrix. We can now apply Lemma 1 to conclude that X_i^C is nonsingular. □

Therefore, if for each voter i we know u_i^* then we can reconstruct the entire vector of utilities u using the strategy profile σ and the system of Eq. (4.3). This allows us to construct a continuous function $(u^*, \sigma) \mapsto (u^*, u^\circ)$ from the set of Nash equilibria with carrier C to the set of utility vectors $\tilde{U}$. The next step of the proof is to apply the following result to such a function. It follows from the Generic Local Triviality Theorem (Bochnak et al. 1998).

Lemma 6 (Govindan and Wilson 2001) *Let X and Y be semialgebraic subsets of $\mathbb{R}^m$ and let $f: X \to Y$ be a continuous semi-algebraic function. If* $\dim(X) \leq \dim(Y)$ *then, for generic $y \in Y$, $f^{-1}(y)$ is a finite or empty set.*

We now have all the necessary ingredients to prove Proposition 1:

Proposition 1 *For generic negative plurality games, the set of probability distributions on candidates induced by Nash equilibria is finite.*

Proof If only one candidate can win under the carrier C (i.e. if $\#\left(\bigcup_{v \in C} W(v)\right) = 1$) then the set of equilibrium distributions induced by Nash equilibria with carrier C is a singleton and, therefore, necessarily finite. Hence, let the carrier C be nondegenerate.

Furthermore, let us first consider those carriers C such that $A_i(C)$, i.e. the set of votes equivalent to abstention, satisfies $\#A_i(C) \leq 1$ for every player i.

Given a utility vector u, the set of Nash equilibria of the corresponding negative plurality game is denoted by $\mathrm{NE}_{\mathrm{np}}(u)$. The graph of the Nash equilibrium subcorrespondence that contains only Nash equilibria with carrier C is

$$GNE_{\mathrm{np}}^C \equiv \left\{(\sigma, u) \in \Sigma \times \tilde{U} : \sigma \in \mathrm{NE}_{\mathrm{np}}(u) \text{ and } C(\sigma) = C\right\}.$$

Write E_{np}^C for the projection of GNE_{np}^C on Σ and on those coordinates of $\tilde{U}$ that contain, for each voter i, her utility to candidates in $K \setminus \hat{K}_i$ (that is, those coordinates of $\tilde{U}$ where we can find the entries of the subvector $u^* = (u_1^*, \ldots, u_n^*)$).

Lemma 5 implies that there is a continuous function $f^C_{np} : E^C_{np} \to \tilde{U}$ mapping (u^*, σ) into $u = (u^\circ, u^*)$. The function f^C_{np} is also semi-algebraic. Since $\dim(\tilde{U}) = nk$, in view of Lemma 4, the only thing remaining to show is $\dim(E^C_{np}) \le nk$.

For each voter i, the dimension of her subset of strategies that have carrier C_i is $\dim(\Delta(C_i)) = \#C_i - 1 = \hat{k}_i$ (recall that we assumed that $\#A_i(C) \le 1$ for every player i). Therefore, the dimension of GNE^C_{np} is at most $\sum_{i\in N} \hat{k}_i + nk$. Consequently,

$$\dim\left(E^C_{np}\right) \le \sum_{i\in N} \hat{k}_i + nk - \sum_{i\in N} \hat{k}_i = nk.$$

Applying Lemma 6 to the function $f^C_{np} : E^C_{np} \to \tilde{U}$ shows that for generic games $u \in \tilde{U}$ the set of Nash equilibria with carrier C is finite.

We now consider those carriers C such that $\#A_i(C) > 1$ for some player i. As argued before, Example 5 shows that there is an open set of utilities for which the negative plurality game has a continuum of Nash equilibria so, clearly, we cannot hope to prove that for generic games $u \in \tilde{U}$ the set of Nash equilibria with carrier C is finite. However, in that example, different Nash equilibria contained in the same continuum only differ on how the strategy of each player i assigns probabilities over the different elements of $A_i(C)$. Moreover, different such assignments do not affect the resulting probability distribution on candidates because elements of $A_i(C)$ are all equivalent to abstention.

Recall that, for each player i, we defined $\hat{K}_i \equiv C_i \setminus (A_i(C) \cup \{v_i^*\})$. (Also recall that our choice of v_i^* was such that $v_i^* \in A_i(C)$ whenever such a set was nonempty.) With abuse of notation, let $\Sigma^{\hat{K}} \equiv \prod_{i=1}^n \mathbb{R}^{\hat{K}_i}$. We define as $GNEO^C_{np}$ the projection of GNE^C_{np} onto $\Sigma^{\hat{K}} \times \tilde{U}$. That is, in this projection, we only eliminate those components of the strategy profile that we do not need to compute the equilibrium outcome $p(\sigma)$.[10] Note that $\dim\left(GNEO^C_{np}\right) \le \sum_{i\in N} \hat{k}_i + nk$.

We write EO^C_{np} for the projection of $GNEO^C_{np}$ on Σ and on those coordinates of $\tilde{U}$ where we can find the entries of the subvector $u^* = (u_1^*, \ldots, u_n^*)$. For each player i, the values that her mixed strategy assumes on $A_i(C)$ are neither needed to construct the system (4.3) nor used to show that it always has a solution. Hence, Lemma 5 implies that there is a continuous function $g^C_{np} : EO^C_{np} \to \tilde{U}$ mapping (u^*, σ) into $u = (u^\circ, u^*)$. The proof now follows the same lines as before. The only difference is that when applying Lemma 6 to the function $g^C_{np} : EO^C_{np} \to \tilde{U}$ we conclude that for generic games $u \in \tilde{U}$ the set $(g^C_{np})^{-1}(u)$ is finite so that $\{p(\sigma) : \sigma \in \mathrm{NE}_{np}(u) \text{ and } C(\sigma) = C\}$ is also finite. □

[10] Note that we do not need to know how any voter i distributes probability among elements in $A_i(C)$. This distribution only affects the distribution of probability between ballot profiles with the same set of winning candidates and different number of negative votes for losing candidates.

Appendix 2: Generic determinacy of Nash equilibrium in approval games with three candidates

In this section we fix the set of candidates $K \equiv \{a, b, c\}$. Hence, we write each player's set of pure strategies as $V_a \equiv \{abc, ab, ac, bc, a, b, c, 0\}$, where each ballot represents the set of candidates approved by the voter.

As in the case of negative plurality, we point out the main complication that we face when trying to prove that for generic approval voting games the set of equilibrium outcomes is finite. The next example is a generic approval game with three candidates and a continuum of Nash equilibria.

Example 6 Consider an approval game with set of candidates $K = \{a, b, c\}$ and five voters with preferences:

$$
\begin{aligned}
u_1 = u_2 &= (4, 2, 1),\\
u_3 &= (1, 4, 3),\\
u_4 = u_5 &= (2, 1, 3)
\end{aligned}
$$

Consider the following continuum of Nash equilibria of this game:

$$\left(\frac{3}{5}a + \frac{2}{5}\gamma\, ab + \frac{2}{5}(1-\gamma)b,\ ab,\ \frac{2}{5}bc + \frac{3}{5}b,\ c,\ c\right) : 0 \le \gamma \le 1 .$$

At every point in this continuum, voter 1 faces with probability $\frac{2}{5}$ a ballot profile where candidates a, b, and c receive, respectively, 1, 2, and 3 approval votes. With the remaining probability $\frac{3}{5}$, voter 1 faces a ballot profile where candidates a, b, and c receive, respectively, 1, 2, and 2 approval votes. Thus, voter 1 can make candidate a win with positive probability by approving that candidate *as long as she does not approve candidate b as well*. In more general terms, we can think of ballots b and ab as being equivalent from the viewpoint of voter 1 in the sense that, given the opponents' behavior, they generate the same probability distribution over winning candidates.

Note that every Nash equilibrium in the continuum induces the same probability distribution $(p_a, p_b, p_c) = (\frac{3}{25}, \frac{11}{25}, \frac{11}{25})$. Voters 2, 4 and 5 are playing a strict best reply while voters 1 and 3 equilibrate each other's expected utilities by playing mixed strategies. Hence, every game in a neighbourhood of this one has a continuum of Nash equilibria with the same characteristics.

Besides $K \equiv \{a, b, c\}$ and the set of possible ballots V_a, we also fix the set of voters N, and a utility vector u that satisfies Assumption 1. Take a Nash equilibrium σ such that every candidate wins with positive probability, i.e. letting $C \equiv C(\sigma)$, we have $W(C) = K$.

Given the carrier C, we partition C_i into equivalence classes. Two ballots $v_i, v_i' \in C_i$ are equivalent if $W(v_{-i}, v_i) = W(v_{-i}, v_i')$ for every $v_{-i} \in C_{-i}$. For instance, in Example 6, strategies b and ab are equivalent for voter 1 because $W(v_{-1}, b) = W(v_{-1}, ab)$ for every $v_{-1} \in \{ab\} \times \{b, bc\} \times \{c\} \times \{c\}$.

For the time being we focus on a voter i with preferences $a \succ_i b \succ_i c$. Furthermore, with slight abuse of notation, we let a denote the ballot that approves candidate a, b denote the ballot that approves candidate b, and ab denote the ballot that approves both candidate a and candidate b.

Lemma 7 *Let C be the carrier of a Nash equilibrium σ of an approval game with set of candidates $K \equiv \{a, b, c\}$. Suppose $W(C) = K$ and let player i have preferences $a \succ_i b \succ_i c$. Then,*

(i) abstention is not in C_i,

(ii) no ballot in C_i approves candidate c, and

(iii) if $b \in C_i$ and $ab \in C_i$ then ab and b belong to the same equivalence class in C_i.

Proof We start with part (i). Looking for a contradiction let $v \equiv (0, v_{-i}) \in C$ be such that $a \notin W(v)$. If, under v, candidate a is tied in the first place with other candidate then we are done because a would be a better response to σ than abstention. If a wins unopposed under v then substitute, for one voter at a time, the ballot where they approve a to some other ballot in their carrier where they do not. Given $W(C) = K$ and the fact that C has a product structure, proceeding in this way we must find a ballot profile $v \equiv (0, v_{-i})$ such that, either a is tied in the first place with some other candidate or candidate a receives just one vote fewer than the winning candidates. In either case, this shows that approving candidate a is a better response to σ than abstention, thus, $v \notin C$.

To prove (ii) by contradiction let $v_i \in C_i$ be a ballot that approves candidate c and let v_i' be the ballot that we obtain from v_i by removing the approval vote for c. Ballot v_i' is also best response against σ_{-i} and it is not a better response than v_i only if the additional approval vote for c in v_i does not increase the probability that c wins. That means that for every $v_{-i} \in C_{-i}$ such that $c \in W(v_{-i}, v_i')$, under (v_{-i}, v_i'), candidate c must receive at least one approval vote more than candidates a and b. At least one such a v_{-i} exists because $c \in W(C)$. In turn, for every $v_{-i} \in C_{-i}$ such that $c \notin W(v_{-i}, v_i')$ candidate c must receive at least two approval votes fewer than a and b. There is at least one such a v_{-i} because $a, b \in W(C)$. Since C_{-i} has a product structure, by changing the ballot of one voter at a time we must find a v_{-i} such that, under (v_{-i}, v_i') candidate c is either tied at the first place or just one approval vote behind the winning candidates. Such a v_{-i} receives positive probability under σ_{-i} and, therefore, v_i' is a better response than v_i against σ_{-i}. We conclude that no ballot in C_i approves candidate c.

Let us move to (iii). If $b \in C_i$ then ab is also a best response against σ_{-i} because C is the carrier of a Nash equilibrium and voter i 's utility can never decrease when her most preferred candidate receives an additional approval vote. Furthermore, ab is not a better response only if that additional approval vote does not increase the probability that a wins. That is, if $W(v_{-i}, b) = W(v_{-i}, ab)$ for every $v_{-i} \in C_{-i}$. $\square$

A consequence of this Lemma is that C_i is partitioned into at most two equivalence classes. When this is indeed the case and voter i has preferences $a \succ_i b \succ_i c$, for any Nash equilibrium σ with carrier C, we can write

$$\sum_{d \in K} p(d \mid \sigma_{-i}, a) u_i(d) = \sum_{d \in K} p(d \mid \sigma_{-i}, ab) u_i(d).$$

(We show below that a and ab cannot belong to the same equivalence class, so we are in fact taking a and ab to be the representatives of the two equivalence classes in C_i.) Rearranging,

$$\left[p(b \mid \sigma_{-i}, ab) - p(b \mid \sigma_{-i}, a)\right] u_i(b) = \left[p(a \mid \sigma_{-i}, a) - p(a \mid \sigma_{-i}, ab)\right] u_i(a) + \left[p(c \mid \sigma_{-i}, a) - p(c \mid \sigma_{-i}, ab)\right] u_i(c).$$

We want to show that if $u_i(a)$ and $u_i(c)$ are known then we can use the equilibrium strategy to find out $u_i(b)$. That is true as long as $p(b \mid \sigma_{-i}, ab) \neq p(b \mid \sigma_{-i}, a)$. The next Lemma establishes that this is indeed the case.

Lemma 8 *Let C be the carrier of a Nash equilibrium of an approval game with set of candidates $K \equiv \{a, b, c\}$. Suppose $W(C) = K$ and take a player i with preferences $a \succ_i b \succ_i c$. If $ab, a \in C_i$ then ab and a do not belong to the same equivalence class.*

Proof The proof follows the same lines as the proof of Lemma 7(ii). Assume that $W(v_{-i}, a) = W(v_{-i}, ab)$ for every $v_{-i} \in C_{-i}$. This implies that for every $v_{-i} \in C_{-i}$ such that $b \in W(v_{-i}, a)$, under (v_{-i}, a), candidate b receives at least one approval vote more than a and c. There is at least one such a v_{-i} because $b \in W(C)$. In turn, in every $v_{-i} \in C_{-i}$ such that $b \notin W(v_{-i}, a)$, under (v_{-i}, a), candidate b receives at least two approval votes fewer than the winner. There is at least one such a v_{-i} because a, $c \in W(C)$. Since C_{-i} has a product structure, by changing the ballot of one voter at a time we must find a v_{-i} such that, under (v_{-i}, a), candidate b is either tied at the first place or just one approval vote behind the winning candidates. For such v_{-i} we have $W(v_{-i}, a) \neq W(v_{-i}, ab)$. ◻

Thus, in more general terms, if we know the utility derived by each voter from her top- and bottom-ranked candidates then there is a semi-algebraic continuous function that, knowing σ, gives us the whole vector of utilities.[11] Paralleling the proof of Proposition 1, in the proof of the next proposition we apply Lemma 6 to such a function.

Lemma 9 *For generic approval voting games with three candidates, the set of Nash equilibrium outcomes is finite.*

Proof Clearly, there are three probability distributions such that just one candidate wins with positive probability. If only two candidates win with positive probability then the strategic interaction reduces to the one in a plurality voting game. In such a case, a similar argument to the one applied in De Sinopoli (2001) proves that for generic utilities the set of Nash equilibria where two candidates win with positive probability is finite.

Thus, take a nondegenerate carrier C such that all three candidates win with positive probability. Given a utility vector u, let $\mathrm{NE}_a(u)$ be the set of Nash equilibria of the

[11] In this case the system of equations is quite simple. For each voter whose set of pure best responses has two or three elements we only have one equation and one unknown.

corresponding approval voting game. The graph of the Nash equilibrium sub-correspondence that contains only Nash equilibria with carrier C is

$$GNE_a^C \equiv \left\{(\sigma, u) \in \Sigma \times \tilde{U} : \sigma \in \mathrm{NE}_a(u) \text{ and } C(\sigma) = C\right\}.$$

We have $\dim(GNE_a^C) \leq \sum_{i=1}^{n} (\#C_i - 1) + nk$. Let $\hat{N}$ be the set of voters i with two equivalence classes in C_i. We decompose $u = (u^*, u^\circ)$ so that u° contains the utility to each voter $i \in \hat{N}$ if her second-ranked candidate wins the election. Write $\tilde{U}^*$ for the projection of $\tilde{U}$ on the corresponding coordinates so that $u^* \in \tilde{U}^*$. Letting $\hat{n} = \#\hat{N}$, we obtain $\dim(\tilde{U}^*) = nk - \hat{n}$. Furthermore, let EO_a^C be the projection of GNE_a^C on $\tilde{U}^*$ and on those coordinates of the strategy space that capture the probability with which each voter only approves her corresponding top-ranked candidate. That is, EO_a^C contains just the part of the strategy profile that we actually need to compute the set of Nash equilibrium outcomes for a given Nash equilibrium with carrier C.[12]

We argued above that there is a semi-algebraic continuous function $f_a^C : EO_a^C \to \tilde{U}$ mapping (u^*, σ) into $u = (u^\circ, u^*)$. We have $\dim(\tilde{U}) = nk$. Hence, in order to apply Lemma 6, we now prove that $\dim(EO_a^C) \leq nk$. However, for every voter i such that C_i has two equivalence classes the set of possible probabilities that she can attach to the ballot that just approves her top-preferred candidate is one dimensional. Thus, we obtain:

$$\dim\left(EO_a^C\right) \leq \hat{n} + \dim(\tilde{U}^*) = \hat{n} + nk - \hat{n} = nk.$$

Applying Lemma 6 to the function $f_a^C : EO_a^C \to \tilde{U}$ shows that for generic games $u \in \tilde{U}$ the set of outcomes induced by Nash equilibria with carrier C is finite. Since there are only finitely many carriers, the desired result follows. □

Remark 1 Extending the result to any number of candidates is challenging. Vaguely speaking, to apply Lemma 6, for each player, we need to recover as many utility values as the dimensionality of the set of probability distributions that that player can induce by changing her strategy. However, note that if the number of candidates is x then the number of strategies is 2^x. Hence, it seems that we need a better understanding about how the set of best replies looks like in an approval voting game. Note, for instance, that Nash equilibrium strategies are not necessarily *sincere* (De Sinopoli et al. 2006), that is, if a voter approves a candidate c she does not necessarily also approve every candidate that she prefers to c.

We now finish the proof of Proposition 2.

Proposition 2 *For generic approval voting games, the set of probability distributions on three or fewer candidates induced by Nash equilibria is finite.*

Proof Take an arbitrary set of candidates K and a Nash equilibrium σ that induces a probability distribution that gives positive probability to exactly three candidates, say, c_1, c_2 and c_3. Construct a three-candidate approval game by choosing those three

[12] Note that we do not need to know how a voter distributes probability between two elements of an equivalence class. This distribution only affects the distribution of probability between ballot profiles with the same set of winning candidates and different number of approval votes for losing candidates.

candidates. Interpreting ballots under approval as subsets of candidates, we define the strategy profile σ of the three-candidate game by $\sigma_i(V_i) = {}_{\{v_i : v_i \subset V_i\}} \sigma_i(v_i)$ for every $i \in N$ and every $V_i \subseteq \{c_1, c_2, c_3\}$.[13] It is not difficult to see that σ is a Nash equilibrium of the three-candidate approval game. A similar thing can be done if two candidates win with positive probabilities. Finally, we note that the set of degenerate distributions on candidates is necessarily finite. □

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[13] The strategy σ is well defined. Candidates c_1, c_2 and c_3 (and only them) all win with positive probability under σ. Hence, if σ is an equilibrium, every voter approves at least one of them in every pure strategy that is played with positive probability.