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Istituto Nazionale di Alta Matematica
Università degli Studi di Pavia
Joint Doctoral Program in Mathematics

# Absorption in Invariant Domains for Quantum Markov Evolutions

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2022 – XXXIV Cycle

# Ringraziamenti

Il primo ringraziamento va alla mia relatrice Raffaella Carbone, che ha accettato di seguirmi durante il dottorato. Mi ha guidato con competenza, infinita pazienza e grande umanità, rendendo questi tre anni un'esperienza arricchente e positiva; nulla sarebbe stato possibile senza di lei.

Devo molto ad Anderson (unico altro probabilista tra i dottorandi) per i momenti trascorsi assieme a lavorare, per le nostre conversazioni e per tutto quello che mi ha insegnato.

Sono grato a tutto il personale del dipartimento di Matematica di Pavia, che si è sempre mostrato estremamente disponibile e accogliente.

Sono grandemente riconoscente verso i miei compagni di dottorato, di ufficio e i miei coinquilini di Pavia, che mi hanno fatto subito sentire a casa, mi hanno aiutato ad affrontare le difficoltá che incontravo e hanno condiviso cone me indimenticabili momenti: in particolare ringrazio Gioia e Christian per i preziosi consigli e per aver guidato i miei primi passi a Pavia, Alberto per le chiacchierate e per aver condiviso il suo gusto raffinato e rétro, Barbara per la sua generosa ed esuberante amicizia, Irene per la sua leggerezza, ma anche per avermi saputo ascoltare e consigliare, Silvia, con cui ho iniziato e con cui concludo questo percorso, Agnese, con cui ci siamo fatti forza a vicenda condividendo le nostre insicurezze, Mai per avermi permesso di far parte del giorno del suo matrimonio e per la forza propulsiva che ha sempre portato nella vita di dipartimento, Zambo, caro amico che ho ritrovato con piacere a Pavia, Antonio per la sua bontà e il suo entusiasmo e Simone, conosciuto troppo tardi.

Un pensiero va alla mia famiglia (Artù compreso e, soprattutto, a mio nonno Giuliano), che mi ha sempre appoggiato in ogni scelta che facevo, mi ha sopportato e sorretto nei momenti difficili ed è stata sempre pronta a festeggiare le mie piccole vittorie.

Anche casa Galleni merita un ringraziamento: è stata il mio secondo ufficio e un posto caro e prezioso durante la pandemia.

# Contents

# Introduction

Quantum physical systems usually interact with an environment (a gas of particles, a beam of photons, a heat bath, a measurement apparatus...). The entire system, consisting of the initial quantum system plus the environment, is an isolated system and, therefore, undergoes a reversible evolution, which is driven by a proper Hamiltonian, but that can be quite hard to describe in practice. A possible approach to the problem is to focus on the dynamic of the system only, at the price of having to deal with evolutions which are not reversible anymore.

We should then consider transformations allowing phenomena like decay, dissipation and decoherence, which are proper of open quantum systems. The reduced dynamic can be described by a one-parameter set of linear operators acting on the set of quantum states, $\mathbf{\Phi}_* = (\Phi_{*t})_{t\geq 0}$; the $\Phi_{*t}$'s are no longer bijective, nor satisfy a group law, as instead usually happens for the reversible dynamic of an isolated system. The linear map $\Phi_{*t}$ transforms the initial state of the system $\rho$ into the state of the system at time $t$, so it needs to preserve the set of quantum states, hence it must be trace preserving and positive (actually the stronger condition of complete positivity will be requested). Such a map $\Phi_{*t}$ is usually called a quantum channel.

When we deal with a memoryless evolution, we can assume that the family $\mathbf{\Phi}_*$ is a semigroup of quantum channels, which additionally implies that

$$\Phi_{*t} \circ \Phi_{*s} = \Phi_{*t+s}, \quad s, t \geq 0. \tag{1}$$

We recall that, by usual duality arguments, one can equivalently consider either the evolution of the states of the system (Schrödinger picture) or of the observables (Heisenberg picture), described by the dual family of maps $\mathbf{\Phi} := (\Phi_t)_{t\geq 0}$. $\mathbf{\Phi}$ turns out to be a semigroup of quantum Markov maps, i.e. (completely) positive and identity preserving operators acting on the observables' space. In our work we will consider semigroups both in continuous and discrete time (where the time index belongs to the set $\mathbb{N}$ instead than $[0, +\infty)$). We shall give more details about the definition of states, observables and semigroups of quantum channels or quantum Markov maps in Chapter 1.

From a purely mathematical perspective, semigroups of quantum Markov maps can be seen as a noncommutative generalization of classical Markov semigroups, hence a natural and successful approach to study such families of maps has been to extend probabilistic ideas and tools introduced for classical Markov semigroups. Much work has already been done in the study of accessibility properties and qualitative long-time behaviour of semigroups of quantum Markov maps: fundamental notions like reducibility, recurrence and transience were introduced in the 70s (see [20, 26]) and have been developed further up to more recent years ( [31, 67]); however there are still some open questions in this area and answering to some of them is the goal of part of this work.

Indeed, Chapter 2 is dedicated to the study of absorption problems, introducing the quantum mathematical objects we need to describe the probability that a quantum evolution is captured in an invariant domain, and then studying some naturally connected questions.

Before proceeding further, we briefly recall the definition of absorption probabilities for a classical discrete time homogeneous Markov chain $(X_n)_{n\geq 0}$ with countable state space $E$ and transition matrix $P = (p_{xy})_{x,y\in E}$. A set of states $C \subseteq E$ is said to be closed (or invariant) if it is not possible for the chain to escape from it and that happens whenever one has $p_{xy} = 0$ for all $x \in C$, $y \notin C$; notice that in general $C$ is not a communication class, nor a set of recurrent states. When we consider such a closed set $C$, we can define the corresponding absorption probabilities by introducing the bounded function $A(C)$ on $E$ as

$$A(C)(x) = \mathbb{P}\left(\bigcup_{n\in\mathbb{N}}\{X_n \in C\}\,\middle|\, X_0 = x\right), \quad \text{for } x \in E,$$

so that, for any $x \in E$, $A(C)(x)$ is equal to the probability that the Markov chain, starting from $x$, is eventually absorbed in $C$.

Absorption probabilities have many remarkable properties: they are related to the communication structure of the Markov chain, recurrence, harmonic functions and ergodic theory. Nevertheless, a generalization of absorption probabilities in the noncommutative context was still missing.

The aim of Chapter 2 is to introduce and study the notion of absorption operator associated to an enclosure (also called invariant domain, it is the noncommutative equivalent of a closed set). It turns out that absorption operators share many remarkable properties with their classical counterpart, and also seem promising for some applications (see for instance [19], where a similar object appeared in an embryonal stage to be used for studying the stability of a quantum information process). We analyze the structure of absorption operators and some basic properties, in particular in relation with the notions of recurrence and transience; as a relevant byproduct, we are able to show that the null recurrent space is an enclosure and to complete the decomposition of semigroups of quantum Markov maps into their positive recurrent, null recurrent and transient restrictions, which were issues left open in [67]. See in particular Theorems 2.2.1, 2.2.3 and 2.2.7.

Another application of absorption operators is in the description of the fixed points set of the semigroup, that is

$$\mathcal{F}(\mathbf{\Phi}) := \{x \in B(\mathfrak{h}) : \Phi_t(x) = x,\, \forall t \geq 0\}.$$

The fixed points set is a fundamental object in ergodic theory. Moreover, whether or not fixed points constitute an algebra is an issue that has raised attention in the physical community: it is related to some Noether theorem-type results in open quantum systems (see [38] and references therein) and to the distinction between observables which are not disturbed by a quantum channel and those which are compatible with it ( [2]); also, a necessary condition for environmental induced decoherence to take place is that $\mathcal{F}(\mathbf{\Phi})$ is an algebra. Absorption operators are fixed points of the semigroup and Theorem 2.3.16 shows that $\mathcal{F}(\mathbf{\Phi})$ is completely described by them whenever the recurrent space is absorbing: this result points out the relationship between fixed points and absorption and accessibility features of the semigroup. In addition, it allows to deduce some remarkable features of fixed points.

Indeed, little is known about fixed points of general semigroups ( [1, 37]), while they have been intensively studied in the positive recurrent case, i.e. when there exists a faithful state $\rho$ which is invariant under the action of $\mathbf{\Phi}_*$; under this assumption, they are an algebra and their structure has been completely understood ( [13, 33]). The study of the fixed points set has allowed to obtain for positive recurrent semigroups a noncommutative analogue of the decomposition of positive recurrent states in communication classes; a natural question is to ask whether this still holds true for recurrent semigroups, as in the case of Markov chains. We provide a negative answer, constructing a noncommutative version of a symmetric random walk on $\mathbb{Z}$ (Example 2.3.7).

The last result of chapter 2 we want to mention is Theorem 2.3.23 and considerations below: employing the seminal results in ergodic theory of quantum Markov semigroups obtained by Frigerio, Verri and Łuczak in the 80s and 90s ( [36, 49, 50]), we are able to generalize to the noncommutative context the mean ergodic theorem for Markov chains ( [52, Theorems 1.8.5 and 3.6.3]). The proof of Theorem 2.3.23 is functional analytic, while in the classical case usually relies on coupling arguments, which are impossible to carry to the noncommutative setting. We remark that, while some results are specific to semigroups of quantum Markov maps acting on the whole algebra of bounded linear operators on a Hilbert space, the proofs of many others can be easily carried to the context of a semigroup acting on a generic $W^*$-algebra (Chapter 1 recalls the definition and some basic properties of $W^*$-algebras).

Chapter 3 is devoted to the study of the asymptotic behaviour of quantum trajectories associated to an open quantum random walk (OQRW); absorption operators turn out to be a key instrument in extending the current knowledge on this mathematical model. OQRWs are quantum channels which generalize classical random walks; they were introduced in [4] and have been intensively studied in the last years: we refer to [62] for a recent survey on the subject and e.g. to [5, 51, 60] for some applications. OQRWs differ from unitary quantum random walks because they take into account the effect of the interaction of the walker/particle with the environment (unitary quantum walks are often of difficult physical implementation due to decoherence effects). The walker moves on the set of vertices $V$ with transition probabilities that depend on the initial and arrival vertices and on some internal degrees of freedom; we focus our attention on the case of homogeneous open quantum random walks (HOQRWs), in which $V \subset \mathbb{R}^d$ is a locally finite lattice (the classical example is $V = \mathbb{Z}^d$), transitions are possible only towards adjacent vertices and transition probabilities only depend on the shift between the initial and the final vertex. The position process $(X_n)_{n\geq 0}$ has been object of many investigations, and results as a strong law of large numbers, a central limit theorem and a large deviations principle were proved under some technical assumptions regarding the finite dimensional local Hilbert space of the system ( [3, 14, 44]). By using reducibility structure, recurrence and absorption operators, we can generalize the previous results removing all these additional assumptions, except for the finite dimension of the local space (while the global Hilbert space of the system will anyway be infinite dimensional).

- Theorem 3.5.2 shows that there exists a random variable $\overline{X}_\infty$ that we are able to completely characterize and such that almost surely
$$\lim_{n\to+\infty} \frac{X_n - X_0}{n} = \overline{X}_\infty.$$
- We can prove (Theorem 3.3.4) that for $n$ large enough, the law of $\frac{X_n-X_0}{\sqrt{n}}$ is close (with respect to every distance that metrizes the weak convergence) to a convex combination of Gaussian random variables. The weights and the parameters of the Gaussian measures are explicitly determined.
- Finally Theorem 3.4.3 provides some explicit large deviation upper an lower bounds for $\frac{X_n-X_0}{n}$. While all the results presented so far are shown in full generality, we have to impose some restrictions on the model in order to ensure that $\frac{X_n-X_0}{n}$ satisfies a large deviation principle (Theorem 3.4.4).

Before starting with the precise description of our results, we recall some facts and results about noncommutative probability spaces and semigroups of quantum Markov maps in Chapter 1.

The contents of Chapter 2 already appeared in [11, 40], while results in Chapter 3 are mainly collected in [12], which are partially in collaboration with Raffaella Carbone and Anderson Melchor Hernandez.

**Recent developments.** While working at the project of this thesis, we could also study some problems that we decided not to include in the final version of the thesis.

1. In collaboration with Anderson Melchor Hernandez, we could find a second proof for the strong law of large numbers and the central limit type theorems for the position process $(X_n)_{n\geq 0}$ in Chapter 3. This other strategy takes inspiration from the techniques introduced in [3], essentially based on the solution of the Poisson equation and the use of limit theorems for martingales. We decided not to include this part of our work in the thesis because these techniques did not allow us to derive the large deviations bounds, but would have requested the introduction of a considerable amount of new concepts and definitions. By now there are two proofs of only part of the results, but we hope this second approach will reveal useful to obtain progresses in different directions we are considering.
2. In a recent collaboration with Raffaella Carbone, Madalin Guţă and Merlijn van Horssen, we proved a strong law of large numbers, a central limit theorem and a large deviation principle for the atom counting process related to the single atom maser, which is described by a quantum dynamical semigroup acting on an infinite

dimensional state space ( [41]). Even if the mathematical objects and techniques are similar to the ones introduced in this thesis, we did not include this part since it is quite long, very recent and has no direct connections with absorption dynamics, which are instead the central theme of this dissertation.

CHAPTER 1

# Preliminaries

The aim of this chapter is to gather the main definitions and results that are used in the next chapters and to introduce the notation used in this work. In Section 1.1 we recall the definition of noncommutative probability space involving $W^*$-algebras and normal states and we gather some properties and results about this objects, while in Section 1.2 we deal with semigroups of quantum Markov maps and their main properties related to the study of ergodic theory and long-time behaviour. Special attention is paid to the case of semigroups acting on the bounded linear operators on a separable Hilbert space.

## 1.1 Noncommutative probability space

### 1.1.1 $W^*$-algebras

The concept of $W^*$-algebra (or Von Neumann algebra) captures the algebraic and topological properties that characterize some fundamental objects of both classical and noncommutative probability. Let us give the precise abstract definition. Let us consider a unital $\mathbb{C}$-algebra $\mathcal{A}$; $\mathcal{A}$ is called a Banach algebra if there exists a positive function $\| \, \| : \mathcal{A} \to [0, +\infty)$ such that for every $x, y \in \mathcal{A}$, $\lambda \in \mathbb{C}$

1. $\|x\| = 0$ if and only if $x = 0$, (positive definiteness)
2. $\|x + y\| \leq \|x\| + \|y\|$, (triangular inequality)
3. $\|\lambda x\| = |\lambda| \|x\|$, (homogeneity)
4. $\|xy\| \leq \|x\| \|y\|$ (product is jointly continuous)

and $\mathcal{A}$ is complete with respect to the norm $\| \, \|$.

If there exists an involution $^* : \mathcal{A} \to \mathcal{A}$ such that for every $x, y \in \mathcal{A}$, $\lambda \in \mathbb{C}$

1. $(x+y)^* = x^* + y^*$ and $(\lambda x)^* = \overline{\lambda} x^*$ (antilinearity),
2. $(xy)^* = y^* x^*$

then $\mathcal{A}$ is called a $^*$-algebra. For every Banach space $X$ we denote by $X^\star$ its topological dual, that is

$$X^\star := \{y : X \to \mathbb{C} \text{ s.t. } y \text{ linear and continuous}\}.$$

$X^\star$ is a Banach space endowed with the dual norm $\|y\| := \sup_{\|x\|=1} |y(x)|$.

**Definition 1.1.1.** *A Banach $^*$-algebra $\mathcal{A}$ is a $W^*$-algebra if*

1. *for every $x \in \mathcal{A}$, $\|x^* x\| = \|x\|^2$,*
2. *there exists a Banach space $\mathcal{A}_\star$ such that $(\mathcal{A}_\star)^\star = \mathcal{A}$ (we mean that they are isomorphic as Banach spaces).*

A first remark is that a Banach $^*$-algebra with only property 1. of the definition is called $C^*$-algebra; such a property implies that for every $x \in \mathcal{A}$, $\|x\| = \|x^*\|$ ($^*$ is an antilinear isometry): indeed

$$\|x\|^2 = \|x^* x\| \leq \|x\| \|x^*\|, \tag{1.1}$$

hence $\|x\| \leq \|x^*\|$ and we conclude since $x = (x^*)^*$.

Since every $W^*$-algebra is the dual of some Banach space, it is natural to consider on it the $w^*$-topology, i.e. the topology induced by the family of seminorms of the type $|\omega(\cdot)|$ for some $\omega \in \mathcal{A}_\star$, where we identify $\omega \in \mathcal{A}_\star$ with the element in $\mathcal{A}^\star$ acting in the following way:

$$\mathcal{A} \ni x \mapsto x(\omega) \in \mathbb{C}.$$

A second important remark is that any $w^*$-closed subalgebra $\mathcal{B} \subset \mathcal{A}$ is again a $W^*$-algebra with predual isomorphic to $\mathcal{A}_\star / \mathcal{B}_\perp$ where $\mathcal{B}_\perp = \{\omega \in \mathcal{A}_\star : \omega(x) = 0, \forall x \in \mathcal{B}\}$. Moreover, given two $W^*$-subalgebras $\mathcal{B}, \mathcal{C} \subset \mathcal{A}$ it is easy to see that their intersection is again a $W^*$-algebra, hence given any subset $S \subset \mathcal{A}$ we can safely define the smallest $W^*$-subalgebra containing $S$ and we denote it by $W^*(S)$.

Let us see two important examples of $W^*$-algebras.

**Example 1.1.2.** Let $(E, \mathscr{E}, \mu)$ be a $\sigma$-finite measure space; for every nonnegative measurable function $f : E \to [0, +\infty)$ we can define its essential supremum as

$$\operatorname{ess\,sup}(f) := \inf\{a \in [0, +\infty) : \mu(f^{-1}(a, +\infty)) = 0\}.$$

If we identify the elements which are almost everywhere equal, the set of essentially bounded functions

$$L^\infty(\mu) := \{f : E \to \mathbb{C} \text{ s.t. measurable}, \operatorname{ess\,sup}(|f|) < +\infty\} / \sim,$$
$$f \sim g \text{ if } \operatorname{ess\,sup}(|f - g|) = 0$$

is indeed a $W^*$-algebra:

- it is a $\mathbb{C}$-algebra with respect to pointwise sum and multiplication,
- it is a Banach algebra with respect to the norm $\|f\|_\infty := \operatorname{ess\,sup}(|f|)$;

- the involution is given by pointwise complex conjugation $f^* := \overline{f}$.

Moreover, the set of integrable functions (with suitable identifications)

$$L^1(\mu) := \left\{ f : E \to \mathbb{C} \text{ s.t. measurable}, \int_E |f| d\mu < +\infty \right\} \Big/ \sim,$$
$$f \sim g \text{ if } \int_E |f - g| d\mu = 0$$

is a Banach space with respect to the usual integral norm $\|f\|_1 := \int_E |f| d\mu$ and by Radon-Nikodym's theorem

$$L^1(\mu)^\star = L^\infty(\mu)$$

in the sense that $L^\infty \ni f \mapsto \int_E f \cdot d\mu \in L^1(\mu)^\star$ is an isometric isomorphism.

**Example 1.1.3.** Let $\mathfrak{h}$ be a Hilbert space and consider the set

$$B(\mathfrak{h}) := \{x : \mathfrak{h} \to \mathfrak{h} \text{ s.t. } x \text{ linear and continuous}\}.$$

It is a $W^*$-algebra:

- it is a $\mathbb{C}$-algebra with respect to pointwise sum and composition,
- it is a Banach algebra with respect to the uniform norm $\|x\|_\infty := \sup_{\|u\|=1} \|xu\|$,
- the involution is given by the adjunction: $x^*$ is the unique bounded operator such that for every $u, v \in \mathfrak{h}$, $\langle x^* u, v\rangle = \langle u, xv\rangle$.

There exists a subspace of operators $L^1(\mathfrak{h}) \subset B(\mathfrak{h})$ such that $L^1(\mathfrak{h})^\star = B(\mathfrak{h})$, but we will provide all the details in Section 1.1.2.

A fundamental difference between $L^\infty(\mu)$ and $B(\mathfrak{h})$ is that the first one is a commutative $W^*$-algebra, and it is indeed possible to prove that every commutative $W^*$-algebra is isomorphic to the set of essentially bounded functions on some measure space, see for instance [58, Proposition 1.18.1]. In general the center of a $W^*$-algebra $\mathcal{A}$, defined as

$$Z(\mathcal{A}) := \{x \in \mathcal{A} : \forall y \in \mathcal{A}, xy = yx\},$$

is a proper $W^*$-subalgebra of $\mathcal{A}$. Thank to the involution $^*$, we can define some natural notions in the framework of $W^*$-algebras: an element $x \in \mathcal{A}$ is said to be

- selfadjoint if $x^* = x$,
- positive if there exists $y \in \mathcal{A}$ such that $x = y^* y$;
- a projection if it is selfadjoint and idempotent, i.e. $x^2 = x$ (we denote by $\mathcal{P}(\mathcal{A})$ the set of projections in $\mathcal{A}$).

In the case of Example 1.1.2, the concepts that we have just introduced coincide with notions that should be familiar: a function $f \in L^\infty(\mu)$ is selfadjoint if it is real valued, it is positive if $f \geq 0$ (of course, both properties need to hold $\mu$-almost everywhere); the projections of $L^\infty(\mu)$ are the indicator functions of measurable sets.

Once we have introduced $W^*$-algebras, it is natural to define $W^*$-homomorphisms.

**Definition 1.1.4.** *A function $\pi : \mathcal{A} \to \mathcal{B}$ between $W^*$-algebras is called a $W^*$-morphism if for every $x, y \in \mathcal{A}$, $\alpha, \beta \in \mathbb{C}$*

1. $\pi(\alpha x + \beta y) = \alpha\pi(x) + \beta\pi(y)$,
2. $\pi(xy) = \pi(x)\pi(y)$,
3. $\pi(x^*) = \pi(x)^*$,
4. $\pi$ *is* $w^* - w^*$*-continuous.*

The structure of $C^*$-algebras is rigid in the sense that a map $\pi$ satisfying $1.$, $2.$ and $3.$ is automatically a contraction and that, if it is injective, it is an isometry. Furthermore, the image of a $W^*$-homomorphism is always $w^*$-closed ( [58, Proposition 1.16.2]). Given any $W^*$-algebra, it is always possible to find a concrete representation (isomorphic $W^*$-algebra) inside the bounded linear operators acting on a Hilbert space: this is trivial in Example 1.1.3, while in Example 1.1.2 a natural isomorphism is obtained noticing that $f \in L^\infty(\mu)$ acts as a multiplication operator $M_f$ on $L^2(\mu)$: $M_f : L^2(\mu) \ni g \mapsto gf \in L^2(\mu)$.

**Theorem 1.1.5** ( [58], Theorem 1.16.7)**.** *Let $\mathcal{A}$ be a $W^*$-algebra; then there exist a Hilbert space $\mathfrak{h}$ and a injective $W^*$-homomorphism $\pi : \mathcal{A} \to B(\mathfrak{h})$.*

Hence it is not reductive to think of a $W^*$-algebra as a $w^*$-closed, selfadjoint subalgebra of $B(\mathfrak{h})$ for some Hilbert space $\mathfrak{h}$; the concrete representation of a $W^*$-algebra as an operator algebra acting on some Hilbert space $\mathfrak{h}$ allows to provide another remarkable characterization of $W^*$-algebras in purely algebraic terms. Given any subset $S \subset \mathcal{B}$ of a $W^*$-algebra $\mathcal{B}$, we define its commutant with respect to $\mathcal{B}$ as the set $S' := \{x \in \mathcal{B} : xy = yx, \forall y \in S\}$.

**Theorem 1.1.6** (Von Neumann Bicommutant Theorem)**.** *Let $\mathcal{A} \subset B(\mathfrak{h})$ be a $^*$-subalgebra containing the identity operator* $\mathbf{1}$*. Then the following are equivalent:*

- $\mathcal{A} = \mathcal{A}''$,
- $\mathcal{A}$ *is* $w^*$*-closed.*

*Proof.* See Theorem 1.20.3 and Proposition 1.15.1 in [58]. □

Theorem 1.1.6 characterizes as bicommutant sets only $W^*$-algebras containing the identity operator $\mathbf{1}$; however, for every $w^*$-closed subalgebra $\mathcal{A} \subset B(\mathfrak{h})$, we can always reduce to this situation considering its action on $\overline{h} = \left(\bigcap_{x\in\mathcal{A}} \ker(x)\right)^\perp$. The concrete representation of a $W^*$-algebra, allows to have a different characterization of the center:

$$Z(\mathcal{A}) = \mathcal{A} \cap \mathcal{A}'.$$

Notice that, while the commutant depends on the bigger $W^*$-algebra that we consider, the center is an intrinsic object of $\mathcal{A}$.

$W^*$-algebras are the right objects to work with if one wants to extend probability to the noncommutative setting for two main reasons: the first one is that if we consider a bounded measurable function $f : \mathbb{R} \to \mathbb{R}$ and a selfadjoint element $x \in \mathcal{A}$, there is a way of defining $f(x)$ and $W^*$-algebras are the minimal topological-algebraic structure such that $f(x)$ is still in $\mathcal{A}$. The second one is that $W^*$-algebras are "rich of projections" in the sense that any $W^*$-algebra is the norm closure of the linear space generated by its projections (see Proposition 1.1.9), hence we can carry to this setting many standard techniques that rely on approximation of measurable functions with projections.

A rigorous meaning for a measurable function of a selfadjoint operator is provided by Borel functional calculus. Let us consider a measurable space $(E, \mathscr{E})$, a resolution of the identity (sometimes called projective valued measure) is a map $e : \mathscr{E} \to \mathcal{P}(\mathcal{A})$ such that the following holds true:

- $e(E) = \mathbf{1}$,
- $e(A \cap B) = e(A)e(B)$ for every $A, B \in \mathscr{E}$,
- $\mathscr{E} \ni A \mapsto \omega(e(A)) =: \mu_{e,\omega}(A)$ is a Borel measure for every $\omega \in \mathcal{A}_\star$.

Given a resolution of the identity $e$, for every $f : E \to [0, +\infty)$ measurable function we can define $\operatorname{ess\,sup}(f) = \inf\{a \in [0, +\infty) : e(f^{-1}(a, +\infty)) = 0\}$ and

$$L^\infty(e) = \{f : E \to \mathbb{C} \text{ s.t. } f \text{ measurable and } \operatorname{ess\,sup}(|f|) < +\infty\}/\sim,$$
$$f \sim g \text{ if } \operatorname{ess\,sup}(|f - g|) = 0.$$

Before stating the next result, we recall that the spectrum of any element $x$ belonging to the $W^*$-algebra $\mathcal{A}$ is defined as the set $\operatorname{Sp}(x) := \{z \in \mathbb{C} : z\mathbf{1} - x \text{ is not invertible in } \mathcal{A}\}$; if $x$ is selfadjoint, then $\operatorname{Sp}(x) \subseteq \mathbb{R}$.

**Theorem 1.1.7** (Borel functional calculus). *Let $\mathcal{A}$ be a $W^*$-algebra and $x \in \mathcal{A}$ a selfadjoint element. There exists a unique resolution of the identity $e$ defined on $\mathscr{B}(\operatorname{Sp}(x))$ the Borel $\sigma$-field of $\operatorname{Sp}(x)$ and a unique isomorphism of $W^*$-algebras $\Psi_x : L^\infty(e) \to W^*(\{x\})$ such that for every polynomial $p(t) \in \mathbb{C}[t]$, $\Psi_x(p) = p(x)$.*

*Remark* 1.1.8. The last equality in the theorem states that the image via functional calculus of a polynomial $p(t) = \sum_{i=1}^n \alpha_i t^i \in \mathbb{C}[t]$ coincides with the expression $\sum_{i=1}^n \alpha_i x^i$, which makes perfect sense because $\mathcal{A}$ is a $\mathbb{C}$-algebra.

*Proof.* First we show that for every resolution of the identity $e$ on $\mathscr{B}(\operatorname{Sp}(x))$, $L^\infty(e)$ can be identified with $L^\infty(\mu)$ for some measure $\mu$ such that $L^1(\mu)^\star = L^\infty(\mu)$ and this turns $L^\infty(e)$ into a $W^*$-algebra. Let us consider $\mathcal{B} := W^*(\{e(A) : A \in \mathscr{B}(\mathrm{Sp(x)})\}) \subset \mathcal{A}$ and $(\omega_\alpha)_\alpha$ a maximal family of normal states in $\mathcal{B}_\star$ with orthogonal supports (see Definition 1.2.1 for the definitions of normal state and support projection); by the definition of resolution of the identity every $\omega_\alpha$ induces a Borel probability measure $\mu_{\omega_\alpha}$ on $\mathrm{Sp(x)}$ and $\operatorname{supp}(\mu_{\omega_\alpha}) \cap \operatorname{supp}(\mu_{\omega\alpha'}) = \emptyset$ for $\alpha \neq \alpha'$. If we consider the measure $\mu := \bigoplus_\alpha \mu_\alpha$, we have that $L^\infty(\mu) = L^1(\mu)^\star$ since it is the sum of a net of probability measures with disjoint support (hence $L^\infty(\mu)$ is a $W^*$-algebra). Moreover, by the maximality of $(\omega_\alpha)_\alpha$, for every $A \in \mathscr{B}(\mathrm{Sp(x)}))$,

$$\begin{aligned} e(A) = 0 &\Leftrightarrow \omega_\alpha(e(A)) = 0 \text{ for every } \alpha \\ &\Leftrightarrow \mu_\alpha(A) = 0 \text{ for every } \alpha \\ &\Leftrightarrow \mu(A) = 0. \end{aligned}$$

We can, therefore, identify $L^\infty(\mu)$ and $L^\infty(e)$.

The proof of Borel functional calculus for (normal, hence also) selfadjoint bounded operators can be found in [56, Section 12.24] and ensures that $\Psi_x : L^\infty(e) \to W^*(\{x\})$ is an isometric isomorphism of $^*$-algebras, hence it is also a $W^*$-isomorphism ( [23, Corollary I.4.1]. $\square$

Hence for every $f : \mathrm{Sp}(x) \to \mathbb{C}$ bounded and measurable, we can define $f(x) := \Psi_x(f)$ and we know that $\|f(x)\| \leq \sup_{y\in E} |f(y)|$. Moreover notice that $W^*(\{x\})$ is the smallest subset of $\mathcal{A}$ closed under Borel funcrional calculus. We have the following important corollary.

**Proposition 1.1.9.** *Let $\mathcal{A}$ be a $W^*$-algebra, then*

$$\mathcal{A} = \overline{\mathrm{span}\mathcal{P}(\mathcal{A})}^{\|\ \|}.$$

*Proof.* Let us consider an arbitrary $x \in \mathcal{A}$, then $\Re(x) = x + x^*/2i$ and $\Im(x) = x - x^*/2$ are selfadjoint elements in $\mathcal{A}$ and $x = \Re(x) + i\Im(x)$; hence without loss of generality we can assume that $x$ is selfadjoint. Since the identity function can be approximated on $\mathrm{Sp}(x)$ uniformly by simple functions, by Theorem 1.1.7 we have the statement. $\square$

The last notion we need to introduce about $W^*$-algebras is that of atomic $W^*$-algebra. Given two projections $p, q \in \mathcal{P}(\mathcal{A})$ we say that $q \leq p$ if $pq = qp = q$; we say that a projection $p$ is minimal if for every $q \leq p$ then either $q = 0$ or $q = p$.

**Definition 1.1.10.** *A $W^*$-algebra is said to be atomic if for every projection $p$, there exists a minimal nonnull projection $q$ such that $q \leq p$.*

Atomic $W^*$-algebras can be characterized in the following ways.

**Theorem 1.1.11** (Theorem 5, [66])**.** *The following statements are equivalent:*

1. *$\mathcal{A}$ is atomic;*
2. *$\mathcal{A}$ is isomorphic to $\bigoplus_{\alpha\in A} B(\mathfrak{h}_\alpha)$ for some collection of Hilbert spaces $\{\mathfrak{h}_\alpha\}_{\alpha\in A}$;*
3. *suppose that $\mathcal{A}$ acts on $\mathfrak{h}$, then there exists a $w^*$-continuous projection of norm one $\pi : B(\mathfrak{h}) \to \mathcal{A}$.*

Notice that by virtue of the characterization in point 2., if $\mathcal{A}$ is an atomic $W^*$-algebra acting on a Hilbert space $\mathfrak{h}$, it is possible to find a decomposition of $\mathfrak{h}$ into ranges of orthogonal minimal projections of $\mathcal{A}$.

### 1.1.2 Normal states

Having in mind Example 1.1.2, the following definition of normal state (which is the noncommutative analogue of a probability density) is completely natural.

**Definition 1.1.12.** *Any $\varphi \in \mathcal{A}_\star$ is said to be a normal state if*

- *it is positive, i.e. $\varphi(x^*x) \geq 0$ for every $x \in \mathcal{A}$,*
- *it is normalized, that is $\varphi(\mathbf{1}) = \mathbf{1}$.*

*We denote by $s(\varphi)$ the support projection of $\varphi$, which is the minimal projection $p \in \mathcal{A}$ such that for every $x \in \mathcal{A}$, $\varphi(x) = \varphi(px) = \varphi(xp)$; $\varphi$ is said to be faithful if $s(\varphi) = \mathbf{1}$.*

We are now in position of giving the definition of noncommutative probability space that we consider in this thesis.

**Definition 1.1.13.** *A noncommutative probability space is a pair* $(\mathcal{A}, \varphi)$ *where* $\mathcal{A}$ *is a* $W^*$*-algebra and* $\varphi \in \mathcal{A}_\star$ *is a normal state.*

Although we introduced noncommutative probability spaces in whole generality and we will meet in the present work more general $W^*$-algebras, our object of study will mainly be the probability space given by $B(\mathfrak{h})$ for some separable Hilbert space $\mathfrak{h}$ and a normal state $\varphi \in B(\mathfrak{h})_\star$. At this point, it is convenient to recall a very concrete representation of the predual of $B(\mathfrak{h})$. Given any operator $x \in B(\mathfrak{h})$ we can define $|x| := (x^*x)^{1/2}$ (for instance via Borel functional calculus); we introduce the set of trace class operators as

$$L^1(\mathfrak{h}) := \{x \in B(\mathfrak{h}) : \mathrm{tr}\,(|x|) < +\infty\}.$$

We recall some properties of $L^1(\mathfrak{h})$ [53, Section I.9]:

- every $x \in L^1(\mathfrak{h})$ is compact and $L^1(\mathfrak{h})$ is a two-sided ideal of $B(\mathfrak{h})$.
- $\|x\|_1 := \mathrm{tr}\,(|x|)$ is a complete norm on $L^1(\mathfrak{h})$.
- $B(\mathfrak{h})_\star = L^1(\mathfrak{h})$ identifying $x \in L^1(\mathfrak{h})$ with $\mathrm{tr}\,(x\cdot)$ ($x$ is called the density operator of $\mathrm{tr}\,(x\cdot)$).
- Normal states on $B(\mathfrak{h})$ correspond to trace class operators $x \in L^1(\mathfrak{h})$ such that $x \geq 0$ and $\mathrm{tr}\,(x) = 1$; this implies that there exists an orthonormal basis $(u_i)$ and a sequence of positive numbers $(\lambda_i)$, $\sum_i \lambda_i = 1$ such that $x = \sum_i \lambda_i \left|u_i\right\rangle \left\langle u_i\right|$.
- Every $x \in L^1(\mathfrak{h})$ is a linear combination of four normal states, hence for every net $(x_\alpha) \subset B(\mathfrak{h})$ and operator $x \in B(\mathfrak{h})$, the following are equivalent:
  - $\lim_\alpha \mathrm{tr}\,(yx_\alpha) = \mathrm{tr}\,(yx)$, $\forall y \in L^1(\mathfrak{h})$,
  - $\lim_\alpha \mathrm{tr}\,(yx_\alpha) = \mathrm{tr}\,(yx)$, for all normal states $y \in L^1(\mathfrak{h})$.

We will usually denote with greek letters operators in $L^1(\mathfrak{h})$ intended as elements of $B(\mathfrak{h})_\star$.

The topological dual of $B(\mathfrak{h})$ admits a remarkable decomposition into disjoint subspaces ( [65]): $B(\mathfrak{h})^\star = B(\mathfrak{h})_\star \bigoplus_{\ell^1} B(\mathfrak{h})_\star^\perp$[1], where $B(\mathfrak{h})_\star^\perp$ is the Banach space of singular linear functionals, that is the linear space generated by those $\varphi \in B(\mathfrak{h})^\star$ such that

- $\varphi(x^*x) \geq 0$ for every $x \in B(\mathfrak{h})$ (positivity),
- if there exists $\psi \in B(\mathfrak{h})_\star$, such that $\psi \geq 0$ and $\psi(x^*x) \leq \varphi(x^*x)$ for every $x \in B(\mathfrak{h})$, then $\psi = 0$ (singularity).

If we want to make use of the language of classical measure theory, $B(\mathfrak{h})_\star$ is the set of finite measures which are absolutely continuous with respect to the trace, while $B(\mathfrak{h})_\star^\perp$ corresponds to those finite measures which are singular with respect to the trace.

[1]It means that every element of $x \in B(\mathfrak{h})^\star$ can be written in a unique way as the sum of two elements $x_1 \in B(\mathfrak{h})_\star$ and $x_2 B(\mathfrak{h})_\star^\perp$ and $\|x\| = \|x_1\| + \|x_2\|$.

We denote by $C(\mathfrak{h})$ the Banach space of compact operators; it is a two sided ideal of $B(\mathfrak{h})$ and we have the isometric isomorphism of Banach spaces $C(\mathfrak{h})^\star = L^1(\mathfrak{h})$. Using the results in [22], one can see that

$$C(\mathfrak{h})^\perp := \{\psi \in B(\mathfrak{h})^\star : \psi(x) = 0, \forall x \in C(\mathfrak{h})\} = B(\mathfrak{h})_\star^\perp.$$

**Definition 1.1.14.** *A linear operator $\Psi : B(\mathfrak{h}) \to B(\mathfrak{h})$ is said to be singular if for every $\psi \in B(\mathfrak{h})_\star$, $\psi \circ \Psi \in B(\mathfrak{h})_\star^\perp$.*

## 1.2 Semigroups of quantum Markov maps

After introducing the noncommutative counterpart of functional spaces, we are interested in studying maps acting on them and generalizing the notions of transition matrix and transition kernel. For this purpose we need to introduce the "correct" notion of positivity for maps acting on $W^*$-algebras.

**Definition 1.2.1.** *Let $\mathcal{A}$, $\mathcal{B}$ be $W^*$-algebras; a linear map $\Phi : \mathcal{A} \to \mathcal{B}$ is said to be*

- *positive if for every $x \in \mathcal{A}$, $\Phi(x^*x) \geq 0$.*
- *completely positive if for every $n \in \mathbb{N}$, $\Phi \otimes \mathrm{Id}_{M_n(\mathbb{C})}$ is positive, where $M_n(\mathbb{C})$ stays for the set of $n \times n$ matrices with complex entries.*

In the following Propositions we recall some useful characterizations and properties of completely positive maps (we refer to [27] and references therein for the proofs).

**Proposition 1.2.2.** *Let $\mathfrak{h}$, $\mathfrak{g}$ be Hilbert spaces, $\mathcal{A}$ a $W^*$-algebra acting on $\mathfrak{h}$ and consider a linear map $\Phi : \mathcal{A} \to B(\mathfrak{g})$. The following are equivalent:*

1. *$\Phi$ is completely positive;*
2. *for every $n \in \mathbb{N}$, $x_1, \ldots, x_n \in \mathcal{A}$ and $y_1, \ldots, y_n \in B(\mathfrak{g})$*

$$\sum_{i,j=1}^n y_i^* \Phi(x_i^* x_j) y_j \geq 0;$$

*If, in addition, either $\mathcal{A}$ or $W^*(\Phi(\mathcal{A}))$ is commutative, then the previous conditions are equivalent to $\Phi$ being positive.*

*If the map $\Phi$ is $w^* - w^*$-continuous, then 1. and 2. are also equivalent to the following:*

3. *there exists a collection of bounded linear operators $V_j : \mathfrak{g} \to \mathfrak{h}$ such that $\sum_{j=1}^{+\infty} V_j^* V_j$ converges in the strong operator topology and for every $x \in \mathcal{A}$*

$$\Phi(x) = \sum_{j=1}^{+\infty} V_j^* x V_j.$$

$V_j$'s in point 3. are called Kraus operators. Proposition 1.2.2 shows how complete positivity is a strictly noncommutative feature.

**Proposition 1.2.3.** *Let $\Phi : \mathcal{A} \mapsto \mathcal{B}$ be a completely positive map; then the following hold true:*

1. *for every $x \in \mathcal{A}$, $\Phi(x^*)\Phi(x) \leq \|\Phi(\mathbf{1})\|\Phi(x^*x)$ (Kadison-Schwarz inequality);*

2. $\|\Phi\| := \sup_{\|x\|=1}\|\Phi(x)\| = \|\Phi(\mathbf{1})\|$.

We remark that the proof of Proposition 1.2.3 only requires $\Phi$ to be 2-positive, which means that $\Phi \otimes \mathrm{Id}_{M_2(\mathbb{C})}$ is positive.

**Definition 1.2.4.** *Let $\mathcal{A}$, $\mathcal{B}$ be two $W^*$-algebras; a linear map $\Phi : \mathcal{A} \to \mathcal{B}$ is said to be a quantum Markov map if*

1. *it is completely positive;*

2. *it is $w^*$-$w^*$-continuous;*

3. *it is unital, i.e. $\Phi(\mathbf{1}) = \mathbf{1}$.*

The map acting on the predual $\Phi_*$ is called quantum channel and it is easy to see that it preserves the set of normal states. If, instead of condition 3., we only have $\Phi(\mathbf{1}) \leq \mathbf{1}$, then $\Phi$ is said to be a quantum subMarkov map and $\Phi_*$ is called quantum operation.

**Definition 1.2.5.** *Let $\mathcal{A}$ be a $W^*$-algebra, a semigroup of quantum Markov maps acting on $\mathcal{A}$ is a collection of quantum Markov maps $\mathbf{\Phi} := (\Phi_t)_{t\in\mathfrak{T}}$ indexed by a semigroup $(\mathfrak{T}, +, 0)$ such that*

- $\Phi_0 = \mathrm{Id}_{\mathcal{A}}$*;*

- $\Phi_t\Phi_s = \Phi_{s+t}$ *for every $t, s \in \mathfrak{T}$.*

*The predual semigroup is given by $\mathbf{\Phi_*} := (\Phi_{*t})_{t\in\mathfrak{T}}$.*

We will consider two cases: $\mathfrak{T} = \mathbb{N}$ (discrete time case) and the semigroup is generated by the powers of the same quantum Markov map $(\Phi^n)_{n\geq 0}$, and $\mathfrak{T} = [0, +\infty)$ (continuous time case). In this second case we require that the map $t \mapsto \Phi_t$ is continuous in the $w^*$-pointwise topology (these are known in the literature as quantum Markov semigroups). The following example shows that classical Markov semigroups are quantum Markov semigroups acting on a commutative $W^*$-algebra.

**Example 1.2.6.** Let us consider again $L^\infty(\mu)$, the set of essentially bounded functions on a $\sigma$-finite measure space $(E, \mathcal{E}, \mu)$, and a family of transition kernels $p : \mathfrak{T}\times E\times\mathcal{E} \to [0, 1]$ such that

1. $E \ni x \mapsto p(t, x, A)$ is $\mathcal{E}$-measurable for every $t \in \mathfrak{T}$, $A \in \mathcal{E}$;

2. for $\mu$-almost every $x \in E$, $\mathcal{E} \ni A \mapsto p(t, x, A)$ is a probability measure for any $t \in \mathfrak{T}$ which coincides with $\delta_x$ for $t = 0$;

3. $p(t + s, x, A) = \int_E p(s, y, A)p(t, x, dy)$ for every $t, s \in \mathfrak{T}$, $A \in \mathcal{E}$, $\mu$-almost every $x \in E$;

4. in case $\mathfrak{T} = [0, +\infty)$, $[0, +\infty) \ni t \mapsto p(t, x, A)$ is $w^*$-continuous for every $x \in E$, $A \in \mathcal{E}$. Then the semigroup $(\Phi_t)_{t\in\mathfrak{T}}$ with

$$\Phi_t : L^\infty(\mu) \to L^\infty(\mu)$$
$$f \mapsto \Phi_t(f)(x) := \int_E f(y)p(t, x, dy)$$

falls into the definition of quantum Markov semigroups, which, therefore, includes classical Markov semigroups. Moreover every quantum Markov semigroup acting on a commutative $W^*$-subalgebra of $B(\mathfrak{h})$ for some separable Hilbert space $\mathfrak{h}$ is of this form (it can be shown putting together [55, Remark 4] and [58, Proposition 1.18.1]).

Some remarkable work has been done in order to carry a number of relevant notions belonging to the theory of classical Markov semigroups to the noncommutative setting and this has provided an invaluable instrument in the study of qualitative long-time behaviour of semigroups of quantum Markov maps. For the rest of the section we will consider a semigroup of quantum Markov maps $\mathbf{\Phi}$ acting on $B(\mathfrak{h})$ for some separable Hilbert space.

### 1.2.1 Reducibility

The first important notion is the one of reducibility for a quantum Markov map and the related concept of enclosure (which is sometimes called invariant domain), which trace back to [20] and since then have been intensively studied and exploited.

**Definition 1.2.7.** *A closed subspace $\mathcal{V}$ of $\mathfrak{h}$ is an enclosure for a quantum Markov map $\Phi$ if, for any normal state $\rho$,*

$$\mathrm{supp}(\rho) \subseteq \mathcal{V} \qquad \textit{implies} \qquad \mathrm{supp}(\Phi_*(\rho)) \subseteq \mathcal{V}.$$

*$\mathcal{V}$ is an enclosure for a semigroup $\mathbf{\Phi} = (\Phi_t)_{t\geq 0}$ when it is an enclosure for any map $\Phi_t$ of the semigroup.*
*$\Phi$ or $\mathbf{\Phi}$ are said to be irreducible if the only enclosures are the trivial ones, i.e. $\{0\}$ and $\mathfrak{h}$.*

In discrete time, i.e. $\mathbf{\Phi} = (\Phi^n)_{n\in\mathbb{N}}$, $\mathcal{V}$ is an enclosure for $\Phi$ if and only if it is an enclosure for $\mathbf{\Phi}$ and, consequently $\Phi$ is irreducible if and only if $\mathbf{\Phi}$ is irreducible.

For every closed subspace $\mathcal{W} \subset \mathfrak{h}$ we denote by $p_\mathcal{W}$ the corresponding orthogonal projection. Moreover we will often identify $p_\mathcal{W} L^1(\mathfrak{h}) p_\mathcal{W}$ ($p_\mathcal{W} B(\mathfrak{h}) p_\mathcal{W}$) with $L^1(\mathcal{W})$ ($B(\mathcal{W})$) when it will not create confusion.

The following ones are equivalent characterizations of the concept of enclosure (see [16, Section 3] and [32, Proposition 5.1]):

- $\mathcal{V}$ is an enclosure for $\Phi$,
- $p_\mathcal{V}$ is a reducing or subharmonic projection for $\Phi$, i.e. $\Phi(p_\mathcal{V}) \geq p_\mathcal{V}$
- $p_\mathcal{V} L^1(\mathfrak{h}) p_\mathcal{V}$ is hereditary for $\Phi_*$, i.e. $\Phi_*(p_\mathcal{V} L^1(\mathfrak{h}) p_\mathcal{V}) \subseteq p_\mathcal{V} L^1(\mathfrak{h}) p_\mathcal{V}$, i.e. a weak-closed face preserved by $\Phi_*$.
- $p_\mathcal{V}^\perp B(\mathfrak{h}) p_\mathcal{V}^\perp$ is hereditary for $\Phi$.

For the previous alternative characterizations only positivity of $\Phi$ is necessary; if, furthermore, $\Phi$ is completely positive and $(V_i)_{i\in I}$ is a choice of Kraus operators, the previous conditions are also equivalent to

- for every $i \in I$, $V_i p_{\mathcal{V}} = p_{\mathcal{V}} V_i p_{\mathcal{V}}$ or equivalently $V_i(\mathcal{V}) \subset \mathcal{V}$.

*Remark* 1.2.8. Let us consider the case of a classical Markov semigroup as in Example 1.2.6; in this framework an enclosure corresponds to a set $C \in \mathcal{E}$ such that $p(t, x, C) = 1$ for every $t \in \mathfrak{T}$, $\mu$-almost every $x \in C$. Such a set $C$ is called a closed set.

When $\mathcal{V}$ is an enclosure, we can define the restricted quantum Markov maps $\Phi^{\mathcal{V}}$ and $\Phi^{\mathcal{V}^\perp}$ and the corresponding predual maps:

$$\begin{aligned} \Phi^{\mathcal{V}}(p_{\mathcal{V}} x p_{\mathcal{V}}) &:= p_{\mathcal{V}} \Phi(p_{\mathcal{V}} x p_{\mathcal{V}}) p_{\mathcal{V}} = p_{\mathcal{V}} \Phi(x) p_{\mathcal{V}} \qquad \forall\, x \in B(\mathfrak{h}), \\ \Phi^{\mathcal{V}}_*(p_{\mathcal{V}} x p_{\mathcal{V}}) &:= p_{\mathcal{V}} \Phi_*(p_{\mathcal{V}} x p_{\mathcal{V}}) p_{\mathcal{V}} = \Phi_*(p_{\mathcal{V}} x p_{\mathcal{V}}) \qquad \forall\, x \in L_1(\mathfrak{h}) \end{aligned} \tag{1.2}$$

and

$$\begin{aligned} \Phi^{\mathcal{V}^\perp}(p^\perp_{\mathcal{V}} x p^\perp_{\mathcal{V}}) &:= p^\perp_{\mathcal{V}} \Phi(p^\perp_{\mathcal{V}} x p^\perp_{\mathcal{V}}) p^\perp_{\mathcal{V}} = \Phi(p^\perp_{\mathcal{V}} x p^\perp_{\mathcal{V}}) \qquad \forall\, x \in B(\mathfrak{h}), \\ \Phi^{\mathcal{V}^\perp}_*(p^\perp_{\mathcal{V}} x p^\perp_{\mathcal{V}}) &:= p^\perp_{\mathcal{V}} \Phi_*(p^\perp_{\mathcal{V}} x p^\perp_{\mathcal{V}}) p^\perp_{\mathcal{V}} = p^\perp_{\mathcal{V}} \Phi_*(x) p^\perp_{\mathcal{V}} \qquad \forall\, x \in L_1(\mathfrak{h}). \end{aligned} \tag{1.3}$$

Thank to Equations (1.2) and (1.3), $\mathbf{\Phi}^{\mathcal{V}}$ and $\mathbf{\Phi}^{\mathcal{V}^\perp}$ mantain the semigroup property. It is well known that the set of all closed subspaces (equivalently projections) is a complete lattice with respect to the following operations: given $\mathcal{V}$ and $\mathcal{W}$ closed subspaces of $\mathfrak{h}$ we define

$$\mathcal{V} \vee \mathcal{W} := \overline{\operatorname{span}\{\mathcal{V}, \mathcal{W}\}}^{\|\,\|}, \quad \mathcal{V} \wedge \mathcal{W} := \mathcal{V} \cap \mathcal{W}.$$

The set of all enclosures behaves well with respect to the lattice structure of closed subspaces of an Hilbert space. We refer to [37, Theorem 3.10] and references therein for the proof of the following result.

**Theorem 1.2.9.** *The set of all enclosures is a complete sublattice of the lattice of the closed subspaces.*

Irreducibility is a fundamental notion in the Perron-Frobenius theory, which has been carried to the setting of positive maps acting on matrix algebras. We denote the spectral radius of a map $\Phi$ as $r(\Phi) := \sup\{|\lambda| : \lambda \in \operatorname{Sp}(\Phi)\}$, where $\operatorname{Sp}(\Phi)$ is the spectrum of $\Phi$.

**Theorem 1.2.10** ( [69, Theorems 6.4 and 6.5])**.** *Let* $\dim(\mathfrak{h}) < +\infty$ *and let* $\Phi$ *be a positive map acting on* $B(\mathfrak{h})$*; then* $r(\Phi)$ *is an eigenvalue and the corresponding eigenvector, called Perron-Frobenius vector, is positive. If in addition* $\Phi$ *is irreducible, then* $r(\Phi)$ *is geometrically simple and the corresponding eigenvector is strictly positive.*

When $\Phi$ is a quantum Markov map, one has $\|\Phi\| = r(\Phi) = 1$.

### 1.2.2 Period

Another relevant notion that draws from the classical theory of Markov processes and which is related to the peripheral point spectrum of irreducible quantum Markov maps is that of period, which was introduced already in [26] in the study of the peripheral spectrum of completely positive maps on finite dimensional $C^*$-algebras.

**Definition 1.2.11.** *A resolution of the identity* $(q_j)_{j=0}^{d-1}$ *for some* $d \geq 1$ *is said to be* $\Phi$*-cyclic if*

$$\Phi(q_j) = q_{j-1} \quad j = 0, \ldots, d-1 \tag{1.4}$$

*where* $j - 1$ *has to be intended modulo* $d$*. The period of an irreducible positive unital map* $\Phi$ *is defined as*

$$d := \sup\{n \in \mathbb{N} : \exists \text{ a } \Phi\text{-cyclic resolution with cardinality } n\}.$$

*If* $d = 1$*,* $\Phi$ *is said to be aperiodic.*

If $\Phi$ is completely positive and $(V_i)_{i \in I}$ is a choice of Kraus operators, it is not difficult to see that equation (1.4) is equivalent to

$$V_i q_j = q_{j+1} V_i \text{ for every } i \in I \text{ and } j = 0, \ldots, d-1. \tag{5'}$$

Moreover notice that if $\Phi$ acts on the bounded linear operators on a finite dimensional Hilbert space, its period must be finite.

**Proposition 1.2.12.** *[Proposition 3.10, [14] and Theorem 5.4, [28]] Let* $\Phi$ *a positive irreducible unital map with period d, let* $(q_j)_{j=0}^{d-1}$ *be a* $\Phi$*-cyclic resolution of the identity and let us denote by* $\mathcal{Q}_j = \mathrm{supp}(q_j)$*. Then*

1. $(\mathcal{Q}_j)_{j=0}^{d-1}$ *are orthogonal enclousures for* $\Phi^d$*;*

2. *the restriction* ${\Phi^d}^{\mathcal{Q}_j}$ *is irreducible and aperiodic.*

There is an equivalent characterization of aperiodic maps in terms of their spectrum.

**Proposition 1.2.13** (Proposition 3.12, [14])**.** *Let* $\Phi$ *a positive irreducible unital map acting on a finite dimensional Hilbert space* $\mathfrak{h}$ *and let* $S^1 \subset \mathbb{C}$ *be the unit circle; the following statements are equivalent:*

1. $\Phi$ *is aperiodic;*

2. $\mathrm{Sp}(\Phi) \cap S^1 = \{1\}$.

### 1.2.3 Recurrence and transience

A fundamental notion for the understanding of long time behaviour and other properties of interest of semigroups of quantum Markov maps is that of invariant normal state.

**Definition 1.2.14.** *A normal state* $\rho$ *is said to be invariant for the semigroup* $\mathbf{\Phi}$ *if* $\Phi_{t*}(\rho) = \rho$ *for every* $t \in \mathfrak{T}$*.*

In analogy with the theory of classical Markov semigroups the following space is called the positive recurrent space:

$$\mathcal{R}_+ := \sup\{\mathrm{supp}(\rho) : \rho \text{ is a normal invariant state}\}. \tag{1.5}$$

It is well known that $\mathcal{R}_+$ is an enclosure ( [15, Proposition 5.1] or [67]); a semigroup $\mathbf{\Phi}$ is said to be positive recurent if $\mathcal{R}_+ = \mathfrak{h}$; in general case, it is not difficult to see that $\mathbf{\Phi}^{\mathcal{R}_+}$ is positive recurrent (if $\mathcal{R}_+ \neq \{0\}$). A semigroup is positive recurrent if and only if it admits a faithful normal invariant state: the separability of $\mathfrak{h}$ always allows to construct an invariant state $\rho$ such that $s(\rho) = p_{\mathcal{R}_+}$ ( [67, Theorem 4]). While the definition of the positive recurrent space for quantum Markov maps showed up already in the 70s ( [26]), the notions of null recurrence and transience (also for infinite dimensional quantum systems) where introduced only in [31,67] (to which we refer for all the details).

For every positive $x \in B(\mathfrak{h})$, we call *form-potential* of $x$ the quadratic form $\mathfrak{U}(x)$

$$\mathfrak{U}(x)[v] = \int_0^\infty \langle v, \Phi_t(x)v\rangle dm(t) = \int_0^\infty \operatorname{tr}\left(\Phi_{*t}(|v\rangle\langle v|)x\right) dm(t), \qquad \forall\, v \in D(\mathfrak{U}(x)),$$

$$\text{where} \qquad D(\mathfrak{U}(x)) = \left\{ v \in \mathcal{H} \,:\, \int_0^\infty \langle v, \Phi_t(x)v\rangle dm(t) < +\infty \right\}.$$

$m$ stays for the counting measure on $\mathbb{N}$ if the semigroup is in discrete time, otherwise is the Lebesgue measure. Notice that $\mathfrak{U}(x)$ is positive, symmetric and closed.

**Definition 1.2.15.** *The set of integrable operators with respect to the semigroup* $\boldsymbol{\Phi}$ *is defined as*

$$B(\mathfrak{h})_{int} := \{x \in B(\mathfrak{h}) : \mathfrak{U}(x) \text{ is bounded}\}.$$

*For* $x \in B(\mathfrak{h})_{\text{int}}$*, one can consider the corresponding selfadjoint bounded linear operator* $\mathcal{U}(x)$ *(the potential of* $x$*) representing the bounded form* $\mathfrak{U}(x)$*.*

The transient subspace $\mathcal{T}$ is defined as

$$\mathcal{T} = \sup\{\operatorname{supp}(\mathcal{U}(x)),\ x \in B(\mathfrak{h})_{\text{int}}\}. \tag{1.6}$$

The recurrent space $\mathcal{R} := \mathcal{T}^\perp$ is given by the orthogonal complement of the transient subspace. It can be proved that $\mathcal{R}$ is an enclosure and that, furthermore, $\mathcal{R}_+ \subset \mathcal{R}$, hence $\mathcal{R}_0 = \mathcal{R}_+^\perp \cap \mathcal{R}$ is a natural definition for the null recurrent space.

*Remark* 1.2.16. Every normal state $\rho$ is a convex combination of pure states $\rho = \sum_i \lambda_i |u_i\rangle\langle u_i|$ for some orthonormal basis $(u_i)$ and a sequence of positive numbers $(\lambda_i)$, $\sum_i \lambda_i = 1$ (see subsection 1.1.2). Hence, for every positive $x \in B(\mathfrak{h})$, $\mathfrak{U}(x)$ induces a positive (possibly unbounded) affine functional $\hat{\mathfrak{U}}(x)$ on the set of normal states:

$$\hat{\mathfrak{U}}(x)[\rho] = \sum_i \lambda_i \mathfrak{U}(x)[u_i].$$

$\mathfrak{U}(x)$ is bounded if and only if $\hat{\mathfrak{U}}(x)$ is bounded (and it extends to the continuous linear functional on $L^1(\mathfrak{h})$ represented by $\mathcal{U}(x)$).

In the case of a classical Markov semigroup, the form potential is a very intuitive object: for any $f \in L^\infty(\mu)$, $g \in L^2(\mu)$ such that $\|g\|_{L^2(\mu)} = 1$ (which means that $|g|^2$ is a probability density), one has

$$\mathfrak{U}(f)[g] = \int_0^{+\infty} \int_E \int_E f(y)p(t,x,dy)|g(x)|^2 d\mu(x)dm(t)$$

and, if we denote by $X_t$ the Markov process with transition probabilities given by $p(t,x,\cdot)$, we see that

$$\mathfrak{U}(f)[g] = \int_0^{+\infty} \mathbb{E}[f(X_t)|X_0 \sim |g|^2]dm(t) = \mathbb{E}\left[ \int_0^{+\infty} f(X_t)dm(t) \,\middle|\, X_0 \sim |g|^2 \right].$$

If $f$ is the characteristic function of any measurable set $A \in \mathscr{E}$, then

$$\mathfrak{U}(1_A)[g] = \mathbb{E}\left[\int_0^{+\infty} 1_A(X_t)dm(t)\middle| X_0 \sim |g|^2\right]$$

is the expected time spent in $A$ by $X_t$ if $X_0$ is distributed according to $|g|^2$ and $\mathfrak{U}(1_A)$ is bounded if the expected time spend in $A$ by $X_t$ is finite for every initial distribution of the process. We remark that, as in the classical case the expected time spent in the set of transient states does not need to be finite, also in the general case $\mathfrak{U}(p_{\mathcal{T}})$ needs not to be bounded.

We say that the semigroup $\mathbf{\Phi}$ is null recurrent (resp. recurrent/transient) if $\mathfrak{h} = \mathcal{R}_0$ (resp. $\mathcal{R}$, $\mathcal{T}$); in general, a semigroup $\mathbf{\Phi}$ does not need to be of one of the above types, but it can always be decomposed in its transient, positive and null recurrent restrictions (see Theorem 2.2.3).

### 1.2.4 Multiplicativity properties and decoherence-free subalgebra

Point 1. of Proposition 1.2.3 tells us that for every quantum Markov map $\Phi$ and operator $x \in B(\mathfrak{h})$, the inequalities $\Phi(x^*)\Phi(x) \leq \Phi(x^*x)$ and $\Phi(x)\Phi(x^*) \leq \Phi(xx^*)$ hold true. It turns out that the set of operators that attains equalities has some remarkable properties: let us define the multiplicative domain of $\Phi$ as

$$\mathcal{M}(\Phi) := \{x \in B(\mathfrak{h}) : \Phi(x^*)\Phi(x) = \Phi(x^*x),\ \Phi(x)\Phi(x^*) = \Phi(xx^*)\}. \tag{1.7}$$

We have the following result (for the proof see [18, Theorem 3.1] and [13, Proposition 2]).

**Theorem 1.2.17.** $\mathcal{M}(\Phi)$ *is a $W^*$-algebra and admits the following alternative characterizations:*

$$\mathcal{M}(\Phi) = \{x \in B(\mathfrak{h}) : \forall y \in B(\mathfrak{h}), \Phi(xy) = \Phi(x)\Phi(y),\ \Phi(yx) = \Phi(y)\Phi(x)\}. \tag{1.8}$$

$\mathcal{M}(\Phi)$ *is the maximal $W^*$-algebra on which $\Phi$ acts as a $^*$-morphism. A projection $p$ belongs to $\mathcal{M}(\Phi)$ if and only if $\Phi(p)$ is a projection.*

We remark that [18, Theorem 3.1] is proved in the context of $C^*$-algebras: if $\Phi$ is not $w^*$-continuous, we still get that $\mathcal{M}(\Phi)$ is the maximal $C^*$-algebra on which $\Phi$ acts as a $^*$-morphism and the characterization in equation (1.8).
In case we are considering a semigroup of quantum Markov maps $\mathbf{\Phi}$ a natural object is the following set, which is known as decoherence-free subalgebra:

$$\mathcal{N}(\mathbf{\Phi}) := \bigcap_{t \in \mathfrak{T}} \mathcal{M}(\Phi_t).$$

Theorem 1.2.17 easily implies the following.

**Theorem 1.2.18.** $\mathcal{N}(\mathbf{\Phi})$ *is a $W^*$-algebra and we have the following alternative characterizations:*

1. $\mathcal{N}(\mathbf{\Phi}) = \{x \in B(\mathfrak{h}) : \forall y \in B(\mathfrak{h}), \forall t \in \mathfrak{T}, \Phi_t(xy) = \Phi_t(x)\Phi_t(y),\ \Phi_t(yx) = \Phi_t(y)\Phi_t(x)\}$.
2. $\mathcal{N}(\mathbf{\Phi})$ *is the maximal $W^*$-algebra on which $\mathbf{\Phi}$ acts as a semigroup of $^*$-morphisms. A projection $p$ belongs to $\mathcal{N}(\mathbf{\Phi})$ if and only if $\Phi_t(p)$ is a projection for every $t \in \mathfrak{T}$.*

# CHAPTER 2

# Absorption operators

When we consider an enclosure (invariant domain) for a semigroup of quantum Markov maps, i.e. a closed subspace which is invariant under the action of the evolution, it is natural to wonder about the probability of being captured/absorbed in this domain. Absorption operators help us to approach the problem in the noncommutative setting.

**Definition 2.0.1.** *Let $\mathcal{V}$ be an enclosure for the semigroup of quantum Markov maps $\mathbf{\Phi}$. We can define the absorption operator associated to $\mathcal{V}$ as*

$$A(\mathcal{V}) := w^* - \lim_{t \to \infty} \Phi_t(p_{\mathcal{V}}). \tag{2.1}$$

While the limit in equation (2.1) is well defined for projections corresponding to enclosures (see point 1. of Proposition 2.1.1), there is no need for it to exist if we replace $p_{\mathcal{V}}$ with an arbitrary bounded operator; nevertheless, we will show that this feature is shared by a wider class of operators (see Theorem 2.3.16).

The notion of absorption operator is inspired by and generalizes the one of absorption probabilty. Let us consider $(X_n)_{n\geq 0}$ a discrete time homogeneous Markov chain on a countable state space $E$ with transition matrix $P = (p_{xy})_{x,y\in E}$. We recall that the transition matrix acts as a bounded linear operator on the Banach space of finite measures on $E$, that is $\ell^1(E)$ (every finite measure on a discrete space is univoquely represented by its density with respect to the counting measure), and on its dual, which is isomorphic to $\ell^\infty(E)$, in the following way:

$$\nu \cdot P(x) = \sum_{y\in E} \nu(y) p_{yx} = \mathbb{P}(X_1 = x | X_0 \sim \nu), \quad \nu \in \ell^1(E),$$
$$P \cdot f(x) = \sum_{y\in E} p_{xy} f(y) = \mathbb{E}[f(X_1)|X_0 = x], \quad f \in \ell^\infty(E).$$

As usual, for every $n \in \mathbb{N}$ we denote by $P^n$ the action of $P$ applied $n$-times ($P^0 = \mathrm{Id}$). The Markov property implies that

$$\nu \cdot P^n(x) = \mathbb{P}(X_n = x | X_0 \sim \nu), \quad \nu \in \ell^1(E),$$
$$P^n \cdot f(x) = \mathbb{E}[f(X_n)|X_0 = x], \quad f \in \ell^\infty(E).$$

As we already mentioned in the introduction, for every closed set $C$, the corresponding absorption probability is the function defined as

$$A(C)(x) := \mathbb{P}\left(\bigcup_{n\in\mathbb{N}} \{X_n \in C\} | X_0 = x\right), \quad x \in E.$$

The notion of closed set implies that $\{X_m \in C\} \subseteq \{X_n \in C\}$, hence

$$\mathbb{P}(\{X_n \in C\}|X_0 = x) = P^n \cdot 1_C(x) \nearrow A(C)(x), \quad x \in E. \tag{2.2}$$

The semigroup corresponding to the Markov chain $(X_n)_{n\geq 0}$ can be thought of as the diagonal restriction of a dilated quantum channel acting on $B(\ell^2(E))$ and the closed linear space of the functions in $\ell^2(E)$ supported on a closed set $C \subseteq E$ turns out to be an enclosure; hence, by equation (2.2), we see that the multiplication operator corresponding to $A(C)$ is the absorption operator corresponding to the enclosure associated to $C$.

Absorption probabilities have important connections with communication structure, recurrence, harmonic functions and ergodic theory; it turns out that absorption operators share some remarkable features with their commutative counterpart. In Section 2.1 we show some first properties of absorption operators: among the others, the relationship between the spectral resolution of absorption operators and accessibility properties of the semigroup and the characterization of absorption operators as solutions of linear systems. We conclude the section with an example where we can compute explicitly absorption operators for a particular semigroup. The main result of Section 2.2 is that the orthogonal complement of any recurrent enclosure inside the recurrent space is again an enclosure; this reflects into the structure of absorption operators and has many important consequences, the most relevant of which is that the null recurrent space is an enclosure. The first motivation for us to introduce absorption operators was the description of fixed points of semigroups of quantum Markov maps and this is treated in detail in Section 2.3: thanks to ergodic theory we are able to show that under the hypothesis that the recurrent space is absorbing, absorption operators completely characterize the set of the fixed points of the semigroup of quantum Markov maps. In the same section we investigate the role played by absorption operators in ergodic theory proving a quantum version of the mean ergodic theorem for classical Markov chains (Theorem 2.3.23) and pointing out their connection to quantum trajectories. Moreover we are able to exhibit an example showing that the quantum analogue of the existence of a decomposition of recurrent states for Markov chains into communication classes, which was proved to hold in case there exists a faithful invariant state, is no longer true for generic semigroups. Finally, Section 2.4 contains some physical examples and applications of the theory presented in this chapter; the examples are both in discrete and continuous time, finite and infinite dimension.

## 2.1 First properties

The parallel with classical Markov chains can suggest some expected properties of absorption operators, provided one replaces bounded functions with bounded operators, complementary sets with orthogonal complement and so on; the first result that we show is a connection between the spectral resolution of absorption operators and accessibility properties of the semigroup of quantum Markov maps.

**Proposition 2.1.1.** *Let $\mathcal{V}$ be an enclosure.*

1. *$A(\mathcal{V})$ is a well defined bounded operator, it is a fixed point for the semigroup and $0 \leq A(\mathcal{V}) \leq \mathbf{1}$.*
2. *If $\lambda$ is an eigenvalue of $A(\mathcal{V})$ with norm one eigenvector $v$, then*

$$\lambda = \lim_{t \to +\infty} \operatorname{tr}\left(p_{\mathcal{V}} \Phi_{*t}(|v\rangle\langle v|)\right).$$

*In particular, the kernel of $A(\mathcal{V})$ consists of all vectors $v$ such that*

$$\operatorname{Enc}(v) := \sup_{t \in \mathfrak{T}} \{\operatorname{supp}(\Phi_{*t}(|v\rangle \langle v|))\} \subseteq \mathcal{V}^{\perp}.$$

3. *The eigenspaces of $A(\mathcal{V})$ corresponding to the eigenvalues 0 and 1 are enclosures.*

We briefly comment on the content of the previous proposition in order to clarify its physical meaning; we can characterize the kernel of $A(\mathcal{V})$ as the space generated by those vectors $v$ such that, if the system starts in the pure state $|v\rangle \langle v|$, zero probability is given to the enclosure $\mathcal{V}$ along the whole evolution. The space $\operatorname{Enc}(v)$ introduced above can be proved to be the smallest enclosure containing $v$ (see [15]). Notice that, when working in a non commutative context, some classical rules are missing, for instance $\operatorname{Enc}(v) \not\perp \mathcal{V}$ is not a sufficiently strong condition to ensure that $v$ is in the support of $A(\mathcal{V})$ and we cannot hope that $\mathcal{V} \subseteq \operatorname{Enc}(v)$ holds for all $v$ in the support of $A(\mathcal{V})$. For a counterexample to the latter fact, we can consider models for which there exist enclosures which do not contain $\mathcal{V}$ and that are not orthogonal to it (see Examples 2.4.1 or 2.4.3).

*Proof.* 1. Notice that, since $p_{\mathcal{V}}$ is subharmonic, $(\Phi_t(p_{\mathcal{V}}))_{t \in \mathfrak{T}}$ is a bounded monotone increasing net of positive operators, hence it admits a limit in the $w^*$-topology, which is also its least upper bound ( [8, Lemma 2.4.19]). Moreover, since $p_{\mathcal{V}}$ is a projection and $\mathbf{\Phi}$ is positivity and identity preserving,

$$0 \leq p_{\mathcal{V}} \leq \mathbf{1} \qquad \Rightarrow \qquad 0 \leq \Phi_t(p_{\mathcal{V}}) \leq \mathbf{1} \quad \forall t \in \mathfrak{T}$$

and passing to the $w^*$-limit, inequalities are preserved. Moreover $A(\mathcal{V})$ is a fixed point due to the $w^*$-continuity of the channels $\Phi_t$:

$$\Phi_t(A(\mathcal{V})) = \Phi_t\left(w^* - \lim_{s \to +\infty} \Phi_s(p_{\mathcal{V}})\right) = w^* - \lim_{s \to +\infty} \Phi_{t+s}(p_{\mathcal{V}}) = A(\mathcal{V}).$$

2. When $v$ is a norm one eigenvector pertaining $\lambda$, then we immediately deduce

$$\lambda = \langle v, A(\mathcal{V})v\rangle = \lim_{t \to +\infty} \operatorname{tr}\left(p_{\mathcal{V}} \Phi_{*t}(|v\rangle\langle v|\right).$$

In particular, when $\lambda = 0$, this implies that, for $v$ in the kernel of $A(\mathcal{V})$,

$$\lim_{t \to +\infty} \mathrm{tr}\,(p_{\mathcal{V}} \Phi_{*t}(|v\rangle\langle v|)) = 0;$$

but, since $\mathcal{V}$ is an enclosure, $\mathrm{tr}\,(p_{\mathcal{V}} \Phi_{*t}(|v\rangle\langle v|))$ is a positive non-decreasing function of $t$, so that the previous gives

$$\mathrm{tr}\,(p_{\mathcal{V}} \Phi_{*t}(|v\rangle\langle v|)) = 0 \qquad \text{for all } t \in \mathfrak{T} \tag{2.3}$$

and the conclusion follows.

3. We first show that the kernel of $A(\mathcal{V})$ is an enclosure. $A(\mathcal{V})$ is positive and harmonic, hence, for an arbitrary normal state $\rho$ such that $\mathrm{supp}(\rho) \subseteq \ker(A(\mathcal{V}))$, we have that for all $t \in \mathfrak{T}$

$$\mathrm{tr}\,(A(\mathcal{V})\Phi_{*t}(\rho)) = \mathrm{tr}\,(\Phi_t(A(\mathcal{V}))\rho) = \mathrm{tr}\,(A(\mathcal{V})\rho) = 0,$$

hence $\mathrm{supp}(\Phi_{*t}(\rho)) \subseteq \ker(A(\mathcal{V}))$ and the kernel of $A(\mathcal{V})$ is an enclosure.

We can repeat similar computations replacing $A(\mathcal{V})$ with the operator $x := \mathbf{1} - A(\mathcal{V})$, and prove that its kernel, which is exactly the eigenspace of $A(\mathcal{V})$ corresponding to the eigenvalue 1, is an enclosure. □

*Remark* 2.1.2. The following simple example shows that there is no reason for the spectrum of $A(\mathcal{V})$ to be discrete; however we remark that point 2. can be easily formulated in a more general way through spectral projections.

Let $\mathfrak{h} = L^2([0,1]) \oplus \mathbb{C}^2$, let $\{e_0, e_1\}$ be an orthonormal basis of $\mathbb{C}^2$ and consider the following quantum Markov map: for every $y \in B(\mathfrak{h})$

$$\Phi(y) = \langle e_0|\, y\, |e_0\rangle\, (x + |e_0\rangle\, \langle e_0|) + \langle e_1|\, y\, |e_1\rangle\, (p_{L^2([0,1])} - x + |e_1\rangle\, \langle e_1|)$$

where $p_{L^2([0,1])}$ is the orthogonal projection onto $L^2([0,1])$ and $x$ is the bounded operactor acting as $x(f(t) + \alpha_0 e_0 + \alpha_1 e_1) = tf(t)$ for every $f \in L^2([0,1])$, $\alpha_0, \alpha_1 \in \mathbb{C}$. Notice that $\Phi^n(|e_0\rangle\, \langle e_0|) = x + |e_0\rangle\, \langle e_0|$ for every $n \geq 1$, hence $|e_0\rangle\, \langle e_0|$ is subharmonic and $A(|e_0\rangle\, \langle e_0|) = x + |e_0\rangle\, \langle e_0|$, whose spectrum is the whole interval $[0,1]$.

Proposition 2.1.1 shows that every enclosure $\mathcal{V}$ induces a relevant decomposition of the Hilbert space; let us denote $\mathcal{V}_1$ the eigenspace of $A(\mathcal{V})$ corresponding to the eigenvalue 1 and let us define $\mathcal{W} = \mathcal{V}_1 \cap \mathcal{V}^{\perp}$, then we have

$$\mathfrak{h} = \mathcal{V} \oplus \mathcal{W} \oplus \mathrm{supp}(A(\mathcal{V}) - p_{\mathcal{V}_1}) \oplus \ker(A(\mathcal{V})). \tag{2.4}$$

We say that a state $\rho$ reaches a linear space $\chi$ if there exists a time $t \in \mathfrak{T}$ such that $\mathrm{tr}\,(p_\chi \Phi_{*t}(\rho)) > 0$. Given two linear spaces $\chi_1$ and $\chi_2$ we say that we can reach $\chi_2$ from $\chi_1$ if there exists a state $\rho$ with $\mathrm{supp}(\rho) \subseteq \chi_1$ such that $\rho$ reaches $\chi_2$. Figure 2.1 highlights the reachability relations between the linear spaces in the decomposition in equation (2.4).

In the sequel we shall repeatedly use the following well known property of positive operators:

**Lemma 2.1.3** (Lemma 1, [67]). *Let $y$ be a positive bounded operator and $p$ an orthogonal projection such that $pyp = 0$; then $py(\mathbf{1} - p) = (\mathbf{1} - p)yp = 0$.*

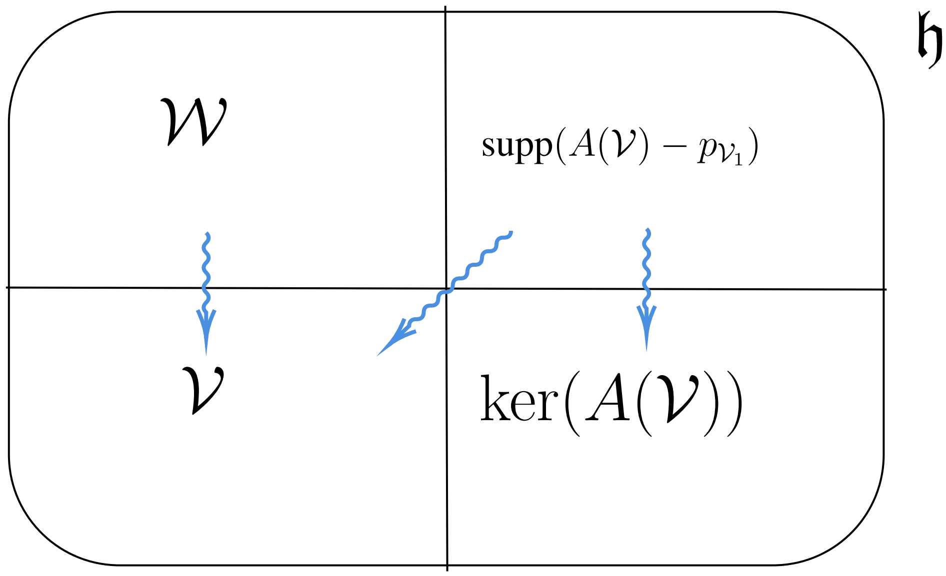



**Figure 2.1:** *Blue arrows represent accessibility relations in the decomposition in equation* (2.4)

**Lemma 2.1.4.** *For any enclosure $\mathcal{V}$ and for any $t \in \mathfrak{T}$,*

$$p_{\mathcal{V}}\Phi_t(p_{\mathcal{V}})p_{\mathcal{V}} = p_{\mathcal{V}} \qquad \textit{and} \qquad p_{\mathcal{V}}\Phi_t(p_{\mathcal{V}})p_{\mathcal{V}^\perp} = p_{\mathcal{V}^\perp}\Phi_t(p_{\mathcal{V}})p_{\mathcal{V}} = 0,$$

*or equivalently*

$$\Phi_t(p_{\mathcal{V}}) = p_{\mathcal{V}} + p_{\mathcal{V}^\perp}\Phi_t(p_{\mathcal{V}})p_{\mathcal{V}^\perp}.$$

*Moreover, for any enclosure $\mathcal{W}$ orthogonal to $\mathcal{V}$,*

$$p_{\mathcal{W}}\Phi_t(p_{\mathcal{V}})p_{\mathcal{W}} = 0.$$

*Proof.* Since $\mathcal{V}$ is an enclosure, $p_{\mathcal{V}} \leq \Phi_t(p_{\mathcal{V}}) \leq \mathbf{1}$, so $p_{\mathcal{V}} = p_{\mathcal{V}}\Phi_t(p_{\mathcal{V}})p_{\mathcal{V}}$. Moreover this implies that, for any $t \in \mathfrak{T}$, $\Phi_t(p_{\mathcal{V}}) - p_{\mathcal{V}}$ is a positive operator such that $p_{\mathcal{V}}(\Phi_t(p_{\mathcal{V}}) - p_{\mathcal{V}})p_{\mathcal{V}} = 0$, and by Lemma 2.1.3

$$p_{\mathcal{V}}(\Phi_t(p_{\mathcal{V}}) - p_{\mathcal{V}})p_{\mathcal{V}^\perp} = p_{\mathcal{V}^\perp}(\Phi_t(p_{\mathcal{V}}) - p_{\mathcal{V}})p_{\mathcal{V}} = 0,$$

which immediately implies $p_{\mathcal{V}}\Phi_t(p_{\mathcal{V}})p_{\mathcal{V}^\perp} = p_{\mathcal{V}^\perp}\Phi_t(p_{\mathcal{V}})p_{\mathcal{V}} = 0$.

For the last assertion, simply notice that $0 \leq p_{\mathcal{V}} \leq \mathbf{1} - p_{\mathcal{W}}$, so the positivity and unitality of $\boldsymbol{\Phi}$ give

$$0 \leq p_{\mathcal{W}}\Phi_t(p_{\mathcal{V}})p_{\mathcal{W}} \leq p_{\mathcal{W}}\Phi_t(\mathbf{1} - p_{\mathcal{W}})p_{\mathcal{W}} = 0,$$

where the last equality follows from the application of the first part of the statement to the enclosure $\mathcal{W}$. □

**Proposition 2.1.5.** *1. For any enclosure $\mathcal{V}$*

$$A(\mathcal{V}) = p_{\mathcal{V}} + p_{\mathcal{V}^\perp}A(\mathcal{V})p_{\mathcal{V}^\perp}.$$

2. *$p_{\mathcal{W}}A(\mathcal{V})p_{\mathcal{W}} = 0$ for any enclosure $\mathcal{W}$ orthogonal to $\mathcal{V}$. In fact the kernel of $A(\mathcal{V})$ is the maximal enclosure orthogonal to $\mathcal{V}$.*

*3. $A(\mathcal{V})$ is the minimal fixed point $x$ of the semigroup $\Phi$ such that*

$$0 \leq x \leq \mathbf{1} \qquad \text{and} \qquad p_{\mathcal{V}} x p_{\mathcal{V}} = p_{\mathcal{V}}. \tag{2.5}$$

*Proof.* The first and second points are evident using the previous lemma and simply passing to the limit. For the last assertion of point 2, also remember that we already know that $\ker A(\mathcal{V})$ is an enclosure by Proposition 2.1.1.

Now consider a fixed point satisfying relations (2.5). Then, since $\mathbf{1} - x$ is a positive operator such that $p_{\mathcal{V}}(\mathbf{1} - x)p_{\mathcal{V}} = 0$, by positivity,

$$p_{\mathcal{V}}(\mathbf{1} - x)p_{\mathcal{V}^\perp} = p_{\mathcal{V}^\perp}(\mathbf{1} - x)p_{\mathcal{V}} = 0$$

so that

$$p_{\mathcal{V}} x p_{\mathcal{V}^\perp} = p_{\mathcal{V}^\perp} x p_{\mathcal{V}} = 0 \quad \text{and} \quad x = p_{\mathcal{V}} + p_{\mathcal{V}^\perp} x p_{\mathcal{V}^\perp} \geq p_{\mathcal{V}}.$$

Applying the semigroup, we obtain for any $t$

$$0 \leq \Phi_t(x - p_{\mathcal{V}}) = x - \Phi_t(p_{\mathcal{V}}) \to x - A(\mathcal{V}).$$

This concludes the proof. □

**Example 2.1.6.** As a first example, we consider a model simple enough to allow an easy intuition and explicit calculations. It displays anyway interesting characteristics of infinite dimensional systems with respect to absorption and fixed points and, even if it seems to behave pretty much like in the classical case, with little variations, it can present features which arise only in the non commutative setting, but we shall come back to this later, in Section 2.4. The model we introduce belongs to the family of homogeneous open quantum random walks (HOQRWs); we refer to Section 3.1 for the definition. Let us consider the Hilbert space $\mathcal{H} := \mathbb{C}^3 \otimes \ell^2(\mathbb{Z})$; the quantum Markov map we want to study is the following:

$$\begin{aligned} \Phi : B(\mathcal{H}) &\to B(\mathcal{H}) \\ x &\mapsto \sum_{k \in \mathbb{Z}} \sum_{\epsilon \in \{+,-\}} L_\epsilon^* \otimes |k\rangle \langle k + \epsilon 1| \, x L_\epsilon \otimes |k + \epsilon 1\rangle \langle k| \end{aligned}$$

and the local transition operators are given by

$$L_\varepsilon = \begin{pmatrix} a_\varepsilon & b_\varepsilon & 0 \\ 0 & c_\varepsilon & 0 \\ 0 & d_\varepsilon & 1/\sqrt{2} \end{pmatrix}, \qquad \varepsilon = -, +.$$

In order for the map to be Markov, we have the following constraints on the coefficients:

$$|a_+|^2 + |a_-|^2 = 1; \quad \sum_{\varepsilon = -,+} |b_\varepsilon|^2 + |c_\varepsilon|^2 + |d_\varepsilon|^2 = 1; \quad \bar{a}_+ b_+ + \bar{a}_- b_- = 0; \quad d_+ = -d_-. \tag{2.6}$$

Let $\{e_i\}_{i=0}^2$ be the canonical basis for $\mathbb{C}^3$ and let $E_i$ be the closed subspace generated by $\{e_i \otimes |k\rangle\}_{k \in \mathbb{Z}}$ for $i = 0, 1, 2$. $E_2$ is a minimal enclosure and the reduced process is essentially a symmetric random walk on $\mathbb{Z}$. $E_0$ is an enclosure too and the corresponding reduced process is again a random walk. So it will be significant to compute the absorption operators $A(E_0)$ and $A(E_2)$. Since we are interested in a model showing non trivial absorption dynamics, we assume that

$$b_+ d_+ b_- d_- \neq 0, \tag{2.7}$$

which means that any state supported in $E_1$ will have the possibility to flow both to $E_0$ and $E_2$. Due to Proposition 2.1.5, we know that the absorption operators are fixed points and have a block diagonal structure, $A(E_k) = p_{E_k} + p_{E_1} A(E_k) p_{E_1}$, $k = 0, 2$. It is immediate to compute explicitly

$$A(E_0) = p_{E_0} + q p_{E_1}, \quad A(E_2) = p_{E_2} + (1-q) p_{E_1},$$

where $q = \frac{\sum_{\varepsilon=-,+} |b_\varepsilon|^2}{\sum_{\varepsilon=-,+} |b_\varepsilon|^2 + |d_\varepsilon|^2} \in (0,1)$: for every $n \geq 1$ by direct computation and induction one obtains that

$$\Phi^n(p_{E_0}) = p_{E_0} + \left( \sum_{\varepsilon=-,+} |b_\varepsilon|^2 \right) \cdot \left( \sum_{k=0}^{n-1} \left( \sum_{\varepsilon=-,+} |c_\varepsilon|^2 \right)^k \right) p_{E_1}$$

and, using the definition of absorption operators, one gets

$$A(E_0) = w^* - \lim_{n \to +\infty} \Phi^n(p_{E_0}) = p_{E_0} + \frac{\sum_{\varepsilon=-,+} |b_\varepsilon|^2}{1 - \sum_{\varepsilon=-,+} |c_\varepsilon|^2} p_{E_1} = p_{E_0} + q p_{E_1}$$

where in the last equality we made use of equation (2.6). Computations for finding $A(E_2)$ are very similar and we do not report them here.

We shall further consider this model at the end of the chapter.

*Remark* 2.1.7. All the properties in this section rely on the (simple) positivity of the semigroup $\mathbf{\Phi}$. This can be significant to remark, because of discussions emerging in the physicists' community, even if the assumption of complete positivity remains surely the most popular. However, if we use complete positivity, we can study how absorption influences the structure of the Kraus operators of the Markov maps. Consider the case $\mathbf{\Phi} = (\Phi^n)_{n \in \mathbb{N}}$ for some quantum Markov map $\Phi$ with Kraus operators $(V_i)_{i \in I}$. We can reformulate the accessibility relations implied by Proposition 2.1.1 in terms of the $V_i$'s. Consider an enclosure $\mathcal{V}$ and let us define $p_\mathcal{K}$ the projection onto $\ker(A(\mathcal{V}))$ and $q$ the range projection of $A(\mathcal{V}) - p_\mathcal{V}$. Indeed we have that

- $p_\mathcal{V} V_i p_\mathcal{K} = p_\mathcal{K} V_i p_\mathcal{V} = 0$ for every $i \in I$;
- $p_\mathcal{V} V_i q \neq 0$ for some $i \in I$ if $q \neq 0$.

In the continuous time setting, when we are treating a quantum dynamical semigroup with a GKLS generator, similar relations can be obtained for the operators appearing in the generator and this has essentially already been observed in [30].

## 2.2 Absorption operators and recurrence

In this section we study the relations of absorption operators with transience and recurrence. We shall see how the classical relations can be generalized in a someway expected manner, reflecting the classical situations in the case when the channel has only orthogonal minimal enclosures, and with appropriate variations when the channel displays the typically quantum feature of non orthogonal minimal enclosures. Absorption operators $A(\mathcal{V})$ turn out to have additional interesting properties in the case when $\mathcal{V}$ is a minimal recurrent enclosure (i.e. the noncommutiative object corresponding to a recurrent class, for a Markov chain). We study this in Theorem 2.2.7.

As a byproduct, absorption can enrich our knowledge about enclosures and transience and recurrence. In particular, we can prove that the orthogonal complement of an enclosure in the recurrent space is again an enclosure (Theorem 2.2.1): this was previously known only in the positive recurrent case and, apart from its intrinsic value, it is usually considered a significant starting step in reduction problems ( [7, 15] for the finite and infinite dimensional cases respectively); however, as Example 2.3.7 shows, the situation is more complicated without assuming the existence of a faithful normal invariant state. In addition, a paramount consequence of Theorem 2.2.1 is that the null recurrent subspace is an enclosure. The fact that positive recurrent configurations are not accessible from null recurrent ones is indeed a well-known fact for classical Markov chains, but still not proved for quantum chains.

**Theorem 2.2.1.** *If $\mathcal{V}, \mathcal{Z}$ are increasing enclosures included in $\mathcal{R}$, i.e. such that $\mathcal{V} \subseteq \mathcal{Z} \subseteq \mathcal{R}$, then $\mathcal{Z} \cap \mathcal{V}^{\perp}$ is an enclosure.*
*In particular $\mathcal{R} \cap \mathcal{V}^{\perp}$ is an enclosure and the null recurrent space $\mathcal{R}_0$ is an enclosure.*

We divide the proof of the theorem in two steps; the first and central point, Lemma 2.2.2, proves that, if a pure state can take mass to an orthogonal enclosure, then the corresponding vector is transient.

**Lemma 2.2.2.** *Let $\mathcal{V}$ be an enclosure, $w$ a norm one vector in $\mathcal{V}^{\perp}$. If there exist $\bar{t} \in \mathfrak{T}$ and $\varepsilon > 0$ such that $\Phi_{\bar{t}}(p_{\mathcal{V}}) \geq \varepsilon |w\rangle\langle w|$, then $w \in \mathcal{T}$.*

*Proof.* Let us call

$$\lambda_t(v) := \mathrm{tr}\,(\Phi_t(|w\rangle\langle w|)|v\rangle\langle v|) = \langle v, \Phi_t(|w\rangle\langle w|)v\rangle, \qquad t \in \mathfrak{T}, v \in \mathfrak{h}, \|v\| = 1,$$

$$\text{so that} \qquad \mathfrak{U}(|w\rangle\langle w|)[v] = \int_0^\infty \lambda_t(v) dm(t).$$

By hypothesis we know that $\Phi_{\bar{t}}(p_{\mathcal{V}}) \geq \epsilon |w\rangle\langle w|$ and, since $A(\mathcal{V}) \geq \Phi_{\bar{t}}(p_{\mathcal{V}})$ by construction, $w$ belongs to $\ker(A(\mathcal{V}))^{\perp}$ and therefore to the space $\mathcal{W} := \mathcal{V}^{\perp} \cap (\ker(A(\mathcal{V})))^{\perp}$ ($w \in \mathcal{V}^{\perp}$ by assumption). Recalling that $\mathcal{V}$ is contained in the range of $A(\mathcal{V})$, $\mathcal{W}^{\perp} := \ker(A(\mathcal{V})) \oplus \mathcal{V}$ is an enclosure as sum of two orthogonal enclosures. Consequently the projection on $\mathcal{W}$, always denoted with the same symbol $\mathcal{W}$ below, is superharmonic and we have, for $t \geq \bar{t}$,

$$\begin{aligned} a_t(v) := \mathrm{tr}\,(\Phi_{*t}(|v\rangle\langle v|)p_{\mathcal{W}}) &= \mathrm{tr}\,(\Phi_{*t-\bar{t}}(|v\rangle\langle v|)\Phi_{\bar{t}}(p_{\mathcal{W}})) \\ (\mathcal{W} \text{ is superharmonic}) &= \mathrm{tr}\,(p_{\mathcal{W}}\Phi_{*t-\bar{t}}(|v\rangle\langle v|)p_{\mathcal{W}}\Phi_{\bar{t}}(p_{\mathcal{W}})) \\ &\leq \mathrm{tr}\,(\Phi_{*t-\bar{t}}(|v\rangle\langle v|)(p_{\mathcal{W}} - p_{\mathcal{W}}\Phi_{\bar{t}}(p_{\mathcal{V}})p_{\mathcal{W}})) \\ &\leq a_{t-\bar{t}}(v) - \mathrm{tr}\,(\Phi_{*t-\bar{t}}(|v\rangle\langle v|)\varepsilon|w\rangle\langle w|) \leq a_{t-\bar{t}}(v) - \varepsilon\lambda_{t-\bar{t}}(v) \end{aligned}$$

Thus, for $s \geq \bar{t}$, we get

$$\int_0^s \lambda_t(v)\, dm(t) \leq \varepsilon^{-1} \int_0^s (a_t(v) - a_{t+\bar{t}}(v))\, dm(t) \leq \varepsilon^{-1} \int_0^{\bar{t}} a_t(v)\, dm(t) \leq \varepsilon^{-1}\bar{t}.$$

This guarantees that $\mathfrak{U}(|w\rangle\langle w|)$ is bounded and its support is contained in the transient subspace, defined in (1.6). In order to conclude, we just have to remark that the support of $\mathfrak{U}(|w\rangle\langle w|)$ contains $w$. Indeed,

$$\begin{aligned} u \in \ker(\mathfrak{U}(|w\rangle\langle w|)) &\Leftrightarrow \int_0^{+\infty} \langle u, \Phi_t(|w\rangle\langle w|)u\rangle dm(t) = 0 \\ &\Leftrightarrow \langle u, \Phi_t(|w\rangle\langle w|)u\rangle = 0 \text{ for } m\text{-a.e. t.} \end{aligned} \tag{2.8}$$

Hence, for any $u \in \ker(\mathfrak{U}(|w\rangle\langle w|))$, we get that $\langle u, w\rangle = 0$ by taking $t = 0$ in the last expression of equation (2.8) (in continuous time we recall that $t \mapsto \langle u, \Phi_t(|w\rangle\langle w|)u\rangle$ is a continuous function). Therefore we have that $w \in \ker(\mathfrak{U}((|w\rangle\langle w|))^\perp = \operatorname{supp}(\mathfrak{U}(|w\rangle\langle w|))$. □

*Proof of Theorem 2.2.1.* We define the subspace $\mathcal{W}$ with corresponding projection $p_\mathcal{W} = p_\mathcal{Z} - p_\mathcal{V} \le p_\mathcal{R}$. By previous lemma, we have, for any time $t$,

$$p_\mathcal{W}\Phi_t(p_\mathcal{V})p_\mathcal{W} = 0. \tag{2.9}$$

Indeed, by contradiction, let us suppose that this is not true. So there would exist a time $\bar{t}$, a norm 1 vector $w \in \mathcal{V}^\perp \setminus \mathcal{W}^\perp$ and $\varepsilon > 0$ such that $\Phi_{\bar{t}}(p_\mathcal{V}) \ge \varepsilon|w\rangle\langle w|$, by Lemma 2.1.4. Then, by Lemma 2.2.2, $w \in \mathcal{T} \perp \mathcal{R} \supset \mathcal{W}$, which is a contradiction.

Now we know that $\mathcal{V}$ and $\mathcal{Z}$ are enclosures included in $\mathcal{R}$, so, by Lemma 2.1.4, we deduce

$$\begin{aligned} \Phi_t(p_\mathcal{W}) &= \Phi_t(p_\mathcal{Z}) - \Phi_t(p_\mathcal{V}) \\ &= p_\mathcal{Z} + (\mathbf{1} - p_\mathcal{Z})\Phi_t(p_\mathcal{Z})(\mathbf{1} - p_\mathcal{Z}) - p_\mathcal{V} - (\mathbf{1} - p_\mathcal{V})\Phi_t(p_\mathcal{V})(\mathbf{1} - p_\mathcal{V}) \\ &= p_\mathcal{W} + (\mathbf{1} - p_\mathcal{Z})\Phi_t(p_\mathcal{W})(\mathbf{1} - p_\mathcal{Z}) + \\ &\quad \underbrace{-(\mathbf{1} - p_\mathcal{Z})\Phi_t(p_\mathcal{V})p_\mathcal{W} - p_\mathcal{W}\Phi_t(p_\mathcal{V})(\mathbf{1} - p_\mathcal{Z}) - p_\mathcal{W}\Phi_t(p_\mathcal{V})p_\mathcal{W}}_{=0} \ge p_\mathcal{W}, \end{aligned}$$

and this guarantees that $\mathcal{W}$ is an enclosure.

Now that the first statement is proven, the others easily follow using first only $\mathcal{Z} = \mathcal{R}$ and then also $\mathcal{V} = \mathcal{R}_+$. □

We remark that in the particular case when $\mathcal{R} = \mathcal{R}_+$, the previous result reduces to the second part of Proposition 5.2 in [15].

It was already shown that $\mathcal{R}_+$ and $\mathcal{R}$ are enclosures (see for instance [26, Section 3] and [67, Corollary 2]). Showing that $\mathcal{R}_0$ is an enclosure, Theorem 2.2.1 provides the answer to two questions left open in [67]:

1. starting from a state supported in the slow recurrent subspace $\mathcal{R}_0$, the semigroup cannot leave $\mathcal{R}_0$;
2. the reduced semigroup $\mathbf{\Phi}^\mathcal{R}$ can be decomposed into a slow recurrent $\mathbf{\Phi}^{\mathcal{R}_0}$ and a fast recurrent $\mathbf{\Phi}^{\mathcal{R}_+}$ semigroups, completing the decomposition of quantum Markov semigroups given in [67, Theorem 9]:

   **Theorem 2.2.3** (Decomposition of semigroups of quantum Markov maps)**.** *If $\mathbf{\Phi}$ is a semigroup of quantum Markov maps acting on $B(\mathfrak{h})$, then the restriction of $\mathbf{\Phi}$ to $p_\mathcal{T}B(\mathfrak{h})p_\mathcal{T}$ is a transient semigroup of quantum subMarkov maps, while $\mathbf{\Phi}^\mathcal{R}$*

*is a recurrent semigroup of quantum Markov maps on $p_{\mathcal{R}}B(\mathfrak{h})P_{\mathcal{R}}$ which contains the positive and null recurrent "sub"-semigroups $\mathbf{\Phi}^{\mathcal{R}_+}$ and $\mathbf{\Phi}^{\mathcal{R}_0}$ acting on $p_{\mathcal{R}_+}B(\mathfrak{h})p_{\mathcal{R}_+}$ and $p_{\mathcal{R}_0}B(\mathfrak{h})p_{\mathcal{R}_0}$, respectively.*

In order to better understand the splitting of the recurrent restriction of the semigroup into the positive and null restrictions, we recall the following result.

**Lemma 2.2.4** (Point 3, Proposition 5.2, [15])**.** *Let $\mathcal{V}$ and $\mathcal{W}$ two orthogonal enclosures and consider a normal state $\rho$ supported in $\mathcal{V} \oplus \mathcal{W}$, then for $p, q \in \{p_{\mathcal{V}}, p_{\mathcal{W}}\}$*

$$\Phi_{*t}(p\rho q) = p\Phi_{*t}(\rho)q \quad \forall t \in \mathfrak{T}.$$

Hence we have that

**Corollary 2.2.5.** *For every normal state $\rho$ supported in $\mathcal{R}$, for $p, q \in \{p_{\mathcal{R}_+}, p_{\mathcal{R}_0}\}$*

$$\Phi^{\mathcal{R}}_{*t}(p\rho q) = p\Phi^{\mathcal{R}}_{*t}(\rho)q.$$

Notice that in the case $\mathbf{\Phi} = (\Phi^n)_{n\in\mathbb{N}}$ for some quantum Markov map $\Phi$ with Kraus operators $(V_i)_{i\in I}$, then it is easy to see that $(p_{\mathcal{R}}V_ip_{\mathcal{R}})_{i\in I}$ is a choice of Kraus operators for $\Phi^{\mathcal{R}}$ (whose powers generate $\mathbf{\Phi}^{\mathcal{R}}$) and

$$p_{\mathcal{R}}V_ip_{\mathcal{R}} = p_{\mathcal{R}_+}V_ip_{\mathcal{R}_+} + p_{\mathcal{R}_0}V_ip_{\mathcal{R}_0}.$$

Theorem 2.2.1 also highlights that the mass collected by an enclosure (not necessarily a recurrent one) always comes from the transient subspace. This fact reflects in the structure of absorption operators.

**Corollary 2.2.6.** *For any enclosure $\mathcal{V}$ and any time $t$,*

$$\Phi_t(p_{\mathcal{V}}) = p_{\mathcal{V}} + p_{\mathcal{T}\cap\mathcal{V}^\perp}\Phi_t(p_{\mathcal{V}})p_{\mathcal{V}^\perp\cap\mathcal{T}}. \tag{2.10}$$

*In particular, $\mathcal{V}$ is a recurrent enclosure for $\Phi$ if and only if the associated projection is harmonic for the reduced semigroup $\Phi^{\mathcal{R}}$.*

*Proof.* We denote by $\mathcal{S}_t$ the space $\operatorname{supp}(p_{\mathcal{V}^\perp}\Phi_t(p_{\mathcal{V}})p_{\mathcal{V}^\perp}) = (\operatorname{supp}\Phi_t(p_{\mathcal{V}})) \cap \mathcal{V}^\perp$. Then, by Lemma 2.1.4, we have

$$\Phi_t(p_{\mathcal{V}}) = p_{\mathcal{V}} + p_{\mathcal{V}^\perp}\Phi_t(p_{\mathcal{V}})p_{\mathcal{V}^\perp} = p_{\mathcal{V}} + p_{\mathcal{S}_t}\Phi_t(p_{\mathcal{V}})p_{\mathcal{S}_t}.$$

But, due to Lemma 2.2.2, $\mathcal{S}_t \subseteq \mathcal{T}$, and this implies relation (2.10).

Now, if $\mathcal{V}$ is a recurrent enclosure for $\mathbf{\Phi}$, then, by (2.10),

$$\Phi^{\mathcal{R}}_t(p_{\mathcal{V}}) = \Phi^{\mathcal{R}}_t(p_{\mathcal{R}}p_{\mathcal{V}}p_{\mathcal{R}}) = p_{\mathcal{R}}\Phi_t(p_{\mathcal{V}})p_{\mathcal{R}} = p_{\mathcal{R}}p_{\mathcal{V}}p_{\mathcal{R}} = p_{\mathcal{V}},$$

so $p_{\mathcal{V}}$ is harmonic for $\Phi^{\mathcal{R}}$. Conversely, if $p_{\mathcal{V}}$ is a harmonic projection for $\Phi^{\mathcal{R}}$, i.e. $\mathcal{V} \subseteq \mathcal{R}$ and $p_{\mathcal{R}}\Phi_t(p_{\mathcal{V}})p_{\mathcal{R}} = p_{\mathcal{V}}$, then

$$p_{\mathcal{R}\cap\mathcal{V}^\perp}\Phi_t(p_{\mathcal{V}})p_{\mathcal{R}\cap\mathcal{V}^\perp} = 0 \quad \text{and so} \quad p_{\mathcal{R}\cap\mathcal{V}^\perp}\Phi_t(p_{\mathcal{V}}) = \Phi_t(p_{\mathcal{V}})p_{\mathcal{R}\cap\mathcal{V}^\perp} = 0$$

by positivity of $\Phi_t(p_{\mathcal{V}})$. Furthermore, since $\|\Phi_t(p_{\mathcal{V}})\| \leq 1$, if we consider any two norm one vectors $v$ in $\mathcal{V}$ and $u$ in $\mathcal{T}$ and any $\varepsilon \in \mathbb{C}$, we have

$$\langle v+\varepsilon u, \Phi_t(p_{\mathcal{V}})(v+\varepsilon u)\rangle = \|v\|^2 + 2\Re e(\varepsilon\langle v, \Phi_t(p_{\mathcal{V}})u\rangle) + |\varepsilon|^2\langle u, \Phi_t(p_{\mathcal{V}})u\rangle \leq \|v+\varepsilon u\|^2 = 1+|\varepsilon|^2.$$

If we pick $\varepsilon \in \mathbb{R}$, we get that $\langle u, \mathbf{1} - \Phi_t(p_\mathcal{V})u\rangle\varepsilon^2 - 2\Re e(\langle v, \Phi_t(p_\mathcal{V})u\rangle)\varepsilon$ is a polynomial in $\varepsilon$ is always non-negative, hence $\Re e(\langle v, \Phi_t(p_\mathcal{V})u\rangle)$ must be 0. Analogously, if we pick $\varepsilon \in i\mathbb{R}$, we get that $\Im m(\langle v, \Phi_t(p_\mathcal{V})u\rangle) = 0$ and so $\langle v, \Phi_t(p_\mathcal{V})u\rangle = 0$. Therefore we can conclude

$$\Phi_t(p_\mathcal{V}) = p_\mathcal{V} + p_\mathcal{T}\Phi_t(p_\mathcal{V})p_\mathcal{T} \geq p_\mathcal{V}$$

i.e. $\mathcal{V}$ is an enclosure for $\Phi$. □

Now we are able to improve Proposition 2.1.5 for absorption operators in relation with transience and recurrence.

**Theorem 2.2.7. Absorption operators.**

1. *For any $\mathcal{V}$ enclosure,*

   $$A(\mathcal{V}) = p_\mathcal{V} + p_\mathcal{T}A(\mathcal{V})p_\mathcal{T} = p_\mathcal{V} + p_{\mathcal{T}\cap\mathcal{V}^\perp}A(\mathcal{V})p_{\mathcal{T}\cap\mathcal{V}^\perp} \quad \text{and} \quad p_{\mathcal{R}\cap\mathcal{V}^\perp}A(\mathcal{V})p_{\mathcal{R}\cap\mathcal{V}^\perp} = 0.$$

   *Further, $x = A(\mathcal{V}) - p_\mathcal{V}$ is a superharmonic operator supported in $\mathcal{T}$ and such that $\Phi_t(x) \searrow 0$ in the $w^*$-topology.*

2. *Suppose that $\mathcal{V}$ is an enclosure included in the recurrent space $\mathcal{R}$. Then $A(\mathcal{V})$ is the minimal fixed point $x$ of the semigroup $\Phi$ such that $0 \leq x \leq \mathbf{1}$, $p_\mathcal{V}xp_\mathcal{V} = p_\mathcal{V}$, $p_{\mathcal{R}\cap\mathcal{V}^\perp}xp_{\mathcal{R}\cap\mathcal{V}^\perp} = 0$. It is the unique fixed point with such features when $\mathfrak{h}$ is finite dimensional or, more in general, when the recurrent projection $p_\mathcal{R}$ is absorbing (i.e. $A(\mathcal{R}) = \mathbf{1}$).*

*Remark* 2.2.8. If not immediately evident, we stress that this theorem proves the quantum counterpart of well-known properties for classical absorption probabilities. Indeed, for a classical Markov chain $X$, recovering the notations at the beginning of Section 2.1, it is known that $A(C)$ is the minimal bounded function on $E$ such that $0 \leq A(C)(x) \leq 1$ for every $x \in E$, verifying

$$\begin{cases} A(C)(x) = 1 \text{ for } x \in C, \\ A(C)(x) = 0 \text{ for a recurrent } x \text{ outside } C \\ A(C)(x) = \sum_{y\in C} p_{xy} + \sum_{y\in\mathcal{T}} p_{xy}A(C)(y), \quad x \in \mathcal{T} \end{cases} \tag{2.11}$$

and the last equation, given the first two conditions, is equivalent to write $A(C) = P \cdot A(C)$, i.e. the function $A(C)$ is harmonic for $P$.

*Proof.* 1. The first part is obtained from equation (2.10) passing to the limit for $t \to +\infty$. For the second part, notice that $x$ is superharmonic being the difference of a harmonic and a subharmonic operators. Hence $(\Phi_t(x))_{t\in\mathfrak{T}}$ is a monotone decreasing net of positive operators whose limit is 0 because of the definition of absorption operator.
2. Consider an operator $x$ as in the statement, then $x = p_\mathcal{V} + p_\mathcal{T}xp_\mathcal{T} = \Phi_t(x)$ so $x = \Phi_t(x) \geq \Phi_t(p_\mathcal{V})$ for any $t \in \mathfrak{T}$ and then passing to the limit we conclude $x \geq A(\mathcal{V})$. Finally, we can also write

$$x - A(\mathcal{V}) = w^* - \lim_{t\to+\infty} \Phi_t(x - p_\mathcal{V}) = w^* - \lim_{t\to+\infty} \Phi_t(p_\mathcal{T}xp_\mathcal{T}) \leq w^* - \lim_{t\to+\infty} \Phi_t(p_\mathcal{T})$$

and the right hand side is null if and only if $\mathcal{R}$ is absorbing. □

*Remark* 2.2.9. Since $\mathcal{R}$ is an enclosure, the absorption operator $A(\mathcal{R})$ is well defined and, proceeding according to a “probabilistic” approach, it is natural to read the operator

$$U := w^* - \lim_{t\to+\infty} \Phi_t(p_{\mathcal{T}}) = \mathbf{1} - A(\mathcal{R}).$$

as representing the probability for the evolution to remain forever in the transient subspace $\mathcal{T}$. By point 2 of the previous theorem, used with $\mathcal{V} = \mathcal{R}$, the operator $U$ is the maximal harmonic operator $y$ such that $0 \leq y \leq \mathbf{1}$ and $p_{\mathcal{T}} y p_{\mathcal{T}} = y$.

*Remark* 2.2.10. Theorem 2.2.7 shows that Lemma 2.2.2 implies that for every enclosure $\mathcal{V}$, one has $\mathrm{supp}(A(\mathcal{V}) - p_{\mathcal{V}}) \subset \mathcal{T}$. In fact, they are equivalent: indeed, for every non null vector $w$ in $\mathcal{V}^{\perp}$ such that there exist $\bar{t} \in \mathfrak{T}$ and $\varepsilon > 0$ for which $\Phi_{\bar{t}}(p_{\mathcal{V}}) \geq \varepsilon |w\rangle\langle w|$, then $w \in \mathrm{supp}(\Phi_{\bar{t}}(p_{\mathcal{V}}) - p_{\mathcal{V}}) \subset \mathrm{supp}(A(\mathcal{V}) - p_{\mathcal{V}})$.

Another consequence of Theorem 2.2.1 is that every recurrent enclosure $\mathcal{V} \subseteq \mathcal{R}$ is diagonal in the block representation induced by $\mathcal{R}_0$ and $\mathcal{R}_+$.

**Corollary 2.2.11.** *Every enclosure $\mathcal{V} \subseteq \mathcal{R}$ is of the form $\mathcal{V} = (\mathcal{R}_+ \cap \mathcal{V}) \oplus (\mathcal{R}_0 \cap \mathcal{V})$.*

*Proof.* By contradiction, let $\mathcal{V} \subseteq \mathcal{R}$ be an enclosure such that $p_{\mathcal{R}_+} p_{\mathcal{V}} p_{\mathcal{R}_0} \neq 0$. We can assume that $\mathcal{V} \cap \mathcal{R}_+ = \{0\}$. Indeed, if this is not the case, we can replace it with the space

$$\mathcal{V}' = \mathcal{V} \cap (\mathcal{V} \cap \mathcal{R}_+)^{\perp} = \mathcal{V} \cap (\mathcal{R} \cap (\mathcal{V} \cap \mathcal{R}_+)^{\perp}),$$

which is an enclosure as intersection of the two enclosures $\mathcal{V}$ and $(\mathcal{R} \cap (\mathcal{V} \cap \mathcal{R}_+)^{\perp})$ (the latter by Theorem 2.2.1), and always included in $\mathcal{R}$. Furthermore, $\mathcal{V}' \cap \mathcal{R}_+ = \{0\}$ and $p_{\mathcal{R}_+} p_{\mathcal{V}'} p_{\mathcal{R}_0} = p_{\mathcal{R}_+} p_{\mathcal{V}} p_{\mathcal{R}_0} \neq 0$. Hence, by taking $\mathcal{V}'$ as $\mathcal{V}$, we can assume that $\mathcal{V} \cap \mathcal{R}_+ = \{0\}$.
Now, since $p_{\mathcal{R}_+} p_{\mathcal{V}} p_{\mathcal{R}_0} \neq 0$, by the definition of $\mathcal{R}_+$, there exists an invariant state $\rho$ with $\mathrm{supp}(\rho) \not\perp \mathcal{V}$, i.e. such that $p_{\mathcal{V}} \rho p_{\mathcal{V}} \neq 0$. For every $t \in \mathfrak{T}$, as $\rho$ is invariant and supported in $\mathcal{R}_+ \subseteq \mathcal{R}$, we can call $\widetilde{\mathcal{V}} = \mathcal{R} \cap \mathcal{V}^{\perp}$ (which is an enclosure by Theorem 2.2.1), and write

$$\begin{aligned} \rho &= p_{\mathcal{V}} \rho p_{\mathcal{V}} + p_{\widetilde{\mathcal{V}}} \rho p_{\mathcal{V}} + p_{\mathcal{V}} \rho p_{\widetilde{\mathcal{V}}} + p_{\widetilde{\mathcal{V}}} \rho p_{\widetilde{\mathcal{V}}} \\ &= p_{\mathcal{V}} \Phi_{*t}(\rho) p_{\mathcal{V}} + p_{\widetilde{\mathcal{V}}} \Phi_{*t}(\rho) p_{\mathcal{V}} + p_{\mathcal{V}} \Phi_{*t}(\rho) p_{\widetilde{\mathcal{V}}} + p_{\widetilde{\mathcal{V}}} \Phi_{*t}(\rho) p_{\widetilde{\mathcal{V}}} = \Phi_{*t}(\rho). \end{aligned} \tag{2.12}$$

$\mathcal{V}$ and $\widetilde{\mathcal{V}}$ are orthogonal enclosures for the completely positive map $\Phi_t$, hence (2.12), together with Lemma 2.2.4, gives, for every $t \in \mathfrak{T}$,

$$\Phi_{*t}(p \rho q) = p \Phi_{*t}(\rho) q, \quad p, q \in \{p_{\mathcal{V}}, p_{\widetilde{\mathcal{V}}}\}$$

and in particular

$$\Phi_{*t}(p_{\mathcal{V}} \rho p_{\mathcal{V}}) = p_{\mathcal{V}} \Phi_{*t}(\rho) p_{\mathcal{V}} = p_{\mathcal{V}} \rho p_{\mathcal{V}};$$

so that

$$\widetilde{\rho} := \frac{p_{\mathcal{V}} \rho p_{\mathcal{V}}}{\mathrm{tr}\,(p_{\mathcal{V}} \rho p_{\mathcal{V}})} \neq 0$$

is an invariant state supported in $\mathcal{V}$. This implies $\mathcal{V} \cap \mathcal{R}_+ \neq \{0\}$ and we get to a contradiction. $\square$

There are some features of absorption operators that can be interpreted in the potential theory framework of [31, 67]:

1. a positive bounded operator $y$ is said to be a potential if there exists $x \in \mathcal{B}(\mathfrak{h})_{\text{int}}$ such that $y = \mathcal{U}(x)$. Together with the following characterization of potentials, the first point of Theorem 2.2.7 shows that for every enclosure $\mathcal{V}$, $A(\mathcal{V}) - p_{\mathcal{V}}$ is a potential:

   **Theorem 2.2.12** (Theorem 4, [31])**.** *A positive operator $x$ is a potential if and only if it is superharmonic and $\Phi_t(x) \searrow 0$ in the $w^*$-topology.*

2. $\ker(A(\mathcal{V})) = \{u \in D(\mathfrak{U}(p_{\mathcal{V}})) : \mathfrak{U}(p_{\mathcal{V}})[u] = 0\}$.

   *Proof.* Proposition 2.1.1 shows that $v \in \ker(A(\mathcal{V}))$ if and only if

   $$\langle v, \Phi_t(p_{\mathcal{V}})v \rangle = 0, \quad \forall t \in \mathfrak{T}, \tag{2.13}$$

   so we need to show that $\int_0^{+\infty} \langle v, \Phi_t(p_{\mathcal{V}})v \rangle < +\infty$ is equivalent to equation (2.13). This is true because since $\mathcal{V}$ is an enclosure, $t \mapsto \langle v, \Phi_t(p_{\mathcal{V}})v \rangle$ is monotone non-decreasing. □

## 2.3 Absorption operators to describe fixed points

In this section, we are mainly concerned with the description of the fixed points of a semigroup of quantum Markov maps. In order to have a full description, we shall need to add the extra condition that the recurrent space is attractive for the evolution. We recall that the set of fixed points (or harmonic operators) of the quantum Markov semigroup $\mathbf{\Phi}$ is defined as

$$\mathcal{F}(\mathbf{\Phi}) = \{x \in \mathcal{B}(\mathfrak{h}) : \Phi_t(x) = x, t \in \mathfrak{T}\}.$$

The structure of the set $\mathcal{F}(\mathbf{\Phi})$ is well known when there exists a faithful normal invariant state (i.e. the semigroup is positive recurrent): in such a case, it is an atomic $W^*$-algebra, since it is in the multiplicative domain and it is the range of a $\mathbf{\Phi}$-invariant $w^*$-$w^*$-continuous conditional expectation (see for instance [34,35,48] and more recent developments in [6, 13, 33]).

When we have no faithful invariant state, we do not have general results and the problem becomes substantially different and more complicated, for instance $\mathcal{F}(\mathbf{\Phi})$ does not need to be an algebra in general. Examples are easy to find, also in classical probability, just considering models with a transient state having access to more than one recurrent class.

### 2.3.1 Recurrent semigroups

Abandoning the condition on the existence of a normal invariant faithful state means considering non purely positive recurrent Markov evolutions. If we try to weaken this condition, one natural first step can be assuming that the semigroup is recurrent, that is $\mathcal{R} = \mathfrak{h}$; notice that, in the case of a generic semigroup $\mathbf{\Phi}$, everything that we prove holds true for the recurrent restriction $\mathbf{\Phi}^{\mathcal{R}}$. In the case of recurrent semigroups, $\mathcal{F}(\mathbf{\Phi})$ is tightly related to communication properties of the semigroup: every projection in $\mathcal{F}(\mathbf{\Phi})$ is by definition the projection onto an enclosure and vice versa by Corollary 2.2.6. Moreover, while in general $\mathcal{F}(\mathbf{\Phi})$ is only a selfadjoint $w^*$-closed linear space, the following result proves that the fixed points set of a recurrent semigroup is a $W^*$-algebra, hence it is completely determined by its projections.

**Proposition 2.3.1.** *If $\mathbf{\Phi}$ is recurrent, then $\mathcal{F}(\mathbf{\Phi})$ is a $W^*$-algebra.*

We remark that this fact was already known in case of positive recurrent semigroups, i.e. when $\mathcal{R} = \mathcal{R}_+$ (see [2, Theorem 2.3]). Before presenting the proof of Proposition 2.3.1, we need to recall the following result.

**Lemma 2.3.2.** *The following are equivalent:*

1. $\mathcal{F}(\mathbf{\Phi})$ *is an algebra,*
2. $\mathcal{F}(\mathbf{\Phi})$ *is a* $W^*$*-algebra,*
3. *if* $x \in \mathcal{F}(\mathbf{\Phi})$*, then* $x^*x \in \mathcal{F}(\mathbf{\Phi})$*,*
4. $\mathcal{F}(\mathbf{\Phi}) \subseteq \mathcal{N}(\mathbf{\Phi})$.

*Proof.* 1. $\Leftrightarrow$ 2. It simply follows from the fact that $\mathcal{F}(\mathbf{\Phi})$ is a $w^*$-closed, $^*$-closed linear subspace of $B(\mathfrak{h})$.
1. $\Rightarrow$ 3. It is a consequence of the fact that $\mathcal{F}(\mathbf{\Phi})$ is closed under $^*$.
3. $\Rightarrow$ 4. For every $x \in \mathcal{F}(\mathbf{\Phi})$, we have that $x^* \in \mathcal{F}(\mathbf{\Phi})$ and by 3. we get that for every $t \geq 0$

$$\Phi_t(x^*x) = x^*x = \Phi_t(x^*)\Phi_t(x) \text{ and } \Phi_t(xx^*) = xx^* = \Phi_t(x)\Phi_t(x^*),$$

hence $x \in \mathcal{N}(\mathbf{\Phi})$.
4. $\Rightarrow$ 1. Let us consider $x \in \mathcal{F}(\mathbf{\Phi}) \subset \mathcal{N}(\mathbf{\Phi})$; Theorem 1.2.18 states that for $t \geq 0$ and for every $y \in \mathcal{F}(\mathbf{\Phi})$

$$\Phi_t(xy) = \Phi_t(x)\Phi_t(y) = xy,$$

which means that $xy \in \mathcal{F}(\mathbf{\Phi})$ and that $\mathcal{F}(\mathbf{\Phi})$ is an algebra. □

*Proof of Proposition 2.3.1.* If $\mathcal{F}(\mathbf{\Phi})$ is not a $W^*$-algebra, Lemma 2.3.2 shows that there exists $x \in \mathcal{F}(\mathbf{\Phi})$ such that $x^*x \notin \mathcal{F}(\mathbf{\Phi})$. Kadison-Schwarz inequality implies that

$$\Phi_t(x^*x) \geq \Phi_t(x^*)\Phi_t(x) = x^*x \text{ for all } t \geq 0, \tag{2.14}$$

which means that $x^*x$ is subharmonic and $(\Phi_t(x^*x))$ is a bounded positive monotone increasing net and we call $y$ its least upper bound. $y$ is a fixed point, since $\forall s \in \mathfrak{T}$

$$\Phi_s(y) = \Phi_s\left(w^* - \lim_{t\to+\infty} \Phi_t(x^*x)\right) = \lim_{t\to+\infty} \Phi_{t+s}(x^*x) = y.$$

Notice that

- $\forall t \in \mathfrak{T}$, $\Phi_t(y - x^*x) = y - \Phi_t(x^*x) \leq y - x^*x$ (superharmonic);
- $\Phi_t(y - x^*x) \searrow 0$.

Moreover $y - x^*x \neq 0$: since $x^*x \notin \mathcal{F}(\mathbf{\Phi})$, there must be a time $\bar{t} > 0$ such that the inequality in equation (2.14) is not an equality, hence

$$y - x^*x \geq \Phi_{\bar{t}}(x^*x) - x^*x \geq 0 \text{ and } \Phi_{\bar{t}}(x^*x) - x^*x \neq 0.$$

By Theorem 2.2.12, there exists a non-null positive operator $z$ such that $\mathfrak{U}(z) = y - x^*x$ is bounded, hence $\{0\} \neq \operatorname{supp}(\mathcal{U}(z)) \subseteq \mathcal{T}$ and the transient space is non trivial. □

**Corollary 2.3.3.** *Let $\mathbf{\Phi}$ be recurrent; then for every enclosure $\mathcal{V}$, $p_{\mathcal{V}} \in \mathcal{F}(\mathbf{\Phi})$ and*

$$\mathcal{F}(\mathbf{\Phi}) = \overline{\mathrm{span}\{p_{\mathcal{V}} : \mathcal{V} \textit{ enclosure}\}}^{\|\,\|}.$$

*Proof.* Corollary 2.2.6 shows that, if $\mathbf{\Phi}$ is recurrent, a projection corresponds to an enclosure if and only if it is harmonic, that is it is a fixed point for the semigroup; since $\mathcal{F}(\mathbf{\Phi})$ is a $W^*$-algebra, it is the norm-closure of the linear span of its projections (Corollary 1.1.9) and we obtain the statement. □

Another immediate consequence is a nice diagonal structure of the elements in $\mathcal{F}(\mathbf{\Phi})$.

**Corollary 2.3.4.** *Let $\mathbf{\Phi}$ be a recurrent semigroup of quantum Markov maps. Then*

$$\mathcal{F}(\mathbf{\Phi}) = \mathcal{F}(\mathbf{\Phi}^{\mathcal{R}_+}) \oplus \mathcal{F}(\mathbf{\Phi}^{\mathcal{R}_0}).$$

*Proof.* First we prove that every $x \in \mathcal{F}(\mathbf{\Phi})$ has the following diagonal form:

$$x = p_{\mathcal{R}_+} x p_{\mathcal{R}_+} + p_{\mathcal{R}_0} x p_{\mathcal{R}_0}. \tag{2.15}$$

By Corollary 2.3.3, it is enough to prove that equation (2.15) holds true for the projections corresponding to enclosures, hence the result follows from Corollary 2.2.11.
Since $\mathbf{\Phi}$ is recurrent and both $\mathcal{R}_+$ and $\mathcal{R}_0$ are enclosures, the corresponding projections are harmonic by Corollary 2.3.3; moreover $\mathcal{F}(\mathbf{\Phi}) \subseteq \mathcal{N}(\mathbf{\Phi})$, which implies that for every $x \in B(\mathfrak{h})$, for every $t \geq 0$

$$\Phi_t(pxq) = \Phi_t(p)\Phi_t(x)\Phi_t(q) = p\Phi_t(x)q \quad p, q \in \{p_{\mathcal{R}_+}, p_{\mathcal{R}_0}\}. \tag{2.16}$$

If we apply equation (2.16) to $x \in \mathcal{F}(\mathbf{\Phi})$, we get that $p_{\mathcal{R}_+} x p_{\mathcal{R}_+} \in \mathcal{F}(\mathbf{\Phi}^{\mathcal{R}_+})$ and $p_{\mathcal{R}_0} x p_{\mathcal{R}_0} \in \mathcal{F}(\mathbf{\Phi}^{\mathcal{R}_0})$, i.e. $\mathcal{F}(\mathbf{\Phi}) \subseteq \mathcal{F}(\mathbf{\Phi}^{\mathcal{R}_+}) \oplus \mathcal{F}(\mathbf{\Phi}^{\mathcal{R}_0})$.
If we instead apply equation (2.16) to any element in $\mathcal{F}(\mathbf{\Phi}^{\mathcal{R}_+}) \oplus \mathcal{F}(\mathbf{\Phi}^{\mathcal{R}_0})$, we get the other inclusion. □

We have already remarked that in the case of positive recurrent semigroup, $\mathcal{F}(\mathbf{\Phi})$ is an atomic $W^*$-algebra. We briefly recall the results about fixed points set when there is a faithful invariant state (the proof can be deduced from Lemma 2 and Theorem 2 in [13]); once again we remark that the same results hold for the positive recurrent restriction $\mathbf{\Phi}^{\mathcal{R}_+}$ of any semigroup of quantum Markov maps $\mathbf{\Phi}$.

**Proposition 2.3.5.** *Let $\mathbf{\Phi}$ be a positive recurrent semigroup, i.e. $\mathcal{R}_+ = \mathfrak{h}$; then there exists a unique decomposition of the Hilbert space as a direct sum of orthogonal enclosures*

$$\mathfrak{h} = \bigoplus_{\alpha \in A} \chi_\alpha$$

*and there exist Hilbert spaces $\mathfrak{h}_\alpha^{(1)}$, $\mathfrak{h}_\alpha^{(2)}$ and unitary operators $U_\alpha : \mathfrak{h}_\alpha^{(1)} \otimes \mathfrak{h}_\alpha^{(2)} \to \chi_\alpha$ such that*

$$\mathcal{F}(\mathbf{\Phi}) = \bigoplus_{\alpha \in A} U_\alpha \left( B(\mathfrak{h}_\alpha^{(1)}) \otimes \mathbf{1}_{\mathfrak{h}_\alpha^{(2)}} \right) U_\alpha^*. \tag{2.17}$$

*Hence $\mathcal{F}(\mathbf{\Phi})$ is an atomic $W^*$-algebra and $p_{\chi_\alpha}$ are its central projections.*

*Moreover, for any $\alpha \in A$, there exists a (unique and faithful) normal state $\rho_\alpha$ on $\mathfrak{h}_\alpha^{(2)}$ such that all the invariant normal states for $\mathbf{\Phi}$ can be written in the form*

$$\sum_{\alpha\in A} \lambda_\alpha U_\alpha(\omega_\alpha \otimes \rho_\alpha)U_\alpha^*, \quad \textit{for some } \omega_\alpha \textit{ normal state on } \mathfrak{h}_\alpha^{(1)},\ \lambda_\alpha \geq 0 \textit{ with } \sum_{\alpha\in A}\lambda_\alpha = 1. \tag{2.18}$$

*For every $\alpha \in A$, the restriction of the semigroup acts in the following way*

$$\Phi_t(U_\alpha(x\otimes y)U_\alpha^*) = U_\alpha(x\otimes \mathcal{Q}_t^\alpha(y))U_\alpha^* \quad x\in B(\mathfrak{h}_\alpha^{(1)}), y \in B(\mathfrak{h}_\alpha^{(2)}). \tag{2.19}$$

*where $\mathcal{Q}_t^\alpha$ is an irreducible positive recurrent semigroup acting on $B(\mathfrak{h}_\alpha^{(2)})$ with faithful normal invariant state $\rho_\alpha$.*

Equation (2.17) tells us that for every $\alpha$, $\chi_\alpha$ is the range of a minimal central projection; when $\dim(\mathfrak{h}_\alpha^{(1)}) = 1$, then $\chi_\alpha$ is also a minimal enclosure, otherwise it admits an infinite number of decompositions as direct sum of minimal orthogonal enclosures.

More precisely, in this case, for any complete orthonormal system $\{e_\beta^\alpha\}_{\beta\in I_\alpha}$ for $\mathfrak{h}_\alpha^{(1)}$, $\mathcal{V}_{\alpha,\beta} := U_\alpha(\mathbb{C}e_\beta^\alpha \otimes \mathfrak{h}_\alpha^{(2)})$ is a minimal enclosure and $\chi_\alpha = \oplus_{\beta\in I_\alpha}\mathcal{V}_{\alpha,\beta}$. Furthermore, for any pair of minimal enclosures $\mathcal{V}_{\alpha,\beta}$ and $\mathcal{V}_{\alpha,\beta'}$, for $\beta \neq \beta'$, we know that there exist a partial isometry $Q_{\alpha,\beta,\beta'} := U_\alpha\left(|e_\beta^\alpha\rangle\langle e_{\beta'}^\alpha| \otimes \mathbf{1}_{\mathfrak{h}_\alpha^{(2)}}\right)U_\alpha^*$ between them; this operator is a fixed point for the fast recurrent semigroup and allows to express the action of the semigroup restricted to $\mathcal{V}_{\alpha,\beta}$ and the minimal invariant state supported on it in terms of the dynamic restricted to $\mathcal{V}_{\alpha,\beta'}$ and the invariant state supported on it (see [15]). The dynamic on $\mathcal{V}_{\alpha,\beta}$ (and all the other isometric enclosures) is described by the semigroup $\mathcal{Q}^\alpha$ (modulo $U_\alpha$) and $U_\alpha(|e_\beta^\alpha\rangle\langle e_\beta^\alpha| \otimes \rho_\alpha)U_\alpha^*$ is the unique extremal invariant state with support $\mathcal{V}_{\alpha,\beta}$.

We remark that Proposition 2.3.5 implies that every positive recurrent semigroup admits a decomposition of the Hilbert space $\mathfrak{h}$ into orthogonal minimal enclosures (DOME). In the case of a recurrent semigroup, thanks to Corollary 2.3.3, the atomicity of $\mathcal{F}(\mathbf{\Phi})$ is equivalent to the fact tha there exists a DOME and whether or not a recurrent semigroup admits a DOME is a natural question, also because it is true for classical Markov chains. However we are able to exhibit an example of a recurrent semigroup which does not admit a DOME, or equivalently, for which $\mathcal{F}(\mathbf{\Phi})$ is not atomic; before, we remark that any DOME of $\mathcal{R}$ must be compatible with $\mathcal{R}_+$ and $\mathcal{R}_0$.

**Lemma 2.3.6.** *Let $(\mathcal{V}_\alpha)$ be a DOME of $\mathcal{R}$, then $(\mathcal{V}_\alpha \cap \mathcal{R}_+)$ and $(\mathcal{V}_\alpha \cap \mathcal{R}_0)$ are DOMEs for $\mathcal{R}_+$ and $\mathcal{R}_0$, respectively.*

*Proof.* Corollary 2.2.11 implies that every (sub)harmonic projection $p_\mathcal{V}$ for $\mathbf{\Phi}$ commutes with $p_{\mathcal{R}_+}$, $p_{\mathcal{R}_0}$ and, since $\mathcal{R}_+$ and $\mathcal{R}_0$ are enclosures and the intersection of two enclosures is again an enclosure, $\mathcal{V}\cap\mathcal{R}_+$ and $\mathcal{V}\cap\mathcal{R}_0$ are enclosures. Therefore a DOME of $\mathcal{R}$ $(\mathcal{V}_\alpha)$ induces DOMEs $(\mathcal{V}_\alpha\cap\mathcal{R}_+)$ and $(\mathcal{V}_\alpha\cap\mathcal{R}_0)$ for $\mathcal{R}_+$ and $\mathcal{R}_0$, respectively (since $\mathcal{V}_\alpha$ is minimal, $\mathcal{V}_\alpha \cap \mathcal{R}_+$ and $\mathcal{V}_\alpha \cap \mathcal{R}_0$ are either $\{0\}$ or $\mathcal{V}_\alpha$). □

We already know that there always exists a DOME of $\mathcal{R}_+$, hence the existence of a DOME of $\mathcal{R}$ and $\mathcal{R}_0$ are equivalent problems.

**Example 2.3.7** (Noncommutative symmetric random walk on $\mathbb{Z}$)**.** Let us consider the group $G$ with two generators $a, b$ satisfying $a^2 = b^2 = e$ ($e$ is the identity element) and the corresponding left and right representations defined on $\mathfrak{h} = \ell^2(G)$. For every $g \in G$, $\lambda(g)$ and $\rho(g)$ are the unitary operators acting in the following way on the canonical basis $\mathscr{C} = \{\delta_g : g \in G\}$ of $\mathfrak{h}$:

$$\lambda(g)\delta_h = \delta_{gh}, \quad \rho(g)\delta_h = \delta_{hg^{-1}} \quad \forall g, h \in G.$$

Notice that $\lambda(g)^* = \lambda(g^{-1})$ and $\rho(g)^* = \rho(g^{-1})$ for every $g \in G$; the fact that $a^2 = b^2 = e$ implies that $\lambda(a)$ and $\lambda(b)$ are selfadjoint operators. We introduce the following notation:

$$L(G) = \{\lambda(g) : g \in G\}'' = \{\lambda(a), \lambda(b)\}'', \quad R(G) = \{\rho(g) : g \in G\}'' = \{\rho(a), \rho(b)\}''.$$

We recall that $L(G)' = R(G)$ ( [63, Section V.7]). We define the quantum channel

$$\Phi(x) = \frac{1}{2}\lambda(a)x\lambda(a) + \frac{1}{2}\lambda(b)x\lambda(b), \quad x \in B(\mathfrak{h})$$

and we consider the semigroup $\mathbf{\Phi} := (\Phi^n)_{n\in\mathbb{N}}$.

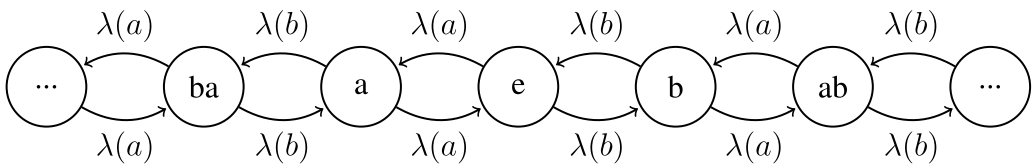
**Invariant commutative subalgebra and null recurrence.** Consider the commutative W*-subalgebra of operators which are diagonal in the canonical basis $\mathscr{C}$ and its predual:

$$\Delta = \{x \in B(\mathfrak{h}) : x = \sum_{g\in G} x_g \left|\delta_g\right\rangle \left\langle\delta_g\right|\} \simeq \ell^\infty(G),$$
$$\Delta_* = \{x \in L^1(\mathfrak{h}) : x = \sum_{g\in G} x_g \left|\delta_g\right\rangle \left\langle\delta_g\right|\} \simeq \ell^1(G).$$

Since $G$ is countable, we can relable its elements with integer numbers:

$$\dots aba \mapsto -3 \quad ba \mapsto -2 \quad a \mapsto -1 \quad e \mapsto 0 \quad b \mapsto 1 \quad ab \mapsto 2 \quad bab \mapsto 3 \dots$$

and this provides isomorphisms between $\ell^\alpha(G)$ and $\ell^\alpha(\mathbb{Z})$ for $\alpha \in \{1, 2, \infty\}$. $\lambda(a), \lambda(b)$ act on $\mathscr{C}$ in the following way (for any $g \in G$, the label $g$ stays for $\delta_g$):




It is easy to see that $\Phi$ preserves $\Delta$ and $\Delta_*$ and its restriction corresponds via the isomorphisms above to the transition matrix of a symmetric random walk on $\mathbb{Z}$: $\Phi(\left|\delta_g\right\rangle \left\langle\delta_g\right|) = \frac{1}{2}(\left|\delta_{ag}\right\rangle \left\langle\delta_{ag}\right| + \left|\delta_{bg}\right\rangle \left\langle\delta_{bg}\right|)$. The symmetric random walk on $\mathbb{Z}$ is null recurrent and this implies that also $\mathbf{\Phi}$ is null recurrent.

**Proposition 2.3.8.** $\mathbf{\Phi}$ *is null recurrent.*

*Proof.* 1. Consider a non-null positive operator $x$, then there must exists some $g \in G$ such that $\langle \delta_g, x\delta_g \rangle \geq c > 0$ for some positive constant $c$. By the symmetry of the semigroup, we can assume $g = e$.

$$+\infty > \mathfrak{U}(x)[\delta_e] = \sum_{k=0}^{+\infty} \operatorname{tr}\left(\Phi_*^k(|\delta_e\rangle\langle\delta_e|)x\right) \geq \sum_{k=0}^{+\infty} p_{0,0}^k \langle \delta_e, x\delta_e \rangle \geq c \sum_{k=0}^{+\infty} p_{0,0}^k = +\infty,$$

where $p_{0,0}^k$ is the probability that a symmetric random walk on $\mathbb{Z}$ that starts in $0$ comes back to $0$ in $k$ steps. Therefore $\mathfrak{U}(x)$ is unbounded and hence $\mathcal{T} = \{0\}$.
2. Suppose there exists an invariant state $\rho$; the action of $\rho$ on $\Delta$ is represented by a state $\tilde{\rho} \in \Delta_*$, which must be invariant for the symmetric random walk on $\mathbb{Z}$, but this is again a contradiction, hence $\mathcal{R}_+ = \{0\}$. $\square$

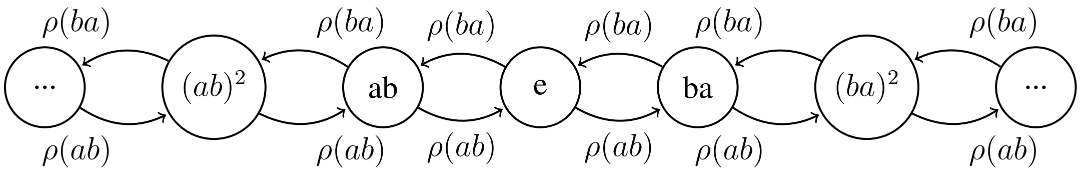
$\mathcal{F}(\mathbf{\Phi})$ **has no minimal projections.** When $\mathcal{F}(\mathbf{\Phi})$ is an algebra (this is the case by Proposition 2.3.1) and the semigroup is generated by a single quantum channel $\Phi$, Proposition 1 in [13] provides us a characterization of $\mathcal{F}(\mathbf{\Phi})$ in terms of Kraus operators of $\Phi$:

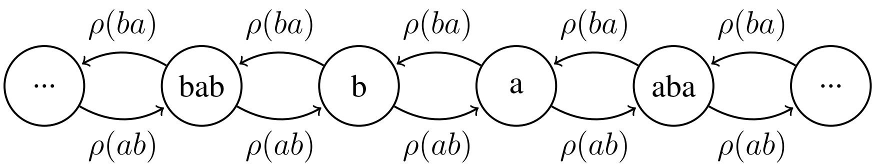
$$\mathcal{F}(\mathbf{\Phi}) = \{\lambda(a), \lambda(b)\}' = L(G)' = R(G).$$

We will show something stronger than the fact that $R(G)$ is not atomic: namely, we will prove that it has no minimal projections. We denote by $Z(G) := R(G) \cap L(G)$ the center of $R(G)$; notice that $(\rho(ab) + \rho(ba))/2 \in Z(G)$: it clearly belongs to $R(G)$ and it commutes with $\rho(a)$ (by symmetry it commutes with $\rho(b)$ too):

$$\rho(a)(\rho(ab) + \rho(ba)) = \rho(b) + \rho(aba) = (\rho(ab) + \rho(ba))\rho(a).$$

Let us focus on the action of $\rho(ab)$ and $\rho(ba) = \rho(ab)^*$ on $\mathscr{C}$:




Hence there exists a unitary operator $U : \ell^2(\mathbb{Z}) \otimes \mathbb{C}^2 \to \ell^2(G)$ such that $U^*\rho(ab)U$ is $S \otimes \mathbf{1}_{\mathbb{C}^2}$, where $S$ is the right shift operator. Consider the Fourier transform between the one dimensional torus $\mathbb{T}$ and $\mathbb{Z}$:

$$\mathscr{F} : L^2(\mathbb{T}) \to \ell^2(\mathbb{Z})$$
$$f(x) \mapsto \mathscr{F}(f)(k) := \frac{1}{2\pi}\int_{\mathbb{T}} f(x)e^{-ikx}dx.$$

By the Fourier transform properties, it is easy to see that $\mathscr{F}^{-1}S\mathscr{F}$ is the multiplication operator $M_{e^{ix}}$ corresponding to the function $e^{ix}$. We have the following chain of equalities

$$M_{\cos(x)} \otimes \mathbf{1}_{\mathbb{C}^2} = \left(\frac{M_{e^{ix}} + M_{e^{-ix}}}{2}\right) \otimes \mathbf{1}_{\mathbb{C}^2} = \mathscr{F}^{-1}(S + S^*)\mathscr{F} \otimes \mathbf{1}_{\mathbb{C}^2} = \\ = (\mathscr{F}^{-1} \otimes \mathbf{1}_{\mathbb{C}^2})U^* \left(\frac{\rho(ab) + \rho(ba)}{2}\right) U(\mathscr{F} \otimes \mathbf{1}_{\mathbb{C}^2}).$$

Therefore the $W^*$-algebras generated by $M_{\cos(x)}$ and $(\rho(ab)+\rho(ba))/2$ are isomorphic.

**Proposition 2.3.9.** *$R(G)$ has no minimal projection.*

*Proof.* Let us consider the following sequence of projections in the $W^*$-algebra generated by $M_{\cos(x)}$: for any $n \in \mathbb{N}$, $j = 0, \ldots, 2^{n+1} - 1$ we define $q_{j,n}$ as the indicator function of the set

$$\cos^{-1}\left(\left[-1 + j2^{-n}, -1 + (j+1)2^{-n}\right]\right);$$

we define $p_{j,n} := U(\mathscr{F} \otimes \mathbf{1}_{\mathbb{C}^2})(q_{j,n} \otimes \mathbf{1}_{\mathbb{C}^2})(\mathscr{F}^{-1} \otimes \mathbf{1}_{\mathbb{C}^2})U^*$, which is the spectral projection of $(\rho(ab) + \rho(ba))/2$ corresponding to the same set and it is in $Z(G)$. Notice that by construction

1. $\sum_{j=0}^{2^{n+1}-1} p_{j,n} = \mathbf{1}$,
2. $p_{j,n}p_{k,n} = \delta_{jk}p_{j,n}$ for every $n \in \mathbb{N}$, $j = 0, \ldots, 2^{n+1} - 1$,
3. for every $p_{j,n}$ with $n \geq 1$, there exists a unique $k$ such that $p_{j,n}p_{k,n-1} = p_{j,n}$.

Let $f \in R(G)$ be a minimal projection; since $(p_{j,n}) \subset Z(G)$, for every $n \in \mathbb{N}$ and $j \in \{0, \ldots, n-1\}$

$$fp_{j,n}f = p_{j,n}fp_{j,n}$$

is a projection dominated by $p_{j,n}$ and $f$. The minimality of $f$ implies that $fp_{j,n}$ is either $0$ or $f$ itself, and, by 1. and 2., for every $n \in \mathbb{N}$, there exists a unique $j \in \{0, \ldots, n-1\}$, which we call $j(n)$, such that $p_{j,n}f = f$. Hence $f \leq p_{j(n),n}$ for every $n \in \mathbb{N}$ and $p_{j(n+1),n+1} \leq p_{j(n),n}$ for every $n \in \mathbb{N}$ and $p_{j(n),n} \downarrow 0$ (in the $w^*$-topology), hence $f = 0$. □

*Remark* 2.3.10. The following is the GKLS generator of a continuous time counterpart of the same example:

$$\mathcal{L}(x) = \frac{1}{2}(\lambda(a)x\lambda(a) - x) + \frac{1}{2}(\lambda(b)x\lambda(b) - x).$$

The analysis above can be carried out also in this case.

Moreover, in an analogous way it is possible to embed any symmetric random walk on $\mathbb{Z}^d$: we only need to consider the group $G$ generated by $\{a_i, b_i\}_{i=1}^d$ such that $a_i^2 = b_i^2 = e$ and $a_ib_ja_i^{-1}b_j^{-1} = a_ia_ja_i^{-1}a_j^{-1} = b_ib_jb_i^{-1}b_j^{-1} = e$ for $i \neq j$.

*Remark* 2.3.11. The structure of the decoherence-free subalgebra $\mathcal{N}(\mathbf{\Phi})$ of a semigroup and wether it is atomic or not has been investigated in [13,33,59], especially in relation to enviromental decoherence. In the present example, even $\mathcal{N}(\mathbf{\Phi})$ has no minimal projections. Indeed, by [13, Proposition 3], we have that

$$\mathcal{N}(\mathbf{\Phi}) = \{\lambda(ab), \lambda(ba)\}'.$$

Let us consider $H$ the subgroup of $G$ generated by $ab, ba$ and the corresponding partition of $G$ into right cosets:

$$G = \bigcup_{s\in S} Hs, \quad Hs := \{hs : h \in H\}$$

where $S$ is a set of representatives of the quotient of $G$ with respect to the equivalence relation that identifies two elements $x, y \in G$ if $xy^{-1} \in H$. Such a partition of $G$ induces a decomposition of the corresponding Hilbert space:

$$\ell^2(G) = \bigoplus_{s\in S} \ell^2(Hs) \simeq \ell^2(H) \otimes \ell^2(S).$$

Notice that $\mathcal{N}(\mathbf{\Phi})$ is $(L(H)\otimes \mathbf{1}_{\ell^2(S)})' = R(H)\otimes B(\ell^2(S))$, hence its center is $L(H)\cap R(H)\otimes \mathbf{1}_{\ell^2(S)} = \{\lambda(ab), \lambda(ba)\}'' \cap \{\rho(ab), \rho(ba)\}''$. $(\rho(ab) + \rho(ba))/2$ is in the center of $\mathcal{N}(\mathbf{\Phi})$ too, hence we can repeat the same proof as for $\mathcal{F}(\mathbf{\Phi})$.

*Remark* 2.3.12. We remark that the unitary group of $\mathcal{F}(\mathbf{\Phi})$ is strictly smaller that the group of symmetries of $\Phi$, i.e. all the unitary operators $U \in B(\mathfrak{h})$ such that

$$\Phi(U^* \cdot U) = U^*\Phi(\cdot)U.$$

For instance the following unitary operators are symmetries of the qunatum Markov map, but they are not in $\mathcal{F}(\mathbf{\Phi}) = \{\lambda(a), \lambda(b)\}'$:

1. $U_1 : \delta_g \mapsto \delta_{u_1(g)}$, where $u_1(g)$ is the word obtained from $g$ changing $a$ into $b$ and vice versa;
2. $U_2 : \delta_g \mapsto (-1)^{u_2(g)}\delta_g$, where $u_2(g)$ is the number of occurencies of the letter $a$ in $g$.

The first case switches the Kraus operators, that means $U_1\lambda(a) = \lambda(b)U_1$, while in the second case $U_2\lambda(a) = -\lambda(a)U_2$.

### 2.3.2 Absorbing recurrent space

We now turn our attention to the fixed points set of semigroups with non trivial transient part; in this case, subharmonic projections need not to be harmonic anymore, hence we cannot hope that Corollary 2.3.3 still holds true. However, Proposition 2.1.1 shows that absorption operators are fixed points of $\mathbf{\Phi}$ and, since the set of fixed points is norm-closed, we know that

$$\overline{\mathrm{span}\{A(\mathcal{V}) :\ \mathcal{V} \text{ enclosure}\}}^{\|\,\|} \subseteq \mathcal{F}(\mathbf{\Phi}).$$

Therefore, a natural question is whether the reverse inclusion holds true and fixed points are completely described by absorption operators. This is indeed the case when the recurrent space is absorbing, that is

$$A(\mathcal{R}) = w^* - \lim_{t \to +\infty} \Phi_t(p_{\mathcal{R}}) = \mathbf{1}. \tag{H1}$$

It means that asymptotically the evolution gets absorbed in the recurrent space, which, therefore, contains all relevant information regarding asymptotic quantities; an immediate consequence is that the evolution passes a finite amount of time in the transient subspace ($U = 0$ by Remark 2.2.9). It is not a too restrictive condition and in some cases it is feasible to check if it holds (see Remark 2.3.22). An important tool that we are going to use is the notion of ergodic projection (for the definition we follow [49]).

**Definition 2.3.13** (Ergodic projection)**.** *A linear operator $\mathcal{E} : B(\mathfrak{h}) \to B(\mathfrak{h})$ is said to be an ergodic projection for the semigroup $\mathbf{\Phi}$ if the following hold true:*

1. *$\mathcal{E}(B(\mathfrak{h})) \subseteq \mathcal{F}(\mathbf{\Phi})$, $\mathcal{E}^2 = \mathcal{E}$,*
2. *$\mathcal{E} \circ \Phi_t = \Phi_t \circ \mathcal{E} = \mathcal{E}$ for every $t \in \mathfrak{T}$,*
3. *$\mathcal{E}$ belongs to the $w^*$-closure of the convex hull of $\{\Phi_t : t \in \mathfrak{T}\}$.*

Point 3. immediately implies that $\mathcal{E}(B(\mathfrak{h})) = \mathcal{F}(\mathbf{\Phi})$ and that $\mathcal{E}$ inherits complete positivity from the semigroup $\mathbf{\Phi}$. Semigroups of quantum Markov maps always admit an ergodic projection.

**Theorem 2.3.14** (Theorem 2.4, [2])**.** *Let $\mathbf{\Phi}$ be a semigroup of quantum Markov maps. There exists a completely positive projection $\mathcal{E} : B(\mathfrak{h}) \to B(\mathfrak{h})$ such that $\mathcal{E}(B(\mathfrak{h})) = \mathcal{F}(\mathbf{\Phi})$ and there exists a net of times $(t_\alpha) \subseteq \mathfrak{T}$ such that*

$$\lim_{t_\alpha \to +\infty} \frac{1}{t_\alpha} \int_0^{t_\alpha} \Phi_s dm(s) = \mathcal{E}$$

*in the pointwise $w^*$-topology.*

At least for all enclosures $\mathcal{V}$, we know that $\{\Phi_t(p_{\mathcal{V}})\}_{t \in \mathfrak{T}}$ converges monotonically to $A(\mathcal{V})$, so the net of Cesaro's means will have the same limit and $\mathcal{E}(p_{\mathcal{V}}) = A(\mathcal{V})$.

Under the assumption $A(\mathcal{R}) = \mathbf{1}$, the set of fixed points of $\mathbf{\Phi}$ is controlled by the set of fixed points of the restricted semigroup $\mathbf{\Phi}^{\mathcal{R}}$.

The content of the following proposition partially appears also in [49, Theorem 3, Corollary 4] and [37, Proposition 8.2, Theorem 8.3], under the stronger (see Remark 2.3.22) assumption that $A(\mathcal{R}_+) = \mathbf{1}$. Our proof is adapted to our setting and makes use of the multiplicative domain of $\mathcal{E}$.

**Proposition 2.3.15.** *Suppose that $A(\mathcal{R}) = \mathbf{1}$. Then*

1. *if $y \in \mathcal{F}(\mathbf{\Phi})$ is such that $p_{\mathcal{R}} y p_{\mathcal{R}} = 0$, then $y = 0$;*
2. *$p_{\mathcal{R}} \mathcal{F}(\mathbf{\Phi}) p_{\mathcal{R}} = \mathcal{F}(\mathbf{\Phi}^{\mathcal{R}})$;*
3. *$\mathcal{F}(\mathbf{\Phi})$ and $\mathcal{F}(\mathbf{\Phi}^{\mathcal{R}})$ are isomorphic as Banach spaces; the isomorphism is given by the restriction $\tilde{\mathcal{E}}$ of $\mathcal{E}$ to $\mathcal{F}(\mathbf{\Phi}^{\mathcal{R}})$ and $\mathcal{E}(p_{\mathcal{R}} x p_{\mathcal{R}}) = \mathcal{E}(x) = x$ for any fixed point $x$.*

We remark that this result is true also for weaker positivity conditions on the semigroup, that is assuming 2-positivity instead of complete positivity.

*Proof.* Let us consider the operators

$$\begin{aligned} p_{\mathcal{R}} \cdot p_{\mathcal{R}} : \mathcal{F}(\mathbf{\Phi}) &\to B(\mathfrak{h}) & \tilde{\mathcal{E}} : \mathcal{F}(\mathbf{\Phi}^{\mathcal{R}}) &\to \mathcal{F}(\mathbf{\Phi}) \\ x &\mapsto p_{\mathcal{R}} x p_{\mathcal{R}}, & x &\mapsto \mathcal{E}(x). \end{aligned} \tag{2.20}$$

Both of them are norm continuous and we shall prove that $p_{\mathcal{R}}\mathcal{F}(\mathbf{\Phi})p_{\mathcal{R}} \subseteq \mathcal{F}(\mathbf{\Phi}^{\mathcal{R}})$ and that $p_{\mathcal{R}} \cdot p_{\mathcal{R}} = \tilde{\mathcal{E}}^{-1}$, in order to obtain statement 3.. First, for any fixed point $y$ of $\mathbf{\Phi}$, we have $\Phi_t^{\mathcal{R}}(p_{\mathcal{R}} y p_{\mathcal{R}}) = p_{\mathcal{R}} \Phi_t(y) p_{\mathcal{R}} = p_{\mathcal{R}} y p_{\mathcal{R}}$; so $p_{\mathcal{R}} y p_{\mathcal{R}}$ is a fixed point for $\mathbf{\Phi}^{\mathcal{R}}$ and $p_{\mathcal{R}}\mathcal{F}(\mathbf{\Phi})p_{\mathcal{R}} \subseteq \mathcal{F}(\mathbf{\Phi}^{\mathcal{R}})$.

$\mathcal{E}(p_{\mathcal{R}}) = A(\mathcal{R}) = \mathbf{1}$, hence $p_{\mathcal{R}} \in \mathcal{M}(\mathcal{E})$ (Theorem 1.2.17) and for any bounded operator $x$

$$\mathcal{E}(p_{\mathcal{R}} x p_{\mathcal{R}}) = \mathcal{E}(p_{\mathcal{R}})\mathcal{E}(x)\mathcal{E}(p_{\mathcal{R}}) = \mathcal{E}(x).$$

When $x$ is a fixed point for $\mathbf{\Phi}$, the previous implies that $\mathcal{E}(p_{\mathcal{R}} x p_{\mathcal{R}}) = \mathcal{E}(x) = x$, which means that $\mathcal{E} \circ (p_{\mathcal{R}} \cdot p_{\mathcal{R}}) = \mathrm{Id}_{\mathcal{F}(\mathbf{\Phi})}$ and that $p_{\mathcal{R}} \cdot p_{\mathcal{R}}$ is injective (1. is proved).

Take now $y = p_{\mathcal{R}} y p_{\mathcal{R}}$ an element in $\mathcal{F}(\mathbf{\Phi}^{\mathcal{R}})$, then

$$p_{\mathcal{R}}\mathcal{E}(y)p_{\mathcal{R}} = w^* - \lim_{t_\alpha \to +\infty} \frac{1}{t_\alpha} \int_0^{t_\alpha} p_{\mathcal{R}} \Phi_s(y) p_{\mathcal{R}} dm(s) = y,$$

where we used the fact that $p_{\mathcal{R}} \cdot p_{\mathcal{R}}$ is $w^*$-continuous. Therefore we proved that $(p_{\mathcal{R}} \cdot p_{\mathcal{R}}) \circ \mathcal{E} = \mathrm{Id}_{\mathcal{F}(\mathbf{\Phi}^{\mathcal{R}})}$ and also in particular point 2.. This concludes the proof. □

We can now give a characterization of the harmonic operators in terms of absorption operators when the recurrent subspace is attractive and prove that the ergodic limit exists for all the fixed points of the restricted semigroup.

**Theorem 2.3.16.** *Suppose $A(\mathcal{R}) = \mathbf{1}$. Then the following facts hold:*

1. *the fixed points are spanned by absorption operators, and more precisely*

$$\overline{\mathrm{span}\{A(\mathcal{V})\}}^{\|\,\|}_{\{\mathcal{V} \subseteq \mathcal{R}\ \textit{enclosure}\}} = \mathcal{F}(\mathbf{\Phi}) = \mathcal{F}(\mathbf{\Phi}^{\mathcal{R}_+}) \oplus \mathcal{F}(\mathbf{\Phi}^{\mathcal{R}_0}) \oplus p_{\mathcal{T}}\mathcal{F}(\mathbf{\Phi})p_{\mathcal{T}};$$

2. *for every $x \in \mathcal{F}(\mathbf{\Phi}^{\mathcal{R}})$, there exists the limit $w^* - \lim_{t \to +\infty} \Phi_t(x) = \mathcal{E}(x)$.*

3. *A recurrent enclosure $\mathcal{W}$ is minimal if and only if for every $x \in \mathcal{F}(\mathbf{\Phi})$ such that $0 \leq x \leq A(\mathcal{W})$, there exists $\lambda \in [0,1]$ such that $x = \lambda A(\mathcal{W})$.*

We point out that, in the case when $\mathbf{\Phi}$ is recurrent, for every enclosure $\mathcal{V}$ one has $A(\mathcal{V}) = p_{\mathcal{V}}$ ($p_{\mathcal{V}}$ is harmonic by Corollary 2.2.6) and $\mathcal{T} = \{0\}$, hence Point 1. of Theorem 2.3.16 becomes exactly Corollary 2.3.3 and Corollary 2.3.4.

*Proof.* 1. Since $\mathcal{F}(\mathbf{\Phi}^{\mathcal{R}})$ is a $W^*$-algebra, it is the norm-closure of the space spanned by its projections (Corollary 1.1.9), which are exactly the projections onto the positive recurrent enclosures of $\mathbf{\Phi}$ by virtue of Corollary 2.2.6, so $\mathcal{F}(\mathbf{\Phi}) = \tilde{\mathcal{E}}(\mathcal{F}(\mathbf{\Phi}^{\mathcal{R}}))$ is the norm-closure of the space

$$\mathrm{span}\, \tilde{\mathcal{E}}\{p \in \mathcal{F}(\mathbf{\Phi}^{\mathcal{R}}), p \text{ projection}\} = \mathrm{span}\{A(\mathcal{V}), \mathcal{V} \text{ rec. enclosure}\}$$

and this directly implies the first equality in the thesis. The second equality follows from the diagonal structure of $\mathcal{F}(\mathbf{\Phi}^{\mathcal{R}})$ proved in Corollary 2.3.4 and the block structure of the absorption operators proven in Theorem 2.2.7.

2. It is enough to prove that the limit is well defined for positive elements of $\mathcal{F}(\mathbf{\Phi}^{\mathcal{R}})$ since it is a $W^*$-algebra. Let us, then, consider a positive $x \in \mathcal{F}(\mathbf{\Phi}^{\mathcal{R}})$, then $x = \lim_K x^{(K)}$ (in operator norm), for some increasing sequence $(x^{(K)})_K$, with $x^{(K)} = \sum_{k\in J_K} x_k p_{\mathcal{V}_k}$ for some finite set of indices $J_K$. Then

$$\mathcal{E}(x^{(K)}) = \sum_{k\in J_K} x_k A(\mathcal{V}_k) = w^* - \lim_{t\to+\infty} \Phi_t(x^{(K)}), \quad \text{and} \quad \lim_K \mathcal{E}(x^{(K)}) = \mathcal{E}(x),$$

where the last equality is due to the boundedness of $\mathcal{E}$. Then, for any operator $\sigma$ in $L^1(\mathfrak{h})$ and for all $K$,

$$\begin{aligned} |\mathrm{tr}\,(\sigma(\Phi_t(x) - \mathcal{E}(x)))| &\le |\mathrm{tr}\left(\sigma(\Phi_t - \mathcal{E})(x^{(K)})\right)| + |\mathrm{tr}\left(\sigma(\Phi_t - \mathcal{E})(x - x^{(K)})\right)| \\ &\le |\mathrm{tr}\left(\sigma(\Phi_t - \mathcal{E})(x^{(K)})\right)| + 2\|\sigma\|_1 \|x - x^{(K)}\|_\infty, \end{aligned}$$

so we can easily conclude.

3. If the recurrent enclosure $\mathcal{W}$ is not minimal, then there exists another non-null enclosure $\mathcal{V} \subsetneq \mathcal{W}$ and $A(\mathcal{V})$ is a fixed point, $0 \le A(\mathcal{V}) \le A(\mathcal{W})$, but not proportional to $A(\mathcal{W})$, since $\mathcal{W} \cap \mathcal{V}^\perp$ is again a non-null enclosure, $\mathcal{W} \cap \mathcal{V}^\perp \subseteq \ker(A(\mathcal{V}))$, while $p_{\mathcal{W}\cap\mathcal{V}^\perp} A(\mathcal{W}) p_{\mathcal{W}\cap\mathcal{V}^\perp} = p_{\mathcal{W}\cap\mathcal{V}^\perp} \neq 0$.

On the other hand, let $\mathcal{W}$ be a minimal recurrent enclosure and suppose there exists a fixed point $x$ such that $0 \le x \le A(\mathcal{W})$; then either $x = 0$ or $0 < p_{\mathcal{R}} x p_{\mathcal{R}} = p_{\mathcal{W}} x p_{\mathcal{W}} \le p_{\mathcal{W}}$. Being $\mathbf{\Phi}^{\mathcal{W}}$ recurrent and irreducible and $p_{\mathcal{W}} x p_{\mathcal{W}}$ a fixed point of the reduced semigroup, there must be a $\lambda \in (0,1]$ such that $p_{\mathcal{W}} x p_{\mathcal{W}} = \lambda p_{\mathcal{W}}$. Hence $x = \mathcal{E}(p_{\mathcal{W}} x p_{\mathcal{W}}) = \lambda A(\mathcal{W})$. ☐

In Corollary 2.2.11 we proved that in the recurrent case subharmonic projections commute with the projections on fast and null recurrent subspaces; now we can improve the result under the assumption $A(\mathcal{R}) = \mathbf{1}$.

**Proposition 2.3.17.** *Assume $A(\mathcal{R}) = \mathbf{1}$ and let $\mathcal{V}$ be an enclosure, then*

$$\mathcal{V} = (\mathcal{R}_+ \cap \mathcal{V}) \oplus (\mathcal{R}_0 \cap \mathcal{V}) \oplus (\mathcal{T} \cap \mathcal{V})$$

*where $\mathcal{R} \cap \mathcal{V}$ is a recurrent enclosure and $p_{\mathcal{T}\cap\mathcal{V}} \le A(\mathcal{R}\cap\mathcal{V}) - p_{\mathcal{R}\cap\mathcal{V}}$.*
*Moreover, when $\mathcal{V}$ is minimal, it is either positive or null recurrent and $A(\mathcal{V})$ is the unique harmonic operator such that*

$$p_{\mathcal{V}} A(\mathcal{V}) p_{\mathcal{V}} = p_{\mathcal{V}} \qquad p_{\mathcal{R}\cap\mathcal{V}^\perp} A(\mathcal{V}) p_{\mathcal{R}\cap\mathcal{V}^\perp} = 0.$$

*Proof.* Consider the block decomposition of $B(\mathfrak{h})$ induced by $\mathcal{R}_+$, $\mathcal{R}_0$, $\mathcal{T}$; $A(\mathcal{V}) - p_{\mathcal{V}}$ and $A(\mathcal{V})$ are diagonal because of points 1 of Theorems 2.2.7 and 2.3.16 respectively; therefore $p_{\mathcal{V}}$ is diagonal too. Since $\mathcal{V}$ and $\mathcal{R}$ are enclosures, $\mathcal{R}\cap\mathcal{V}$ is an enclosure too. Notice that

$$A(\mathcal{R}\cap\mathcal{V}) = \mathcal{E}(p_{\mathcal{R}} p_{\mathcal{V}} p_{\mathcal{R}}) = \mathcal{E}(p_{\mathcal{R}})\mathcal{E}(p_{\mathcal{V}})\mathcal{E}(p_{\mathcal{R}}) = \mathcal{E}(p_{\mathcal{V}}) = A(\mathcal{V})$$

and hence $A(\mathcal{R}\cap\mathcal{V}) = A(\mathcal{V}) \ge p_{\mathcal{V}} = p_{\mathcal{R}\cap\mathcal{V}} + p_{\mathcal{T}\cap\mathcal{V}}$. Therefore $A(\mathcal{R}\cap\mathcal{V}) - p_{\mathcal{R}\cap\mathcal{V}} \ge p_{\mathcal{T}\cap\mathcal{V}}$ and if $\mathcal{V} \neq \{0\}$, $\mathcal{R}\cap\mathcal{V}$ cannot be $\{0\}$, otherwise $0 = A(\mathcal{R}\cap\mathcal{V}) = A(\mathcal{V}) \ge p_{\mathcal{V}}$. Hence, if $\mathcal{V}$ is minimal, it must be $\mathcal{V} = \mathcal{R}\cap\mathcal{V}$ and the conclusion follows by point 2 of Theorem 2.2.7. ☐

*Remark* 2.3.18. The second point in the previous proposition is particularly useful from a computational viewpoint, especially when there is not an analytic way to compute the limit defining absorption operators (see for instance Example 2.4.1). In the parallel with the classical case, already described in Remark 2.2.8, we can now add that the solution of the system (2.11) is unique. Even if it can be not immediate to recognize, similar forms of this result for particular models already appeared in [19, Proposition 7] for finite dimensional quantum systems.

Moreover Proposition 2.3.15 shows that every fixed point $x \in \mathcal{F}(\mathbf{\Phi})$ is completely determined by its recurrent restriction $p_{\mathcal{R}} x p_{\mathcal{R}}$; in addition Theorem 2.3.16 proves that $x = p_{\mathcal{R}} x p_{\mathcal{R}} + p_{\mathcal{T}} x p_{\mathcal{T}}$. It is therefore possible to express $p_{\mathcal{T}} x p_{\mathcal{T}}$, and consequently $x$, as a function of $p_{\mathcal{R}} x p_{\mathcal{R}}$: there exists a superoperator

$$\Psi : \mathcal{F}(\mathbf{\Phi}^{\mathcal{R}}) \to p_{\mathcal{T}} \mathcal{F}(\mathbf{\Phi}) p_{\mathcal{T}}$$

such that, for every $x \in \mathcal{F}(\mathbf{\Phi})$, we can write

$$p_{\mathcal{T}} x p_{\mathcal{T}} = \Psi(p_{\mathcal{R}} x p_{\mathcal{R}}). \tag{2.21}$$

Since

$$p_{\mathcal{T}} x p_{\mathcal{T}} = \mathcal{E}(p_{\mathcal{R}} x p_{\mathcal{R}}) - p_{\mathcal{R}} x p_{\mathcal{R}} = p_{\mathcal{T}} \mathcal{E}(p_{\mathcal{R}} x p_{\mathcal{R}}) p_{\mathcal{T}},$$

we can find a first expression for $\Psi$:

$$\Psi(y) = \tilde{\mathcal{E}}(y) - y, \quad y \in \mathcal{F}(\mathbf{\Phi}^{\mathcal{R}}).$$

As $\mathcal{E}$ is not commonly given in a model, it can be useful to find alternative expressions for $\Psi$ using directly the semigroup. If the semigroup is continuous time, we consider the infinitesimal generator $\mathcal{L}$, while, if it is discrete time, we replace it by $\Phi - \mathrm{Id}_{B(\mathfrak{h})}$. For any fixed point $x$, we know that $p_{\mathcal{T}} x p_{\mathcal{T}}$ is the unique solution $y \in p_{\mathcal{T}} B(\mathfrak{h}) p_{\mathcal{T}}$ of

$$\mathcal{L}(y) = -p_{\mathcal{T}} \mathcal{L}(p_{\mathcal{R}} x p_{\mathcal{R}}) p_{\mathcal{T}},$$

due to either Proposition 2.3.15 or 2.3.17. Then the operator $\mathcal{L}$, though not invertible in general, admits a unique inverse image if applied to the right hand side of the previous equation and we can provide a further expression for $\Psi$,

$$\Psi(y) = -\mathcal{L}^{-1}(p_{\mathcal{T}} \mathcal{L}(y) p_{\mathcal{T}}), \quad y \in \mathcal{F}(\mathbf{\Phi}^{\mathcal{R}}),$$

which is the same already found in [1, Proposition 3] for the finite dimensional case.

### 2.3.3 The role of absorption operators in ergodic theory

Absorption operators, by construction, register features of the limit behavior of the semigroup, therefore it is not surprising that they have some explicit relations with ergodic theory. First we need to recall some further results about ergodic projections (for the proofs see Theorem 1 and Theorem 2 in [50]).

**Theorem 2.3.19.** *Let $\mathcal{E}$ be an ergodic projection for the semigroup of quantum Markov maps $\mathbf{\Phi}$. Then there exists a unique decomposition $\mathcal{E} = \mathcal{E}_n + \mathcal{E}_s$ that satisfies*

1. *$\mathcal{E}_n$ and $\mathcal{E}_s$ are completely positive $\mathbf{\Phi}$-invariant projections;*

2. *$\mathcal{E}_n$ is $w^*$-$w^*$-continuous, $\mathcal{E}_s$ is singular (see Definition 1.1.14);*

*3.* $\mathcal{E}_n \circ \mathcal{E}_s = \mathcal{E}_s \circ \mathcal{E}_n = 0$.

*Moreover*

$$\mathcal{F}(\mathbf{\Phi}_*) = \{\omega \circ \mathcal{E}_n : \omega \in L^1(\mathfrak{h})\} \tag{2.22}$$

*and for any two ergodic projections $\mathcal{E}$ and $\mathcal{E}'$, it is true that $\mathcal{E}_n = \mathcal{E}'_n$.*

The last fact of Theorem 2.3.19 allows us to denote $\mathcal{E}_n$ the unique $w^*$-$w^*$-continuous ergodic projection. The following result highlights the relationship between $\mathcal{E}_n$ and absorption operators; with the usual abuse of notation we identify $B(\mathcal{R}_+)$ with $p_{\mathcal{R}_+} B(\mathfrak{h}) p_{\mathcal{R}_+}$.

**Lemma 2.3.20.** *For every $x \in \mathcal{F}(\mathbf{\Phi}^{\mathcal{R}_+})$,*

$$\mathcal{E}_n(x) = \mathcal{E}(x) = w^* - \lim_{t \to +\infty} \Phi_t(x).$$

*In particular, for every positive recurrent enclosure $\mathcal{V} \subseteq \mathcal{R}_+$, $\mathcal{E}_n(p_\mathcal{V}) = A(p_\mathcal{V})$.*

*Proof.* Corollary 2.3.4 applied to $\mathbf{\Phi}^{\mathcal{R}}$ tells us that $\mathcal{F}(\mathbf{\Phi}^{\mathcal{R}_+}) \subseteq \mathcal{F}(\mathbf{\Phi}^{\mathcal{R}})$ and we already know by point 2. of Theorem 2.3.16 that for every $x \in \mathcal{F}(\mathbf{\Phi}^{\mathcal{R}})$, the limit

$$w^* - \lim_{t \to +\infty} \Phi_t(x)$$

exists and is equal to $\mathcal{E}(x)$. Therefore, since $\mathcal{E}_n$ and $\mathcal{E}$ are norm-continuous and $\mathcal{F}(\mathbf{\Phi}^{\mathcal{R}_+})$ is the norm-closure of the linear span of the projection corresponding to positive recurrent enclosures (apply Corollary 2.3.3 to $\mathbf{\Phi}^{\mathcal{R}_+}$), we only need to prove that for every positive recurrent enclosure $\mathcal{V} \subseteq \mathcal{R}_+$, $\mathcal{E}_n(p_\mathcal{V}) = \mathcal{E}(p_\mathcal{V}) = A(\mathcal{V})$.
$\mathcal{E}_n$ is $w^*$-$w^*$-continuous and $\mathbf{\Phi}$-invariant, hence we get

$$\mathcal{E}_n(p_\mathcal{V}) = w^* - \lim_{t \to +\infty} \mathcal{E}_n \circ \Phi_t(p_\mathcal{V}) = \mathcal{E}_n\left(w^* - \lim_{t \to +\infty} \Phi_t(p_\mathcal{V})\right) = \mathcal{E}_n(A(\mathcal{V})),$$

hence we need to show that $\mathcal{E}_n(A(\mathcal{V})) = A(\mathcal{V})$. Notice that, since $A(\mathcal{V}) \in \mathcal{F}(\mathbf{\Phi})$,

$$A(\mathcal{V}) = \mathcal{E}(A(\mathcal{V})) = \mathcal{E}_n(A(\mathcal{V})) + \mathcal{E}_s(A(\mathcal{V})) \tag{2.23}$$

and by the positivity of $\mathcal{E}_s$ we have that $\mathcal{E}_n(A(\mathcal{V})) \leq A(\mathcal{V})$.
The proof of [49, Theorem 5] shows that $\mathcal{E}_s(\mathbf{1}) \leq \mathbf{1} - p_{\mathcal{R}_+}$, hence

$$0 \leq \mathcal{E}_s(A(\mathcal{V})) \leq \mathcal{E}_s(\mathbf{1}) \leq \mathbf{1} - p_{\mathcal{R}_+} \leq \mathbf{1} - p_\mathcal{V},$$

where the last inequality is due to the fact that $\mathcal{V}$ is positive recurrent. Using Lemma 2.1.3, we get that $\mathcal{E}_s(A(\mathcal{V})) = p_{\mathcal{V}^\perp} \mathcal{E}_s(A(\mathcal{V})) p_{\mathcal{V}^\perp}$. Together with equation (2.23) and point 1. of Proposition 2.1.5, this implies that

$$p_\mathcal{V} + p_{\mathcal{V}^\perp} A(\mathcal{V}) p_{\mathcal{V}^\perp} = \mathcal{E}_n(A(\mathcal{V})) + p_{\mathcal{V}^\perp} \mathcal{E}_s(A(\mathcal{V})) p_{\mathcal{V}^\perp},$$

which can be rewritten as

$$0 \leq \mathcal{E}_n(A(\mathcal{V})) = p_\mathcal{V} + p_{\mathcal{V}^\perp} \mathcal{E}_n(A(\mathcal{V})) p_{\mathcal{V}^\perp},$$

hence $\mathcal{E}_n(A(\mathcal{V})) \geq p_\mathcal{V}$. Now recall that $\mathcal{E}_n$ is $\mathbf{\Phi}$-invariant and $\Phi_t$ is positive for every $t \in \mathfrak{T}$, hence

$$\mathcal{E}_n(A(\mathcal{V})) = \Phi_t \circ \mathcal{E}_n(A(\mathcal{V})) \geq \Phi_t(p_\mathcal{V}), \quad \forall t \in \mathfrak{T}$$

and we get that $\mathcal{E}_n(A(\mathcal{V})) \geq A(\mathcal{V})$. □

An immediate consequence is a clean characterization of when $\mathcal{E}_n$ is the unique ergodic projection in terms of the absorption operator of the positive recurrent subspace.

**Corollary 2.3.21.** *$\mathcal{E}_n$ is is the unique ergodic projection if and only if $A(\mathcal{R}_+) = \mathbf{1}$.*

*Proof.* Equation (2.22) and the definition of $\mathcal{R}_+$ imply that $\mathcal{E}_n(\mathbf{1}) = \mathcal{E}_n(p_{\mathcal{R}_+})$ (for details see [50, Corollary 2]). For every ergodic projection $\mathcal{E}$, thank to Lemma 2.3.20, we get

$$\mathbf{1} = \mathcal{E}(\mathbf{1}) = A(\mathcal{R}_+) + \mathcal{E}_s(\mathbf{1}).$$

Since $\mathcal{E}_s$ is positive,

$$\mathcal{E}_s = 0 \text{ if and only if } 0 = \mathcal{E}_s(\mathbf{1}) = \mathbf{1} - A(\mathcal{R}_+)$$

and we are done. □

*Remark* 2.3.22. Corollary 2.3.21 shows that assuming that the positive recurrent space is absorbing, i.e.

$$A(\mathcal{R}_+) = w^* - \lim_{t\to+\infty} \Phi_t(p_{\mathcal{R}_+}) = \mathbf{1}, \tag{H2}$$

is a very natural hypothesis in the study of ergodic theory, and, in fact, it is quite popular and it has been extensively used ( [36, 37]). Moreover it is not too restrictive: for instance, it holds true if $\mathfrak{h}$ is finite dimensional (see [67, Section 6]).

We point out some immediate consequences of condition (H2).

- There exists at least one invariant state, since $\mathcal{R}_+ \neq 0$ (otherwise we would have $A(\mathcal{R}_+) = 0$).
- Similarly as in the commutative case, this assumption implies that there are no null recurrent vectors i.e. $\mathcal{R}_+ = \mathcal{R}$. Indeed, since $\mathcal{R}_0$ is an enclosure and, by definition, $\mathcal{R}_0 \subseteq \mathcal{R}_+^\perp$, for every $t \in \mathfrak{T}$, we have that $\Phi_t(p_{\mathcal{R}_0}) \leq \Phi_t(p_{\mathcal{R}_+^\perp})$ and consequently

  $$0 \leq A(\mathcal{R}_0) = w^* - \lim_{t\to+\infty} \Phi_t(p_{\mathcal{R}_0}) \leq w^* - \lim_{t\to+\infty} \Phi_t(p_{\mathcal{R}_+^\perp}) = 1 - A(\mathcal{R}_+) = 0,$$

  which implies $A(\mathcal{R}_0) = 0$. So $\mathcal{R}_0 = 0$ since $0 \leq p_{\mathcal{R}_0} \leq A(\mathcal{R}_0) = 0$.
  Notice that (H2) is equivalent to assuming that (H1) holds true and $\mathcal{R}_0 = 0$.
- $\mathcal{R}_+$ must be contained in any other attractive projection; indeed consider a closed subspace $Q$, then $\mathcal{E}_n(p_Q) = \mathbf{1}$ if and only if $Q$ contains $\mathcal{R}_+$ (i.e. $\mathcal{R} \subseteq Q$). Notice that $Q$ is not necessarily an enclosure. One implication is obvious, for the other, just notice that, if $\mathcal{E}_n(p_Q) = \mathbf{1}$, then, for any invariant density $\sigma$, we have $\sigma = \mathcal{E}_{n*}(\sigma)$ so

  $$\mathbf{1} = \operatorname{tr}(\sigma) = \operatorname{tr}(\sigma\mathcal{E}_n(p_Q)) = \operatorname{tr}(\sigma Q);$$

  and this implies that $\mathcal{R}_+ \subseteq Q$ because $\mathcal{R}_+$ is the supremum of all supports of invariant densities.

As pointed out already in [36], there is a wide class of quantum Markov semigroups for which checking the validity of (H1) or (H2) reduces to an analogous problem for a classical Markov chains.

We are ready to state an ergodic theorem that generalizes [36, Theorem 2.1] (for semigroups acting on $B(\mathfrak{h})$). We recall that $L^1(\mathfrak{h})$ is isomorphic to the topological dual of the Banach space of compact operators $C(\mathfrak{h})$, hence the $w^*$-topology on $L^1(\mathfrak{h})$ is the smallest topology with respect to which $C(\mathfrak{h})$ are continuous linear functionals on $L^1(\mathfrak{h})$ (it is weaker than the $w$-topology).

**Theorem 2.3.23.** *Let $\mathbf{\Phi}$ be a semigroup of quantum Markov maps; then we have that*

$$w^* - \lim_{t\to+\infty} \frac{1}{t}\int_0^t \Phi_{*s}(\rho)dm(s)$$

*exists for every normal state $\rho \in L^1(\mathfrak{h})$ and it is equal to $\mathcal{E}_{n*}(\rho)$. In general $\mathcal{E}_{n*}(\rho)$ is not a state and* $\mathrm{tr}\,(\mathcal{E}_{n*}(\rho)) = \mathrm{tr}\,(\rho A(\mathcal{R}_+))$.

*The following are equivalent:*

*(i)* $A(\mathcal{R}_+) = \mathbf{1}$*;*

*(ii) $\mathcal{E}_n$ is the unique ergodic projection;*

*(iii)* $w - \lim \frac{1}{t}\int_0^t \Phi_{*s}(\phi)dm(s)$ *exists for every $\phi \in L^1(\mathfrak{h})$.*

*If the above conditions are satisfied, then $\mathcal{E}_n$ is given by*

$$\mathcal{E}_n(x) := w^* - \lim_{t\to\infty} \frac{1}{t}\int_0^t \Phi_s(x)\, dm(s).$$

*Proof.* The net $\left(\frac{1}{t}\int_0^t \Phi_s dm(s)\right)$ is compact in the pointwise $w^*$-topology and every accumulation point $\mathcal{E}$ is an ergodic projection (see [2, Theorem 2.4,]), hence, in order to prove the first part of the theorem, it is enough to notice that $\mathcal{E}_{|C(\mathfrak{h})} = \mathcal{E}_{n|C(\mathfrak{h})}$, since $\mathcal{E}_{s|C(\mathfrak{h})} = 0$: indeed, for every $x \in C(\mathfrak{h})$, for every $\phi \in L^1(\mathfrak{h})$, by definition of singular operator we have that

$$\phi \circ \mathcal{E}_s \in B(\mathfrak{h})_\star^\perp = C(\mathfrak{h})^\perp,$$

hence $\phi(\mathcal{E}_s(x)) = 0$ and this implies that $\mathcal{E}_s(x) = 0$.
By Lemma 2.3.20, for every normal state $\rho$

$$\mathrm{tr}\,(\mathcal{E}_{n*}(\rho)) = \mathrm{tr}\,(\rho\mathcal{E}_n(\mathbf{1})) = \mathrm{tr}\,\left(\rho\mathcal{E}_n(p_{\mathcal{R}_+})\right) = \mathrm{tr}\,(\rho A(\mathcal{R}_+))\,.$$

The equivalence between $(i)$ and $(ii)$ is the content of Corollary 2.3.21, while the equivalence between $(i)$ and $(iii)$ is proved in [36, Theorem 2.1]. □

Since for every normal state $\phi$, $\mathcal{E}_{n*}(\phi)$ is an invariant positive functional for $\mathbf{\Phi}$, it will be written in the form of equation (2.18), i.e.

$$\mathcal{E}_{n*}(\phi) = \sum_{\alpha\in A} \lambda_\alpha(\phi)U_\alpha^*(\omega_\alpha(\phi)\otimes\rho_\alpha)U_\alpha,$$

for some $\omega_\alpha(\phi)$ state on $\mathfrak{h}_\alpha^{(1)}$, and real numbers $\lambda_\alpha(\phi) \geq 0$ with $\sum_{\alpha\in A}\lambda_\alpha(\phi) = \mathrm{tr}\,(\phi A(\mathcal{R}_+))$. The coefficients $\omega_\alpha(\phi)$, $\lambda_\alpha(\phi)$, $\alpha \in A$, are immediately identified knowing absorption operators and the operators $\mathcal{E}_n(Q_{\alpha,\beta,\beta'}) = w^* - \lim_{t\to+\infty}\Phi_t(Q_{\alpha,\beta,\beta'})$. Indeed the mass flowing to the enclosure $\chi_\alpha$ (introduced in Proposition 2.3.5) is

$$\lambda_\alpha(\phi) = \mathrm{tr}\,(\mathcal{E}_{n*}(\phi)U_\alpha^*U_\alpha) = \mathrm{tr}\,(\mathcal{E}_{n*}(\phi)p_{\chi_\alpha}) = \mathrm{tr}\,(\phi A(\chi_\alpha))$$

and, introducing as before an orthonormal basis $\{e^\alpha_\beta\}_{\beta\in I_\alpha}$ for $\mathfrak{h}^{(1)}_\alpha$, we can represent $\omega_\alpha(\phi)=\sum_{\beta,\beta'}(\omega_\alpha(\phi))_{\beta\beta'}|e_\beta\rangle\langle e_{\beta'}|$, where

$$\lambda_\alpha(\phi)(\omega_\alpha(\phi))_{\beta\beta'} \quad = \quad \mathrm{tr}\left(\phi\mathcal{E}_n(U^*_\alpha(|e_\beta\rangle\langle e_{\beta'}|\otimes \mathbf{1}_{\mathfrak{h}^{(2)}_\alpha})U_\alpha)\right)=\mathrm{tr}\left(\phi\mathcal{E}_n(Q_{\alpha,\beta,\beta'})\right);$$

since $\omega_\alpha(\phi)$ is meaningful only when $\lambda_\alpha(\phi)\neq 0$, the previous completely identifies $\omega_\alpha(\phi)$.

Passing to the Heisenberg picture and considering $x\in B(\mathfrak{h})$, we shall similarly obtain an expression for $\mathcal{E}_n$ of the form

$$\mathcal{E}_n(x)=\sum_{\alpha\in A}\sum_{\beta,\beta'}\mathrm{tr}\left(U^*_\alpha(|e^\alpha_\beta\rangle\langle e^\alpha_{\beta'}|\otimes\rho^\alpha)U_\alpha p_{\mathcal{R}_+}xp_{\mathcal{R}_+}\right)\ \mathcal{E}_n(Q_{\alpha,\beta,\beta'})$$

(once again it is evident that $\mathcal{E}_n(x)$ depends only on $p_{\mathcal{R}_+}xp_{\mathcal{R}_+}$). In the easiest case, when there is a unique decomposition in orthogonal minimal enclosures of the fast recurrent subspace, the isometries $U_\alpha$ are trivial and the invariant states are convex combinations of the invariant states $\rho_\alpha$ supported in the minimal enclosures, with weights $\mathrm{tr}\,(\phi A(\chi_\alpha))$; thus we have

$$\mathcal{E}_{n*}(\phi)=\sum_{\alpha\in A}\mathrm{tr}\left(\phi A(\chi_\alpha)\right)\rho_\alpha.$$

*Remark* 2.3.24. Theorem 2.3.23 is a generalization of the following well known fact for classical Markov chains on a discrete state space $E$ (we recover the same notation we used in the introduction and at the beginning of Section 2.1): consider the unique decomposition of the set of positive recurrent states into communication classes $\mathcal{R}_+=\bigcup_{\alpha\in A}C_\alpha$ and denote by $\pi_\alpha$ the unique invariant density supported on $C_\alpha$. Then for every probability density $\nu\in\ell^1(E)$ and for every state $x\in E$,

$$\lim_{n\to+\infty}\{\nu\cdot P^n(x)=\mathbb{P}(X_n=x|X_0\sim\nu)\}=\pi(\nu)(x) \tag{2.24}$$

where $\pi(\nu)=\sum_{\alpha\in A}\mathbb{E}_\nu[A(C_\alpha)]\pi_\alpha$ and $\mathbb{E}_\nu[A(C_\alpha)]:=\sum_{y\in E}\nu(y)A(C_\alpha)_y$. Moreover the following limit

$$\lim_{n\to+\infty}\mathbb{E}[f(X_n)|X_0\sim\nu]$$

exists for every $\nu\in\ell^1(E)$ and $f\in\ell^\infty(E)$ if and only if $\pi(\nu)$ is a probability measure for every intial probability density $\nu$.

The Banach space corresponding to $C(\mathfrak{h})$ in the commutative setting is $c(E)\subset\ell^\infty(E)$, defined as the closure with respect to the uniform norm of the set of the bounded functions which are different from zero only on finitely many elements of $E$ and one has

$$c(E)^\star\simeq\ell^1(E).$$

Therefore, the previous result can be rephrased in a functional analytical flavour that brings it in a form closer to Theorem 2.3.23: for every probability density $\nu\in\ell^1(E)$,

$$w^*-\lim_{n\to+\infty}\nu\cdot P^n=\pi(\nu).$$

Moreover the following limit

$$w-\lim_{n\to+\infty}\nu\cdot P^n$$

exists for every $\nu \in \ell^1(E)$ if and only if $\pi(\nu)$ is a probability measure for every intial probability density $\nu$.

**Quantum trajectories.** If we look at quantum trajectories and at the pathwise versions of the ergodic theorems ( [47]), we shall obtain similar relations, but absorption operators register a someway "mean" behavior anyway. For simplicity, assume here the time set to be discrete, i.e. $\mathfrak{T} = \mathbb{N}$, and the semigroup to be generated by a quantum channel $\Phi$. The same ideas will work for the continuous time case. We consider an associated state valued stochastic process $(\Theta_n)_n$ which is a homogeneous Markov process describing the quantum trajectories (we follow the same notations as in [47]). Once chosen a set of Kraus operators $\{V_i\}_{i\in I}$ for $\Phi$, every initial state $\phi \in L^1(\mathfrak{h})$ uniquely determines a probability measure on the set $\Omega = I^\infty$ through the condition

$$\mathbb{P}_\phi(\{\omega \in \Omega : w_1 = i_1, \omega_2 = i_2, \ldots, \omega_n = i_n\}) = \mathrm{tr}\left(V_{i_n} \cdots V_{i_1} \phi V_{i_1}^* \cdots V_{i_n}^*\right).$$

The Markov chain $(\Theta_n)$ is defined in the following way:

$$\Theta_0 = \phi, \quad \Theta_n(\omega) = \frac{V_{\omega_n} \cdots V_{\omega_1} \phi V_{\omega_1}^* \cdots V_{\omega_n}^*}{\mathrm{tr}\left(V_{\omega_n} \cdots V_{\omega_1} \phi V_{\omega_1}^* \cdots V_{\omega_n}^*\right)},$$

and it verifies by construction that, for every $n < m$, $n, m \in \mathbb{N}$, $\mathbb{E}_\phi[\Theta_m | \mathcal{F}_n] = \Phi_*^{m-n}(\Theta_n)$, where $(\mathcal{F}_n)_{n\in\mathbb{N}}$ is the filtration generated by the coordinate process. If $\mathfrak{h}$ is finite dimensional (hence $A(\mathcal{R}_+) = \mathbf{1}$), Kummerer and Maassen proved that

$$\lim_{N\to+\infty} \frac{1}{N} \sum_{k=0}^{N-1} \Theta_k = \Theta_\infty \quad \mathbb{P}_\phi \text{ a. s.}$$

where $\Theta_\infty$ is a random equilibrium state such that $\mathbb{E}_\phi[\Theta_\infty] = \lim_{N\to+\infty} \frac{1}{N} \sum_{k=0}^{N-1} \Phi_*^k(\Theta_0)$. So, for any enclosure $\mathcal{V}$ and state $\phi \in L^1(\mathfrak{h})$ we have

$$\mathrm{tr}\,(\phi A(\mathcal{V})) = \mathrm{tr}\left(\lim_{N\to+\infty} \frac{1}{N} \sum_{k=0}^{N-1} \int_\Omega \Theta_k(\omega) d\mathbb{P}_\phi(\omega) \mathcal{V}\right) = \mathbb{E}_\phi[\mathrm{tr}\,(\Theta_\infty \mathcal{V})]. \tag{2.25}$$

Being a random invariant state for $\Phi$, $\Theta_\infty$ is of the form

$$\sum_{\alpha\in A} \lambda_\alpha U_\alpha^* (\omega_\alpha \otimes \rho_\alpha) U_\alpha,$$

with $\lambda_\alpha$ and $\omega_\alpha$ random variables such that $\lambda_\alpha \geq 0$, $\sum_{\alpha\in A} \lambda_\alpha = 1$ and $\omega_\alpha \in L^1(\mathfrak{h}_\alpha^{(1)})$. Moreover computations as in 2.25 show that

$$\mathrm{tr}\,(\phi A(\chi_\alpha)) = \mathbb{E}_\phi[\lambda_\alpha], \qquad \mathrm{tr}\,(\phi \mathcal{E}_{n*}(Q_{\alpha,\beta,\beta'})) = \mathbb{E}_\phi[\lambda_\alpha (\omega_\alpha)_{\beta\beta'}]$$

where again $(\omega_\alpha)_{\beta\beta'}$ are the matrix entries of $\omega_\alpha$.

### 2.3.4 Multiplicative properties of fixed points

If the fixed points space $\mathcal{F}(\mathbf{\Phi})$ is an algebra, there are some constraints on absorption operators which have a nice probabilistic interpretation.

**Proposition 2.3.25.** *If $\mathcal{F}(\mathbf{\Phi})$ is an algebra, the following facts hold true:*

1. *for every enclosure $\mathcal{V}$, $A(\mathcal{V})$ is a projection;*
2. *for every pair of orthogonal enclosures $\mathcal{V}$ and $\mathcal{W}$, the supports of $A(\mathcal{V})$ and $A(\mathcal{W})$ are two orthogonal enclosures.*

By the second point, we can informally say that, when $\mathcal{F}(\mathbf{\Phi})$ is an algebra, no transient state can reach two different orthogonal enclosures.

*Proof.* 1. Since $\mathcal{F}(\mathbf{\Phi})$ is a $W^*$-algebra, $q := 1_{\{1\}}(A(\mathcal{V}))^1$ is a fixed point too. Notice that $0 \leq q \leq A(\mathcal{V}) \leq \mathbf{1}$ and that $p_{\mathcal{V}} q p_{\mathcal{V}} = p_{\mathcal{V}}$, therefore, by point 3 in Proposition 2.1.5, $q = A(\mathcal{V})$ and $A(\mathcal{V})$ is a projection.

2. If $\mathcal{V}$ and $\mathcal{W}$ are orthogonal enclosures, then $A(\mathcal{V}) + A(\mathcal{W}) = A(\mathcal{V} + \mathcal{W})$, which are projections by point 1. If the sum of two projections is again a projection, they must be orthogonal. Since $A(\mathcal{V})$ is a projection, $\mathrm{supp}(A(\mathcal{V}))$ coincides with the eigenspace corresponding to the eigenvalue 1 and is an enclosure by Proposition 2.1.1. The same will hold for $A(\mathcal{W})$. □

The proof of Proposition 2.3.25 points out that 1. implies 2.; if assumption (H1) holds true, we can also prove that 1. and 2. are equivalent and each one of them implies that $\mathcal{F}(\mathbf{\Phi})$ is an algebra. However, in general, we cannot hope to get the same result, since we can easily find an example even in the commutative case where 2. holds, but 1. does not (and hence $\mathcal{F}(\mathbf{\Phi})$ is not an algebra).

**Example 2.3.26.** Consider the Markov chain on $\mathbb{N} \cup \{a_i\}_{i=0}^k$ for some $k \in \mathbb{N}$ with the following transition probabilities:

$$p(a_i, a_i) = 1 \text{ for } 0 \leq i \leq k, \quad p(n, a_0) = \frac{1}{2^{n+2}}, \quad p(n, n+1) = 1 - \frac{1}{2^{n+2}}, \quad n \in \mathbb{N}.$$

The only non trivial enclosures are the singletons $\{a_i\}$ for $0 \leq i \leq k$ ; $A(\{a_i\}) = 1_{\{a_i\}}$ for $i > 0$, while $\mathrm{supp}(A(\{a_0\})) = \{a_0\} \cup \mathbb{N}$, hence 2. holds true. However $A(\{a_0\})$ is not a projection; denoting with $f = 1_{\{a_0\}}$, it is easy to see that for $N \geq 1$:

$$0 \leq P^N \cdot f(0) = \sum_{k=1}^{N} \Pi_{i=1}^{k-1} \left(1 - \frac{1}{2^{i+1}}\right) \frac{1}{2^{k+1}} \leq \sum_{k=1}^{N} \frac{1}{2^{k+1}} \leq \frac{1}{2}.$$

**Proposition 2.3.27.** *Assume that $A(\mathcal{R}) = \mathbf{1}$. If, for every pair of orthogonal enclosures $\mathcal{V}, \mathcal{W} \subseteq \mathcal{R}$, it is true that $A(\mathcal{V})A(\mathcal{W}) = 0$, then $\mathcal{F}(\mathbf{\Phi})$ is an algebra and $\tilde{\mathcal{E}}$ is an isomorphism of $W^*$-algebras.*

*Proof.* For every enclosure $\mathcal{V} \subseteq \mathcal{R}$, $A(\mathcal{V})^2 = A(\mathcal{V})(A(\mathcal{R}) - A(\mathcal{R} - \mathcal{V})) = A(\mathcal{V})$, thus $A(\mathcal{V})$ is a projection. In particular it is a harmonic projection, hence it is in the dechoerence-free subalgebra $\mathcal{N}(\mathbf{\Phi})$. $\mathcal{N}(\mathbf{\Phi})$ is a $W^*$-algebra, so it is $w^*$-closed and, since by Theorem 2.3.16, $\mathcal{F}(\mathbf{\Phi})$ is the $w^*$-closure of the linear span of the absorption operators, $\mathcal{F}(\mathbf{\Phi}) \subseteq \mathcal{N}(\mathbf{\Phi})$. Hence fixed points enjoy the multiplication property (see Theorem 1.2.18), $\Phi_t(x_1^* x_2) = \Phi_t(x_1^*)\Phi_t(x_2) = x_1^* x_2$ for $x_1, x_2 \in \mathcal{F}(\mathbf{\Phi})$, and thus

[1] $1_{\{1\}}(A(\mathcal{V}))$ has to be inteded in the sense of functional calculus.

$\mathcal{F}(\mathbf{\Phi})$ is a $^*$-algebra. Moreover, always for $x_1, x_2 \in \mathcal{F}(\mathbf{\Phi})$, by Proposition 2.3.15 and Theorem 2.3.16, $x_i = p_{\mathcal{R}} x_i p_{\mathcal{R}} + p_{\mathcal{T}} x_i p_{\mathcal{T}} = \tilde{\mathcal{E}}(p_{\mathcal{R}} x_i p_{\mathcal{R}})$ for $i = 1, 2$, and we have

$$\begin{aligned}\tilde{\mathcal{E}}((p_{\mathcal{R}} x_1 p_{\mathcal{R}})(p_{\mathcal{R}} x_2 p_{\mathcal{R}})) &= \tilde{\mathcal{E}}(p_{\mathcal{R}} x_1 x_2 p_{\mathcal{R}}) = \tilde{\mathcal{E}}(p_{\mathcal{R}})\tilde{\mathcal{E}}(x_1 x_2)\tilde{\mathcal{E}}(p_{\mathcal{R}}) \\ &\stackrel{(*)}{=} x_1 x_2 = \tilde{\mathcal{E}}(p_{\mathcal{R}} x_1 p_{\mathcal{R}})\tilde{\mathcal{E}}(p_{\mathcal{R}} x_2 p_{\mathcal{R}})\end{aligned}$$

($(*)$ because $x_1 x_2$ is a fixed point too). Therefore $\tilde{\mathcal{E}}$ preserves multiplication and is an isomorphism of $W^*$-algebras. □

*Remark* 2.3.28. The example in Subsection 2.4.3 shows that when the decomposition in orthogonal minimal enclosures of the fast recurrent space is non-unique, it is not enough to check that the absorption operators are projections only for a chosen decomposition to ensure that the fixed points set is an algebra.

*Remark* 2.3.29. We remark that in [17] some results of the present section were proved for a specific class of quantum Markov semigroups (sometimes called generic semigroups) under the stronger hypotesis that the semigroup is ergodic and not only mean ergodic. The techniques used there are quite different and rely on a link between this special family of quantum models and the associated classical Markov chains, which allows to use classical probability tools.

Proposition 2.3.5 points out that, in the case of a positive recurrent semigroup, the dynamic of some minimal enclosures is strictly related by symmetries of the semigroup $\mathbf{\Phi}$ encoded in the structure of $\mathcal{F}(\mathbf{\Phi})$; the following result shows that, if assumption $A(\mathcal{R}_+) = \mathbf{1}$ holds and $\mathcal{F}(\mathbf{\Phi})$ is an algebra, the symmetries of the reduced positive recurrent dynamic extend to the whole semigroup $\Phi$.

**Proposition 2.3.30.** *If $A(\mathcal{R}_+) = \mathbf{1}$ and $\mathcal{F}(\mathbf{\Phi})$ is an algebra then*

1. *$\mathcal{F}(\mathbf{\Phi})$ is atomic and it is isomorphic as a $W^*$-algebra to $\mathcal{F}(\mathbf{\Phi}^{\mathcal{R}})$;*
2. *its central projections are the absorption operators corresponding to the central projections of $\mathcal{F}(\mathbf{\Phi}^{\mathcal{R}})$;*
3. *let $\mathcal{K}$ be a minimal central projection of $\mathcal{F}(\mathbf{\Phi})$, then there exist Hilbert spaces $\mathfrak{h}^{(1)}$, $\mathfrak{h}^{(2)}$ and a unitary transformation $U : \mathfrak{h}^{(1)} \otimes \mathfrak{h}^{(2)} \to \mathcal{K}$ such that the restriction of the semigroup $\mathbf{\Phi}^{\mathcal{K}}$ acts in the following way:*

$$\Phi_t^{\mathcal{K}}(U(A \otimes B)U^*) = U(A \otimes \mathcal{Q}_t(B))U^* \quad A \in B(\mathfrak{h}^{(1)}), B \in B(\mathfrak{h}^{(2)})$$

   *where $\mathcal{Q}$ is a quantum Markov semigroup acting on $B(\mathfrak{h}^{(2)})$ with a unique invariant state $\rho \in L^1(\mathfrak{h}^{(2)})$ and such that $w^* - \lim_{t \to +\infty} \mathcal{Q}_t(\mathrm{supp}(\rho)) = \mathbf{1}_{\mathfrak{h}^{(2)}}$.*

*With an abuse of notation, we use the same symbol $\mathrm{supp}(\rho)$ for the space and for the corresponding projection.*

In general $\mathcal{F}(\mathbf{\Phi}^{\mathcal{R}_+})$ and $\mathcal{F}(\mathbf{\Phi})$ are not spatially isomorphic.

*Proof.* 1. $\mathcal{F}(\mathbf{\Phi}^{\mathcal{R}})$ is an atomic $W^*$-algebra and it can be proved to be isomorphic to $\mathcal{F}(\mathbf{\Phi})$ as in the proof of Proposition 2.3.27.
2. Central projections are preserved by isomorphism of $W^*$-algebras and we showed that the isomorphism maps projections of $\mathcal{F}(\mathbf{\Phi}^{\mathcal{R}})$ into the corresponding absorption

operators.
3. $p_{\mathcal{K}}\mathcal{F}(\mathbf{\Phi})p_{\mathcal{K}}$ is a type I factor, hence there exist Hilbert spaces $\mathfrak{h}^{(1)}$, $\mathfrak{h}^{(2)}$ and a unitary operator $U : \mathfrak{h}^{(1)} \otimes \mathfrak{h}^{(2)} \to \mathcal{K}$ such that

$$p_{\mathcal{K}}\mathcal{F}(\mathbf{\Phi})p_{\mathcal{K}} = U(B(\mathfrak{h}^{(1)}) \otimes \mathbf{1}_{\mathfrak{h}^{(2)}})U^*.$$

For every $t \in \mathfrak{T}$, $A \in B(\mathfrak{h}^{(1)})$ and $B \in B(\mathfrak{h}^{(2)})$,

$$\Phi_t(U(A\otimes B)U^*) = U(A\otimes \mathbf{1}_{\mathfrak{h}^{(2)}})U^*\Phi_t(U(\mathbf{1}_{\mathfrak{h}^{(2)}}\otimes B)U^*) = \Phi_t(U(\mathbf{1}_{\mathfrak{h}^{(2)}}\otimes B)U^*)U(A\otimes \mathbf{1}_{\mathfrak{h}^{(2)}})U^*.$$

It implies that $\Phi_t(\mathbf{1}_{\mathfrak{h}^{(1)}} \otimes B) \in B(\mathfrak{h}^{(1)}) \otimes \mathbf{1}_{\mathfrak{h}^{(2)}}{}' = \mathbf{1}_{\mathfrak{h}^{(1)}} \otimes B(\mathfrak{h}^{(2)})$ and so $\Phi_t^{\mathcal{K}} = id_{\mathfrak{h}^{(1)}} \otimes \mathcal{Q}_t$ with $\mathcal{Q}$ quantum Markov semigroup acting on $B(\mathfrak{h}^{(2)})$. Since $\mathcal{F}(\mathbf{\Phi})$ is the range of a $w^*$-$w^*$-continuous conditional expectation from a type I factor, $\mathcal{Q}$ admits a unique invariant state $\rho \in L^1(\mathfrak{h}^{(2)})$ (see Lemma 2 in [13]) and it is easy to see that $\mathcal{V} = U(\mathfrak{h}^{(1)} \otimes \operatorname{supp}(\rho))$, where $p_{\mathcal{V}}$ is the central projection of $\mathcal{F}(\mathbf{\Phi}^{\mathcal{R}})$ such that $p_{\mathcal{K}} = A(\mathcal{V})$. Therefore

$$\mathbf{1}_{\mathfrak{h}^{(1)}} \otimes \mathcal{Q}_t(\operatorname{supp}(\rho)) = U^*\Phi_t(p_{\mathcal{V}})U \to_{t\to+\infty} U^*A(\mathcal{V})U = \mathbf{1}_{\mathfrak{h}^{(1)}} \otimes \mathbf{1}_{\mathfrak{h}^{(2)}}$$

and this concludes the proof. □

*Remark* 2.3.31. For every $p_{\mathcal{K}}$ minimal central projection of $\mathcal{F}(\mathbf{\Phi})$, consider $\mathfrak{h}^{(1)}$, $\mathfrak{h}^{(2)}$ as in point 3 of Proposition 2.3.30. For every $w, v \in \mathfrak{h}^{(1)}$, $Q_{w,v} = U^*(|w\rangle\langle v| \otimes \mathbf{1}_{\mathfrak{h}^{(2)}})U$ is a symmetry of the semigroup $\mathbf{\Phi}$, i.e.

$$\Phi_t(Q_{w,v}^* x Q_{w,v}) = Q_{w,v}^*\Phi_t(x)Q_{w,v} \quad \forall t \in \mathfrak{T},\ \forall x \in B(\mathfrak{h}).$$

If we drop the hypothesis that $\mathcal{F}(\mathbf{\Phi})$ is an algebra, many easy examples show that in general the symmetries of the reduced positive recurrent dynamic do not extend to the whole semigroup $\mathbf{\Phi}$ and, consequently, that it is not possible to recover $A(\mathcal{V}_{\alpha,\beta})$ from $A(\mathcal{V}_{\alpha,\beta'})$ (see examples in Section 2.4).

Nevertheless the following result holds. Recall that by definition $Q_{\alpha,\beta,\beta}$ coincides with the projection $p_{\mathcal{V}_{\alpha,\beta}}$.

**Proposition 2.3.32.** *Assume that $A(\mathcal{R}_+) = \mathbf{1}$ and consider the decomposition in equation* (2.17) *and, for any orthonormal system $\{e_\beta^\alpha\}_{\beta\in I_\alpha}$ for $\mathfrak{h}_\alpha^{(1)}$, define $Q_{\alpha,\beta,\beta'} := U_\alpha(|e_\beta^\alpha\rangle\langle e_{\beta'}^\alpha| \otimes \mathbf{1}_{\mathfrak{h}_\alpha^{(2)}})U_\alpha^*$ as before. Then*

$$\mathcal{F}(\mathbf{\Phi}) = \overline{\operatorname{span}\{\mathcal{E}(Q_{\alpha,\beta,\beta'}) : \alpha \in A, \beta, \beta' \in I_\alpha\}}^{w^*}.$$

*In particular, if $\mathcal{W}$ is a positive recurrent minimal enclosure, then $p_{\mathcal{W}} = U_\alpha(|f\rangle\langle f| \otimes \mathbf{1}_{\mathfrak{h}_\alpha^{(2)}})U_\alpha^*$ for some index $j$ and for some $f$ in $\mathfrak{h}_\alpha^{(1)}$ and*

$$A(\mathcal{W}) = \mathcal{E}(\mathcal{W}) \overset{(w^*)}{=} \sum_{i,k} \overline{\langle f, e_\beta^\alpha\rangle}\langle f, e_{\beta'}^\alpha\rangle\mathcal{E}(Q_{\alpha,\beta,\beta'}).$$

*Proof.* The first part easily follows from the normality of $\tilde{\mathcal{E}}$ and from the fact that the space spanned by $\{|e_\beta^\alpha\rangle\langle e_{\beta'}^\alpha|\}_{\beta,\beta'\in I_\alpha}$ is $w^*$-dense in $B(\mathfrak{h}_\alpha^{(1)})$ for any $\alpha \in A$.

Every positive recurrent minimal enclosure $\mathcal{W}$ is contained in a subspace of $\mathfrak{h}_\alpha^{(1)}$ for some index $\alpha \in A$, hence we can drop the index $\alpha$ for simplicity. $\mathcal{W}$ is then trivially of the form $U(|f\rangle\langle f|\otimes \mathbf{1}_{\mathfrak{h}^{(2)}})U^*$ for some $f\in\mathfrak{h}^{(1)}$ and $|f\rangle\langle f|\otimes\mathbf{1}_{\mathfrak{h}^{(2)}} \overset{(w^*)}{=} \sum_{\beta,\beta'}\overline{\langle f,e^\alpha_\beta\rangle}\langle f,e^\alpha_{\beta'}\rangle\,|e^\alpha_\beta\rangle\langle e^\alpha_{\beta'}|\otimes\mathbf{1}_{\mathfrak{h}^{(2)}}$. The second statement follows again by linearity and $w^*$- continuity of $\tilde{\mathcal{E}}$. □

*Remark* 2.3.33. Many results in this chapter can be extended to the case when $\mathbf{\Phi}$ acts on more general $W^*$- algebras. This is not true anyway for instance for Theorem 2.3.23 and for Propositions 2.3.32 and 2.3.30 because the proofs rely on the fact that $\mathcal{F}(\mathbf{\Phi}^{\mathcal{R}_+})$ is atomic, which is a consequence of the existence of a $w^*$-$w^*$-continuous conditional expectation $\mathcal{E}$ from a type I factor onto $\mathcal{F}(\mathbf{\Phi}^{\mathcal{R}_+})$ (see the proof of [66, Theorem 5]).

## 2.4 Examples and applications

In this section we have collected some models where absorption operators can be studied. We searched for processes with various characteristics: continuous or discrete time, finite and infinite dimensional system spaces, different situations for the existence and structure of the positive recurrent projection.

### 2.4.1 Excitation transport in a unitary quantum walk

The physical model discussed in [68] describes the coherent transport of excitation along a finite ring of coupled quantum systems with a sink located at one vertex of the ring which absorbes the excitation; it presents both a non trivial transient subspace and multiple recurrent enclosures, hence it is an interesting example for studying absorption dynamic. Although the simple structure, we shall see that the explicit expression for absorption operators is not easy to write.

The state space $\mathfrak{h}=\mathfrak{h}_P\otimes\mathfrak{h}_C$ is the tensor product of the position space $\mathfrak{h}_P=\mathrm{span}\{e_m : m=-N,\dots,N\}$, for some $N=1,2,3,\dots$, and the internal state or coin space $\mathfrak{h}_C=\mathrm{span}\{u_0,u_1\}$. The graph on which the excitation moves is a finite line with two sinks in $N$ and $-N$. The evolution of the system can be described by the quantum channel

$$\Phi: B(\mathfrak{h})\to B(\mathfrak{h}),\qquad \Phi(x)=\sum_{j=1}^{2}V_j^* x V_j,$$

where $V_1=(|e_{-N}\rangle\langle e_{-N}|+|e_N\rangle\langle e_N|)\otimes\mathbf{1}_C$ and $V_2=S(\mathbf{1}_P\otimes C)$ is the composition of the change of position

$$S=\sum_{m=1-N}^{N-1}|e_{m-1}\rangle\langle e_m|\otimes|u_0\rangle\langle u_0|+|e_{m+1}\rangle\langle e_m|\otimes|u_1\rangle\langle u_1|$$

and the coin toss, which in the basis $\{u_0,u_1\}$ is represented by the following matrix:

$$C=\begin{pmatrix}\rho & \tilde\rho\\ \tilde\rho & -\rho\end{pmatrix},\quad \rho\in(0,1),\quad \tilde\rho=\sqrt{1-\rho^2}.$$

The model described in [68] really specifies only the Kraus operator $V_2$ (the action on the transient space), while the choice of $V_1$ is not uniquely determined.

We can roughly say that, if the system starts from the pure state $u_0\otimes e_k$ [resp.ly $u_1\otimes e_k$], $k\neq\pm N$, it will move to the left [right] with probability $1$. $\rho^2$ represents the probability of keeping the same direction of the previous step, while $\tilde{\rho}^2$ is the probability of changing direction with respect to the previous shift on the lattice.

First, we remark that the recurrent space $\mathcal{R}$ coincides with the subspace $\mathcal{W} := \mathrm{span}\{e_{-N}, e_N\} \otimes \mathfrak{h}_C$. Indeed, by direct computation, we have that, for $\psi \in \mathcal{W}$,

$$\Phi_*^n(|\psi\rangle\langle\psi|) = |\psi\rangle\langle\psi|, \quad \text{for all } n,$$

and this proves both that $\psi$ is recurrent and that $\mathrm{span}\{\psi\}$ is an enclosure. Then $\mathcal{W}$ is an enclosure too and it is contained in $\mathcal{R}$.

Moreover, if $\psi$ is instead in $\mathcal{W}^\perp = \mathrm{span}\{e_{-N+1}, ..., e_{N-1}\} \otimes \mathfrak{h}_C$, then

$$p_{\mathcal{W}}\Phi_*^n(|\psi\rangle\langle\psi|)p_{\mathcal{W}} \geq p_{\mathcal{W}}|V_2^n\psi\rangle\langle V_2^n\psi|p_{\mathcal{W}}, \quad \text{for all } n, \tag{2.26}$$

and the last operator is non null for $n$ big enough. This proves that $\mathcal{R} = \mathcal{W}$: indeed, if we had that $\mathcal{W} \subsetneq \mathcal{R}$, then Theorem 2.2.1 would imply that $\mathcal{R} \cap \mathcal{W}^\perp$ is an enclosure and for any $\psi \in \mathcal{R} \cap \mathcal{W}^\perp$, $\psi \neq 0$ we would have that $p_{\mathcal{W}}\Phi_*^n(|\psi\rangle\langle\psi|)p_{\mathcal{W}} = 0$ for every $n \in \mathbb{N}$, which contradicts equation (2.26).

Since the model is finite dimensional, there is no null recurrent subspace and the (fast) recurrent space is attractive, i.e. $A(\mathcal{R}_+) = \mathbf{1}$. As mentioned above, the action of the quantum channel on $\mathcal{R}$ is trivial, hence any one dimensional subspace of $\mathcal{R}$ is an enclosure. These are the only minimal enclosures of the channel (but not the only non trivial enclosures).

In this physical model an explicit formula for the absorption operators for every dimension and every enclosure of the chain seems hard to be found. However, there is a general way of computing them: given an enclosure $\mathcal{V}$, we know that $A(\mathcal{V}) = A(\mathcal{R} \cap \mathcal{V}) = p_{\mathcal{R}}p_{\mathcal{V}}p_{\mathcal{R}} + p_{\mathcal{T}}A(\mathcal{V})p_{\mathcal{T}}$ is the only fixed point of the channel with this structure (Theorem2.2.7) so, recalling Remark 2.3.18, $p_{\mathcal{T}}A(\mathcal{V})p_{\mathcal{T}}$ is the unique solution $x \in B(\mathcal{T})$ of the linear system

$$x - p_{\mathcal{T}}V_2^*xV_2p_{\mathcal{T}} = p_{\mathcal{T}}V_2^*p_{\mathcal{V}}V_2p_{\mathcal{T}}. \tag{2.27}$$

Notice that the latter is a discrete Lyapunov equation and that the operator $V_2$ is a "sparse matrix".

For some particular enclosures, determining the absorption operators is immediate. Indeed, we can easily see that

$$\begin{aligned} &A(|e_{-N}\rangle\langle e_{-N}| \otimes |u_1\rangle\langle u_1|) = |e_{-N}\rangle\langle e_{-N}| \otimes |u_1\rangle\langle u_1| \text{ and} \\ &A(|e_N\rangle\langle e_N| \otimes |u_0\rangle\langle u_0|) = |e_N\rangle\langle e_N| \otimes |u_0\rangle\langle u_0|, \end{aligned}$$

hence such enclosures do not collect any mass coming from the transient space.

In general, equation 2.27 gives us an easy inductive procedure to find the solution, but, as we wrote before, giving the precise formula of the absorption operators is complicated. It is anyway immediate to see that, if we consider enclosures which are not contained in $\mathrm{span}\{e_{N-1} \otimes Cu_1, e_{1-N} \otimes Cu_0\}$, then the support of the corresponding absorption operator will spread all over the ring.

In particular, we can also show that some absorption operators have eigenspaces corresponding to the eigenvalue 1 which are bigger than the associated recurrent enclosure:

$$B_N = \mathrm{span}\{e_{N-1} \otimes Cu_1, e_N \otimes u_1\}, \quad B_{-N} = \mathrm{span}\{e_{1-N} \otimes Cu_0, e_{-N} \otimes u_0\}.$$

are the eigenspaces corresponding to the eigenvalue $1$ of $A(|e_N\rangle\langle e_N|\otimes|u_1\rangle\langle u_1|)$ and $A(|e_{-N}\rangle\langle e_{-N}|\otimes|u_0\rangle\langle u_0|)$ respectively (recall that $B_N$ and $B_{-N}$ will be enclosures, by Proposition 2.1.1).

Already in the simplest case $N = 1$ it is anyway evident that symmetries among minimal enclosures of the recurrent dynamic do not extend to the whole channel $\Phi$; for instance there is no way of reconstructing $A(|e_N\otimes u_1\rangle\langle e_N\otimes u_1|)$ from $A(|e_N\otimes u_0\rangle\langle e_N\otimes u_0|)$ knowing only the action of the quantum channel on the recurrent space: changing the value of $\rho$, $C$ varies accordingly and so does $A(|e_N\otimes u_1\rangle\langle e_N\otimes u_1|)$, while $A(|e_N\otimes u_0\rangle\langle e_N\otimes u_0|)$ and $\Phi^{\mathcal{R}}$ stay the same.

### 2.4.2 A homogeneous Open Quantum Random Walk

We reconsider the model in Example 2.1.6. Now that we have recalled transience and recurrence, we can discuss more properties of this model. $E_2$ is always a minimal slow recurrent enclosure, but, according to the different choices of the involved parameters, the situations and the possible communication and so absorption phenomena can significantly change. $E_0$, for instance, will be always an enclosure, as already written, but it is not necessarily minimal (only if $a_-, a_+ \neq 0$) and it will be slow recurrent when $|a_+|^2 = |a_-|^2$, otherwise it is transient. As for $E_1$, in general, under the mentioned conditions (2.7), it is not an enclosure, but it could be when all the parameters $b_\varepsilon$ and $d_\varepsilon$ are null, and in that case, it will not necessarily be minimal. So, this simple model, can really be instructive to understand what happens for different reduction of the space, because it offers examples of both transient and recurrent minimal enclosures, situations with a more complex structure of the recurrent subspace. Different absorption operators can then be considered. In particular, it is anyway never true that $A(\mathcal{R}_+) = \mathbf{1}$ and $A(\mathcal{R}) = \mathbf{1}$ if and only if $|a_+|^2 = |a_-|^2$.

Whenever $a_-, a_+ \neq 0$ and (2.7) hold, it is easy to show that absorption operators are able to completely describe the fixed points set of the quantum channel: even when the recurrent space is not attractive, and so the hypotheses of Theorem 2.3.16 are not satisfied, the conclusion holds true and the fixed points set coincides with the closure of the set of absorption operators $A(\mathcal{V})$, for $\mathcal{V}$ enclosure. In particular, we have the following result.

**Proposition 2.4.1.** *If $a_+ = e^{i\theta}/\sqrt{2}$ and $a_- = e^{-i\theta}/\sqrt{2}$ with $\theta \in [0, 2\pi)$, the recurrent restriction of the quantum channel has infinitely many non orthogonal minimal enclosures and the fixed points set of the restricted channel is isomorphic to $M_{2\times 2}(\mathbb{C})$.*

*For all the other values of $a_\pm$, $E_0$ and $E_2$ are the only minimal enclosures and the corresponding absorption operators generate the fixed points.*

*Proof. Step 1.* Let $\mathcal{V}$ be an enclosure, then by [16, Proposition 6.1] there exists an enclosure $\mathcal{V}' \subset \mathcal{V}$ which is diagonal with respect to the position observable, i.e. commutes with the projections $(\mathbf{1}_{\mathfrak{h}} \otimes |i\rangle\langle i|)_{i\in\mathbb{Z}^d})$. We will show that, in the case

$$a_+a_- = \frac{1}{2} \iff a_+ = \frac{e^{i\theta}}{\sqrt{2}} \text{ and } a_- = \frac{e^{-i\theta}}{\sqrt{2}} \text{ for some } \theta \in [0, 2\pi), \qquad (*)$$

the only minimal enclosures are $E_0$ and $E_1$, otherwise they are the linear space $\mathbb{C}u$ for $u = (\alpha, 0, \gamma)$, with $\alpha$ and $\gamma$ which are not both $0$. Let us consider a generic non-null vector $u = (\alpha, \beta, \gamma) \in \mathbb{C}^3 \setminus \{e_0, e_2\}$ and suppose that $u \otimes |i\rangle\langle i| \in \mathcal{V}'$. We have to consider several cases.

1. If $\beta = 0$, then $L_-L_+u = L_+L_-u = (a_-a_+\alpha, 0, 1/2\beta)$, hence, if hypothesis $(*)$ holds true, $L_-L_+u$ and $u$ are linearly independent and also $e_0 \otimes |i\rangle\langle i|$ and $e_2 \otimes |i\rangle\langle i|$ are in $\mathcal{V}'$, which means that $E_0 \vee E_1 \subseteq \mathcal{V}'$. Otherwise $\mathbb{C}u$ is a minimal enclosure.

2. If $\beta \neq 0$, we want to show that $u$ cannot be an eigenvector of $A := L_-L_+$ and $B := L_+L_-$ simultaneously. This implies that the enclosure containing $u$ is at least of dimension 2 and, hence, has a non trivial intersection (which is again an enclosure) with $\mathrm{span}\{e_0, e_2\}$. We already know that $A$ and $B$ have as common eigenvectors $e_0$ and $e_2$ with corresponding eigenvalues $a_+a_-$ and $1/2$, respectively. By explicit computations one can see that the third eigenvalue is $c_+c_-$ and to find the corresponding eigenvectors we need to compute the kernel of the following matrices:

$$A - c_+c_- = \begin{pmatrix} a_+a_- - c_+c_- & a_-b_+ + b_-c_+ & 0 \\ 0 & 0 & 0 \\ 0 & d_-c_+ + \frac{1}{\sqrt{2}}d_+ & \frac{1}{2} - c_+c_- \end{pmatrix}$$

$$B - c_+c_- = \begin{pmatrix} a_+a_- - c_+c_- & a_+b_- + b_+c_- & 0 \\ 0 & 0 & 0 \\ 0 & d_+c_- + \frac{1}{\sqrt{2}}d_- & \frac{1}{2} - c_+c_- \end{pmatrix}.$$

Before proceeding, we make the following remarks:

(a) $c_+c_- \neq \frac{1}{2}$, since $|c|^2 := |c_+|^2 + |c_-|^2 < 1$ (equation (2.6));

(b) $a_-b_+ + b_-c_+ \neq a_+b_- + b_+c_-$: if equality was true instead, we would have $a_-b_+ - a_+b_- = b_+c_- - b_-c_+$; however by (2.6), we know that $b_+ = \frac{\overline{a}_-b_-}{\overline{a}_+}$, hence we get

$$|a_-b_+ - a_+b_-|^2 = \left|\frac{|a_-|^2b_-}{\overline{a}_+} + a_+b_-\right|^2 = \frac{|b_-|^2}{|a_+|^2}$$

and

$$|b_+c_- - b_-c_+|^2 \leq (|b_+|^2 + |b_-|^2)|c|^2 = \left(\frac{|a_-|^2|b_-|^2}{|a_+|^2} + |b_-|^2\right)|c|^2 = \frac{|b_-|^2}{|a_+|^2}|c|^2 < \frac{|b_-|^2}{|a_+|^2},$$

hence we proved that $a_-b_+ + b_-c_+$ and $a_+b_- + b_+c_-$ must be different.

We consider three cases:

(a) if $a_+a_- = c_+c_-$ and $a_-b_+ + b_-c_+ = 0 \neq a_+b_- + b_+c_-$, we get that for $x \in \mathbb{R}$, $y \in \mathbb{R}$

$$\ker(A - c_+c_-) = \begin{pmatrix} x \\ y \\ z = -\frac{d_-c_+ + \frac{1}{\sqrt{2}}d_+}{\frac{1}{2} - c_+c_-}y \end{pmatrix}, \quad \ker(B - c_+c_-) = \mathbb{C}e_0$$

and their intersection is $\mathbb{C}e_0$. We arrive to the same conclusion if $a_-b_+ + b_-c_+ \neq 0 = a_+b_- + b_+c_-$.

(b) if $a_+a_- = c_+c_-$ and $a_-b_+ + b_-c_+ \neq 0 \neq a_+b_- + b_+c_-$, then
$$\ker(A - c_+c_-) = \ker(B - c_+c_-) = \mathbb{C}e_0.$$

(c) if $a_+a_- \neq c_+c_-$, then
$$\ker(A - c_+c_-) = \mathbb{C}\begin{pmatrix} \frac{a_-b_+ + b_-c_+}{a_+a_- - c_+c_-} \\ 1 \\ -\frac{d_-c_+ + \frac{1}{\sqrt{2}}d_+}{\frac{1}{2} - c_+c_-} \end{pmatrix}, \quad \ker(B - c_+c_-) = \mathbb{C}\begin{pmatrix} \frac{a_+b_- + b_+c_-}{a_+a_- - c_+c_-} \\ 1 \\ -\frac{d_+c_- + \frac{1}{\sqrt{2}}d_-}{\frac{1}{2} - c_+c_-} \end{pmatrix},$$
hence $\ker(A - c_+c_-) \cap \ker(B - c_+c_-) = \{0\}$.

*Step 2.* Since $A(E_0 \vee E_1) = \mathbf{1}$, we can apply the same reasoning as in Proposition 2.3.15 and, if we denote by $q$ the projection onto $E_0 \vee E_1$, we get that for every $x \in \mathcal{F}(\Phi)$ we have
$$x = \mathcal{E}(x) = \mathcal{E}(qxq).$$
Hence in the case where $|a_+| = |a_-| = \frac{1}{\sqrt{2}}$ and both $E_0$ and $E_1$ are recurrent, we can apply Theorem 2.3.16 to characterize the fixed points set. For all the other choices of parameters, we will prove that the only fixed points of $\Phi$ restricted to $E_0 \vee E_1$ are $\mathbb{C}p_{E_0} + \mathbb{C}p_{E_1}$, hence $\mathcal{F}(\Phi) = \mathcal{E}(\mathbb{C}p_{E_0} + \mathbb{C}p_{E_1}) = \mathbb{C}A(E_0) + \mathbb{C}A(E_1)$: let us consider such a fixed point $x = \sum_{i \in \mathbb{Z}} x^{(i)} \otimes |i\rangle\langle i|$. The sequences $x_{11}^{(i)}$ and $x_{22}^{(i)}$ are fixed points for the asymmetric and symmetric ramdom walks, respectively, hence they are constant sequences. The relations the non-diagonal parts must satisfy are the following:
$$x_{21}^{(i)} = \frac{a_+}{\sqrt{2}}x_{21}^{(i+1)} + \frac{a_-}{\sqrt{2}}x_{21}^{(i-1)}, \quad x_{12}^{(i)} = \frac{\overline{a}_+}{\sqrt{2}}x_{12}^{(i+1)} + \frac{\overline{a}_-}{\sqrt{2}}x_{12}^{(i-1)}.$$
Hence, if we show that $(x_{21}^{(i)})$ must be zero, we are done. Without loss of generality, we can assume that $|a_+| > \frac{1}{\sqrt{2}} > |a_-|$. If $p, q \in \mathbb{C}$ are the solutions of $z^2 - \frac{\sqrt{2}}{a_-}z + \frac{a_+}{a_-}$ (since the equation is symmetric in $p$ and $q$, we can pick $q$ such that $|q| \geq |p|$; notice that we have $|q| > 1$, since $|qp| > 1$), we can rewrite the condition for the $(x_{21}^{(i)})$ as
$$x_{21}^{(i-1)} = qx_{21}^{(i)} + p(x_{21}^{(i)} - qx_{21}^{(i+1)}) \text{ or } x_{21}^{(i+1)} = \frac{1}{p}x_{21}^{(i)} + \frac{1}{q}\left(x_{21}^{(i)} - \frac{1}{p}x_{21}^{(i-1)}\right). \tag{2.28}$$
If we call $\alpha := qx_{21}^{(1)}$ and $\beta = x_{21}^{(0)} - qx_{21}^{(1)}$, we can show by induction that for $n \geq 0$
$$x_{21}^{(-n)} = q^n\left(\alpha + \beta\sum_{k=0}^{n}\left(\frac{p}{q}\right)^k\right). \tag{2.29}$$
Let us study the situation case by case:

1. if $|q| = |p|$, then $p/q = e^{i\theta}$ for $\theta \in [0, 2\pi)$, hence
$$|x_{21}^{(-n)}| = \begin{cases} |q|^n|\alpha + (n+1)\beta| & \text{if } \theta = 0, \\ |q|^n\left|\alpha + \frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}}\beta\right| & \text{if } \theta \neq 0 \end{cases}$$
and in both cases $|x_{21}^{(-n)}|$ is unbounded unless $\alpha = \beta = 0$.

2. if $|q| > |p|$, then $|x_{21}^{(-n)}| = |q|^n \left|\alpha + (1-(p/q)^{n+1})\frac{q}{q-p}\beta\right|$, which explodes unless $\alpha + \frac{q}{q-p}\beta = 0$. Let us consider this instance: we introduce the following parameters:
$$\gamma := \frac{1}{p}x_{21}^{(0)} = \frac{\alpha+\beta}{p} = \frac{\beta}{p}\left(1-\frac{q}{q-p}\right) = -\frac{\beta}{q-p},$$
$$\delta := x_{21}^{(1)} - \frac{1}{p}x_{21}^{(0)} = \frac{\alpha}{q} + \frac{\beta}{q-p} = \frac{\beta}{q-p}(1-1) = 0.$$
By equation (2.28), we get for any $n \in \mathbb{N}$
$$|x_{21}^{(1)}| = \left|\frac{1}{p}\right|^n |\gamma|$$
which, if $|p| < 1$, diverges unless $\gamma = 0$, which means that $\alpha = \beta = 0$. Let us show that $|p| \neq 1$, i.e. it cannot be of the form $p = e^{-i\theta}$ $\theta \in [0, 2\pi)$. In this case, since $x_{21}^{(i)}$ satisfies equation (2.28), we get that
$$\frac{1}{p^n} = \frac{a_+}{\sqrt{2}}\frac{1}{p^{n+1}} + \frac{a_-}{\sqrt{2}}\frac{1}{p^{n-1}}, \iff 1 = \frac{a_+}{\sqrt{2}}e^{i\theta} + \frac{a_-}{\sqrt{2}}e^{i\theta}.$$
Let us introduce $b_1 = a_+e^{i\theta}$, $b_2 = a_-e^{-i\theta}$, which satisfy
$$\begin{cases} b_1 + b_2 = \sqrt{2}, \\ |b_1|^2 + |b_2|^2 = 1. \end{cases}$$
The second equation and the concavity of $f(x) = \sqrt{x}$ imply that
$$\frac{|b_1|+|b_2|}{2} \leq \sqrt{\frac{|b_1|^2+|b_2|^2}{2}} \leq \frac{1}{\sqrt{2}}.$$
Therefore we have $\sqrt{2} = b_1+b_2 = |b_1+b_2| \leq |b_1|+|b_2| \leq \sqrt{2}$, hence $|a_+|+|a_-| = \sqrt{2}$, which implies that $|a_+| = |a_-| = 1/\sqrt{2}$, which is a choice of parameters that we excluded.

□

In the example considered above, $p_{\mathcal{R}_+}$, $p_{\mathcal{R}_0}$ and $p_{\mathcal{T}}$ belong to the $W^*$-algebra of the operators which are diagonal with respect to the position observable, i.e.
$$\Delta := \left\{ x = \sum_{i\in\mathbb{Z}^d} x(i) \otimes |i\rangle\langle i|, x(i) \in B(\mathfrak{h}) \right\}$$
where we denote by $\mathfrak{h}$ the Hilbert space corresponding to the internal degrees of freedom of the walker. The following Proposition shows that this is always the case for a generic OQRWs on an arbitrary countable set of vertices $V \subset \mathbb{R}^d$ (which is not required to be a lattice anymore), which is a quantum Markov map of this form
$$\Phi : B(\mathcal{H}) \to B(\mathcal{H})$$
$$x \mapsto \sum_{i,j\in V} L_{i,j}^* \otimes |i\rangle\langle j| x L_{i,j} \otimes |j\rangle\langle i|$$

where $\mathcal{H} = \mathfrak{h} \otimes \ell^2(V)$ for some separable Hilbert space $\mathfrak{h}$, $\{|i\rangle\}_{i \in V}$ is an orthonormal basis of $\ell^2(V)$ and the operators $L_{i,j} : \mathfrak{h} \to \mathfrak{h}$ satisfy $\sum_{j \in V} L_{i,j}^* L_{i,j} = \mathbf{1}_{\mathfrak{h}}$ for every $i \in V$ (see [4, 16] for more details); OQRWs are a wider family of quantum Markov maps than include HOQRWs and which describe the evolution of a walker that moves on an arbitrary set of vertices and such that the transition between any two vertices is a priori possible (not only between nearest neighbours).

**Proposition 2.4.2.** *Let $\Phi$ be a OQRW; then $p_{\mathcal{R}_+}$, $p_{\mathcal{R}_0}$, $p_{\mathcal{T}}$ belong to $\Delta$.*

*Proof.* Every normal invariant state belongs to $\Delta$ and if an operator belongs to $\Delta$, then also its support projection does, hence by definition $p_{\mathcal{R}_+} \in \Delta$. Considering any pair of vertices in $V$, by path from $i$ to $j$ we mean a finite sequence $(i_1, \ldots, i_n)$ of vertices such that $L_{i_k, i_{k+1}} \neq 0$ for $k = 1, \ldots, n-1$ and $i_1 = i$, $i_n = j$; if $\pi(i,j) = (i_1, \ldots, i_n)$ is a path from $i$ to $j$, we denote $L_{\pi(i,j)} = L_{i_{n-1}, i_n} \cdots L_{i_1, i_2}$. Let us define $q$ as the orthogonal projection onto

$$\bigoplus_{j \in V} \overline{\operatorname{span}\{L_{\pi(i,j)} u : u \in \operatorname{supp}(p_{\mathcal{R}_0}), \pi(i,j) \text{ is any path from } i \text{ to } j \text{ , } i \in V\}};$$

by the definition it is clear that $q \in \Delta$, it is again subharmonic and it is such that $q \leq p_{\mathcal{R}_0}$. We are going to prove that $q = p_{\mathcal{R}_0}$. Assume there exists $u$ in $\operatorname{supp}(p_{\mathcal{R}_0})$ and such that $qu = 0$, then consider $\mathfrak{U}[|u\rangle\langle u|](u)$: since for every $k > 0$, $\Phi_*^k(|u\rangle\langle u|) \leq q$, $\mathfrak{U}(|u\rangle\langle u|)[u] = \|u\|^2$, hence, by the definition of recurrence, $u = 0$. □

Hence $\mathbf{\Phi}^{\mathcal{R}_+}$, $\mathbf{\Phi}^{\mathcal{R}_0}$ and $\mathbf{\Phi}^{\mathcal{T}}$ are again OQRW on the same set of vertices $V$, but with smaller internal space $\mathfrak{h}$. Now let us come back to the particular case of HOQRWs on $\mathbb{Z}^d$; for this class of OQRW one can show that $p_{\mathcal{R}_+} = 0$ (and so $\mathbf{\Phi}^{\mathcal{R}_+} = 0$): indeed, suppose there exists a unique normal invariant state $\rho$ (we do not lose in generality, since, if $\mathcal{R}_+ \neq \{0\}$, we can always consider the restriction of the HOQRW to an irreducible positive recurrent enclosure), then it is of the form $\rho = \sum_{i \in \mathbb{Z}^d} \rho(i) \otimes |i\rangle\langle i|$ and it must be traslational invariant, i.e. for every $i, j \in \mathbb{Z}^d$, $\rho(i) = \rho(j)$ and so $\operatorname{tr}(\rho) < +\infty$ implies that $\rho = 0$, which is a contradiction. Hence HOQRWs, reflecting the situation of classical random walks, represents very natural examples where we have non trivial transient and null recurrent spaces and where the positive recurrent space is null.

We can consider the decomposition of the local space $\mathfrak{h}$ into the transient $\tilde{\mathcal{T}}$ and recurrent space $\tilde{\mathcal{R}}$ with respect to the action of the local map $\mathcal{L}(\cdot) = \sum_{s=1}^{2d} L_s^* \cdot L_s$ (we refer to the next chapter for more details) and a natural question is what is their relationship with $\mathcal{T}$ and $\mathcal{R}_0$; we have the following inclusion.

**Proposition 2.4.3.** $\tilde{\mathcal{T}} \otimes \ell^2(\mathbb{Z}^d) \subseteq \mathcal{T}$.

*Proof.* If $\tilde{\mathcal{T}} = \{0\}$ we are done, otherwise consider a nonnull integrable positive operator $x \in B(\mathfrak{h})$ and the associated operator $x \otimes \mathbf{1}_{\ell^2(\mathbb{Z}^d)}$. It is an immediate calculation that shows that it is integrable too and hence $\operatorname{supp}(x \otimes \mathbf{1}_{\ell^2(\mathbb{Z}^d)}) = \operatorname{supp}(x) \otimes \ell^2(\mathbb{Z}^d) \leq p_{\mathcal{T}}$:

$$\sum_{k=0}^{+\infty} \Phi^n(x \otimes \mathbf{1}_{\ell^2(\mathbb{Z}^d)}) = \sum_{k=0}^{+\infty} \mathfrak{L}^n(x) \otimes \mathbf{1}_{\ell^2(\mathbb{Z}^d)} = \mathcal{U}(x) \otimes \mathbf{1}_{\ell^2(\mathbb{Z}^d)}.$$

□

However, in general we do not have the reverse set inclusion, as one can see considering a HOQRW on $\mathbb{Z}$ with one dimensional local space $\mathfrak{h} = \mathbb{C}$ and with local operators equal to $L_1 = p$ and $L_{-1} = 1-p$, for $p \neq 1/2$ (which is a dilation of the nonsymmetric random walk on the integer line): the local map is positive recurrent, but the HOQRW is transient.

### 2.4.3 A continuous time model: 2-photons absorption

We describe here an infinite dimensional model, with a not uniquely reducible attractive fast recurrent space, but for which we can explicitly compute any absorption operator relative to a fast recurrent enclosure. We consider the Hilbert space $\mathfrak{h} = l^2(\mathbb{N})$ with orthonormal basis $(e_n)_{n\geq 0}$ and denote the creation and annihilation operators by $a^\dagger$ and $a$ respectively,

$$ae_n = \sqrt{n}e_{n-1}, \qquad a^\dagger e_n = \sqrt{n+1}e_{n+1}, \qquad n \geq 0.$$

The 2-photons absorption model described in [29] can be represented with the Lindblad generator $\mathcal{L}$, defined on a dense subspace of $B(\mathfrak{h})$ by

$$\mathcal{L}(x) \;=\; i\omega\left[a^{\dagger 2}a^2, x\right] - \frac{\mu^2}{2}\left(a^{\dagger 2}a^2 x - 2a^{\dagger 2}xa^2 + xa^{+2}a^2\right), \tag{2.30}$$

where $\omega$ is a real constant and $\mu^2$ is the absorption rate. Such a master equation arises as the weak coupling limit of a one-mode electromagnetic field with a Bosonic Gaussian zero-temperature reservoir of two-photon absorbing atoms ( [29, 39]). In order to have an intuition about some properties of this model, it can be useful to recall that the diagonal restriction of the generator $\mathcal{L}$ coincides with the generator of a classical continuous time Markov chains with values in $\mathbb{N}$ for which the only possible jumps are the ones from $k$ to $k-2$ for all $k \geq 2$. More precisely, using (2.30) for a diagonal $x$, $x = \sum_{k\geq 0} f_k|e_k\rangle\langle e_k|$ in the domain of $\mathcal{L}$, we have

$$\mathcal{L}\left(\sum_{k\geq 0} f_k|e_k\rangle\langle e_k|\right) \;=\; \mu^2\sum_{k\geq 2} k(k-1)|e_k\rangle\langle e_k|(f_{k-2} - f_k))$$

This tells that the diagonal restriction of $\mathcal{L}$ is a classical generator $Q$ with transition rates

$$q(k, k-2) = \mu^2 k(k-1) = -q(k,k) \qquad \text{for } k \geq 2, \qquad q(k,j) = 0 \qquad \text{otherwise.}$$

Many properties of this model have been deeply investigated in [29], so we know that the normal invariant states of this generator consist of all the density operators with support included in the positive recurrent space $\mathcal{R}_+ = \text{span}\{e_0, e_1\}$. It is immediate to see that any vertex $k$ is accessible from all vertices $k + 2j$ with $j \geq 0$ and this implies that all the $e_k$ are transient for $k \geq 2$. Moreover, it is an easy exercise on classical processes to verify that the Markov chain $X$ with generator $Q$ is absorbed in $\{0, 1\}$ and consequently

$$A(\mathcal{R}_+) = \lim_{t\to\infty}\sum_k (e^{tQ}(e_0 + e_1))_k|e_k\rangle\langle e_k| = \mathbf{1};$$

in particular

$$A(|e_0\rangle\langle e_0|) = \sum_k |e_{2k}\rangle\langle e_{2k}| =: p_e, \qquad A(|e_1\rangle\langle e_1|) = \sum_k |e_{2k+1}\rangle\langle e_{2k+1}| =: p_o. \tag{2.31}$$

Really we know even more, because due to [29, Proposition 3.2 and Corollary 7.3], we have

**Proposition 2.4.4.** *The invariant states of this generator are all the density operators of the form*

$$\rho_{\alpha,z} = \alpha|e_0\rangle\langle e_0| + (1-\alpha)|e_1\rangle\langle e_1| + z|e_0\rangle\langle e_1| + \bar{z}|e_1\rangle\langle e_0|$$

*with* $\alpha \in [0,1]$, $|z|^2 \leq \alpha(1-\alpha)$.

*For any state* $\sigma = \sum_{k,l\geq 0} \sigma_{kl}|e_k\rangle\langle e_l|$, *we denote*

$$\alpha(\sigma) = \sum_k \sigma_{2k,2k}, \qquad z(\sigma) = \sum_k c_k \sigma_{2k,2k+1},$$

*where* $c_k = 2^{-2k}\sqrt{2k+1}\frac{(2k)!}{k!}\prod_{j=1}^k (j - \frac{i\omega}{\mu^2})^{-1}$. *The attraction domain of* $\rho_{\alpha,z}$ *is given by*

$$\begin{aligned} D(\rho_{\alpha,z}) &:= \left\{ \sigma \textit{ state } s.t. \lim_{t\to+\infty} \langle e_i, \Phi_{*t}(\sigma)e_j\rangle = \langle e_i, \rho_{\alpha,z}e_j\rangle, \forall i,j \in \mathbb{N} \right\} \\ &= \{\sigma \textit{ state } s.t.\ \alpha = \alpha(\sigma),\ z = z(\sigma)\}. \end{aligned}$$

For any minimal enclosure $\mathcal{V}$ in $\mathcal{R}_+$, $\mathcal{V}$ is spanned by a vector $u$ which is linear combination of $e_0$ and $e_1$, $u = u_0e_0 + u_1e_1$, and the previous proposition allows us to conclude that, for any state $\sigma$,

$$\begin{aligned} \mathrm{tr}\,(\sigma A(|u\rangle\langle u|)) &= \lim_{t\to+\infty} \mathrm{tr}\,(\Phi_{*t}(\sigma)(|u\rangle\langle u|)) = \mathrm{tr}\,\big(\rho_{\alpha(\sigma),z(\sigma)}(|u\rangle\langle u|)\big) \\ &= \alpha(\sigma)|u_0|^2 + (1-\alpha(\sigma))|u_1|^2 + 2\mathfrak{Re}\{z(\sigma)\bar{u}_0u_1\} \end{aligned}$$

so that

$$\begin{aligned} A(|u\rangle\langle u|) &= \sum_k (|u_0|^2|e_{2k}\rangle\langle e_{2k}| + |u_1|^2|e_{2k+1}\rangle\langle e_{2k+1}|) \\ &\quad + \sum_k c_k u_0\bar{u}_1|e_{2k}\rangle\langle e_{2k+1}| + \bar{c}_k\bar{u}_0u_1|e_{2k+1}\rangle\langle e_{2k}|. \end{aligned}$$

It is immediate to see some properties of absorption operators.

- The operator $A(|u\rangle\langle u|)$ is a projection if and only if $u$ is proportional to $e_0$ or $e_1$ (2.31). When $u$ is not proportional to $e_0$ or to $e_1$, i.e. when $u_0u_1 \neq 0$, the precise spectral representation of $A(|u\rangle\langle u|)$ is easy to write explicitly and its support contains the entire transient space in this case. Indeed, for $u$ as before with $u_0u_1 \neq 0$, $A(|u\rangle\langle u|)$ has eigenvalues 0 and 1, which are simple: $u$ is eigenvector for 1 and $u^\perp := \bar{u}_1e_0 - \bar{u}_0e_1$ generates the kernel. While, when $\lambda \neq 0,1$, $v = \sum_j v_je_j$, we trivially have

$$A(|u\rangle\langle u|)v = \lambda v \Leftrightarrow \begin{cases} v_{2k+1} = \frac{(\lambda-|u_0|^2)}{c_ku_0\bar{u}_1}v_{2k} \\ ((|u_1|^2-\lambda)(\lambda-|u_0|^2) + |\bar{c}_k\bar{u}_0u_1|^2)v_{2k} = 0 \end{cases} \qquad \text{for all } k,$$

$$\text{which gives} \qquad A(|u\rangle\langle u|) = \sum_k \lambda_k |w^k\rangle\langle w^k|, \quad \text{where}$$

- $\lambda_{2k}, \lambda_{2k+1}$ are the two distinct solutions of equations

$$\lambda^2 - \lambda + |u_0 u_1|^2(1 - |c_k|^2) = 0$$

- and the pertaining eigenvectors are such that $w^{2k}$ and $w^{2k+1}$ are orthogonal norm one vectors in span$\{e_{2k}, e_{2k+1}\}$ completely determined by the previous linear system.

- $(A(|u\rangle\langle u|))^2$ is not a fixed point when $u_0 u_1 \neq 0$ (direct computation), so the fixed points do not form an algebra.
- $A(\mathcal{R}_+) = \mathbf{1}$, so $\mathcal{R}_0 = \{0\}$ and all results of Section 2.3 for semigroups with absorbing recurrent space can be applied to this model and in particular the structure of fixed points is described by Theorem 2.3.16.
- Since $A(|e_0\rangle\langle e_0|)$ and $A(|e_1\rangle\langle e_1|)$ are projections, the set of fixed points for the classical diagonal process is instead an algebra.
- As we already remarked in previous sections and as in Example 2.4.1, when the decomposition in minimal enclosures is not unique, the behaviour of the absorption operators related to different minimal enclosures cannot be controlled by the absorption operators associated with a generating basis, as $A(|e_0\rangle\langle e_0|)$ and $A(|e_1\rangle\langle e_1|)$ here.

### 2.4.4 Dissipation-Induced Decomposition

The definition of absorption operators recalls the Dissipation-Induced Decomposition (DID) introduced in [19, 64] for finite dimensional quantum Markov semigroups in order to takle the problem of establishing whether an invariant subspace is attractive (which is a relevant issue for instance for initializing a quantum system in a certain state). In [19], for instance, the authors consider a single quantum channel $\Phi$; given an enclosure $\mathcal{V}$, they provide an algorithm that, using a choice of Kraus operators for $\Phi$, decomposes the state space

$$\mathfrak{h} = \mathcal{V} \oplus \bigoplus_{n=1}^{N} \mathfrak{h}_n \oplus \mathfrak{h}_R$$

for some finite $N \in \mathbb{N}$. DID highlights the one-step accessibility relations: starting from a state supported in $\mathfrak{h}_n$, $\mathcal{V}$ is reached in exactly $n$-steps, while $\mathfrak{h}_R$ does not have access to $\mathcal{V}$. Then, exploiting the obvious relations

$$\operatorname{supp}(\Phi^m(\mathcal{V})) = \mathcal{V} \oplus \bigoplus_{n=1}^{m} \mathfrak{h}_n \text{ for } m = 1, \dots, N-1,$$

$$\operatorname{supp}(\Phi^m(\mathcal{V})) = \operatorname{supp}(A(\mathcal{V})) = \mathcal{V} \oplus \bigoplus_{n=1}^{N} \mathfrak{h}_n \text{ for } m \geq N,$$

we could rewrite the previous decomposition in our notations as

$$\mathfrak{h}_m = \operatorname{supp}(\Phi^m(\mathcal{V})) \cap \operatorname{Ker}(\Phi^{m-1}(\mathcal{V})) \text{ for } m = 1, \dots, N,$$
$$\mathfrak{h}_R = \ker(A(\mathcal{V})).$$

Notice that, since the dimension of the state space is finite,

$$\mathcal{V} \text{ is attractive, i.e. } A(\mathcal{V}) = \mathbf{1} \Leftrightarrow \mathrm{supp}(A(\mathcal{V})) = \mathfrak{h} \Leftrightarrow \mathfrak{h}_R = \{0\} \text{ in the DID.}$$

CHAPTER 3

# Asymptotics of the position process associated to an homogeneous open quantum random walk

We start this chapter by defining the object of our study, namely homogeneous open quantum random walks (HOQRWs). Let $V \subseteq \mathbb{R}^d$ denote the locally finite lattice on which the particle moves; without loss of generality we assume that it contains $0$, and is positively generated by a set $S = \{s_1, \ldots, s_v\} \neq \{0\}$ for some $v \in \mathbb{N}$, hence $V = \{\sum_{i=1}^{v} \alpha_i s_i : \alpha_i \in \mathbb{N}\}$. The canonical example is $V = \mathbb{Z}^d$ and $S = \{\pm e_1, \ldots, \pm e_d\}$ where $(e_1, \ldots, e_d)$ is the canonical basis of $\mathbb{R}^d$. We denote by $\ell^2(V)$ the Hilbert space of square summable sequences indexed by $V$, describing the position of the particle in the quantum evolution, and we introduce a finite-dimensional Hilbert space $\mathfrak{h}$ describing the internal degrees of freedom of the particle. We fix $\{|\underline{k}\rangle\}_{\underline{k}\in V}$ an orthonormal basis for $\ell^2(V)$. We consider a quantum system described by the separable complex Hilbert space $\mathcal{H} = \mathfrak{h} \otimes \ell^2(V)$, which takes into account both the internal degrees of freedom of the particle and its position.

A HOQRW ( [4, 14]) $\mathfrak{M}$ is a particular quantum Markov map acting on $B(\mathcal{H})$ in such a way that, at each time step, the position of the evolution can go only to nearest neighbors and also the change in the local state only depends on the position shift. More precisely, $\mathfrak{M}$ is defined through its Kraus form as

$$\begin{aligned} \mathfrak{M} &: B(\mathcal{H}) \to B(\mathcal{H}) \\ \omega &\mapsto \sum_{\underline{k}\in V} \sum_{i=1}^{v} (L_i^* \otimes |\underline{k}\rangle \langle \underline{k} + s_i|)\, \omega\, (L_i \otimes |\underline{k} + s_i\rangle \langle \underline{k}|), \end{aligned} \tag{3.1}$$

where $\{L_i\}_{i=1}^{v}$ are operators in $B(\mathfrak{h})$ such that $\sum_{i=1}^{v} L_i^* L_i = 1_{\mathfrak{h}}$. The auxiliary (or local) map is the quantum channel $\mathfrak{L}$ on the space $L^1(\mathfrak{h})$ defined by

$$\mathfrak{L} : L^1(\mathfrak{h}) \to L^1(\mathfrak{h}), \qquad \mathfrak{L}(\sigma) = \sum_{i=1}^{v} L_i \sigma L_i^*.$$

We shall see that this auxiliary map is of primary importance in our study: it completely characterizes $\mathfrak{M}$ and so it contains all essential information. In this chapter we will mainly work in the Schrödinger picture, hence, in order to keep the notation as simple as possible, we make the choice to denote by $\mathfrak{L}$ the quantum channel acting on $L^1(\mathfrak{h})$ and by $\mathfrak{L}^*$ the dual map; the same will be done regarding any other quantum channel acting on trace class operators on the local space $\mathfrak{h}$.

Notice that, no matter what is the initial state $\rho$, the evolved state after one time step $\mathfrak{M}_*(\rho)$ commutes with the projections corresponding to the position measurement $(\mathbf{1}_\mathfrak{h} \otimes |\underline{k}\rangle\langle\underline{k}|)_{\underline{k}\in V}$ and it only depends on the "diagonal elements" of $\rho$: $\rho(\underline{k}) = \mathrm{tr}_{\ell^2(V)}(\mathbf{1}_\mathfrak{h} \otimes |\underline{k}\rangle\langle\underline{k}|\,\rho\mathbf{1}_\mathfrak{h} \otimes |\underline{k}\rangle\langle\underline{k}|)$, for $\underline{k} \in V$; without loss of generality, then, we can assume that the initial state is of the form $\rho = \sum_{\underline{k}\in V} \rho(\underline{k}) \otimes |\underline{k}\rangle\langle\underline{k}|$ (even more so since we are interested in the behaviour of the system for long times).

Given the open quantum random walk $\mathfrak{M}$, we can then fix an initial state $\rho$, and, following the usual construction for quantum trajectories, we can introduce the process $(X_n, \rho_n)_{n\geq 0}$, keeping track of the position $X_n$, valued in $V$, and of the internal state $\rho_n$ of the particle (a positive unit-trace operator in $L^1(\mathfrak{h})$). See Section 3.1 for more precise definitions.

In Section 3.2, we determine a family of probability measures under which the position process verifies a central limit theorem (the techinque of the proof, inspired by [14], uses spectral and deformation tecniques in order to apply Bryc's Theorem). These probability measures are absolutely continuous with respect to the standard measure $\mathbb{P}_\rho$, induced by the initial state $\rho$ of the evolution, and are naturally associated with the recurrent enclosures of the local map. The densities of these measures and the parameters of the limit Gaussian are explicitly written in terms of the initial state and of the particular enclosure.

Then, in Section 3.3, we shall go to the general case using the decomposition of the space $\mathfrak{h}$ already introduced in Proposition 2.3.5 and deducing an expression of $\mathbb{P}_\rho$ as convex combinations of probability measures described in the previous section. The main result of the section is the "generalized CLT" for the law of the position process (Theorem 3.3.4).

In Section 3.4 we study large deviations of $\frac{X_n - X_0}{n}$ and we can prove a large deviation principle in the case the local channel is positive recurrent (Theorem 3.4.4), while, when there is a non trivial transient subspace, we can only find upper and lower bounds through Gärtner-Ellis' theorem (Theorem 3.4.3). In both cases we can explicitly write the rate functions using the same ingredients as for the "generalized CLT".

Section 3.5 we show that $\frac{X_n - X_0}{n}$ converges $\mathbb{P}_\rho$-almost surely; in order to describe the limit random variable, we introduce a further decomposition of the recurrent space induced by the local map $\mathcal{R}_+ = \bigoplus_{\gamma\in C} \mathcal{W}_\gamma$ which is such that there exists a $C$-valued random variable $\Gamma$ and $\mathbb{P}_\rho$-almost surely the process $\rho_n$ gets absorbed in $\mathcal{W}_\Gamma$. Conditioned to the event that $\rho_n$ is absorbed in $\mathcal{W}_\gamma$, $\frac{X_n - X_0}{n}$ converges to a fixed value $m_\gamma$ or, in other words, $\mathbb{P}_\rho$-almost surely

$$\lim_{n\to+\infty} \frac{X_n - X_0}{n} = m_\Gamma.$$

$m_\gamma$ is related to the action of the local map restricted to $\mathcal{W}_\gamma$.

Finally, we discuss some examples and numerical simulations in Section 3.6.

## 3.1 Quantum trajectories

Quantum trajectories associated to $\mathfrak{M}$ are the realizations of the stochastic process obtained measuring at every time step the position of a particle on the lattice, knowing that the state of the particle between every time step undergoes an evolution described by $\mathfrak{M}$. The position measurement corresponds to the following resolution of the identity

$$\begin{aligned} e : \mathscr{P}(V) &\to \mathcal{P}(B(\mathcal{H})) \\ \{\underline{k}\} &\mapsto \mathbf{1}_{\mathfrak{h}} \otimes |\underline{k}\rangle\langle\underline{k}|, \end{aligned}$$

where $\mathscr{P}(V)$ stays for the power set of $V$ (since $V$ is denumerable, $e$ is completely determined by its action on singletons). Hence, if the particle starts in the state $\rho = \sum_{\underline{k}\in V} \rho(\underline{k}) \otimes |\underline{k}\rangle\langle\underline{k}|$, at time 0 we find the particle in the position $\underline{k}$ with probability $\operatorname{tr}(\mathbf{1}_{\mathfrak{h}} \otimes |\underline{k}\rangle\langle\underline{k}| \rho \mathbf{1}_{\mathfrak{h}} \otimes |\underline{k}\rangle\langle\underline{k}|) = \operatorname{tr}(\rho(\underline{k}))$ and after the measurement the new state of the particle is $\frac{\rho(\underline{k})}{\operatorname{tr}(\rho(\underline{k}))} \otimes |\underline{k}\rangle\langle\underline{k}|$. Then the state evolves according to $\mathfrak{M}$ and it becomes $\sum_{s\in S}(L_s\rho(\underline{k})L_s^*) \otimes |\underline{k}+s\rangle\langle\underline{k}+s|$ and the position measurement gives the outcome $\underline{k}+s$ with probability $\operatorname{tr}(L_s\rho(\underline{k})L_s^*)$ and so on. Hence the stochastic process is given by a sequence of random variables of the form $\rho_n \otimes |X_n\rangle\langle X_n|$, where $\rho_n$ is the internal state of the particle and $X_n$ is the position of the particle on the lattice, both at time $n$. It is an easy computation to see that the expected value of $\rho_n \otimes |X_n\rangle\langle X_n|$ is given by $\mathfrak{M}_*^n(\rho)$; moreover we remark that $\rho_n \otimes |X_n\rangle\langle X_n|$ is completely determined by the pair $(X_n, \rho_n)$ taking values in $V$ and the convex set of states on $\mathfrak{h}$, respectively, so we will consider this other process instead.

Let us formally define the stochastic process of interest. The stochastic evolution of the system depends on the initial state $\rho$ and we shall denote by $\mathbb{P}_\rho$ the associated probability measure. Let us first define the probability space. We denote by $J = \{1, \dots, v\}$ the set of indices for all possible shifts in $S = \{s_1, \dots, s_v\}$ and we choose the sample set $\Omega = V \times J^{\mathbb{N}}$. On $V$ and $J$ we consider the $\sigma$-algebras of the power sets, and on $\Omega$ we then consider the $\sigma$-algebra $\mathcal{F}$ generated by cylindrical sets. If $\mathcal{F}_0$ is the power set of $V$ and $\mathcal{F}_n$ is the $\sigma$-algebra generated by the projections on the first $n$ components of $\Omega$ for $n \geq 1$, notice that $(\mathcal{F}_n)_{n\geq 0}$ is a filtration and $\mathcal{F} = \sigma\left(\bigcup_{n\geq 0} \mathcal{F}_n\right)$.

We define a family of compatible finite dimensional laws which univoquely determines a measure $\mathbb{P}_\rho$ on $(\Omega, \mathcal{F})$ by Kolmogorov extension theorem: for all $\underline{k} \in V$, $n \geq 1$, $\underline{j} = (j_1, \dots, j_n) \in J^n$,

$$\begin{aligned} \mathbb{P}_\rho(\{\underline{k}\} \times J^{\mathbb{N}}) &= \operatorname{tr}(\rho(\underline{k})), \\ \mathbb{P}_\rho(\{(\underline{k}, \underline{j})\} \times J^{\mathbb{N}}) &= \operatorname{tr}\left(L_{j_n} \cdots L_{j_1} \rho L_{j_1}^* \cdots L_{j_n}^*\right). \end{aligned}$$

The quantum trajectory is the process $(X_n, \rho_n)_{n\geq 0}$ defined, for $\omega = (\underline{k}, j_1, \dots)$, as

$$X_0(\omega) = \underline{k}, \quad \rho_0(\omega) = \frac{\rho(\underline{k})}{\operatorname{tr}(\rho(\underline{k}))},$$

$$X_{n+1}(\omega) = X_n(\omega) + s_{j_{n+1}}, \quad \rho_{n+1}(\omega) = \frac{L_{j_{n+1}}\rho_n(\omega)L^*_{j_{n+1}}}{\operatorname{tr}\left(L_{j_{n+1}}\rho_n(\omega)L^*_{j_{n+1}}\right)} \qquad \forall n \geq 0.$$

$(X_n, \rho_n)_{n\geq 0}$ is a Markov process on the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_{n\geq 0}, \mathbb{P}_\rho)$, with initial law given by

$$\mathbb{P}_\rho\left\{(X_0, \rho_0) = \left(\underline{k}, \frac{\rho(\underline{k})}{\operatorname{tr}(\rho(\underline{k}))}\right)\right\} = \operatorname{tr}(\rho(\underline{k})), \quad \underline{k} \in V$$

and transition probabilities

$$\mathbb{P}_\rho\left(X_{n+1} = X_n + s_j, \rho_{n+1} = \frac{L_j\rho_n L_j^*}{\operatorname{tr}\left(L_j\rho_n L_j^*\right)}\middle| X_n, \rho_n\right) = \operatorname{tr}\left(L_j\rho_n L_j^*\right), \quad j \in J, \quad n \geq 1.$$

Notice that $(\mathcal{F}_n)_n$ coincides with the natural filtration of $(X_n, \rho_n)_{n\geq 0}$ and that the the average behaviour of $(\rho_n)$ is completely determined by the auxiliary map $\mathfrak{L}$, since we have $\mathbb{E}_\rho[\rho_n] = \sum_{\underline{k}\in V} \mathfrak{L}^n(\rho(\underline{k}))$.

In order to fix some ideas about the definition of OQRW and on the behavior of the related position process, we introduce two simple examples, both on the lattice $V = \mathbb{Z}$, for which we provide the simulated trajectories of the rescaled position process in the next figures. In this case ($V = \mathbb{Z}$), the HOQRW has two possible movements at each time step, i.e. v=2, and the walk is completely determined once we fix the two Kraus operators $L_1, L_2$ describing the action on the internal state when moving to the right or to the left. For convenience, we shall call them $R$ and $L$ respectively.

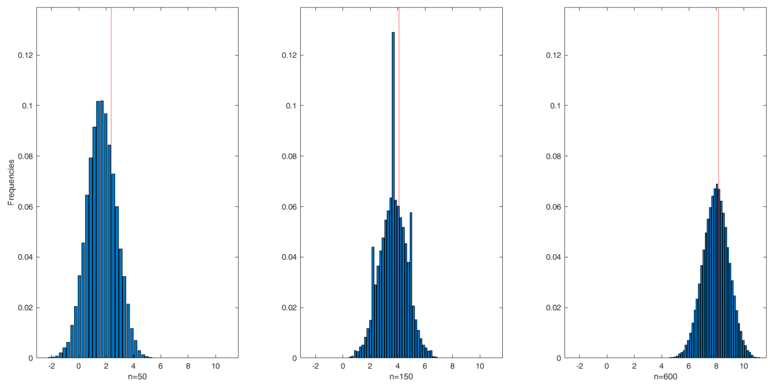
**Example 3.1.1.** Let us consider a HOQRW on $V = \mathbb{Z}$ with local space $\mathfrak{h} = \mathbb{C}^2$ (we denote by $\{e_0, e_1\}$ the canonical basis) and the following local operators:

$$L = \begin{pmatrix} \sqrt{\frac{1}{2}} & 0 \\ -\frac{\sqrt{2}}{3} & \sqrt{\frac{1}{3}} \end{pmatrix}, \quad R = \begin{pmatrix} \sqrt{\frac{1}{6}} & 0 \\ \frac{1}{3} & \sqrt{\frac{2}{3}} \end{pmatrix}$$

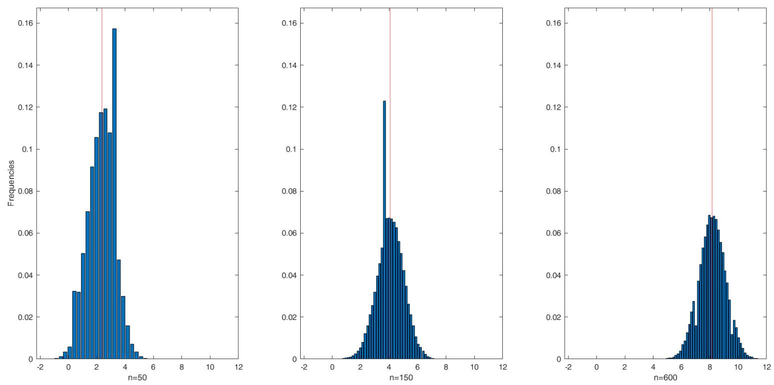
corresponding to going to the left and the right respectively. In this case the local map $\mathfrak{L}(\cdot) = L \cdot L^* + R \cdot R^*$ admits a unique invariant state $\tau_0 = |e_1\rangle\langle e_1|$. For every initial state $\rho$, simulations show that, for increasing values of $n$, the law of $\frac{X_n - X_0}{\sqrt{n}}$ becomes approximately Gaussian, with fixed variance, and mean growing as $\sqrt{n}$ (see Figure 3.1). Moreover, if we look at the trajectories of $\frac{X_n - X_0}{n}$ they tend to a fixed constant (equal to the mean of the Gaussian divided by $\sqrt{n}$), as it is shown in Figure 3.2.

**Example 3.1.2.** Consider now always $V = \mathbb{Z}$, but local space $\mathfrak{h} = \mathbb{C}^4$ and local Kraus operators

$$L = \begin{pmatrix} \frac{1}{2\sqrt{2}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ -\sqrt{\frac{1}{6}} & 0 & 0 & \frac{\sqrt{2}}{3} \end{pmatrix}, \quad R = \begin{pmatrix} \sqrt{\frac{3}{8}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{3}} & 0 & 0 & \frac{1}{\sqrt{3}} \end{pmatrix}.$$




**(a)** $\rho = \tau_0 \otimes |0\rangle\langle 0|$




**(b)** $\rho = |e_0\rangle\langle e_0| \otimes |0\rangle\langle 0|$

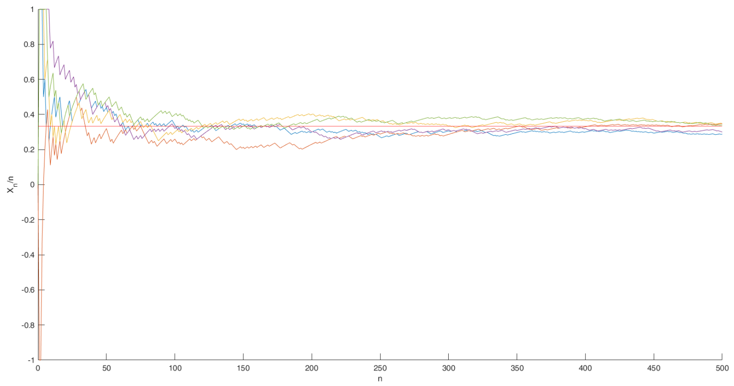
**Figure 3.1:** *Panels (a)-(b) show the apperance of the same Gaussian distribution for two different initial states in the model introduced in Example 3.1.1. We used* $N = 5 \times 10^4$ *samples of* $\frac{X_n}{\sqrt{n}}$ *for* $n = 50, 150, 600$ *in order to draw its profile. The vertical red line corresponds to the mean value of the Gaussian.*

The invariant states of the local map are of the following form: $x\sigma + (1-x)|e_3\rangle\langle e_3|$ where $\sigma$ is any state supported in $\mathrm{span}\{e_1, e_2\}$ and $x \in [0,1]$. In this case simulations show that, as $n$ increases, the law of $\frac{X_n - X_0}{\sqrt{n}}$ can approach either a Gaussian or the mixture of two Gaussians, whose parameters will be easy to compute using the results of next sections ($\mathcal{N}(0,1)$ and $\mathcal{N}(-\sqrt{n}/3, 8/9)$). Figure 3.3 shows that the profile of such a mixture (especially the weights of the mixture) strongly depends on the initial state. As Figure 3.4 shows, the trajectories of $\frac{X_n - X_0}{n}$ have now two limit values to which they tend in proportions which are equal to the weights of the two limit Gaussians.

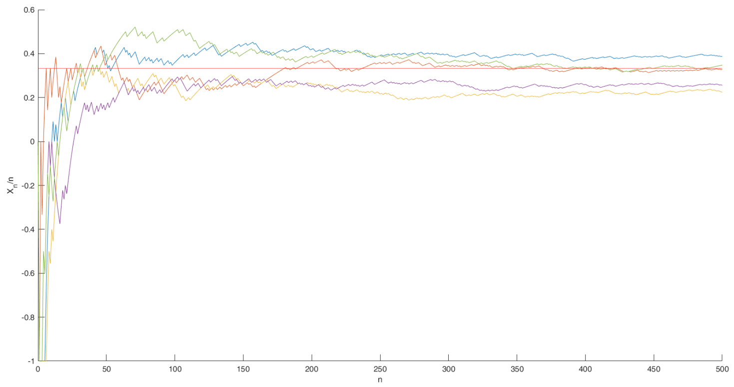
We shall recover this example in the last section, considering a slightly more general family of Kraus operators.

## 3.2 Selecting a single Gaussian

In this section, we shall concentrate on a single minimal enclosure $\mathcal{V}$ of the local channel $\mathfrak{L}$ and we shall introduce a proper associated probability measure $\mathbb{P}'_\rho$, which is




**(a)** $\rho = \tau_0 \otimes |0\rangle\langle 0|$




**(b)** $\rho = |e_0\rangle\langle e_0| \otimes |0\rangle\langle 0|$

**Figure 3.2:** *Panels (a)-(b) show the first $n = 500$ steps of $N = 5$ trajectories of $\frac{X_n}{n}$ evolving according to the system described in Example 3.1.1; notice that they concentrate around the red line. The difference between the two pictures is the initial state; notice how in panel (b) the convergence is slower.*

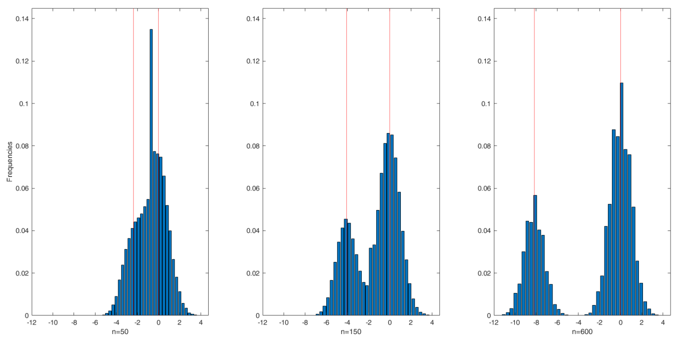
absolutely continuous with respect to $\mathbb{P}_\rho$, with a relative density which assigns weights according to the absorption in $\mathcal{V}$. We shall prove that the position process $(X_n)_n$ always satisfies a central limit theorem under this measure (Theorem 3.2.5). Previous CLT results can be seen as a consequence of the case of a single enclosure (see Remark 3.2.6 below).

According to the notations introduced in the previous chapter, we shall call $A(\mathcal{V}) = \lim_{n\to+\infty} \mathfrak{L}^{*n}(p_\mathcal{V})$ the absorption operator of the enclosure $\mathcal{V}$ for $\mathfrak{L}$ and we denote by $\tilde{p}_\mathcal{V}$ the support projection of $A(\mathcal{V})$.

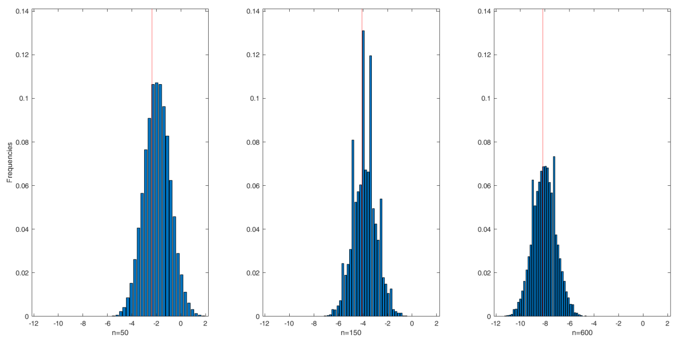
**Lemma 3.2.1.** *Let $(Y_n)_{n\geq 0}$ be the process defined by $Y_n = \operatorname{tr}(A(\mathcal{V})\rho_n)$ for any $n \geq 0$.*

1. *Then $(Y_n)_{n\geq 0}$ is a positive and bounded $\mathbb{P}_\rho$-martingale, converging (almost surely and $L^1$) to a random variable $Y_\infty$ valued in $[0, 1]$.*

2. *If $\mathbb{E}_\rho[Y_0] = \mathbb{E}_\rho[\operatorname{tr}(A(\mathcal{V})\rho_0)] > 0$, we can define a new probability measure $\mathbb{P}'_\rho$*

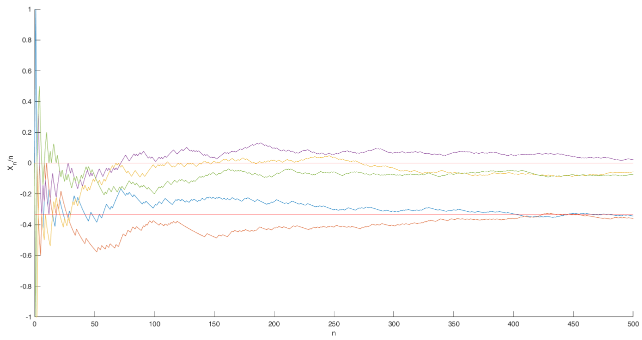



**(a)** $\rho = \frac{1}{3}(|e_1\rangle\langle e_1| + |e_2\rangle\langle e_2| + |e_3\rangle\langle e_3|) \otimes |0\rangle\langle 0|$

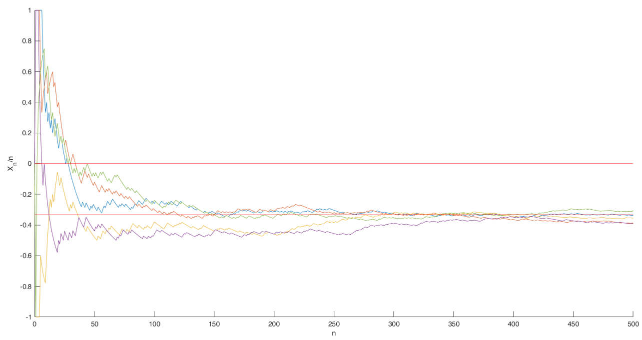



**(b)** $\rho = |e_0\rangle\langle e_0| \otimes |0\rangle\langle 0|$

**Figure 3.3:** *Panel (a) shows a mixture of two Gaussian distributions for a particular choice of the initial state, while panel (b) shows a single Gaussian for another initial state for the model considered in Example 3.1.2. We used* $N = 5 \times 10^4$ *samples of* $\frac{X_n}{\sqrt{n}}$ *for* $n = 50, 150, 600$ *in order to draw their profile. The vertical red lines correspond to the mean values of the Gaussians.*




**(a)** $\rho = \frac{1}{3}(|e_1\rangle\langle e_1| + |e_2\rangle\langle e_2| + |e_3\rangle\langle e_3|) \otimes |0\rangle\langle 0|$




**(b)** $\rho = |e_0\rangle\langle e_0| \otimes |0\rangle\langle 0|$

**Figure 3.4:** *Panel (a) shows the convergence of the trajectories of* $\frac{X_n}{n}$ *to two different values for a particular choice of the initial state, while panel (b) shows how picking a different initial state, trajectories converge to only one of the two limit values. We used* $N = 5$ *samples of* $\frac{X_n}{n}$ *followed along* $n = 500$ *steps and evolving according to the model considered in Example 3.1.2. Red lines correspond to the mean values of the Gaussians divided by* $\sqrt{n}$*.*

*such that*
$$\frac{d\mathbb{P}'_\rho}{d\mathbb{P}_\rho} = \frac{Y_\infty}{\mathbb{E}_\rho[Y_0]}, \qquad \frac{d\mathbb{P}'_\rho}{d\mathbb{P}_\rho}\bigg|_{\mathcal{F}_n} = \frac{Y_n}{\mathbb{E}_\rho[Y_0]}.$$
*Moreover the density* $\frac{d\mathbb{P}'_\rho}{d\mathbb{P}_\rho}$ *is valued in* $[0, \mathbb{E}_\rho[Y_0]^{-1}]$ *and*
$$\begin{aligned}\left\{\frac{d\mathbb{P}'_\rho}{d\mathbb{P}_\rho} = \frac{1}{\mathbb{E}_\rho[Y_0]}\right\} &= \left\{\lim_{n\to+\infty} \|p_\mathcal{V}\rho_n p_\mathcal{V} - \rho_n\| = 0\right\}, \\ \left\{\frac{d\mathbb{P}'_\rho}{d\mathbb{P}_\rho} = 0\right\} &= \left\{\lim_{n\to+\infty} \|\tilde{p}_\mathcal{V}^\perp \rho_n \tilde{p}_\mathcal{V}^\perp - \rho_n\| = 0\right\}.\end{aligned} \tag{3.2}$$

We remark that, for this lemma, it is not necessary for $\mathcal{V}$ to be minimal. The last sentence of the statement gives a mathematical meaning to the intuition that, given any enclosure $\mathcal{V}$, the corresponding $\mathbb{P}'_\rho$ encodes the notion of conditioning to the "absorption in $\mathcal{V}$". Nevertheless, $Y_\infty$ is not a Bernoulli random variable in general, hence there does not need to exist any measurable set $B \in \mathcal{F}$ such that $\mathbb{P}'_\rho(\cdot) = \mathbb{P}_\rho(\cdot|B)$, even if it can happen in some cases (see Example 3.6.2 and in particular the simulations in Figure 3.7); we will treat the random variable $Y_\infty$ in more in detail in Section 3.5.

*Proof.* $Y_n$ is trivially positive and bounded and
$$\mathbb{E}_\rho[Y_{n+1}|\mathcal{F}_n] = \sum_{i=1}^{v} \mathrm{tr}\,(L_i\rho_n L_i^*) \frac{\mathrm{tr}\,(A(\mathcal{V})L_i\rho_n L_i^*)}{\mathrm{tr}\,(L_i\rho_n L_i^*)} = \mathrm{tr}\,(\mathfrak{L}^*(A(\mathcal{V}))\rho_n) = \mathrm{tr}\,(A(\mathcal{V})\rho_n) = Y_n.$$

Since $(Y_n)$ is a positive and bounded martingale, it converges almost surely and in $L^1 := L^1(\Omega, \mathbb{P}_\rho)$ to a positive random variable $Y_\infty$. When $\mathcal{V}$ and $\rho$ are such that $\mathbb{E}_\rho[\mathrm{tr}\,(A(\mathcal{V})\rho_0)] = \sum_{\underline{k}\in V} \mathrm{tr}\,(A(\mathcal{V})\rho(\underline{k})) > 0$, we can introduce the random variables
$$0 \le Z_n := \frac{Y_n}{\mathbb{E}_\rho[\mathrm{tr}\,(A(\mathcal{V})\rho_0)]},$$
and the sequence $(Z_n)$ is a $\mathbb{P}_\rho$-martingale with expected value equal to 1 and converges almost surely to
$$0 \le Z_\infty := \frac{Y_\infty}{\mathbb{E}_\rho[\mathrm{tr}\,(A(\mathcal{V})\rho_0)]}. \tag{3.3}$$
Note that $Z_\infty \in L^1$. Therefore we can consider the new measure $\mathbb{P}'_\rho$ which has density $Z_\infty$ with respect to $\mathbb{P}_\rho$, so that
$$\frac{d\mathbb{P}'_\rho}{d\mathbb{P}_\rho} = Z_\infty, \qquad \frac{d\mathbb{P}'_\rho}{d\mathbb{P}_\rho}\bigg|_{\mathcal{F}_n} = Z_n.$$

The range of $\frac{d\mathbb{P}'_\rho}{d\mathbb{P}_\rho}$ trivially follows from the fact that $0 \le Y_\infty \le 1$.
The only thing left is the proof of Equation (3.2); first notice the following set equivalence:
$$\left\{\frac{d\mathbb{P}'_\rho}{d\mathbb{P}_\rho} = \frac{1}{\mathbb{E}_\rho[Y_0]}\right\} = \{Y_\infty = 1\}, \quad \left\{\frac{d\mathbb{P}'_\rho}{d\mathbb{P}_\rho} = 0\right\} = \{Y_\infty = 0\}.$$
Let us denote by $q$ the orthogonal projection onto the eigenspace corresponding to the eigenvaule 1 of $A(\mathcal{V})$; since $0 \le A(\mathcal{V}) \le \mathbf{1}_\mathfrak{h}$, $Y_\infty = 0$ ($Y_\infty = 1$) if and only if

$\lim_{n\to+\infty}\|\tilde{p}_{\mathcal{V}}^{\perp}\rho_n\tilde{p}_{\mathcal{V}}^{\perp}-\rho_n\|=0$ ($\lim_{n\to+\infty} q\rho_n q-\rho_n\|=0$). By Theorem 2.2.7, we know that $q-p_{\mathcal{V}}\leq p_{\mathcal{T}}$, hence to conclude we only need to show that $\lim_{n\to+\infty}\|p_{\mathcal{T}}\rho_n p_{\mathcal{T}}\|=0$. Since $p_{\mathcal{T}}$ is superharmonic, $T_n:=\operatorname{tr}(p_{\mathcal{T}}\rho_n)$ is a supermartingale:

$$\mathbb{E}_\rho[T_{n+1}|\mathcal{F}_n]=\sum_{i=1}^{v}\operatorname{tr}(L_i\rho_n L_i^*)\frac{\operatorname{tr}(p_{\mathcal{T}}L_i\rho_n L_i^*)}{\operatorname{tr}(L_i\rho_n L_i^*)}=\operatorname{tr}(\mathfrak{L}^*(p_{\mathcal{T}})\rho_n)\leq\operatorname{tr}(p_{\mathcal{T}}\rho_n)=T_n.$$

Furthermore $0\leq T_n\leq 1$, hence $T_n$ converges $\mathbb{P}_\rho$-a.s. to a certain limit $T_\infty$. Notice that $\mathbb{E}_\rho[T_\infty]=\lim_{n\to+\infty}\mathbb{E}_\rho[T_n]=\lim_{n\to+\infty}\mathfrak{L}^{*n}(p_{\mathcal{T}})=0$, hence $T_\infty=0$, which implies that $\lim_{n\to+\infty}\|p_{\mathcal{T}}\rho_n p_{\mathcal{T}}\|=0$. □

We shall use spectral analysis and deformation techniques in order to prove the central limit theorem for the position process $(X_n)_{n\geq 0}$ under the measure $\mathbb{P}'_\rho$. For all $u\in\mathbb{R}^d$, let us define the following operators:

$$L_i^{(u)}=e^{\frac{u\cdot s_i}{2}}L_i,\qquad \tilde{L}_i^{(u)}=e^{\frac{u\cdot s_i}{2}}\tilde{p}_{\mathcal{V}}L_i\tilde{p}_{\mathcal{V}},\quad i=1,\dots,v$$

and we call $\mathfrak{L}_u$ and $\tilde{\mathfrak{L}}_u$ the analytic perturbations of $\mathfrak{L}$ and $\tilde{\mathfrak{L}}=\tilde{\mathfrak{L}}_0$ respectively, defined as the completely positive operators

$$\mathfrak{L}_u(\sigma)=\sum_{i=1}^{v}L_i^{(u)}\sigma L_i^{(u)*},\qquad \tilde{\mathfrak{L}}_u(\sigma)=\sum_{i=1}^{v}\tilde{L}_i^{(u)}\sigma\tilde{L}_i^{(u)*}.$$

We denote by $\lambda_u$ the spectral radius of $\tilde{\mathfrak{L}}_u$, that is $\lambda_u=r(\tilde{\mathfrak{L}}_u)$. Theorem 1.2.10 ensures that $\lambda_u\in\mathrm{Sp}(\tilde{\mathfrak{L}}_u)$ with corresponding positive eigenvector $\tau_u$. Notice that $\lambda_0=1$ and $\tau_0$ is the unique minimal invariant state supported on $\mathcal{V}$. Moreover, $\mathfrak{L}_u$ and $\tilde{\mathfrak{L}}_u$ can be extended for complex values of $u$ and form two analytic families of matrices: $\mathfrak{L}_u(\sigma)=\sum_{i=1}^{v}\mathrm{e}^{u\cdot s_i}L_i\sigma L_i^*$ and $\tilde{\mathfrak{L}}_u(\sigma)=\sum_{i=1}^{v}\mathrm{e}^{u\cdot s_i}\tilde{p}_{\mathcal{V}}L_i\tilde{p}_{\mathcal{V}}\sigma\tilde{p}_{\mathcal{V}}L^*{}_i\tilde{p}_{\mathcal{V}}$. In Lemma 3.2.2 we shall prove that also the perturbed eigenvalue $\lambda_u$ and eigenvector $\tau_u$ are analytic at least in a neighborhood of the origin.

Notice that all previous mathematical objects depend on the enclosure $\mathcal{V}$, so it would be more precise to highlight this and denote them $\tilde{L}_i^{(u,\mathcal{V})}$, $\tilde{\mathfrak{L}}_u^{\mathcal{V}}$, ..., $\mathbb{P}_\rho^{\prime(\mathcal{V})}$. Since the notations are already quite heavy, we drop the dependence on $\mathcal{V}$ in this section, since we shall use only one enclosure and we shall recover it when necessary, treating the general case. Moreover, given any map $\Phi$ acting on $L^1(\mathfrak{h})$ and any closed subspace $\mathcal{W}\subset\mathfrak{h}$, in order to avoid confusion with the other indices, in the rest of this chapter we denote by $\Phi_{|\mathcal{W}}(\cdot)$ the restriction $p_{\mathcal{W}}\Phi(p_{\mathcal{W}}\cdot p_{\mathcal{W}})p_{\mathcal{W}}$.

**Lemma 3.2.2.** *Let $\mathcal{V}$ be a minimal enclosure. The operators $\tilde{\mathfrak{L}}$ and $\tilde{\mathfrak{L}}_{|\mathcal{V}}=\mathfrak{L}_{|\mathcal{V}}$ have the same peripheral eigenvalues and eigenvectors with the same multiplicities.*

*Moreover in a complex neighborhood of the origin the following hold true:*

1. $u\mapsto\lambda_u$ *and* $u\mapsto\tau_u$ *are analytic;*

2. $\operatorname{supp}(\tau_u)\subset\mathcal{V}$.

*Hence $\lambda_u$ and $\tau_u$ coincide with the analogous quantities for the restricted deformation $\tilde{\mathfrak{L}}_{u|\mathcal{V}}=\mathfrak{L}_{u|\mathcal{V}}$ (i.e. $\lambda_u:=r(\mathfrak{L}_{u|\mathcal{V}})$, $\mathfrak{L}_{u|\mathcal{V}}(\tau_u)=\lambda_u\tau_u$).*

*Proof.* Let $\theta \in [0, 2\pi)$ and $\sigma \in L^1(\mathfrak{h})$ such that

$$\tilde{\mathfrak{L}}(\sigma) = \mathrm{e}^{i\theta}\sigma. \tag{3.4}$$

In order to prove that the peripheral eigenvectors and eigenvalues of $\tilde{\mathfrak{L}}$ are the same as those of $\tilde{\mathfrak{L}}_{|\mathcal{V}}$ we need to prove that $\sigma = p_{\mathcal{V}}\sigma p_{\mathcal{V}}$. Let us consider the orthogonal decomposition $\mathrm{supp}(A(\mathcal{V})) = \mathcal{V} \oplus \mathcal{W}$; by definition $\mathcal{W} = \mathrm{supp}(A(\mathcal{V}) - p_{\mathcal{V}})$ and, since $\dim(\mathfrak{h}) < +\infty$, we know that there exists a constant $\gamma > 0$ such that $p_{\mathcal{W}} \leq \gamma(A(\mathcal{V}) - p_{\mathcal{V}})$, hence by Theorem 2.2.7 we have that

$$\tilde{\mathfrak{L}}^{*n}(p_{\mathcal{W}}) = \tilde{p}_{\mathcal{V}}\mathfrak{L}^{*n}(p_{\mathcal{W}})\tilde{p}_{\mathcal{V}} \leq \gamma\tilde{p}_{\mathcal{V}}\mathfrak{L}^{*n}(A(\mathcal{V}) - p_{\mathcal{V}})\tilde{p}_{\mathcal{V}} \to 0.$$

This implies that $\lim_{n\to+\infty}\|\tilde{\mathfrak{L}}^n(\sigma) - p_{\mathcal{V}}\tilde{\mathfrak{L}}^n(\sigma)p_{\mathcal{V}}\| = 0$, which, together with equation 3.4, implies that $\sigma = p_{\mathcal{V}}\sigma p_{\mathcal{V}}$. If we consider $\sigma$ as above and $\xi$ is such that $\tilde{\mathfrak{L}}(\xi) = e^{i\theta}\xi + \sigma$, with the same reasoning as before we can deduce that also $\xi = p_{\mathcal{V}}\xi p_{\mathcal{V}}$ and hence the algebraic multiplicity of $\mathrm{e}^{i\theta}$ is the same for $\tilde{\mathfrak{L}}$ and $\tilde{\mathfrak{L}}_{|\mathcal{V}}$.
1. By perturbation theory of linear matrices (see [42]), we only need to show that $\lambda_0 = 1$ is an algebraically simple eigenvalue of $\tilde{\mathfrak{L}}$, which, by virtue of point 1, is equivalent to prove it for $\tilde{\mathfrak{L}}_{|\mathcal{V}} = \mathfrak{L}_{|\mathcal{V}}$ and this follows for instance from [69, Proposition 6.2].
2. Notice that by definition $\tilde{\mathfrak{L}}_u$ preserves the set $p_{\mathcal{V}}L^1(\mathfrak{h})p_{\mathcal{V}}$ and eigenvalues and eigenvectors of $\tilde{\mathfrak{L}}_{u|\mathcal{V}}$ are also eigenvalues and eigenvectors of $\tilde{\mathfrak{L}}_u$. Let $\lambda_u^{\mathcal{V}}$ be the perturbation of 1 for $\tilde{\mathfrak{L}}_{|\mathcal{V}}$; by [42, Theorem VII.1.7] and the proof of point 2 in the present Lemma, for small values of $u$, $\lambda_u$ is the unique eigenvalue of $\tilde{\mathfrak{L}}_u$ in a neighborhood of 1 and it is algebraically simple, however $\lambda_u^{\mathcal{V}}$ is another eigenvalue of $\tilde{\mathfrak{L}}_u$ and $\lambda_0^{\mathcal{V}} = 1$ too, hence they must coincide in a neighborhood of the origin (remember that $u \mapsto \lambda_u^{\mathcal{V}}$ is continuous, see [42, Theorem 5.1]). Therefore we have that $\mathrm{supp}(\tau_u) \subset \mathcal{V}$. □

In Theorem 3.2.5 below, we shall apply Bryc's theorem to prove the central limit theorem for the position process. We quote it for the reader's convenience:

**Theorem 3.2.3** (Bryc, [10, Proposition 1])**.** *Let $(T_n)_{n\geq 0}$ be a sequence of random variables defined on the probability spaces $(\Omega_n, \mathcal{F}_n, \mathbb{P}_n)$, $T_n : \Omega_n \to \mathbb{R}^d$ and suppose there exists $\epsilon > 0$ such that*

$$h(u) = \lim_{n\to+\infty}\frac{1}{n}\log(\mathbb{E}_n[\mathrm{e}^{u\cdot T_n}])$$

*exists for every complex $u$ with $|u| < \epsilon$. Then*

$$\frac{(T_n - \mathbb{E}_n[T_n])}{\sqrt{n}} \longrightarrow \mathcal{N}(0, D) \qquad \textit{(in law)},$$

*where $\mathcal{N}(0, D)$ denotes a centered Gaussian measure with covariance $D = H(h)(0) \geq 0$ ( $H$ is the hessian of $h$ at $u = 0$), and*

$$\lim_{n\to+\infty}\frac{\mathbb{E}_n[T_n]}{n} = \nabla h(0).$$

The gradient and the hessian of the limit function $h$ will then describe the asymptotic mean and covariance matrix and the following lemma proves that they are related to the spectral radius of the perturbed operator restricted to the minimal enclosure $\mathcal{V}$.

**Lemma 3.2.4.** *The function $c : \mathbb{R}^d \ni u \mapsto \log(\lambda_u)$ is infinitely differentiable in 0. For every $u \in \mathbb{R}^d$, we introduce the operators $\mathfrak{L}'_{|\mathcal{V},u}$ and $\mathfrak{L}''_{|\mathcal{V},u}$ by*

$$\mathfrak{L}'_{|\mathcal{V},u}, \mathfrak{L}''_{|\mathcal{V},u} : L^1(\mathcal{V}) \longrightarrow L^1(\mathcal{V})$$

$$\mathfrak{L}'_{|\mathcal{V},u}(\sigma) = \sum_{i=1}^{v} u \cdot s_i L_i \sigma L_i^*, \qquad \mathfrak{L}''_{|\mathcal{V},u}(\sigma) = \sum_{i=1}^{v} (u \cdot s_i)^2 L_i \sigma L_i^*.$$

*Denoting $\lambda'_u = \frac{d\lambda_{tu}}{dt}\big|_{t=0}$, $\lambda''_u = \frac{d^2\lambda_{tu}}{dt^2}\big|_{t=0}$, we have*

$$\lambda'_u = \mathrm{tr}\left(\mathfrak{L}'_{|\mathcal{V},u}(\tau_0)\right), \qquad \lambda''_u = \mathrm{tr}\left(\mathfrak{L}''_{|\mathcal{V},u}(\tau_0)\right) + 2\mathrm{tr}\left(\mathfrak{L}'_{|\mathcal{V},u}(\eta_u)\right)$$

*where $\eta_u \in L^1(\mathcal{V})$ is the unique solution with zero trace of the equation*

$$(\mathrm{Id} - \mathfrak{L}_{|\mathcal{V}})(\eta_u) = \mathfrak{L}'_{|\mathcal{V},u}(\tau_0) - \mathrm{tr}\left(\mathfrak{L}'_{|\mathcal{V},u}(\tau_0)\right)\tau_0.$$

*This implies immediately that*

$$dc(0)(u) = \lambda'_u, \quad d^2c(0)(u) = \lambda''_u - {\lambda'_u}^2,$$

*where we denote by $dc(0)$ the differential of $c$ in 0 and by $d^2c(0)$ the bilinear form induced by the Hessian of $c$ in 0.*

*Proof.* Notice that

$$\mathfrak{L}'_{|\mathcal{V},u}(\sigma) = \sum_{i=1}^{v} u \cdot s_i \{p_{\mathcal{V}} L_i p_{\mathcal{V}}\} \sigma \{p_{\mathcal{V}} L_i^* p_{\mathcal{V}}\},$$

due to the fact that $\mathcal{V}$ is an enclosure (and similarly for $\mathfrak{L}''_{|\mathcal{V},u}(\sigma)$). This fact, together with Lemma 3.2.2, allows us to reduce the analysis to the irreducible channel $\mathfrak{L}_{|\mathcal{V}}$ and the proof is the same as in [14, Corollary 5.9], which we report below.
If we consider equation

$$\mathfrak{L}_{|\mathcal{V},tu}(\tau_{tu}) = \lambda_{tu}\tau_{tu}$$

and we differentiate it once in $t$ and we evaluate it at $t = 0$, we obtain

$$\mathfrak{L}'_{|\mathcal{V},u}(\tau_0) + \mathfrak{L}_{|\mathcal{V}}(\eta_u) = \lambda'_u\tau_0 + \eta_u, \tag{3.5}$$

where $\eta_u = \frac{d\tau_{tu}}{dt}\big|_{t=0}$. If we differentiate once more, we get

$$\mathfrak{L}''_{|\mathcal{V},u}(\tau_0) + 2\mathfrak{L}'_{|\mathcal{V},u}(\eta_u) + \mathfrak{L}_{|\mathcal{V}}(\sigma_u) = \lambda''_u\tau_0 + 2\lambda'_u\eta_u + \sigma_u \tag{3.6}$$

and $\sigma_u = \frac{d\eta_{tu}}{dt}\big|_{t=0}$. Remember that $\mathrm{tr}\,(\tau_u) = 1$, hence $\mathrm{tr}\,(\eta_u) = \mathrm{tr}\,(\sigma_u) = 0$ and that $\mathfrak{L}_{|\mathcal{V}}$ is trace preserving, hence if we trace equation (3.5) we obtain

$$\mathrm{tr}\left(\mathfrak{L}'_{|\mathcal{V},u}(\tau_0)\right) = \lambda'_u.$$

Substituting this new expression for $\lambda'_u$ into equation (3.5) we see that $\eta_u$ is the unique $(\dim(\mathrm{Id} - \ker(\mathfrak{L}_{|\mathcal{V}})) = 1)$ traceless solution of

$$(\mathrm{Id} - \mathfrak{L}_{|\mathcal{V}})(\eta_u) = \mathfrak{L}'_{|\mathcal{V},u}(\tau_0) - \mathrm{tr}\left(\mathfrak{L}'_{|\mathcal{V},u}(\tau_0)\right)\tau_0.$$

Finally, tracing out also equation (3.6) we obtain

$$\lambda''_u = \mathrm{tr}\left(\mathfrak{L}''_{|\mathcal{V},u}(\tau_0)\right) + 2\mathrm{tr}\left(\mathfrak{L}'_{|\mathcal{V},u}(\eta_u)\right).$$

□

**Theorem 3.2.5.** *Consider a minimal enclosure $\mathcal{V}$, and $\tau_0$ and $\lambda_u$ defined as before. We introduce the vector*

$$m = \sum_{i=1}^{v} \operatorname{tr}(L_i \tau_0 L_i^*)\, s_i$$

*and the matrix $D$ which is the unique matrix satisfying the following formula for every $u \in \mathbb{R}^d$:*

$$\langle u, Du \rangle = \lambda_u'' - {\lambda_u'}^2.$$

*Then, under $\mathbb{P}'_\rho$,*

$$\frac{(X_n - X_0) - nm}{\sqrt{n}} \to \mathcal{N}(0, D) \tag{3.7}$$

*where the convergence is in law. Moreover*

$$\left| \frac{\mathbb{E}'_\rho[X_n - X_0]}{n} - m \right| = O\left(\frac{1}{n}\right),$$

*where $\mathbb{E}'_\rho[\cdot]$ stays for the expected value with respect to $\mathbb{P}'_\rho$.*

*Remark* 3.2.6. We point out that, when there is a unique minimal enclosure $\mathcal{V}$, then $A(\mathcal{V}) = \mathbf{1}_{\mathfrak{h}}$, $\mathbb{P}'_\rho = \mathbb{P}_\rho$, and Theorem 3.2.5 is the central limit theorem for the position process (see [3, Theorem 5.2] and [14, Theorem 5.12]).

Notice that $m$ is the mean shift if the particle starts in the unique invariant state $\tau_0$ supported in $\mathcal{V}$, while the covariance matrix $D$ is the sum of the covariance matrix of the shift (always starting in $\tau_0$) plus another term taking into account the fact that the shifts are not independent one from the other.

*Proof.* With some calculation we get the expression for the moment generating function of the process $(X_n - X_0)$ for every $u \in \mathbb{C}^d$:

$$\mathbb{E}'_\rho[e^{u\cdot(X_n - X_0)}] = \frac{1}{\mathbb{E}_\rho[\operatorname{tr}(A(\mathcal{V})\rho_0)]} \sum_{\underline{k}\in V} \sum_{s_{j_1},\dots,s_{j_n}} e^{u\cdot\sum_{k=1}^{n} s_{j_k}} \operatorname{tr}\left(A(\mathcal{V})\tilde{L}_{j_n}\cdots\tilde{L}_{j_1}\rho(\underline{k})\tilde{L}_{j_1}^*\cdots\tilde{L}_{j_n}^*\right) =$$

$$= \sum_{\underline{k}\in V} \frac{\operatorname{tr}\left(A(\mathcal{V})\tilde{\mathfrak{L}}_u^n(\rho(\underline{k}))\right)}{\mathbb{E}_\rho[\operatorname{tr}(A(\mathcal{V})\rho_0)]}.$$

We are interested in the functions of the form

$$h_n(u) = \frac{1}{n}\log(\mathbb{E}'_\rho[e^{\langle u, X_n - X_0\rangle}]).$$

In order to apply Bryc's theorem (Theorem 3.2.3), we need to show the existence of $\lim_{n\to+\infty} h_n(u)$ for $u$ in a complex neighborhood of $0$. Let us first consider the case where $\tilde{\mathfrak{L}}_{|\mathcal{V}}$ is aperiodic. In this case we have

$$\delta = \sup\{|\lambda| : \lambda \in \operatorname{Sp}(\tilde{\mathfrak{L}}) \setminus \{1\}\} < 1$$

and so, considering the Jordan form of $\tilde{\mathfrak{L}}$, there exists $\epsilon > 0$ such that $\delta + \epsilon < 1$ and for $u$ in a neighbourhood of $0$, for $n \in \mathbb{N}$ we have

$$\tilde{\mathfrak{L}}_u^n(\cdot) = \lambda_u^n(\varphi_u(\cdot)\tau_u + O((\delta+\epsilon)^n))$$

where $\varphi_u$ is a linear form on $L^1(\mathfrak{h})$, analytic in $u$ in the considered neighbourhood of the origin and $O$ is with respect to any norm (remember that in finite dimension all the operator norms are equivalent). Therefore

$$\begin{aligned} h_n(u) &= \log(\lambda_u) \\ &\quad + \frac{1}{n}\left[-\log(\mathbb{E}_\rho[\mathrm{tr}\,(A(\mathcal{V})\rho_0)]) + \log\left(\sum_{\underline{k}\in V}\varphi_u(\rho(\underline{k}))\mathrm{tr}\,(A(\mathcal{V})\tau_u) + O((\delta+\epsilon)^n\right)\right] \\ &\to \log(\lambda_u). \end{aligned}$$

From the proof of Theorem 3.2.3 we know that all $h_n$ are analytic in a neighborhood of the origin. Further, these functions converge uniformly on compact sets to $h$ and $\sup_{u\in K}|h_n(u)-\log(\lambda_u)| = O(1/n)$ where $K$ is a compact set in the considered neighborhood of the origin. Hence, by Cauchy integral formula we can deduce, since

$$\frac{\mathbb{E}'_\rho[X_n - X_0]}{n} = \nabla h_n(0) \text{ and } m = \nabla h(0)$$

that

$$\left|\frac{\mathbb{E}'_\rho[X_n - X_0]}{n} - m\right| = O\left(\frac{1}{n}\right)$$

and this allows us to put $nm$ instead of $\mathbb{E}'_\rho[X_n - X_0]$ in equation (3.7).

On the other hand, if $\mathfrak{L}_{|\mathcal{V}}$ has period $l > 1$ with cyclic resolution $p_0, \ldots, p_{l-1}$, we can write for $n = ql + r$ and $0 \le r < l$

$$\mathbb{E}'_\rho[e^{u\cdot(X_n - X_0)}] = \sum_{j=0}^{l-1}\underbrace{\sum_{\underline{z}\in V}\frac{\mathrm{tr}\,(A(p_j)\rho(\underline{z}))}{\mathbb{E}_\rho[\mathrm{tr}\,(A(\mathcal{V})\rho_0)]}}_{w_j}\underbrace{\sum_{\underline{k}\in V}\frac{\mathrm{tr}\left(A(p_j)\tilde{\mathfrak{L}}_u^n(\rho(\underline{k}))\right)}{\sum_{\underline{z}\in V}\mathrm{tr}\,(A(p_j)\rho(\underline{z}))}}_{II}.$$

We can safely define $A(p_j)$ using $\tilde{\mathfrak{L}}^l$, for which $p_0, \ldots, p_{l-1}$ are minimal enclosures (see Proposition 1.2.12). Furthermore we can express $II$ as

$$II = \sum_{\underline{k}\in V}\frac{\mathrm{tr}\left(A(p_j)\tilde{\mathfrak{L}}_u^{lq}(\tilde{\mathfrak{L}}_u^r(\rho(\underline{k})))\right)}{\sum_{\underline{z}\in V}\mathrm{tr}\,(A(p_j)\rho(\underline{z}))}.$$

The support projection of $A(p_j)$, which we call $P_j$, is superharmonic for $\tilde{\mathfrak{L}}^l$, hence, if we consider $\tilde{\mathfrak{L}}^l_{j,u} := P_j\tilde{\mathfrak{L}}^l(P_j\cdot P_j)P_j$, we can write

$$\mathrm{tr}\left(A(p_j)\tilde{\mathfrak{L}}_u^{lq}(\tilde{\mathfrak{L}}_u^r(\rho(\underline{k})))\right) = \mathrm{tr}\left(A(p_j)\tilde{\mathfrak{L}}_{j,u}^{lq}(P_j\tilde{\mathfrak{L}}_u^r(\rho(\underline{k}))P_j)\right)$$

and we are back to the aperiodic case. Furthermore the perturbation of $1$ for every reduction $\tilde{\mathfrak{L}}^l_{j,u}$ is the same as the one of $\tilde{\mathfrak{L}}^l_u$ since $P_j\tau_uP_j$ is an eigenvector of $\tilde{\mathfrak{L}}^l_{j,u}$ for the eigenvalue $\lambda_u^l$:

$$\tilde{\mathfrak{L}}^l_{j,u}(P_j\tau_uP_j) = P_j\tilde{\mathfrak{L}}^l_u(\tau_u)P_j = \lambda_u^lP_j\tau_uP_j.$$

Therefore we can again prove the statement. □

## 3.3 General case: mixture of Gaussians

In order to tackle the general case, we now need to consider different enclosures and to handle the simultaneous appearance of different Gaussians. The description of the general context is tightly related to the decomposition of $\mathfrak{h}$ provided by Proposition 2.3.5, which is the noncommutative counterpart of the decomposition in communication classes for classical Markov chains. Such a splitting of the Hilbert space will induce a decomposition of the measure $\mathbb{P}_\rho$ in terms of measures of the form $\mathbb{P}'_\rho$ as defined in Lemma 3.2.1. Below we shall briefly recall some notation and the main properties related to the decomposition of $\mathfrak{h}$, since we will need them in the present section.

**Decomposition of the local Hilbert space and of the recurrent subspace.**
Since we are working under the assumption that $\mathfrak{h}$ is finite dimensional, there is no null recurrent space and the positive recurrent space is non trivial and absorbing, i.e. we can write

$$\mathfrak{h} = \mathcal{R} \oplus \mathcal{T}, \qquad A(\mathcal{R}) = \mathbf{1}_{\mathfrak{h}} - \lim_{n\to+\infty} \mathfrak{L}^{*n}(p_{\mathcal{T}}) = \mathbf{1}_{\mathfrak{h}}.$$

Moreover any minimal enclosure is included in $\mathcal{R}$ and is the support of a unique extremal invariant state. $\mathfrak{L}$ induces a unique decomposition of $\mathcal{R}$ of the form

$$\mathcal{R} = \bigoplus_{\alpha\in A} \chi_\alpha, \tag{3.8}$$

where $(\chi_\alpha)_{\alpha\in A}$ is a finite set of mutually orthogonal enclosures and every $\chi_\alpha$ is minimal in the set of enclosures verifying the property:

for any minimal enclosure $\mathcal{W}$ either $\mathcal{W} \perp \chi_\alpha$ or $\mathcal{W} \subset \chi_\alpha$.

Every $\chi_\alpha$ either is a minimal enclosure or can be further decomposed (but not in a unique way!) as the sum of mutually orthogonal isomorphic minimal enclosures, i.e.

$$\chi_\alpha = \bigoplus_{\beta\in I_\alpha} \mathcal{V}_{\alpha,\beta}, \qquad \mathcal{R} = \bigoplus_{\alpha\in A} \chi_\alpha = \bigoplus_{\alpha\in A} \bigoplus_{\beta\in I_\alpha} \mathcal{V}_{\alpha,\beta}, \tag{3.9}$$

for some finite set $\mathcal{V}_{\alpha,\beta}$, $\beta \in I_\alpha$ of minimal enclosures and, if we fix a particular $\bar\beta \in I_\alpha$, there exists a unitary transformation $U_\alpha$ such that

$$U_\alpha : \mathbb{C}^{|I_\alpha|} \otimes \mathcal{V}_{\alpha,\bar\beta} \to \chi_\alpha. \tag{3.10}$$

Moreover one can define an irreducible quantum channel $\psi$ on $B(\mathcal{V}_{\alpha,\bar\beta})$ which completely describes the restriction of the channel to $\chi_\alpha$

$$\mathfrak{L}^*_{|\mathcal{R}}(U_\alpha(a\otimes b)U_\alpha^*) = U_\alpha(a\otimes\psi(b))U_\alpha^* \qquad a \in B(\mathbb{C}^{|I_\alpha|}),\, b \in B(\mathcal{V}_{\alpha,\bar\beta}). \tag{3.11}$$

*Remark* 3.3.1. $\chi_\alpha$ is a minimal enclosure if and only if $|I_\alpha| = 1$. Otherwise, it is not minimal and it admits infinite possible decompositions in orthogonal minimal enclosures of the form $U_\alpha(\mathbb{C}v \otimes \mathcal{V}_{\alpha,\bar\beta})$ for $v \in \mathbb{C}^{|I_\alpha|}$. In this case, however, a rigid structure of the channel essentially reduces the action on any minimal enclosure inside $\chi_\alpha$ to be the same up to a unitary transform.

**Lemma 3.3.2.** *The parameters $m = m(\mathcal{V})$ and $D = D(\mathcal{V})$ introduced in Theorem 3.2.5 are independent of the particular minimal enclosure $\mathcal{V}$ in $\chi_\alpha$. Then we define*

$$m_\alpha := \sum_{i=1}^{v} \operatorname{tr}\left(L_i \tau_0^{\mathcal{V}} L_i^*\right) s_i, \qquad \langle u, D_\alpha u\rangle = \lambda''_u - {\lambda'_u}^2,$$

*where $\lambda'_u, \lambda''_u$ are defined as in Lemma 3.2.4 for $\mathfrak{L}_{|\mathcal{V}}$.*

*Proof.* Let us consider two minimal enclosures $\mathcal{V}$ and $\mathcal{W}$ contained in a same $\chi_\alpha$. We just have to prove that the parameters $m$ and $D$ are equal for the two enclosures.

Relations (3.10) and (3.11) imply that there exist two vectors $v, w$ in $\mathbb{C}^{|I_\alpha|}$ such that

$$\mathcal{V} = U_\alpha((\mathbb{C}v) \otimes \mathcal{V}_{\alpha,\bar{\beta}})U_\alpha^*, \qquad \mathcal{W} = U_\alpha((\mathbb{C}w) \otimes \mathcal{V}_{\alpha,\bar{\beta}})U_\alpha^*, \tag{3.12}$$

and we can define a partial isometry $Q = U_\alpha((|w\rangle\langle v|) \otimes 1_{\mathcal{V}_{\alpha,\bar{\beta}}})U_\alpha^*$, from $\mathcal{V}$ to $\mathcal{W}$, such that

$$Q^*Q = p_{\mathcal{V}}, \quad QQ^* = p_{\mathcal{W}} \quad \text{and} \quad \mathfrak{L}^*_{|\mathcal{V}}(x) = Q^*\mathfrak{L}^*_{|\mathcal{W}}(QxQ^*)Q \qquad \forall x \in B(\mathcal{V}), \tag{3.13}$$

where $\mathfrak{L}_{|\mathcal{V}}$ and $\mathfrak{L}_{|\mathcal{W}}$ are the restrictions of $\mathfrak{L}$ to $\mathcal{V}$ and $\mathcal{W}$ respectively, following the notations introduced before. Due to relation (3.11), $Q$ (and $Q^*$) is also a fixed point for the dual channel $\mathfrak{L}^*$, so that it commutes with the Kraus operators $L_i, L_i^*$ for all $i$ (see for instance [13], in particular Proposition 1 applied to the fast recurrent channel $\mathfrak{L}$ restricted to $\chi_\alpha$).

Moreover, since $\mathcal{V}$ and $\mathcal{W}$ are minimal, they are the support of two invariant states, that we can denote by $\tau_0^{\mathcal{V}}$ and $\tau_0^{\mathcal{W}}$ and will verify

$$\tau_0^{\mathcal{V}} = Q^*\tau_0^{\mathcal{W}}Q.$$

Then we have

$$\mathrm{tr}\left(L_i\tau_0^{\mathcal{W}}L_i^*\right) = \mathrm{tr}\left(L_iQ\tau_0^{\mathcal{V}}Q^*L_i^*\right) = \mathrm{tr}\left(QL_i\tau_0^{\mathcal{V}}L_i^*Q^*\right) = \mathrm{tr}\left(p_{\mathcal{V}}L_i\tau_0^{\mathcal{V}}L_i^*\right) = \mathrm{tr}\left(L_i\tau_0^{\mathcal{V}}L_i^*\right)$$

so that

$$m(\mathcal{W}) = \sum_i \mathrm{tr}\left(L_i\tau_0^{\mathcal{W}}L_i^*\right)s_i = \sum_i \mathrm{tr}\left(L_i\tau_0^{\mathcal{V}}L_i^*\right)s_i = m(\mathcal{V}).$$

Similarly we deduce, for any $u \in \mathbb{R}^d$,

$$\mathfrak{L}'_{|\mathcal{V},u}(Q^* \cdot Q) = Q^*\mathfrak{L}'_{|\mathcal{W},u}(\cdot)Q, \quad \mathfrak{L}''_{|\mathcal{V},u}(Q^* \cdot Q) = Q^*\mathfrak{L}''_{|\mathcal{W},u}(\cdot)Q.$$

Therefore

$$\mathrm{tr}\left(\mathfrak{L}'_{|\mathcal{V},u}(\tau_0^{\mathcal{V}})\right) = \mathrm{tr}\left(\mathfrak{L}'_{|\mathcal{W},u}(\tau_0^{\mathcal{W}})\right) \qquad \text{and} \qquad \mathrm{tr}\left(\mathfrak{L}''_{|\mathcal{V},u}(\tau_0^{\mathcal{V}})\right) = \mathrm{tr}\left(\mathfrak{L}''_{|\mathcal{W},u}(\tau_0^{\mathcal{W}})\right).$$

By the same arguments, for all $u \in \mathbb{R}^d$,

$$\eta_u^{\mathcal{V}} = Q^*\eta_u^{\mathcal{W}}Q \qquad \text{and} \qquad \mathrm{tr}\left(\mathfrak{L}'_{|\mathcal{V},u}(\eta_u^{\mathcal{V}})\right) = \mathrm{tr}\left(\mathfrak{L}'_{|\mathcal{W},u}(\eta^{\mathcal{W}})\right)$$

and we can conclude that $D(\mathcal{V}) = D(\mathcal{W})$. □

**Decomposition of the measure $\mathbb{P}_\rho$.**

In Lemma 3.2.1, we fixed an enclosure $\mathcal{V}$ and we introduced the probability measure denoted by $\mathbb{P}'_\rho$. Now we need to handle different enclosures, the ones appearing in the decomposition of $\mathcal{R}$ given in relations (3.9). We need to highlight the dependence on the enclosure and we shall denote from now on by $\mathbb{P}^\alpha$ (resp.ly $\mathbb{P}^{\alpha,\beta}$) the measure $\mathbb{P}'_\rho$ obtained with $\mathcal{V} = \chi_\alpha$ (resp.ly $\mathcal{V} = \mathcal{V}_{\alpha,\beta}$), i.e. with densities

$$\left.\frac{d\mathbb{P}_\rho^\alpha}{d\mathbb{P}_\rho}\right|_{\mathcal{F}_n} = \frac{\mathrm{tr}\left(A(\chi_\alpha)\rho_n\right)}{\mathbb{E}_\rho[\mathrm{tr}\left(A(\chi_\alpha)\rho_0\right)]}, \qquad \left.\frac{d\mathbb{P}_\rho^{\alpha,\beta}}{d\mathbb{P}_\rho}\right|_{\mathcal{F}_n} = \frac{\mathrm{tr}\left(A(\mathcal{V}_{\alpha,\beta})\rho_n\right)}{\mathbb{E}_\rho[\mathrm{tr}\left(A(\mathcal{V}_{\alpha,\beta})\rho_0\right)]}. \tag{3.14}$$

We can then decompose $\mathbb{P}_\rho$ into a mixture of $\mathbb{P}_\rho^\alpha$ and $\mathbb{P}_\rho^{\alpha,\beta}$.

**Lemma 3.3.3.** *For any $\alpha \in A$, $\beta \in I_\alpha$ let us define*

$$\begin{aligned} a_\alpha(\rho) &:= \mathbb{E}_\rho[Y_0^\alpha] = \mathbb{E}_\rho[\mathrm{tr}\,(A(\chi_\alpha)\rho_0)] = \sum_{\underline{k}\in V} \mathrm{tr}\,(A(\chi_\alpha)\rho(\underline{k})) \\ \text{and} \qquad a_{\alpha,\beta}(\rho) &:= \mathbb{E}_\rho[Y_0^{\alpha,\beta}] = \mathbb{E}_\rho[\mathrm{tr}\,(A(\mathcal{V}_{\alpha,\beta})\rho_0)] = \sum_{\underline{k}\in V} \mathrm{tr}\,(A(\mathcal{V}_{\alpha,\beta})\rho(\underline{k}))\,. \end{aligned}$$

*Then we can write $\mathbb{P}_\rho$ as convex combination*

$$\mathbb{P}_\rho = \sum_{\alpha\in A} a_\alpha(\rho)\mathbb{P}_\rho^\alpha = \sum_{\alpha\in A}\sum_{\beta\in I_\alpha} a_{\alpha,\beta}(\rho)\mathbb{P}_\rho^{\alpha,\beta}. \tag{3.15}$$

*Proof.* Indeed, for every $\underline{k} \in V$, $n \geq 0$, $\underline{j} \in J^n$

$$\begin{aligned} \mathbb{P}_\rho(\{(\underline{k},\underline{j})\}\times J^{\mathbb{N}}) &= \mathrm{tr}\left(L_{\underline{j}}\rho(\underline{k})L_{\underline{j}}^*\right) = \sum_{\alpha\in A}\mathrm{tr}\left(A(\chi_\alpha)L_{\underline{j}}\rho(\underline{k})L_{\underline{j}}^*\right) \\ &= \sum_{\alpha\in A} a_\alpha(\rho)\cdot\mathrm{tr}\left(L_{\underline{j}}\rho(\underline{k})L_{\underline{j}}^*\right)\frac{1}{\mathbb{E}_\rho[\mathrm{tr}\,(A(\chi_\alpha)\rho_0)]}\mathrm{tr}\left(A(\chi_\alpha)\frac{L_{\underline{j}}\rho(\underline{k})L_{\underline{j}}^*}{\mathrm{tr}\left(L_{\underline{j}}\rho(\underline{k})L_{\underline{j}}^*\right)}\right) \\ &= \sum_{\alpha\in A} a_\alpha(\rho)\mathbb{P}_\rho^\alpha(\{(\underline{k},\underline{j})\}\times J^{\mathbb{N}}). \end{aligned}$$

where the second equality follows because $\sum_{\alpha\in A} A(\chi_\alpha) = 1_{\mathfrak{h}}$. Similarly one can further decompose the probability measure in $\mathbb{P}_\rho^{\alpha,\beta}$ because for every $\alpha \in A$, $\sum_{\beta\in B\alpha} A(\mathcal{V}_{\alpha,\beta}) = A(\chi_\alpha)$. Equation (3.15) is then true because sets of the form $\{(\underline{k},\underline{j})\}\times J^{\mathbb{N}}$ generate $\mathcal{F}$. □

**Generalized Central Limit Theorem**

The convergence in law is metrizable by different distances. On this subject, we refer for instance to [24]. Among them, we choose the Fortet-Mourier metric, but the convergence results keep holding true also with a different choice. Let us denote by BL the set of bounded Lipschitz functions on $\mathbb{R}^d$ equipped with the norm

$$\|f\|_{BL} = \sup_{x\in\mathbb{R}^d}|f(x)| + \sup_{x\neq y}\frac{|f(x)-f(y)|}{|x-y|};$$

we introduce the Fortet-Mourier distance between two probability laws $P, Q$ on $\mathbb{R}^d$,

$$\mathrm{dist}(P,Q) := \sup\left\{\Big|\int_{\mathbb{R}^d} f dP - \int_{\mathbb{R}^d} f dQ\Big| : f\in BL, \|f\|_{BL}\leq 1\right\}.$$

We recall that [24, Theorem 11.3.3], for $P_n$, $P$ probability measures on $\mathbb{R}^d$, the following fact holds

$$P_n \to P \text{ in law} \qquad \text{if and only if} \qquad \mathrm{dist}(P_n, P)\to 0.$$

We are now in a position to state the "generalized Central Limit Theorem".

**Theorem 3.3.4. Convergence to mixture of Gaussians.**
*Take $m_\alpha$ and $D_\alpha$ as in Lemma 3.3.2 and let $\mathbb{P}_{\rho,n}$ be the law of the process $\frac{X_n-X_0}{\sqrt{n}}$ under $\mathbb{P}_\rho$. Then*

$$\lim_{n\to+\infty} \mathrm{dist}\left(\mathbb{P}_{\rho,n}, \sum_{\alpha\in A} a_\alpha(\rho)\mathcal{N}(\sqrt{n}m_\alpha, D_\alpha)\right) = 0,$$

*where $a_\alpha(\rho) = \mathbb{E}_\rho[\mathrm{tr}\,(A(\chi_\alpha)\rho_0)]$ and $\mathcal{N}(\sqrt{n}m_\alpha, D_\alpha)$ denotes the Gaussian measure with mean $\sqrt{n}m_\alpha$ and covariance matrix $D_\alpha$.*

*Proof.* By Theorem 3.2.5, we know that the process $\frac{X_n-X_0-nm_\alpha}{\sqrt{n}}$ converges in law to a centered normal distribution with covariance matrix $D_\alpha$ under the measure $\mathbb{P}_\rho^{\alpha,\beta}$, so that we can write

$$\lim_{n\to+\infty} \mathrm{dist}\left(\mathbb{P}_\rho^{\alpha,\beta}\left(\frac{X_n-X_0-nm_\alpha}{\sqrt{n}}\right), \mathcal{N}(0, D_\alpha)\right) = 0.$$

By definition, the Fortet-Mourier distance is invariant with respect to translations and consequently we deduce

$$\mathrm{dist}\left(\mathbb{P}_\rho^{\alpha,\beta}\left(\frac{X_n-X_0}{\sqrt{n}}\right), \mathcal{N}(\sqrt{n}m_\alpha, D_\alpha)\right) \longrightarrow 0, \qquad \text{as } n\to+\infty.$$

Now, since this limit does not depend on $\beta$ and, by equation (3.15) $\mathbb{P}_\rho^\alpha = \sum_{\beta\in I_\alpha} a_{\alpha,\beta}(\rho)\mathbb{P}_\rho^{\alpha,\beta}$ (we denote by $\mathcal{N}_\alpha$ the law $\mathcal{N}(\sqrt{n}m_\alpha, D_\alpha)$ to shorten the expressions in this proof),

$$\begin{aligned}
&\mathrm{dist}\left(\mathbb{P}_\rho^{\alpha}\left(\frac{X_n-X_0}{\sqrt{n}}\right), \mathcal{N}_\alpha\right) = \\
&= \sup\left\{\Big|\int_{\mathbb{R}^d} f\left(\frac{X_n-X_0}{\sqrt{n}}\right)d\mathbb{P}_\rho^\alpha - \int_{\mathbb{R}^d} f d\mathcal{N}_\alpha\Big| : f\in BL, \|f\|_{BL}\le 1\right\} \\
&\le \sum_{\beta\in I_\alpha} a_{\alpha,\beta}(\rho)\sup\left\{\Big|\int_{\mathbb{R}^d} f\left(\frac{X_n-X_0}{\sqrt{n}}\right)d\mathbb{P}_\rho^{\alpha,\beta} - \int_{\mathbb{R}^d} f d\mathcal{N}_\alpha\Big| : f\in BL, \|f\|_{BL}\le 1\right\} \\
&= \sum_{\beta\in I_\alpha} a_{\alpha,\beta}(\rho)\mathrm{dist}\left(\mathbb{P}_\rho^{\alpha,\beta}\left(\frac{X_n-X_0}{\sqrt{n}}\right), \mathcal{N}_\alpha\right) \longrightarrow 0, \qquad \text{as } n\to+\infty.
\end{aligned}$$

Similarly, always by relation (3.15), $\mathbb{P}_\rho = \sum_{\alpha\in A} a_\alpha(\rho)\mathbb{P}_\rho^\alpha$ and by triangular inequality for any $f$ in $BL$, we can call $\nu_n = \sum_{\alpha\in A} a_\alpha(\rho)\mathcal{N}_\alpha$ and write

$$\Big|\int_{\mathbb{R}^d} f\left(\frac{X_n-X_0}{\sqrt{n}}\right)d\mathbb{P}_\rho - \int_{\mathbb{R}^d} f d\nu_n\Big| \le \sum_{\alpha\in A} a_\alpha(\rho)\Big|\int_{\mathbb{R}^d} f\left(\frac{X_n-X_0}{\sqrt{n}}\right)d\mathbb{P}_\rho^\alpha - \int_{\mathbb{R}^d} f d\mathcal{N}_\alpha\Big|$$

and we then conclude

$$\mathrm{dist}\left(\mathbb{P}_{\rho,n}, \nu_n\right) \le \sum_{\alpha\in A} a_\alpha(\rho)\mathrm{dist}\left(\mathbb{P}_{\rho,n}^\alpha, \mathcal{N}_\alpha\right),$$

which converges to 0 as $n\to+\infty$. □

Notice that, while the weights $a_\alpha(\rho)$ depend on the initial state and on the transient part of $\mathfrak{L}$, the parameters of the Gaussian measures only depend on the restriction to the fast recurrent part. Theorem 3.3.4 has the following direct consequence on the convergence of the empirical means.

**Corollary 3.3.5.** *Let $\hat{\mathbb{P}}_{\rho,n}$ the law of the process $\frac{X_n - X_0}{n}$ under $\mathbb{P}_\rho$, then*

$$\lim_{n\to+\infty} \operatorname{dist}\left(\hat{\mathbb{P}}_{\rho,n}, \sum_{\alpha\in A} a_\alpha(\rho)\delta_{m_\alpha}\right) = 0,$$

*where $a_\alpha(\rho)$ are defined as in previous theorem and $\delta_{m_\alpha}$ denotes the Dirac measure concentrated in $m_\alpha$.*

*Remark* 3.3.6. **Possible Extensions**. As for previous versions of central limit theorems for HOQRWs, we can extend our results to more general cases.

1. There is an immediate generalization of HOQRW obtained considering a change in the local state after a shift $s_i$ given by a quantum operation $\mathfrak{L}_j$ with more than one Kraus operator, which is the case we considered ($\mathfrak{L}_j(\cdot) = L_j \cdot L_j^*$). In this case it suffices to change the notation in the proof of Theorem 3.3.4 to see that it still holds true.

2. Open quantum random walks have been defined also in continuous time ( [54]) and the central limit theorem for the position process has already been proved in [9], under the assumption of irreducibility of $\mathcal{R}$. Theorem 3.3.4 can be carried with some technical adaptations to the continuous time case.

*Remark* 3.3.7. **Comparison with previous results.** The first CLT for HOQRWs appeared in [3] where the authors proved it by the use of Poisson equation and martingale techniques in the case $\mathcal{R}$ irreducible. Indeed, in [3, Theorem 7.3] they showed the convergence to different Gaussian measures under proper conditional probabilities and under assumptions which can be translated in our language to be

- $\mathcal{T} = \{0\}$,
- $\chi_\alpha$ is minimal for every $\alpha \in A$,
- $m_\alpha \neq m_{\alpha'}$ if $\alpha \neq \alpha'$.

These techniques revealed to be successful to treat also other walks and in particular have recently been exploited also in [43] to obtain a CLT for the so-called lazy OQWs. Successively, in [14], an alternative proof of the central limit theorem for an irreducible fast recurrent local channel $\mathfrak{L}$ could be deduced from a large deviation principle, proved by deformation techniques. Finally the results in [44–46] (which are formulated in the setting of homogeneous open quantum walks on crystal lattices) state a kind of convergence to a mixture of Gaussian measures, under some conditions, always essentially implying that the local channel is fast recurrent.

Here, with Theorem 3.3.4, we can find an improvement of all these previous results since we can drop any condition about recurrence or transience or reducibility of the local channel and we can specify the form of convergence to the mixture of Gaussians introducing a metric on the set of probability measures. Moreover we can specify the weights of the limit mixture in terms of the initial state and of the decomposition of the local space.

We refer the reader to [62] for other hints on the existing literature until 2019 and to [9, 54, 57] for CLT results for different families of open walks.

## 3.4 Large Deviations

We start this section recalling the basic notions about large deviations that we will need. If we consider a $\mathbb{R}^d$ -valued stochastic process $(Z_n)$, it is said to satisfy a large deviation principle with good rate function if there exists a function $\Lambda : \mathbb{R}^d \to [0, +\infty]$ (the rate function) with compact sublevel sets and such that for every Borel set $B \subset \mathbb{R}^d$ the following holds true

$$-\inf_{x\in\mathring{B}} \Lambda(x) \leq \liminf_{n\to+\infty} \frac{1}{n} \log(\mathbb{P}(Z_n \in B)) \leq \limsup_{n\to+\infty} \frac{1}{n} \log(\mathbb{P}(Z_n \in B)) \leq -\inf_{x\in\overline{B}} \Lambda(x),$$

where $\mathring{B}$ denotes the interior and $\overline{B}$ is the closure of $B$. Gärtner-Ellis' theorem provides a powerful tool in order to get explicit asymptotic upper and lower bounds on the laws of $Z_n$ and, in some cases, a proof of the large deviation principle. Before recalling the statement of the theorem, we need to provide a definition.

**Definition 3.4.1** (Exposing hyperplane)**.** *Let $\Lambda : \mathbb{R}^d \to [0, +\infty]$ be a convex function, then $y \in \mathbb{R}^d$ is an exposed point of $\Lambda$ if for some $\eta \in \mathbb{R}^d$ and all $x \neq y$*

$$\langle \eta, y\rangle - \Lambda(y) > \langle \eta, x\rangle - \Lambda(x).$$

*$\eta$ is called an exposing hyperplane.*

The geometric meaning of the previous condition is that the hyperplane orthogonal to $\eta$ and passing by $(y, \Lambda(y))$ layes below the graph of $\Lambda$ and touches it only in $(y, \Lambda(y))$.

**Theorem 3.4.2** (Gärtner-Ellis, Theorem 2.3.6, [21])**.** *Let $(Z_n)_{n\in\mathbb{N}}$ be a $\mathbb{R}^d$-valued stochastic process. Suppose that the (limiting) logarithmic moment generating function*

$$\lambda(u) = \lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}\left[e^{nu\cdot Z_n}\right], \quad u \in \mathbb{R}^d$$

*exists as an extended real number and is finite in a neighbourhood of the origin, and let $\Lambda$ denote the Fenchel-Legendre transform of $\lambda$, given by*

$$\Lambda(x) = \sup_{u\in\mathbb{R}^d} \{ \langle x, u\rangle - \lambda(u) \}.$$

*Then for any measurable $B \in \mathcal{B}(\mathbb{R}^d)$*

*(a)* $\limsup_{n\to+\infty} \frac{1}{n} \log(\mathbb{P}(Z_n \in B)) \leq -\inf_{x\in\overline{B}} \Lambda(x)$,

*(b)* $\liminf_{n\to+\infty} \frac{1}{n} \log(\mathbb{P}(Z_n \in B)) \geq -\inf_{x\in\mathring{B}\cap\mathcal{S}} \Lambda(x)$, *where $\mathcal{S}$ is the set of exposed points of $\Lambda$ whose exposing hyperplane belongs to the domain of $\lambda$.*

*(c) If $\lambda$ is smooth, then the large deviation principle holds with good rate function $\Lambda$.*

When the central limit theorem is approached by Bryc's theorem, it is often treated together with large deviations, and this was indeed the idea in [14], where the proof of the central limit theorem in the particular case of an irreducible fast recurrent subspace was a byproduct of the large deviation principle. Similarly, it is here natural to wonder whether a large deviation principle can hold in general for the position process of a

HOQRW, always under the measure $\mathbb{P}_\rho$ induced by the initial state $\rho$. We shall prove that Gärtner-Ellis' theorem can be applied and thus large deviations hold when the local map is recurrent. Moreover, the rate function is related to the spectrum of the deformed map $\mathfrak{L}_u$. When instead there is a non trivial transient subspace for the local channel $\mathfrak{L}$, the limit of the moment generating functions is not smooth in general, as [14, Example 7.3] shows, and Gärtner-Ellis' theorem will simply provide lower and upper bounds.

As for the results in the previous section, only the minimal enclosures in the decomposition of $\mathcal{R}$ that are "reachable" by a initial state $\rho$ will play a role in the large deviations results. For this reason, it is useful to remember the definition of the quantities $a_\alpha(\rho)$, $a_{\alpha,\beta}(\rho)$ (introduced in Lemma 3.3.3), which are a kind of quantum absorption probabilities of the evolution in the enclosures $\chi_\alpha$, or $\mathcal{V}_{\alpha,\beta}$ respectively, when the initial state is $\rho$. Differently from the central limit type results, here also the index $\beta$, and so the particular enclosures $\mathcal{V}_{\alpha,\beta}$ selected inside $\chi_\alpha$ are important, and this is related to the fact that the evolution on the transient subspace affects large deviations results.

Since we need to define restrictions of the channel $\mathfrak{L}$ which take into account only proper subspaces reachable by the local initial states $\rho(\underline{k})$, we define the subspace

$$\mathcal{E}(\rho) := \operatorname{span}\{\operatorname{supp}(\mathfrak{L}^n(\rho(\underline{k}))), \underline{k} \in V, n \geq 0\} \subset \mathfrak{h},$$

which is an enclosure due to [15, Propositions 4.1 and 4.2].

We recall that by $\hat{\mathbb{P}}_{\rho,n}$ we denote the law of $\frac{X_n - X_0}{n}$ under $\mathbb{P}_\rho$ and, given any enclosure $\mathcal{V}$, $\tilde{p}_\mathcal{V}$ is the orthogonal projection onto $\operatorname{supp}(A(\mathcal{V}))$.

**Theorem 3.4.3. Large deviation principle.** *Suppose that the local map $\mathfrak{L}$ is recurrent, i.e. $\mathcal{R} = \mathfrak{h}$. Then $(\hat{\mathbb{P}}_{\rho,n})_{n\geq 1}$ satisfies a large deviation principle with good rate function*

$$\Lambda_\rho(x) = \min_{\alpha\,:\, a_\alpha(\rho)\neq 0} \Lambda_\alpha(x),$$

*where $\Lambda_\alpha$ is the Fenchel-Legendre transform of the logarithm of the spectral radius $\lambda_{\alpha,u}$ of $\mathfrak{L}_{|\chi_\alpha,u}$, i.e.*

$$\lambda_{\alpha,u} = r(\mathfrak{L}_{|\chi_\alpha,u}), \qquad \Lambda_\alpha(x) = \sup_{u\in\mathbb{R}^d}\{\langle u, x\rangle - \log(\lambda_{\alpha,u})\} \quad x \in \mathbb{R}^d.$$

**Theorem 3.4.4. Large deviations upper and lower bounds.** *For any measurable $B \in \mathcal{B}(\mathbb{R}^d)$*

- $\limsup_{n\to+\infty} \frac{1}{n}\log(\hat{\mathbb{P}}_{\rho,n}(B)) \leq -\inf_{x\in\overline{B}} \min_{\alpha,\beta\,:\, a_{\alpha,\beta}(\rho)\neq 0} \Lambda^\rho_{\alpha,\beta}(x)$,
- $\liminf_{n\to+\infty} \frac{1}{n}\log(\hat{\mathbb{P}}_{\rho,n}(B)) \geq -\min_{\alpha,\beta\,:\, a_{\alpha,\beta}(\rho)\neq 0} \inf_{x\in\mathring{B}\cap\mathcal{S}_{\alpha,\beta}} \Lambda^\rho_{\alpha,\beta}(x)$

*where*

- $\lambda^\rho_{\alpha,\beta,u} = r(\mathfrak{L}_{|\mathcal{Q}^\rho_{\alpha,\beta},u})$ *for* $\mathcal{Q}^\rho_{\alpha,\beta} := \tilde{p}_{\mathcal{V}_{\alpha,\beta}}\mathcal{E}(\rho)$,
- $\Lambda^\rho_{\alpha,\beta}(x) = \sup_{u\in\mathbb{R}^d}\{\langle u, x\rangle - \log(\lambda^\rho_{\alpha,\beta,u})\}$ *is the Fenchel-Legendre transform of* $\log(\lambda^\rho_{\alpha,\beta,u})$,
- $\mathcal{S}_{\alpha,\beta} = \mathbb{R}^d$ *if $\lambda^\rho_{\alpha,\beta,u}$ is smooth, otherwise $\mathcal{S}_{\alpha,\beta}$ is the set of exposed points of $\Lambda^\rho_{\alpha,\beta}$.*

*Remark* 3.4.5. We remark that, whenever $a_{\alpha,\beta}(\rho) \neq 0$, $\mathcal{Q}^{\rho}_{\alpha,\beta}$ is non trivial and

$$\mathcal{Q}^{\rho}_{\alpha,\beta} = \mathcal{V}_{\alpha,\beta} \oplus (\mathcal{T} \cap \mathcal{Q}^{\rho}_{\alpha,\beta}) \subset \operatorname{supp}(A(\mathcal{V}_{\alpha,\beta}))$$

(see the first step in the proof of Theorem 3.4.4).

We shall prove the two theorems in inverse order. The proof will request different steps and we shall proceed similarly as we did for central limit theorems, first considering the measure $\mathbb{P}'_\rho$ associated with the absorption in a single minimal enclosure (Lemma 3.2.1), and then generalizing using the expression of $\mathbb{P}_\rho$ as a convex combination given in Lemma 3.3.3.

*Proof of Theorem 3.4.4.* **Step 1.** We fix the initial state $\rho$ and a minimal enclosure $\mathcal{V}$, whose corresponding absorption operator is denoted as usual by $A(\mathcal{V})$. If $\mathbb{E}_\rho[\operatorname{tr}(A(\mathcal{V})\rho_0)] > 0$, we introduce the measure $\mathbb{P}'_\rho$ as previously in Lemma 3.2.1. This first step consists in proving large deviations bounds for the position process under the measure $\mathbb{P}'_\rho$.

We need to consider a restriction of the channel $\mathfrak{L}$ which takes into account only the subspace of $\operatorname{supp} A(\mathcal{V})$ which is someway reachable by the local initial states $\rho(\underline{k})$. To this aim we use the enclosure $\mathcal{E}(\rho)$ and define the subspace

$$\mathcal{Q} = \tilde{p}_{\mathcal{V}} \mathcal{E}(\rho).$$

1. $\mathcal{Q} \oplus (\mathcal{E}(\rho)^\perp \cap \operatorname{supp}(A(\mathcal{V}))) = \operatorname{supp}(A(\mathcal{V}))$.
   Indeed, $v \in \mathcal{Q}^\perp \cap \operatorname{supp}(A(\mathcal{V}))$ if and only if
   $$v \in \operatorname{supp}(A(\mathcal{V})) \text{ and, } \forall w \in \mathcal{E}(\rho),\ 0 = \langle v, \tilde{p}_{\mathcal{V}}(w)\rangle = \langle \tilde{p}_{\mathcal{V}}(v), w\rangle = \langle v, w\rangle,$$
   i.e. $v \in \operatorname{supp}(A(\mathcal{V})) \cap \mathcal{E}(\rho)^\perp$.
2. $\mathbb{E}_\rho[\operatorname{tr}(A(\mathcal{V})\rho_0)] = 0$ if and only if $\mathcal{Q} = \{0\}$.
   Since $\operatorname{tr}(A(\mathcal{V})\rho_0)$ is a non negative random variable, it has zero mean if and only if it is almost surely null, that is
   $$\begin{aligned} &\Leftrightarrow\ \ 0 = \operatorname{tr}(A(\mathcal{V})\rho(\underline{k})) = \operatorname{tr}(A(\mathcal{V})\mathfrak{L}^n(\rho(\underline{k}))) \quad \forall \underline{k} \in V \\ (\text{since } \mathfrak{L}^*(A(\mathcal{V})) = A(\mathcal{V}))\ \ &\Leftrightarrow\ \ \operatorname{tr}(A(\mathcal{V})\mathfrak{L}^n(\rho(\underline{k}))) = 0 \quad \forall \underline{k} \in V, n \geq 0 \\ &\Leftrightarrow\ \ \tilde{p}_{\mathcal{V}}(\operatorname{supp}(\mathfrak{L}^n(\rho(\underline{k})))) = \{0\} \quad \forall \underline{k}, n \end{aligned}$$
3. Otherwise $\mathbb{E}_\rho[\operatorname{tr}(A(\mathcal{V})\rho_0)] > 0$ and $\mathcal{V} \subset \mathcal{Q}$.
   By using the same ideas as before, if $\mathbb{E}_\rho[\operatorname{tr}(A(\mathcal{V})\rho_0)] > 0$, $\mathcal{Q}$ is non trivial and there exist some $\underline{k} \in V, n \geq 0$ such that $\operatorname{tr}(p_{\mathcal{V}}\mathfrak{L}^n(\rho(\underline{k}))) \neq 0$ and this implies
   $$\{0\} \neq p_{\mathcal{V}}(\mathcal{E}(\rho)) = p_{\mathcal{V}}(p_{\mathcal{R}}(\mathcal{E}(\rho))) = p_{\mathcal{V}}(\mathcal{R} \cap \mathcal{E}(\rho))$$
   where the last equality follows from Proposition 2.3.17. So $(\mathcal{R} \cap \mathcal{E}(\rho))$ is a non null positive recurrent enclosure (as intersection of enclosures) and it is non orthogonal to $\mathcal{V}$, hence it contains a minimal enclosure $\mathcal{W}$ which is in the same $\chi_\alpha$ as $\mathcal{V}$ and is not orthogonal to $\mathcal{V}$. Then, by using the representation of $\mathcal{V}$ and $\mathcal{W}$ given by the partial isometry $U_\alpha$ as in relation (3.12), we deduce that
   $$\mathcal{V} = p_{\mathcal{V}}(\mathcal{W}) \subset \tilde{p}_{\mathcal{V}}(\mathcal{E}(\rho)) = \mathcal{Q}.$$

We call $\Phi$ the restriction of $\mathfrak{L}$ to the subspace $\mathcal{Q}$, $\Phi(\sigma) = p_{\mathcal{Q}}\mathfrak{L}(p_{\mathcal{Q}}\sigma p_{\mathcal{Q}})p_{\mathcal{Q}}$ and $\Phi_u$ its deformation. $\mathcal{Q}$ and consequently $\Phi$ obviously depend on the enclosure $\mathcal{V}$ and on the initial state $\rho$, but we do not need to highlight this in the notations.

**Lemma 3.4.6.** *Suppose* $\mathbb{E}_\rho[\mathrm{tr}\,(A(\mathcal{V})\rho_0)] > 0$. *For any measurable* $B \in \mathcal{B}(\mathbb{R}^d)$

- $\limsup_{n\to+\infty} \frac{1}{n}\log(\hat{\mathbb{P}}'_{\rho,n}(B)) \leq -\inf_{x\in\overline{B}} \Lambda(x)$*;*
- $\liminf_{n\to+\infty} \frac{1}{n}\log(\hat{\mathbb{P}}'_{\rho,n}(B)) \geq -\inf_{x\in\mathring{B}\cap\mathcal{S}} \Lambda(x)$

*where*

- $\Lambda$ *is the Fenchel-Legendre transform of* $\log(\lambda_u^\rho)$*,*
- $\lambda_u^\rho$ *is the spectral radius of* $\Phi_u$ *,*
- $\mathcal{S} = \mathbb{R}^d$ *if* $\lambda_u^\rho$ *is smooth, otherwise it corresponds to the set of exposed points of* $\Lambda$*.*

*Proof.* In order to apply Theorem 3.4.2, we need to prove that for every $u \in \mathbb{R}^d$ we have

$$\lim_{n\to+\infty} \frac{1}{n}\log(\mathbb{E}'_\rho[\mathrm{e}^{u\cdot(X_n-X_0)}]) = \log(\lambda_u^\rho).$$

Notice that we computed the same limit in the proof of Theorem 3.2.5, but for $u$ in a complex neighborhood of the origin.

For any $n \in \mathbb{N}$, by construction $\Phi_u^n(\rho(\underline{k})) = \tilde{\mathfrak{L}}_u^n(\rho(\underline{k}))$ for all $\underline{k}$ and $u$, so we can write

$$\begin{aligned}
\mathbb{E}_\rho[\mathrm{tr}\,(A(\mathcal{V})\rho_0)]\cdot\mathbb{E}'_\rho[\mathrm{e}^{u\cdot(X_n-X_0)}] &= \sum_{\underline{k}\in V}\mathrm{tr}\left(A(\mathcal{V})\tilde{\mathfrak{L}}_u^n(\rho(\underline{k}))\right) = \sum_{\underline{k}\in V}\mathrm{tr}\,(A(\mathcal{V})\Phi_u^n(\rho(\underline{k})))\\
&= \sum_{\underline{k}\in V}\mathrm{tr}\,(\rho(\underline{k})\Phi_u^{*n}(A(\mathcal{V})))\\
&\leq \|\sum_{\underline{k}\in V}\rho(\underline{k})\|_{L^1}\,\|\Phi_u^{*n}(A(\mathcal{V})))\|_\infty \leq \|\Phi_u^{*n}\|_\infty.
\end{aligned}$$

Because of Gelfand formula, we get

$$\limsup_{n\to+\infty}\frac{1}{n}\log(\mathbb{E}'_\rho[\mathrm{e}^{u\cdot(X_n-X_0)}]) \leq \log\left(\lim_{n\to+\infty}\|\Phi_u^{*n}\|_\infty^{1/n}\right) = \log(\lambda_u^\rho).$$

Now consider $w_u \in B(\mathfrak{h})$ the Perron-Frobenius eigenvector for $\Phi_u^*$, i.e. such that $\Phi_u^*(w_u) = \lambda_u^\rho w_u$. $w_u$ is a non null positive operator supported in $\mathcal{Q}$, so there exist $N \in \mathbb{N}$ and $\hat{\underline{k}}$ in $V$ such that $\mathrm{tr}\left(\tilde{\mathfrak{L}}^N(\rho(\hat{\underline{k}}))w_u\right) \neq 0$. Therefore $\mathrm{tr}\left(\Phi_u^N(\rho(\hat{\underline{k}}))w_u\right) = \mathrm{tr}\left(\tilde{\mathfrak{L}}_u^N(\rho(\hat{\underline{k}}))w_u\right) \neq 0$.

Since $\mathcal{Q}$ is finite dimensional, there exists a constant $M > 0$ such that $p_{\mathcal{Q}}A(\mathcal{V})p_{\mathcal{Q}} \geq$

$Mw_u$, hence for every $n \geq N$ we have

$$\begin{aligned}\mathbb{E}_\rho[\mathrm{tr}\,(A(\mathcal{V})\rho_0)] \cdot \mathbb{E}'_\rho[\mathrm{e}^{u\cdot(X_n-X_0)}] &= \sum_{\underline{k}\in V} \mathrm{tr}\left(A(\mathcal{V})\tilde{\mathfrak{L}}_u^n(\rho(\underline{k}))\right)\\ &\geq \mathrm{tr}\left(A(\mathcal{V})\tilde{\mathfrak{L}}_u^n(\rho(\hat{\underline{k}}))\right)\\ &= \mathrm{tr}\left(A(\mathcal{V})\Phi_u^n(\rho(\hat{\underline{k}}))\right)\\ &\geq M\mathrm{tr}\left(\Phi_u^N(\rho(\hat{\underline{k}}))\Phi_u^{*(n-N)}(w_u)\right)\\ &= M\mathrm{tr}\left(\Phi_u^N(\rho(\hat{\underline{k}}))w_u\right)(\lambda_u^\rho)^{n-N}.\end{aligned}$$

Therefore

$$\liminf_{n\to+\infty}\frac{1}{n}\log(\mathbb{E}'_\rho[\mathrm{e}^{u\cdot(X_n-X_0)}]) \geq \log(\lambda_u^\rho).$$

This allows to compute the desired limit and the statement follows by direct application of the Gärtner-Ellis' theorem. Notice that we do not have to worry about the domain of $\log(\lambda_u^\rho)$ since it is easy to see that $\lambda_u^\rho$ is a strictly positive real number for every $u \in \mathbb{R}^d$. □

**Step 2.** We complete the proof of the statement of the theorem by using the expression of $\mathbb{P}_\rho$ as convex combinations of the $\mathbb{P}_\rho^{\alpha,\beta}$ deduced in relation (3.15). This implies that a similar decomposition holds for $\hat{\mathbb{P}}_{\rho,n}$ in terms of $(\hat{\mathbb{P}}_{\rho,n}^{\alpha,\beta})_{\alpha,\beta}$, i.e.

$$\hat{\mathbb{P}}_{\rho,n} = \sum_{\alpha\in A}\sum_{\beta\in I_\alpha} a_{\alpha,\beta}(\rho)\hat{\mathbb{P}}_{\rho,n}^{\alpha,\beta}. = \sum_{j\in J_\rho} a_j(\rho)\hat{\mathbb{P}}_{\rho,n}^j,$$

$$\text{where} \qquad J_\rho := \{(\alpha,\beta) : \alpha \in A, \beta \in I_\alpha : a_{\alpha,\beta}(\rho) > 0\}.$$

Since, for any $j \in J_\rho$ and $B \in \mathcal{B}(\mathbb{R}^d)$, $\hat{\mathbb{P}}_{\rho,n}(B) \geq a_\alpha \hat{\mathbb{P}}_{\rho,n}^j(B)$, we trivially have

$$\begin{aligned}\liminf_{n\to+\infty}\frac{1}{n}\log(\hat{\mathbb{P}}_{\rho,n}(B)) &\geq \max_{j\in J_\rho}\liminf_{n\to+\infty}\frac{1}{n}\log(\hat{\mathbb{P}}_{\rho,n}^j(B)),\\ \limsup_{n\to+\infty}\frac{1}{n}\log(\hat{\mathbb{P}}_{\rho,n}(B)) &\geq \max_{j\in J_\rho}\limsup_{n\to+\infty}\frac{1}{n}\log(\hat{\mathbb{P}}_{\rho,n}^j(B)).\end{aligned}$$

Then we have

$$\begin{aligned}\limsup_{n\to+\infty}\frac{1}{n}\log(\hat{\mathbb{P}}_{\rho,n}(B)) &\leq \underbrace{\limsup_{n\to+\infty}\frac{1}{n}\log(|J_\rho|)}_{=0} + \limsup_{n\to+\infty}\frac{1}{n}\log\left(\max_{j\in J_\rho}\hat{\mathbb{P}}_{\rho,n}^j(B)\right)\\ &= \max_{j\in J_\rho}\limsup_{n\to+\infty}\frac{1}{n}\log(\hat{\mathbb{P}}_{\rho,n}^j(B))\end{aligned}$$

and we are done. □

*Proof of Theorem 3.4.3.* Under the hypothesis $\mathfrak{h} = \mathcal{R}$, we have that $A(\mathcal{V}_{\alpha,\beta}) = p_{\mathcal{V}_{\alpha,\beta}}$, which implies $\mathcal{Q}_{\alpha,\beta} = \mathcal{V}_{\alpha,\beta}$ and $\tilde{\mathfrak{L}}_{\alpha,\beta,u} = \mathfrak{L}_{|\mathcal{V}_{\alpha,\beta},u}$.

Since $\mathfrak{L}_{|\mathcal{V}_{\alpha,\beta}}$ is irreducible, $\lambda_{\alpha,\beta,u}$ is an analytic function of $u \in \mathbb{R}^d$ ( [14, Lemma 5.3]) and consequently $S_{\alpha,\beta} = \mathbb{R}^d$.

Moreover recall (equation (3.11)) that $\mathfrak{L}^*_{|\chi_\alpha}$ is unitarily equivalent to $\mathrm{Id}_{B(\mathbb{C}^{|I_\alpha|})} \otimes \psi$ where $\psi$ is equal to $\mathfrak{L}^*_{|\mathcal{V}_{\alpha,\beta}}$, hence $\mathfrak{L}_{|\chi_\alpha}$ and $\mathfrak{L}_{|\mathcal{V}_{\alpha,\beta}}$ have the same spectral radius.

Therefore the following equality holds:

$$\min_{(\alpha,\beta)\in J_\rho} \Lambda_{\alpha,\beta} = \min_{\alpha:a_\alpha(\rho)\neq 0} \Lambda_\alpha.$$

Theorem 3.4.4 ensures that for any measurable $B \in \mathcal{B}(\mathbb{R}^d)$

- $\limsup_{n\to+\infty} \frac{1}{n} \log(\hat{\mathbb{P}}_{\rho,n}(B)) \leq -\inf_{x\in\overline{B}} \min_{\alpha:a_\alpha(\rho)\neq 0} \Lambda_\alpha(x)$,
- $\liminf_{n\to+\infty} \frac{1}{n} \log(\hat{\mathbb{P}}_{\rho,n}(B)) \geq -\inf_{x\in\mathring{B}} \min_{\alpha:a_\alpha(\rho)\neq 0} \Lambda_\alpha(x)$,

which is exactly the definition of large deviation principle with rate function $\Lambda_\rho(x) := \min_{\alpha:a_\alpha(\rho)\neq 0} \Lambda_\alpha(x)$, $x \in \mathbb{R}^d$. Note that $\Lambda_\rho$ has compact sublevel sets because every $\Lambda_\alpha$ does (it is a consequence of Gärtner-Ellis' theorem). $\square$

Consider a minimal enclosure $\mathcal{V}$ such that $\mathbb{E}_\rho[\mathrm{tr}\,(A(\mathcal{V})\rho_0)] \neq 0$; taking the notations of the first step in the proof of Theorem 3.4.4, the following proposition states that $\lambda_u^\rho$ can be seen as the result of two contributions: one depending on the recurrent dynamic on $\mathcal{V}$ and the other one on the transient dynamic on its orthogonal complement in $\mathcal{Q}$, which we denote by $\mathcal{W} := \mathcal{Q} \cap \mathcal{T}$.

**Proposition 3.4.7.** *Let $\lambda_u^{\mathcal{V}}$ and $\lambda_u^{\mathcal{W}}$ be the spectral radii of $\Phi_{|\mathcal{V},u}$ and $\Phi_{|\mathcal{W},u}$ respectively. Then $\lambda_u^\rho = r(\tilde{\mathfrak{L}}_u) = \max\{\lambda_u^{\mathcal{V}}, \lambda_u^{\mathcal{W}}\}$.*

*Proof.* We only need to prove that if $\lambda_u^\rho > \lambda_u^{\mathcal{V}}$, then $\lambda_u^\rho = \lambda_u^{\mathcal{W}}$. Theorem 1.2.10 tells us that there exists a positive $\omega_u \in L^1(\mathcal{Q})$ such that $\Phi_u(\omega_u) = \lambda_u^\rho \omega_u$; since $\lambda_u^\rho > \lambda_u^{\mathcal{V}}$, it must be true that $p_{\mathcal{W}}\omega_u p_{\mathcal{W}} \neq 0$ and we have the following:

$$p_{\mathcal{W}}\Phi_u(p_{\mathcal{W}}\omega_u p_{\mathcal{W}})p_{\mathcal{W}} = p_{\mathcal{W}}\Phi_u(\omega_u)p_{\mathcal{W}} = \lambda_u p_{\mathcal{W}}\omega_u p_{\mathcal{W}}.$$

The first equality follows from the fact that for any $\rho \in L^1(\mathfrak{h})$

$$\Phi_u(p_{\mathcal{V}}\rho) = p_{\mathcal{Q}}\mathfrak{L}_u(p_{\mathcal{V}}\rho p_{\mathcal{Q}})p_{\mathcal{Q}} \overset{(\mathcal{V}\text{ is an enclosure})}{=} p_{\mathcal{V}}\mathfrak{L}_u(p_{\mathcal{V}}\rho p_{\mathcal{Q}})p_{\mathcal{Q}} = p_{\mathcal{V}}\Phi_u(p_{\mathcal{V}}\rho)$$

and analogously $\Phi_u(\rho p_{\mathcal{V}}) = \Phi(\rho p_{\mathcal{V}})p_{\mathcal{V}}$.

$\square$

## 3.5 Strong Law of Large Numbers

The main result of this section is the proof of the almost sure convergence of $\frac{X_n - X_0}{n}$ to a limit random variable $\overline{X}_\infty$, which can be explicitly described. As a first step, we need to introduce a coarser (with respect to the one presented in equation (3.8)) decomposition of the positive recurrent space, in which we group together all the $\chi_\alpha$ with the same value of $m_\alpha$. More precisely, let us consider on $A$ the equivalent relation that identifies $\alpha, \alpha' \in A$ if $m_\alpha = m_{\alpha'}$ and let us denote by $[\alpha]$ the equivalence class containing $\alpha$, $m_{[\alpha]}$ the corresponding value of the asymptotic mean and by $C$ a set of representatives of the equivalence classes, i.e.

- $C \subseteq A$,
- $m_\gamma \neq m_{\gamma'}$ (hence $[\gamma] \neq [\gamma']$) for $\gamma, \gamma' \in C$, $\gamma \neq \gamma'$,
- for every $\alpha \in A$, there exists $\gamma \in C$ such that $\alpha \in [\gamma]$.

The partition in equivalence classes of $A$ induces a decomposition in enclosures of the recurrent space for the local map:

$$\mathcal{R} = \bigoplus_{\gamma \in C} \mathcal{W}_{[\gamma]}, \quad \mathcal{W}_{[\gamma]} = \bigoplus_{\alpha \in [\gamma]} \chi_\alpha. \tag{3.16}$$

We remark that, being the sum of orthogonal enclosures, $\mathcal{W}_{[\gamma]}$ is an enclosure for every $\gamma \in C$, hence we can consider the corresponding absorption operator $A(\mathcal{W}_{[\gamma]})$ and the measure $\mathbb{P}_\rho^{[\gamma]}$, whose Radon-Nikodym derivative with respect to $\mathbb{P}_\rho$ is given by $Z_\infty^{[\gamma]} = \frac{Y_\infty^{[\gamma]}}{\mathbb{E}_\rho[\mathrm{tr}\left(A(\mathcal{W}_{[\gamma]})\rho_0\right)]}$, where $Y_\infty^{[\gamma]}$ is the $\mathbb{P}_\rho$-almost sure limit of the sequence $Y_n^{[\gamma]} = \mathrm{tr}\left(A(\mathcal{W}_{[\gamma]})\rho_n\right)$ (we assume that $\mathbb{E}_\rho[\mathrm{tr}\left(A(\mathcal{W}_{[\gamma]})\rho_0\right)] > 0$). As in Lemma 3.3.3, one can see that

$$\mathbb{P}_\rho = \sum_{\gamma \in C} a_{[\gamma]}(\rho)\mathbb{P}_\rho^{[\gamma]}, \quad \mathbb{P}_\rho^{[\gamma]} = \frac{1}{a_{[\gamma]}(\rho)} \sum_{\alpha \in [\gamma]} a_\alpha(\rho)\mathbb{P}_\rho^\alpha = \frac{1}{a_{[\gamma]}(\rho)} \sum_{\alpha \in [\gamma]} \sum_{\beta \in I_\alpha} a_{\alpha,\beta}(\rho)\mathbb{P}_\rho^{\alpha,\beta}, \tag{3.17}$$

where $a_{[\gamma]}(\rho) := \mathbb{E}_\rho[\mathrm{tr}\left(A(\mathcal{W}_{[\gamma]})\rho_0\right)]$, $a_\alpha(\rho) = \mathbb{E}_\rho[\mathrm{tr}\left(A(\chi_\alpha)\rho_0\right)]$ and $a_{\alpha,\beta}(\rho) = \mathbb{E}_\rho[\mathrm{tr}\left(A(\mathcal{V}_{\alpha,\beta})\rho_0\right)]$.

Lemma 3.4.6 allows us to derive the following law of large numbers under the measure $\mathbb{P}_\rho^{[\gamma]}$.

**Corollary 3.5.1.** *For every $\gamma \in C$,*

$$\lim_{n \to +\infty} \frac{X_n - X_0}{n} = m_\gamma \quad \mathbb{P}_\rho^{[\gamma]} - a.\ s..$$

*Proof.* Since by equation (3.17)we know that

$$\mathbb{P}_\rho^{[\gamma]} = \frac{1}{a_{[\gamma]}(\rho)} \sum_{\alpha \in [\gamma]} \sum_{\beta \in I_\alpha} a_{\alpha,\beta}(\rho)\mathbb{P}_\rho^{\alpha,\beta},$$

if we prove that for every $\alpha \in [\gamma]$ and $\beta \in I_\alpha$,

$$\lim_{n \to +\infty} \frac{X_n - X_0}{n} = m_\alpha = m_{[\gamma]} \quad \mathbb{P}_\rho^{\alpha,\beta}\text{-a.s.}, \tag{3.18}$$

we are done. Standard theory ( [25, Theorem II.6.3 and Theorem II.6.4] shows that the convergence in equation (3.18) can be proved using the regularity of $\lim_{n \to +\infty} \frac{1}{n} \log(\mathbb{E}_\rho^{\alpha,\beta}[e^{u \cdot (X_n - X_0)}])$ in $u = 0$ (which follows from the proof of Theorem 3.2.5) and the properties of the rate function $\Lambda^\rho_{\alpha,\beta}$ and the upper bound

$$\limsup_{n \to +\infty} \frac{1}{n} \log(\hat{\mathbb{P}}^{\alpha,\beta}_{\rho,n}(B)) \leq - \inf_{x \in \overline{B}} \Lambda^\rho_{\alpha,\beta}(x), \quad B \in \mathscr{B}(\mathbb{R}^d),$$

which follow from Lemma 3.4.6. □

We remark that Corollary 3.5.1 could be reformulated without making use of the decomposition in equation (3.16), since it is equivalent to say that for every $\alpha \in A$, for every $\beta \in I_\alpha$

$$\lim_{n\to+\infty} \frac{X_n - X_0}{n} = m_\alpha \quad \mathbb{P}_\rho^{\alpha,\beta} - \text{a. s.}.$$

The following theorem explains the reason why $(\mathbb{P}_\rho^{[\gamma]})_{\gamma\in C}$ is a natural family of probability measures to consider when studying the almost sure convergence of $(X_n - X_0)/n$. Theorem 3.5.2 is a generalization of Theorem 7.3 in [3]: the proof shares the same ideas, but the result is obtained in full generality.

**Theorem 3.5.2.** *Let us define* $E_{[\gamma]} = \left\{\lim_{n\to+\infty}\|\rho_n - p_{\mathcal{W}_{[\gamma]}}\rho_n p_{\mathcal{W}_{[\gamma]}}\| = 0\right\}$ *for every* $\gamma \in C$. *Then*

1. $(E_{[\gamma]})$ *is a partition of* $\Omega$, *modulo* $\mathbb{P}_\rho$*-null sets;*
2. $Y_\infty^{[\gamma]} = 1_{E_{[\gamma]}}$*;*
3. $\lim_{n\to+\infty} \frac{X_n - X_0}{n} = \overline{X}_\infty$ $\mathbb{P}_\rho$*-a.s. and* $\overline{X}_\infty = \sum_{\gamma\in C} m_{[\gamma]} Y_\infty^{[\gamma]}$*;*
4. $\lim_{n\to+\infty}\|\rho_n - \sum_{\gamma\in C} Y_\infty^{[\gamma]} p_{\mathcal{W}_{[\gamma]}}\rho_n p_{\mathcal{W}_{[\gamma]}}\| = 0$ $\mathbb{P}_\rho$*-a.s..*

We remark that point 2 shows that $\mathbb{P}_\rho^{[\gamma]}$ is indeed obtained from $\mathbb{P}_\rho$ conditioning on the event $E_{[\gamma]}$, which is the event that $\rho_n$ gets absorbed in $\mathcal{W}_{[\gamma]}$. In general this is not true for an arbitrary enclosure, as Example 3.6.2 shows.

*Proof.* Let us define $\tilde{E}_{[\gamma]} := \left\{\lim_{n\to+\infty} \frac{X_n - X_0}{n} = m_{[\gamma]}\right\}$; we will prove points 1. and 2. with $\tilde{E}_{[\gamma]}$ instead of $E_{[\gamma]}$ and the conclusion will follow by the fact that $\{Y_\infty^{[\gamma]} = 1\} = E_{[\gamma]}$ (equation (3.2)).

1. Since, by construction, $m_{[\gamma]} \neq m_{[\gamma']}$ for $\gamma \neq \gamma'$, we have that $E_{[\gamma]} \cap E_{[\gamma']} = \emptyset$. Corollary 3.5.1 implies that $\mathbb{P}_\rho^{[\gamma]}(E_{[\gamma]}^C) = 0$ for every $\gamma \in C$, hence

$$\mathbb{P}_\rho\left(\left(\bigcup_{\gamma\in C} E_{[\gamma]}\right)^C\right) = \mathbb{P}_\rho\left(\bigcap_{\gamma\in C} E_{[\gamma]}^C\right) = \sum_{\gamma'\in C} a_{[\gamma']}(\rho)\mathbb{P}_\rho^{[\gamma']}\left(\bigcap_{\gamma\in C} E_{[\gamma]}^C\right)$$
$$\leq \sum_{\gamma'\in C} a_{[\gamma']}(\rho)\mathbb{P}_\rho^{[\gamma']}\left(E_{[\gamma']}^C\right) = 0.$$

2. As we have done before, we use the notation $Z_\infty^{[\gamma]} = \frac{d\mathbb{P}_\rho^{[\gamma]}}{d\mathbb{P}_\rho}$. By Corollary 3.5.1 we have that

$$1 = \mathbb{P}_\rho^{[\gamma]}(E_{[\gamma]}) = \int_{E_{[\gamma]}} Z_\infty^{[\gamma]} d\mathbb{P}_\rho,$$

hence $0 = \mathbb{P}_\rho\left(\operatorname{supp}(Z_\infty^{[\gamma]}) \cap E_{[\gamma]}^C\right) \geq \mathbb{P}_\rho\left(\operatorname{supp}(Z_\infty^{[\gamma]}) \cap E_{[\gamma']}\right)$ for $\gamma \neq \gamma'$. By the fact that $\mathbb{P}_\rho$ is a convex combination of $\mathbb{P}_\rho^{[\gamma]}$, one can deduce that $\mathbb{P}_\rho\left(\bigcup_{\gamma\in C}\operatorname{supp}(Z_\infty^{[\gamma]})\right) = 1$ and so

$$\mathbb{P}_\rho(E_{[\gamma]}) = \sum_{\gamma'\in C}\mathbb{P}_\rho\left(\bigcup_{\gamma'\in C} E_{[\gamma]} \cap \operatorname{supp}(Z_\infty^{[\gamma']})\right) = \mathbb{P}_\rho(E_{[\gamma]} \cap \operatorname{supp}(Z_\infty^{[\gamma]})),$$

hence $\mathbb{P}_\rho(E_{[\gamma]} \,\Delta\, \mathrm{supp}(Z_\infty^{[\gamma']})) = 0$, where $\Delta$ stays for the symmetric difference between sets. Since $\mathrm{supp}(Z_\infty^{[\gamma]}) = \mathrm{supp}(Y_\infty^{[\gamma]})$ and $\sum_{\gamma \in C} Y_\infty^{[\gamma]} = \lim_{n\to+\infty} \sum_{\gamma\in C} Y_n^{[\gamma]} = 1$, we can conclude that $Y_\infty^{[\gamma]} = 1_{E_{[\gamma]}}$ and we are done.
3. and 4. They follow directly from points 1. and 2..

□

## 3.6 Examples and numerical simulations

### 3.6.1 Commuting normal local Kraus operators

As a first family of examples, we consider some HOQRWs studied in [61]: take $V = \mathbb{Z}^d$ and a local channel with normal commuting Kraus operators $\{L_j\}_{j=1}^{2d}$. In this case, there exists an orthonormal basis $\{\phi_i\}_{i=1}^h$ that simultaneously diagonalizes the Kraus operators and we can write $L_j = \sum_{i=1}^h \zeta_{i,j} \left|\phi_i\right\rangle\left\langle\phi_i\right|$. The normalization condition for the operators $L_j$ given by equation (3.1) implies that $\sum_{j=1}^{2d} |\zeta_{i,j}|^2 = 1$ for any $i = 1,\dots,h$.

It is easy to verify by direct computation that, for every $i = 1,\dots,h$, $\omega_i = \left|\phi_i\right\rangle\left\langle\phi_i\right|$ is a minimal invariant state for $\mathfrak{L}$, and consequently $\mathcal{V}_i := \mathrm{span}\{\phi_i\}$ is a minimal recurrent enclosure. Hence $\mathfrak{L}$ is positive recurrent and $\mathfrak{h} = \oplus_i \mathcal{V}_i$ is a decomposition of the local space $\mathfrak{h}$ in minimal orthogonal enclosures.

However, for our study, we are interested in a decomposition of the form described in (3.9) and in particular we should identify the enclosures $\chi_\alpha$, which will be given by the direct sum of some of the $\mathcal{V}_i$'; indeed, we can see that $\mathcal{V}_i$ and $\mathcal{V}_l$ are in the same $\chi_\alpha$ if and only if for every $j = 1,\dots,2d$, $\zeta_{i,j} = \zeta_{l,j} =: \zeta_{\alpha,j}$. This reflects on the structure of the Kraus operators, that will also be written as $L_j = \sum_{\alpha\in A} \zeta_{\alpha,j} p_{\chi_\alpha}$, $j = 1,\dots,2d$.

In this simple example, the probability law of the shift $X_n - X_0$ is a convex combination of $|A|$ multinomial distributions with parameters $(|\zeta_{\alpha,1}|^2,\dots,|\zeta_{\alpha,2d}|^2)$: for every $n \geq 1$

$$\mathbb{P}_\rho(X_1 - X_0 = e_{j_1},\dots,X_n - X_{n-1} = e_{j_n}) = \sum_{\alpha=1}^{|A|} \underbrace{\sum_{\underline{k}\in\mathbb{Z}^d} \mathrm{tr}\left(p_{\chi_\alpha}\rho(\underline{k})\right)}_{=:a_\alpha(\rho)} \prod_{k=1}^n |\zeta_{\alpha,j_k}|^2,$$

where $e_1,\dots,e_d$ is the canonical basis of $\mathbb{R}^d$ and $e_{2j} = -e_j$ for $j = 1,\dots,d$. Applying the central limit theorem for the mean of i.i.d. random variables, we see that

$$\lim_{n\to+\infty} \mathrm{dist}\left(\mathbb{P}_{\rho,n}, \sum_{\alpha=1}^{|A|} a_\alpha(\rho)\mathcal{N}\left(\sqrt{n}m_\alpha, D_\alpha\right)\right) = 0 \tag{3.19}$$

where $m_\alpha = \sum_{j=1}^{2d} |\zeta_{\alpha,j}|^2 e_j$ and $D_\alpha = \sum_{j=1}^d (|\zeta_{\alpha,j}|^2 + |\zeta_{\alpha,2j}|^2) \left|e_j\right\rangle\left\langle e_j\right|$.

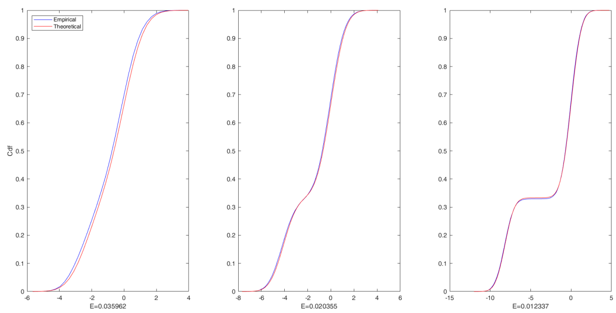
Similarly, if we apply Theorem 3.3.4, we find again relation (3.19) (in this case computations for the asymptotic means and covariance matrices are very easy).

Also, by applying Theorem 3.4.3, we can state that a large deviations' principle holds for the process $\frac{X_n - X_0}{n}$ and the rate function is given by

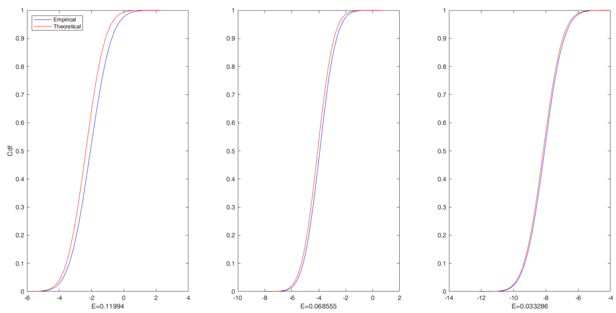
$$\Lambda_\rho(x) := \min_{\alpha: a_\alpha(\rho)\neq 0} \Lambda_\alpha(x), \quad x \in \mathbb{R}^d$$

where $\Lambda_\alpha(x) = \sup_{u\in\mathbb{R}^d}\{\langle u, x\rangle - \log(\lambda_{\alpha,u})\}$ and $\lambda_{\alpha,u} = \sum_{j=1}^{2d} |\zeta_{\alpha,j}|^2 \mathrm{e}^{u\cdot e_j}$.

### 3.6.2 An example with non trivial transient space




**(a)** $\rho = \frac{1}{3}(|e_1\rangle\langle e_1| + |e_2\rangle\langle e_2| + |e_3\rangle\langle e_3|)$.




**(b)** $\rho = |e_0\rangle\langle e_0|$.

**Figure 3.5:** *Panels (a)-(b) report the plots of the empirical cumulative function of* $(X_n - X_0)/\sqrt{n}$ *and the cumulative function of the mixture of Gaussians for* $n = 50, 150, 600$ *(we chose* $p_3 = 1/2$*) for two different choices of initial state. Below the graphs we reported the maximum difference in modulus of the values of the two functions.*

We consider a family of HOQRW with local Hilbert space $\mathfrak{h} = \mathbb{C}^4$, including the walk defined in Example 3.1.2. We introduce the parameters $p_1, p_2, p_3 \geq 0$ such that $\sum_{i=1}^3 p_i = \frac{1}{2}$ and define left and right Kraus operators

$$L = \begin{pmatrix} \frac{1}{2\sqrt{2}} & 0 & 0 & 0 \\ \sqrt{\frac{p_1}{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \sqrt{\frac{p_2}{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ -\sqrt{\frac{p_3}{3}} & 0 & 0 & \frac{2}{\sqrt{3}} \end{pmatrix}, \quad R = \begin{pmatrix} \sqrt{\frac{3}{8}} & 0 & 0 & 0 \\ -\sqrt{\frac{p_1}{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ -\sqrt{\frac{p_2}{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ \sqrt{\frac{2p_3}{3}} & 0 & 0 & \frac{1}{\sqrt{3}} \end{pmatrix}.$$

Notice that Example 3.1.2 corresponds to the case $p_1 = p_2 = 0,\ p_3 = 1/2$.

This family of local channels revealed to be very useful since, though with a low dimensional local Hilbert space, it can display already a more sophisticated structure of the decomposition of the local space. Indeed, the transient subspace is non trivial and the recurrent subspace is reducible as a sum of two $\chi_\alpha$, one which is a minimal enclosure and one which is not.

Let $\{e_i\}_{i=0}^3$ be the canonical basis of $\mathfrak{h}$. It is immediate to see, for instance by computing explicitly the invariant states of the corresponding local channel $\mathfrak{L}$, that $\mathcal{T} = \mathrm{span}\{e_0\}$, $\mathcal{R} = \mathrm{span}\{e_1, e_2, e_3\}$ and the decomposition of the recurrent space is the following:

$$\mathcal{R} = \underbrace{\mathrm{span}\{e_1, e_2\}}_{\chi_2} \oplus \underbrace{\mathrm{span}\{e_3\}}_{\chi_1}.$$

With simple direct computations one can find the parameters of the limit Gaussians: for the enclosure $\chi_2$ it has mean $m_2 = 0$ and variance $D_2 = 1$, while for $\chi_1$ it has parameters $m_1 = -\frac{1}{3}$ and $D_1 = \frac{8}{9}$.

For this walk, depending on the different choice of the initial state $\rho$, we can observe either only one of the two Gaussians or various mixtures of the two Gaussians. When the $\rho(\underline{k})$'s are all contained in a same $\chi_\alpha$, then we shall see only the Gaussian associated with the same $\chi_\alpha$, $\alpha = 1, 2$.

In order to consider the asymptotic behavior, we need the following absorption operators:

$$A(\chi_1) = 2p_3 \left|e_0\right\rangle \left\langle e_0\right| + \left|e_3\right\rangle \left\langle e_3\right|, \quad A(\chi_2) = \mathbf{1}_{\mathfrak{h}} - A(\chi_1).$$

We can take for simplicity $X_0 = 0$ and it will be particularly interesting to consider an initial state $\rho$ supported in the transient subspace, and so of the form $\rho = \rho_0 \otimes |0\rangle\langle 0|$, with $\rho_0 = (\rho_0(i,j))_{i,j=0,\dots 3}$ a non negative unit-trace matrix in $M_4(\mathbb{C})$. Then we can explicitly compute the weights of the Gaussian mixture appearing in the generalized CLT, which will be given by the quantum absorption probabilities

$$\begin{aligned} a_1(\rho) &= 2p_3\rho_0(0,0) + \rho_0(3,3), \\ a_2(\rho) &= 1 - a_1(\rho) = 2(p_1 + p_2)\rho_0(0,0) + \rho_0(1,1) + \rho_0(2,2). \end{aligned}$$

We illustrate our result also by numerical simulations. We used $N = 5 \times 10^4$ samples of $\frac{X_n}{\sqrt{n}}$ for $n = 50, 150, 600$ in order to estimate their probability distribution and we compared it with the expected convex combination of Gaussian measures. Figures 3.1 and 3.3 show the histograms of $\frac{X_n - X_0}{\sqrt{n}}$ at the three different times ($n = 50, 150, 600$) for the choice $p_3 = \frac{1}{2}$ and for two different choices of the local initial state $\rho_0$. In Figure 3.5 we reported the empirical and the expected cumulative function. The same plots for the choice $p_3 = \frac{1}{6}$ are reported in Figure 3.6. Once again we remark that, tuning initial state and absorption rates the Gaussian laws in the mixture do not change, but only their weights.

Finally, numerical simulations can also help us to have a better intuition of the behavior of the processes $(Y_n)_n$ used to introduce the laws of the family $\mathbb{P}'_\rho$ (recall Lemma 3.2.1). In this example, since $m_1 \neq m_2$, the decompositions of the recurrent space in equation (3.8) and in equation (3.17) coincide. For the enclosure $\chi_1$, for instance, the corresponding process $Y_n^1 = \mathrm{tr}\,(\chi_1 \rho_n)$ helps us to select the trajectories absorbed in $\chi_1$. In Figure 3.7 we trace the trajectories of $(Y_n^1)_n$ along 800 steps, which show how $Y_\infty^1$ is a Bernoulli random variable with parameter $\mathbb{E}_\rho[\mathrm{tr}\,(A(\chi_1)\rho_0)]$; hence in this case $\mathbb{P}_\rho^1(\cdot)$ (defined as in relation (3.14)) is equal to $\mathbb{P}_\rho(\cdot|E_1)$ where $E_1 = \{Y_\infty^1 = 1\} = \{\lim_{n\to+\infty} \|\rho_n - p_{\chi_1}\rho_n p_{\chi_1}\| = 0\}$ and it represents the probability obtained conditioning $\mathbb{P}_\rho$ to the event of "being absorbed in $\chi_1$".

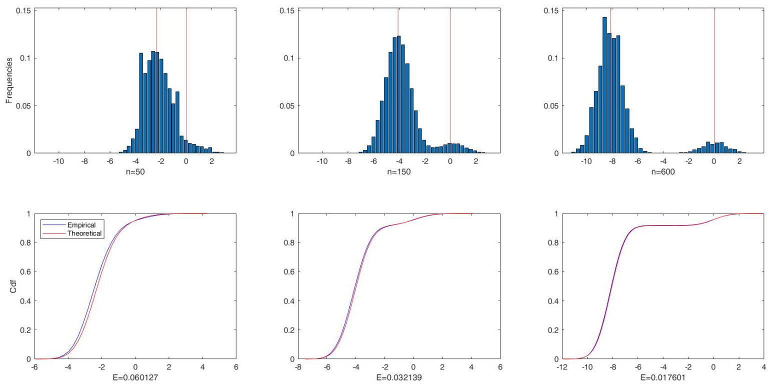
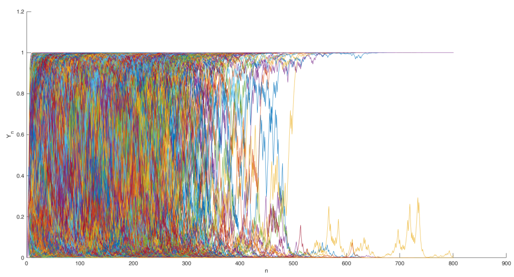
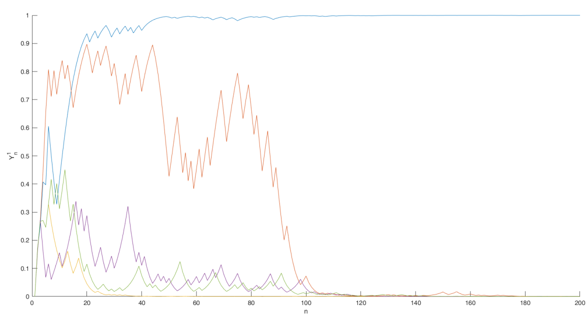



**Figure 3.6:** *We consider the trajectories obtained choosing $p_3 = \frac{1}{6}$ and starting in the state $\rho_0 = \frac{1}{8}\,|e_0\rangle\,\langle e_0| + \frac{7}{8}\,|e_3\rangle\,\langle e_3|$. Panel (a) reports the histograms of $(X_n - X_0)/n$ for $n = 50, 150, 600$ (red lines correspond to the meas of the Gaussians in the mixture). Panel (b) shows the empirical cumulative function of $(X_n - X_0)/n$ and the cumulative function of the mixture of Gaussians; under the plots we report the maximum difference in modulus between these functions.*



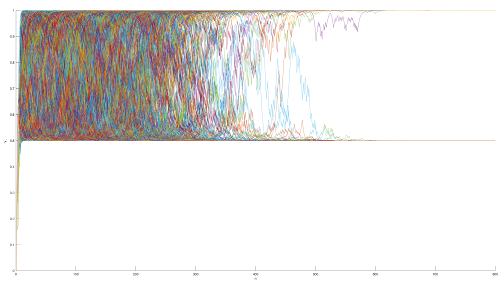
**(a)** *The graph represents $N = 10^4$ trajectories of $Y_n^1$ along $800$ steps ($\rho_0 = |e_0\rangle\langle e_0|$ and $p_3 = \frac{1}{6}$).*



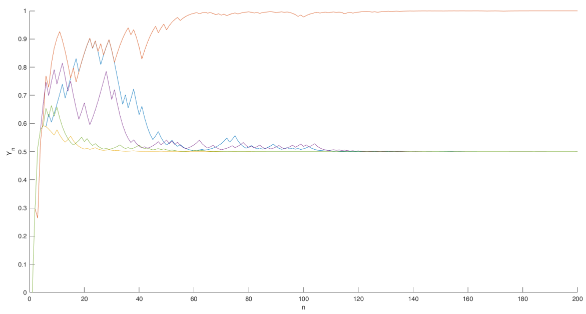
**(b)** *The graph represents $N = 5$ among the previous trajectories.*

**Figure 3.7:** *The behavior of $Y_n^1$.*

Note that the frequence of trajectories such that $Y^1_{800} > 0.99$ is equal to $0.3388$, and the frequence of trajectories for which $Y^1_{800} < 0.01$ is $0.6612$. This is in agreement with $a_1(\rho) = \mathbb{E}_\rho[\mathrm{tr}\,(A(\chi_1)\rho_0)] = \frac{1}{3}$. On the other hand, if we consider the enclosure $\mathcal{V} := \mathrm{span}\{e_1, e_3\}$, it is easy to see that the corresponding measure $\mathbb{P}'_\rho$ cannot be obtained from $\mathbb{P}_\rho$ conditioning on some event, since its density is $\frac{d\mathbb{P}'_\rho}{d\mathbb{P}_\rho} = \frac{3}{2}1_{E_1} + \frac{3}{4}1_{E_1^C}$: Figure 3.8 shows the trajectories of $Y_n$ up to $n = 800$ which converge to $1$ on $E_1$ (in a percentage of $0.3355$) and to $1/2$ otherwise.



**(a)** *The graph represents $N = 10^4$ trajectories of $Y_n$ along $800$ steps ($\rho_0 = |e_0\rangle\langle e_0|$ and $p_1 = p_2 = p_3 = \frac{1}{6}$).*




**(b)** *The graph represents $N = 5$ among the previous trajectories.*

**Figure 3.8:** *The behavior of $Y_n$.*

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