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On totally geodesic subvarieties in the Torelli locus and their uniformizing symmetric spaces

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Contents

Introduction

Let M_g denote the moduli space of smooth, complex algebraic curves of genus g, A_g the moduli space of principally polarized abelian varieties of dimension g over $\mathbb{C}$ and consider the Torelli map

$$j : \mathsf{M}_g \to \mathsf{A}_g, \quad [C] \longmapsto [JC]$$

which associates to the class of a curve the class of its Jacobian, which is a principally polarized abelian variety with polarization induced by the cup product. By the Torelli theorem, the map j is injective. It is a natural and interesting problem to understand the geometry of the image of M_g in A_g.

In this thesis, we address this problem with the following focus. The moduli space A_g is the quotient of the Siegel space $\mathfrak{S}_g$ by the action of $\mathrm{Sp}(2g, \mathbb{Z})$. Since the Siegel space $\mathfrak{S}_g$ is a Riemannian symmetric space, it follows that A_g is endowed with a locally symmetric Riemannian metric (in the orbifold sense). We call the induced metric on A_g the *Siegel metric.* We are interested in studying the local properties of the inclusion $j(\mathsf{M}_g) \subset \mathsf{A}_g$ with respect to the Siegel metric. In simple words, the expectation on this problem is that the geometry of $j(\mathsf{M}_g)$ should drift apart from the one of the environment space A_g. Let us make this idea more precise.

Definition. *A closed algebraic subvariety $Z \subset \mathsf{A}_g$ is* ***totally geodesic*** *if $Z = \pi(X)$ for some connected totally geodesic submanifold $X \subset \mathfrak{S}_g$. We say that X is the symmetric space* ***uniformizing*** *Z.*

A totally geodesic subvariety of a Riemannian manifold is a submanifold with identically vanishing second fundamental form. It follows that every geodesic of the submanifold is a geodesic in the manifold itself.

Conjecture 1. *For $g >> 0$, there are no positive-dimensional totally geodesic subvarieties Z of A_g such that $Z \subset \overline{j(\mathsf{M}_g)}$ and $Z \cap j(\mathsf{M}_g) \neq \emptyset$.*

This conjecture expresses the expectation that $j(\mathsf{M}_g)$ should be very "curved" with respect to the Siegel metric on A_g. A very interesting aspect is that this conjecture, which comes from differential geometric considerations, agrees with another conjecture, whose origin is linked to arithmetical considerations.

Conjecture 2 (Coleman-Oort). *For $g >> 0$, there are no positive dimensional Shimura subvarieties Z of A_g such that $Z \subset \overline{j(\mathsf{M}_g)}$ and $Z \cap j(\mathsf{M}_g) \neq \emptyset$.*

Shimura subvarieties of A_g are defined as Hodge loci for the natural variation of Hodge structure on A_g. They are totally geodesic (see Section 1.4 and references therein). In particular, Conjecture 1 implies conjecture 2. Note that the statement of Conjecture 2 is completely algebraic. These conjectures motivate the study of totally geodesic subvarieties of A_g, which is the object of this thesis. The approach will mainly be the study of totally geodesic subvarieties of A_g through the study of their uniformizing symmetric spaces.

Riemannian symmetric spaces can be described both in terms of Lie groups and in terms of Lie algebras. We will briefly recall this aspect in the **first Chapter**, together with some other **preliminaries**. Now let us just recall that the tangent space of a symmetric space X is endowed with a kind of operation coming from the Lie bracket on the Lie algebra associated with X. This can be thought of as section β of a vector bundle over X. Since the differential geometry of symmetric spaces can be studied in terms of Lie theory, the tensor β is important because it encodes a lot of information on the geometry of the space. Applying these considerations to the case of the moduli space A_g, which is locally symmetric, we get that A_g is endowed with a tensor β that describes its local geometry. One thing one can do is to consider the pull-back of this tensor $\mathbf{B} = j^*\beta$ to the moduli space of curves via the Torelli map. The study of this object should give important information on the geometry of the inclusion $j(\mathsf{M}_g) \subset \mathsf{A}_g$ and on Conjectures 1 and 2. The first step in this direction is the computation of $\mathbf{B}$ at a moduli point $[C] \in \mathsf{M}_g$ in terms of the geometry of the curve C. This is the first main result of **Chapter 2**. Let $\bar{C}$ denote the conjugate curve, i.e., with the opposite complex structure. We first show that the dual map of $\mathbf{B}$ can be seen as a map

$$\mathbf{B}^* : H^0(K_C) \otimes H^0(K_{\overline{C}}) \to H^0(2K_C) \otimes H^0(2K_{\overline{C}}).$$

Secondly, we consider the algebraic surface $Z = C \times \overline{C}$. By Künneth formula $H^0(Z, K_Z) \simeq H^0(K_C) \otimes H^0(K_{\overline{C}})$ and $H^0(2K_Z) \simeq H^0(2K_C) \otimes H^0(2K_{\overline{C}})$. With these identifications, we prove the following (see Theorem 2.4.2).

Theorem A. *The map*

$$\mathbf{B}^* : H^0(Z, K_Z) \longrightarrow H^0(Z, 2K_Z)$$

coincides with the multiplication by $-i\mathbf{K}$, where $\mathbf{K} \in H^0(Z, K_Z)$ is the Bergman kernel of the curve C.

In other words, Theorem A says that the Bergman kernel $\mathbf{K}$ governs the restriction of the Lie bracket to $dj(T\mathsf{M}_g)$. The definition of the Bergman kernel in the sense we need (see 2.2.1) is due to Kobayashi, and it generalizes the classical Bergman kernel on open domains in $\mathbb{C}^n$. It can be either thought of as a $(2,0)$ form on Z, or as a $(1,1)$ form on $S := C \times C$. Let $\Delta \subset S$ be the diagonal. The second main result of the Chapter links the Bergman kernel of a curve C with the meromorphic form $\hat{\eta} \in H^0(C, K_S(2\Delta))$ constructed in [6] using results from [7]. It is proved in [6, Theorem 3.13] that the form $\hat{\eta} \in H^0(C, K_S(2\Delta))$ governs the second fundamental form of the Torelli map with respect to the Siegel metric. Also, the form has been studied in [3] in relation with projective structures on compact Riemann surfaces. Section 5 of [3] is dedicated to the study of the cohomology class of the form $\hat{\eta}$ and contains a characterization of $\hat{\eta}$ as the unique (up to multiples) element of $H^0(S, K_S(2\Delta))$ with cohomology class in $H^2(S - \Delta, \mathbb{C})$ of pure type $(1,1)$. Our second result makes this more precise. (See Theorem 2.8.2).

Theorem B. *The Bergman kernel is the* $(1,1)-$*harmonic representative of the cohomology class of* $\hat{\eta} \in H^0(S, K_S(2\Delta))$ *in* $H^2(S - \Delta)$. *More precisely, there exists* $\alpha \in H^0(S, \mathcal{A}^{1,0}(\Delta))$ *such that*

$$\hat{\eta} - 2\pi\mathbf{K} = d\alpha,$$

that is $\partial\alpha = \hat{\eta}$ *and* $\bar{\partial}\alpha = -2\pi\mathbf{K}$.

The material concerning Theorems A and B has been published in [24]. Besides these results, Chapter 2 also presents an analogue of Theorem A for the triple Lie bracket. The latter is interpreted as a multiplication map too. This result is accompanied by some comments explaining the motivation for the study of the triple Lie bracket and the link with Conjectures 1 and 2. (see Sections 2.5 and 2.6).

What makes the study of the Conjectures above even more interesting is that for low genus examples of (positive-dimensional) Shimura subvarieties Z of A_g such that $Z \subset \overline{j(\mathsf{M}_g)}$ and $Z \cap j(\mathsf{M}_g) \neq \emptyset$ do exists. All the examples known so far are in genus $g \leq 7$ and can be divided in three classes:

1. those obtained as families of Galois covers of $\mathbb{P}^1$;
2. those obtained as families of Galois covers of elliptic curves;
3. those obtained via fibrations constructed on the examples in (2) .

We briefly recall some known aspects of these examples. Roughly speaking, the examples in (1) and (2) are constructed as follows: families of G-covers are identified by data of combinatorial and group-theoretical nature. A numerical condition $(\star)$ on the datum ensures that the family's image in A_g is a Shimura variety. Table 2 in [15] lists all examples of type (1) with $g \leq 9$. Table 2 in [16] lists all examples of type (2) with $g \leq 9$. There are some overlaps between (1) and (2). Moreover, it is proved in [19] that there are no families of Galois covers of curves with genus ≥ 2 satisfying $(\star)$. Recently, it has been proved in [8] that, indeed, the one listed in the references above are the only positive-dimensional families of Galois coverings satisfying $(\star)$ with $2 \leq g \leq 100$. Note that, in general, it is not known whether $(\star)$ is also necessary for a family to yield a Shimura variety. More generally, very little is known about other counterexamples. In **Chapter 3**, dedicated to explain the results from [59], we determine which symmetric space uniformizes each of the counterexamples via the computation of the associated Lie algebra decomposition. More precisely, it is first computed the uniformizing symmetric space for the examples belonging to only (1), next for those in (2), including 4 examples that also appear in (1). Finally, the main result of the Chapter discusses the relationship between the fibrations and the uniformizing symmetric space of the examples in (2). The idea is the following. Let $f_t : C_t \to C'_t$, $t \in B$ be one of the six families of Galois covers of elliptic curve found in [16]. Associated with this family there are two maps: the first is the generalized Prym map

$$P : B \to \mathsf{A}^{\delta}_{g-1}, \quad [C_t \to C'_t] \mapsto Prym(C_t, C_t)$$

and the second is the map

$$\varphi : B \to \mathsf{A}_1, \quad [C_t \to C'_t] \mapsto [JC'_t].$$

It is proved in [19] that the connected components of the fibers of both maps have a totally geodesic image in A_g. Therefore, in an orbifold sense, the image of the family in A_g admits two different fibrations in totally geodesic subvarieties. Moreover, it is proved there that countably many of these totally geodesic fibers are Shimura. Linked to the study of these maps is the decomposition, up to isogeny, of the Jacobian JC of C, as $JC \sim JC' \times Prym(C, C')$. The main result of the Chapter studies this decomposition at the level of the Siegel space. (See Theorem 3.2.3).

Theorem C. *Let X be the uniformizing symmetric space associated to one of the families in (2). Then*

i) X decomposes as $B_1(\mathbb{C}) \times M$, where M is an hermitian symmetric space of codimension 1 and $B_1(\mathbb{C})$ is the open unit ball in $\mathbb{C}^n$.

ii) M uniformizes the irreducible components of the fibers of φ.

iii) $B_1(\mathbb{C})$ uniformizes the irreducible components of the fiber of the Prym map.

In particular, the result also shows how to uniformize the examples in (3).

In **Chapter 4** we focus on the study of families of Galois covers for their own sake. It is known (see e.g.[26]) that, given a finite group G and a G-covering $C \to \mathbb{P}^1$, there exists a holomorphic family of algebraic curves

$$\pi : \mathscr{C} \to \mathsf{B},$$

such that

1. every curve C' in the family is *topologically equivalent* to C, i.e. there are $\eta \in \operatorname{Aut} G$ and an orientation preserving homeomorphism $f : C' \cong C$ such that $f(g \cdot x) = \eta(g) \cdot f(x)$ for any $x \in C'$ and any $g \in G$;

2. every curve that is topologically equivalent to C is G-isomorphic to some fiber of the family and at most to finitely many of the fibers.

The construction in [26] uses Teichmüller theory. In the Chapter, we describe an alternative, explicit, and completely topological construction of such a family. We proceed as follows. Let $\mathbf{F}_{0,n}\,\mathbb{P}^1$ denote the configuration space of n distinct points in $\mathbb{P}^1$

$$\mathbf{F}_{0,n}\,\mathbb{P}^1 := \{(x_1, \ldots, x_n) \in (\mathbb{P}^1)^n : x_i \neq x_j \text{ for } i \neq j\}.$$

For any integer $n \geq 3$ set

$$\mathsf{M}_{0,n} := \{X = (x_1, \ldots, x_n) \in \mathbf{F}_{0,n}\,\mathbb{P}^1 : x_{n-2} = 0, x_{n-1} = 1, x_n = \infty\}.$$

Finally, set $\Gamma_n = \langle \gamma_1, ..., \gamma_n : \prod_{k=1}^n \gamma_k = 1 \rangle$. Now let G be a finite group and $\theta : \Gamma_n \to G$ an epimorphism. Each point $X \in \mathsf{M}_{0,n}$ corresponds to a G-covering of $\mathbb{P}^1$ with monodromy θ and branch locus X. We want a holomorphic family parametrizing these coverings. Consider the map

$$p : \mathsf{M}_{0,n+1} \to \mathsf{M}_{0,n}, \quad p(x_0, x_1, ..., x_n) = (x_1, ..., x_n).$$

The projection p has fiber $p^{-1}(X) = \mathbb{P}^1 - X$. This can be thought as the universal family of genus 0 curves with n marked points. The basic idea of our construction is that we can obtain the total space of our family as a suitable G-covering of $\mathsf{M}_{0,n+1}$. For the construction of this covering, choose

i) an element $x = (x_0, X) \in \mathsf{M}_{0,n+1}$;

ii) an isomorphism $\chi : \Gamma_n \to \pi_1(\mathbb{P}^1 - X, x_0)$.

The following sequence is exact and splits

$$1 \to \pi_1(\mathbb{P}^1 - X, x_0) \to \pi_1(\mathsf{M}_{0,n+1}, x) \to \pi_1(\mathsf{M}_{0,n}, X) \to 1.$$

Set $f := \chi^{-1} \circ \theta$ and $H := \pi_1(\mathsf{M}_{0,n}, X)$. Assume that we can prove that f extends to a morphism $\tilde{f} : \pi_1(\mathsf{M}_{0,n+1}, x) = \pi_1(\mathbb{P}^1 - X, x_0) \rtimes H \to G$. Then we are in the following situation:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(\mathbb{P}^1 - X, x_0) & \longrightarrow & \pi_1(\mathsf{M}_{0,n+1}, x) & \longrightarrow & H & \longrightarrow & 1. \\ & & & \searrow^{f} & \downarrow{\scriptstyle \tilde{f}} & \swarrow_{\tilde{f}|_H} & & & \\ & & & & G & & & & \end{array}$$

The morphism $\tilde{f}$ gives rise to a topological covering

$$\pi' : \mathscr{C}^* \to \mathsf{M}_{0,n+1}.$$

By Grauert-Remmert Extension Theorem π' compactifies to a branched covering $\pi' : \mathscr{C} \to \mathbb{P}^1 \times \mathsf{M}_{0,n}$. Composing π' with the projection over $\mathsf{M}_{0,n}$ we get a holomorphic family

$$\pi : \mathscr{C} \to \mathbb{P}^1 \times \mathsf{M}_{0,n} \to \mathsf{M}_{0,n}$$

satisfying Property 1.

In general there may not exist any morphism $\tilde{f} : \pi_1(\mathbb{P}^1 - X, x_0) \rtimes H \to G$ extending f. We show that one can always find a finite index subgroup $H' \subset H$ such that f extends to $\tilde{f} : \pi_1(\mathbb{P}^1 - X, x_0) \rtimes H' \to G$. Thus, in the general case, the base of the family will be a finite cover B of $\mathsf{M}_{0,n}$ instead of $\mathsf{M}_{0,n}$ itself, and the construction requires more details. Finally, we prove that our construction does not depend on the choices *i* and *ii* (see Section 4.6). In particular, this also implies that Property 2 from above holds for our family.

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Chapter 1

Preliminaries

In this Chapter, we briefly recall some concepts from different topics that we will need in the sequel of the thesis. Section 1.1 is dedicated to basics about symmetric spaces, whereas in Section 1.2 we review some definitions and results from abstract Hodge theory. In Section 1.3 we review the definition of the Siegel upper-half space, which will play a key role in the thesis, and recall some of its interpretations. Section 1.4 is devoted to the definitions of totally geodesic and Shimura subvarieties of the moduli space A_g of principally polarized abelian varieties. Finally, in Section 1.5 we describe the problem that motivates the study of this thesis.

1.1 Symmetric spaces

For the material presented in this section we refer to [33, 39, 61].

1.1.1 Basic definitions

Definition 1.1.1. *Let (M, g) be a connected Riemannian manifold. We say that*

a) M is a ***symmetric space*** *if for every $p \in M$ there exists an isometry $s_p : M \to M$ such that $s_p(p) = p$ and $d(s_p)_p = -Id$.*

b) M is a ***locally simmetric space*** *if for every $p \in M$ there exist $r > 0$ and an isometry $s_p : B_r(p) \to B_r(p)$ such that $s_p(p) = p$ and $d(s_p)_p = -Id$.*

Where $B_r(p) = \{q \in M| \quad d(p, q) < r\}$.

Using some basic facts from Riemannian geometry, it is easy to see that a (global) symmetric space M has the following geometric properties:

1. The isometry s_p flips the geodesics passing though p: if γ is a geodesic with $\gamma(0) = 0$, then $s_p(\gamma(t)) = \gamma(-t)$.

2. M is a complete Riemannian manifold.

3. For every $p, q \in M$, there exists an isometry $f : M \to M$ such that $f(p) = q$.

Let $I(M)$ denote the isometry group of M. By Property 3, $I(M)$ acts transitively on M. Moreover, it follows by Property 2 that, firstly, $I(M)$ is a Lie group and, secondly, that for every $p \in M$ the stabilizer $I(M)_p = \{g \in I(M) : \; g.p = p\}$ is compact (see e.g. [38, pp. 45-50]). For a fixed point $p \in M$, we thus have that $M = G/K$, where $G = I(M)$ and $K = I(M)_p$ is a compact subgroup of G. Note that, with this identification, the action of G on M corresponds to left translations $L_g : G/K \to G/K$, where $L_g(aK) = (ga)K$. Since isometries in K are determined by thier differential in the point p, we get that the isotropy representation of K on T_pM, $h \mapsto d(L_h)_p$, is effective. It follows that we can think $K \subset O(T_pM)$. Denote by K^0 the connected component of K containing the identity.

Definition 1.1.2. *A symmetric space M is called **irreducible** if the action of K^0 on T_pM is irreducible.*

Irreducible symmetric spaces have additional properties, e.g., one can show that the Riemannian metric of an irreducible symmetric space is uniquely determined (up to multiple). The following decomposition result says that the notion of irreducibility leads the study of a symmetric space back to easier atoms:

Proposition 1.1.3. *If M is a simply connected symmetric space, then it is isometric to $M_1 \times ... \times M_k$, where M_i is an irreducible symmetric space.*

By going to the universal cover, the proposition implies that a reducible symmetric space is locally isometric to a product of irreducible symmetric spaces. We point out that irreducible symmetric spaces are completely classified (see [33, Chapter X]).

Remark 1.1.4. Let ∇ be the Levi-Civita connection of a locally symmetric space (M, g). This induces a connection on every tensor bundle $TM^{\otimes h} \otimes (TM^*)^{\otimes k}$ of M, which we still denote by ∇. If $R \in \Gamma(M, TM^{\otimes 3} \otimes TM^*)$ is the curvature tensor of M, then ∇R is a tensor of odd order on M, which is invariant by isometries. In particular, ∇R is invariant by the local isometries s_p. Since by definition $d(s_p)_p = -id$, it follows that $\nabla R = 0$. One can show that, in fact, this property characterizes locally simmetric spaces, i.e. that a Riemannian manifold (M, g) is locally symmetric if and only if $\nabla R = 0$ (see [33, p. 201]).

1.1.2 Characterization via Lie algebras

Let M be a Riemannian symmetric space. By the connectedness of M follows that, since the isometry group acts transitively on M, so does its connected component of the identity $I_0(M)$. As a consequence, we can replace $I(M)$ with $I_0(M)$ when writing M as homogeneus space, and for a fixed point $p \in M$, we get $M = G/K$, where $G = I_0(M)$ and $K = G_p$ is the stabilizer of p. Let us consider the conjugation by s_p in $I(M)$, i.e. the map $\sigma : g \mapsto s_p g s_p$. Since G is connected and $\sigma(id) = id$, we can consider its restriction

$$\sigma = \sigma_p : G \to G, \quad g \mapsto s_p g s_p.$$

The involution σ of G is called the **Cartan involution** of the symmetric space $M = G/K$. The following property is satisfied (See e.g. [39, Theorem 1.5, Chapter XI]).

Proposition 1.1.5. *Let $G^\sigma = \{g \in G : \ \sigma(g) = g\}$ be the subgroup of G of the elements fixed by σ, and G^σ_0 its connected component of the identity. Then $G^\sigma_0 \subset K \subset G^\sigma$.*

Indeed, this property characterizes symmetric spaces. To be more precise, the following holds (see [61, Proposition 6.25]).

Proposition 1.1.6. *Let G be a connected Lie group with an involution $\sigma : G \to G$ such that G^σ_0 is compact. Then, for any compact subgroup $K \subset G$ such that $G^\sigma_0 \subset K \subset G^\sigma$,*

1. *there always exists a $G-$invariant metric on G/K;*

2. *the homogeneus space G/K, endowed with any $G-$invariant metric, is a symmetric space.*

To syntetize, symmetric spaces are characterized in terms of Lie groups by three ingredients (G, K, σ), where:

1. G is a connected Lie group;

2. $\sigma : G \to G$ is an involutive automorphism;

3. $K \subset G$ is a compact subgroup such that $G^\sigma_0 \subset K \subset G^\sigma$.

We call (G, K, σ) a **symmetric couple** or **symmetric pair**. A symmetric pair (G, K, σ) is said to be *effective* (resp. *almost effective*) if G acts effectively (resp. almost effectively, i.e. with finite stabilizer) on G/K.

This description via Lie groups is the starting point to prove an algebraic characterization of symmetric spaces in terms of Lie algebras. Indeed, let (G, K, σ) be a symmetric pair. Denote by $\mathfrak{k}$ and $\mathfrak{p}$, respectively, the ± 1 eigenspaces of the linear involution $d\sigma$. The Lie algebra $\mathfrak{g}$ is thus decomposed, as a vector space, as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. This decomposition has the following properties (see e.g. [39, Section 2 of Chapter X, and Propositions 2.1, 2.2 of Chapter XI])

i) $[\mathfrak{k},\mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}$, $[\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p}$;

ii) $\mathfrak{k}$ is the Lie algebra of K;

iii) $\mathfrak{p} \cong T_{eK}G/K, \quad X \mapsto \frac{d}{dt}\big|_{t=0} \exp(tX).e$;

iv) $Ad_K(\mathfrak{p}) \subset \mathfrak{p}$.

where $e \in G$ is the identity element of G. We say that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the **Cartan decomposition** associated with the symmetric space. Conversely, starting by such a decomposition of a Lie algebra, one can reconstruct a symmetric space (see [61, Proposition 6.27]):

Proposition 1.1.7. *Let $\mathfrak{g}$ be a Lie algebra with a vector space decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Suppose that*

$$[\mathfrak{k},\mathfrak{k}] \subset \mathfrak{k} \qquad [\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k} \qquad [\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p}.$$

Denote by G the simply connected Lie group with Lie algebra $\mathfrak{g}$ and by $K \subset G$ its connected subgroup with Lie algebra $\mathfrak{k}$. Then

1. *there exists an automorphism $\sigma : G \to G$ such that $K = G_0^\sigma$;*

2. *if K is compact, then every $G-$invariant metric on G/K is symmetric;*

3. *G/K is almost effective if and only if $\mathfrak{g}$ and $\mathfrak{k}$ have no common ideals.*

It is possible to study differential geometry of symmetric spaces through Lie theory starting from the described correspondences. This aspect is, clearly, of central importance in the study of symmetric space, and, in particular, it will be important in the sequel of the thesis.

1.1.3 Hermitian symmetric spaces

An important subclass of symmetric spaces is that of Hermitian symmetric spaces, which are those that preserve a complex structure. These provide important examples of Kähler manifolds.

Definition 1.1.8. *A symmetric space M is a* ***Hermitian symmetric space*** *if M is a Hermitian manifold and the symmetries s_p are holomorphic.*

Proposition 1.1.9. *Let (G, K) be a symmetric couple with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and let Q be any $G-$invariant metric on G/K. If there exists $J_o : \mathfrak{p} \longrightarrow \mathfrak{p}$ such that*

1. *J_o is $Q-$orthogonal and $J_o^2 = -Id$.*

2. *$J_o \circ Ad(h) = Ad(h) \circ J_o$ for every $h \in K$.*

Then $M = G/K$ is a Hermitian symmetric space. Moreover, it is a Kähler manifold.

In other words, if a symmetric space M is endwed with an orthogonal complex structure in a point and the isotropy representation is linear with respect to this structure, then M is Hermitian.

1.1.4 Types of symmetric spaces

Definition 1.1.10. *Let (G, K) be a symmetric pair with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, and let B be the Killing form on $\mathfrak{g}$. The symmetric space G/K is said to be*

- *of the **compact** type if $B|_{\mathfrak{p}} < 0$*
- *of the **non-compact** type if $B|_{\mathfrak{p}} > 0$*
- *of the **euclidean** type if $B|_{\mathfrak{p}} = 0$.*

Every irreducible symmetric space falls into one of these three types. Moreover (cf. Proposition 1.1.3), if M a simply connected symmetric space, then

$$M \cong M_0 \times M_1 \times M_2$$

with M_0 euclidean, M_1 of the non-compact type, and M_2 of the compact type. In particular, one says that a symmetric space M has **no euclidean factors** if its universal cover does not contain factors of euclidean type. Symmetric spaces of the three types have special geometric properties.

Type	Curvature	G	G/K
Non-compact	Negative	Semisimple, non-compact	Non-compact
Compact	Positive	Semisimple, compact	Compact
Euclidean	Zero		Loc. isometric to $\mathbb{R}^n$

1.2 Hodge theory

In this section we review some concepts from Hodge theory. For the main definitions and facts presented in this section we refer to [48, 53, 57].

1.2.1 Basic definitions

Let R be a ring such that $\mathbb{Z} \subseteq R \subseteq \mathbb{R}$, and $n \in \mathbb{N}$.

Definition 1.2.1. *A **pure** $R-$**Hodge structure of weight** n is a finite and free $R-$module V with a decomposition*

$$V_{\mathbb{C}} := V \otimes_R \mathbb{C} = \bigoplus_{\substack{p,q \in \mathbb{Z} \\ p+q=n}} V^{p,q}$$

such that $\overline{V^{p,q}} = V^{q,p}$. *It is called* ***effective*** *if* $V^{p,q} = 0$ *for all* $p, q < 0$.

A $R-$Hodge structure is a finite direct sum $\oplus_n V_n$, where V_n is a $R-$Hodge strucure of weight n.

Remark 1.2.2. Given a pure $R-$Hodge structure V of weight n, define

$$F^p V_{\mathbb{C}} := \bigoplus_{\substack{r \geq p \\ r+q=n}} V^{r,q} \subset V_{\mathbb{C}}, \quad \forall p \in \mathbb{Z} \tag{1.2.1}$$

Then $\{F^\bullet\}$ is a decreasing filtration of $V_{\mathbb{C}}$, and satisfies $F^p \oplus \overline{F^{n+1-p}} = V_{\mathbb{C}}$. It is called **Hodge filtration**. Hodge filtrations allow to give an equivalent definition of Hodge structure. Indeed, if V is a finite and free $R-$module and $\{F^\bullet\}$ is a decreasing filtration of $V_{\mathbb{C}}$ such that $F^p \oplus \overline{F^{n+1-p}} = V_{\mathbb{C}}$, it is immediate to check that, setting $V^{p,q} := F^p \cap \overline{F^{n-p}}$, one gets a pure $R-$Hodge structure of weight n on V.

Definition 1.2.3. *Let* $(V, \{V^{p,q}\})$ *be a* $R-$*Hodge structure. A Hodge substructure* $W \subset V$ *is a* $R-$*submodule having an induced Hodge decomposition*

$$W_{\mathbb{C}} := \bigoplus_{p,q} W^{p,q}$$

where $W^{p,q} := W \cap V^{p,q}$.

Definition 1.2.4. *A* ***polarization*** *of a pure Hodge structure* V *of weight* n *is a bilinear form* $Q : V \times V \to R$ *which is symmetric for* n *even and skew-symmetric for* n *odd, and whose complex extension satisfies:*

1. *The Hodge decomposition is orthogonal for the hermitian form* $h : V_{\mathbb{C}} \times V_{\mathbb{C}} \to \mathbb{C}$ *defined as* $h(a,b) := i^n Q(a, \overline{b})$.

2. *For all* $a \in V^{p,q}$, $a \neq 0$, *one has* $(-1)^{\frac{n(n-1)}{2}+q} h(a,a) > 0$.

One says that a Hodge structure is polarisable if it admits a polarisation. What makes polarizable Hodge structure specially convenient to work with, is that, in the case $R = \mathbb{Q}$, if $V' \subset V$ is a Hodge substructure of a polarizad Hodge stucture (V, Q), then the orthogonal complement V'' of V' with respect to Q is again a Hodge substructure and $V = V' \oplus V''$.

A third equivalent way, due to Deligne, of viewing Hodge structures, is as representations of a certain algebraic group, as we now see. To begin with, define the **Deligne torus** $\mathbb{S}$ as the linear algebraic group, defined over $\mathbb{Q}$, given by the matrices

$$\mathbb{S} = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Q}) : \ a = d, \ b + c = 0\}.$$

It is the affine variety $V((x^2 + y^2)t - 1) \subset \mathbb{A}^3_{\mathbb{Q}}$. Since $\mathbb{S}$ is defined over $\mathbb{Q}$, it makes sense to consider its set of points $\mathbb{S}(F)$ over any $\mathbb{Q} \subset F \subset \mathbb{C}$. In particular, note that

$$\mathbb{C}^* \to \mathbb{S}(\mathbb{R}), \quad a + ib \mapsto \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

yields an isomorphism between $\mathbb{C}^*$ and $\mathbb{S}(\mathbb{R})$. Now let $(H, \{H^{p,q}\})$ be a $R-$Hodge structure of weight n. This determines a representation of $\mathbb{S}(\mathbb{R})$ on $H_{\mathbb{C}}$ as follows: $z \in \mathbb{S}(\mathbb{R}) = \mathbb{C}^*$ acts on $H^{p,q}$ as multiplication by $z^{-p}\overline{z}^{-q}$. One sees easily from the definition that z sends $H_{\mathbb{R}}$ in $H_{\mathbb{R}}$, i.e. that we get a representation

$$\rho_H : \mathbb{S}(\mathbb{R}) \to GL(H_{\mathbb{R}}).$$

Note that for $t \in \mathbb{R} \subset \mathbb{S}(\mathbb{R})$, $\rho_H(t)$ is the multiplication by $t^{-(p+q)} = t^{-n}$ on every $H^{p,q}$, i.e. $\rho_H(t) = t^{-n}\,\mathrm{id}$, so that one can read the weight of the Hodge structure from the corresponding representation. Conversely, given any a representation $h : \mathbb{S}(\mathbb{R}) \to GL(H_{\mathbb{R}})$, defining $H^{p,q}$ as the eigenspace for the character $z^{-p}\overline{z}^{-q}$, one gets a decompostion

$$H_{\mathbb{C}} = \bigoplus_{\substack{p,q\in\mathbb{Z} \\ p+q=n}} H^{p,q}, \quad \overline{H^{p,q}} = H^{q,p} \quad \forall p, q.$$

Hence to give a representation $h : \mathbb{S}(\mathbb{R}) \to GL(H_{\mathbb{R}})$ is the same as giving $H_{\mathbb{R}}$ and an $\mathbb{R}-$Hodge structure, and this structure is pure of weight n precisely if $h|_{\mathbb{R}} : \mathbb{R} \to GL(H_{\mathbb{R}})$ is given by $t \mapsto t^{-n}\,\mathrm{id}$. More generally, an $R-$Hodge structure of weight n consist in a finite and free $R-$module H and a representation $h : \mathbb{S}(\mathbb{R}) \to GL(H_{\mathbb{R}})$ such that $h|_{\mathbb{R}} : \mathbb{R} \to GL(H_{\mathbb{R}})$ is given by $t \mapsto t^{-n}\,\mathrm{id}$. All operations on Hodge structures (tensor products, Homs, direct sums, duals, etc.) can be performed in terms of the corresponding representations.

Remark 1.2.5. We point out that the algebraic representation ρ_H associated with a $\mathbb{Q}-$Hodge structure H is only defined over $\mathbb{R}$, though both the algebraic groups $\mathbb{S}$ and $GL(H)$ are defined over $\mathbb{Q}$.

1.2.2 Hodge classes and Mumford-Tate groups

Definition 1.2.6. *Let H be a $\mathbb{Q}-$Hodge structure of weight $2n$. The space*

$$H \cap H^{n,n}$$

is called the space of ***Hodge classes*** *on H.*

Let H be a $R-$Hodge structure of weight n and $\rho_H : \mathbb{S}(\mathbb{R}) \to GL(H_\mathbb{R})$ be its corresponding representation. We can restrict h to $S^1(\mathbb{R}) = \{z \in \mathbb{C} : z\overline{z} = 1\} \subset \mathbb{S}(\mathbb{R})$

$$S^1 \hookrightarrow \mathbb{S}(\mathbb{R}) \xrightarrow{h} GL(H_\mathbb{R})$$

and there $h(z) = z^{-(p-q)}$, since for $z \in S^1$, $\overline{z} = z^{-1}$. Note that, once the weight $p + q = n$ is known, we can determine p and q from $p - q$. In other words, given a representation $u : S^1 \to GL(H_\mathbb{R})$ and a weight n, one can reconstruct, by definining $h(z) = |z|^{-n}u(z/|z|)$, an Hodge structure $h : \mathbb{S}(\mathbb{R}) \to GL(H_\mathbb{R})$ of weight n. Roughly speaking, looking at $\rho_H|_{S^1}$ corresponds to forget about the weight of the Hodge structure.

Definition 1.2.7. *Let H be a $R-$Hodge structure of weight n and $\rho_H : \mathbb{S}(\mathbb{R}) \to GL(H_\mathbb{R})$ be its corresponding representation.*

1. *The* ***Mumford-Tate group*** *of H is the smallest algebraic subgroup $\mathrm{MT}(H)$ of $GL(H)$, defined over $\mathbb{Q}$, whose set of real points contains the image of ρ_H, i.e. $\rho_H(\mathbb{S}(\mathbb{R})) \subset \mathrm{MT}(H)(\mathbb{R})$.*

2. *The* ***Hodge group*** *(or* ***special Mumford-Tate group****) of H is the smallest algebraic subgroup $\mathrm{Hg}(H)$ of $GL(H)$, defined over $\mathbb{Q}$, whose set of real points contains $\rho_H(S^1)$, i.e. $\rho_H(S^1) \subset \mathrm{Hg}(H)(\mathbb{R})$.*

Note that $\mathrm{Im}\,\rho_H$ is an $\mathbb{R}-$algebraic subgroup of $GL(H_\mathbb{R})$. Thus, equivalently, the Mumford-Tate group of H is the Zariski closure of $\mathrm{Im}\,\rho_H$ in $GL(H_\mathbb{Q})$.

Remark 1.2.8. By the definitions and the previous considerations one gets that the Mumford-Tate group and the Hodge group differ by a factor $\mathbb{G}_m$. More precisely, if $n \neq 0$ and D denotes the subgroup of diagonal matrices in $GL(H_\mathbb{Q})$, then $\mathrm{MT}(H) = D \cdot \mathrm{Hg}(H)$.

Remark 1.2.9. It is known that $\mathrm{MT}(H)$ is a connected algebraic group. Moreover, if the Hodge structure H is polarizable, $\mathrm{MT}(H)$ is a reductive group. (See [53, Proposition 2])).

The importance of the Mumford-Tate group is due to its relationship with Hodge classes in Hodge structures constructed as tensor products of

H.

Let H be a $\mathbb{Q}-$Hodge structure of weight n. For any pair of multi-indeces $d, e \in \mathbb{N}^m$, let

$$T^{d,e}(H) := \bigoplus_{j=1}^{m} H^{\otimes_{\mathbb{Q}} d_j} \otimes_{\mathbb{Q}} (H^*)^{\otimes_{\mathbb{Q}} e_j}. \tag{1.2.2}$$

It is direct sum of Hodge structures of weight $n(d_j - e_j)$.

Proposition 1.2.10. *Let $V \subset T^{d,e}(H)$ be any rational subspace. Then V is a sub-Hodge structure if and only if it is stable under the action of* $\mathrm{MT}(H)$*. Similarly, a rational vector $t \in T^{d,e}(H)$ is a $(0,0)-$Hodge class if and only if it is invariant under* $\mathrm{MT}(H)$*.*

(See [53, Proposition 1]). Note that we can only have nonzero $(0,0)-$Hodge classes in $T^{d,e}(H)$ if the weight is zero. The proposition says that Hodge classes in $T^{d,e}(H)$ (i.e. the elements in $T^{d,e}(H) \cap T^{d,e}(H)^{0,0}_{\mathbb{C}}$) are the invariants in $T^{d,e}(H)$ by the action of $\mathrm{MT}(H)$. The crucial fact is that this property characterizes the Mumford-Tate group uniquely. Namely, (see [53, Proposition 3])

Proposition 1.2.11. *Let $G \subset GL(H)$ be the subgroup of elements that fixes every $(0,0)-$Hodge class in every tensor space $T^{d,e}(H)$. Then $G = \mathrm{MT}(H)$.*

In other words, knowing the Mumford-Tate group $\mathrm{MT}(H)$ is the same as knowing Hodge classes in every Hodge structure costructed from H.

1.2.3 Mumford-Tate group in families

Definition 1.2.12. *Let S be a complex connected manifold and $\mathbb{Z} \subseteq R \subseteq \mathbb{R}$ a ring. An $R-$VHS (variation of Hodge structure) of weight n consists of a local system $\mathcal{V}$ of finite and free $R-$modules and a decreasing filtration of the associated holomorphic vector bundle $\mathbb{V} = \mathcal{V} \otimes_R \mathcal{O}_S$:*

$$... \subset \mathbb{F}^p \subset \mathbb{F}^{p-1} \subset ...$$

by holomorphic subbundles $\mathbb{F}^p$ satisfying:

1. *$\mathbb{V} = \mathbb{F}^p \oplus \overline{\mathbb{F}^{n-p+1}}$ as $\mathcal{C}^\infty$ bundles, where the conjugation is taken relative to the local system of real vector spaces $\mathcal{V}_{\mathbb{R}} := \mathcal{V} \otimes_R \mathbb{R}$.*
2. *$\nabla(\mathcal{F}^p) \subset \Omega^1_S \otimes \mathcal{F}^{p-1}$, where ∇ denotes the flat connection on $\mathbb{V}$ and $\mathcal{F}^p$ denotes the sheaf of holomorphic sections of $\mathbb{F}^p$.*

Remark 1.2.13. Note that condition *1* in the definition implies that, for each $s \in S$, $(\mathbb{V}_s, \mathbb{F}^\bullet_s)$ is a $R-$Hodge structure of weight n. (Cf. Remark 1.2.1)

Let $(\mathcal{V}, \mathcal{F}^\bullet)$ be a $\mathbb{Q}-$variation of Hodge structure of weight n over a complex connected manifold S. For each $s \in S$ we have a Mumford-Tate group $\mathrm{MT}_s \subset GL(\mathbb{V}_s)$. By the considerations of the previous section, understanding how this group varies with s is the same as studying how Hodge classes in tensor constructions $T^{d,e}(\mathbb{V}_s)$ vary with s.

Let $\pi : \tilde{S} \to S$ be the universal cover of S and set $(\tilde{\mathcal{V}}, \tilde{\mathcal{F}^\bullet}) := \pi^*(\mathcal{V}, \mathcal{F}^\bullet)$. Then $(\tilde{\mathcal{V}}, \tilde{\mathcal{F}^\bullet})$ is a $\mathbb{Q}-$variation of Hodge structure on $\tilde{S}$ and, since $\tilde{S}$ is simply connected, $\tilde{\mathcal{V}}$ is a constant local system. We thus have a trivialization $\tilde{\mathbb{V}} \xrightarrow{\sim} V \times \tilde{S}$, where V is a finite and free $R-$module. Using this trivialization we may view the Mumford-Tate group MT_x of a point $x \in \tilde{S}$ as an algebraic subgroup of $GL(V)$. In other words, by going to the universal cover we can describe the Mumford-Tate groups MT_x as subgroups of the same group. Given $d, e \in \mathbb{N}^m$, we get a variation of Hodge structures $T^{d,e}(\tilde{\mathbb{V}})$ on $\tilde{S}$. Denote by $\{\mathbb{F}^p(T^{d,e}(\tilde{\mathbb{V}}))\}$ its associated filtration. For $t \in T^{d,e}(V)$, set

$$\begin{aligned} Y(t) = \{x \in \tilde{S} :\ & t \text{ is a (0,0)-Hodge class for } T^{d,e}(\tilde{\mathbb{V}})_x\} = \\ &= \{x \in \tilde{S} : t \in \mathbb{F}^0(T^{d,e}(\tilde{\mathbb{V}}))_x\}. \end{aligned}$$

As $\mathbb{F}^0(T^{d,e}(\tilde{\mathbb{V}})) \subset T^{d,e}(V)_{\mathbb{C}}$ is a holomorphic subbundle, $Y(t) \subset \tilde{S}$ is an analytic subset, hence a countable union of irreducible analytic subsets of $\tilde{S}$. Set

$$\tilde{\Sigma} := \bigcup_{t:\ Y(t) \neq \tilde{S}} Y(t).$$

It is a countable union of proper analytic subsets of $\tilde{S}$. By definition, Hodge classes are constant on $\tilde{S} - \tilde{\Sigma}$. Hence we get

Proposition 1.2.14. *Outside of a countable union of analytic subvarieties of $\tilde{S}$, the Mumford-Tate group is constant.*

The subgroup of $GL(V)$ thus obtained is called **generic Mumford-Tate group** and is denoted by MT^{gen}. Note that, for any $x \in \tilde{S}$, we have an inclusion $\mathrm{MT}_x \subset \mathrm{MT}^{gen}$. The subset $\tilde{\Sigma} \subset \tilde{S}$ is stable under the action of the covering group of $\pi : \tilde{S} \to S$ and therefore defines a subset $\Sigma \subset S$. It follows from the construction that Σ is a countable union of proper analytic subspaces of S.

Proposition 1.2.15. *Outside of a countable union of analytic subvarieties of S, the Mumford-Tate group* MT_s *is locally constant.*

Definition 1.2.16. *A point $s \in S - \Sigma$ is called* ***Hodge generic***.

By definition, $Y(t)$ is the locus of points $x \in \tilde{S}$, where t is an Hodge class for the Hodge structure at the point x. By considering the locus in $\tilde{S}$ of points where some given $t_1, ..., t_r$, taken in various tensor constructions, are Hodge classes, one gets the following definition.

Definition 1.2.17. *An analytic subspace $Z \subset S$ is called a* ***Hodge locus*** *for the variation of Hodge stucture $(\mathcal{V}, \mathcal{F}^\bullet)$ if there exist nonzero classes $t_1, ..., t_r$ in some tensor construction $T^{d,e}(V)$ such that Z is an irreducible component of the image of $Y(t_1) \cap ...Y(t_r)$ in S.*

A Hodge locus is hence the image in S of an irreducible component of the locus in $\tilde{S}$ of points where some given $t_1, ..., t_r$ are Hodge classes.

1.3 Siegel space

We dedicate this section to recalling the definition and some properties of the Siegel upper-half space, which will play a key role in the thesis.

Lemma 1.3.1. *Let (V, Q) be a real symplectic vector space and $J \in EndV$ with $J^2 = -id_V$. The following are equivalent:*

1. *$J^*Q = Q$, i.e. $Q(Jx, Jy) = Q(x, y)\ \forall x, y \in V$;*

2. *g_J, defined as $g_J(x, y) := Q(x, Jy)$, is symmetric.*

Definition 1.3.2. *The Siegel space associated with the real symplectic vector space (V, Q) is*

$$\mathfrak{S}(V, Q) = \{J \in EndV :\ J^2 = -id_V,\ J^*Q = Q,\ g_J \text{ is positive definite}\}.$$

In the following, we recall the main properties of the Siegel space. In particular, we see it is a hermitian symmetric space of the non-compact type and recall its Cartan decomposition.

Given a complex structure $J \in End(V)$, we denote by V_J the complex vector space obtained using V as underlying real vector space and letting multiplication by i act as J. If $J \in \mathfrak{S}(V, Q)$, since g_J is positive definite, $H_J(x, y) := g_J(x, y) - iQ(x, y)$ is a Hermitian product on V_J.

Proposition 1.3.3. *$Sp(V, Q)$ acts transitively by conjugation on $\mathfrak{S}(V, Q)$.*

Proof. See e.g. [22, Section 3.2] □

Note that, if $a \in Sp(V, Q)$ and $J \in \mathfrak{S}(V, Q)$, then $a.J = J$ if and only if $a^*H_J = H_J$. Thus the stabilizer of $J \in \mathfrak{S}(V, Q)$ is $Sp(V, Q)_J = U(V_J, H_J) = O(V, g_J) \cap Sp(V, Q)$ and $\mathfrak{S}(V, Q) \cong Sp(V, Q)/U(V_J, H_J)$.

For $L \in End(V)$ and $J \in \mathfrak{S}(V, Q)$, denote by L^{T_J} the trasposed operator with respect to the scalar product g_J.

Lemma 1.3.4. *If $a \in Sp(V, Q)$, then $a^{T_J} = Ja^{-1}J^{-1} = -Ja^{-1}J$.*

In particular, if $a \in Sp(V,Q)$ and $J \in \mathfrak{S}(V,Q)$, then also $a^{T_J} \in \mathrm{Sp}(V,Q)$. Let us consider the involution

$$\sigma_J : Sp(V,Q) \to Sp(V,Q), \quad \sigma_J(a) = (a^{-1})^{T_J} = JaJ^{-1}. \tag{1.3.1}$$

The subgroup of $Sp(V,Q)$ of the elements fixed by σ_J is $Sp(V,Q)^{\sigma_J} = \{a \in Sp(V,Q) : \; JaJ^{-1} = a\} = U(H_J, V_J)$. By Proposition 1.1.6, this proves that $\mathfrak{S}(V,Q) \cong Sp(V,Q)/U(V_J,H_J)$ is a symmetric space.

At the level of the Lie algebra, $d\sigma_J$ is the involution

$$d\sigma_J : \mathfrak{sp}(V,Q) \longrightarrow \mathfrak{sp}(V,Q), \quad d\sigma_J(X) = -X^{T_J}.$$

We get that the Cartan decomposition associated with the Siegel space is $\mathfrak{g} := \mathfrak{sp}(V,Q) = \mathfrak{k} \oplus \mathfrak{p}$, with

$$\mathfrak{k} = \mathfrak{sp}^{d\sigma_J} = \{X \in \mathfrak{sp}(V,Q) : \; X = -X^{T_J}\} = \mathfrak{sp}(V,Q) \cap \mathfrak{o}(V, g_J);$$
$$\mathfrak{p} = \{X \in \mathfrak{sp}(V,Q) : \; X = X^{T_J}\} = \mathfrak{sp}(V,Q) \cap \Sigma,$$

where $\Sigma = \{L \in End(V) : \; L = L^{T_J}\}$. Now, recall that the Killing form of the Lie algebra $\mathfrak{sp}(V,Q)$ is the bilinear symmetric form $B(X,Y) = (2g+2)tr(XY)$ (see [33, p. 190]). Since the trace operator is symmetric and positive definite on Σ, we get that $B|_{\mathfrak{p}}$ is also positive definite, and thus that the Siegel space is of the non-compact type. Finally, $\hat{I} := (1/2)\mathrm{ad}J$ is a complex structure on $T_J\mathfrak{S}$ that makes the Siegel space Hermitian symmetric.

1.3.1 Siegel space as bounded symmetric domain

Let R be a ring with $\mathbb{Z} \subseteq R \subseteq \mathbb{R}$. Recall that

Proposition 1.3.5. *There is a bijection between Hodge structures of type $(-1,0),(0,-1)$ on a finititely generated and free $R-$module V and complex structures on $V_{\mathbb{R}}$.*

$R-$Hodge structures of type $(-1,0),(0,-1)$ on V	$\longleftrightarrow$	*Complex structures on $V_{\mathbb{R}}$*

Proof. Given a complex structure $J \in End(V_{\mathbb{R}})$, $J^2 = -I_{V_{\mathbb{R}}}$, denoted by $V_{-1,0}(J)$ and $V_{0,-1}(J)$ the $\pm i$- eigenspaces of J, we get a decomposition

$$V_{\mathbb{C}} = V_{-1,0}(J) \oplus V_{0,-1}(J) \text{ such that } V_{0,-1}(J) = \overline{V_{-1,0}(J)}$$

and hence a Hodge strucure on V as stated. Conversely, given a decomposition $V_{\mathbb{C}} = V_{-1,0} \oplus V_{0,-1}$ with $V_{0,-1} = \overline{V_{-1,0}}$, one can define $J : V_{\mathbb{C}} \to V_{\mathbb{C}}$ as the multiplication by i on $V_{-1,0}$ and as the multiplication by $-i$ on $V_{0,-1}$. Since J commutes with conjugation $J(H_{\mathbb{R}}) \subset H_{\mathbb{R}}$ and thus one gets a complex structure on $H_{\mathbb{R}}$. □

Now let (V, Q) be a real symplectic vector space. The above preposition says that every complex structure on V (in particular, every element in $\mathfrak{S}(V, Q)$) indentifies a decomposition $V_{\mathbb{C}} = V_{-1,0}(J) \oplus V_{0,-1}(J)$ such that $V_{0,-1}(J) = \overline{V_{-1,0}(J)}$ and viceversa. Define

$$h_Q : V_{\mathbb{C}} \times V_{\mathbb{C}} \longrightarrow \mathbb{C}, \quad h_Q(u, v) := -iQ(u, \bar{v}).$$

Lemma 1.3.6. *Let $J \in EndV$ with $J^2 = -id_V$. The following properties hold:*

1. *$J^*Q = Q$ if and only if $V_{0,-1}(J)$ is lagrangian;*
2. *If $J^*Q = Q$, then g_J is positive definite if and only if $h_Q|_{V_{0,-1}(J)}$ is negative definite.*

Set

$$\Omega' := \{W \in \mathbb{G}(g, V_{\mathbb{C}}) : W \cap \overline{W} = \{0\}, Q|_W = 0, h_Q|_W \ll 0\}$$

Ω' is an open subset of the Lagrangian Grassmannian, which is the compact dual of the Siegel space. The previous Lemma proves that

Proposition 1.3.7. *The map*

$$F : \mathfrak{S} \to \Omega', \quad F(J) := V_{0,-1}(J)$$

is a biholomorphism.

Let $V = L \oplus L'$ be a decomposition in Lagrangian subspaces. Denote by p_L, $p_{L'}$ the projections. Since L is real, we have $h_Q|_{L_{\mathbb{C}}} = h_Q|_{L'_{\mathbb{C}}} = 0$. Hence for $W \in \Omega'$ we have $W \cap L_{\mathbb{C}} = W \cap L'_{\mathbb{C}} = \{0\}$, so W is the graph of the linear map

$$f = p_{L'} \circ (p_L|_W)^{-1} : L_{\mathbb{C}} \to L'_{\mathbb{C}}.$$

This means that Ω' is contained in the domain of the holomorphic chart of $\mathbb{G}(g, V_{\mathbb{C}})$ corresponding to the decomposition $V_{\mathbb{C}} = L_{\mathbb{C}} \oplus L'_{\mathbb{C}}$. Now given $W = \Gamma_f \in \Omega'$ set

$$B_f : V_{\mathbb{C}} \times V_{\mathbb{C}} \longrightarrow \mathbb{C}, \quad B_f(u, v) := Q(u, fv).$$

Set

$$\mathscr{H} := \{B \in S^*L^*_{\mathbb{C}} : (\operatorname{Im} B)|_L \gg 0\}.$$

Proposition 1.3.8. *The map*

$$\Phi : \Omega' \longrightarrow \mathscr{H}, \quad W = \Gamma_f \mapsto B_f$$

is a biholomorphism.

Remark 1.3.9. Notice that the map $W = \Gamma_f \mapsto f$ also yields a chart of the Lagrangian Grassmannian $GLag(g, V_{\mathbb{C}}, Q) = \{W \subset V_{\mathbb{C}}$ con $\dim W = g$ e $Q|_W \equiv 0\}$ that maps to $\mathrm{Hom}^S(L_{\mathbb{C}}, L'_{\mathbb{C}}) = \{f \in \mathrm{Hom}(L_{\mathbb{C}}, L'_{\mathbb{C}}) : B_f$ is symmetric$\}$.

Proof. Let $\{e_1, \dots, e_{2g}\}$ be a symplectic basis of (V, Q), i.e.

$$Q(e_i, e_j) = Q(e_{g+i}, e_{g+j}) = 0, \quad Q(e_i, e_{g+J}) = \delta_{ij}$$

for any $i, j \leq g$. (I.e.

$$Q = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$$

in this basis.) If $W = \Gamma_f \in \Omega'$, $f(e_j) = \sum_{k=1}^{g} z_{kj} e_{g+k}$ and $B_f(e_i, e_j) = z_{ij}$. Thus $Z = Z^T$ and $\mathrm{Im}\, Z \gg 0$. □

Remark 1.3.10. By Propositions 1.3.7 and 1.3.8, and by fixing a symplectic basis as in the proof of Proposition 1.3.8, we get that the Siegel space $\mathfrak{S}(V, \omega)$ is biholomorphic to the space $\mathcal{H}_g$ of symmetric complex $g \times g$ matrices $Z = X + iY$ with positive-definite imaginary part Y (where $2g = \dim V$). This is the most common definition of the Siegel upper-half space. The map $Z = (z_{ij}) \mapsto (z_{ij})_{j \geq i}$ identifies $\mathcal{H}_g$ with an open subset of $\mathbb{C}^{g(g+1)/2}$. Finally, let $\mathcal{D}_g$ be the set of symmetric complex matrices such that $I_g - \overline{Z}^t Z$ is positive-definite. The map $(z_{ij}) \mapsto (z_{ij})_{j \geq i}$ idenfities $\mathcal{D}_g$ as a bounded domain in $\mathbb{C}^{g(g+1)/2}$, and $Z \mapsto (Z - iI_g)(Z + iI_g)^{-1}$ permits to identify $\mathcal{H}_g$ to $\mathcal{D}_g$. Therefore $\mathfrak{S}(V, \omega)$ is a bounded symmetric domain.

1.3.2 Siegel space and abelian varieties

Let $H \simeq \mathbb{Z}^{2g}$ be a free $\mathbb{Z}-$module. Fix an alternating bilinear form

$$Q : H \times H \to \mathbb{Z}.$$

By definition, Q is a polarization for a Hodge strucure of type $(-1, 0), (0, -1)$ on H if and only if, denoted by $h : H_{\mathbb{C}} \times H_{\mathbb{C}} \to \mathbb{C}$ the hermitian form defined as $h(a, b) = -iQ(a, \bar{b})$, the following conditions are satisfied (cf. Definition 1.2.4):

1. $H_{-1,0}$ and $H_{0,-1}$ are $h-$orthogonal.
2. The form h is positive definite on $H_{-1,0}$ and is negative definite on $H_{0,-1}$.

Proposition 1.3.11. *The form Q is a polarization for the Hodge structure associated with a complex structure J on $H_{\mathbb{R}}$ if and only if:*

i. $J^*Q = Q$, that is, $Q(Ju, Jv) = Q(u, v)$ for all $u, v \in H_{\mathbb{R}}$;

ii. $Q(u, Ju) > 0$ *for all* $0 \neq u \in H_{\mathbb{R}}$.

Proof. See e.g. [52, p. 23] □

Note that a complex structure $J \in End(H_{\mathbb{R}})$ satisfies condition *i.* if and only if the bilinear form $g_J := Q(\cdot, J\cdot)$ is symmetric. The previous considerations thus prove the following:

Proposition 1.3.12. *Let* $H \simeq \mathbb{Z}^{2g}$ *be a free* $\mathbb{Z}-$*module and* $Q : H \times H \to \mathbb{Z}$ *be an alternating bilinear form, then the Siegel space*

$$\mathfrak{S}(H_{\mathbb{R}}, Q) = \{J \in End(H_{\mathbb{R}}),\ J^2 = -id_{H_{\mathbb{R}}},\ J^*Q = Q,\ g_J > 0\}$$

parametrizes $\mathbb{Z}-$*Hodge structures of type* $(-1, 0), (0, -1)$ *on* H *with polarization* Q.

Now recall that

Proposition 1.3.13. *There is an equivalence of categories*

$\mathbb{Z}-$*Hodge structures of type* $(-1, 0), (0, -1)$ *on* $H \simeq \mathbb{Z}^{2g}$.	$\longleftrightarrow$	*Complex tori of complex dimension* g.

Proof. A Hodge structure of type $(-1, 0), (0, -1)$ on H corresponds to a complex structure J on $H_{\mathbb{R}}$, which thus inhearites the structure of a $\mathbb{C}-$vector space. Since H is a full rank lattice in $H_{\mathbb{R}}$, we get that $(H_{\mathbb{R}}/H, J)$ is a complex torus. Conversely, let $X = V/\Lambda$ be a complex torus, where V is a complex vector space of dimension g and $\Lambda \simeq H_1(X, \mathbb{Z}) \simeq \mathbb{Z}^{2n}$ is a full rank lattice. Then $\Lambda \otimes \mathbb{C} \simeq V \oplus \overline{V}$ is a Hodge structure of type $(-1, 0), (0, -1)$. □

In terms of the complex torus $X = (H_{\mathbb{R}}/H, J)$ associated with the Hodge structure (H, J), the existence of a form $Q : H \times H \to \mathbb{Z}$ which satisfies *i.* and *ii.* in Proposition 1.3.11 corresponds precisely to the fact that Q satisfies Riemann's bilinear relations and hence that X is an abelian variety with polarization (in the sense of the theory of abelian variety) Q. In other words

Proposition 1.3.14. *The map* $(X, Q) \mapsto (H_1(X, \mathbb{Z}), Q)$ *gives an equivalence of categories*

Polarized $\mathbb{Z}-$*Hodge structures of type* $(-1, 0), (0, -1)$ *on* $H \simeq \mathbb{Z}^{2g}$.	$\longleftrightarrow$	*Polarized complex abelian varieties of dimension* g.

In an appropriate basis of H, the matrix of Q has the form

$$\begin{pmatrix} 0 & T \\ -T & 0 \end{pmatrix}, \qquad \text{with } T = \operatorname{diag}(d_1, ..., d_g)$$

and $0 < d_1|d_2|...|d_g$ (see e.g. [21, p. 391]). The vector $(d_1, ..., d_g)$ is called the *type* of the form Q. We assume from now on that Q has type $(1, ..., 1)$. In this case Proposition 1.3.14, together with Proposition 1.3.12, imply

Proposition 1.3.15. *Let $H \simeq \mathbb{Z}^{2g}$ be a free $\mathbb{Z}-$module and $Q : H \times H \to \mathbb{Z}$ be an alternating bilinear form of type $(1, ..., 1)$, then the Siegel space*

$$\mathfrak{S}(H_{\mathbb{R}}, Q) = \{J \in End(H_{\mathbb{R}}),\ J^2 = -id_{H_{\mathbb{R}}},\ J^*Q = Q,\ g_J > 0\}$$

parametrizes principally polarized abelian varieties of dimension g over $\mathbb{C}$.

For $J \in \mathfrak{S}(H_{\mathbb{R}}, Q)$, the corresponding principally polarized abelian variety is simply $A_J = (T_J, Q)$, where T_J is the real torus $T := H_{\mathbb{R}}/H$ equipped with the complex structure J, and Q is the polarization.

The symplectic group $\Gamma := Sp(H, Q)$ is a discrete subgroup of $Sp(H_{\mathbb{R}}, Q)$ and acts properly discontinuously on $\mathfrak{S}(H_{\mathbb{R}}, Q)$. Two principally polarized abelian varieties $A_J = (T_J, Q)$ and $A_{J'} = (T_{J'}, Q)$ with $J, J' \in \mathfrak{S}(H_{\mathbb{R}}, Q)$ are isomorphic if and only if there exists a biholomorphism $f : T_J \to T_{J'}$ that preserves the polarization, i.e. if and only if A_J and $A_{J'}$ are in the same orbit under the action of Γ. Set

$$\mathsf{A}_g := \Gamma \backslash \mathfrak{S}(H_{\mathbb{R}}, Q). \tag{1.3.2}$$

By our considerations, A_g parametrizes isomorphism classes of principally polarized abelian varieties. By a theorem of H. Cartan this quotient is a normal complex analytic space. It has also a natural structure of complex analytic orbifold. Moreover, it follows from the theorem of Baily and Borel that A_g is a quasiprojective variety. Since Γ is a group of isometries, the symmetric Riemannian metric on $\mathfrak{S}(H_{\mathbb{R}}, Q)$ induces a locally symmetric orbifold metric on A_g, which we call *Siegel metric.*

Remark 1.3.16. As we have seen, we can associate to every $J \in \mathfrak{S}(H_{\mathbb{R}}, Q)$ a principally polarized abelian variety $A_J = (T_J, Q)$, where T_J is the real torus $T := H_{\mathbb{R}}/H$ equipped with the complex structure J, and Q is the polarization. Indeed, what is true is that over $\mathfrak{S} = \mathfrak{S}(H_{\mathbb{R}}, Q)$ we have a universal family of abelian varieties: $\pi : \mathfrak{T} \to \mathfrak{S}$ (see e.g. [41, Chapter 8, Section 8.7]). The fibre being precisely the complex torus $T_J := \pi^{-1}(J)$ with the polarization Q. This is a family of Kähler manifolds and the Siegel space is thus naturally endowed with a polarised variation of the Hodge structure on $R^1\pi_*\mathbb{Q}$. Now $H^1(T, \mathbb{C}) = H^*_{\mathbb{C}}$ and $H^{1,0}(T_J) = \operatorname{Ann} H_{0,-1}(J)$. Hence, if we identify $\mathfrak{S}$ with Ω' as in Proposition 1.3.7, the corresponding period map is

$$\begin{array}{c} \mathfrak{T} \\ \pi \big\downarrow \\ \mathfrak{S} \end{array} \qquad \mathscr{P} : \mathfrak{S} \longrightarrow \mathbb{G}(g, H^1(T, \mathbb{C})), \quad \mathscr{P}(W) = \operatorname{Ann} W.$$

Set $\mathbf{H}_{\mathbb{C}} := R^1\pi_*\mathbb{C} \to \mathfrak{S}$. We consider $\mathbf{H}_{\mathbb{C}}$ as a holomorphic vector bundle, i.e. $\mathbf{H}_{\mathbb{C}} = \mathfrak{S} \times H^*_{\mathbb{C}}$. Then the Hodge bundle $\mathbf{F}^1 \subset \mathbf{H}_{\mathbb{C}}$ is the subbundle with fiber $\mathbf{F}^1_J = H^{1,0}(T_J) = \operatorname{Ann}(H_{0,-1}(J)) \subset H^1(T, \mathbb{C}) = H^*_{\mathbb{C}}$. This variation of Hodge structure descends to an (orbifold) variation over A_g. We will study with more detail this variation of Hodge structure in the next section.

1.4 Totally geodesic and Shimura subvarieties in A_g

We dedicate this section to recall the definitions and the relations between totally geodesic and Shimura subvarieties in A_g.

1.4.1 Totally geodesic subvarieties

Let

$$\pi : \mathfrak{S} \to \mathsf{A}_g$$

denote the canonical projection.

Definition 1.4.1. *A totally geodesic subvariety of A_g is a closed algebraic subvariety $Z \subset \mathsf{A}_g$, such that $Z = \pi(X)$ for some connected totally geodesic submanifold $X \subset \mathfrak{S}(V, Q)$. We say that X is the symmetric space **uniformizing** Z.*

We recall the following characterization of totally geodesic submanifolds in symmetric spaces (see [11, p.19]), that we will use several times in the sequel of the thesis.

Theorem 1.4.2. *Let X be a symmetric space with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and let $X' \subset X$ be a closed connected submanifold of X. Let $p \in X$ and set $G := Isom(X)^0$ and $K = G_p$. The following conditions are equivalent:*

1. *$X' \subset X$ is totally geodesic;*
2. *$X' \subset X$ is a symmetric subspace, i.e. $s_x(X') = X'$ for all $x \in X'$;*
3. *$X' = \exp_p(\mathfrak{m})$, where $\mathfrak{m} \subset \mathfrak{p}$ is a Lie triple system, i.e. $[[\mathfrak{m}, \mathfrak{m}], \mathfrak{m}] \subset \mathfrak{m}$.*

1.4.2 Shimura subvarieties

Let Λ be a $2g$ rank lattice, $Q : \Lambda \times \Lambda \to \mathbb{Z}$ a symplectic form of type $(1, ..., 1)$, and set $V := \Lambda_{\mathbb{R}}$.

We consider on $\mathfrak{S}(V, Q)$ the natural polarized integral variation of Hodge structure, whose fiber over $J \in \mathfrak{S}(V, Q)$ is $V_{\mathbb{C}}$ equipped with the tautological Hodge structure $V_{\mathbb{C}} = V_{-1,0}(J) \oplus V_{0,-1}(J)$. (Note that this is just the dual of the VHS defined in 1.3.16). Since this Hodge structure only depends on J, we denote it simply by J, and we denote by $\rho_J : \mathbb{S}(\mathbb{R}) \to GL(V_{\mathbb{R}})$ its associated representation, that is $\rho_J(z)v = zv$ for $v \in V_{-1,0}(J)$ and $\rho_J(z)v = \overline{z}v$ for $v \in V_{0,-1}(J)$ (cf. Section 1.2.1).

We denote by $\mathrm{MT}(J)$ and by $\mathrm{Hg}(J)$, respectively, the Mumford-Tate group and the Hodge group associated with J. Note that, since Q is a polarization for J, it follows from Proposition 1.3.11 that $\mathrm{Hg}(J) \subset Sp(V, Q)$.

Lemma 1.4.3. *The orbit* $\mathrm{Hg}(J) \cdot J \subset \mathfrak{S}(V, Q)$ *is a complex totally geodesic submanifold of* $\mathfrak{S}(V, Q)$*. With the induced metric it is a Hermitian symmetric space of the non-compact type.*

Proof. See [22, Lemma 3.5]. □

The tautological variation of Hodge structure on $\mathfrak{S}(V, Q)$ descends to an (orbifold) variation over A_g.

Definition 1.4.4. *The* ***special or Shimura subvarieties*** *of* A_g *are defined as Hodge loci for the natural variation of Hodge structure over* $\mathbb{Q}$ *on* A_g*.*

Let us recall from Section 1.2.3 that these are defined as loci in A_g where we have some given collection of Hodge classes. More precisely, following Section 1.2.3, these are costructed as follows. Given $d, e \in \mathbb{N}^m$ and $t \in T^{d,e}(\Lambda_{\mathbb{Q}})$ (cf. (1.2.2)), we consider

$$Y(t) := \{J \in \mathfrak{S}(V, Q) : t \in T^{d,e}(V_J)^{0,0}\}.$$

Since the tensor space $T^{d,e}(V_J)^{0,0}$ is the space of Hodge classes for the Hodge structure at the point J, $Y(t)$ is the locus of points in $\mathfrak{S}$, where t is an Hodge class. It is an analytic subset of $\mathfrak{S}(V, Q)$. If $t_1, ..., t_r$ are rational vectors in various tensor constructions, set $Y(t_1, ..., t_r) := Y(t_1) \cap ... \cap Y(t_r)$. $Y(t_1, ..., t_r)$ is *proper* if $Y(t_1, ..., t_r) \neq \mathfrak{S}(V, Q)$. By definition 1.2.17, a Hodge locus of A_g is an irreducible component of a proper $\pi(Y(t_1, ..., t_r)) \subsetneq \mathsf{A}_g$. It can be shown that Hodge loci are exactly the subset of the form $\pi(Z)$, where Z is an irreducible component of some proper $Y(t_1..., t_r)$. The irreducible components of proper subsets $Y(t_1, ..., t_r)$ form a countable family $\{Z_i\}_{i \in \mathbb{N}}$ of proper subsets of $\mathfrak{S}(V, Q)$. Set

$$Z_i^0 := Z_i \setminus \bigcap_{j : Z_i \not\subseteq Z_j} Z_j$$

Theorem 1.4.5. *For* $J \in Z_i^0$ *we have* $Z_i = \mathrm{Hg}(J) \cdot J$*. In particolar* Z_i *is a totally geodesic submanifold of* $\mathfrak{S}(V, Q)$*.*

Proof. See [22, Theorem 3.7] □

Corollary 1.4.6. *Shimura subvarieties of* A_g *are totally geodesic subvarieties.*

Remark 1.4.7. The simplest Shimura subvarieties of A_g are the Shimura subvarieties of **PEL type**. They are defined as follows (see [15, Section 3] and references therein). Given $J \in \mathfrak{S}(V, Q)$, set

$$\mathrm{End}_{\mathbb{Q}}(A_J) := \{f \in \mathrm{End}_{\mathbb{Q}}(\Lambda_{\mathbb{Q}}) : \ Jf = fJ\}$$

Fix a point $J_0 \in \mathfrak{S}(V, Q)$ and set $D := \mathrm{End}_{\mathbb{Q}}(A_{J_0})$. Consider the locus $Y(D)$ locus of points in $\mathfrak{S}$, where every element in D is an Hodge class

$$Y(D) = \{J \in \mathfrak{S}(V, Q) : \ D \subset \mathrm{End}_{\mathbb{Q}}(A_J)\}.$$

The PEL type Shimura subvariety $\mathsf{Z}(D)$ is defined as the image in A_g of the connected component of $Y(D)$ that contains J_0.

1.4.3 Complex multiplication

Definition 1.4.8. *An abelian variety A over $\mathbb{C}$ is of CM-type if A is isogenous to a product $X_1 \times ... \times X_k$ of simple abelian varieties and there are fields $K_i \subset End(X_i) \otimes_{\mathbb{Z}} \mathbb{Q}$ such that $[K_i : \mathbb{Q}] \geq 2\dim(X_i)$.*

Let A be an abelian variety and $J \in \mathfrak{S}$ such that $A = A_J$. An equivalent characterization of CM-type is (see [49, Section 2]):

Proposition 1.4.9. *An abelian variety A is of CM-type if and only if* $\mathrm{Hg}(J)$ *is a torus algebraic group.*

Definition 1.4.10. *A CM point of A_g is a moduli point $[A]$, where A is an abelian variety of CM-type.*

The notion of complex multiplication allows to characterize Shimura subvarieties of A_g among totally geodesic subvarieties. The following characterization is due to Mumford and Moonen (see [49, 46]).

Theorem 1.4.11. *An algebraic subvariety of A_g is Shimura if and only if it is totally geodesic and contains a CM point.*

1.5 The Torelli map and the geometry of M_g

In this Section we describe the problem that motivates the study of this thesis.

1.5.1 Torelli morphism

Let C be a compact Riemann surface of genus g. By Hodge decomposition theorem, the cohomology group $H^1(X, \mathbb{Z})$ carries a natural integral Hodge structure of type $(1, 0)$, $(0, 1)$:

$$H^1(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \cong H^1(X, \mathbb{C}) = H^{1,0}(X) \oplus H^{0,1}(X)$$

where $H^{p,q}(X)$ is the space of classes rapresentable by a closed form of type (p, q). The skew-symmetric bilinear form

$$Q : H^1(X, \mathbb{Z}) \times H^1(X, \mathbb{Z}) \to \mathbb{Z}, \ Q([\alpha], [\beta]) = \langle [\alpha] \cup [\beta], [X] \rangle = \int_X \alpha \wedge \beta$$

defines a polarization on $H^1(X,\mathbb{C}) = H^{1,0}(X) \oplus H^{0,1}(X)$. Indeed, the induced hermitian form $h : H^1(X,\mathbb{C}) \times H^1(X,\mathbb{C}) \to \mathbb{C}$, $h(\alpha,\beta) = i\int_X \alpha \wedge \overline{\beta}$ satisfies the Riemann's bilinear relations:

1. For $\alpha \in H^{1,0}(X)$ and $\beta \in H^{0,1}(X)$, $h(\alpha,\beta) = 0$. Indeed, $\alpha \wedge \overline{\beta}$ is a $(2,0)$ form on X, which has complex dimension 1.
2. For $0 \neq \alpha \in H^{1,0}(X)$, with local expression $\alpha = f(z)dz$, $h(\alpha,\alpha) = 2\int_X |f|^2 dx \wedge dy > 0$.

One can see that Q is indeed a polarization of type $(1, ..., 1)$. Dualizing, we get a polarized Hodge structure of type $(-1,0)$, $(0,-1)$ on $H_1(X,\mathbb{Z})$.

$$H_1(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \cong H_1(X,\mathbb{C}) = Ann(H^{1,0}(X)) \oplus Ann(H^{0,1}(X)).$$

Let $J(C) \in \mathfrak{S}(H_1(X,\mathbb{R}),Q)$ denote the element of the Siegel space corresponding to $(Ann(H^{1,0}),Q)$ in the sense of Proposition 1.3.7. As above, let $\pi : \mathfrak{S} \to \mathsf{A}_g$ denote the canonical projection. Then $\pi(J(C))$ is the principally polarized abelian variety

$$Jac(X) = H^{1,0}(X)^*/H_1(X,\mathbb{Z}).$$

$Jac(X)$ is called the **Jacobian variety** associated with C.

Definition 1.5.1. *Let M_g denote the moduli space of smooth complex algebraic curves of genus g and A_g the moduli space of principally polarized abelian varieties of dimension g over $\mathbb{C}$. The **Torelli map** is the morphism*

$$j : \mathsf{M}_g \to \mathsf{A}_g$$

which associates to $[C] \in \mathsf{M}_g$ its Jacobian variety $[JC] \in \mathsf{A}_g$.

Similarly to what we saw for A_g, the moduli space M_g is a quasi-projective variety and a complex analytic orbifold. It can be defined as the quotient of the Teichmüller space by the holomorphic and properly discontinuous action of the mapping class group. We recall that the Teichmüller space T_g has a natural structure of a complex manifold which is homeomorphic to the unit ball in $\mathbb{C}^{3g-3}$ (i.e., it is a bounded domain). Thinking M_g and A_g with their natural structures of quasi-projective varieties, j is a regular map. On the other hand, working over the complex numbers, both M_g and A_g can be provided with the structure of complex analytic orbifold, and the map j is an orbifold map, i.e., it lifts to a holomorphic map of the uniformizers. The following fundamental theorem lies at the basis of the problems studied in this thesis.

Theorem 1.5.2 (Torelli)**.** *The map j is injective.*

The problem of understanding the geometry of the image of M_g inside A_g is interesting and classical. We dedicate the following section to discuss some aspects and conjectures related to it.

1.5.2 Geometry of M_g inside A_g

The expectation about the geometry of the inclusion of M_g in A_g is that the way in which M_g sits inside of A_g should be "complicated". One way to make this statement more precise is the following. Recall that on A_g there is a natural metric (called Siegel metric) coming from the unique $Sp(2g, \mathbb{R})$ invariant metric on the Siegel space $\mathfrak{S}_g$. The idea is that the image $j(\mathsf{M}_g) \subset \mathsf{A}_g$ should be very "curved" with respect to this metric. In particular, it should contain very few totally geodesic submanifolds of A_g.

Definition 1.5.3. *A subvariety $Z \subset \mathsf{A}_g$ is generically contained in $j(\mathsf{M}_g)$ if $Z \subset \overline{j(\mathsf{M}_g)}$ and $Z \cap j(\mathsf{M}_g) = \emptyset$.*

Definition 1.5.4. *The closure $\overline{j(\mathsf{M}_g)}$ is called **Torelli locus**.*

Conjecture 1.5.5. *For large genus, there are no positive dimensional totally geodesic subvarieties of A_g generically contained in $j(\mathsf{M}_g)$.*

By the characterization 1.4.11, due to Mumford and Moonen, an algebraic subvariety of A_g is Shimura if and only if it is totally geodesic and contains a CM point. In the light of this, the expectation above agrees with the following conjecture, whose origin is linked to arithmetical considerations

Conjecture 1.5.6 (Coleman-Oort)**.** *For large genus, there are no positive dimensional totally Shimura subvarieties of A_g generically contained in $j(\mathsf{M}_g)$.*

The Conjectures 1.5.5 and 1.5.6 motivate the study of totally geodesic subvarieties of A_g with the aim of obtaining some non-existence results for totally geodesic subvarieties of A_g contained in M_g. This study is still largely open (see e.g. [6, 9, 15, 16, 17, 18, 19, 29, 23, 31, 42, 43, 47, 48] for related results) and motivates the main results presented in this thesis. We dedicate the next section to describe one of the approaches to the problem that will be addressed in the following of the thesis. We finally point out that, for low genus, there are examples of Shimura subvarieties of A_g generically contained in $j(\mathsf{M}_g)$. All the examples known so far are in genus $g \leq 7$ and arise from families of Galois coverings. Chapter 3 deals with these examples and contains more details on their construction.

1.5.3 Second fundamental form of $j(\mathsf{M}_g^*)$

Oort and Steenbrink [50] proved that the restriction of j to the set of non-hyperelliptic curves is an orbifold immersion. In particular, $j(\mathsf{M}_g^*)$ is a complex analytic suborbifold of the Riemannian orbifold A_g. For $x = [X] \in \mathsf{M}_g^*$, we thus have an exact sequence

$$0 \to T_x M_g \stackrel{dj_x}{\to} T_{j(x)} \mathsf{A}_g \stackrel{\pi}{\to} N_x \to 0$$

where N denotes normal bundle of $j(\mathsf{M}_g^*) \subset \mathsf{A}_g$. We get

$$0 \to H^1(X, T_X) \stackrel{dj_x}{\to} S^2H^0(X, K_X)^* \stackrel{\pi}{\to} N_x \to 0. \tag{1.5.1}$$

Denoted by ∇ the Levi-Civita connection of the symmetric metric on A_g induced by the Siegel space, the map

$$\mathrm{II} : S^2T\mathsf{M}_g^* \to N, \quad \mathrm{II}(X \odot Y) = \pi(\nabla_X(Y))$$

is the second fundamental form of the Torelli map with respect to the Siegel metric on A_g. This has been the object of study of [7, 6]. One can express Conjecture 1.5.5 in terms of II by saying that it should be highly non-degenerate, i.e., it should most of the time be non-zero. This aspect is not yet understood. Nevertheless, the study of II has lead to significan results. We briefly recall the ones we will need in the sequel.

For $x = [X] \in \mathsf{M}_g$, consider the exact sequece

$$0 \to I_2(K_X) \to S^2H^0(X, K_X) \stackrel{m}{\to} H^0(X, 2K_X) \to 0$$

where m denotes the multiplication map. This sequence is the dual of the sequence (1.5.1). This comes from the fact that $m = dj_x^*$. It follows that $N_x^* = I_2(K_X)$ and one can identify II_x with

$$\rho_x := \mathrm{II}_x^* : I_2(K_X) \to S^2H^0(X, K_X).$$

Using results from [7], it is proved in [6, Theorem 3.13] an interpretation of ρ_x as multiplication map between spaces of sections on $S = X \times X$. To be more precise, let $\Delta \subset S$ denote the diagonal. One can see $I_2(K_X) \hookrightarrow H^0(S, K_S(-2\Delta))$. The main result on the second fundamental form can be stated as follows: there exists a section $\hat{\eta} \in H^0(S, K_S(2\Delta))$ such that the following diagram commutes

$$\begin{array}{ccc} H^0(K_S(2\Delta)) & \xrightarrow{\cdot\hat{\eta}} & H^0(2K_S) \\ \uparrow & & \uparrow \\ I_2(K_X) & \xrightarrow{\rho_x} & S^2H^0(K_X). \end{array}$$

In other words, the second fundamental form of the Torelli map in the point $[X] \in \mathsf{M}_g$ can be interpreted as the multiplication by $\hat{\eta}$. Thanks to this characterization, in [6] and [17] the authors obtained bounds on the dimension of totally geodesic subvarieties in M_g, which only depends on g. This goes in the direction of the Coleman-Oort conjecture.

Chapter 2 contains the proof of the characterization of another important map (the Lie bracket map on the tangent to M_g) as a multiplication map. The form $\hat{\eta}$ will be object of our interest in Section 2.7. There we will also recall from [6] the definition of $\hat{\eta} \in H^0(X, K_S(2\Delta))$.

Chapter 2

Bergman kernel and period map for curves

The following Chapter is divided into two parts. Both link the Bergman kernel form associated with an algebraic curve with the geometry of the period map $j : \mathsf{M}_g \to \mathsf{A}_g$. The first part studies the restriction to $dj(T\mathsf{M}_g)$ of the Lie bracket on the tangent of A_g. The motivations for this study are described in Section 2.1. In Section 2.2 we recall the definition and give some property of the Bergman kernel form associated with an algebraic curve. Section 2.3 is dedicated to giving a suitable description of the Lie bracket on the tangent of the Siegel upper half-space. In Section 2.4 we prove the first main result of the Chapter, i.e., Theorem 2.4.2 (that is Theorem A in the Introduction). It characterizes the Lie bracket on the tangent to M_g in a moduli point $[C]$ as the multiplication map, between spaces of sections on the algebraic surface $Z := C \times \overline{C}$, by the Bergman kernel form. The material in Sections 2.1 to 2.4 has been published in [24]. Section 2.5 presents an analogue of Theorem 2.4.2 for the triple Lie bracket, which is also interpreted as a multiplication map. We conclude the first part of the Chapter presenting some comments and remarks related to the triple Lie bracket in Section 2.6.

Let $S := C \times C$ and $\Delta \subset S$ be the diagonal. The second part of the Chapter links the Bergman kernel of a curve C with the meromorphic form $\hat{\eta} \in H^0(C, K_S(2\Delta))$ constructed in [6], which governs the second fundamental form of the Torelli map with respect to the Siegel metric. In Section 2.7 we first recall the definition of $\hat{\eta} \in H^0(C, K_S(2\Delta))$. Next we recall from [3] the analysis of its cohomology class: $\hat{\eta}$ is characterized as the unique element (up to multiples) of $H^0(S, K_S(2\Delta))$ with cohomology class in $H^2(S - \Delta)$ of pure type $(1, 1)$. Finally, in Section 2.8 we prove the second main result of the Chapter, i.e., that the Bergman kernel is the harmonic representative of the cohomology class of $\hat{\eta} \in H^0(S, K_S(2\Delta))$ in $H^2(S - \Delta)$. (See Theorem 2.8.2 / Theorem B in the Introduction). The material in Sections 2.7 and

2.8 is again in [24].

2.1 Motivations

Let X be a Riemannian symmetric space. Let us recall from Chapter 1 (Section 1.1) that, for a fixed point $x \in X$ we have $X = G/K$, where G is a Lie group (independent of x) and $K = G_x$ is the stabilizer of x. Moreover there is a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ such that $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$ and $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$. Since $\mathfrak{p} \cong T_x X$, the Lie bracket on $\mathfrak{g}$ gives rise to a kind of operation

$$B_x : T_x X \times T_x X \cong \mathfrak{p} \times \mathfrak{p} \to \mathfrak{k} = \mathfrak{g}_x. \tag{2.1.1}$$

Let $\mathbf{S} = G \times_K \mathfrak{k}$ be the homogeneous bundle over X corresponding to the adjoint representation of K. Then $\mathbf{S}_x = \mathfrak{g}_x$ for any x, so B is a section of $\Lambda^2 T^* X \otimes \mathbf{S}$. Since the differential geometry of X can be studied by means of Lie theory, the tensor B, which reflects Lie bracket, is of central importance.

Since the tensor B is invariant by the action of G it makes sense also on any locally symmetric space X. In this Chapter we consider the case where X is A_g, the moduli space of principally polarized abelian varieties of dimension g over $\mathbb{C}$, which is a locally symmetric space obtained as a quotient of the Siegel upper half-space $\mathfrak{S}_g$. Denote by M_g the moduli space of curves of genus g and consider the Torelli map $j : \mathsf{M}_g \to \mathsf{A}_g$. As explained in Section 1.5, we are interested in the study of totally geodesic subvarieties of A_g in relation with the Conjectures 1.5.5 and 1.5.6. Since the tensor B controls the local geometry of A_g, its pull-back

$$\mathbf{B} := j^* B \tag{2.1.2}$$

to M_g should give important information on the extrinsic geometry of the inclusion $j(\mathsf{M}_g) \subset \mathsf{A}_g$. This Chapter is dedicated to the study of the pull-back tensor $\mathbf{B}$.

To be more precise, the idea of studying $\mathbf{B}$ has its motivation in the Characterization 1.4.2 of totally geodesic submanifolds in symmetric spaces, which says that totally geodesic submanifolds in a symmetric space X are exactly those submanifolds of the form $\exp(\mathfrak{a})$, where $\mathfrak{a} \subset \mathfrak{p}$ is a Lie triple system, i.e. $[[\mathfrak{a}, \mathfrak{a}], \mathfrak{a}] \subset \mathfrak{a}$. Since we are interested in totally geodesic submanifolds of A_g contained in the Torelli locus, this result led us to the study of the Lie bracket on the tangent space to A_g on vectors that are image, through the differential of the Torelli map, of vectors tangent to M_g. In the light of the Coleman-Oort conjecture, one would like to obtain some non-existence results for totally geodesic subvarieties of A_g contained in M_g. The study of $\mathbf{B}$ should give constraints on the existence of Lie triples tangent to M_g. Sections 2.5 and 2.6 are dedicated to some considerations in this direction.

2.2 Bergman kernel

2.2.1 Definition

Let C be a smooth complex projective curve of genus $g \geq 1$. Set $S := C \times C$ and let $p, q : S \to C$ be the projections $p(x, y) = x$, $q(x, y) = y$. Let $\bar{C}$ denote the conjugate variety and set $Z := C \times \bar{C}$. Z coincides with S as a real manifold, but has a different complex structure. The projections $p : Z \to C$, $q : Z \to \bar{C}$ are holomorphic. Denote by h the Hodge Hermitian product on $H^0(C, K_C)$, defined by

$$h(\alpha, \beta) := i \int \alpha \wedge \overline{\beta}.$$

Definition 2.2.1. *Let $\omega_1, ..., \omega_g$ be a unitary basis for $H^0(C, K_C)$. Then*

$$\mathbf{K} := \sum_{j=1}^{g} p^* \omega_j \wedge q^* \overline{\omega}_j$$

*is a well-defined $(1, 1)-$form on S independent of the choice of the unitary basis. It is called the **Bergman kernel form** of the algebraic curve C.*

This is the definition of the Bergman kernel form on an arbitrary complex manifold due to Kobayashi [36]. It generalizes the classical Bergman kernel on open domains in $\mathbb{C}^n$. $\mathbf{K}$ can also be seen as a holomorphic 2-form on Z. In particular it is a harmonic form with respect to any Kähler metric on Z. If we consider it as a (1,1)-form on S, it is harmonic for any Kähler metric on S which is Kähler also on Z. In particular it is harmonic for any product metric.

If $x, y \in C$, $T_{(x,y)}S = T_x C \oplus T_y C$. Thus elements of $T_{(x,y)}S$ are pairs (u, v) with $u \in T_x C$ and $v \in T_y C$. Since $\mathbf{K}$ is a (1,1)-form, its behaviour is controlled by the values $\mathbf{K}((u, 0), (0, \overline{v}))$ for $u \in T_x^{1,0} C$, $v \in T_y^{1,0} C$.

Remark 2.2.2. Although not needed in the following, it is interesting to point out the following relation between the Bergman kernel and the period matrix associated to the algebraic curve C (cf. [58, eq. (2.4)]). Let Q denote the intersection form on $H_1(C, \mathbb{Z})$ and let $\{a_i, b_i\}$ be a symplectic basis for $(H_1(C, \mathbb{Z}), Q)$. Consider a basis $\omega_1, ..., \omega_g$ of $H^0(C, K_C)$ normalized with respect to $\{a_i, b_i\}$ and the period matrix $Z = (z_{ij})$ with $z_{ij} = \int_{b_j} \omega_i$. Then, with respect to this basis, the Bergman kernel has the form

$$\mathbf{K} = \frac{1}{2} \sum_{i,j} (\operatorname{Im} Z)^{ij} \, p^* \omega_j \wedge q^* \overline{\omega}_j \tag{2.2.1}$$

where $(\operatorname{Im} Z)^{ij}$ denote the coefficients of $(\operatorname{Im} Z)^{-1}$.

To check this observe first that $h(\omega_i, \omega_j) = 2\operatorname{Im} z_{ij}$. Indeed let $\mathscr{B} = \{a_i^*, b_j^*\}$ be the dual basis. If $D : H^1(C) \to H_1(C)$ is Poincaré duality, then $Da_i^* = b_i$ and $Db_i^* = -a_i$, so $\mathscr{B}$ is symplectic for $Q^*(\alpha, \beta) = \int_C \alpha \cup \beta$. Since $\omega_i = a_i^* + \sum_{k=1}^n z_{ik} b_k^*$, the result follows.

Now (2.2.1) is a consequence of the following general fact: if $\alpha_1, ..., \alpha_g$ is a basis of $H^0(C, K_C)$ and A is the matrix with entries $a_{ij} := h(\alpha_i, \alpha_j)$, then $\mathbf{K} = \sum_{i,j} a^{ij} p^*\alpha_i \wedge q^*\overline{\alpha}_j$.

2.2.2 Bergman kernel and elementary potentials

Next we show how to recover the Bergman kernel using the so-called elementary potentials. Let (U, z) be a chart centered at $x \in C$ and set $u = \frac{\partial}{\partial z}(x)$. Classical results ensure the existence of a harmonic function $f_u \in C^\infty(C - \{x\})$ such that $f_u = -\frac{1}{z} + g(z)$ on $U - \{x\}$ for some $g \in C^\infty(U)$. The function f_u is unique up to an additive constant and it is called *elementary potential* (see [6, §3]). We recall some of its properties that will be relevant in our analysis. It follows from the definition that $\bar\partial f_u$ is smooth on C and that

$$\int_C \omega \wedge (-\bar\partial f_u) = 2\pi i \omega(u), \quad \text{for } \omega \in H^0(C, K_C). \tag{2.2.2}$$

(See [6, Section 3] for more details.) This shows that elementary potentials are related to evaluation and the canonical map. Therefore they are clearly related to Bergman kernel as we show now.

For $x \in C$ and $u \in T_x C$, let

$$\operatorname{ev}_u : H^0(C, K_C) \to \mathbb{C}, \qquad \operatorname{ev}_u(\omega) = \omega_x(u)$$

be the evaluation map and let $k_u \in H^0(K_C)$ be such that

$$\operatorname{ev}_u = h(\cdot, k_u). \tag{2.2.3}$$

Lemma 2.2.3. *For $u \in T_x^{1,0}C$, $v \in T_y^{1,0}C$, with $x, y \in C$, we have*

$$\mathbf{K}((u, 0), (0, \overline{v})) = h(k_v, k_u) = k_v(u).$$

Proof. Let $\omega_1, ..., \omega_g$ be a unitary basis for $H^0(K_C)$. Then $k_v = \sum_j \lambda_j \omega_j$, with $\lambda_j = h(k_v, \omega_j) = \overline{\omega_j(v)}$. Thus $h(k_v, k_u) = k_v(u) = \sum_j \overline{\omega_j(v)} \omega_j(u) = \mathbf{K}((u, 0), (0, \overline{v}))$. □

Lemma 2.2.4. *Let $x, y \in C$, and $u \in T_x^{1,0}C$, $v \in T_y^{1,0}C$. If f_u is an elementary potential, then $\bar\partial f_u = 2\pi \overline{k_u}$. In particular*

$$\mathbf{K}((u, 0), (0, \overline{v})) = \frac{1}{2\pi} \bar\partial f_u(\overline{v}).$$

Proof. From (2.2.2) and (2.2.3) it follows that $\int_C \omega \wedge (\bar\partial f_u) = -2\pi i \omega(u) = -2\pi i\ h(\omega, k_u) = 2\pi \int_C \omega \wedge \overline{k_u}$. So $\bar\partial f_u$ and $2\pi \overline{k_u}$ have the same cohomology class. Since both are harmonic they coincide. Next by the previous Lemma $\mathbf{K}((u,0),(0,\overline{v})) = h(k_v, k_u) = \overline{h(k_u, k_v)} = \overline{k_u(v)} = \overline{k_u}(\overline{v}) = \frac{1}{2\pi}\bar\partial f_u(\overline{v})$. □

2.3 Lie bracket

In this section we study the Lie bracket i.e. the tensor B on Siegel space, introduced in (2.1.1), and prove Theorem A. We start by recalling something about Siegel upper half-space. Next we go through several identifications of the tangent space to $\mathfrak{S}_g$ and write down B in terms of them (Proposition 2.3.5). Given a curve, we apply this to its Jacobian, i.e. we consider $\mathbf{B}$ as in (2.1.2) (Proposition 2.4.1). We recall further identifications using the conjugate curve and the surface $Z = C \times \bar{C}$. This allows to understand the dual map $\mathbf{B}^*$ as a map $H^0(Z, K_Z) \to H^0(Z, 2K_Z)$. Finally using the elementary potentials we prove our main result Theorem 2.4.2.

Let (V, Q) be a real symplectic vector space and consider the Siegel space

$$\mathfrak{S} := \mathfrak{S}(V,Q) := \{J \in \operatorname{End} V : J^2 = -I_V, J^*Q = Q, g_J \text{ is positive def.}\}.$$

For every J we denote $V_{-1,0}(J)$ and $V_{0,-1}(J)$ the $\pm i$-eigenspaces of J on $V_{\mathbb{C}}$. We also set

$$H^{1,0}_J := \operatorname{Ann} V_{0,-1} \quad H^{0,1}_J := \operatorname{Ann} V_{-1,0}(J).$$

We usually drop J in the notation. When $V = \mathbb{R}^{2g}$ and Q is the standard form, we write $\mathfrak{S}_g$. Recall that the symplectic group $\mathrm{Sp} := \mathrm{Sp}(V,Q)$ acts on $\mathfrak{S}$ by conjugation. This action is transitive and $\mathfrak{S}$ is a Hermitian symmetric space. For $X \in \operatorname{End} V$ set $Q_X := Q(\cdot, X\cdot)$. Then $\mathfrak{sp} = \mathfrak{sp}(V,Q) = \{X \in \operatorname{End} V : Q_X \text{ is symmetric}\}$. If $\mathfrak{sp} := \mathfrak{sp}_J \oplus \mathfrak{p}$ is the Cartan decomposition at $J \in \mathfrak{S}$, then

$$\mathfrak{p} = \{X \in \mathfrak{sp} : XJ + JX = 0\}, \quad \mathfrak{sp}_J = \{X \in \mathfrak{sp} : [J, X] = 0\}.$$

We endow $\mathfrak{p} \cong T_J\mathfrak{S}$ with the complex structure $\hat{I} := (1/2)\mathrm{ad}J$. Then

$$\begin{aligned} &\mathfrak{p}_{\mathbb{C}} = \{X \in \mathfrak{sp}_{\mathbb{C}} : X(V_{-1,0}) \subset V_{0,-1} \text{ and } X(V_{0,-1}) \subset V_{-1,0}\} \\ &\mathfrak{p}^{1,0} = \{X \in \operatorname{Hom}(V_{0,-1}, V_{-1,0}) : Q_X \text{ is symmetric}\}, \end{aligned} \tag{2.3.1}$$

Remark 2.3.1. We have an isomorphism

$$\varphi_Q : V_{\mathbb{C}} \longrightarrow V^*_{\mathbb{C}}, \quad \varphi_Q(v) := Q(\cdot, v).$$

Its inverse is denoted by $\psi_Q := \varphi_Q^{-1}$. For any Lagrangian subspace $L \subset V_{\mathbb{C}}$ the isomorphism φ_Q maps L onto $\operatorname{Ann}(L)$. Therefore φ_Q gives an isomorphism $V_{0,-1} \cong H^{1,0}$.

As mentioned in 2.1 we are interested in the Lie bracket which can be seen as a section of a bundle over the symmetric space $\mathfrak{S}$. We wish to compute $B_J \in \Lambda^2 T_J^*\mathfrak{S} \otimes \mathfrak{sp}_J$. As usual it is useful to look at B thourgh its complexification

$$B_J : (T_J\mathfrak{S})_{\mathbb{C}} \times (T_J\mathfrak{S})_{\mathbb{C}} \longrightarrow (\mathfrak{sp}_J)_{\mathbb{C}}.$$

Recall that $(T_J\mathfrak{S})_{\mathbb{C}} = \mathfrak{p}_{\mathbb{C}} = \mathfrak{p}^{1,0} \oplus \mathfrak{p}^{0,1}$, where $\mathfrak{p}^{1,0}$ is given by (2.3.1) and

$$\begin{aligned}(\mathfrak{sp}_J)_{\mathbb{C}} =&\{X \in \operatorname{End} V_{\mathbb{C}} : X(V_{-1,0}) \subset V_{-1,0},\\ &X(V_{0,-1}) \subset V_{0,-1}, Q_X \text{ is symmetric}\}.\end{aligned} \tag{2.3.2}$$

Lemma 2.3.2. *The map B_J is of type $(1,1)$, i.e. it vanishes on vectors of the same type.*

Proof. Since B_J is real, it is enough to show that it vanishes on pairs of $(1,0)$-vectors. If $X, Y \in \mathfrak{p}^{1,0}$, then $X(V_{\mathbb{C}}) \subset V_{-1,0}$ and $Y|_{V_{-1,0}} = 0$, thus $YX = 0$. For the same reason also $XY = 0$. Thus $B_J(X,Y) = XY - YX = 0$. □

If $X \in \operatorname{End} V_{\mathbb{C}}$, let $X^* \in \operatorname{End} V_{\mathbb{C}}^*$ denote the transpose. The transposition map $X \mapsto X^*$ is a canonical isomorphism $\operatorname{End} V_{\mathbb{C}} \cong \operatorname{End} V_{\mathbb{C}}^*$. It is useful to reinterpret everything in terms of $\operatorname{End} V_{\mathbb{C}}^*$ rather than $\operatorname{End} V_{\mathbb{C}}$. Set

$$Q^* := \psi_Q^* Q.$$

(Notation as in 2.3.1.) Then Q^* is a symplectic form on $V_{\mathbb{C}}^*$.

Lemma 2.3.3. *If $X \in \operatorname{End} V_{\mathbb{C}}$, then Q_X is symmetric iff $Q^*_{X^*}$ is symmetric.*

Proof. Define $\tilde{X} \in \operatorname{End} V_{\mathbb{C}}$ by $Q(Xa,b) = Q(a, \tilde{X}b)$. Then

$$X^*\varphi_Q a = X^* Q(\cdot, a) = Q(X\cdot, a) = Q(\cdot, \tilde{X}a) = \varphi_Q \tilde{X} a.$$

Given $\alpha, \beta \in V_{\mathbb{C}}^*$ let $a = \psi_Q \alpha, b = \psi_Q \beta$. Then

$$\begin{aligned}-Q^*_{X^*}(\alpha,\beta) = Q^*(X^*\alpha, \beta) = Q(\psi_Q X^* \varphi_Q a, \psi_Q \varphi_Q b) =\\ = Q(\tilde{X}a, b) = Q(a, Xb) = Q_X(a,b).\end{aligned}$$

So $-Q^*_{X^*}(\alpha,\beta) = Q_X(a,b)$. The statement follows. □

Lemma 2.3.4. *If $X \in \operatorname{End} V_{\mathbb{C}}$, then*

$$\begin{aligned}X|_{V_{-1,0}} = 0 &\Longrightarrow \operatorname{Im} X^* \subset H^{0,1}\\ \operatorname{Im} X \subset V_{-1,0} &\Longrightarrow X^*|_{H^{0,1}} = 0.\end{aligned}$$

Proof. Recall that for any linear map $L : E \to F$ of vector spaces $\operatorname{Ann}\operatorname{Im} L = \ker L^*$. Now $X|_{V_{-1,0}} = 0 \Longrightarrow V_{-1,0} \subset \ker X \Longrightarrow H^{0,1} = \operatorname{Ann} V_{-1,0} \supset \operatorname{Ann}\ker X = \operatorname{Im} X^*$. And $\operatorname{Im} X \subset V_{-1,0} \Longrightarrow \ker X^* = \operatorname{Ann}\operatorname{Im} X \supset \operatorname{Ann} V_{-1,0} = H^{0,1}$. □

Proposition 2.3.5. *There are canonical isomorphisms*

$$\mathfrak{p}^{1,0} \cong \{t \in \operatorname{Hom}(H^{1,0}, H^{0,1}) : Q^*_t \text{ is symmetric}\}, \qquad (2.3.3)$$
$$(\mathfrak{sp}_J)_{\mathbb{C}} \cong \operatorname{End} H^{1,0}.$$

Using these isomorphisms B_J gets identified with the map

$$B_J : \mathfrak{p}^{1,0} \times \overline{\mathfrak{p}^{1,0}} \longrightarrow \operatorname{End} H^{1,0} \quad (s, \bar{t}) \mapsto \bar{t}s.$$

Proof. The first isomorphism is simply the restriction to $\mathfrak{p}^{1,0}$ of the map $X \mapsto X^*$. Lemmata 2.3.3 and 2.3.4 show that indeed the image of $\mathfrak{p}^{1,0}$ is the set of $X^* \in \operatorname{End} V^*_{\mathbb{C}}$ that vanish on $H^{0,1}$, have image in $H^{0,1}$ and such that $Q^*_{X^*}$ is symmetric. To describe the second isomorphism start from (2.3.2). Again the Lemmata show that the map $X \mapsto X^*$ sends $(\mathfrak{sp}_J)_{\mathbb{C}}$ to the set of $X^* \in \operatorname{End} V^*_{\mathbb{C}}$ that preserve each $H^{p,q}$ and such that $Q^*_{X^*}$ is symmetric. The latter means that $Q^*(u, X^*v) = Q^*(v, X^*u)$. This identity is trivial if u and v have the same type, since in that case both term vanish. Hence $X^*|_{H^{1,0}}$ is an arbitrary endomorphism of $H^{1,0}$. On the contrary the identity shows that for any $v \in H^{0,1}$ the value X^*v is determined by $X^*|_{H^{1,0}}$. Hence

$$(\mathfrak{sp}_J)_{\mathbb{C}} \longrightarrow \operatorname{End} H^{1,0}, \quad X \mapsto X^*|_{H^{1,0}} \qquad (2.3.4)$$

is the desired isomorphism. Now let $X, Y \in \mathfrak{p}^{1,0}$ and set $s := X^*$, $t := Y^*$. Then $B(X, \overline{Y})^* = \bar{t}s - s\bar{t}$. Since $\bar{t}|_{H^{1,0}} = 0$, in the isomorphism (2.3.4) $B(X, Y) \in (\mathfrak{sp}_J)_{\mathbb{C}}$ corresponds to $B(X, \overline{Y})^*|_{H^{1,0}} = \bar{t}s$. □

Since $H^{1,0} = \operatorname{Ann} V_{0,-1}$, we have $\operatorname{Ann} H^{1,0} = V_{0,-1}$. So there is a canonical isomorphism $(H^{1,0})^* \cong V_{\mathbb{C}} / \operatorname{Ann} H^{1,0} = V_{\mathbb{C}}/V_{0,-1} \cong V_{-1,0}$. We treat this isomorphism as an identity. By 2.3.1 φ_Q maps $V_{-1,0}$ isomorphically onto $H^{0,1}$. Thus $\psi_Q = \varphi_Q^{-1}$ restricts to an isomorphism

$$\psi_Q : H^{0,1} \xrightarrow{\cong} (H^{1,0})^*. \qquad (2.3.5)$$

Lemma 2.3.6. *For $\bar{\omega} \in H^{0,1}$, we have $\psi_Q(\bar{\omega}) = Q^*(\bar{\omega}, \cdot)$.*

Proof. First we claim that $\varphi_{Q^*}\varphi_Q = -\operatorname{id}_{V_{\mathbb{C}}}$ i.e. $\psi_Q = -\varphi_{Q^*}$. Indeed fix $v \in V_{\mathbb{C}}$. That $\varphi_{Q^*}\varphi_Q(v) = -v$ means that $Q^*(\cdot, \varphi_Q(v)) = -v$, i.e. that $Q^*(\lambda, \varphi_Q(v)) = -\lambda(v)$ for any $\lambda \in V^*_{\mathbb{C}}$. Assume $\lambda = \varphi_Q(w)$ for $w \in V_{\mathbb{C}}$. Then $\lambda(v) = Q(v, w)$ and $Q^*(\lambda, \varphi_Q(v)) = Q^*(\varphi_Q(w), \varphi_Q(v)) = Q(w, v)$. □

2.4 The computation of B at a moduli point

Now consider the period map $j : \mathsf{M}_g \longrightarrow \mathsf{A}_g$. Let $x \in \mathsf{M}_g$ be the moduli point of a curve C: $x = [C]$. In this case, $V = H_1(C, \mathbb{Z})$ and the symplectic form Q on V is the intersection form. In particular, $V^* = H^1(C, \mathbb{Z})$ and $Q^* : H^1(C, \mathbb{C}) \times H^1(C, \mathbb{C}) \to \mathbb{C}$ is

$$Q^*(\alpha, \beta) = \int \alpha \wedge \beta. \tag{2.4.1}$$

If we fix a symplectic basis of $H_1(C, \mathbb{Z})$ we get a sympletic isomorphism of $H_1(C, \mathbb{R})$ with the intersection form onto $(\mathbb{R}^{2g}, Q)$. Thus the Hodge decomposition $H^1(C, \mathbb{C}) = H^{1,0}(C) \oplus H^{0,1}(C)$ gives a complex structure on $H^1(C, \mathbb{C})$, hence a point $J \in \mathfrak{S}_g$.

In the following we use $T_x\mathsf{M}_g$ to denote the *real* tangent space i.e. the tangent space of M_g as a differentiable manifold (and similarly for A_g). Thus $(T_x\mathsf{M}_g)_\mathbb{C} = T_x^{1,0}\mathsf{M}_g \oplus T_x^{0,1}\mathsf{M}_g$ and $T_x^{1,0}\mathsf{M}_g = H^1(C, T_C)$, while $(T_x\mathsf{A}_g)_\mathbb{C} = (T_J\mathfrak{S}_g)_\mathbb{C} = \mathfrak{p}^{1,0} \oplus \mathfrak{p}^{0,1}$. By a theorem of Griffiths the map $dj_x : H^1(C, T_C) \longrightarrow \mathfrak{p}^{1,0}$ using the interpretation (2.3.3) is given by

$$dj_x(\xi) = \xi \cup \cdot : H^{1,0} \longrightarrow H^{0,1}, \quad dj_x(\xi)(\omega) = \xi \cup \omega. \tag{2.4.2}$$

(See e.g. [56, pp. 234ff].) Now $T_x^{0,1}\mathsf{M}_g = \overline{H^1(C, T_C)}$ is the conjugate vector space, i.e. it has the same underlying real vector space as $H^1(C, T_C)$ but multiplication by i is replaced with multiplication by $-i$. Since j is holomorphic, its differential is a direct sum of the map $dj_x : H^1(C, T_C) \to \mathfrak{p}^{1,0}$ and its conjugate. Hence for $\bar{\eta} \in \overline{H^1(C, T_C)}$ and $\omega \in H^{1,0}(C)$ we have

$$dj_x(\bar{\eta}) = \bar{\eta} \cup \cdot : H^{1,0} \longrightarrow H^{0,1}, \quad dj_x(\bar{\eta})(\bar{\omega}) = \overline{\eta \cup \omega}.$$

As mentioned in the Introduction our goal is to study the map

$$\mathbf{B}_x := dj_x^* B_J : H^1(C, T_C) \times \overline{H^1(C, T_C)} \longrightarrow \operatorname{End} H^{1,0}(C).$$

The following is a consequence of Proposition 2.3.5.

Proposition 2.4.1. *For $\xi, \eta \in H^1(C, T_C)$ and $\omega \in H^{1,0}(C)$, we have*

$$\mathbf{B}(\xi, \bar{\eta})(\omega) = \bar{\eta} \cup (\xi \cup \omega).$$

Once again it is useful to dualize. This time we dualize the map $\mathbf{B}$ itself. Using (2.3.5) we can describe the domain of $\mathbf{B}^*$ as follows:

$$(\operatorname{End} H^{1,0})^* = (H^{1,0\,*} \otimes H^{1,0})^* = H^{1,0} \otimes H^{1,0\,*} \cong H^{1,0} \otimes H^{0,1}.$$

More explicitly, let $\omega, \omega' \in H^{1,0}$ and $t \in \operatorname{End} H^{1,0}$. Then $\omega \otimes \bar{\omega}' \in H^{1,0} \otimes H^{0,1}$. Recalling Lemma 2.3.6 one easily verifies that the corresponding element of $(\operatorname{End} H^{1,0})^*$ is the linear functional mapping t to $Q^*(\bar{\omega}', t\omega)$.

The dual of $H^1(C, T_C)$ is $H^0(C, 2K_C)$. Thus the dual of $\mathbf{B}$ is defined on $H^0(C, 2K_C) \otimes \overline{H^0(C, 2K_C)}$ and maps to $H^{1,0}(C) \otimes \overline{H^{1,0}(C)}$.

Denoting by $\bar{C}$ the conjugate variety we have

$$\overline{H^0(C, 2K_C)} = H^0(\bar{C}, 2K_{\bar{C}}), \quad \overline{H^{1,0}(C)} = H^{1,0}(\bar{C}).$$

Thus $\mathbf{B}^*$ is a map from $H^{1,0}(C) \otimes H^{1,0}(\bar{C})$ to $H^0(C, 2K_C) \otimes H^0(\bar{C}, 2K_{\bar{C}})$. We further reinterpret domain and target of $\mathbf{B}^*$ as spaces of sections of appropriate bundles on $Z = C \times \bar{C}$. Denoting by $p : Z \to C$ and $q : Z \to \bar{C}$ the projections and given bundles $L \to C$ and $M \to \bar{C}$, set $L \boxtimes M := p^*L \otimes q^*M$. The map

$$H^0(C, L) \otimes H^0(\bar{C}, M) \longrightarrow H^0(Z, L \boxtimes M), \quad s \otimes t \mapsto p^*s \otimes q^*t,$$

is an isomorphism. For any positive integer n there is a canonical isomorphism

$$K_Z^n \cong K_C^n \boxtimes K_{\bar{C}}^n, \tag{2.4.3}$$

obtained as follows: if $\alpha \in K_{C,x}$ and $\beta \in K_{\bar{C},y}$, denote by $\alpha^n \in (K_{C,x})^{\otimes n}$ and $\beta^n \in (K_{\bar{C},y})^{\otimes n}$ the tensor powers. Then $\alpha^n \otimes \beta^n \in (K_C^n \boxtimes K_{\bar{C}}^n)_{(x,y)}$, while $(p^*\alpha \wedge q^*\beta)^n \in K_{Z,(x,y)}$. The isomorphism (2.4.3) maps $\alpha^n \otimes \beta^n$ to $(p^*\alpha \wedge q^*\beta)^n$.

We now prove Theorem A.

Theorem 2.4.2. *The map*

$$\mathbf{B}^* : H^0(Z, K_Z) \longrightarrow H^0(Z, 2K_Z)$$

coincides with the multiplication by $-i\mathbf{K}$.

Proof. Fix a point $x \in C$ and a chart (U, z) centered in x. Set $u = \frac{\partial}{\partial z}(x)$ and consider the Schiffer variation ξ_u at $x \in C$. We recall that

$$\xi_u \cup = -2\pi \operatorname{ev}_u \otimes \overline{k}_u. \tag{2.4.4}$$

Indeed fix a Dolbeault representative $\varphi = \frac{\bar{\partial} b}{z} \frac{\partial}{\partial z}$, where $b \in C^\infty$ is a bump function which is equal to 1 in a neighbourhood of x. For $\omega \in H^0(K_C)$, with local expression $\omega = h(z)dz$ on U, it holds that

$$dj_x(\xi_u)(\omega) = \xi_u \cup \omega = [\varphi \cdot \omega] = \left[\frac{\bar{\partial}(bh)}{z} \right].$$

If f_u is an elementary potential, the functions

$$\frac{b}{z} + f_u \quad \text{and} \quad b \cdot \frac{h - h(0)}{z}$$

are smooth on C. Hence

$$\bar{\partial} \left(\frac{bh}{z} \right) = \bar{\partial} \left(b \cdot \frac{h - h(0)}{z} \right) + h(0) \cdot \bar{\partial} \left(\frac{b}{z} + f_u \right) - h(0) \bar{\partial} f_u.$$

Thus $dj_x(\xi_u)(\omega) = h(0)\cdot[-\bar\partial f_u]$. As usual we identify $H^{0,1}(C)$ with the space of antiholomorphic forms. Thus since $h(0) = \omega(u)$, using Lemma 2.2.4 and the fact that $\overline{k}_u$ is antiholomorphic, we get (2.4.4).

We also recall (see [6, Lemma 2.3]) that for $\beta \in H^0(C, 2K_C) = H^1(C, T_C)^*$, we have $\beta(\xi_u) = 2\pi i\beta(u)$. It follows that for $\Phi \in H^0(Z, 2K_Z)$

$$\Phi((u,0),(0,\bar v)) = -\frac{1}{4\pi^2}\Phi(\xi_u \otimes \xi_{\bar v}).$$

Now we can prove the statement. Without loss of generality we can assume that $\Omega = p^*\omega \wedge q^*\overline{\omega'}$, with $\omega, \omega' \in H^0(K_C)$. Then

$$(\mathbf{B}^*\Omega)_{(x,y)}((u,0),(0,\overline{v})) = -\frac{1}{4\pi^2}\mathbf{B}^*(p^*\omega \wedge q^*\overline{\omega'})(\xi_u \otimes \xi_{\overline{v}}) = \\ = -\frac{1}{4\pi^2}(p^*\omega \wedge q^*\overline{\omega'})(\mathbf{B}(\xi_u \otimes \xi_{\overline{v}})).$$

It follows from (2.4.4) that

$$\xi_{\overline{v}} \cup (\xi_u \cup \omega) = -2\pi\omega(u)\cdot \xi_{\overline{v}} \cup \overline{k}_u = -2\pi\omega(u)\cdot \overline{\xi_v \cup k_u} = \\ = 4\pi^2\omega(u)\cdot \overline{k_u(v)}\cdot k_v.$$

Using this and Lemma 2.3.6 we get

$$(\mathbf{B}^*\Omega)_{(x,y)}((u,0),(0,\overline{v})) = -\frac{1}{4\pi^2}Q^*(\overline{\omega'}, \mathbf{B}(\xi_u \otimes \xi_{\overline{v}})\omega) = \\ = -\frac{1}{4\pi^2}Q^*(\overline{\omega'}, \xi_{\overline{v}} \cup (\xi_u \cup \omega)) = -\omega(u)\cdot \overline{k_u(v)}\cdot Q^*(\overline{\omega'}, k_v).$$

By 2.2.3 and (2.4.1) we have that $Q^*(\overline{\omega'}, k_v) = i\cdot \overline{\omega'(v)}$. Using this and Lemma 2.2.3 we finally get

$$(\mathbf{B}^*\Omega)_{(x,y)}((u,0),(0,\overline{v})) = -i\omega(u)\overline{\omega'(v)}\cdot \overline{k_u(v)} = \\ = -i\cdot(\Omega\cdot \mathbf{K})_{(x,y)}((u,0),(0,\overline{v}))$$

□

Remark 2.4.3. Since $\mathbf{K}$ is nonzero, it follows from Theorem 2.4.2 that $\mathbf{B}^*$ is injective, and thus that $\mathbf{B}$ is surjective. As a consequence we get that the complexification $(T_x\mathsf{M}_g)_{\mathbb{C}} \times (T_x\mathsf{M}_g)_{\mathbb{C}} \to (\mathfrak{sp}_J)_{\mathbb{C}}$ of the restriction of Lie bracket on the tangent space to M_g is surjective. Thus the Lie bracket $[\cdot,\cdot] : T_x\mathsf{M}_g \times T_x\mathsf{M}_g \to \mathfrak{sp}_J$ on the tangent space to M_g is surjective.

2.5 Analogous results for the triple Lie bracket

We dedicate this section to the computation of the triple Lie bracket on the tangent space to M_g. As mentioned in Section 2.1, the study of the Lie bracket on M_g was motivated by the intention to obtain some constraints on the existence of Lie triples systems tangent to M_g. Intending to present some considerations in this direction, we dedicate this section to the computation of the triple Lie bracket on the tangent to M_g. The arguments are analogous to those of the previous sections. As above, we use $T_x\mathsf{M}_g$ to denote the real tangent space of M_g, and similarly for A_g. By definition, a real vector subspace $V \subset T_x\mathsf{M}_g \subset T_x\mathsf{A}_g$ is a Lie triple system if $[[V,V],V] \subset V$. We set $V^{1,0} = V_{\mathbb{C}} \cap T_x^{1,0}\mathsf{M}_g$ and $V^{0,1} = V_{\mathbb{C}} \cap T_x^{0,1}\mathsf{M}_g$.

Lemma 2.5.1. *Let $V \subset T_x\mathsf{M}_g$ be a real vector subspace. Then $[[V,V],V] \subset T_x\mathsf{M}_g$ if and only if $[[V^{1,0},V^{0,1}],V^{1,0}] \subset T_x^{1,0}\mathsf{M}_g$.*

Proof. Since V is real, for a subspace $W \subset \mathfrak{p}$ we have that $[[V,V],V] \subset W$ if and only if $[[V_{\mathbb{C}},V_{\mathbb{C}}],V_{\mathbb{C}}] \subset W_{\mathbb{C}}$. Now $[[V_{\mathbb{C}},V_{\mathbb{C}}],V_{\mathbb{C}}]$ would be the sum of eight pieces, depending on the types. But $[V^{1,0},V^{1,0}] = 0$ and the same for $V^{0,1}$. Furthermore

$$\overline{[V^{1,0},V^{0,1}]} = [V^{0,1},V^{1,0}] = [V^{1,0},V^{0,1}].$$

In particular, $[V_{\mathbb{C}},V_{\mathbb{C}}] = [V^{1,0},V^{0,1}]$. It follows that $[[V_{\mathbb{C}},V_{\mathbb{C}}],V_{\mathbb{C}}]$ is the sum of only two pieces, namely $[[V^{1,0},V^{0,1}],V^{1,0}]$ and its conjugate. The statement follows by observing that $[[\mathfrak{p}^{1,0},\mathfrak{p}^{0,1}],\mathfrak{p}^{1,0}] \subset \mathfrak{p}^{1,0}$ since $\mathfrak{p}^{1,0} \subset \mathrm{Hom}(V_{0,-1},V_{-1,0})$ by (2.3.1). ☐

The Lemma says that we can check whether $V \subset T_x\mathsf{M}_g$ is a Lie triple system or not by looking at $[[V^{1,0},V^{0,1}],V^{1,0}]$. For $J \in \mathfrak{S}$, we focus our attention on the map

$$T_J : T_J^{1,0}\mathfrak{S} \times \overline{T_J^{1,0}\mathfrak{S}} \times T_J^{1,0}\mathfrak{S} \to T_J^{1,0}\mathfrak{S}, \qquad (X,\overline{Y},Z) \mapsto [[X,\overline{Y}],Z].$$

With the notations of (2.3.1), we obtain.

Proposition 2.5.2. *Using the isomorphism* (2.3.3)*, T_J gets identified with the map*

$$T_J : \mathfrak{p}^{1,0} \times \overline{\mathfrak{p}^{1,0}} \times \mathfrak{p}^{1,0} \to \mathfrak{p}^{1,0}, \qquad (s,\bar{t},r) \mapsto r\bar{t}s + s\bar{t}r.$$

Proof. Let $X,Y,Z \in T_J^{1,0}\mathfrak{S}$. We have that $T(X,\overline{Y},Z) = [[X,\overline{Y}],Z] = X\overline{Y}Z - \overline{Y}XZ - ZX\overline{Y} + Z\overline{Y}X$. Since the endomorphisms of $T_J^{1,0}\mathfrak{S}$ vanish on $V_{-1,0}$ and have image in $V_{-1,0}$ (cf. (2.3.1)), it follows that $\overline{Y}XZ = ZX\overline{Y} = 0$. Now set $s := X^*$, $t := Y^*$, and $r := Z^*$. Then $T(X,\overline{Y},Z)^* = r\bar{t}s + s\bar{t}r$. By the isomorphism (2.3.3), the statement follows. ☐

Now let $x \in \mathsf{M}_g$ be the moduli point of a curve C. We are interested in the restriction of the triple Lie bracket to the tangent space to M_g, which is given by

$$\mathbf{T}_x := dj_x^* T_J : H^1(C,T_C) \times \overline{H^1(C,T_C)} \times H^1(C,T_C) \longrightarrow \mathfrak{p}^{1,0}.$$

As a consequence of (2.4.2) and of Proposition 2.5.2 we get

Proposition 2.5.3. *For $\xi, \eta, \alpha \in H^1(C, T_C)$ and $\omega \in H^{1,0}(C)$, we have*

$$\mathbf{T}(\xi, \bar{\eta}, \alpha)(\omega) = \alpha \cup (\bar{\eta} \cup (\xi \cup \omega)) + \xi \cup (\bar{\eta} \cup (\alpha \cup \omega)).$$

As we did with $\mathbf{B}$, we want now to dualize the map $\mathbf{T}$. Reasoning as in (2.3.5) and Lemma 2.3.6, we get that $\psi_Q = \varphi_Q^{-1}$ restricts to an isomorphism

$$\psi_Q : H^{1,0} \xrightarrow{\cong} (H^{0,1})^*, \quad \psi_Q(\omega) = Q^*(\omega, \cdot). \tag{2.5.1}$$

We use this isomorphism to describe the domain of $\mathbf{T}^*$ as follows.

$$(\mathrm{Hom}(H^{1,0}, H^{0,1}))^* = \left((H^{1,0})^* \otimes H^{0,1}\right)^* = H^{1,0} \otimes (H^{0,1})^* \cong H^{1,0} \otimes H^{1,0}. \tag{2.5.2}$$

The isomorphism is the following. Let $\omega, \omega' \in H^{1,0}$. By (2.5.1), the element of $(\mathrm{Hom}(H^{1,0}, H^{0,1}))^*$ which corresponds to $\omega \otimes \omega' \in H^{1,0} \otimes H^{1,0}$ is the linear functional mapping $t \in \mathrm{Hom}(H^{1,0}, H^{0,1})$ to $Q^*(\omega', t\omega)$. One easily checks that, through the described isomorphism, $\mathfrak{p}^{1,0} \subset \mathrm{Hom}(H^{1,0}, H^{0,1})$ gets identified to $S^2 H^{1,0} \subset H^{1,0} \otimes H^{1,0}$. The dual of $\mathbf{T}$ is thus defined on $S^2 H^0(C, K_C)$ and maps to $H^0(C, 2K_C) \otimes \overline{H^0(C, 2K_C)} \otimes H^0(C, 2K_C)$. Denoting by $\bar{C}$ the conjugate variety we get

$$\mathbf{T}^* : S^2 H^0(C, K_C) \longrightarrow H^0(C, 2K_C) \otimes H^0(\bar{C}, 2K_{\bar{C}}) \otimes H^0(C, 2K_C)$$

In this case we reinterpret domain and target of $\mathbf{T}^*$ as spaces of sections of bundles on $W = C \times \bar{C} \times C$. We fix some notations. Let us denote by $\pi_1 : W \to C$ and $\pi_2 : W \to \bar{C}$ and $\pi_3 : W \to C$ the projections. Also, let $S = C \times C$ with projections $p : S \to C$ and $q : S \to C$, and similarly for $Z = C \times \bar{C}$. Finally denote by π_{ij} the projection from W on the factors i, j (e.g., $\pi_{13} : W \to S, (x, y, z) \mapsto (x, z)$). Consider the map

$$H^0(C, K_C) \otimes H^0(C, K_C) \to H^0(W, \pi_{13}^* K_S), \tag{2.5.3}$$

$$\omega \otimes \omega' \longmapsto \pi_{13}^*(p^*\omega \wedge q^*\omega') = \pi_1^*\omega \wedge \pi_3^*\omega'.$$

As remarked in (2.4.3), $p^* K_C \otimes q^* K_C \cong K_S$. The isomorphism maps $\alpha \otimes \beta$ to $p^*\alpha \wedge q^*\beta$. It follows that $\pi_{13}^* K_S \cong \pi_{13}^*(p^* K_C \otimes q^* K_C) \cong \pi_1^* K_C \otimes \pi_3^* K_C$. Thus we also have $H^0(W, \pi_{13}^* K_S) \cong H^0(W, \pi_1^* K_C \otimes \pi_3^* K_C)$ and the map above is an isomorphism. We denote by $\mathbf{K}^C$ the Bergman kernel associated with the curve C and by $\mathbf{K}^{\bar{C}}$ the one associated with $\bar{C}$.

Theorem 2.5.4. *With the above identifications, $\mathbf{T}^* : S^2 H^0(C, K_C) \longrightarrow H^0(C, 2K_C) \otimes H^0(\bar{C}, 2K_{\bar{C}}) \otimes H^0(C, 2K_C)$ is the restriction to $S^2 H^0(C, K_C)$ of the multiplication map*

$$H^0(W, \pi_{13}^* K_S) \longrightarrow H^0(W, 2K_W), \quad \Omega \mapsto \Omega \cdot \alpha$$

where $\alpha = -2 \cdot \pi_{12}^* \mathbf{K}^C \cdot \pi_{23}^* \mathbf{K}^{\bar{C}} \in H^0(W, \pi_1^* K_C \otimes \pi_2^* K_{\bar{C}}^{\otimes 2} \otimes \pi_3^* K_C)$.

Proof. Let $\tilde{\Omega} \in S^2H^0(K_C)$. Without loss of generality we can assume that $\tilde{\Omega} = \omega \odot \omega'$, with $\omega, \omega' \in H^0(K_C)$. The isomorphism (2.5.3) maps $\omega \odot \omega' \in S^2H^0(C, K_C)$ to $\Omega = \pi_1^*\omega \wedge \pi_3^*\omega' + \pi_1^*\omega' \wedge \pi_3^*\omega \in H^0(W, \pi_{13}K_S)$.
As in the proof of Theorem 2.4.2 we recall that that for $\beta \in H^0(C, 2K_C) = H^1(C, T_C)^*$, we have $\beta(\xi_u) = 2\pi i\beta(u)$. It follows that for $\Phi \in H^0(W, 2K_W)$

$$\Phi((u,0,0),(0,\bar{v},0),(0,0,t))) = -\frac{1}{8\pi^3 i}\Phi(\xi_u \otimes \xi_{\bar{v}} \otimes \xi_t).$$

Thus for $u \in T_x^{1,0}C$, $v \in T_y^{1,0}C$, and $t \in T_z^{1,0}C$ with $x, y, z \in C$, we have

$$(\mathbf{T}^*\Omega)_{x,y,z}((u,0,0),(0,\overline{v},0),(0,0,t)) = \tag{2.5.4}$$

$$= -\frac{1}{8\pi^3 i}[(\pi_1^*\omega \wedge \pi_3^*\omega')\mathbf{T}(\xi_u \otimes \xi_{\bar{v}} \otimes \xi_t) + (\pi_1^*\omega' \wedge \pi_3^*\omega)\mathbf{T}(\xi_u \otimes \xi_{\bar{v}} \otimes \xi_t)].$$

We focus on the first summand. By (2.5.2)

$$-\frac{1}{8\pi^3 i}[(\pi_1^*\omega \wedge \pi_3^*\omega')\mathbf{T}(\xi_u \otimes \xi_{\bar{v}} \otimes \xi_t)] = -\frac{1}{8\pi^3 i}Q^*(\omega', \mathbf{T}(\xi_u \otimes \xi_{\bar{v}} \otimes \xi_t)(\omega)).$$

Now using Proposition 2.5.3 and (2.4.4) we get that

$$\mathbf{T}(\xi_u \otimes \xi_{\bar{v}} \otimes \xi_t)(\omega) = \xi_t \cup (\xi_{\bar{v}} \cup (\xi_u \cup \omega)) + \xi_u \cup (\xi_{\bar{v}} \cup (\xi_t \cup \omega)) =$$
$$= -8\pi^3 \left(\omega(u)\overline{k_u(v)}k_v(t)\overline{k_t} + \omega(t)\overline{k_t(v)}k_v(u)\overline{k_u}\right).$$

Now recall that, by 2.2.3 and (2.4.1), $iQ^*(\omega', \overline{k_t}) = \omega'(t)$. Thus $Q^*(\omega', \overline{k_t}) = -i\omega'(t)$ and we get

$$-\frac{1}{8\pi^3 i}[(\pi_1^*\omega \wedge \pi_3^*\omega')\mathbf{T}(\xi_u \otimes \xi_{\bar{v}} \otimes \xi_t)] =$$
$$= -\left(\omega(u)\omega'(t)\overline{k_u(v)}k_v(t) + \omega(t)\omega'(u)\overline{k_t(v)}k_v(u)\right) =$$

Since $k_u(v) = \overline{k_v(u)}$

$$= -[\omega(u)\omega'(t) + \omega(t)\omega'(u)]\ \overline{k_u(v)}k_v(t).$$

Now note that $K^{\bar{C}}((\overline{u}, 0), (0, v)) = k_v(u)$. We have

$$-\frac{1}{8\pi^3 i}[(\pi_1^*\omega \wedge \pi_3^*\omega')\mathbf{T}(\xi_u \otimes \xi_{\bar{v}} \otimes \xi_t)] =$$
$$= -[(\pi_1^*\omega \wedge \pi_3^*\omega')(u,\overline{v},t) + (\pi_1^*\omega' \wedge \pi_3^*\omega)(u,\overline{v},t)]\pi_{12}^*\mathbf{K}^C(u,\overline{v},t)\pi_{23}^*\mathbf{K}^{\bar{C}}(u,\overline{v},t) =$$
$$= -\Omega(u,\overline{v},t)(\pi_{12}^*\mathbf{K}^C \cdot \pi_{23}^*\mathbf{K}^{\bar{C}})(u,\overline{v},t)$$

where we are writing $(u, \overline{v}, t)$ for $((u,0,0),(0,\overline{v},0),(0,0,t))$. The same calculation for the second summand of (2.5.4) implies

$$(\mathbf{T}^*\Omega)_{x,y,z}(u,\overline{v},t) = -2\Omega(u,\overline{v},t)(\pi_{12}^*\mathbf{K}^C \cdot \pi_{23}^*\mathbf{K}^{\bar{C}})(u,\overline{v},t).$$

□

Remark 2.5.5. Since we will use it in the sequel, we stress out that by the previus Theorem we get the following formula: if $\Omega \in S^2H^0(K_C)$ and $\xi_u, \xi_{\overline{v}}, \xi_t$ are Schiffer variations for $u \in T_x^{1,0}C$, $v \in T_y^{1,0}C$, and $t \in T_z^{1,0}C$ with $x, y, z \in C$, then

$$\langle \mathbf{T}^* Q, \xi_u \otimes \xi_{\bar{v}} \otimes \xi_t \rangle = -2\Omega(u,t)\mathbf{K}(u,\bar{v})\mathbf{K}(t,\bar{v}) \tag{2.5.5}$$

2.6 Some considerations on Lie triple systems

We mentioned in 1.5.3 that, through the study of the second fundamental form of the Torelli map with respect to the Siegel metric, in [6] and [17] the authors obtained some bounds on the dimension of a totally geodesic subvariety of A_g contained in M_g, excluding the existence of totally geodesic (and, in particular, Shimura) subvarieties of very large dimension. A natural question is if a similar work can be done using the Lie bracket. In other words, is it possible to use the characterization of Theorems 2.4.2 and 2.5.4 to improve the known upper bounds on the dimension of a totally geodesic subvariety Y of A_g generically contained in the Torelli locus? We dedicate this last section to some considerations in this direction.

2.6.1 Moderate vectors

A real vector subspace $V \subset T_x\mathsf{M}_g \subset T_x\mathsf{A}_g$ is a Lie triple system if

$$[[V,V],V] \subset V.$$

In particular, if V is a Lie triple system, then $[[V,V],V] \subset T_x\mathsf{M}_g$. For this reason, a first step towards the investigation of Lie triple systems could be to search for conditions on subspaces $V \subset T_x\mathsf{M}_g$ such that $[[V,V],V] \subset T_x\mathsf{M}_g$.

Definition 2.6.1. *Let $x \in \mathsf{M}_g$ be the moduli point of a curve C. We say that a vector $\xi \in H^1(C,T_C)$ is **moderate** if $\mathbf{T}(\xi \otimes \bar{\xi} \otimes \xi) \in H^1(C,T_C)$.*

By the previous considerations and by Lemma 2.5.1, if V is a Lie triple system and $\xi \in V^{1,0} \subset T_x\mathsf{M}_g$, then ξ is moderate.

Let $\xi \in H^1(C,T_C)$. We write ξ as linear combination of Schiffer variations

$$\xi = \sum_{i=1}^{n} x_i \xi_{u_i}, \tag{2.6.1}$$

with $p_i \in C$, $p_i \neq p_j$ for $j \neq i$, $u_i \in T_{p_i}C - \{0\}$, $x_i \in \mathbb{C}$. Since $H^1(C,T_C) = Ann(I_2(K_C))$, the vector $\xi \in H^1(C,T_C)$ is moderate if and only if

$$\langle \mathbf{T}(\xi \otimes \bar{\xi} \otimes \xi), Q \rangle = 0 \qquad \forall Q \in I_2(K_C). \tag{2.6.2}$$

Let us compute $\langle \mathbf{T}(\xi \otimes \bar{\xi} \otimes \xi), Q\rangle$:

$$\langle \mathbf{T}(\xi \otimes \bar{\xi} \otimes \xi), Q\rangle = \sum_{i,j,k} x_i x_j \overline{x_k} \ \langle \mathbf{T}(\xi_{u_i} \otimes \bar{\xi}_{u_k} \otimes \xi_{u_j}), Q\rangle =$$

$$= \sum_{i,j,k} x_i x_j \overline{x_k} \ \langle (\xi_{u_i} \otimes \bar{\xi}_{u_k} \otimes \xi_{u_j}), \mathbf{T}^* Q\rangle.$$

By Theorem 2.5.4 and, in particular, by (2.5.5) we get

$$\langle \mathbf{T}(\xi \otimes \bar{\xi} \otimes \xi), Q\rangle = -2 \cdot \sum_{i,j=1}^{n} x_i x_j Q(u_i, u_j) \cdot \left(\sum_{k=1}^{n} \overline{x}_k \mathbf{K}(u_i, \bar{u}_k) \mathbf{K}(u_j, \bar{u}_k) \right),$$

where we are writing $(u_i, \bar{u}_k)$ for $((u, 0), (0, \bar{u}_k))$. Note that, since $Q \in I_2(K_C)$, $Q(u, u) = 0$ and thus we have

$$\langle \mathbf{T}(\xi \otimes \bar{\xi} \otimes \xi), Q\rangle = -4 \cdot \sum_{i<j} x_i x_j Q(u_i, u_j) \cdot \left(\sum_{k=1}^{n} \overline{x}_k \mathbf{K}(u_i, \bar{u}_k) \mathbf{K}(u_j, \bar{u}_k) \right).$$

By condition (2.6.2) we get the following. If ξ is of the form (2.6.1), then ξ is moderate if and only if for every $Q \in I_2(K_C)$

$$\sum_{i<j} x_i x_j Q(u_i, u_j) \cdot \left(\sum_{k=1}^{n} \overline{x}_k \mathbf{K}(u_i, \bar{u}_k) \mathbf{K}(u_j, \bar{u}_k) \right) = 0. \tag{2.6.3}$$

Case n=1. If $\xi = \xi_{u_1}$ is a Schiffer variation, then ξ is moderate.

Case n=2. For $n = 2$, (2.6.3) becomes

$$x_1 x_2 \, Q(u_1, u_2) \cdot \left(\overline{x}_1 \mathbf{K}_{11} \mathbf{K}_{21} + \overline{x}_2 \mathbf{K}_{12} \mathbf{K}_{22} \right) = 0$$
where $\mathbf{K}_{ij} := \mathbf{K}(u_i, \bar{u}_j)$.

By Enriques-Babbage Theorem (see [1, pag. 124]), if C is a smooth curve of genus $g > 3$, which is non-hyperelliptic, non-trigonal and not isomorphic to a plane quintic, it is always possibile to find $Q \in I_2(K_C)$ such that $Q(u_1, u_2) \neq 0$. Searching for solutions with $x_1 x_2 \neq 0$, this allows to lead back (2.6.3) to the resolution of a linear system in x_1, x_2. We obtain that ξ is moderate if and only if

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \cdot \begin{pmatrix} -\mathbf{K}_{21} \mathbf{K}_{22} \\ \mathbf{K}_{11} \mathbf{K}_{12} \end{pmatrix}.$$

In particular, if $\mathbf{K}_{12} = \overline{\mathbf{K}_{21}} = 0$, then all vectors of the form $\xi = x_1 \xi_{u_1} + x_2 \xi_{u_2}$ are moderate. If $\mathbf{K}_{12} \neq 0$ instead, then the only moderate vector of the form $\xi = x_1 \xi_{u_1} + x_2 \xi_{u_2}$ (with $x_1 x_2 \neq 0$) is

$$\xi := -\mathbf{K}_{21} \mathbf{K}_{22} \xi_{u_1} + \mathbf{K}_{11} \mathbf{K}_{12} \xi_{u_2}. \tag{2.6.4}$$

Syntetizing

Proposition 2.6.2. *Let C be a smooth curve of genus $g > 3$, which is non-hyperelliptic, non-trigonal and not isomorphic to a plane quintic. Given two Schiffer variantions ξ_{u_1}, ξ_{u_2} with $p_i \in C$, $u_i \in T_{p_i}C - \{0\}$, there are only two possibility:*

1. *If $\mathbf{K}_{12} = 0$, then every $\xi \in \operatorname{span}(\xi_{u_1}, \xi_{u_2})$ is moderate.* (Note that, fixing p_1, the points which satisfy $\mathbf{K}_{12} = 0$ are $2g-2$).
2. *If $\mathbf{K}_{12} \neq 0$, then there are exactly three moderate vectors in $[\xi] \in \mathbb{P}\operatorname{span}(\xi_{u_1}, \xi_{u_2})$. These are $\xi = \xi_{u_1}, \xi = \xi_{u_2}$ and the one given by* (2.6.4).

Case n≥3. For $n \geq 3$, condition (2.6.3) is quite complicated. One could try to work as follows. In analogy with the case $n = 2$, one would like to lead back (2.6.3) to a system of linear equations.

Proposition 2.6.3. *Assume that C is a smooth curve of genus $g > 3$, which is non-hyperelliptic, non-trigonal and not isomorphic to a plane quintic. Let $p_1, ..., p_{g-2}$ be general points on C. Then for every $i < j$ there exists a quadric $Q_{ij} \in I_2(K_C)$ such that $Q_{ij}(p_i, p_j) \neq 0$ and $Q_{ij}(p_s, p_k) = 0$ when $\{i, j\} \neq \{s, k\}$.*

Proof. Let $p_1, ..., p_g$ be g general points on C. Let $\omega_1, ..., \omega_g$ be a basis of $H^0(C, K_C)$ such that $\omega_i(p_j) = \delta_{ij}$ and such that the supports of the divisors $\operatorname{div}(\omega_i)$ are pairwise disjoint (see [1, pag. 125]). Petri's construction of an explicit basis of $I_2(K_X)$ produces quadrics $\{Q_{ij}\}$ in $I_2(K_X)$ such that (see [1, pag. 130])

$$Q_{ij}(p_a, p_b) = \delta_{ia}\delta_{jb} + \delta_{ja}\delta_{ib}, \qquad \text{for } a, b \geq 2.$$

That is, the quadrics assume arbitrary values on $\{p_3, ..., p_g\}$. □

Corollary 2.6.4. *Let C be a smooth non-hyperelliptic curve of genus $g > 3$. If $n \leq g-2$ and $p_1, \ldots, p_n$ are points in general position on C, then the map*

$$I_2(K_C) \longrightarrow \mathbb{C}^{n(n-1)/2}, \quad Q \mapsto Q(u_i, u_j), \quad i < j \tag{2.6.5}$$

is surjective.

In the hypotesis of the corollary, it follows from (2.6.3) that

$$x_i x_j \cdot \left(\sum_{k=1}^{n} \overline{x}_k \mathbf{K}(u_i, \bar{u}_k) \mathbf{K}(u_j, \bar{u}_k) \right) = 0, \quad \forall i < j.$$

Supposing $x_i \neq 0$ (i.e., n is minimal for ξ) and conjugating, we get

$$\sum_{s=1}^{n} x_s \mathbf{K}(u_s, \bar{u}_i) \mathbf{K}(u_s, \bar{u}_j) = 0, \quad \forall i < j. \tag{2.6.6}$$

This is a linear system with $n(n-1)/2$ equations and n unknown. We denote by $A' \in M_{n(n-1)/2 \times n}(\mathbb{C})$ the matrix of the coefficients. We syntetize our considerations as follows. Let $\xi \in H^1(C, T_C)$ be a vector such that

1. $\xi = \sum_{i=1}^{n} x_i \xi_i$ with $p_i \in C$, $p_i \neq p_j$ for $j \neq i$, $u_i \in T_{p_i}C - \{0\}$, $x_i \in \mathbb{C} - \{0\}$;
2. $n \leq g-2$ and the points p_i are such the map (2.6.5) is surjective.

Then ξ is moderate if and only if $\operatorname{rank} A' < n$.

2.6.2 Final remarks on the triple Lie bracket

Remark 2.6.5. Our idea was to use moderate vectors with the aim to get a bound on the dimension of a totally geodesic subvariety Y of M_g. However, the approach used did not bring satisfactory results. The kind of argument we had in mind was the following. Let $Y \subset \mathsf{M}_g$ be a totally geodesic subvariety of codimension c, and let $x \in Y$ be a smooth point. Let X be a curve such that $[X] = x$. Consider the bicanonical image $Bic(X) := \varphi_{|2K_X|}(X) \subset \mathbb{P}(H^1(X, T_X))$. Denote by S^kX the variety of k-secants of X. Points of S^kX are limits of points lying on k-dimensional linear subspaces of $\mathbb{P}(H^1(X, T_X))$ spanned by $k+1$ points of X. Since $\dim S^kX = 2k+1$, we get that if $2k+1 \geq c$ then $\emptyset \neq S^kX \cap \mathbb{P}(T_xY) \subset \mathbb{P}(H^1(X, T_X))$. Hence $S^kX \cap \mathbb{P}(T_xY)$ contains at least some point $[\xi_0]$. In particular, $[\xi_0] \in \langle D \rangle$ for some effective divisor D on X of degree $k+1$. Now suppose that we can prove that a point $[\xi] \in \langle D \rangle$ cannot be moderate if $\deg D \leq k_0$, then we would get $c \geq 2k_0 + 1$, and thus a bound on the maximal dimension of a totally geodesic subvariety in M_g. However, from our analysis, we only got a sufficient condition for $\xi_0 \in \langle D \rangle$ not to be moderate; that is, conditions (1) and (2) above and (3) $\operatorname{rank} A' = n$. The problem is that, of course, there is no reason why ξ_0 should satisfy these conditions. The study of the dimension of the subspace $\tilde{S}^kX$ of elements in S^kX which satifisy the three conditions has also led no satisfactory result.

Remark 2.6.6. Already for $n = 3$, we have not succeded in describing which vectors $\xi \in \operatorname{span}(\xi_{u_1}, \xi_{u_2}, \xi_{u_3})$ are moderate with no additional assumptions on the points p_i. Even in the case of points in general positions, the study of the rank of the matrix A' is not manageable either. Since the study of moderate vectors has not led to the hoped-for results, we do not further describe the carried out analyses. We only report some final observations. Recall that $\mathbf{K}(u_j, \bar{u}_i) = h(k_{u_i}, k_{u_j})$, where h is the Hodge Hermitian product on $H^0(X, K_X)$. A first easy observation is that, if all the k_{u_i}'s are othogonal, then every $\xi \in \operatorname{span}(\xi_{u_1}, ..., \xi_{u_n})$ is moderate. This case is analogous to case 1 in Proposition 2.6.2. A second observation is that, since we are working under the assumption that the points p_i are in general position, we get that the matrix $A = (a_{ij})$, with $a_{ij} = \mathbf{K}(u_j, \bar{u}_i)$, is hermitian and positive definite. What it should be true is that, if the p_i's are very general, then $\operatorname{rank} A' = n$, and thus ξ is **not** moderate.

Remark 2.6.7. An important point to stress is the following. Let $Y \subset \mathsf{M}_g$ be a subvariety and let $[X] = x \in Y$ be a smooth point. As said, if Y is totally geodesic, then its tangent space $V = T_x Y$ is a Lie triple system. On the other hand, if $V = T_x Y$ is a Lie triple system, we can only conclude by characterization 1.4.2 that V is the tangent space of a totally geodesic subvariety of A_g and not that Y itself is totally geodesic. Thus V being a Lie triple system is only a necessary condition for Y to be totally geodesic. For example, the tangent space of every subvariety of $\mathbb{R}^n$ is a Lie triple system. On the other hand, the case of the Siegel space is much more complicated and the condition for a subspace V of the tangent space of being a Lie triple system is quite strong. In particular, we expect there should be obstructions to the existence of Lie triple systems that are contained in the tangent of M_g. It would be interesting to use some kind of property of the Bergman kernel in this direction.

2.7 Bergman kernel and the form $\hat{\eta}$

The following sections are dedicated to prove the second main result of the Chapter, which links the Bergman kernel of a curve C with the meromorphic form $\hat{\eta}$ constructed in [6].

2.7.1 Definition of $\hat{\eta}$

The construction of the form $\hat{\eta}$ goes as follows. For $x \in C$, let

$$j_x : H^0(C, K_C(2x)) \hookrightarrow H^1(C - \{x\}, \mathbb{C}) = H^1(C, \mathbb{C})$$

be the map that associates to $\omega \in H^0(C, K_C(2x))$ its de Rham cohomology class. This map is an injection since $C \neq \mathbb{P}^1$. As $H^{1,0}(C) \subset j_x(H^0(C, K_C(2x)))$ and $h^0(C, K_C(2x)) = g + 1$, the preimage $j_x^{-1}(H^{0,1}(C))$ is a line. Thus, fixed a chart (U, z) centered at $x \in C$, there exists a unique element φ in this line such that on $U - \{x\}$

$$\varphi = \left(\frac{1}{z^2} + h(z)\right) dz$$

with $h \in \mathcal{O}_C(U)$. Set $u = \frac{\partial}{\partial z}(x)$, and define the map

$$\eta_x : T_x^{1,0} C \to H^0(C, K_C(2x)), \quad \lambda u \mapsto \eta_x(\lambda u) := \lambda \varphi.$$

It is easy to see that η_x does not depend on the choice of the local coordinate. In the following we will also use the fact that if f_u is an elementary potential, then $\partial f_u = \eta_u$, see [6, Lemma 3.1].

Next consider the line bundle $L := K_S(2\Delta)$ on S and set

$$V := p_*(q^*K_C(2\Delta)), \quad E := p_*L.$$

By the projection formula $E = K_C \otimes V$. Also, since $q^*K_C(2\Delta)|_{\{x\}\times C} = q^*K_C(2x)$, we have that $H^0(p^{-1}(x), q^*K_C(2\Delta)) \simeq H^0(C, K_C(2x))$ and the fiber of the holomorphic vector bundle $V \to C$ on $x \in C$ is isomorphic to $H^0(C, K_C(2x))$. Thus $\eta_x \in E_x$. More precisely, the map $x \mapsto \eta_x$ is a holomorphic section of E ([6, Proposition 3.4]).

Finally, since $E = p_*L$, there is an isomorphism between $H^0(C, E)$ and $H^0(S, L)$ that associates to $\alpha \in H^0(C, E)$ the section $\hat{\alpha}$ of L such that $\alpha_x = \hat{\alpha}|_{\{x\}\times C} \in E_x$. The form $\hat{\eta} \in H^0(S, K_S(2\Delta))$ is defined as the holomorphic section of L corresponding to $\eta \in H^0(C, E)$. Note that, in particular, for $u \in T^{1,0}_x C$ and $v \in T^{1,0}_x C$ with $x \neq y$, it holds

$$\eta_x(u)(v) = \hat{\eta}(u, v). \tag{2.7.1}$$

Remark 2.7.1. The form $\hat{\eta}$ also appears in an unpublished book of Gunning [30] under the name of *intrinsic double differential of the second kind.*

2.7.2 Cohomology of $\hat{\eta}$

As explained is Section 1.5.3, the importance of $\hat{\eta}$ comes from the fact that the second fundamental form of the Torelli map outside the hyperelliptic locus coincides with the multiplication by $\hat{\eta}$ [7, 6]. The form $\hat{\eta}$ has been further studied in [3] in relation with projective structures on compact Riemann surfaces. Moreover Section 5 in [3] contains an analysis of the cohomology class of the form $\hat{\eta}$. Denoting by $j : S - \Delta \hookrightarrow S$ the inclusion map, it follows from the exact sequence of homology groups for the pair (S, Δ) and Poincaré and Lefschetz dualities, that the homomorphism $j^* : H^2(S, \mathbb{Z}) \to H^2(S - \Delta, \mathbb{Z})$ is surjective and its kernel is generated by the (pure) class of the diagonal. Consequently, the Hodge decomposition of $H^2(S)$ induces a decomposition of $H^2(S-\Delta)$. In particular, for any $\zeta \in H^0(C, K_S(2\Delta))$ there is $[\gamma] \in H^2(S)$ such that $[\zeta] = j^*[\gamma] \in H^2(S - \Delta)$. Moreover the $(0, 2)$ part of $[\gamma]$ vanishes. So

$$[\gamma] = \gamma^{2,0} + \gamma^{1,1}$$

where $\gamma^{2,0}$ is holomorphic and $\gamma^{1,1}$ is harmonic of type $(1, 1)$. (Harmonicity is for any product metric.) In this context, the fundamental result on the cohomology of $\hat{\eta}$ is that

$$[\hat{\eta}] = j^*[\gamma^{1,1}] \in H^2(S - \Delta),$$

where $\gamma^{1,1}$ is a harmonic $(1, 1)-$form on S. That is, $\hat{\eta}$ has cohomology class in $H^2(S-\Delta)$ of pure type $(1, 1)$. In fact, this implies the characterization of $\hat{\eta}$ as the unique element (up to multiples) of $H^0(S, K_S(2\Delta))$ with cohomology class in $H^2(S - \Delta)$ of pure type $(1, 1)$. In the following we give an explicit description of the harmonic representative $\gamma^{1,1}$ of $\hat{\eta}$.

2.8 Explicit description of the harmonic representative

Denote by $\mathcal{A}^{p,q}$ the sheaf of smooth differential forms of type (p,q) on S. Denote by $\mathcal{A}^{p,q}(n\Delta)$ the sheaf of $(p,q)-$forms having a pole of order at most n on Δ, i.e. those forms ω such that $x^n\omega$ is smooth of type (p,q), where $x=0$ is a local equation of Δ.

The elementary potentials (cf. Section 2.2.2) are defined up to an additive constant. We can normalize them as follows: if $f_u=-1/z+g(z)$, we impose that $g(0)=0$. In the following we fix this normalization. For $u\in(T_xC)_{\mathbb{C}}$, denote by $u^{1,0}$ the $(1,0)-$component of u and set $f_u:=f_{u^{1,0}}$. For $u\in(T_xC)_{\mathbb{C}}$ and $v\in(T_yC)_{\mathbb{C}}$, set

$$\alpha(u,v):=2f_v(p)+f_u(q).$$

Lemma 2.8.1. *The form α is a section of $\mathcal{A}^{1,0}(\Delta)$.*

Proof. We need to show that $f_u(q)$ depends smoothly from u and q when $u\in T_pC$ and $p\neq q$. Fix a chart (U,z) containing both p and q. Define $\varphi: U\times U\to\mathbb{C}$ by

$$f_{\frac{\partial}{\partial z_1}(z_1)}=-\frac{1}{z_2-z_1}+\varphi(z_1,z_2).$$

By the normalization we have $\varphi(z_1,z_1)=0$. We show that $\partial\varphi/\partial z_2$ and $\partial\varphi/\partial\bar z_2$ are smooth functions on $U\times U$. This will prove that $\varphi\in C^\infty(U\times U)$ and hence the lemma. Indeed

$$\frac{\partial\varphi}{\partial\bar z_2}=\frac{\partial}{\partial\bar z_2}\left(f_{\frac{\partial}{\partial z_1}(z_1)}+\frac{1}{z_2-z_1}\right)=2\pi\,\mathbf{K}\left(\left(\frac{\partial}{\partial z_1},0\right),\left(0,\frac{\partial}{\partial\bar z_2}\right)\right),$$

$$\frac{\partial\varphi}{\partial z_2}=\frac{\partial}{\partial z_2}\left(f_{\frac{\partial}{\partial z_1}(z_1)}+\frac{1}{z_2-z_1}\right)=\eta_{\frac{\partial}{\partial z_1}(z_1)}\left(\frac{\partial}{\partial z_2}\right)-\frac{1}{(z_2-z_1)^2}.$$

The second term is smooth since $\hat\eta\in H^0(S,K_S(2\Delta))$.

□

We can now prove Theorem B.

Theorem 2.8.2. *The Bergman kernel is the $(1,1)-$harmonic representative of the cohomology class of $\hat\eta\in H^0(S,K_S(2\Delta))$ in $H^2(S-\Delta)$. More precisely,*

$$\hat\eta-2\pi\mathbf{K}=d\alpha,$$

that is $\partial\alpha=\hat\eta$ and $\bar\partial\alpha=-2\pi\mathbf{K}$.

Proof. We first observe that for $u \in (T_xC)_{\mathbb{C}}$ and $v \in (T_yC)_{\mathbb{C}}$, denoted by U, V two vector fields on C such that $U_x = u$ and $V_y = v$, since $[(U,0),(0,V)] = 0$, we have that

$$d\alpha((u,0),(0,v)) = (U,0)(\alpha(0,v)) - (0,V)(\alpha(u,0)).$$

Now assume $u \in T_x^{1,0}C$, $v \in T_y^{1,0}C$ and that U and V are (1,0). For $x \neq y$, $\partial f_u = \eta_x(u)$ and we get

$$\partial\alpha((u,0),(0,v)) = 2(U,0)f_v - (0,V)f_u =$$

$$= 2\partial f_v(u) - \partial f_u(v) = 2\eta_y(v)(u) - \eta_x(u)(v) = \hat{\eta}((u,0),(0,v)),$$

where for the last equality we used (2.7.1) and the symmetry of $\hat{\eta}$ (see [6, Lemma 3.5]). Similarly using Lemma 2.2.4

$$\bar{\partial}\alpha((u,0),(0,\overline{v})) = -(0,\overline{V})f_u = -\bar{\partial}f_u(\overline{v}) = -2\pi\mathbf{K}((u,0),(0,\overline{v})).$$

□

Remark 2.8.3. It is quite hard to control the behaviour of $\hat{\eta}$ outside of the diagonal. Only along Δ its behaviour admits an algebraic description, via the second Gaussian map μ_2, see [7]. Our result goes in the direction of a better understanding of $\hat{\eta}$ and the second fundamental form.

Chapter 3

Uniformizing symmetric spaces of the counterexamples

In this Chapter we present the results of [59]. Here the objects of study are the known counterexamples to the Coleman-Oort conjecture 1.5.6. Indeed, for low genus there exists examples of (positive-dimensional) Shimura subvarieties Z of A_g such that $Z \subset \overline{j(\mathsf{M}_g)}$ and $Z \cap j(\mathsf{M}_g) \neq \emptyset$. All the examples known so far are in genus $g \leq 7$ and can be divided in three classes:

1. those obtained as families of Galois covers of $\mathbb{P}^1$ (for a complete list see [15]);
2. those obtained as families of Galois covers of elliptic curves (see [16]);
3. those obtained via fibrations constructed on the examples in (2) (see [19]).

There are some overlaps between (1) and (2). The idea of the construction of the examples in (1) and (2) is the following: let $\mathcal{C} \to B$ be the family of all Galois covers $C_t \to C'_t = C_t/G$, with fixed genera $g = g(C_t)$, $g' = g(C'_t)$, ramification and monodromy. (See e.g. [26], cf. Chapter 4). Let Z denote the closure in A_g of the locus described by $[JC_t]$ for C_t varying in the family. The group G acts holomorphically on every curve C of the family, so it maps injectively into $Sp(2g, \mathbb{R})$. Denote by Γ the image of G in $Sp(2g, \mathbb{R})$. The image of $\mathfrak{S}^\Gamma$ in A_g is a Shimura variety (See e.g. [15, Proposition 3.7]). Moreover, $Z \subset \pi(\mathfrak{S}^\Gamma)$ ([15, Theorem 3.9]). Then, the condition

$$(\dim Z =) \dim(S^2 H^0(C, K_C))^G = \dim H^0(C, 2K_C)^G (= \dim \pi(\mathfrak{S}^\Gamma)), \quad (\star)$$

implies that $Z = \pi(\mathfrak{S}^\Gamma)$. Thus, under the condition $(\star)$, Z provides an example of a Shimura subvariety of A_g contained in the Torelli locus, whose uniformizing symmetric space is $\mathfrak{S}^{\Gamma}$. In particular, it is a Shimura subvariety of A_g generically contained in $j(\mathsf{M}_g)$. Table 2 in [15] lists all examples with $g' = 0$ and $g \leq 9$. Table 2 in [16] lists all examples with $g' = 1$ and $g \leq 9$. Recently, it has been proven in [8] that these families are the only positive-dimensional families of Galois covers satisfying $(\star)$ with $2 \leq g \leq 100$. Note that, in general, it is not known whether $(\star)$ is also necessary for a family to yield a Shimura variety. More generally, very little is known about other counterexamples.

In this Chapter we determine which symmetric space uniformizes each of the known counterexamples (cf. Definition 1.4.1). Since all $1-$dimensional irreducible hermitian symmetric spaces of the non-compact type are isomorphic to the Poincaré disc, we will focus on counterexamples of dimension greater than one. We dedicate Section 3.1 to some considerations on Lie algebras and symmetric spaces that will be relevant in the sequel. In Section 3.2 it is computed the uniformizing symmetric space first for the examples belonging only to (1), next for the six examples in (2) (actually five, since one of them has dimension 1), including the four examples that also appear in (1). Denote by $B_n(\mathbb{C}) = \{w \in \mathbb{C}^n : \sum_{k=1}^n w^i \overline{w}^i < 1\}$ the open unit ball in $\mathbb{C}^n$. Our results are the following:

Theorem 3.0.1. *The uniformizing symmetric spaces for the examples in [15] of dimension ≥ 2 are the following:*

Family	g	G	$\dim_{\mathbb{C}}$	*Uniformizer*
(2)	*2*	$\mathbb{Z}/2$	*3*	$\mathfrak{S}_2$
(6)	*3*	$\mathbb{Z}/3$	*2*	$B_2(\mathbb{C})$
(8)	*3*	$\mathbb{Z}/4$	*2*	$B_2(\mathbb{C})$
(10)	*4*	$\mathbb{Z}/3$	*3*	$B_3(\mathbb{C})$
(14)	*4*	$\mathbb{Z}/6$	*2*	$B_2(\mathbb{C})$
(16)	*6*	$\mathbb{Z}/5$	*2*	$B_2(\mathbb{C})$
(26)	*2*	$\mathbb{Z}/2 \times \mathbb{Z}/2$	*2*	$B_1(\mathbb{C}) \times B_1(\mathbb{C})$
(27)	*3*	$\mathbb{Z}/2 \times \mathbb{Z}/2$	*3*	$B_1(\mathbb{C}) \times B_1(\mathbb{C}) \times B_1(\mathbb{C})$
(31)	*3*	S_3	*2*	$B_1(\mathbb{C}) \times B_1(\mathbb{C})$
(32)	*3*	D_4	*2*	$B_1(\mathbb{C}) \times B_1(\mathbb{C})$

Theorem 3.0.2. *The uniformizing symmetric spaces for the examples in [16] of dimension ≥ 2 are the following:*

Family	*g*	*G*	$\dim_{\mathbb{C}}$	*Uniformizer*
(1e)=(26)	*2*	$\mathbb{Z}/2$	*2*	$B_1(\mathbb{C}) \times B_1(\mathbb{C})$
(2e)	*3*	$\mathbb{Z}/2$	*4*	$B_1(\mathbb{C}) \times \mathfrak{S}_2$
(3e)=(31)	*3*	$\mathbb{Z}/3$	*2*	$B_1(\mathbb{C}) \times B_1(\mathbb{C})$
(4e)=(32)	*3*	$\mathbb{Z}/4$	*2*	$B_1(\mathbb{C}) \times B_1(\mathbb{C})$
(6e)	*4*	$\mathbb{Z}/3$	*3*	$B_1(\mathbb{C}) \times B_2(\mathbb{C})$

The six families of Galois covers of elliptic curves have been further studied in [19], where it is shown that these families admit two fibrations in totally geodesic subvarieties, countably many of which in each family are Shimura. If $f : C \to C'$ is a Galois covering of an elliptic curve C' with group G, these fibrations are given by the maps

$$\begin{aligned} P:\ [C \to C'] &\mapsto Prym(C, C') \in \mathcal{A}^{\delta}_{g-1} \\ \varphi:\ [C \to C'] &\mapsto [JC'] \in \mathcal{A}_1. \end{aligned}$$

In Section 3.2.3 we discuss the relationship between the fibrations and the uniformizing symmetric space of the examples. More precisely, we prove Theorem 3.2.3 (that is Theorem C in the Introduction). In particular, the result also shows how to uniformize the examples in (3).

3.1 Cartan decomposition

We devote this section to some considerations on Lie algebras and symmetric spaces that will be relevant in the sequel. For the basic definitions and facts related to symmetric spaces (e.g., the definition of symmetric pair and of euclidean factor), see Section 1.1.

Fix a rank $2g$ lattice Λ and an alternating form $Q : \Lambda \times \Lambda \longrightarrow \mathbb{Z}$ of type $(1, ..., 1)$. Let $G := \mathrm{Sp}(\Lambda_{\mathbb{R}}, Q)$ and $\mathfrak{S} = \mathfrak{S}(\Lambda_{\mathbb{R}}, Q)$ the Siegel space:

$$\mathfrak{S} = \{J \in GL(\Lambda_{\mathbb{R}}) :\ J^2 = -I,\ J^*Q = Q,\ Q(x, Jx) > 0,\ \forall x \neq 0\}. \quad (3.1.1)$$

Fix a finite subgroup $\Gamma \subset G$ and consider $\mathfrak{S}^{\Gamma}$. As a consequence of the Cartan fixed point theorem, $\mathfrak{S}^{\Gamma}$ is nonempty and hence it is a connected complex submanifold of $\mathfrak{S}$ (see e.g. [15, Lemma 3.3]). For $J \in \mathfrak{S}$, denote by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the Cartan decomposition associated to $\mathfrak{S}$ and by $K = Stab_J(G)$. Set $\mathfrak{Z}_{\mathfrak{g}}(\Gamma) = \{X \in \mathfrak{g} : \mathrm{Ad}_{\Gamma}(X) = X\}$, and define similarly $\mathfrak{Z}_{\mathfrak{k}}(\Gamma)$ and $\mathfrak{Z}_{\mathfrak{p}}(\Gamma)$. For $J \in \mathfrak{S}^{\Gamma}$, $T_J\mathfrak{S}^{\Gamma} = (T_J\mathfrak{S})^{\Gamma} = \mathfrak{Z}_{\mathfrak{p}}(\Gamma)$. The following is standard, we include the proof for the reader's convenience.

Lemma 3.1.1. *Let* $J \in \mathfrak{S}^{\Gamma}$. *Then* $\mathfrak{Z}_{\mathfrak{g}}(\Gamma) = \mathfrak{Z}_{\mathfrak{k}}(\Gamma) \oplus \mathfrak{Z}_{\mathfrak{p}}(\Gamma)$. *In particular,* $\mathfrak{Z}_{\mathfrak{p}}(\Gamma) = \mathfrak{Z}_{\mathfrak{g}}(\Gamma) \cap \mathfrak{p}$.

Proof. Let $X \in \mathfrak{Z}_{\mathfrak{g}}(\Gamma)$, and write $X = u + v$ with $u \in \mathfrak{k}$ and $v \in \mathfrak{p}$. Since $J \in \mathfrak{S}^\Gamma = \{J \in \mathfrak{S} : \ \gamma J \gamma^{-1} = J \ \forall \gamma \in \Gamma\}$, we have that $\Gamma \subset Stab_J = K$. It follows that $\mathrm{Ad}_\Gamma(\mathfrak{p}) \subset \mathfrak{p}$ and $\mathrm{Ad}_\Gamma(\mathfrak{k}) \subset \mathfrak{k}$. Thus $X = \mathrm{Ad}_\Gamma(X) = \mathrm{Ad}_\Gamma(u) + \mathrm{Ad}_\Gamma(v)$, with $\mathrm{Ad}_\Gamma(u) \in \mathfrak{k}$ and $\mathrm{Ad}_\Gamma(v) \in \mathfrak{p}$. We conclude that $u = \mathrm{Ad}_\Gamma(u)$ and $\mathrm{Ad}_\Gamma(v) = v$. □

Observe that, since Γ is a group of isometries of $\mathfrak{S}$, $\mathfrak{S}^\Gamma$ is a totally geodesic submanifold of $\mathfrak{S}$ (see e.g. [37, Theorem 5.1, p. 59]). In particular, it is an hermitian symmetric space of the non-compact type. The image of $\mathfrak{S}^\Gamma$ in A_g is a Shimura variety (See e.g. [15, Proposition 3.7]).

Lemma 3.1.2. *Let Z be a complex totally geodesic submanifold of $\mathfrak{S}$. Then Z has no euclidean factor.*

Proof. Since $\mathfrak{S}$ is an hermitian symmetric space, the complex structure $\widehat{I}$ on $\mathfrak{S}$ is induced by $\mathrm{Ad}(z)|_{\mathfrak{p}} : \mathfrak{p} \to \mathfrak{p}$ with $z \in Z(K)$. More precisely, one can show that the matrix representing z in an appropriate basis is given by $z = \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ -I & I \end{pmatrix} \in Sp(2g, \mathbb{R}) \simeq G$. With this choice it easy is to see that for $X \in \mathfrak{p}, X \neq 0$, we have $[X, \widehat{I}X] \neq 0$. □

Lemma 3.1.3. *Let (G, K) be a symmetric pair with no euclidean factor and Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Suppose that G acts almost effectively on the coset space $M = G/K$. Then $\mathfrak{k} = [\mathfrak{p}, \mathfrak{p}]$.*

For a proof see e.g. [33, Theorem 4.1, p. 243]. As a consequence of Lemma 3.1.2, $\mathfrak{S}^\Gamma$ has no euclidean factor. We can thus apply Lemma 3.1.3 to an almost effective symmetric pair associated with $\mathfrak{S}^\Gamma$. Recalling also Lemma 3.1.1, we get the following.

Corollary 3.1.4. *If (G', K') is an almost effective symmetric pair associated to $\mathfrak{S}^\Gamma$, then its Cartan decomposition $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{p}'$ is given by*

$$\mathfrak{p}' = \mathfrak{Z}_{\mathfrak{p}}(\Gamma) = \mathfrak{Z}_{\mathfrak{g}}(\Gamma) \cap \mathfrak{p}, \qquad \mathfrak{k}' = [\mathfrak{Z}_{\mathfrak{p}}(\Gamma), \mathfrak{Z}_{\mathfrak{p}}(\Gamma)].$$

Notice that since $\mathfrak{S}^\Gamma$ is of the noncompact type, the Cartan decomposition in Corollary 3.1.4 determines the symmetric space up to isomorphism.

The following table exhausts the list of irreducible Hermitian symmetric spaces of the non-compact type up to dimension 3. (See [33, Table V, p. 518]).

	Space	Complex Dimension	
A III(p=1,q)	$SU(1,q)/S(U(1) \times U(q))$	q	$1 \leq q \leq 3$
C I(n=2)	$Sp(4, \mathbb{R})/U(4)$	3	

The possibilities are thus the following: $B_1(\mathbb{C})$ in dimension 1 (type A III($p = 1, q = 1$)), $B_2(\mathbb{C})$ in dimension 2 (A III ($p = 1, q = 2$)), $B_3(\mathbb{C})$ (A III ($p = 1, q = 3$)) and $\mathfrak{S}_2$ (CI(3)) in dimension 3.

3.2 Uniformizing symmetric spaces

In this section we compute the Cartan decomposition associated with the symmetric pairs uniformizing every known counter-example to the Coleman-Oort conjecture. By the previous section, it determines the symmetric spaces themself.

If $f : C \to C'$ is one of the coverings in a family of G-coverings, the action of G on C induces the following action of G on holomorphic $1-$forms

$$\rho : G \to GL(H^0(C, K_C)), \qquad \rho(g)(\omega) = g.\omega := (g^{-1})^*(\omega).$$

Notice that the equivalence class of the representation ρ only depends on the family (not on the point of the family). The homomorphism ρ maps G injectively into $Sp(\Lambda, Q)$, where $\Lambda = H^1(C, \mathbb{Z})$ and Q is the cup product (see e.g. [14, p. 270]). Denote by Γ the image of G in $Sp(\Lambda, Q)$ and $\mathfrak{S} = \mathfrak{S}(\Lambda_{\mathbb{R}}, Q)$, defined as in (3.1.1). In the sequel, our technique will be the following: we think $Sp(\Lambda_{\mathbb{R}}, Q) \subset GL(H^1(C, \mathbb{C}))$. Through the study of the action of Γ on $H^1(C, \mathbb{C})$, we fix an appropriate basis of $H^1(C, \mathbb{C})$ so that we can nicely describe first $\Gamma \subset Sp(\Lambda_{\mathbb{R}}, Q)$, and next the spaces $\mathfrak{Z}_{\mathfrak{p}}(\Gamma)$, and $[\mathfrak{Z}_{\mathfrak{p}}(\Gamma), \mathfrak{Z}_{\mathfrak{p}}(\Gamma)]$. By Corollary 3.1.4, these spaces give the Lie algebra decomposition associated with the symmetric space $\mathfrak{S}^{\Gamma}$ that uniformizes the family.

Notation: We will denote by $M(n, m, \mathbb{C})$ the space of $n \times m$ complex matrices.

3.2.1 Galois coverings of the line. Cyclic case.

We start with the case of a cyclic group G. Fix a family of covers, and let a_i be an element of order m_i in G that represents the local monodromy of the covering $\pi : C \to \mathbb{P}^1$ at the $i-$th branch point. In the case the group G is cyclic, the study of the multipilicity of a given irreducible representation of G in $H^0(C, K_C)$ reduces to the study of the eigenspaces of the generator of the group. The following is the Chevalley-Weil formula in the cyclic case.

Lemma 3.2.1. *Let $\mathbb{Z}/m = \langle \zeta \rangle$ be the cyclic group of order m. Consider the $\mathbb{Z}/m-$cover $\pi : C_t \to \mathbb{P}^1$, with $t = (t_1, ..., t_N)$ branch points in $\mathbb{P}^1$ with* *local monodromy a_i about t_i. For $n \in \mathbb{Z}/m$ we write $H^0(C_t, K_{C_t})_{(n)} := \{\omega \in H^0(C_t, K_{C_t}) : \zeta.\omega = \zeta^n\omega\}$. Then*

$$\dim H^0(C_t, K_{C_t})_{(n)} = -1 + \sum_{i=1}^{N} \frac{[na_i]_m}{m},$$

where $[a]_m$ denotes the unique representative of $a \in \mathbb{Z}/m$ in $\{0, ..., m-1\}$.

Let $\langle\ ,\ \rangle$ be the Hodge Hermitian product on $H^0(C, K_C)$, defined by $\langle \omega, \omega' \rangle := i \int_X \omega \wedge \overline{\omega'}$. Fixing a unitary basis of $H^0(C, K_C)_{(n)}$ for each $n \in \mathbb{Z}/m$, one obtains a unitary basis $\{\omega_1, ..., \omega_g\}$ of $H^0(C, K_C)$ with respect to which the matrix A_0 that represents the generator $\rho(\zeta)$ of Γ is diagonal. The multiplicity of each $m-$th root of unity on the diagonal is given by Lemma 3.2.1. Consider now the action $\tilde{\rho} : G \to GL(H^1(C, \mathbb{C}))$ of G on the whole $H^1(C, \mathbb{C})$ by pullback. Notice that if $\omega \in H^0(C, K_C)$, then $\tilde{\rho}(\gamma)(\overline{\omega}) = \overline{\rho(\gamma)(\omega)}$. Thus, with respect to the basis $\{\omega_1, ..., \omega_g, \overline{\omega_1}, ..., \overline{\omega_g}\}$ of $H^1(C, \mathbb{C})$, $\tilde{\rho}(\zeta)$ is represented by the matrix

$$A = \begin{pmatrix} A_0 & 0 \\ 0 & \overline{A_0} \end{pmatrix}. \tag{3.2.1}$$

Lemma 3.2.2. *Let $\mathfrak{g}' = \mathfrak{p}' \oplus \mathfrak{k}'$ be the Cartan decomposition of the uniformizing symmetric space $\mathfrak{S}^\Gamma$ associated to a family of cyclic covering of $\mathbb{P}^1$. Then, with the notation above, $\mathfrak{k}' = [\mathfrak{p}', \mathfrak{p}']$ and*

$$\mathfrak{p}' = \mathfrak{Z}_{\mathfrak{p}}(A) = \{\begin{pmatrix} 0 & \overline{D} \\ D & 0 \end{pmatrix} : D \in M(2, 2, \mathbb{C}),\ D = D^t,\ DA_0 = \overline{A_0}D\}.$$

Proof. Clearly, $\mathfrak{S}^\Gamma = \mathfrak{S}^A = \{J \in \mathfrak{S} : JA = AJ\}$. For $J \in \mathfrak{S}^\Gamma$, denote by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the Cartan decomposition associated to $\mathfrak{S}$. Fix the unitary basis $\{\omega_1, ..., \omega_g, \overline{\omega_1}, ..., \overline{\omega_g}\}$ of $H^1(C, \mathbb{C})$ defined above. Observe that, since $\langle \omega, \omega' \rangle = iQ(\omega, \overline{\omega'})$, the matrix representing Q in this basis is $Q = \begin{pmatrix} 0 & iI_g \\ -iI_g & 0 \end{pmatrix}$. An element $U \in \mathfrak{gl}(H^1(X, \mathbb{C}))$ belongs to $\mathfrak{g} \simeq \mathfrak{sp}(2g, \mathbb{R})$ if and only if it is real, and thus is a block matrix of the form $\begin{pmatrix} C & \overline{D} \\ D & \overline{C} \end{pmatrix}$, where $C, D \in M(g, g, \mathbb{C})$, and satisfies

$$0 = U^tQ + QU = i \begin{pmatrix} D - D^t & C^t + \overline{C} \\ -\overline{C}^t - C & \overline{D}^t - D \end{pmatrix}.$$

Thus $D = D^t$ and $C \in \mathfrak{u}(g)$. One immediately gets the Cartan decomposition

$$U = \begin{pmatrix} C & 0 \\ 0 & \overline{C} \end{pmatrix} + \begin{pmatrix} 0 & \overline{D} \\ D & 0 \end{pmatrix}, \qquad \text{with } \begin{pmatrix} C & 0 \\ 0 & \overline{C} \end{pmatrix} \in \mathfrak{k},\ \begin{pmatrix} 0 & \overline{D} \\ D & 0 \end{pmatrix} \in \mathfrak{p}. \tag{3.2.2}$$

Now $U \in \mathfrak{Z}_{\mathfrak{g}}(A)$ if and only if $UA = AU$, i.e.

$$\begin{pmatrix} C & \overline{D} \\ D & \overline{C} \end{pmatrix} \begin{pmatrix} A_0 & 0 \\ 0 & \overline{A_0} \end{pmatrix} = \begin{pmatrix} A_0 & 0 \\ 0 & \overline{A_0} \end{pmatrix} \begin{pmatrix} C & \overline{D} \\ D & \overline{C} \end{pmatrix}.$$

In other words, U must preserve the eigenspaces of A. This concludes the study of $\mathfrak{S}^\Gamma$. Indeed, if $\mathfrak{S}^\Gamma$ has Cartan decomposition $\mathfrak{g}' = \mathfrak{p}' \oplus \mathfrak{k}'$, it follows from Corollary 3.1.4, that $\mathfrak{k}' = [\mathfrak{p}', \mathfrak{p}']$ and

$$\mathfrak{p}' = \mathfrak{Z}_{\mathfrak{p}}(A) = \mathfrak{Z}_{\mathfrak{g}}(A) \cap \mathfrak{p} = \{ \begin{pmatrix} 0 & \overline{D} \\ D & 0 \end{pmatrix}, \ D = D^t, \ DA_0 = \overline{A_0} D \}.$$

□

With this Lemma, we can determine the uniformizing symmetric space for all the cyclic cases. We will follow the same outline of the argument above. In a few words, the procedure is the following: first, consider the decomposition (3.2.2) and see which conditions C and D must satisfy to preserve the eigenspaces of the generator A. This gives $\mathfrak{p}'$ as in Lemma 3.2.2. Next calculate $\mathfrak{k}' = [\mathfrak{p}', \mathfrak{p}']$. Finally, identify $\mathfrak{S}^\Gamma$ in terms of the classification of Hermitian symmetric spaces of the non-compact type (see Section 3.1). Note that $\mathfrak{S}^\Gamma$ may be a reducible symmetric space. For the reader's convenience, we will carry out all the explicit calculations for the first family we present. The obtained results are summarized in Theorem 3.0.1.

Family (8) This is the family of cyclic covers $\pi : C \to \mathbb{P}^1$ of $\mathbb{P}^1$ with group $G = \mathbb{Z}/4$, ramification data $\mathbf{m} = (2^3, 4^2)$, $g = 3$ and dimension 2. In this case $a = (2^3, 1^2)$. Consider the generator $\zeta = e^{2\pi i/4}$ of $G = \langle \zeta \rangle$. Applying Lemma 3.2.1, we get $\dim H^0(C, K_C)_{(1)} = 1$, $\dim H^0(C, K_C)_{(2)} = 0$, $\dim H^0(C, K_C)_{(3)} = 2$, $\dim H^0(C, K_C)_{(4)} = 0$; that is, the eigenspace of ζ^3 has dimension 2, and the eigenspace of ζ has dimension 1. Now fix $\{\omega_1, \omega_2\}$ an $\langle\ ,\ \rangle$−unitary basis of $H^0(C, K_C)_{(3)}$ and $\{\omega_3\}$ a generator of $H^0(C, K_C)_{(1)}$ with $\langle \omega_3, \omega_3 \rangle = 1$. Since ω_3 and ω_1, ω_2 are eigenvectors of $\rho(\zeta)$ with respect to different eigenvalues, $\{\omega_1, \omega_2, \omega_3\}$ is a $\langle\ ,\ \rangle$−unitary basis of $H^0(C, K_C)$. With respect to this basis $\rho(\zeta)$ is represented by the matrix

$$A_0 = \begin{pmatrix} \zeta^3 & 0 & 0 \\ 0 & \zeta^3 & 0 \\ 0 & 0 & \zeta \end{pmatrix}.$$

As above, consider now the action $\tilde{\rho} : G \to GL(H^1(C, \mathbb{C}))$ of G on the whole $H^1(C, \mathbb{C})$. With respect to the basis $\beta = \{\omega_1, \omega_2, \omega_3, \overline{\omega_1}, \overline{\omega_2}, \overline{\omega_3}\}$ of $H^1(C, \mathbb{C})$, $\tilde{\rho}(\zeta)$ is represented by the matrix

$$A = \begin{pmatrix} A_0 & 0 \\ 0 & \overline{A_0} \end{pmatrix}.$$

Let Γ denote the image of G inside $Sp(H^1(C,\mathbb{R}),Q) \subset GL(H^1(C),\mathbb{C})$. Γ is generated by the matrix A. Thus the symmetric space uniformizing the image in A_g of the family is $\mathfrak{S}^A = \{J \in \mathfrak{S} : \; JA = AJ\}$. For $J \in \mathfrak{S}^\Gamma$, denote by $\mathfrak{g} = \mathfrak{sp}(6,\mathbb{R})$, and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the Cartan decomposition associated to $\mathfrak{S}$. Recall that $\mathfrak{p} = \mathfrak{g} \cap \{X \in \mathfrak{g} : \; X = X^t\}$ and $\mathfrak{k} = \mathfrak{u}(3)$. An element $U \in \mathfrak{gl}(H^1(C,\mathbb{C}))$ that belongs to $\mathfrak{g}$ is of the form (3.2.2):

$$U = \begin{pmatrix} C & 0 \\ 0 & \overline{C} \end{pmatrix} + \begin{pmatrix} 0 & \overline{D} \\ D & 0 \end{pmatrix}, \qquad \text{with } \begin{pmatrix} C & 0 \\ 0 & \overline{C} \end{pmatrix} \in \mathfrak{k}, \quad \begin{pmatrix} 0 & \overline{D} \\ D & 0 \end{pmatrix} \in \mathfrak{p}.$$

Now $U \in \mathfrak{Z}_{\mathfrak{g}}(A)$ if and only if $UA = AU$, i.e.

$$\begin{pmatrix} C & \overline{D} \\ D & \overline{C} \end{pmatrix} \begin{pmatrix} A_0 & 0 \\ 0 & \overline{A_0} \end{pmatrix} = \begin{pmatrix} A_0 & 0 \\ 0 & \overline{A_0} \end{pmatrix} \begin{pmatrix} C & \overline{D} \\ D & \overline{C} \end{pmatrix}.$$

Thus, in this case, $U \in \mathfrak{Z}_{\mathfrak{g}}(A)$ satisfies

$$C = \begin{pmatrix} E & 0 \\ 0 & i\lambda \end{pmatrix}, \; E \in \mathfrak{u}(2), \; \lambda \in \mathbb{R}, \quad D = \begin{pmatrix} 0 & d \\ d^t & 0 \end{pmatrix}, \; d \in M(2,1,\mathbb{C}).$$

Denoted by $\mathfrak{g}' = \mathfrak{p}' \oplus \mathfrak{k}'$ the Cartan decomposition of $\mathfrak{S}^\Gamma$, it follows from Lemma 3.2.2 that

$$\mathfrak{p}' = \mathfrak{Z}_{\mathfrak{p}}(A) = \{\begin{pmatrix} 0 & \overline{D} \\ D & 0 \end{pmatrix}, \; D = \begin{pmatrix} 0 & d \\ d^t & 0 \end{pmatrix}, \; d \in M(2,1,\mathbb{C})\}.$$

Moreover, $\mathfrak{k}' = [\mathfrak{p}', \mathfrak{p}']$. We now compute $\mathfrak{k}'$. Let $X, Y \in \mathfrak{p}'$, $X = \begin{pmatrix} 0 & \overline{D} \\ D & 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 0 & \overline{E} \\ E & 0 \end{pmatrix}$. Then

$$[X,Y] = XY - YX = D = \begin{pmatrix} \overline{D}E - \overline{E}D & 0 \\ 0 & D\overline{E} - E\overline{D} \end{pmatrix}.$$

If $D = \begin{pmatrix} 0 & d \\ d^t & 0 \end{pmatrix}$, with $d \in M(2,1,\mathbb{C})$ and $E = \begin{pmatrix} 0 & e \\ e^t & 0 \end{pmatrix}$, with $e \in M(2,1,\mathbb{C})$, we have

$$\overline{D}E - \overline{E}D = \begin{pmatrix} \overline{d}e^t - \overline{e}d^t & 0 \\ 0 & \overline{d}^t e - \overline{e}^t d \end{pmatrix}.$$

Set $F = \overline{d}e^t - \overline{e}d^t$ and notice that $\overline{d}^t e - \overline{e}^t d = tr(F) = 2iIm(\langle e, d \rangle)$, where $\langle \; , \; \rangle$ denotes the standard hermitian product in $\mathbb{C}^2$. We conclude

$$\mathfrak{k}' = [\mathfrak{p}', \mathfrak{p}'] \subset \{\begin{pmatrix} C & 0 \\ 0 & \overline{C} \end{pmatrix}, \; C = \begin{pmatrix} F & 0 \\ 0 & \mathrm{tr}(F) \end{pmatrix}, \; F \in \mathfrak{u}(2)\}.$$

The choices $\{d = (-i/2, 0)^t,\ e = (1, 0)^t\}$, $\{d = (0, -i/2)^t,\ e = (0, 1)^t\}$, $\{d = (0, 1)^t,\ e = (-1, 0)^t\}$, $\{d = (0, 1)^t,\ e = (i, 0)^t\}$ for D and E, give as F the following matrices:

$$\begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

which constitute a basis for $\mathfrak{u}(2)$. Thus the equality sign holds:

$$\mathfrak{k}' = [\mathfrak{p}', \mathfrak{p}'] = \{\begin{pmatrix} C & 0 \\ 0 & \overline{C} \end{pmatrix},\ C = \begin{pmatrix} F & 0 \\ 0 & \mathrm{tr}(F) \end{pmatrix},\ F \in \mathfrak{u}(2)\}.$$

Consider now the adjoint representation $\mathrm{ad} : \mathfrak{k}' \to \mathfrak{gl}(\mathfrak{p}')$ of $\mathfrak{k}'$ on $\mathfrak{p}'$. If $F \in \mathfrak{k}' = \mathfrak{u}(2)$, and $d \in \mathbb{C}^2 = \mathfrak{p}'$, then $\mathrm{ad}_F(d) = \overline{E}d - Tr(E)d$. One easily checks it is an irreducible representation and this proves that $\mathfrak{S}^A$ is an irreducible hermitian symmetric space of complex dimension 2. We conclude that $\mathfrak{S}^\Gamma$ is of type A III ($p = 1, q = 2$).

By the same procedure one can treat all the other cyclic cases. We list here the results of all the computations.

Family (10) $A_0 = \mathrm{diag}(\zeta^2, \zeta^2, \zeta^2, \zeta)$, with $\zeta = e^{2\pi i/3}$.
$\mathfrak{p}' = \mathfrak{Z}_{\mathfrak{p}}(A) = \{\begin{pmatrix} 0 & \overline{D} \\ D & 0 \end{pmatrix},\ D = \begin{pmatrix} 0 & d \\ d^t & 0 \end{pmatrix},\ d \in M(3, 1, \mathbb{C})\}$.
Let $X, Y \in \mathfrak{p}'$, $X = \begin{pmatrix} 0 & \overline{D} \\ D & 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 0 & \overline{E} \\ E & 0 \end{pmatrix}$. Then

$$[X, Y] = XY - YX = \begin{pmatrix} \overline{D}E - \overline{E}D & 0 \\ 0 & D\overline{E} - E\overline{D} \end{pmatrix}.$$

If $D = \begin{pmatrix} 0 & d \\ d^t & 0 \end{pmatrix}$, with $d \in M(3, 1, \mathbb{C})$ and $E = \begin{pmatrix} 0 & e \\ e^t & 0 \end{pmatrix}$, with $e \in M(3, 1, \mathbb{C})$, we have

$$\overline{D}E - \overline{E}D = \begin{pmatrix} \overline{d}e^t - \overline{e}d^t & 0 \\ 0 & \overline{d}^t e - \overline{e}^t d \end{pmatrix}.$$

Set $F = \overline{d}e^t - \overline{e}d^t$ and notice that $\overline{d}^t e - \overline{e}^t d = tr(F) = 2iIm(\langle e, d \rangle)$, where $\langle\ ,\ \rangle$ denotes the standard hermitian product in $\mathbb{C}^3$. Thus, similarly to the previous family, we get

$$\mathfrak{k}' = [\mathfrak{p}', \mathfrak{p}'] = \{\begin{pmatrix} C & 0 \\ 0 & \overline{C} \end{pmatrix},\ C = \begin{pmatrix} F & 0 \\ 0 & \mathrm{tr}(F) \end{pmatrix},\ F \in \mathfrak{u}(3)\}.$$

In particular $\dim \mathfrak{k}' = 9$. Looking at the adjoint representation $\mathrm{ad} : \mathfrak{k}' \to \mathfrak{gl}(\mathfrak{p}')$, one checks that $\mathfrak{S}^A$ is irreducible. Thus it is an irreducible hermitian symmetric space of dimension 3. Looking at $\dim \mathfrak{k}'$ and comparing it with the possibilities in Section 3.1, we conclude that $\mathfrak{S}^\Gamma$ is of type A III ($p=1, q=3$).

Family (2) Here $A_0 = -I_2$. Thus $\mathfrak{S}^\Gamma = \mathfrak{S}^A = \mathfrak{S} = \mathfrak{S}_2$. In fact here $Z = \overline{\mathsf{M}_2} = \mathsf{A}_2$.

Family (6) $A_0 = \mathrm{diag}(\zeta^2, \zeta^2, \zeta)$, with $\zeta = e^{2\pi i/3}$.
$\mathfrak{p}' = \{\begin{pmatrix} 0 & \overline{D} \\ D & 0 \end{pmatrix},\ D = \begin{pmatrix} 0 & d \\ d^t & 0 \end{pmatrix},\ d \in M(2,1,\mathbb{C})\}$.
$\mathfrak{k}' = [\mathfrak{p}', \mathfrak{p}'] = \{\begin{pmatrix} C & 0 \\ 0 & \overline{C} \end{pmatrix},\ C = \begin{pmatrix} F & 0 \\ 0 & \mathrm{tr}(F) \end{pmatrix},\ F \in \mathfrak{u}(2)\}$.
$\mathfrak{S}^\Gamma$ is of type A III ($p=1, q=2$).

Family (14) $A_0 = \mathrm{diag}(\zeta^5, \zeta^5, \zeta^2, \zeta)$, with $\zeta = e^{2\pi i/6}$.
$\mathfrak{p}' = \{\begin{pmatrix} 0 & \overline{D} \\ D & 0 \end{pmatrix},\ D = \begin{pmatrix} 0 & 0 & d \\ 0 & 0 & 0 \\ d^t & 0 & 0 \end{pmatrix},\ D \in M(4,\mathbb{C}),\ d \in M(2,1,\mathbb{C})\}$.
$\mathfrak{k}' = [\mathfrak{p}', \mathfrak{p}'] = \{\begin{pmatrix} C & 0 \\ 0 & \overline{C} \end{pmatrix},\ C = \begin{pmatrix} F & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathrm{tr}(F) \end{pmatrix},\ F \in \mathfrak{u}(2)\}$.
$\mathfrak{S}^\Gamma$ is of type A III ($p=1, q=2$).

Family (16) $A_0 = \mathrm{diag}(\zeta^4.\zeta^4, \zeta^4, \zeta^3, \zeta^3, \zeta^2)$, with $\zeta = e^{2\pi i/5}$.
$\mathfrak{p}' = \{\begin{pmatrix} 0 & \overline{D} \\ D & 0 \end{pmatrix},\ D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & d \\ 0 & d^t & 0 \end{pmatrix},\ D \in M(6,\mathbb{C}),\ d \in M(2,1,\mathbb{C})\}$.
$\mathfrak{k}' = [\mathfrak{p}', \mathfrak{p}'] = \{\begin{pmatrix} C & 0 \\ 0 & \overline{C} \end{pmatrix},\ C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & F & 0 \\ 0 & 0 & \mathrm{tr}(F) \end{pmatrix},\ F \in \mathfrak{u}(2)\}$.
$\mathfrak{S}^\Gamma$ is of type A III ($p=1, q=2$).

3.2.2 Galois coverings of the line. Non-cyclic case.

Now we turn to the non-cyclic cases. In this section, we will deal with just one case, that is family **(27)**. Indeed, we postpone the analysis of the uniformizing symmetric space $\mathfrak{S}^\Gamma$ of the other families of non-cyclic coverings of the line, namely of families **(26, 31, 32)**, to the next section, where we present the study of these varieties as families of Galois covers of elliptic curves. In the case the group G is not cyclic, we cannot use Lemma 3.2.1

to study the action ρ of G on $H^0(C, K_C)$. Instead, we will use the general Chevalley-Weil formula to get the multiplicity of a given irreducible representation of G in $H^0(C, K_C)$ (see e. g. [25, Theorem 1.3.3]). Another difference is that, of course, we will deal with more than one generator of Γ.

Family (27) This is the family of covers of $\mathbb{P}^1$ with group $G = \mathbb{Z}/2 \times \mathbb{Z}/2$, ramification data $\mathbf{m} = (2^6)$, $g = 3$ and dimension 3. Label the characters of $\mathbb{Z}/2 \times \mathbb{Z}/2$ as follows:

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
χ_1	1	1	1	1
χ_2	1	-1	1	-1
χ_3	1	1	-1	-1
χ_4	1	-1	-1	1

From the Chevalley-Weil formula follows that the character χ_ρ of the action ρ of G on $H^0(C, K_C)$ is given by $\chi_\rho = \chi_2 + \chi_3 + \chi_4$. Thus there exists $V_i \subset H^0(C, K_C)$, $i = 2, 3, 4$, $\dim V_i = 1$, such that $H^0(C, K_C) = V_2 \oplus V_3 \oplus V_4$ and

$$\rho : G \to GL(V_2 \oplus V_3 \oplus V_4), \quad \rho(g) = \begin{pmatrix} \rho_2(g) & 0 & 0 \\ 0 & \rho_3(g) & 0 \\ 0 & 0 & \rho_4(g) \end{pmatrix},$$

where ρ_i denotes the irreducible representation of G with character χ_i. The choice of norm one vectors that span V_i, $i = 2, 3, 4$, gives a unitary basis $\{\omega_1, \omega_2, \omega_3\}$ of $H^0(C, K_C)$, with respect to which

$$\rho(0,1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} =: A_0, \quad \rho(1,0) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} =: B_0.$$

Consider now the action $\tilde{\rho} : G \to GL(H^1(C, \mathbb{C}))$ of G on $H^1(C, \mathbb{C})$, and let A, B denote the matrices that represent $\tilde{\rho}(1,0)$ and $\tilde{\rho}(0,1)$ with respect to the basis of $H^1(C, \mathbb{C})$ induced by $\{\omega_1, \omega_2\}$:

$$A = \begin{pmatrix} A_0 & 0 \\ 0 & \overline{A_0} \end{pmatrix}, \quad B = \begin{pmatrix} B_0 & 0 \\ 0 & \overline{B_0} \end{pmatrix}.$$

The matrices A and B are the generators of the injective image Γ of H in $Sp(H^1(C, \mathbb{Z}), Q)$. Thus $\mathfrak{S}^\Gamma = \mathfrak{S}^{A,B} = \{J \in \mathfrak{S} : \ JA = AJ, \ JB = BJ\}$. With the notation (3.2.2), $U \in \mathfrak{g}$ is of the form $U = \begin{pmatrix} C & \overline{D} \\ D & \overline{C} \end{pmatrix}$, with $D = D^t$

and $C \in \mathfrak{u}(2)$, and $U \in \mathfrak{Z}_{\mathfrak{g}}(A,B)$ satisfies

$$C = \begin{pmatrix} i\lambda_1 & 0 & 0 \\ 0 & i\lambda_2 & 0 \\ 0 & 0 & i\lambda_3 \end{pmatrix}, \ \lambda_i \in \mathbb{R} \quad D = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}, \ d_i \in \mathbb{C}.$$

Thus, from Lemma 3.1.1, the Cartan decomposition associated to $\mathfrak{S}^\Gamma$ is $\mathfrak{g}' = \mathfrak{p}' \oplus \mathfrak{k}'$, with

$$\mathfrak{p}' = \mathfrak{Z}_{\mathfrak{p}}(A) = \{\begin{pmatrix} 0 & \overline{D} \\ D & 0 \end{pmatrix}, \ D = \operatorname{diag}(d_1, d_2, d_3), \ d_i \in \mathbb{C}\}$$

and $\mathfrak{k}' = [\mathfrak{p}', \mathfrak{p}']$. To calculate $\mathfrak{k}'$, consider $X, Y \in \mathfrak{p}'$. If $X = \begin{pmatrix} 0 & \overline{D} \\ D & 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 0 & \overline{E} \\ E & 0 \end{pmatrix}$, with $D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$, and $E = \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}$, we have

$$[X, Y] = \begin{pmatrix} \overline{D}E - \overline{E}D & 0 \\ 0 & D\overline{E} - E\overline{D} \end{pmatrix},$$

where

$$\overline{D}E - \overline{E}D = \operatorname{diag}(\overline{d_1}e_1 - \overline{e_1}d_1, \overline{d_2}e_2 - \overline{e_2}d_2, \overline{d_3}e_3 - \overline{e_3}d_3).$$

Thus $\mathfrak{k}' = [\mathfrak{p}', \mathfrak{p}'] = \mathfrak{Z}_{\mathfrak{g}}(A) \cap \mathfrak{k} = \{\begin{pmatrix} C & 0 \\ 0 & \overline{C} \end{pmatrix}, \ C = \operatorname{diag}(i\lambda_1, i\lambda_2, i\lambda_3), \lambda_i \in \mathbb{R}\}$. In this case, the adjoint representation $\operatorname{ad} : \mathfrak{k}' \to \mathfrak{gl}(\mathfrak{p}')$ is not irreducible. Indeed, let $C = \operatorname{diag}(i\lambda_1, i\lambda_2) \in \mathfrak{k}'$ and $D = \operatorname{diag}(d_1, d_2) \in \mathfrak{p}'$. We have

$$\left[\begin{pmatrix} C & 0 \\ 0 & \overline{C} \end{pmatrix}, \begin{pmatrix} 0 & \overline{D} \\ D & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & C\overline{D} - \overline{DC} \\ \overline{C}D - DC & \end{pmatrix}.$$

Thus $\operatorname{ad}_C(D) = \overline{C}D - DC = \operatorname{diag}(-2i\lambda_1 d_1, -2i\lambda_2 d_2, -2i\lambda_3 d_3)$ and hence $W_1 = span(\operatorname{diag}(1,0,0))$, $W_2 = span(\operatorname{diag}(0,1,0))$, $W_3 = span((0,0,1))$ are invariant subspaces of $\mathfrak{p}'$. Hence $\mathfrak{S}^\Gamma$ is of type A III(1,1) $\times$ A III(1,1) $\times$ A III(1,1).

3.2.3 Galois coverings of elliptic curves.

Let $f_t : C_t \to C'_t$, $t \in B$ be one of 6 families of Galois covers of elliptic curve found in [16]. Associated with this family there are two maps: the first is the generalized Prym map

$$P : B \to \mathsf{A}^{\delta}_{g-1}, \quad [C_t \to C'_t] \mapsto Prym(C_t, C_t)$$

and the second is the map

$$\varphi : B \to \mathsf{A}_1, \quad [C_t \to C'_t] \mapsto [JC'_t].$$

It is proved in [19] that the connected components of the fibers of both maps have a totally geodesic image in A_g. Therefore, in an orbifold sense, the image of the family in A_g admits two different fibrations in totally geodesic subvarieties. Moreover, it is proved there that countably many of these totally geodesic fibers are Shimura. Linked to the study of these maps is the decomposition, up to isogeny, of the Jacobian JC of C, as $JC \sim JC' \times Prym(C, C')$. The aim of this section is to study this decomposition at the level of the Siegel space, relating it to the study of the uniformizing symmetric space of the examples. Consider the natural projection map $\pi : \mathfrak{S} \to \mathsf{A}_g$ and let

$$B \xrightarrow{h} \mathsf{M}_g \xrightarrow{j} \mathsf{A}_g$$

be, respectively, the natural map associated to the family, and the Torelli map. Recall that h is generically finite.

Theorem 3.2.3. *Let $\mathfrak{S}^\Gamma$ be the uniformizing symmetric space associated to one of the families (1e), (2e) (3e), (4e), (6e). Then*

i) $\mathfrak{S}^\Gamma$ decomposes as $B_1(\mathbb{C}) \times M$, where M is an hermitian symmetric space of codimension 1.

ii) $\pi(M) = \overline{(j \circ h)(F)}$, where F is an irreducible component of the fiber of φ. In particular, M uniformizes $\overline{(j \circ h)(F)}$.

iii) $\pi(B_1(\mathbb{C})) = \overline{(j \circ h)(F)}$, where F is an irreducible component of the fiber of the Prym map.

Remark 3.2.4. It is known that the fibers of the Prym map are not irreducible for the family (1e) and irreducible for the family (2e) (see [20] and references therein). In particular, the statement of Theorem 3.2.3 can be made more precise as follows: the fibers of the Prym map of the family (2e) are irreducible and $\pi(B_1(\mathbb{C})) = \overline{(j \circ h)(F)}$, where F is the fiber of the Prym map. The fibers of the Prym map of family (1e) are not irreducible and $\pi(B_1(\mathbb{C})) = \overline{(j \circ h)(F)}$, where F is an irreducible component of the non-irreducible fiber of P.

Consider as usual the action of G on the space of holomorphic one−forms and let $H^0(C, K_C) = \oplus_\chi V_\chi$ be the associated decomposition in isotypic components. The isotypic component V_χ is the direct sum of all the copies of the irreducible representation associated with χ that appear in $H^0(C, K_C)$. If χ_0 is the character of the trivial representation, then $V_{\chi_0} = H^0(C, K_C)^G$. Denote by V_- the direct sum of the other isotypic components. We get $H^0(C, K_C) = H^0(C, K_C)^G \oplus V_-$. We point out that $H^0(C, K_C)^G$ and V_- are the subspaces of $H^0(C, K_C)$ corresponding, respectively, to the construction of JC' and of $Prym(C, C')$. In fact, since the pullback map f^* : $H^0(C', K_{C'}) \to H^0(C, K_C)$ is injective, $H^0(C', K_{C'}) \simeq f^*(H^0(C', K_{C'})) = H^0(C, K_C)^G$. Note that this implies $\dim H^0(C, K_C)^G = 1$. Moreover recall that, for a certain sublattice $\Lambda \subset H_1(C, \mathbb{Z})$, we have $Prym(C, C') = (V_-)^*/\Lambda$.

For $J \in \mathfrak{S}^\Gamma$ denote as usual by $\mathfrak{g} = \mathfrak{sp}(2g, \mathbb{R})$, and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the Cartan decomposition associated to $\mathfrak{S}$. We are again in the situation of Lemma 3.1.1. As usual, we are interested in $T_J\mathfrak{S}^\Gamma \subset T_J\mathfrak{S} = \mathfrak{p}$. The proof of Theorem 3.2.3 is based on the following two Propositions.

Proposition 3.2.5. *Let $\mathfrak{g}' = \mathfrak{p}' \oplus \mathfrak{k}'$ be the Cartan decomposition of the uniformizing symmetric space $\mathfrak{S}^\Gamma$ of one of the families (1e), (2e) (3e), (4e), (6e). Then*

$$\mathfrak{p}' = W_1 \oplus W_2, \quad W_1 = S^2(H^0(K_C)^G)^*, \quad W_2 \subset S^2(V_-)^*$$

where W_i is $\mathrm{ad}_{\mathfrak{k}'}-$invariant. In other words, $\mathfrak{S}^\Gamma = B_1(\mathbb{C}) \times M$, where M is a symmetric space with $T_JM = W_2$ and $T_JB_1(\mathbb{C}) = W_1$.

Proof. From the decomposition $H^0(K_C) = H^0(K_C)^G \oplus V_-$, we get

$$(S^2H^0(K_C))^G = S^2H^0(K_C)^G \oplus (S^2V_-)^G.$$

Recalling that $\mathfrak{p} \simeq S^2H^0(K_C)^*$, we conclude

$$\mathfrak{p}' = W_1 \oplus W_2, \quad W_1 = S^2(H^0(K_C)^G)^*, \; W_2 = (S^2(V_-)^*)^G \subset S^2(V_-)^*. \tag{3.2.3}$$

We want now to show that the subspaces W_i are $ad_{\mathfrak{k}'}-$invariant and hence that we get a decomposition of $\mathfrak{S}^\Gamma$ as symmetric space. Choose a norm one vector that spans $H^0(K_C)^G$, choose a unitary basis of V_-, and consider the induced basis of $H^1(C, \mathbb{C})$. With the choice of this basis, (3.2.3) implies that, in terms of the usual matrix notation 3.2.2, an element $D \in \mathfrak{p}' = \mathfrak{Z}_\mathfrak{p}(\Gamma)$ is a block matrix of the form

$$D = \begin{pmatrix} d_{11} & 0 \\ 0 & F \end{pmatrix}.$$

Since $\mathfrak{k}' = [\mathfrak{p}', \mathfrak{p}']$, the elements in $\mathfrak{k}'$ also inherit this property. Hence

$$W_1 = \{\begin{pmatrix} d_{11} & 0 \\ 0 & 0 \end{pmatrix}, \; d_{11} \in \mathbb{C}\} \quad \text{and} \quad W_2 = \{\begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} \in \mathfrak{p}'\}$$

are $\mathrm{ad}_{\mathfrak{k}'}-$invariant subspaces of $\mathfrak{p}'$. That is, the adjoint representation is not irreducible and $\mathfrak{S}^\Gamma \simeq B_1(\mathbb{C}) \times M$, where M is a symmetric space with tangent space W_2, and $T_J B_1(\mathbb{C}) = W_1$. □

The following Lemma is probably well-known but, since we were not able to locate it in the literature, we recall it with its proof.

Lemma 3.2.6. *Let $M_1, M_2 \subset \mathfrak{S}$ be closed connected complex submanifolds such that $\pi(M_1) \subset \pi(M_2)$. Then there exists $a \in Sp(2g, \mathbb{Z})$ such that $a.(M_1) \subset M_2$.*

Proof. For $a \in Sp(2g, \mathbb{Z})$, let $f_a : \mathfrak{S} \to \mathfrak{S}$, $f_a(J) = a.J = aJa^{-1}$. This is the usual action of the symplectic group on the Siegel space in the model of (3.1.1) (see e.g.). By assumption

$$M_1 \subset \bigcup_{a \in Sp(2g,\mathbb{Z})} f_a^{-1}(M_2).$$

Indeed, for $J \in M_1$ there exists $a \in Sp(2g, \mathbb{Z})$ such that $f_a(J_1) = J_2$. Thus

$$M_1 = \bigcup_{a \in Sp(2g,\mathbb{Z})} M_1 \cap f_a^{-1}(M_2).$$

Since $M_1 \cap f_a^{-1}(M_2)$ is closed in M_1 and a vary in a countable set, Baire theorem [4, p. 57] implies that there exists a and an open subset $U \subset M_1$, such that $U \subset f_a^{-1}(M_2)$, i.e. $f_a(U) \subset M_2$. Therefore $f_a^{-1}(M_2) \cap M_1$ is an analytic subset of M_1, which contains an open subset $U \subset M_1$. By the Identity Lemma [27, p. 167], this implies that $f_a^{-1}(M_2) \cap M_1 = M_1$ and hence that $f_a(M_1) \subset M_2$. □

Proposition 3.2.7. *Let $\mathfrak{S}^\Gamma = B_1(\mathbb{C}) \times M$ be the uniformizing symmetric space of one of the families (1e), (2e) (3e), (4e), (6e). Then*

i) $\pi(M) = \overline{(j \circ h)(F)}$, *where F is an irreducible component of the fiber of φ.*

ii) $\pi(B_1(\mathbb{C})) = \overline{(j \circ h)(F)}$, *where F is an irreducible component of the fiber of the Prym map.*

Proof. i) For $t \in B$ generic, $dh_t(T_t B) = H^0(C_t, T_{C_t})^G$, and the codifferential of φ at t is given by the composition

$$S^2(H^0(K_C)^G) \xrightarrow{m \circ i} H^0(2K_C)^G \xrightarrow{dh^*} (T_t B)^*,$$

where $i : S^2H^0(K_C)^G \hookrightarrow S^2H^0(K_C)$ is the inclusion and m is the multiplication map. Since $m^* = dj$, we get $d\varphi = i^* \circ dj \circ dh$ and we conclude that

$$d\varphi \circ d(j \circ h)^{-1} = i^* : (S^2H^0(K_C)^*)^G \to S^2(H^0(K_C)^G)^*, \quad \lambda \mapsto \lambda|_{S^2H^0(K_C)}.$$

Since $W_2 \subset S^2(V_-)^*$, it follows that $W_2 \subset \ker(d\varphi \circ d(j \circ h)^{-1}) = d(j \circ h)(\ker d\varphi) = d(j \circ h)(T_t F)$. So $\pi(M) \subset \overline{(j \circ h)(F)}$. By [19, Theorem 3.11], $\overline{(j \circ h)(F)} \subset \mathsf{A}_g$ is totally geodesic of dimension $\dim \mathfrak{S}^\Gamma - 1$. Denote by M' its uniformizing symmetric space. Applying Lemma 3.2.6 to M and M', we obtain that $a.M \subset M'$ for some $a \in Sp(2g, \mathbb{Z})$. Since they have the same dimension, the equality sign holds and we get $\pi(M) = \pi(a.M) = \pi(M') = \overline{(j \circ h)(F)}$.

ii) For $t \in B$ generic the codifferential of the Prym mat at t is given by the composition

$$S^2(V_-) \xrightarrow{pr} (S^2(V_-))^G \xrightarrow{m \circ i} H^0(2K_{C_t})^G \xrightarrow{dh^*} (T_t B)^*$$

where pr denotes the natural projection and now $i : (S^2 V_-)^G \hookrightarrow S^2 H^0(K_C)$. Thus $dP = pr^* \circ i^* \circ dj \circ dh$ and we conclude that

$$dP \circ d(j \circ h)^{-1} = pr^* \circ i^* : (S^2 H^0(K_C)^*)^G \to S^2(V_-)^*.$$

Since $i^* : S^2 H^0(K_C)^* \to (S^2 V_-)^{G*}$ is the restriction and $W_1 = S^2 H^0(K_C)^G$, we get $W_1 \subset \ker(dP \circ d(j \circ h)^{-1}) = d(j \circ h)(\ker dP) = d(j \circ h)(T_t F)$. So $\pi(B_1(\mathbb{C})) \subset \overline{(j \circ h)(F)}$. By [19, Theorem 3.9], $\overline{(j \circ h)(F)} \subset \mathsf{A}_g$ is totally geodesic of dimension 1. Applying again Lemma 3.2.6, the same argument as above implies $\pi(B_1(\mathbb{C})) = \overline{(j \circ h)(F)}$. $\square$

We present in the following the analysis of the various examples, reporting the isomorphisms with the families of Galois covers of the line. The obtained results are summarized in Theorem 3.0.2. For $G = \mathbb{Z}/m$, we will denote by χ_j the character of the irreducible representation ρ_j of G defined as

$$\rho_j(1)(\omega) = \zeta^j \omega, \quad \forall \omega \in H^0(C, K_C),$$

where $\zeta = e^{2\pi i/m}$.

Family (1e)=(26) The Chevalley-Weil formula gives $\chi_\rho = \chi_0 + \chi_1$. It follows that $H^0(C, K_C)$ decomposes as $H^0(C, K_C) = H^0(C, K_C)^G \oplus V_-$, with $H^0(C, K_C)^G = V_0$ and $V_- = V_1$. Both V_0 and V_1 have dimension 1. Thus the action G on $H^0(C, K_C)$ is given by

$$\rho : \mathbb{Z}/2\mathbb{Z} \to GL(V_0 \oplus V_1), \quad \rho(g) = \begin{pmatrix} \rho_0(g) & 0 \\ 0 & \rho_1(g) \end{pmatrix}.$$

We get $A_0 = \mathrm{diag}(1, -1)$.

$\mathfrak{p}' = \mathfrak{Z}_{\mathfrak{p}}(A) = \{ \begin{pmatrix} 0 & \overline{D} \\ D & 0 \end{pmatrix}, \; D = \mathrm{diag}(d_1, d_2), \; d_i \in \mathbb{C} \}$

$\mathfrak{k}' = [\mathfrak{p}', \mathfrak{p}'] = \mathfrak{Z}_{\mathfrak{g}}(A) \cap \mathfrak{k} = \{\begin{pmatrix} C & 0 \\ 0 & \overline{C} \end{pmatrix},\ C = \mathrm{diag}(i\lambda_1, i\lambda_2),\ \lambda_i \in \mathbb{R}\}.$
$W_1 = span(\mathrm{diag}(1,0)) = S^2(H^0(C, K_C)^G)^*,$
$W_2 = span(\mathrm{diag}(0,1)) = S^2(V_-)^*.$
$\mathfrak{S}^\Gamma$ decomposes in the product of two irreducible hermitian symmetric spaces of complex dimension 1.

Family (2e) The Chevalley-Weil formula implies $\chi = \chi_0 + 2\chi_1$. Thus there exists 1−dimensional subspaces $V_0, V_1, V_1' \subset H^0(C, K_C)$, such that $H^0(C, K_C) = V_0 \oplus V_1 \oplus V_1'$ and

$$\rho : \mathbb{Z}/2 \to GL(V_0 \oplus V_1 \oplus V_1'), \quad \rho(g) = \begin{pmatrix} \rho_0(g) & 0 & 0 \\ 0 & \rho_1(g) & 0 \\ 0 & 0 & \rho_1(g) \end{pmatrix}.$$

$A_0 = \mathrm{diag}(1, -1, -1).$
$\mathfrak{p}' = \mathfrak{Z}_{\mathfrak{p}}(A) = \{\begin{pmatrix} 0 & \overline{D} \\ D & 0 \end{pmatrix},\ D = \begin{pmatrix} d & 0 \\ 0 & F \end{pmatrix},\ d \in \mathbb{C},\ F = F^t\}.$
$\mathfrak{k}' = \mathfrak{Z}_{\mathfrak{g}}(A) \cap \mathfrak{k} = \{\begin{pmatrix} C & 0 \\ 0 & \overline{C} \end{pmatrix},\ C = \begin{pmatrix} i\lambda & 0 \\ 0 & E \end{pmatrix},\ \lambda \in \mathbb{R},\ E \in \mathfrak{u}(2)\}.$
For $C = \begin{pmatrix} i\lambda & 0 \\ 0 & E \end{pmatrix} \in \mathfrak{k}'$ and $D = \begin{pmatrix} d & 0 \\ 0 & F \end{pmatrix} \in \mathfrak{p}'$, we have

$$\mathrm{ad}_C(D) = \overline{C}D - DC = \begin{pmatrix} -2\lambda d & 0 \\ 0 & \overline{E}F - FE \end{pmatrix}.$$

$W_1 = span(\mathrm{diag}(1,0,0)) = S^2(H^0(C, K_C)^G)^*,$
$W_2 = \{\begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix},\ F = F^t\} = S^2(V_-)^*.$
$\mathfrak{S}^\Gamma \simeq B_1(\mathbb{C}) \times M$, where M is a 3−dimensional irreducible symmetric space with Cartan decomposition $\mathfrak{g}'' = \mathfrak{k}'' \oplus \mathfrak{p}''$ with $\mathfrak{k}'' = \mathfrak{u}(2)$ and $\mathfrak{p}'' = \{F \in \mathfrak{gl}(2, \mathbb{C}),\ F = F^t\}$. We conclude that M is of type CI ($n = 2$) and $\mathfrak{S}^\Gamma \simeq$ A III (1,1)× CI ($n = 2$).

Family (3e)=(31) The Chevalley-Weil formula gives $\chi = \chi_0 + \chi_1 + \chi_2$. Thus there exists $V_i \subset H^0(C, K_C)$, $i = 0, 1, 2$, $\dim V_i = 1$, such that $H^0(C, K_C) = V_0 \oplus V_1 \oplus V_2$ and

$$\rho : \mathbb{Z}/3\mathbb{Z} \to GL(V_0 \oplus V_1 \oplus V_2), \quad \rho(1) = \begin{pmatrix} \rho_0(1) & 0 & 0 \\ 0 & \rho_1(1) & 0 \\ 0 & 0 & \rho_2(1) \end{pmatrix}.$$

To get the decomposition $H^0(C, K_C) = H^0(C, K_C)^G \oplus V_-$, recall that $V_0 = H^0(C, K_C)^G$, and $V_- = V_1 \oplus V_2$. Denoted by $\zeta = e^{2\pi i/3}$, we have

$A_0 = \mathrm{diag}(1, \zeta, \zeta^2)$.

$\mathfrak{p}' = \mathfrak{Z}_{\mathfrak{p}}(A) = \{\begin{pmatrix} 0 & \overline{D} \\ D & 0 \end{pmatrix},\ D = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & 0 & d_2 \\ 0 & d_2 & 0 \end{pmatrix},\ d_i \in \mathbb{C}\}$

$\mathfrak{k}' = \mathfrak{Z}_{\mathfrak{g}}(A) \cap \mathfrak{k} = \{\begin{pmatrix} C & 0 \\ 0 & \overline{C} \end{pmatrix},\ C = \mathrm{diag}(i\lambda_1, i\lambda_2, i\lambda_3),\ \lambda_i \in \mathbb{R}\},\ \lambda_i \in \mathbb{R}\}$.

In this case $\mathrm{ad}_C(D) = \overline{C}D - DC = \begin{pmatrix} -2i\lambda_1 d_1 & 0 & 0 \\ 0 & 0 & -2i\lambda_2 d_2 \\ 0 & -2i\lambda_2 d_2 & 0 \end{pmatrix}$.

$W_1 = span(\mathrm{diag}(1,0,0)) = S^2(H^0(C, K_C)^G)^*$,

$W_2 = span(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}) \subset S^2(V_-)^*$

$\mathfrak{S}^\Gamma$ is product of two irreducible hermitian symmetric spaces of complex dimension 1.

Family (4e)=(32) The Chevalley-Weil formula gives $\chi_\rho = \chi_0 + \chi_1 + \chi_3$. Thus there exists $V_i \subset H^0(C, K_C)$, $i = 0, 1, 3$, $\dim V_i = 1$, such that $H^0(C, K_C) = V_0 \oplus V_1 \oplus V_3$ and

$$\rho : \mathbb{Z}/4\mathbb{Z} \to GL(V_0 \oplus V_1 \oplus V_3), \quad \rho(1) = \begin{pmatrix} \rho_0(1) & 0 & 0 \\ 0 & \rho_1(1) & 0 \\ 0 & 0 & \rho_3(1) \end{pmatrix}.$$

$A_0 = \mathrm{diag}(1, i, -i)$.

$\mathfrak{p}' = \mathfrak{Z}_{\mathfrak{p}}(A) = \{\begin{pmatrix} 0 & \overline{D} \\ D & 0 \end{pmatrix},\ D = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & 0 & d_2 \\ 0 & d_2 & 0 \end{pmatrix},\ d_i \in \mathbb{C}\}$

$\mathfrak{k}' = [\mathfrak{p}', \mathfrak{p}'] = \{\begin{pmatrix} C & 0 \\ 0 & \overline{C} \end{pmatrix},\ C = \mathrm{diag}(i\lambda_1, i\lambda_2, i\lambda_3),\ \lambda_i \in \mathbb{R}\},\ \lambda_i \in \mathbb{R}\}$.

The adjoint representation is given by

$$\mathrm{ad}_C(D) = \overline{C}D - DC = \begin{pmatrix} -2i\lambda_1 d_1 & 0 & 0 \\ 0 & 0 & -2i\lambda_2 d_2 \\ 0 & -2i\lambda_2 d_2 & 0 \end{pmatrix}.$$

$W_1 = span(\mathrm{diag}(1,0,0)) = S^2(H^0(C, K_C)^G)^*$,

$W_2 = span(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}) \subset S^2(V_-)^*$.

$\mathfrak{S}^\Gamma$ is product of two irreducible hermitian symmetric spaces of complex dimension 1.

Family (6e) This family was originally studied in [51]. The Chevalley-Weil formula gives $\chi_\rho = \chi_0 + \chi_1 + 2\chi_2$.
$A_0 = \mathrm{diag}(1, \zeta^2, \zeta^2, \zeta)$, with $\zeta = e^{2\pi i/3}$.

$$\mathfrak{p}' = \{\begin{pmatrix} 0 & \overline{D} \\ D & 0 \end{pmatrix},\ D = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & 0 & d \\ 0 & d^t & 0 \end{pmatrix},\ d_1 \in \mathbb{C},\ d \in M(2,1,\mathbb{C})\}.$$

$$\mathfrak{k}' = [\mathfrak{p}', \mathfrak{p}'] = \{\begin{pmatrix} C & 0 \\ 0 & \overline{C} \end{pmatrix},\ C = \begin{pmatrix} i\lambda & 0 & 0 \\ 0 & F & 0 \\ 0 & 0 & \mathrm{tr}(F) \end{pmatrix},\ F \in \mathfrak{u}(2)\}.$$

If $C \in \mathfrak{k}'$ and $D \in \mathfrak{p}'$, then $\mathrm{ad}_C(D) = \begin{pmatrix} -2i\lambda d_1 & 0 & 0 \\ 0 & 0 & \overline{E}d - tr(E)d \\ 0 & (\overline{E}d - tr(E)d)^t & 0 \end{pmatrix}$.

$\mathfrak{S}^\Gamma \simeq B_1(\mathbb{C}) \times B_2(\mathbb{C})$.

3.3 Comments

We recall that all known counter-examples to the Coleman-Oort conjecture can be constructed via families of Galois covers satisfying ($\star$) and that it has been proven in [8] that the one studied in this Chapter are the only positive-dimensional families of Galois covers satisfying ($\star$) with $2 \le g \le 100$. This suggests that either further counter-examples do not exist, or they are of a completely different nature. The study of the uniformizing symmetric spaces presented in this Chapter is aimed to get a better understanding of these examples. We point out here some related ideas:

1. One could try to work towards the Coleman-Oort conjecture in the case of a fixed type of uniformizing symmetric space. That is, obtain some non-existence results for totally geodesic subvarieties of A_g contained in M_g with a certain uniformizing symmetric space. A result of this kind is [5], in which the authors exclude from the Torelli locus certain Shimura varieties of unitary and orthogonal types.

2. Condition ($\star$) is sufficient for a family to yield a Shimura variety. In [47] Moonen proved that when $g' = 0$ and the group G is cyclic ($\star$) is also necessary for Z to be Shimura. Mohajer and Zuo [45] extended this to the case where $g' = 0$, the group G is abelian, and the family is one-dimensional. In general, it is unknown whether ($\star$) is necessary for a family of covers to yield a Shimura subvariety or not. One could try to investigate the necessity of the condition ($\star$) imposing some condition on the uniformizing symmetric space.

Chapter 4

A topological construction of families of G-covers of the line

Let G be a finite group and let Σ_1 and Σ_2 be oriented topological surfaces both endowed with an action of G. We say that the two actions are *topologically equivalent* if there are $\eta \in \operatorname{Aut} G$ and an orientation preserving homeomorphism $f : \Sigma_1 \cong \Sigma_2$ such that $f(g \cdot x) = \eta(g) \cdot f(x)$ for any $x \in \Sigma_1$ and any $g \in G$, see [26]. An equivalence class is called a *topological type* of G-action (sometimes this is called *unmarked topological type*).

Definition 4.0.1. *Fix on S^2 the orientation by the outer normal. We let $\mathscr{T}^n(G)$ denote the set of topological types of G-actions on a topological surface Σ such that $\Sigma/G \cong S^2$ (as oriented surfaces) and the projection $\pi : \Sigma \to \Sigma/G$ has n branch points.*

It is known that a topological type of $G-$action $[\Sigma \to S^2] \in \mathscr{T}^n(G)$ is identified by the monodromy morphism associated with the covering $\Sigma \to S^2$. In the first part of this Chapter we review in detail this aspect. The main result of the fist part is Theorem 4.2.4, where we give a combinatorial description of $\mathscr{T}^n(G)$.

Given a finite group G and a topological type of G-action, it is known (see e.g.[26]) that there exists a holomorphic family of algebraic curves

$$\pi : \mathscr{C} \to B,$$

such that

1. every curve C' in the family is *topologically equivalent* to C;
2. every curve with an action of the given topological type is G-isomorphic to some fiber of the family and at most to a finite number.

The construction in [26] uses Teichmüller theory. In the second part of the Chapter we describe an alternative, explicit and completely topological construction of such family.

4.1 Geometric bases

In this section we fix some notation and recall some basic definitions and results that we will use in the sequel.

4.1.1. If M is a manifold, $\mathbf{F}_{0,n} M$ denotes its configuration space:

$$\mathbf{F}_{0,n} M := \{(x_1, \dots, x_n) \in M^n : x_i \neq x_j \text{ for } i \neq j\}.$$

The group $\pi_1(\mathbf{F}_{0,n} M)$ is called the *pure braid group* with n strings of the manifold M. The symmetric group Σ_n acts freely on $\mathbf{F}_{0,n} M$ and the quotient is denoted by $\mathbf{B}_{0,n} M$. The group $\pi_1(\mathbf{B}_{0,n} M)$ is called the *(full) braid group* of M. In what follows we will write $x \in \mathbf{F}_{0,n+1} M$ as $x = (x_0, X)$, with $X \in \mathbf{F}_{0,n} M$. For $X \in \mathbf{F}_{0,n} M$, let $supp(X) = \{x_1 \cdots, x_n\}$. In the following, we will use the notation $M - X$ to indicate $M - supp(X)$.

4.1.2. Let Σ be an oriented surface and set $\Sigma^* := \Sigma - \{y\}$ for some $y \in \Sigma$. Given $b_0, b_1 \in \Sigma$ let $\Omega(\Sigma, b_0, b_1)$ denote the set of all paths α in Σ with $\alpha(0) = b_0$ and $\alpha(1) = b_1$. Fix $x_0 \in \Sigma^*$. Let $\tilde{\alpha} \in \Omega(\Sigma, x_0, y)$ be such that $\tilde{\alpha}(t) = y$ only for $t = 1$ and let D be a small disk around y. Let α be the loop that starts at x_0, travels along $\tilde{\alpha}$ till it reaches ∂D, then makes a complete tour of ∂D counterclockwise and finally goes back to x_0 again along $\tilde{\alpha}$. An important observation is that the conjugacy class of $[\alpha]$ in $\pi_1(\Sigma^*, x_0)$ is well defined. Indeed the choice of the disk does not change $[\alpha]$, while if a different path $\tilde{\beta} \in \Omega(\Sigma, x_0, y)$ is chosen, then $[\beta]$ and $[\alpha]$ are conjugate by the class of a loop in Σ^* that starts at x_0 travels along $\tilde{\alpha}$ up to ∂D, then along a piece of ∂D and finally goes back along $\tilde{\beta}$.

4.1.3. Fix a point $(x_0, X) \in \mathbf{F}_{0,n} S^2$. Consider n smooth regular arcs $\tilde{\alpha}_i$ in S^2 each one joining x_0 to x_{σ_i} (for some permutation σ), and such that for $i \neq j$ $\tilde{\alpha}_i$ and $\tilde{\alpha}_j$ intersect only at x_0. Assume also that the tangent vectors at x_0 of the $\tilde{\alpha}_i$'s are all distinct and follow each other in counterclockwise order (we orient S^2 by the outer normal). Now consider loops α_i based at x_0 constructed as in 4.1.2: α_i starts at x_0, goes along $\tilde{\alpha}_i$ until near x_{σ_i}, there travels counterclockwise along a small circle around x_{σ_i}, and finally goes back to x_0 again along $\tilde{\alpha}_i$. The circles have to be pairwise disjoint and the intersection of the interior of the i-th circle with X must reduce to x_{σ_i}.

Definition 4.1.4. *We call a set of generators $\{[\alpha_1], \dots, [\alpha_n]\}$ obtained as above a **geometric basis** of $\pi_1(S^2 - X, x_0)$.*

Notice that, thanks to the permutation, the definition of geometric basis depends only on the set $\{x_1, \dots, x_n\}$, not on the ordering of the points. On the other hand the classes $\{[\alpha_i]\}$ have a fixed order.

4.1.5. For $n \geq 3$, set

$$\Gamma_n := \langle \gamma_1, \dots, \gamma_n \mid \prod_{i=1}^{n} \gamma_i = 1 \rangle.$$

From a geometric basis $\mathscr{B} = \{[\alpha_i]\}_{i=1}^n$ we get an isomorphism

$$\chi : \Gamma_n \to \pi_1(S^2 - X, x_0)$$

such that $\chi(\gamma_i) = [\alpha_i]$. Assume that $\mathscr{B} = \{[\alpha_i]\}_{i=1}^n$ and $\overline{\mathscr{B}} = \{[\bar{\alpha}_i]\}_{i=1}^n$ are two geometric bases for $\pi_1(S^2 - X, x_0)$. It follows from 4.1.2 that every $[\alpha_i]$ is conjugate to some $[\bar{\alpha}_j]$. If we denote by $\chi, \overline{\chi} : \Gamma_n \to \pi_1(S^2 - X, x_0)$ the isomorphisms corresponding to the two bases, then $\mu := \overline{\chi} \circ \chi^{-1} \in \operatorname{Aut} \pi_1(S^2 - X, x_0)$ has the following properties:

1. for every $i = 1, \dots, n$, $\mu([\alpha_i])$ is conjugate to $[\alpha_j]$ for some j;
2. the induced homomorphism on $H^2(\pi_1(S^2 - X, x_0), \mathbb{Z})$ is the identity.

Definition 4.1.6. *Denote by* $\operatorname{Aut}^* \pi_1(S^2 - X, x_0)$ *the subgroup of elements of* $\operatorname{Aut} \pi_1(S^2 - X, x_0)$ *satisfying properties (1) and (2) above. By 4.1.5 this definition does not depend on the choice of the geometric basis* $\mathscr{B}$*.*

Definition 4.1.7. *Let* $x = (x_0, X) \in \mathbf{F}_{0,n+1} S^2$*. We say that a geometric basis* $\mathscr{B} = \{[\alpha_i]\}_{i=1}^n$ *is* ***adapted*** *to* x *if it respects the order of the points in* X*, that is,* α_i *turns around* x_i*, i.e., the permutation* $\sigma = id$*.*

4.1.8. Assume that $\mathscr{B} = \{[\alpha_i]\}_{i=1}^n$ and $\overline{\mathscr{B}} = \{[\bar{\alpha}_i]\}_{i=1}^n$ are two geometric bases for $\pi_1(S^2 - X, x_0)$ adapted to X. In this case, for every $i = 1, \dots, n$, $[\alpha_i]$ is conjugate to $[\bar{\alpha}_i]$. As a consequence, denoting by $\chi, \overline{\chi} : \Gamma_n \to \pi_1(S^2 - X, x_0)$ the isomorphisms corresponding to the two bases, the automorphism $\mu := \overline{\chi} \circ \chi^{-1}$ of $\pi_1(S^2 - X, x_0)$ belongs to the subgroup $\operatorname{Aut}^{**} \pi_1(S^2 - X, x_0)$ defined as follows

Definition 4.1.9. *We denote by* $\operatorname{Aut}^{**} \pi_1(S^2 - X, x_0)$ *the subgroup of* $\operatorname{Aut}^* \pi_1(S^2 - X, x_0)$ *of elements that map* $[\alpha_i]$ *to a conjugate of* $[\alpha_i]$ *for every* $i = 1, \dots, n$*. The definition does not depend on the choice of the geometric basis* $\mathscr{B}$ *adapted to* x*.*

Definition 4.1.10. *Similarly, we denote by* $\operatorname{Aut}^* \Gamma_n \subset \operatorname{Aut} \Gamma_n$ *the subgroup of automorphisms* ν *satisfying:*

1. *for* $i = 1, \dots, n$ *the element* $\nu(\gamma_i)$ *is conjugate to* γ_j *for some* j*;*

2. the automorphism of $H^2(\Gamma_n, \mathbb{Z})$ *induced by* ν *is the identity.*

We denote by $\operatorname{Aut}^{**}\Gamma_n \subset \operatorname{Aut}^*\Gamma_n$ *the subgroup of automorphisms* ν *such that*

(1') for $i = 1, \dots, n$ *the element* $\nu(\gamma_i)$ *is conjugate to* γ_i*.*

If $\chi : \Gamma_n \to \pi_1(S^2 - X, x_0)$ is induced from a geometric basis (not necessarily adapted to x), then $\nu \in \operatorname{Aut}^*\Gamma_n$ (resp., $\nu \in \operatorname{Aut}^{**}\Gamma_n$) if and only if $\chi\nu\chi^{-1} \in \operatorname{Aut}^*\pi_1(S^2 - X, x_0)$ (resp. $\chi\nu\chi^{-1} \in \operatorname{Aut}^{**}\pi_1(S^2 - X, x_0)$).

4.1.11. If G is a group and $a \in G$, then $\operatorname{inn}_a : G \to G$ denotes conjugation by a: $\operatorname{inn}_a(x) = axa^{-1}$. Notice that if $f : G \to H$ is a morphism, then $f \circ \operatorname{inn}_a = \operatorname{inn}_{f(a)} \circ f$. The group of inner automorphisms of G is denoted $\operatorname{Inn} G$. It is a normal subgroup of $\operatorname{Aut} G$. We set $\operatorname{Out} G := \operatorname{Aut} G / \operatorname{Inn} G$. For $(x_0, X) \in \mathbf{F}_{0,n+1} S^2$, we define by $\operatorname{Out}^*\pi_1(S^2 - X, x_0) := \operatorname{Aut}^*(\pi_1(S^2 - X, x_0))/(\operatorname{Inn}(\pi_1(S^2 - X, x_0)))$ and similarly for $\operatorname{Out}^{**}(\pi_1(S^2 - X, x_0))$

4.1.12. If $S_{g,n}$ is a topological surface of genus g with n punctures, the mapping class group of $S_{g,n}$ is denoted by $\operatorname{Mod}(S_{g,n})$, while $\operatorname{PMod}(S_{g,n})$ denotes the pure mapping class group of $S_{g,n}$, which is defined to be the subgroup of $\operatorname{Mod}(S_{g,n})$ of elements that fix each punctures individually.

4.1.13. In the sequel we will need the following variants of the Dehn-Nielsen-Baer Theorem, for which see [13, Thm. 8.8 p. 234], [60, Thm. 5.7.1 p. 197 and Thm. 5.13.1 p. 214] and [35, §2.9].

Theorem 4.1.14 (Dehn-Nielsen-Baer)**.** *Let* $x = (x_0, X) \in \mathbf{F}_{0,n+1} S^2$*. Then* $\varphi \in \operatorname{Aut}^*\pi_1(S^2 - X, x_0)$ *if and only if there exists* $\sigma \in \operatorname{Inn}\pi_1(S^2 - X, x_0)$ *and an orientation-preserving homeomorphism* $h : S^2 - X \to S^2 - X$ *such that* $h(x_0) = x_0$ *and* $\varphi = \sigma \circ h_*$*. In other words,* $\operatorname{Mod}(S^2 - X) \cong \operatorname{Out}^*(\pi_1(S^2 - X, x_0))$*.*

Corollary 4.1.15. *Let* $x, y \in \mathbf{F}_{0,n+1} S^2$ *and* $\varphi : \pi_1(S^2 - X, x_0) \to \pi_1(S^2 - Y, y_0)$ *be a homomorphism that sends geometric bases to geometric bases. Then there exists* $\sigma \in Inn(\pi_1(S^2 - Y, y_0))$ *and an orientation-preserving homeomorphism* $h : S^2 - X \to S^2 - Y$ *such that* $h(x_0) = y_0$ *and* $\varphi = \sigma \circ h_*$*.*

Proof. Fix an orientation preserving homeomorphism $f : S^2 - Y \to S^2 - X$ such that $f(y_0) = x_0$ and apply the Dehn-Nielsen-Baer theorem to $f_* \circ \varphi$. □

Corollary 4.1.16. *Let* $x = (x_0, X) \in \mathbf{F}_{0,n+1} S^2$*. Then* $\varphi \in \operatorname{Aut}^{**}\pi_1(S^2 - X, x_0)$ *if and only if there exists* $\sigma \in Inn(\pi_1(S^2 - X, x_0))$ *and an orientation-preserving homeomorphism* $h : S^2 - X \to S^2 - X$ *such that (1)* $h(x_0) = x_0$ *and* $\varphi = \sigma \circ h_*$ *and (2)* h *extends to a self-homeomorphism of* S^2 *fixing every* x_i *individually. In other words,* $\operatorname{PMod}(S^2 - X) \cong \operatorname{Out}^{**}(S^2 - X, x_0)$*.*

Proof. Applying the Dehn-Nielsen-Baer theorem we get the homeomorphism h of $S^2 - X$ and σ. It is elementary that h extends to a homeomorphism of S^2. Next assume $h(x_1) = x_j$ and fix a geometric basis $\mathscr{B} = \{[\alpha_i]\}$ adapted to x. Here α_i is a loop at x_0 that makes a counterclockwise turn aroung x_i as in 4.1.2. Hence $[h\alpha_1]$ is a loop making a turn around $h(x_1) = x_j$. But $[h\alpha_1]$ is conjugate to $\sigma h_*([\alpha_1]) = \varphi([\alpha_1])$ which is conjugate to $[\alpha_1]$ since $\varphi \in \mathrm{Aut}^{**} \pi_1(S^2 - X, x_0)$. Since α_1 makes a turn around x_1 it follows that $h(x_1) = x_j = x_1$. Similarly $h(x_i) = x_i$ for any i. $\square$

4.2 Topological types of actions

Let G be a finite group. In this section we give a combinatorial description of the set $\mathscr{T}^n(G)$ of topological type of G-actions with n branch points, defined in 4.0.1. More precisely, we put in relation the set $\mathscr{T}^n(G)$ with the set $\mathscr{D}^n(G)$ of all n-data associated with G, defined as follows.

Definition 4.2.1. *If G is a finite group an n-datum is an epimorphism $\theta : \Gamma_n \to G$ is such that $\theta(\gamma_i) \neq 1$ for $i = 1, \dots, n$. We let $\mathscr{D}^n(G)$ denote the set of all n-data associated with the group G.*

4.2.2. Fix a point $x = (x_0, X) \in \mathbf{F}_{0,n+1} S^2$ and a geometric basis $\mathscr{B} = \{[\alpha_i]\}_{i=1}^n$ of $\pi_1(S^2 - X, x_0)$. Denote by $\chi : \Gamma_n \cong \pi_1(S^2 - X, x_0)$ the corresponding isomorphism. Given an n-datum $\theta : \Gamma_n \to G$, the epimorphism $\theta \circ \chi^{-1}$ gives rise to a topological G-covering $p : \Sigma_0^\theta \to S^2 - X$. By the topological part of Riemann Existence Theorem this can be completed to a branched G-cover $p : \Sigma^\theta \to S^2$. By taking the equivalence class of Σ^θ we get a topological type of G-action. We get a map

$$\mathscr{F}_{x,\mathscr{B}} : \mathscr{D}^n(G) \to \mathscr{T}^n(G), \quad (\theta : \Gamma_n \to G) \longmapsto [\Sigma^\theta]$$

4.2.3. The group $\mathrm{Aut}^* \Gamma_n \times \mathrm{Aut}\, G$ acts on the set $\mathscr{D}^n(G)$ by the rule

$$(\nu, \eta) \cdot \theta := \eta \circ \theta \circ \nu^{-1},$$

where $(\nu, \eta) \in \mathrm{Aut}^* \Gamma_n \times \mathrm{Aut}\, G$ and $\theta \in \mathscr{D}^n(G)$ is a datum.

Theorem 4.2.4. *Let G be a finite group. Choose*

1. *an element $x = (x_0, X) \in \mathbf{F}_{0,n+1} S^2$;*
2. *a geometric basis $\mathscr{B} = \{[\alpha_i]\}_{i=1}^n$ of $\pi_1(S^2 - X, x_0)$.*

Then the map $\mathscr{F}_{x,\mathscr{B}}$ induces a bijection between $\mathscr{D}^n(G)/(\mathrm{Aut}^* \Gamma_n \times \mathrm{Aut}\, G)$ and the set $\mathscr{T}^n(G)$ of topological types of G-actions. The bijection does not depend on the choices of the point $x \in \mathbf{F}_{0,n+1} S^2$ and of the geometric basis $\mathscr{B}$.

The proof of Theorem 4.2.4 is based on the following two Propositions.

Proposition 4.2.5. *The map $\mathscr{F}_{x,\mathscr{B}}$ is constant on the orbits of the action of* $\operatorname{Aut}^*\Gamma_n \times \operatorname{Aut} G$.

Proof. Let $\theta : \Gamma_n \to G$ be a datum and $(\nu, \eta) \in \operatorname{Aut}^*\Gamma_n \times \operatorname{Aut} G$. Let $\theta' = \eta \circ \theta \circ \nu^{-1}$. We want to show that Σ^θ and $\Sigma^{\theta'}$ have the same topological type of $G-$action. Set $\overline{\nu} = \chi \circ \nu \circ \chi^{-1}$. Observe that $\overline{\nu} \in \operatorname{Aut}^*(\pi_1(S^2 - X, x_0))$ since $\nu \in \operatorname{Aut}^*\Gamma_n$. By the Dehn-Nielsen-Baer Theorem 4.1.14, there is $\sigma \in \operatorname{Inn}(\pi_1(S^2 - X, x_0))$ and an orientation-preserving diffeomorphism $h : (S^2 - X, x_0) \to (S^2 - X, x_0)$ such that $h(x_0) = x_0$ and $\sigma \circ h_* = \overline{\nu}$. Let p, p' denote the projections:

$$\begin{array}{ccc} \Sigma_0^\theta & & \Sigma_0^{\theta'} \\ \big\downarrow p & & \big\downarrow p' \\ (S^2 - X, x_0) & \xrightarrow{\;h\;} & (S^2 - X, x_0). \end{array}$$

Choose $\tilde{x}_0 \in \Sigma_0^\theta$ and $\tilde{x}_0' \in \Sigma_0^{\theta'}$ both over x_0. We have that $h_*(p_*(\pi_1(\Sigma_0^\theta, \tilde{x}_0))) = (\sigma^{-1} \circ \overline{\nu})(\ker(\theta \circ \chi^{-1})) = \sigma^{-1}(\ker(\theta \circ \chi^{-1} \circ \overline{\nu}^{-1})) = \ker(\theta \circ \chi^{-1} \circ \overline{\nu}^{-1})$, where the last equality holds because σ is an inner automorphism. Morever, since $\eta \in \operatorname{Aut} G$, $\ker(\theta \circ \chi^{-1} \circ \overline{\nu}^{-1}) = \ker(\eta \circ \theta \circ \chi^{-1} \circ \overline{\nu}^{-1})$. Thus $h_*(p_*(\pi_1(\Sigma_0^\theta, \tilde{x}_0))) = \ker(\eta \circ \theta \circ \chi^{-1} \circ \overline{\nu}^{-1}) = (p')_*(\pi_1(\Sigma_0^{\theta'}))$. By the lifting theorem we get an oriented homeomorphism $\tilde{h} : \Sigma_0^\theta \to \Sigma_0^{\theta'}$, such that the diagram commutes and which extends to the compactifications. Hence the G-actions on Σ^θ and $\Sigma^{\theta'}$ have the same topological type. □

Proposition 4.2.6. *Let $\theta \in \mathscr{D}^n(G)$, then $\mathscr{F}_{x,\mathscr{B}}(\theta)$ does not depend on the choices of the point $x \in \mathbf{F}_{0,n+1} S^2$ and of the geometric basis $\mathscr{B}$.*

Proof. First fix x and consider two geometric bases $\mathscr{B}$ and $\overline{\mathscr{B}}$. Let $\chi, \bar{\chi} : \Gamma_n \to \pi_1(S^2 - X, x_0)$ denote the corresponding isomorphisms. Then $\nu := \chi^{-1} \circ \bar{\chi} \in \operatorname{Aut}^*\Gamma_n$. For a datum θ, we have $\theta \circ \bar{\chi}^{-1} = \theta \circ \nu^{-1} \circ \chi^{-1}$. So $\mathscr{F}_{x,\overline{\mathscr{B}}}(\theta) = \mathscr{F}_{x,\mathscr{B}}(\theta \circ \nu^{-1})$. By Proposition 4.2.5, $\mathscr{F}_{x,\mathscr{B}}(\theta \circ \nu^{-1}) = \mathscr{F}_{x,\mathscr{B}}(\theta)$. Hence $\mathscr{F}_{x,\overline{\mathscr{B}}}(\theta) = \mathscr{F}_{x,\mathscr{B}}(\theta)$ as desired.

Now suppose that $x, y \in \mathbf{F}_{0,n+1} S^2$. Let $\chi : \Gamma_n \to \pi_1(S^2 - X, x_0)$ and $\bar{\chi} : \Gamma_n \to \pi_1(S^2 - Y, y_0)$ be the isomorphisms associated with two geometric bases $\mathscr{B}$ and $\overline{\mathscr{B}}$. Then $\nu := \bar{\chi} \circ \chi^{-1} : \pi_1(S^2 - X, x_0) \to \pi_1(S^2 - Y, y_0)$ sends a geometric basis to a geometric basis. Hence by Corollary 4.1.15, there is $\sigma \in \operatorname{Inn}(\pi_1(S^2 - Y, y_0)))$ and an orientation preserving homeomorphism $h : (S^2 - X, x_0) \to (S^2 - Y, y_0)$ such that $h(x_0) = y_0$ and $\sigma \circ h_* = \nu$. Given a datum θ, h_* maps the kernel of $\theta \circ \chi^{-1}$ to the kernel of $\theta \circ \bar{\chi}^{-1}$. By the lifting theorem there is an oriented diffeomorphism $\tilde{h}$ that extends to the compactifications. Hence the G-actions on Σ_x^θ and Σ_y^θ have the same topological type. □

In other words, to construct a $G-$topological covering from a datum $\theta \in \mathscr{D}^n(G)$ we need to make some choices: we fix a point $x \in \mathbf{F}_{0,n+1} S^2$ and a geometric basis $\mathscr{B}$ of $\pi_1(S^2 - X, x_0)$. Proposition 4.2.6 says that the topological type of the obtained covering does not depend on these choices. In the follwing proof we will also use some basic facts about monodromy maps that we recall here. Let $p : E \to B$ be a topological G-covering. For $b \in B$ and $e \in p^{-1}(b)$, we denote by $\mu_{p,e}$ the monodromy map $\mu_{p,e} : \pi_1(B, b) \to G$ such that $g = \mu_{p,e}[\alpha]$ maps e to $\alpha_e(1)$, where α_e is the lifting of α with initial point e.

Lemma 4.2.7. *Let $p : E \to B$ be a topological G-covering. Fix $b_0, b_1 \in B$ and $e_i \in p^{-1}(b_i)$. Let δ be a path from e_0 to e_1 and $\gamma = p \circ \delta$. Then $\mu_{p,e_0} = \mu_{p,e_1} \circ \gamma_\#$. In particular, if $b_0 = b_1$, μ_{p,e_0} and μ_{p,e_1} differ by an inner automorphism of $\pi_1(B, b_0)$ or - equivalently - of G.*

Lemma 4.2.8. *Let $p : E \to B$ and $p' : E' \to B'$ be G-coverings. Let $\tilde{h} : E \to E'$ be a $G-$equivariant homeomorphism and denote by $h : B \to B'$ the induced homeomorphism. Fix $e_0 \in E$. Then $\mu_{p,e_0} = \mu_{p',\tilde{h}(e_0)} \circ h_*$.*

Proof of Theorem 4.2.4. By Proposition 4.2.5, $\mathscr{F}_{x,\mathscr{B}}$ induces a map between $\mathscr{D}^n(G)/(\operatorname{Aut}^* \Gamma_n \times \operatorname{Aut} G)$ and $\mathscr{T}^n(G)$. To prove the statement we have to check the following:

1. if two epimorphisms $\theta, \theta' : \Gamma_r \to G$ give rise to the same topological type of G-action, then θ and θ' are in the same orbit for the action of $\operatorname{Aut}^* \Gamma_n \times \operatorname{Aut} G$;

2. every topological type of G-action with n branch points can be constructed from a datum in $\mathscr{D}^n(G)$.

To prove (1), consider the branched covers $p : \Sigma \to S^2$ and $p' : \Sigma' \to S^2$ associated with $\theta \circ \chi^{-1}$ and $\theta' \circ \chi^{-1}$ and suppose that there exists $\eta \in \operatorname{Aut} G$ and an orientation preserving homeomorphism $\tilde{h} : \Sigma \to \Sigma'$ such that $\tilde{h}(g \cdot e) = \eta(g)\tilde{h}(e)$. We get an induced homeomorphism $h : \Sigma/G \to \Sigma'/G$ and an isomorphism $h_* : \pi_1(S^2 - X, x_0) \to \pi_1(S^2 - X, h(x_0))$. Fix $e_0 \in p^{-1}(x_0)$. From Lemma 4.2.8 it follows that $\mu_{p,e_0} = \eta \circ \mu_{p',\tilde{h}(e_0)} \circ h_*$. Now fix $e_0' \in (p')^{-1}(x_0)$ and a path in Σ' from $\tilde{h}(e_0)$ to e_0'. Finally let $\gamma = p' \circ \delta$. By Lemma 4.2.7 we get that $\mu_{p',\tilde{h}(e_0)} = \mu_{p',e_0'} \circ \gamma_\#$. Thus

$$\mu_{p,e_0} = \eta \circ \mu_{p',e_0'} \circ \gamma_\# \circ h_*. \tag{4.2.1}$$

Observe that, since $\tilde{h}$ preserves the orientation, so does h, hence $\gamma_\# \circ h_* : \pi_1(S^2 - X, x_0) \to \pi_1(S^2 - X, x_0)$ lies in $\operatorname{Aut}^*(\pi_1(S^2 - X, x_0))$. Let $\nu := \chi^{-1} \circ (\gamma_\# \circ h_*) \circ \chi \in \operatorname{Aut}^* \Gamma_n$ be the corresponding automorphism in $\operatorname{Aut}^* \Gamma_n$. (Again we are using that χ comes from a geometric basis.) Also, observe that $\theta \circ \chi^{-1}$ coincides with μ_{p,e_0} up to an inner automorphism of G, and the same holds for $\theta' \circ \chi^{-1}$ and for μ_{p',e'_0}. We get that there exists $\eta \in \operatorname{Aut} G$ such that (4.2.1) becomes

$$\theta \circ \chi^{-1} = \eta \circ \theta' \circ \chi^{-1} \circ (\gamma_{\#} \circ h_*) = \eta \circ \theta' \circ \nu^{-1} \circ \chi^{-1}.$$

Thus $(\eta, \nu).\theta' = \theta$; that is, they are in the same orbit for the action of $\operatorname{Aut}^* \Gamma_n \times \operatorname{Aut} G$.

To prove (2) assume that G acts effectively on a surface Σ in such a way that $\Sigma/G \cong S^2$. Up to diffeomorphism we can assume that the set of critical values of $p : \Sigma \to S^2$ coincides with X. Fix a point $\tilde{x}_0 \in p^{-1}(x_0)$. Let $\theta := \mu_{p,\tilde{x}_0} \circ \chi : \Gamma_n \to G$ be the monodromy of the unramified cover. Since Σ_0^θ is connected θ is surjective, and $\theta(\gamma_i) \neq 1$ since all the points of X are branch points. So it is an n-datum. By construction the associated cover coincides with Σ.

Finally, it follows from Proposition 4.2.6 that the bijection induced by $\mathscr{F}_{x,\mathscr{B}}$ does not depend on x and $\mathscr{B}$. □

4.3 A result on groups

This section is dedicated to some considerations from group theory, which we will need in the following.

Lemma 4.3.1. *Let N, H and G be groups and let $\varepsilon : H \to \operatorname{Aut} N, h \mapsto \varepsilon_h$ be a morphism. Denote by $K := N \rtimes_\varepsilon H$ the semidirect product. Let $f : N \to G$ and $\varphi : H \to G$ be morphisms. Then there is a (necessarily unique) morphism $\tilde{f} : K \to G$ extending f and φ if and only if for any $h \in H$ we have*

$$\operatorname{inn}_{\varphi(h)} \circ f = f \circ \varepsilon_h. \tag{4.3.1}$$

Lemma 4.3.2. *Let N, H, G, $\varepsilon : H \to \operatorname{Aut} N, h \mapsto \varepsilon_h$ and $f : N \to G$ be as above. Assume that f is surjective, that N is finitely generated and that G is finite. Then*

$$H'' := \{h \in H : \varepsilon_h(\ker \theta) = \ker \theta\}$$

is a finite index subgroup of H; there is a morphism $\tilde{\varepsilon} : H'' \to \operatorname{Aut} G$ such that the diagram

$$\begin{array}{ccc} N & \xrightarrow{\varepsilon_h} & N \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ G & \xrightarrow{\tilde{\varepsilon}_h} & G \end{array}$$

commutes; $H' := \ker \tilde{\varepsilon}$ is a finite index subgroup of H and there is a unique morphism $\tilde{f} : K' := N \rtimes_\theta H' \to G$ that extends f and such that $\tilde{f}|_{H'} \equiv 1$.

Proof. The subgroup $\ker f$ has index $d := |G| < \infty$ in N. Since N is finitely generated, there are a finite number of index d subgroups of N (see e.g. [40, p. 56] or [32, p. 128].) The natural action of $\operatorname{Aut} N$ on the subgroups of N preserves the index. Therefore the orbit of $\operatorname{Aut} N$ through $\ker f$ is finite. Hence $(\operatorname{Aut} N)_{\ker f}$ has finite index in $\operatorname{Aut} N$. Since H/H'' injects in $\operatorname{Aut} N/(\operatorname{Aut} N)_{\ker f}$, also H'' has finite index in H. The existence of $\tilde{\varepsilon}_h$ follows immediately from the inclusion $\varepsilon_h(\ker f) \subset \ker f$ for $h \in H''$. Since $\operatorname{Aut} G$ is finite H' has finite index in H'' and in H. By construction for any $h \in H'$ we have $f = \tilde{\varepsilon}_h \circ f = f \circ \varepsilon_h$, i.e. (4.3.1) holds with $\varphi : H' \to G$ the trivial morphism. □

4.4 Families of G-curves

Fix a finite group G. Given a topological type $\theta \in \mathscr{D}^n(G)$, we describe in this section a topological construction of a base B_θ and a family of curves of the given topological type

$$\pi_\theta : \mathscr{C}_\theta \to \mathsf{B}_\theta,$$

such that every curve with an action of the given topological type is G-isomorphic to some fiber.

4.4.1. If $n \geq 3$, the group $\mathrm{PGL}(2, \mathbb{C})$ acts freely and holomorphically on $\mathbf{F}_{0,n}\,\mathbb{P}^1$. The quotient $\mathbf{F}_{0,n}\,\mathbb{P}^1/\mathrm{PGL}(2, \mathbb{C})$ is the moduli space of smooth curves of genus 0 with n marked points. Set $\mathbb{C}^{**} := \mathbb{C} - \{0, 1\}$. The map

$$\mathbf{F}_{0,n-3}\,\mathbb{C}^{**} \longrightarrow \mathbf{F}_{0,n}\,\mathbb{P}^1, \quad (z_1, \ldots, z_{n-3}) \mapsto (z_1, \ldots, z_{n-3}, 0, 1, \infty)$$

is a section for the action of $\mathrm{PGL}(2, \mathbb{C})$, i.e. its image intersects each orbit in exactly one point and it induces a biholomorphism of $\mathbf{F}_{0,n-3}\,\mathbb{C}^{**}$ onto the moduli space $\mathbf{F}_{0,n}\,\mathbb{P}^1/\mathrm{PGL}(2, \mathbb{C})$. We *define* $\mathsf{M}_{0,n}$ as the image of the section i.e. we set

$$\mathsf{M}_{0,n} := \mathbf{F}_{0,n-3}\,\mathbb{C}^{**} \times \{(0, 1, \infty)\} = \\ = \{X = (x_1, \ldots, x_n) \in \mathbf{F}_{0,n}\,\mathbb{P}^1 : x_{n-2} = 0, x_{n-1} = 1, x_n = \infty\}.$$

Note that the map $\mathrm{PGL}(2, \mathbb{C}) \times \mathsf{M}_{0,n} \to \mathbf{F}_{0,n}\,\mathbb{P}^1$, mapping (g, X) to $g(X)$ gives a biholomorphism $\mathsf{M}_{0,n} \times \mathrm{PGL}(2, \mathbb{C}) \cong \mathbf{F}_{0,n}\,\mathbb{P}^1$. In particular

$$\pi_1(\mathsf{M}_{0,n}) \subset \pi_1(\mathbf{F}_{0,n}\,\mathbb{P}^1) \quad \text{and} \quad \pi_1(\mathbf{F}_{0,n}\,\mathbb{P}^1) \cong \pi_1(\mathsf{M}_{0,n}) \times \mathbb{Z}/2\mathbb{Z}.$$

Points of $\mathsf{M}_{0,n}$ will be denoted by capital letters: $X = (x_1, \ldots, x_n)$ and it is understood that $x_{n-2} = 0, x_{n-1} = 1, x_n = \infty$. Similarly we set

$$\mathsf{M}_{0,n+1} := \{x = (x_0, \ldots, x_{n+1}) \in \mathbf{F}_{0,n+1}\,\mathbb{P}^1 : x_{n-2} = 0, x_{n-1} = 1, x_n = \infty\}.$$

Points of $\mathsf{M}_{0,n+1}$ will be denoted by lowercase letters $x = (x_0, x_1, \ldots, x_n)$ and we will also write $x = (x_0, X)$ with $X := (x_1, \ldots, x_n) \in \mathsf{M}_{0,n}$. The map

$$p : \mathsf{M}_{0,n+1} \to \mathsf{M}_{0,n}, \quad p(x_0, X) := X, \tag{4.4.1}$$

is a fiber bundle [2] and the fiber over X is $\mathbb{P}^1 - \{x_1, \cdots, x_n\} = \mathbb{C}^{**} - \{x_1, \cdots, x_{n-3}\}$. Hence (4.4.1) is the universal family of genus 0 curves with n marked points.

4.4.2. Denote by $\mathrm{T}(2,\mathbb{C}) \subset \mathrm{PGL}(2,\mathbb{C})$ the subset of elements in $\mathrm{PGL}(2,\mathbb{C})$ fixing ∞. The group $\mathrm{T}(2,\mathbb{C})$ acts on the configuration space $\mathbf{F}_{0,n-1}\,\mathbb{C}$ and the map

$$\mathbf{F}_{0,n-3}\,\mathbb{C}^{**} \longrightarrow \mathbf{F}_{0,n-1}\,\mathbb{C}, \quad (z_1, \ldots, z_{n-3}) \mapsto (z_1, \ldots, z_{n-3}, 0, 1)$$

is a section for this action. We get a biholomorphism

$$\mathsf{M}_{0,n} \cong \{X = (x_1, \ldots, x_{n-1}) \in \mathbf{F}_{0,n-1}\,\mathbb{C} : x_{n-2} = 0, x_{n-1} = 1\}. \tag{4.4.2}$$

Similarly as before, $\mathsf{M}_{0,n} \times \mathrm{T}(2,\mathbb{C}) \cong \mathbf{F}_{0,n-1}\,\mathbb{C}$ via the map $(g, (x_1, ..., x_n)) \mapsto g(x_1, ..., x_{n-1})$. The elements in $\mathrm{T}(2,\mathbb{C})$ are tringular matrices of the form $\mathrm{T}(2,\mathbb{C}) = \{\begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \in M(2,\mathbb{C}),\ a \neq 0\}$. In particular,

$$\pi_1(\mathsf{M}_{0,n}) \subset \pi_1(\mathbf{F}_{0,n-1}\,\mathbb{C}) \quad \text{and} \quad \pi_1(\mathbf{F}_{0,n-1}\,\mathbb{C}) \cong \pi_1(\mathsf{M}_{0,n}) \times \mathbb{Z}.$$

This tell us that, when dealing with $\pi_1(\mathsf{M}_{0,n})$, we can work in terms of the classical groups of braids of the plane and avoid spherical braid groups, for which some of the results that follow do not work.

4.4.3. Fix $x = (x_0, X) \in \mathsf{M}_{0,n+1}$ and let $\tilde{x} = (x_0, \tilde{X}) \in \mathbf{F}_{0,n}\,\mathbb{C}$ be the corresponding point via (4.4.2). The following sequences are exact and split (see e.g. [2, Corollary 1.8.1] and [12, Theorem 3.1])

$$1 \to \pi_1(\mathbb{P}^1 - X, x_0) \to \pi_1(\mathbf{F}_{0,n+1}\,\mathbb{P}^1, x) \to \pi_1(\mathbf{F}_{0,n}\,\mathbb{P}^1, X) \to 1 \tag{4.4.3}$$

$$1 \to \pi_1(\mathbb{C} - \tilde{X}, x_0) \to \pi_1(\mathbf{F}_{0,n}\,\mathbb{C}, \tilde{x}) \to \pi_1(\mathbf{F}_{n-1}\,\mathbb{C}, \tilde{X}) \to 1. \tag{4.4.4}$$

These sequences are the long exact sequences associated with the fibrations

$$\mathbf{F}_{0,n+1}\,\mathbb{P}^1 \to \mathbf{F}_{0,n}\,\mathbb{P}^1, \quad (x_0, X) \mapsto X$$

$$\mathbf{F}_{0,n}\,\mathbb{C} \to \mathbf{F}_{0,n-1}\,\mathbb{C}, \quad (x_0, \tilde{X}) \mapsto \tilde{X}.$$

By our previous considerations, the restriction of the fibration (4.4.3) to $\mathsf{M}_{0,n}$ (resp. the restriction of (4.4.4) to the image of $\mathsf{M}_{0,n}$ via (4.4.2)), gives the exact spitting sequence

$$1 \to \pi_1(\mathbb{P}^1 - X, x_0) \to \pi_1(\mathsf{M}_{0,n+1}, x) \to \pi_1(\mathsf{M}_{0,n}, X) \to 1. \tag{4.4.5}$$

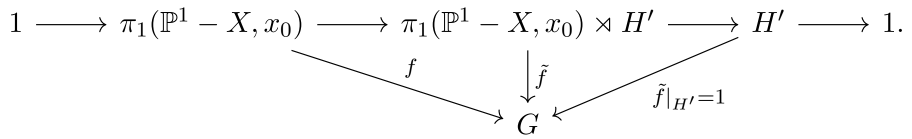
This is the exact sequence associated with the fibration 4.4.1 and it can be thought of as the restriction any of the two sequences (4.4.3) or (4.4.4).

4.4.4. Let G be a finite group and $\theta : \Gamma_n \to G$ a datum. Choose

1. an element $x = (x_0, X) \in \mathsf{M}_{0,n+1}$;
2. a geometric basis $\mathscr{B} = \{[\alpha_i]\}_{i=1}^n$ of the fundamental group $\pi_1(\mathbb{P}^1 - X, x_0)$ of the fiber $p^{-1}(X) = \mathbb{P}^1 - X$.

We apply the results of Section 4.3 to the sequence (4.4.5) with $f = \theta \circ \chi^{-1}$. We get

Lemma 4.4.5. *There exists a finite index subgroup H' of $\pi_1(\mathsf{M}_{0,n}, X)$ and a unique morphism $\tilde{f} : N \rtimes_\theta H' \to G$ that extends f and such that $\tilde{f}|_{H'} = 1$.*



Remark 4.4.6. Let

$$\varepsilon : \pi_1(\mathsf{M}_{0,n}) \to \mathrm{Aut}(\pi_1(\mathbb{P}^1 - X, x_0))$$

denote the morphism giving the semidirect product in the splitting exact sequence (4.4.5). By the considerations in 4.4.3, ε is just the restriction to $\pi_1(\mathsf{M}_{0,n})$ of the morphism $\tilde{\varepsilon}$ giving the splitting of the exact sequence (4.4.4). In [2, Corollary 1.8.3] it is explicitely described the image via $\tilde{\varepsilon}$ of the generators of the pure braid group of $n-1$ strings of the plane. To be more precise, the notation in [2] corresponds to identify

$$\mathsf{M}_{0,n} \cong \{(x_1, ..., x_{n-1}) \in \mathbf{F}_{0,n-1}\,\mathbb{C} : \ x_1 = 0, \ x_2 = 1\}$$

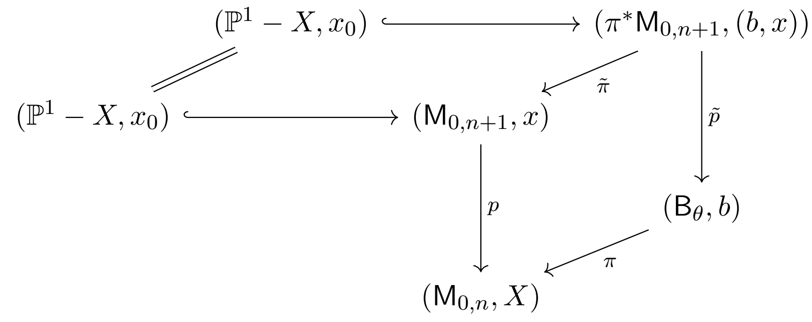
instead of (4.4.2). By this description, it is easy to conclude that, in the case where G is abelian, then $H' = H$. In general however, there may not exist any morphism $\tilde{f} : \pi_1(\mathbb{P}^1 - X, x_0) \rtimes H \to G$ extending f. Thus, in general $H' \neq H$. For example, consider $\theta : \Gamma_4 \to S_3$ given by $\theta(\gamma_1) = (12)$, $\theta(\gamma_2) = (23)$, $\theta(\gamma_3) = (23)$, $\theta(\gamma_4) = (12)$. Then, with the notation in [2], $\pi_1(\mathsf{M}_{0,4})$ is free on the generators A_{12}, A_{13} and $f(\varepsilon(A_{12})\gamma_1) = (23)$, $f(\varepsilon(A_{12})\gamma_2) = (13)$, $f(\varepsilon(A_{12})\gamma_3) = (23)$, $f(\varepsilon(A_{12})\gamma_4) = (12)$. Now note that on one side $\gamma_1\gamma_2\gamma_3\gamma_4 = 1$ and thus lies in $\ker\theta$, but on the other side $f(\varepsilon(A_{12})\gamma_1\gamma_2\gamma_3\gamma_4) = (23)(13) = (123) \neq 1$. It follows that, with the notation of Lemma 4.3.2, $H'' \neq H$. In particular, $H' \subset H'' \neq H$. We stress the following point: if $\tilde{f} : \pi_1(\mathbb{P}^1 - X, x_0) \rtimes H \to G$ is a morphism extending f, by Lemma 4.3.1 we should have $H'' = H$. Thus the example not only shows that there do not exist any morphism $\tilde{f}$ extending f with $\tilde{f}|_H = 1$, but more generally that there do not exist any morphism $\tilde{f} : \pi_1(\mathbb{P}^1 - X, x_0) \rtimes H \to G$ extending f. Geometrically, one can interpret the fact that $\tilde{f}$ does not extend as follows. On $\mathsf{M}_{0,4} \cong \mathbb{C}^{**}$ there is the universal family of elliptic curves $\mathscr{E} \to \mathsf{M}_{0,4}$. We denote by E_λ the fiber of $\mathscr{E} \to \mathsf{M}_{0,4}$ over $\lambda \in \mathbb{C}^{**}$. The existence of $\tilde{f} : \pi_1(\mathsf{M}_{0,5}) \to S_3$ corresponds to the existence of a family of lines $l_\lambda \subset E_\lambda[3] \cong H_1(E_\lambda, \mathbb{Z}/3\mathbb{Z})$. That is, the existence of a line $l_{\lambda_0} \subset E_{\lambda_0}$ which is stable under the action of the monodromy of the family $\mathscr{E}$. But the image of this monodromy is Γ_2, the congrunce subgroup of level 2, which does fixes no line in $H_1(E_{\lambda_0}, \mathbb{Z}/3\mathbb{Z})$.

Denote by B_θ the finite cover of $\mathsf{M}_{0,n}$ with fundamental group given by the finite-index subgroup H' of $\pi_1(\mathsf{M}_{0,n}, X)$. Let $\pi : \mathsf{B}_\theta \to \mathsf{M}_{0,n}$ denote the projection and consider the pullback $\tilde{p} : \pi^*\mathsf{M}_{0,n+1} \to \mathsf{B}_\theta$ of the fibration $p : \mathsf{M}_{0,n+1} \to \mathsf{M}_{0,n}$

$$\begin{array}{ccc} \pi^*\mathsf{M}_{0,n+1} & \xrightarrow{\tilde{\pi}} & \mathsf{M}_{0,n+1} \\ \downarrow{\scriptstyle \tilde{p}} & & \downarrow{\scriptstyle p} \\ \mathsf{B}_\theta & \xrightarrow{\pi} & \mathsf{M}_{0,n}. \end{array}$$

Lemma 4.4.7. *$\tilde{\pi} : \pi^*\mathsf{M}_{0,n+1} \to \mathsf{M}_{0,n+1}$ is the finite cover of $\mathsf{M}_{0,n+1}$ corresponding to the subgroup $\pi_1(\mathbb{P}^1 - X, x_0) \rtimes H'$ of $\pi_1(\mathsf{M}_{0,n+1}, x)$.*

Proof. Fix $z = (z_0, Z) \in \mathsf{M}_{0,n+1}$. We want to show that there exists an open neighborhood of z evenly covered by $\tilde{\pi}$. Let $V \subset \mathsf{M}_{0,n}$ be an open neighborhood of Z evenly covered by π, i.e. such that $\pi^{-1}(V) = \sqcup V_i$ and each V_i is mapped homeomorphically to V by π. Also, let $V_0 \subset \mathbb{P}^1$ be an open neighborhood of z_0 such that $U := V_0 \times V \subset \mathsf{M}_{0,n+1}$. Set $U_i := \pi^*\mathsf{M}_{0,n+1} \cap (V_i \times U)$. By construction $\tilde{\pi}^{-1}(U) \subset \sqcup U_i$. Moreover $U_i = \{(b, (p_0, \pi(b))) : p_0 \in V_0,\ b \in V_i\}$ is homeomorphic to $V_0 \times V_i$ and $\tilde{\pi}|_{U_i} : U_i \cong V_0 \times V_i \to V_0 \times U$ coincides with $\mathrm{id}_{V_0} \times \pi|_{V_i}$. In particular, $\tilde{\pi}|_{U_i}$ is an homeomorphism. This proves that U is evenly covered and hence that $\tilde{\pi}$ is a covering map. Now recall that we fixed a point $x = (x_0, X) \in \mathsf{M}_{0,n+1}$ and that the projection p has fiber $p^{-1}(X) = \mathbb{P}^1 - X$. Let $b \in \pi^{-1}(X) \subset \mathsf{B}_\theta$. The fiber of $\tilde{p}$ over b is $\mathbb{P}^1 - X$.



We want to show that $\pi_1(\pi^*\mathsf{M}_{0,n+1}, (b, x)) = \pi_1(\mathbb{P}^1 - P, x_0) \rtimes H'$. Since $\pi_2(\mathsf{B}_\theta) = \pi_2(\mathsf{M}_{0,n}) = 1$, the commutative diagram induces the following sequences

$$\begin{array}{ccccccc} 1 \longrightarrow \pi_1(\mathbb{P}^1 - X, x_0) & \longrightarrow & \pi_1(\pi^*\mathsf{M}_{0,n+1}\mathbb{P}^1, (b,x)) & \longrightarrow & H' & \longrightarrow & 1 \\ \| & & \downarrow g & & \downarrow & & \\ 1 \longrightarrow \pi_1(\mathbb{P}^1 - X, x_0) & \longrightarrow & \pi_1(\mathsf{M}_{0,n+1}, x) & \longrightarrow & \pi_1(\mathsf{M}_{0,n}, X) & \longrightarrow & 1 \end{array}$$

Recalling that $\pi_1(\mathsf{M}_{0,n+1}, x) = \pi_1(\mathbb{P}^1 - X, x_0) \rtimes \pi_1(\mathsf{M}_{0,n}, X)$, it follows by the commutativity of the right part of the diagram that the image of the map g lies in $\pi_1(\mathbb{P}^1 - X, x_0) \rtimes H'$. Thus we get

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(\mathbb{P}^1 - X, x_0) & \longrightarrow & \pi_1(\mathsf{M}_{0,n+1}, (b,x)) & \longrightarrow & H' & \longrightarrow & 1 \\ & & \| & & \downarrow g & & \| & & \\ 1 & \longrightarrow & \pi_1(\mathbb{P}^1 - X, x_0) & \longrightarrow & \pi_1(\mathbb{P}^1 - X, x_0) \rtimes H' & \longrightarrow & H' & \longrightarrow & 1 \end{array}$$

Applying the short five lemma [10, p. 16], we get the statement. □

Now consider the extension $\tilde{f} : \pi_1(\mathbb{P}^1 - X, x_0) \rtimes H' \to G$ of the morphism f from Lemma 4.4.5. By Lemma 4.4.7, $\tilde{f}$ gives rise to a topological covering $\pi'_\theta : \mathscr{C}^*_\theta \to \pi^*\mathsf{M}_{0,n+1}$. By composing with $\tilde{p} : \pi^*\mathsf{M}_{0,n+1} \to \mathsf{B}_\theta$, one gets $\pi_\theta : \mathscr{C}^*_\theta \to \mathsf{B}_\theta$. Denote by Σ^*_b the fiber of π_θ over $b \in \mathsf{B}_\theta$.

$$\begin{array}{ccccc} & & \Sigma^*_b & \hookrightarrow & \mathscr{C}^*_\theta \\ & \swarrow & & \swarrow {\scriptstyle \pi'_\theta} & \\ \mathbb{P}^1 - \pi(b) & \hookrightarrow & \pi^*\mathsf{M}_{0,n+1} & & \\ \downarrow \tilde{p} & & \downarrow \tilde{p} & & \\ b & \hookrightarrow & \mathsf{B}_\theta & & \end{array}$$

We recall the Grauert-Remmert Extension Theorem, which generalizes Riemann's Existence Theorem. (See [28, Chapter XII, Theorem 5.4])

Theorem 4.4.8 (Grauert-Remmert Extension Theorem)**.** *Let Y be a connected complex manifold and $Z \subset Y$ a closed analytic subspace such that $Y^\circ := Y - Z$ is dense in Y . Let $f^\circ : X^\circ \to Y^\circ$ be a finite unramified cover. Then there exists a unique normal analytic space X and a unique analytic covering $f : X \to Y$ such that $X^\circ \subset X$ and $f^\circ = f|_{X^\circ}$. (Uniqueness is up to isomorphism.)*

Corollary 4.4.9. *In the hypoteses above, if Z is a smooth divisor, then X is smooth.*

Proof. Let $D = \{|z| < 1\} \subset \mathbb{C}$ be the unit disc. Using a local chart $U \cong D^n$ of Y such that $U \cap Z = D^n \cap \{z_1 = 0\}$ we get a finite cover of $D^* \times D^{n-1}$. By the topological classification of coverings disc, it is of the form $(z_1, ..., z_n) \mapsto (z_1^m, z_2, ..., z_n)$ for some $m \geq 1$, hence extends to an analytic cover $D^n \to D^n$. So by uniqueness $f^{-1}(U) \cong D^n$. In particular $f^{-1}(U)$ is smooth. □

Lemma 4.4.10. *The topological covering $\pi'_\theta : \mathscr{C}^*_\theta \to \pi^*\mathsf{M}_{0,n+1}$ extends uniquely to a ramified cover $\pi'_\theta : \mathscr{C}_\theta \to \mathbb{P}^1 \times \mathsf{B}_\theta$, with $\mathscr{C}_\theta$ smooth.*

Proof. Consider $\mathbb{P}^1 \times \mathsf{M}_{0,n}$. Let $x_0 \in \mathbb{P}^1$ and $(x_1, ..., x_n) \in \mathsf{M}_{0,n}$ (i.e., $x_{n-2} = 0, x_{n-1} = 1, x_n = \infty$ and $(x_1, \ldots, x_{n-3}) \in \mathbf{F}_{0,n-3}\,\mathbb{C}^{**}$). Let $Z_i \subset \mathbb{P}^1 \times \mathsf{M}_{0,n}$ be the smooth divisor $Z_i := \{x_0 = x_i\}$ for $i = 1, ..., n$. The $Z_i's$ are pairwise disjoint and their union, which we denote by Z, is a smooth divisor of $\mathbb{P}^1 \times \mathsf{M}_{0,n}$. As a consequence, π^*Z is a smooth divisor of $\mathbb{P}^1 \times \pi^*\mathsf{M}_{0,n+1} = \mathbb{P}^1 \times \mathsf{B}_\theta$ and, since $\mathsf{M}_{0,n+1} = (\mathbb{P}^1 \times \mathsf{M}_{0,n}) - Z$, $\pi^*\mathsf{M}_{0,n+1} = (\mathbb{P}^1 \times \mathsf{B}_\theta) - \pi^*Z$. It follows that we can apply Grauert-Remmert Extension Theorem to the topological covering $\pi'_\theta : \mathscr{C}^*_\theta \to \pi^*\mathsf{M}_{0,n+1}$, which can be thus completed to a ramified cover $\pi'_\theta : \mathscr{C}_\theta \to \mathbb{P}^1 \times \mathsf{B}_\theta$, with $\mathscr{C}_\theta$ smooth. □

4.4.11. Composing with the projection over B_θ we get

$$\pi_\theta : \mathscr{C}_\theta \xrightarrow{\pi'_\theta} \mathbb{P}^1 \times \mathsf{B}_\theta \xrightarrow{pr_2} \mathsf{B}_\theta,$$

Note that π_θ is a submersion. Indeed, let $U \cong D^n$ be a local chart in $\mathbb{P}^1 \times \mathsf{B}_\theta$ such that $U \cap \pi^*Z = U \cap \pi^*Z_i = \{x_0 - x_i = 0\}$ for some $i = 1, ..., n$ (with $x_{n-2} = 0$, $x_{n-1} = 1$, $x_n = \infty$). Denoted by $w = x_0 - x_i$, we get that $w, x_1, ..., x_n$ are local coordinates on U and $\pi'|_{\pi'^{-1}(U)} : \pi'^{-1}(U) \to U$ is of the form $(w, x_1, ..., x_n) \mapsto (w^m, x_1, ..., x_n)$, for some $m \geq 2$. Composing with the projection pr_2 over B_θ we conclude that we can locally write π_θ as $(w, x_1, ..., x_n) \mapsto (x_1, ..., x_n)$. Thus π_θ is a submersion. In particular, its fibers are smooth curves. We call

$$\pi_\theta : \mathscr{C}_\theta \to \mathsf{B}_\theta$$

the family of $G-$curves associated with the datum $\theta \in \mathscr{D}^n(G)$.

Given $b \in \mathsf{B}_\theta$, denote by $C_b = \pi_\theta^{-1}(b)$. The restriction $\pi'_\theta|_{C_b} : C_b \to \mathbb{P}^1$ is the unique smooth compactification of the topological G-cover $C^*_b \to \mathbb{P}^1 - \pi(b)$. Hence it is exactly the one given by Riemann's Existence Theorem.

4.5 Parallel transport

In this section we recall a notion of parallel transport up to homotopy on any fiber bundle. In the sequel, we will use parallel transport for the bundle $p : \mathsf{M}_{0,n+1} \to \mathsf{M}_{0,n}$ to study the dependence of the construction of Section 4.4 from the choices made.

4.5.1. Let $p : E \to B$ be a fiber bundle (in the sense of [54, p.90] i.e. a locally trivial bundle). Assume that the base B is Hausdorff and paracompact. Then p is a fibration [54, Cor. 14, p. 96] i.e. it has the homotopy lifting property for every topological space Z: if $H : Z \times [0,1] \to B$ is any map and $f : Z \to E$ lifts $H(\cdot, 0)$, then there is a lifting $\tilde{H}$ of H with $\tilde{H}(\cdot, 0) = f$ (see e.g. [54, p. 66]).

4.5.2. Given $b_0, b_1 \in B$ let $\Omega(B, b_0, b_1)$ denote the set of all paths α in B with $\alpha(0) = b_0$ and $\alpha(1) = b_1$. We write $\alpha \sim \beta$ if $\alpha \simeq \beta \operatorname{rel}\{0,1\}$. Let $\Pi_1(B)$ denote the fundamental groupoid of B: this is the small category whose objects are the points of B and with morphisms from b_0 to b_1 equal to $\Omega(B, b_0, b_1)/\sim$, composition being given by $[\alpha] \cdot [\beta] = [\alpha * \beta]$.

4.5.3. For any fiber bundle $p : E \to B$ one can define a sort of parallel transport up to homotopy, which is a contravariant functor T from $\Pi_1(B)$ to the homotopy category of topological spaces, denoted by $\mathrm{h} - \mathrm{TOP}$. For $b \in B$ set $T(b) := E_b = p^{-1}(b)$. Given $[\alpha] \in \Pi_1(B)(b_0, b_1)$ consider the map $H : E_{b_0} \times [0,1] \to B, H(e,t) := \alpha(t)$. The inclusion $i : E_{b_0} \hookrightarrow E$ is a lifting of $H(\cdot, 0)$. By the homotopy lifting property there is $\tilde{H} : E_{b_0} \times [0,1] \to E$ with $p\tilde{H} = H$ and $\tilde{H}(\cdot, 0) = i$. Then we set $T([\alpha]) = [\tilde{H}(\cdot, 1)] \in [E_{b_0}, E_{b_1}]$. See e.g. [54, Thm. 12, p. 101] and [44, p. 54].

4.5.4. If $p : E \to B$ is a differentiable fiber bundle one can say more. Recall the following basic fact from differential topology. Let M and N be smooth manifolds. An *isotopy* of M in N is a smooth map $f : M \times [0,1] \to N$ such that $f(\cdot, t)$ is an embedding for any t. If $M = N$, $f(\cdot, t)$ is a diffeomorphism of M for any t and $f(\cdot, 0) = \mathrm{id}_M$, we say that f is a *ambient isotopy.*

Theorem 4.5.5. *If M is a compact submanifold of N, any isotopy $f : M \times [0,1] \to N$ such that $f(\cdot, 0)$ is the inclusion $M \hookrightarrow N$ extends to an ambient isotopy.*

(See e.g [34, Thm. 1.3 p. 180].)

Lemma 4.5.6. *Assume that $p : E \to B$ is a differentiable bundle. Let α be a path in B from b_0 to b_1. Let σ be a path in E with $p\sigma = \alpha$ and set $x_0 = \sigma(0) \in E_{b_0}$, $x_0' = \sigma(1) \in E_{b_1}$. Then there is a map $\tilde{H} : E_{b_0} \times [0,1] \to E$ such that*

1. *$\tilde{H}(\cdot, 0)$ is the inclusion $E_{b_0} \hookrightarrow E$;*
2. *$\tilde{H}(\cdot, t)$ is a diffeomorphism of E_{b_0} onto $E_{\alpha(t)}$;*
3. *$\tilde{H}(x_0, t) = \sigma(t)$.*

In particular the map $f^\alpha := \tilde{H}(\cdot, 1)$ is a diffeomorphism of E_{b_0} onto E_{b_1} such that $f(x_0) = x_0'$ and $T([\alpha]) = [f^\alpha]$.

Proof. Denote by $\tilde{\alpha} : \alpha^*E \to E$ the bundle map covering α. Since $[0,1]$ is contractible, there is a (smooth) trivialization $\psi : E_{b_0} \times [0,1] \to \alpha^*E$ such that $\psi(x,0) = x$, see [55, Cor. 11.6 p. 53]. Given any such ψ the composition $\tilde{\alpha}\circ\psi : E_{b_0}\times[0,1] \to E$ is a possible choice for the map $\tilde{H}$ in 4.5.3. We now modify ψ so that it matches the conditions (a)-(c). First notice that if $\{h_t\}_{t\in[0,1]}$ is any path in $\mathrm{Diff}(E_{b_0})$ starting at the the identity, then $\psi'_t := \psi_t h_t$ is a new trivialization of α^*E. Next observe that $t \mapsto \psi_t^{-1}(\sigma(t))$ is a path in E_{b_0} from x_0 to $\psi_1^{-1}(x'_0)$, i.e. an isotopy of $\{x_0\}$ in E_{b_0}. By Theorem 4.5.5 there is $\{h_t\}$ that extends this isotopy. Then $\psi'_t := \psi_t h_t$ is a trivialization and $\tilde{H} := \tilde{\alpha} \circ \psi'$ satisfies (a)-(c). □

Proposition 4.5.7. *Let $x, x' \in \mathsf{M}_{0,n+1}$. Let $\beta : [0,1] \to \mathsf{M}_{0,n}$ be a path such that $\beta(0) = X$ and $\beta(1) = X'$. Let $\tilde{H}$, f^β and $T([\beta])$ be as in Lemma 4.5.6. Assume that $f^\beta(x_0) = x'_0$. Set $\sigma(t) := \tilde{H}(t, x_0)$. Then for $[\gamma] \in \pi_1(\mathbb{P}^1 - X, x_0)$ we have $f^\beta_*[\gamma] = \sigma_\#([\gamma])$.*

Proof. Take $[\gamma] \in \pi_1(\mathbb{P}^1 - X, x_0)$. Consider the map

$$F : [0,1] \times [0,1] \to \mathsf{M}_{0,n+1} \quad F(t,s) = \tilde{H}(\gamma(s), t).$$

Then $F(0,s) = \tilde{H}(\gamma(s),0) = \gamma(s)$, $F(0,1) = \tilde{H}(\gamma(s),1) = f^\beta \circ \gamma(s)$ and $F(t,0) = F(t,1) = \tilde{H}(x_0,t) = \sigma(t)$. The map F is an homotopy between γ and $f^\beta \circ \gamma$ and we get $f^\beta_*[\gamma] = \sigma_\#([\gamma]) = [\sigma * \gamma * i(\sigma)]$ for any $[\gamma] \in \pi_1(\mathbb{P}^1 - X, x_0)$. □

4.5.8. Let us continue by making some considerations related to the fiber bundle $p : \mathsf{M}_{0,n+1} \to \mathsf{M}_{0,n}$. A simple way to produce a cross section is as follows: Given $x = (x_1, \ldots, x_n) \in \mathsf{M}_{0,n}$ we set

$$f(x) := \frac{1}{2}\min\{1, |x_1|, \ldots, |x_{n-3}|\}.$$

Then $s(x) := (f(x), x_1, \ldots, x_n)$ is a section of $p : \mathsf{M}_{0,n+1} \to \mathsf{M}_{0,n}$. (A similar idea is used in [12, Thm. 3.1].) Since p admits cross sections, the exact sequence (4.4.5) splits. More precisely, let $s : \mathsf{M}_{0,n} \to \mathsf{M}_{0,n+1}$ be a section of $p : \mathsf{M}_{0,n+1} \to \mathsf{M}_{0,n}$. Fix $X \in \mathsf{M}_{0,n}$. Then $s(X)$ equals $x = (x_0, X)$ for some $x_0 \in \mathbb{P}^1 - X$. We get $s_* : \pi_1(\mathsf{M}_{0,n+1}, X) \to \pi_1(\mathsf{M}_{0,n}, x)$ and the morphism giving the semidirect product in (4.4.5) is

$$\begin{aligned} &\varepsilon : \pi_1(\mathsf{M}_{0,n}, X) \to \mathrm{Aut}(\pi_1(\mathbb{P}^1 - X, x_0)) \\ &\varepsilon([\alpha])([\gamma]) = s_*[\alpha] \cdot [\gamma] \cdot s_*[\alpha]^{-1} = [s \circ \alpha * \gamma * s \circ i(\alpha)]. \end{aligned} \tag{4.5.1}$$

Proposition 4.5.7 gives a geometric intepretation of the morphism (4.5.1) in terms of parallel transport.

Proposition 4.5.9. *Let $[\alpha] \in \pi_1(\mathsf{M}_{0,n}, X)$ and let $\tilde{H}$, f^α and $T([\alpha])$ be as in Lemma 4.5.6. Assume that $\sigma(t) := \tilde{H}(t, x_0) = s \circ \alpha$. Then $\varepsilon([\alpha]) = f^\alpha_*$.*

Proof. By Proposition 4.5.7, we get $f^\alpha_*[\gamma] = \sigma_\#([\gamma]) = [\sigma * \gamma * i(\sigma)]$ for any $[\gamma] \in \pi_1(\mathbb{P}^1 - X, x_0)$. Hence f^α satisfies $f^\alpha_*[\gamma] = [s\circ\alpha*\gamma*s\circ i(\alpha)] = \varepsilon([\alpha])([\gamma])$ for every $[\gamma] \in \pi_1(\mathbb{P}^1 - X, x_0)$. $\square$

Corollary 4.5.10. $\operatorname{Im}\varepsilon \subset \operatorname{Aut}^{**}(\pi_1(\mathbb{P}^1 - X, x_0))$.

Proof. By the continuity of $\tilde{H}$, f^α extends to a homeomorphism $f^\alpha : \mathbb{P}^1 \to \mathbb{P}^1$ that fixes every x_i individually. By the Corollary 4.1.16 of the Dehn-Nielsen-Baer Theorem, the statement follows. $\square$

4.5.11. We now consider the (pure version of the) generalized Birman exact sequence associated with $\mathbb{C}^{**} = \mathbb{P}^1 - \{0, 1, \infty\}$, see [13, Thm. 9.1, p. 245].

$$1 \to \pi_1(\mathsf{M}_{0,n}, X) \xrightarrow{Push} \mathrm{PMod}(\mathbb{P}^1 - X) \xrightarrow{Forget} \mathrm{PMod}(\mathbb{C}^{**}) \to 1. \qquad (4.5.2)$$

The map $Forget$ is the natural homeomorphism obtained by filling in the puntures, i.e., it is the map induced by the inclusion $\mathbb{P}^1 - X \hookrightarrow \mathbb{C}^{**}$. The map $Push$ is defined as follows (see [13, Section 4.2.1]). Let $\alpha = (\alpha_1, ..., \alpha_n) : [0, 1] \to \mathsf{M}_{0,n}$ be a pure braid in $\mathbb{P}^1$, with $\alpha(0) = \alpha(1) = X$. Thinking of α as an isotopy from X to X (sending each x_i to x_i) we get that it can be extended to an isotopy of the whole $\mathbb{P}^1$ by Theorem 4.5.5. Denoting by Φ_α the homeomorphism of $\mathbb{P}^1$ obtained at the end of the isotopy, we have that $\Phi_\alpha(x_i) = \alpha_i(1) = x_i$, and thus Φ_α can be reguarded as an homeomorphism of $\mathbb{P}^1 - X$. Taking its isotopy class we get $Push(\alpha) = [\Phi_\alpha] \in \mathrm{PMod}(\mathbb{P}^1 - X)$. This map is well defined, i.e., it does not depend on the choice of α within its homotopy class nor on the choice of the isotopy extension.

4.5.12. Let $\tilde{\varepsilon} : \pi_1(\mathsf{M}_{0,n}, X) \to \mathrm{Out}^{**}(\pi_1(\mathbb{C}^{**} - X, x_0))$ denote the composition of ε with the natural projection $\mathrm{Aut}^{**} \to \mathrm{Out}^{**}$

$$\tilde{\varepsilon} : \pi_1(\mathsf{M}_{0,n}, X) \xrightarrow{\varepsilon} \mathrm{Aut}^{**}(\pi_1(\mathbb{C}^{**} - X, x_0)) \to \mathrm{Out}^{**}(\pi_1(\mathbb{C}^{**} - X, x_0)).$$

Also, denote by $F : \mathrm{PMod}(\mathbb{C}^{**} - X) \to \mathrm{Out}^{**}(\pi_1(\mathbb{C}^{**} - X, x_0))$ the isomorphism $F : [h] \mapsto [h_*]$ coming from Corollary 4.1.16 of the Dehn-Nielsen-Baer Theorem.

Proposition 4.5.13. *The following diagram commutes*

$$\begin{array}{ccc} & & \mathrm{PMod}(\mathbb{P}^1 - X) \\ & \overset{Push}{\nearrow} & \big\downarrow F \\ \pi_1(\mathsf{M}_{0,n}, X) & & \\ & \underset{\tilde{\varepsilon}}{\searrow} & \\ & & \mathrm{Out}^{**}(\pi_1(\mathbb{P}^1 - X, x_0)) \end{array}$$

Proof. Let $\alpha : [0, 1] \to \mathsf{M}_{0,n}$ be a pure braid in $\mathbb{P}^1$, with $\alpha(0) = \alpha(1) = X$, that we think as an isotopy from X to X. Let $\tilde{H} : (\mathbb{P}^1 - X) \times [0, 1] \to \mathsf{M}_{0,n+1}$

and f^α be as in Lemma 4.5.6. Define a map $\psi : \mathbb{P}^1 \times [0,1] \to \mathbb{P}^1$ by $\psi(u,t) := \tilde{H}(u,t)$ for $u \notin X$ and $\psi(x_i,t) := \alpha_i(t)$. So ψ is an ambient isotopy of $\mathbb{P}^1$ extending the isotopy α. This proves the result, since by Proposition 4.5.9 $\varepsilon([\alpha]) = f^\alpha_*$, so $\tilde{\varepsilon}([\alpha]) = f^\alpha_* \mod \operatorname{Inn}\pi_1(\mathbb{P}^1 - X, x_0)$, while $Push([\alpha]) = [f^\alpha]$. □

Remark 4.5.14. Since PMod($\mathbb{C}^{**}$) is trivial (see [13, Proposition 2.3]), it follows from (4.5.2) that $Push$ (and thus $\tilde{\varepsilon}$) is an isomorphism. In particular, for every $\overline{\nu} \in \operatorname{Aut}^{**}(\pi_1(\mathbb{P}^1 - X, x_0))$, there exists $g \in \pi_1(\mathsf{M}_{0,n}, X)$ and $\sigma \in \operatorname{Inn}(\pi_1(\mathbb{P}^1 - X, x_0))$ such that $\varepsilon(g) = \overline{\nu} \circ \sigma$.

Remark 4.5.15. Considering configuations of points in $\mathbb{C}$ instead of $\mathbb{C}^{**}$, Proposition 4.5.13 corresponds to Theorem 1.10 in [2].

4.6 Independence from the choices

Given a finite group G and a datum $\theta : \Gamma_n \to G$, we have constructed in Section 4.4 a smooth family $\pi_\theta : \mathscr{C}_\theta \to \mathsf{B}_\theta$ of $G-$curves associated with $\theta \in \mathscr{D}^n(G)$. For the construction, we made some choices. In particular, we fixed

1. an element $x = (x_0, X) \in \mathsf{M}_{0,n+1}$;
2. a geometric basis $\mathscr{B} = \{[\alpha_i]\}_{i=1}^n$ of $\pi_1(\mathbb{P}^1 - X, x_0)$ adapted to X.

In this Section we want to show that, indeed, the construction does not depend on these choices (up to isomorphism).

4.6.1. We will need some consideration on universal coverings that we recall here. Assume that X is a connected and semilocally 1-connected topological space. Fix $x \in X$. Denote by $C([0,1], 0, X, x)$ the set of paths in X starting at x. Let $U(X,x)$ denote the quotient of $C([0,1], 0, X, x)$ by the relation $\sim$, i.e. homotopy rel ∂I. It is well-known that with the appropriate topology on $U(X,x)$, the map $u_{(X,x)} : U(X,x) \to X$, defined by $u_{(X,x)}([\alpha]) := \alpha(1)$ is a simply-connected hence universal covering of X, that there is a distinguished point in $U(X,x)$ corresponding to the class of the constant path and that this point is mapped by $u_{(X,x)}$ to x [4, p. 155-156]. We remark that given $x, x' \in X$ and $\alpha \in \Omega(X, x, x')$ the map

$$U([\alpha]) : U(X,x) \to U(X,y), \quad [\beta] \mapsto [i(\alpha) * \beta]$$

is an isomorphism of coverings over X, i.e. $u_{(X,x')} \circ U([\alpha]) = u_{(X,x)}$. One could rephrase this by saying that the maps $x \mapsto U(X,x)$ and $[\alpha] \mapsto U([\alpha])$ form a functor U from the fundamental groupoid of X to the category of coverings of X. On the other hand one can also associate to a map $f:(X,x)\to(Y,y)$ the map

$$U([f]) := f_* : U(X,x) \to U(Y,x'), \quad U([f])([\alpha]) = [f\circ\alpha].$$

In this case the maps $(X,x)\mapsto U(X,x)$, $[f]\mapsto U([f])$ give a functor from $\mathrm{h}-\mathrm{TOP}^+$, the homotopy category of pointed spaces, to the category of pointed coverings (with varying base).

At a practical level, this means that after fixing a point $x\in X$ we have a pointed universal covering space $U(X,x)$ which is really well-defined, not just up to isomorphism. Moreover $\pi_1(X,x)$ acts on $U(X,x)$. Given $H\subset\pi_1(X,x)$, there is a well-defined pointed covering associated to H, namely $U(X,x)/H$. If $g\in\pi_1(X,x)$ and $H\subset\pi_1(X,x)$, the pointed coverings associated with H and with gHg^{-1} are canonically isomorphic. Indeed the action of g on $U(X,x)$ descends to an isomorphism $\tilde{g}:U(X,x)/H\to U(X,x)/(gHg^{-1})$.

4.6.2. We first show that if $\theta,\theta'\in\mathscr{D}^n(G)$ are in the same orbit for the action of $\operatorname{Aut}^{**}\Gamma_n\times\operatorname{Aut}G$, then the family associated with (G,θ) is isomorphic to the family associated with (G,θ'). Since changing geometric bases of $\pi_1(\mathbb{P}^1-X,x_0)$ adapted to X corresponds to acting with $\operatorname{Aut}^{**}\Gamma_n$, this implies that the costruction does not depend on the choice of the geometric basis.

Theorem 4.6.3. *Let G be a finite group. Choose an element $x=(x_0,X)\in\mathsf{M}_{0,n+1}$ and a geometric basis $\mathscr{B}=\{[\alpha_i]\}_{i=1}^n$ of $\pi_1(\mathbb{P}^1-X,x_0)$ adapted to X. Then the family $\pi_\theta:\mathscr{C}_\theta\to\mathsf{B}_\theta$ of $G-$curves associated with a datum $\theta\in\mathscr{D}^n(G)$ remains the same, up to isomorphism, under the action of* $\operatorname{Aut}^{**}\Gamma_n\times\operatorname{Aut}G$ *on $\mathscr{D}^n(G)$.*

Proof. Denote by $\chi:\Gamma_n\to\pi_1(\mathbb{P}^1-X,x_0)$ the isomorphism associated with the basis $\mathscr{B}$. Let $(\nu,\eta)\in\operatorname{Aut}^{**}\times\operatorname{Aut}G$ and set $\theta':=\eta\circ\theta\circ\nu^{-1}$. Let $f:=\theta\circ\chi^{-1}:\pi_1(\mathbb{P}^1-X,x_0)\to G$ and $f':=\theta'\circ\chi^{-1}:\pi_1(\mathbb{P}^1-X,x_0)\to G$ be the corresponding monodromy morphisms. Note that $f'=\eta\circ f\circ\overline{\nu}^{-1}$, where $\overline{\nu}:=\chi\circ\nu\circ\chi^{-1}\in\operatorname{Aut}^{**}(\pi_1(\mathbb{P}^1-X,x_0))$. We want to show that the families associated with f and with f' are isomorphic. We need to compare the finite index subgroups H'_f and $H'_{f'}$ of $\pi_1(\mathsf{M}_{0,n},X)$ of Lemma 4.4.5, on which the extensions of f and f', respectively, are defined. We first consider H''_f and $H''_{f'}$ defined as in 4.3.2 by:

$$H''_f = \{h\in\pi_1(\mathsf{M}_{0,n},X):\ \varepsilon(h)(\ker f)=\ker f\};$$
$$H''_{f'} = \{\overline{h}\in\pi_1(\mathsf{M}_{0,n},X):\ \varepsilon(\overline{h})(\ker f')=\ker f'\}.$$

Since $\overline{\nu}\in\operatorname{Aut}^{**}(\pi_1(\mathbb{P}^1-X,x_0))$, by Remark 4.5.14 there exists $g\in\pi_1(\mathsf{M}_{0,n},X)$ and $\sigma\in\operatorname{Inn}(\pi_1(\mathbb{P}^1-X,x_0))$ such that $\varepsilon(g)=\overline{\nu}\circ\sigma$. Since $\ker f'=\overline{\nu}(\ker f)=\varepsilon(g)(\ker f)$, we get that $\overline{h}\in H''_{f'}$ if and only if $\varepsilon(g)^{-1}\circ\varepsilon(\overline{h})\circ\varepsilon(g)=\varepsilon(g^{-1}\overline{h}g)\in$

H''_f. Thus $H''_{f'} = gH''_f g^{-1}$. Next we move to consider H'_f and $H'_{f'}$. Recall that

$$H'_f = \{h \in H''_f : \; f \circ \varepsilon(h) = f\};$$
$$H'_{f'} = \{\overline{h} \in H''_{f'} : \; f' \circ \varepsilon(\overline{h}) = f'\}.$$

Using that $f' = \eta \circ f \circ \overline{\nu}^{-1}$ and $H''_{f'} = gH''_f g^{-1}$, it is immediate to check that the same relation holds for H'_f and $H'_{f'}$, namely that $H'_{f'} = gH'_f g^{-1}$. By 4.6.1, the pointed covering associated with H'_f and $H'_{f'}$ are canonically isomorphic. We get a biholomorphism $\beta : \mathsf{B}_\theta \to \mathsf{B}_{\theta'}$ induced by g and a commutative diagram

$$\begin{array}{ccc} \mathsf{B}_\theta & \xrightarrow{\beta} & \mathsf{B}_{\theta'} \\ & {\scriptstyle\gamma}\searrow \quad \swarrow{\scriptstyle\delta} & \\ & \mathsf{M}_{0,n}. & \end{array}$$

Now observe that, since $g \in \pi_1(\mathsf{M}_{0,n}, X)$, then $H'_{f'} = gH'_f g^{-1}$ also imply that $\pi_1(\mathbb{P}^1 - X, x_0) \rtimes H'_{f'} = g(\pi_1(\mathbb{P}^1 - X, x_0) \rtimes H'_f)g^{-1}$. Again we get a canonical biholomorphism $\mathscr{C}^*_\theta \to \mathscr{C}^*_{\theta'}$ and the commutative diagram

$$\begin{array}{ccc} \mathscr{C}^*_\theta & \xrightarrow{\alpha} & \mathscr{C}^*_{\theta'} \\ & \searrow \quad \swarrow & \\ & \mathsf{M}_{0,n+1}. & \end{array}$$

We obtain that the following commutes

$$\begin{array}{ccc} \mathscr{C}^*_\theta & \xrightarrow{\alpha} & \mathscr{C}^*_{\theta'} \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi'} \\ \mathsf{B}_\theta & & \mathsf{B}_{\theta'} \\ & {\scriptstyle\gamma}\searrow \quad \swarrow{\scriptstyle\delta} & \\ & \mathsf{M}_{0,n}. & \end{array}$$

We conclude that $\delta \circ \beta \circ \pi = \delta \circ \pi' \circ \alpha$. It follows that there exists $\psi \in \mathrm{Aut}(\delta : \mathsf{B}_{\theta'} \to \mathsf{M}_{0,n})$ such that $\pi' \circ \alpha = \psi \circ \beta \circ \pi$. Note that the composition with the automorphism ψ of the covering $\mathsf{B}_{\theta'} \to \mathsf{M}_{0,n}$ corresponds to changing the central fiber of the covering. We conclude that, up to this change of the central fiber, the families are isomorphic. That is, they are isomorphic as non-pointed families. □

Corollary 4.6.4. *Let G be a finite group. Choose an element $x = (x_0, X) \in \mathsf{M}_{0,n+1}$ and a geometric basis $\mathscr{B} = \{[\alpha_i]\}_{i=1}^n$ of $\pi_1(\mathbb{P}^1 - X, x_0)$ adapted to*

X. Then the family $\pi_\theta : \mathscr{C}_\theta \to \mathsf{B}_\theta$ of $G-$curves associated with a datum $\theta \in \mathscr{D}^n(G)$ does not depend (up to isomorphism) on the choice of the geometric basis.

4.6.5. Let us now focus on the dependence of the family on the point in $\mathsf{M}_{0,n+1}$. Let $x \in \mathsf{M}_{0,n+1}$. Let $\mathscr{B} = \{[\alpha_i]\}_{i=1}^n$ be a geometric basis of $\pi_1(\mathbb{P}^1 - X, x_0)$ adapted to X. Denote by $\chi : \Gamma_n \to \pi_1(\mathbb{P}^1 - X, x_0)$ the associated isomorphism. Now let $\overline{x} \neq x \in \mathsf{M}_{0,n+1}$. Let $\beta : [0,1] \to \mathsf{M}_{0,n+1}$ be a path such that $\beta(0) = X$ and $\beta(1) = \overline{X}$, and let f^β be the parallel transport along β such that $f^\beta(x_0) = \overline{x}_0$. We choose $\overline{\mathscr{B}}$ to be the image of $\mathscr{B}$ through $f_*^\beta : \pi_1(\mathbb{P}^1 - X, x_0) \to \pi_1(\mathbb{P}^1 - \overline{X}, \overline{x}_0)$ and set $\overline{\chi} := f_*^\beta \circ \chi$. Also, we define $\Phi : \mathrm{Aut}^{**}(\pi_1(\mathbb{P}^1 - X, x_0)) \to \mathrm{Aut}^{**}(\pi_1(\mathbb{P}^1 - \overline{X}, \overline{x}_0))$ as $\Phi(y) = f_*^\beta \circ y \circ (f_*^\beta)^{-1}$.

Lemma 4.6.6. *The following diagram is commutative*

$$\begin{array}{ccc}
\pi_1(\mathsf{M}_{0,n}, X) & \xrightarrow{\quad\varepsilon\quad} & \mathrm{Aut}^{**}(\pi_1(\mathbb{C}^{**} - X, x_0)) \\
\downarrow{\scriptstyle \beta_\#} & & \downarrow{\scriptstyle \Phi} \\
\pi_1(\mathsf{M}_{0,n}, \overline{X}) & \xrightarrow{\quad\overline{\varepsilon}\quad} & \mathrm{Aut}^{**}(\pi_1(\mathbb{C}^{**} - \overline{X}, \overline{x}_0)).
\end{array}$$

Proof. Let $[\alpha] \in \pi_1(\mathsf{M}_{0,n}, X)$. We recall that, by Proposition 4.5.9, ε is given by the parallel transport. We get $\overline{\varepsilon}(\beta_\#([\alpha])) = f_*^{\beta_\#([\alpha])} = (f^{\beta^{-1} * \alpha * \beta})_*$. On the other side $f_*^\beta \circ \varepsilon([\alpha]) \circ (f_*^\beta)^{-1} = f_*^\beta \circ f_*^\alpha \circ f_*^{\beta^{-1}}$. Since $f^{\beta^{-1} * \alpha * \beta} = f^\beta \circ f^\alpha \circ f^\beta$ the statement follows. □

Theorem 4.6.7. *Let $\theta \in \mathscr{D}^n(G)$ be a datum. Fix $x, \bar{x} \in \mathsf{M}_{0,n+1}$. Denote by $f := \theta \circ \chi^{-1}$ and $\overline{f} := \theta \circ \overline{\chi}^{-1} = f \circ (f_*^\beta)^{-1}$ be the associated monodromies. Let H' and $\overline{H'}$ be the subgroups of $\pi_1(\mathsf{M}_{0,n}, X)$, resp. $\pi_1(\mathsf{M}_{0,n}, \overline{X})$, on which the extensions of f and f' are defined. Then $\beta_\#(H') = \overline{H'}$.*

Proof. Similarly as in the proof of Theorem 4.6.3, one checks with an easy calculation first that $\beta_\#(H'') = \overline{H''}$ and secondly that the statement holds using $\ker \overline{f} = f_*^\beta \ker f$, and Lemma 4.6.6. □

Corollary 4.6.8. *Let $\tilde{H}$ denote the parallel transport along β. Set $\sigma(t) := \tilde{H}(t, x_0)$. Then $\sigma_\#(\pi_1(\mathbb{P}^1 - X, x_0)) \rtimes H') = \pi_1(\mathbb{P}^1 - \overline{X}, \overline{x}_0) \rtimes \overline{H'}$.*

Proof. The statement follows from Proposition 4.5.7 and Theorem 4.6.7. □

Theorem 4.6.9. *Let G be a finite group. Choose an element $x = (x_0, X) \in \mathsf{M}_{0,n+1}$ and a geometric basis $\mathscr{B} = \{[\alpha_i]\}_{i=1}^n$ of $\pi_1(\mathbb{P}^1 - X, x_0)$ adapted to X. Then the family $\pi_\theta : \mathscr{C}_\theta \to \mathsf{B}_\theta$ of $G-$curves associated with a datum $\theta \in \mathscr{D}^n(G)$ does not depend (up to isomorphism) on the choice of the point $x \in \mathsf{M}_{0,n+1}$.*

Proof. Let $\overline{x}, x \in \mathsf{M}_{0,n+1}$. Using the notation above, we need to show that the families constructed with $f : \pi_1(\mathbb{P}^1 - X, x_0) \to G$ and with $\overline{f} : \pi_1(\mathbb{P}^1 - \overline{X}, \overline{x}_0)$ are isomorphic. Let $(\tilde{\mathsf{M}}_{0,n}, \tilde{x})$ be the pointed universal covering space of $\mathsf{M}_{0,n}$, and similarly $(\tilde{\mathsf{M}}_{0,n}, \tilde{y})$. We have a natural biholomorphism $F_\beta : (\tilde{\mathsf{M}}_{0,n}, \tilde{x}) \to (\tilde{\mathsf{M}}_{0,n}, \tilde{y})$ induced by β. Quotienting $(\tilde{\mathsf{M}}_{0,n}, \tilde{x})$ by H' and $(\tilde{\mathsf{M}}_{0,n}, \tilde{y})$ by $\beta_\#(H') = \overline{H'}$, F_β induces an isomorphism $\mathsf{B}_f \to \mathsf{B}_{\overline{f}}$ and a commutative diagram

$$\begin{array}{ccc} \mathsf{B}_f & \longrightarrow & \mathsf{B}_{\overline{f}} \\ & \searrow \quad \swarrow & \\ & \mathsf{M}_{0,n}. & \end{array}$$

The same argument made with σ instead of β and $\sigma_\#(\pi_1(\mathbb{P}^1 - X, x_0)) \rtimes H') = \pi_1(\mathbb{P}^1 - \overline{X}, \overline{x}_0) \rtimes \overline{H'}$ instead of $\beta_\#(H') = \overline{H'}$ gives a biholomorphism $\mathscr{C}^*_f \to \mathscr{C}^*_{\overline{f}}$ and the commutative diagram

$$\begin{array}{ccc} \mathscr{C}^*_f & \longrightarrow & \mathscr{C}^*_{\overline{f}} \\ & \searrow \quad \swarrow & \\ & \mathsf{M}_{0,n+1}. & \end{array}$$

We conclude as in the proof of Theorem 4.6.3. □

Theorem 4.6.10. *Let G be a finite group and $\theta \in \mathscr{D}^n(G)$. Let $\pi_\theta : \mathscr{C}_\theta \to \mathsf{B}_\theta$ be the family of G-curves associated with θ. Then every algebraic curve with an action of the topological type given by $[\theta]$ is G-isomorphic to some fiber of the family.*

Proof. Suppose that the family $\pi_\theta : \mathscr{C}_\theta \to \mathsf{B}_\theta$ is constructed by fixing the point $x \in \mathsf{M}_{0,n+1}$ and the geometric basis $\mathscr{B}$. Let C be an algebraic curve such that G acts effectively on C in such a way that $C/G \cong \mathbb{P}^1$. We get the ramified covering $\pi : C \to \mathbb{P}^1$. By acting via $\mathrm{PGL}(2, \mathbb{C})$, one can move any three branch points of π to $0, 1$ and ∞. We can thus assume that the set of critical values of $\pi : C \to \mathbb{P}^1$ coincides with $Y \in \mathsf{M}_{0,n}$. Set $C^* := \pi^{-1}(\mathbb{P}^1 - Y)$. Fix a point $y_0 \in \mathbb{P}^1 - Y$ and consider the monodromy $f : \pi_1(\mathbb{P}^1 - Y, y_0) \to G$ associated with $\pi|_{C^*} : C^* \to \mathbb{P}^1 - Y$. Finally fix a basis $\mathscr{B}'$ of $\pi_1(\mathbb{P}^1 - Y, y_0)$ adapted to Y. Let $\chi : \Gamma_n \to \pi_1(\mathbb{P}^1 - Y, y_0)$ denote the associated isomorphism. Denote by $\theta' = f \circ \chi : \Gamma_n \to G$ the datum associated with C. Assume that C has the same topological type of G-action as $[\theta]$, namely that $[\theta] = [\theta'] \in \mathscr{D}^n(G)/\operatorname{Aut}^{**}\Gamma_n \times \operatorname{Aut} G$. Since the family associated to a datum does not depend (up to isomorphism) on the choice of the datum within its orbit for the action of $\operatorname{Aut}^{**}\Gamma_n \times \operatorname{Aut} G$, nor on the choice of the point in $\mathsf{M}_{0,n+1}$, we get that the family $\pi'_\theta : \mathscr{C}'_\theta \to \mathsf{B}'_\theta$ associated with θ', constructed by fixing the point $y \in \mathsf{M}_{0,n}$ and the geometric basis

$\mathscr{B}'$, is isomorphic to $\pi_\theta : \mathscr{C}_\theta \to \mathsf{B}_\theta$. In particular C, which is the central fiber for π'_θ, is G-isomorphic to some fiber of π_θ. $\square$

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