

INFINITE GEODESICS OF SUB-FINSLER DISTANCES IN HEISENBERG GROUPS

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ABSTRACT. We consider Heisenberg groups equipped with a sub-Finsler metric. Using methods of optimal control theory we prove that in this geometric setting the infinite geodesics are horizontal lines under the assumption that the sub-Finsler metric is defined by a strictly convex norm. This answers a question posed in [10] and has applications in the characterisation of isometric embeddings into Heisenberg groups.

1. INTRODUCTION

In the recent paper [8] the problem of classification of isometric embeddings of Heisenberg groups \mathbb{H}^m into \mathbb{H}^n for $m \leq n$ has been considered. Here both groups \mathbb{H}^m and \mathbb{H}^n were endowed with a homogeneous distance. By such a distance, we mean a left-invariant metric induced by a gauge function which is homogeneous with respect to a one-parameter family of ‘Heisenberg dilations’ adapted to the stratification of the underlying Lie algebra. These type of geometric structures are interesting objects and have been in the focus of several recent studies as shown in [1, 10, 11, 15, 16, 18, 22]. In the above referenced papers the more local aspects of the geometry were mainly considered. Our purpose is to concentrate on more global geometrical aspects as we study the behaviour of infinite geodesics in sub-Finsler distances. For additional results on sub-Riemannian and sub-Finsler structures we refer to the recent papers [4, 3, 19].

The main motivation of our work is the following result of [10]: consider \mathbb{H}^m and \mathbb{H}^n , $n \geq m$, to be equipped with left-invariant homogeneous distances d and d' , respectively. Assuming that every infinite geodesic in the target space (\mathbb{H}^n, d') is a line, then every isometric embedding $f : (\mathbb{H}^m, d) \rightarrow (\mathbb{H}^n, d')$ is the composition of a left translation and a homogeneous homomorphism.

The above result raises the natural question of characterising homogeneous distances d defined on \mathbb{H}^n such that the space (\mathbb{H}^n, d) has the property that every infinite geodesic in (\mathbb{H}^n, d) is a line. In [10] this property was called the geodesic linearity property or *GLP* of the space (\mathbb{H}^n, d) . It was conjectured in [10] that (\mathbb{H}^n, d) satisfies *GLP* if and only if the underlying Euclidean norm N associated to d is strictly convex. In [10] this conjecture has been verified for several concrete examples. The purpose of this paper is to prove this conjecture for general homogeneous distances on \mathbb{H}^n . In order to formulate our result we need to fix notation and recall some preliminaries.

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The n -th Heisenberg group \mathbb{H}^n is the set $\mathbb{R}^{2n} \times \mathbb{R}$ equipped with the multiplication

$$g * g' = (z, t) * (z', t') := (z + z', t + t' + 2\langle z, J_n z' \rangle), \text{ where } J_n = \begin{pmatrix} 0 & -E_n \\ E_n & 0 \end{pmatrix} \in \mathbb{R}^{2n \times 2n},$$

is the standard symplectic matrix on \mathbb{R}^{2n} and E_n denotes the $(n \times n)$ identity matrix. Sometimes it is convenient to write in coordinates

$$z = (x_1, \dots, x_n, y_1, \dots, y_n).$$

It can be easily verified that $(\mathbb{H}^n, *)$ satisfies all properties of a group with neutral element $e := (0, 0)$ and inverse $(z, t)^{-1} := (-z, -t)$. Denoting the nonlinear term $\langle z, J_n z' \rangle$ by $\omega_n(z, z')$, we remark that this expression defines a skew-symmetric bilinear form on \mathbb{R}^{2n} , and that two elements (z, t) and (z', t') in \mathbb{H}^n commute if and only if the term $\omega_n(z, z')$ is zero. Since this does not hold for all elements in \mathbb{H}^n , it turns out that the Heisenberg group is non-abelian. For $\lambda > 0$, the map $\delta_\lambda : \mathbb{H}^n \rightarrow \mathbb{H}^n$, $(z, t) \mapsto (\lambda z, \lambda^2 t)$ is called λ -dilation. It can be easily verified that any λ -dilation defines a group isomorphism with inverse $\delta_{\lambda^{-1}}$. It plays an analogous role as the usual scalar multiplication in \mathbb{R}^n .

We say that a *norm* on \mathbb{H}^n is a map $\hat{N} : \mathbb{H}^n \rightarrow \mathbb{R}_{\geq 0}$ which satisfies

$$\begin{aligned} i) \quad & \hat{N}(g) = 0 \Leftrightarrow g = e, \quad \forall g \in \mathbb{H}^n, \\ ii) \quad & \hat{N}(g^{-1}) = \hat{N}(g), \quad \forall g \in \mathbb{H}^n, \\ iii) \quad & \hat{N}(g * g') \leq \hat{N}(g) + \hat{N}(g'), \quad \forall g, g' \in \mathbb{H}^n. \end{aligned}$$

A metric $d : \mathbb{H}^n \times \mathbb{H}^n \rightarrow \mathbb{R}_{\geq 0}$ is called *left-invariant*, if for every $g_o \in \mathbb{H}^n$, the map $L_{g_o} : (\mathbb{H}^n, d) \rightarrow (\mathbb{H}^n, d)$, $g \mapsto g_o * g$ is an isometry, that is, $d(g_o * g, g_o * g') = d(g, g')$, for all $g, g' \in \mathbb{H}^n$.

Every norm $\hat{N} : \mathbb{H}^n \rightarrow \mathbb{R}_{\geq 0}$ induces a left-invariant metric $d_{\hat{N}} : \mathbb{H}^n \times \mathbb{H}^n \rightarrow \mathbb{R}_{\geq 0}$, and vice versa. More precisely, we can establish the following bijection

$$\begin{aligned} \{ \hat{N} : \mathbb{H}^n \rightarrow \mathbb{R}_{\geq 0} : \hat{N} \text{ is a norm} \} & \rightarrow \{ d : \mathbb{H}^n \times \mathbb{H}^n \rightarrow \mathbb{R}_{\geq 0} : d \text{ is a left-invariant metric} \} \\ \hat{N} & \mapsto d_{\hat{N}} : \mathbb{H}^n \times \mathbb{H}^n \rightarrow \mathbb{R}_{\geq 0}, (g, g') \mapsto \hat{N}(g^{-1} * g'), \end{aligned}$$

$$\begin{aligned} \{ d : \mathbb{H}^n \times \mathbb{H}^n \rightarrow \mathbb{R}_{\geq 0} : d \text{ is a left-invariant metric} \} & \rightarrow \{ \hat{N} : \mathbb{H}^n \rightarrow \mathbb{R}_{\geq 0} : \hat{N} \text{ is a norm} \} \\ d & \mapsto \hat{N}_d : \mathbb{H}^n \rightarrow \mathbb{R}_{\geq 0}, g \mapsto d(g, e). \end{aligned}$$

A norm $\hat{N} : \mathbb{H}^n \rightarrow \mathbb{R}_{\geq 0}$ on the Heisenberg group is called *homogeneous* if

$$\hat{N}(\delta_\lambda(g)) = \lambda \hat{N}(g), \text{ for all } \lambda > 0, \text{ for all } g \in \mathbb{H}^n.$$

It is easy to see that a norm \hat{N} on \mathbb{H}^n is homogeneous if and only if its associated left-invariant metric is homogeneous in the sense that $d_{\hat{N}}(\delta_\lambda(g), \delta_\lambda(g')) = \lambda d_{\hat{N}}(g, g')$. Every left-invariant distance on \mathbb{H}^n induced by a homogeneous norm is a homogeneous distance. From now on, we will use the expression "*homogeneous distance on \mathbb{H}^n* " to talk about the left-invariant metric induced by a homogeneous norm. It follows from Proposition 2.26 in [16] that the topology induced by any homogeneous distance on \mathbb{H}^n coincides with the Euclidean topology on \mathbb{R}^{2n+1} , and that any homogeneous norm is continuous with respect to the Euclidean topologies of \mathbb{R}^{2n+1} and \mathbb{R} . In particular, we note that any two homogeneous distances on \mathbb{H}^n induce the same topology.

On the other hand, the metric structure induced by a homogeneous norm \hat{N} on \mathbb{H}^n is very different from \mathbb{R}^{2n+1} endowed with the Euclidean distance d_{eucl} . The two distances $d_{\hat{N}}$ and d_{eucl} are not bi-Lipschitz equivalent for any choice of homogeneous norm \hat{N} on \mathbb{H}^n . However, we can associate to \hat{N} a norm N by restricting it to \mathbb{R}^{2n} by:

$$N(z) := \hat{N}(z, 0), \quad \text{for } z \in \mathbb{R}^{2n}. \quad (1.1)$$

Indeed, one can easily check (see Proposition 2.8 [10]) that $N : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ satisfies the axioms of a norm on the vector space \mathbb{R}^{2n} . The main result of the paper is the following:

Theorem 1.2. *Let (\mathbb{H}^n, d) be the Heisenberg group endowed with the homogeneous distance d . Denote by $\hat{N} : \mathbb{H}^n \rightarrow \mathbb{R}_{\geq 0}$ the associated homogeneous norm such that $d = d_{\hat{N}}$ and consider the associated norm $N : \mathbb{R}^{2n} \rightarrow \mathbb{R}_{\geq 0}$ defined by (1.1). Then the space (\mathbb{H}^n, d) has the GLP if and only if N is strictly convex.*

Let us recall that by definition, the norm $N : \mathbb{R}^{2n} \rightarrow \mathbb{R}_{\geq 0}$ is strictly convex iff its closed unit ball B_N (see Definition 2.1 below) is a strictly convex set, that is, if $z_1, z_2 \in B_N$ and $\alpha \in (0, 1)$ then $N(\alpha z_1 + (1 - \alpha)z_2) < 1$.

Let $I \subset \mathbb{R}$ be closed and convex; recall also that by a *geodesic* $\gamma : I \rightarrow (\mathbb{H}^n, d)$, we mean an isometric embedding of I into (\mathbb{H}^n, d) , that is,

$$d(\gamma(s), \gamma(s')) = |s - s'|, \quad \text{for all } s, s' \in I. \quad (1.3)$$

If we have I unbounded in the above definition, we say that γ is an *infinite geodesic*.

To comment on the statement of Theorem 1.2 let us mention that in the setting of normed spaces the strict convexity of the norm is equivalent to the fact that all (and not just the infinite geodesics) are straight line segments or lines (see e.g. [24]). This fact is definitely false in the sub-Finsler setting of the Heisenberg group. Let us recall ([22]) that in the standard sub-Riemannian metric the underlying norm is the Euclidean one, that is strictly convex. On the other hand there are a multitude of finite geodesics that are not line segments.

The statement of Theorem 1.2 no longer holds for higher step groups such as in the case of the Engel groups (see [5, 6, 7]). Moreover the strict convexity assumption on the norm is crucial in Theorem 1.2 and, as we will show in section 5, without such requirement our main result does not hold.

The statement of Theorem 1.2 in the case $n = 1$ follows from a theorem of Busemann [13] who solved the isoperimetric problem in the Minkowski plane. In order to prove Theorem 1.2 for general n we shall consider the *sub-Finsler distance* associated to N . The definition of the sub-Finsler distance is based on the notion of horizontal curves.

Let $I \subset \mathbb{R}$ be closed and convex; a *horizontal curve* in \mathbb{H}^n is an absolutely continuous curve $\gamma : I \rightarrow \mathbb{R}^{2n+1}$ with the property that

$$\dot{\gamma}(s) \in H_{\gamma(s)}, \quad \text{for almost every } s \in I,$$

where for $g \in \mathbb{H}^n$, we set

$$H_g := \text{span} \{X_{1,g}, \dots, X_{n,g}, Y_{1,g}, \dots, Y_{n,g}\}.$$

Here X_i and Y_i , $i = 1, \dots, n$, are the left-invariant vector fields (with respect to $*$) which at the origin agree with the standard basis vectors: $X_{i,e} = e_i$ and $Y_{i,e} = e_{n+i}$. These

left-invariant vector fields can be written as first order differential operators as follows

$$\begin{aligned} X_j &= \partial_{x_j} + 2y_j \partial_t, & j &= 1, \dots, n, \\ Y_j &= \partial_{y_j} - 2x_j \partial_t, & j &= 1, \dots, n. \end{aligned} \quad (1.4)$$

Denoting the $(2n + 1)$ components of an absolutely continuous curve $\gamma : I \rightarrow \mathbb{H}^n$ by $\gamma_i, i = 1, \dots, 2n + 1$, it follows that γ is horizontal if and only if

$$\dot{\gamma}_{2n+1}(s) = 2 \sum_{i=1}^n (\dot{\gamma}_i(s) \gamma_{n+i}(s) - \dot{\gamma}_{n+i}(s) \gamma_i(s)), \quad \text{for almost every } s \in I. \quad (1.5)$$

It is well known that a horizontal curve $\gamma : I \rightarrow \mathbb{H}^n$ is rectifiable and admits a Lipschitz parametrization (see for instance [18, Proposition 1.1] for a proof and note that this statement holds for any homogeneous norm on \mathbb{H}^n). In converse direction, every rectifiable curve admits a 1-Lipschitz parametrization and this parametrization is horizontal, see [23]. We say that a horizontal curve $\gamma : I \rightarrow \mathbb{H}^n$, with $I \subset \mathbb{R}$ closed, convex and unbounded (bounded), is a *horizontal line* (*horizontal segment*) if $\dot{\gamma}$ is nonvanishing and constant on I .

Given a norm $N : \mathbb{R}^{2n} \rightarrow \mathbb{R}_{\geq 0}$, the *sub-Finsler distance* associated to N on \mathbb{H}^n is given by

$$d_{SF}(g, g') := \inf_{\gamma} \int_a^b N(\dot{\gamma}_I(s)) \, ds, \quad (1.6)$$

where the infimum is taken over all horizontal curves $\gamma = (\gamma_I, \gamma_{2n+1}) : [a, b] \rightarrow \mathbb{H}^n$ with $\gamma(a) = g$ and $\gamma(b) = g'$, $\gamma_I = (\gamma_1, \dots, \gamma_{2n})$. Here, and in the remaining part of the paper we shall refer to γ_I as the projection of the horizontal curve γ and if γ_I is given the curve $\gamma = (\gamma_I, \gamma_{2n+1})$ satisfying (1.5) is referred to as the horizontal lift of γ_I . It is important to mention that given $g, g' \in \mathbb{H}^n$ and $a < b \in \mathbb{R}$ with $b - a = d_{SF}(g, g')$ then the above minimisation problem (1.6) has always a solution and horizontal curves solving this problem are (up to a reparametrization) geodesics in the space (\mathbb{H}^n, d_{SF}) .

Coming back to Theorem 1.2 let us mention that by Proposition 3.14, [10] we already have one direction of the statement, namely that *GLP* of $(\mathbb{H}^n, d_{\hat{N}})$ implies that N is strictly convex. Our purpose here is to prove the other implication that is substantially more difficult. Namely, we need to show that if N is strictly convex, then every infinite geodesic in $(\mathbb{H}^n, d_{\hat{N}})$ is a straight line. Notice first that according to Proposition 2.19 in [10] if $\gamma : \mathbb{R} \rightarrow \mathbb{H}^n$ is an infinite geodesic with respect to $d_{\hat{N}}$, then it is also an infinite geodesic with respect to d_{SF} . Therefore we shall only need to consider infinite geodesics associated to d_{SF} . In order to study properties of geodesics related to the sub-Finsler distance (1.6) we reformulate this minimisation problem as a problem of optimal control theory.

For the sake of simplicity we assume that $a = 0, b = T$, the starting point $\gamma(0) = e$ is the neutral element of \mathbb{H}^n and the final point is $\gamma(T) = g \neq e$. Then the length-minimising property of the geodesic from (1.6) will be equivalent to the following optimal control

problem with fixed time and fixed end-point:

$$\left\{ \begin{array}{l} \inf_v \int_0^T N(v(s)) \, ds \\ \dot{x}_i = -v_i, \quad i = 1, \dots, n \\ \dot{y}_i = -v_{n+i}, \quad i = 1, \dots, n \\ \dot{t} = -2 \sum_{i=1}^n (y_i v_i - x_i v_{n+i}) \\ \gamma(0) = e \\ \gamma(T) = g \end{array} \right. . \quad (1.7)$$

Notice, that in the above formulation, the curve $\gamma = (x_1, \dots, x_n, y_1, \dots, y_n, t) : [0, T] \rightarrow \mathbb{H}^n$ is automatically a horizontal curve, and the control function $-v : [0, T] \rightarrow \mathbb{R}^{2n}$ is in fact the horizontal velocity of γ , $\dot{\gamma}_I = -v$. It is clear that the two minimisation problems (1.6) and (1.7) are equivalent to each other.

Our approach is to apply Pontryagin's Maximum Principle in order to obtain useful information about solutions of (1.7). We should mention here that the study of sub-Finsler metrics by this approach has also been undertaken by [10]. However, in the aforementioned work, properties of infinite geodesics were not addressed. In fact, the main technical difficulty of our paper is precisely to pass from finite to infinite geodesics because of the possible ambiguity of the multiplier arising in Pontryagin's theorem since such multiplier, in general, is not unique.

The paper is organized as follows: in Section 2 we review some necessary background on convex analysis that is going to be used in the sequel. In Section 3 we apply Pontryagin's theorem to our situation. Section 4 contains the proof of our main result and Section 5 is for final remarks and examples.

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2. PRELIMINARY RESULTS FROM CONVEX ANALYSIS

Our goal is to study a class of norms on \mathbb{H}^n that are generated by convex norms coming from \mathbb{R}^{2n} . In this section we collect some basic results from the Euclidean convex analysis to be used in the sequel.

Let us start with a general norm $N : \mathbb{R}^{2n} \rightarrow \mathbb{R}_{\geq 0}$ and let us denote by B_N its unit ball, i.e. the set

$$B_N := \{z \in \mathbb{R}^{2n} : N(z) \leq 1\}. \quad (2.1)$$

It is clear that B_N is compact, convex with 0 in its interior. Since for every $\lambda \in [0, 1]$ and z, z' in \mathbb{R}^{2n}

$$N((1 - \lambda)z + \lambda z') \leq N((1 - \lambda)z) + N(\lambda z') = (1 - \lambda)N(z) + \lambda N(z'),$$

every norm is a convex function in \mathbb{R}^{2n} . We define the *dual norm* N_* of N as usual by

$$N_*(p) = \sup\{p \cdot z : N(z) = 1, z \in \mathbb{R}^{2n}\}.$$

The unitary ball B_{N_*} of N_* is the polar set B_N° of B_N , i.e.

$$B_{N_*}^\circ := \{p \in \mathbb{R}^{2n} : p \cdot z \leq 1, z \in B_N\}.$$

Given a function $f : \mathbb{R}^{2n} \rightarrow \mathbb{R}$, the *subdifferential* is a set valued mapping $\partial f : \mathbb{R}^{2n} \rightrightarrows \mathbb{R}^{2n}$ defined by

$$\partial f(z) = \{p \in \mathbb{R}^{2n} : f(z') \geq f(z) + p \cdot (z' - z), \forall z' \in \mathbb{R}^{2n}\}.$$

A well-known result due to Rockafellar characterizes the convexity of a function via the subdifferential; more precisely, $f : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ is convex if and only if $\partial f(z) \neq \emptyset$ for all $z \in \mathbb{R}^{2n}$ (see for instance [25]). Moreover, for a convex function f , the subdifferential set valued mapping ∂f is upper semicontinuous, and it is compact and convex valued (see [14, Proposition 2.1.5 and Proposition 2.2.7] and also [12]). These properties of the map ∂f will play a fundamental role in the case of non-smooth norm (see Proposition 4.8 below).

The *Legendre transform* of f is $f^* : \mathbb{R}^{2n} \rightarrow (-\infty, +\infty]$ defined by

$$f^*(p) = \sup_{z \in \mathbb{R}^{2n}} (p \cdot z - f(z))$$

Then f^* is always a convex function; if f is superlinear, then f^* is real valued. For every convex function f , the Legendre transform f^* is related to the subdifferential via the following equality (see for example [25, Proposition 11.3])

$$p \in \partial f(z) \iff z \in \partial f^*(p) \iff f(z) + f^*(p) = z \cdot p. \quad (2.2)$$

In all that follows, given a norm N we associate the function $F_N : \mathbb{R}^{2n} \rightarrow \mathbb{R}_{\geq 0}$ by

$$F_N(z) = \frac{1}{2}(N(z))^2. \quad (2.3)$$

As we shall see, F_N will play a crucial role in this paper.

The previous notions and properties are closely intertwined with the homogeneity and the convexity of the norm N as follows:

Proposition 2.4. *Let N be a norm in \mathbb{R}^{2n} . The following properties hold:*

i. for all $z \in \mathbb{R}^{2n}$

$$p \in \partial N(z) \iff p \in \partial N(\alpha z), \quad \forall \alpha > 0; \quad (2.5)$$

ii. for every $z \in \mathbb{R}^{2n}$, with $z \neq 0$, we have

$$p \in \partial N(z) \iff N(z) = z \cdot p. \quad (2.6)$$

iii. F_N is a convex function and hence $\partial F_N(z) \neq \emptyset$, for every $z \in \mathbb{R}^{2n}$; moreover, for every $z \neq 0$

$$\partial F_N(z) = N(z)\partial N(z); \quad (2.7)$$

iv. The Legendre transform $(F_N)^*$ of F_N is denoted by F_N^* and satisfies the relation $F_N^* = F_{N^*}$, i.e.

$$F_N^*(p) = \frac{1}{2}(N_*(p))^2.$$

Proof: The proof of (2.5) is an easy exercise.

Let us pass to ii: based on (2.5) we can assume that $N(z) = 1$. Thus we have to prove that $p \in \partial N(z)$ if and only if $p \cdot z = 1$.

According to (2.2) we have that $p \in \partial N(z)$ if and only if $N(z) + N^*(p) = p \cdot z$. In our case this relation reads as $1 + N^*(p) = p \cdot z$. Let us consider the Legendre transform N^* of N . Since

$$\begin{aligned} N^*(p) &= \sup_{z \in \mathbb{R}^{2n}} (p \cdot z - N(z)) \\ &= \sup_{\alpha \geq 0} \alpha \left(\sup_{\{z: N(z)=1\}} (p \cdot z - 1) \right). \end{aligned}$$

By the above relation it follows that N^* takes only values in the set $\{0, +\infty\}$. Moreover, $N^*(p) = 0$ if and only if $p \in B_N^\circ$. This implies that

$$p \in \partial N(z) \iff z \cdot p = 1,$$

which proves ii.

Let us prove iii. For z_0 and z_1 in \mathbb{R}^{2n} and $\alpha \in [0, 1]$ we have

$$\begin{aligned} F_N((1-\alpha)z_0 + \alpha z_1) &\leq \frac{1}{2} (N((1-\alpha)z_0) + N(\alpha z_1))^2 \\ &= \frac{1}{2} ((1-\alpha)N(z_0) + \alpha N(z_1))^2 \end{aligned} \quad (2.8)$$

$$\leq \frac{1}{2} ((1-\alpha)(N(z_0))^2 + \alpha(N(z_1))^2) \quad (2.9)$$

The inequality (2.8) follows from the homogeneity of the norm N and the fact that $s \mapsto s^2$ is an increasing function in $[0, \infty)$. Inequality (2.9) follows from the convexity of the function $s \mapsto s^2$.

Now consider a convex function $f : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ and a non decreasing, convex and regular function $\varphi : I \rightarrow \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval such that $f(\mathbb{R}^{2n}) \subset I$, then the following version of the chain rule holds (see [12, Proposition 4.2.5]):

$$\partial(\varphi \circ f)(z) = \varphi'(f(z))\partial f(z).$$

Applying this result to our case with $\varphi(s) = \frac{s^2}{2}$ and $f = N$, we have (2.7) and the proof of iii.

For the proof of iv. see Proposition 11.21 in [25], taking into account that the function $\theta(s) = \frac{1}{2}s^2$ is convex and $\theta^* = \theta$. \square

Let us introduce the following fundamental notion:

Definition 2.10. We say that a norm $N : \mathbb{R}^{2n} \rightarrow \mathbb{R}_{\geq 0}$ is *strictly convex* if z and z' in $\mathbb{R}^{2n} \setminus \{0\}$ are such that $N(z + z') = N(z) + N(z')$, then $z' = \alpha z$ for some $\alpha > 0$.

We say that a norm is *smooth* if at every point $z \in \partial B_N$ the ball B_N has a unique supporting hyperplane.

As an equivalent notion to require that a norm N is strictly convex if and only if ∂B_N does not contain line-segments. Furthermore, the smoothness of N is equivalent to the fact that, for every $z \neq 0$, N is differentiable, or that $\partial N(z)$ is singleton, i.e.

$$\partial N(z) = \{\nabla N(z)\}, \quad \forall z \neq 0.$$

We will denote by $\|\cdot\|$ the classical Euclidean norm. Finally, let us recall that a norm N is strictly convex if and only if N_* is smooth. See for instance Chapter 5 in [21] for more details.

Every norm N is a convex function however, the strict convexity of N as a norm does not imply its strict convexity as a function: in fact

$$N(\lambda z) = \lambda N(z) + (1 - \lambda)N(0), \quad \forall z \in \mathbb{R}^{2n}, \lambda \in [0, 1].$$

Taking the square of N and by defining the function F_N as in (2.3) will get us around this difficulty and, starting with a strictly convex norm, we obtain a useful strictly convex function. Hence we have the following result:

Proposition 2.11. *Let N be a strictly convex norm. Then F_N is a strictly convex function and $\partial F_N : \mathbb{R}^{2n} \rightrightarrows \mathbb{R}^{2n}$ is injective, i.e.*

$$z \neq z' \quad \Rightarrow \quad \partial F_N(z) \cap \partial F_N(z') = \emptyset.$$

Proof: Observe first that since $s \mapsto s^2$ is a strictly increasing function in $[0, \infty)$, in (2.8) we have an equality if and only if $N((1 - \alpha)z_0 + \alpha z_1) = N((1 - \alpha)z_0) + N(\alpha z_1)$. Since N is a strictly convex norm, this implies that there exists $\theta > 0$ such that

$$(1 - \alpha)z_0 = \theta \alpha z_1. \quad (2.12)$$

Since $s \mapsto s^2$ is a strictly convex function, in (2.9) we have an equality if and only if

$$N(z_0) = N(z_1). \quad (2.13)$$

Relations (2.12) and (2.13) give $z_0 = z_1$. Hence F_N is a strictly convex function.

To prove injectivity of ∂F_N we argue by contradiction: suppose that there exist two different points z and z' in \mathbb{R}^{2n} such that $p \in \partial F_N(z) \cap \partial F_N(z') \neq \emptyset$. The strict convexity of F_N implies that

$$F_N(z) > F_N(z') + p \cdot (z - z') \quad \text{and} \quad F_N(z') > F_N(z) + p \cdot (z' - z).$$

Clearly this is impossible. □

The strictly convexity of the function F_N plays a fundamental role in what follows. Indeed, let us recall that if $f : \mathbb{R}^{2n} \rightarrow [0, \infty)$ is a strictly convex function, then it may happen that f does not have a minimum point, i.e. it is not possible to guarantee in general that there exists z^* such that

$$z^* \in \arg \min_{z \in \mathbb{R}^{2n}} f(z).$$

However, if such z^* exists, by the strict convexity of f , it is unique.

3. OPTIMAL CONTROL APPROACH TO GEODESICS IN \mathbb{H}^n .

Let $N : \mathbb{R}^{2n} \rightarrow \mathbb{R}_{\geq 0}$ be a norm. The Heisenberg geodesics $\gamma = (x_1, \dots, x_n, y_1, \dots, y_n, t) : [0, T] \rightarrow \mathbb{H}^n$ with respect to the norm N , joining the points $e, g \neq e \in \mathbb{H}^n$ will be determined by the optimal control problem (1.7) with fixed final time $T > 0$ and fixed end-point g (which is equivalent to the problem (1.6)). Let us only recall that in the formulation (1.7), the curve γ is automatically a horizontal curve, and the control function $-v = [0, T] \rightarrow \mathbb{R}^{2n}$ is in fact the horizontal velocity of γ , $\dot{\gamma}_I = -v$.

3.1. Equivalence of optimal control problems related to N and F_N . If we want to apply the methods of optimal control theory to the problem (1.7) we shall run into problems due to the fact that the norm N has a linear growth at infinity. Therefore, our purpose is to change the above setting to an equivalent problem related to F_N where the integrand has a quadratic growth at infinity. The first step in this direction is the following normalization lemma where the homogeneity of N is crucial.

Proposition 3.1. *Let $N : \mathbb{R}^{2n} \rightarrow \mathbb{R}_{\geq 0}$ be a norm. Let us consider the problem (1.7). Without loss of generality we can reparametrize by arc length the horizontal curve γ such that*

$$N(v(s)) = K, \quad \text{a.e. } s \in [0, T],$$

where $v = -\dot{\gamma}_I$ and K is a positive constant depending only on T and γ .

Proof: Consider a horizontal curve $\gamma : [0, T] \rightarrow \mathbb{H}^n$ from 0 to g , with

$$\ell(\gamma) = \int_0^T N(v(s)) \, ds$$

and $v = -\dot{\gamma}_I$. If the two endpoints are different: $g \neq e$ we can assume without loss of generality that $v(s) \neq 0$ for all $s \in [0, T]$. Then we can define an absolutely continuous homeomorphism $\tau : [0, T] \rightarrow [0, T]$ by

$$\tau(s) = \frac{T}{\ell(\gamma)} \int_0^s N(v(u)) \, du.$$

The map τ provides the required reparametrization by defining the horizontal curve $\tilde{\gamma} : [0, T] \rightarrow \mathbb{H}^n$ with $\tilde{\gamma}(\sigma) = \gamma(\tau^{-1}(\sigma))$. To see this notice that $\tilde{\gamma}(0) = e$, $\tilde{\gamma}(T) = g$ and $\gamma(s) = \tilde{\gamma}(\tau(s))$.

Differentiating this relation with respect to s we obtain:

$$\frac{d\gamma_I(s)}{ds} = \frac{d\tilde{\gamma}_I}{d\sigma}(\tau(s)) \cdot \tau'(s) = \frac{d\tilde{\gamma}_I}{d\sigma}(\tau(s)) \frac{T}{\ell(\gamma)} N(v(s)),$$

for a.e. $s \in [0, T]$. Applying the function N to both side of this equality and reordering we obtain:

$$N(\tilde{v}(\sigma)) = N\left(-\frac{d\tilde{\gamma}_I}{d\sigma}(\sigma)\right) = \frac{\ell(\gamma)}{T}$$

for a.e. $\sigma \in [0, T]$. □

Now, let us consider the problem

$$\begin{cases} \inf_v \int_0^T F_N(v(s)) \, ds \\ \dot{x}_i = -v_i, & i = 1, \dots, n \\ \dot{y}_i = -v_{n+i}, & i = 1, \dots, n \\ \dot{t} = -2 \sum_{i=1}^n (y_i v_i - x_i v_{n+i}) \\ \gamma(0) = e \\ \gamma(T) = g \end{cases} \quad (3.2)$$

where F_N is defined by N via (2.3).

The following proposition states that the two optimal control problems (1.7) and (3.2) are in fact equivalent:

Proposition 3.3. *Let $N : \mathbb{R}^{2n} \rightarrow \mathbb{R}_{\geq 0}$ be a norm. The problems (1.7) and (3.2) are equivalent; more precisely, the control v^* is optimal for (1.7) if and only if v^* is optimal for (3.2).*

Proof: Without loss of generality, let us assume $T = 1$. First, let us suppose that v^* is optimal for (1.7), i.e.

$$\int_0^1 N(v^*) \, ds \leq \int_0^1 N(v) \, ds,$$

for every admissible control v . By Proposition 3.1, we are in the position to assume that $N(v^*(s))$ is constant. Hence, by Cauchy–Schwartz inequality, we obtain

$$\int_0^1 (N(v^*))^2 \, ds = \left(\int_0^1 N(v^*) \, ds \right)^2 \leq \left(\int_0^1 N(v) \, ds \right)^2 \leq \int_0^1 (N(v))^2 \, ds.$$

This shows that v^* is optimal for the problem (3.2).

Conversely, let us suppose that v^* is optimal for (3.2) and by contradiction let us assume that there exists an admissible control \tilde{v} such that

$$\int_0^1 N(\tilde{v}) \, ds < \int_0^1 N(v^*) \, ds.$$

Again by Proposition 3.1, we may assume $N(\tilde{v}(s))$ constant. The previous inequality and Cauchy–Schwartz inequality give

$$\int_0^1 (N(\tilde{v}))^2 \, ds = \left(\int_0^1 N(\tilde{v}) \, ds \right)^2 < \left(\int_0^1 N(v^*) \, ds \right)^2 \leq \int_0^1 (N(v^*))^2 \, ds;$$

this contradicts the optimality of v^* for the problem (3.2), proving the claim. \square

Let us remark that, as in the classical case where N is the Euclidean norm, there exists an optimal control for the two problems (1.7) and (3.2). To be precise (see for example Theorem 4.1 in [17])

Remark 3.4. Let N be a norm. The convexity of N and the superlinearity of F_N guarantee the existence of the optimal control for (3.2).

3.2. The Pontryagin Maximum Principle with a general norm N . In what follows we shall focus our attention to the study of the problem (3.2). Our approach is based on Pontryagin’s Maximum Principle. In order to do that, we shall introduce the Hamiltonian:

$$H : \mathbb{R}^{2n+1} \times \mathbb{R}^{2n+2} \times \mathbb{R}^{2n} \rightarrow \mathbb{R}, \quad (x, \lambda, v) \rightarrow H(x, \lambda, v)$$

as usual by

$$H(x, \lambda, v) = \lambda_0 F_N(v) - \sum_{i=1}^{2n} \lambda_i v_i - 2\lambda_{2n+1} \sum_{i=1}^n (y_i v_i - x_i v_{n+i})$$

where $x = (x_1, \dots, x_n, y_1, \dots, y_n, t)$, $\lambda = (\lambda_0, \lambda_1, \dots, \lambda_{2n+1})$ and $v = (v_1, \dots, v_{2n})$. Let us reorganize the Hamiltonian function by

$$\begin{aligned} H(x, \lambda, v) &= \lambda_0 F_N(v) - \sum_{i=1}^n [(\lambda_i + 2y_i \lambda_{2n+1}) v_i + (\lambda_{n+i} - 2x_i \lambda_{2n+1}) v_{n+i}] \\ &= \lambda_0 F_N(v) - a \cdot v, \end{aligned}$$

where the function $a : [0, T] \rightarrow \mathbb{R}^{2n}$ is given by

$$a = [\lambda_1 + 2y_1\lambda_{2n+1}, \dots, \lambda_n + 2y_n\lambda_{2n+1}, \lambda_{n+1} - 2x_1\lambda_{2n+1}, \dots, \lambda_{2n} - 2x_n\lambda_{2n+1}]. \quad (3.5)$$

The Pontryagin Maximum Principle gives a necessary condition for a control v to be optimal for a control problem. Let us only recall that by giving a control problem with a cost functional

$$J(v) = \int_0^T L(s, w(s), v(s)) ds$$

where w is the unique trajectory associated to the admissible control v via the dynamics and the initial/final points, the classical assumptions on such cost functional J require that the running cost function L is continuous in (s, w, v) , differentiable in w for every fixed (s, v) , and the derivatives $\frac{\partial L}{\partial w}$ and $\frac{\partial L}{\partial s}$ are continuous as function of all variables (see for example [20, Theorem 2.2.1] or [2, Theorem 12.10]). In our case the dynamics is given by the horizontality condition for the curve/trajectory and the running cost function is $L(s, w, v) = F_N(v)$. Since F_N is a convex function, clearly it is continuous in v and we need no other assumptions in order to apply the Pontryagin Maximum Principle. This is the reason that allows us to study the problem (3.2) without additional regularity assumptions on N .

Pontryagin's Maximum Principle applied to our situation (see the mentioned books above) gives the following statement:

Theorem 3.6. *Let $N : \mathbb{R}^{2n} \rightarrow \mathbb{R}_{\geq 0}$ be a norm and let us consider the problem (3.2). If v is an optimal control, then there exists a multiplier where*

- * $\lambda_0^v = \lambda_0$ is a constant in $\{0, 1\}$
- * $\lambda^v = \lambda = (\lambda_1, \dots, \lambda_{2n+1}) : [0, T] \rightarrow \mathbb{R}^{2n+1}$ is an absolutely continuous function
- * $(\lambda_0^v, \lambda^v) \neq (0, 0)$

such that, for a.e. $s \in [0, T]$ the following properties hold:

$$v(s) \in \arg \min_{u \in \mathbb{R}^{2n}} (\lambda_0 F_N(u) - a(s) \cdot u) \quad (3.7)$$

$$\dot{\lambda}_{2n+1}(s) = \frac{\partial H}{\partial x_{2n+1}} = 0 \quad \Rightarrow \quad \lambda_{2n+1} = k \quad (3.8)$$

$$\dot{\lambda}_i(s) = -\frac{\partial H}{\partial x_i} = -2k v_{n+i}(s), \quad i = 1, \dots, n \quad (3.9)$$

$$\dot{\lambda}_{n+i}(s) = -\frac{\partial H}{\partial x_{n+i}} = 2k v_i(s), \quad i = 1, \dots, n \quad (3.10)$$

$$s \mapsto \lambda_0 F_N(v(s)) - a(s) \cdot v(s) = c \quad (3.11)$$

where k and c are constants, and a is as in (3.5).

Relation (3.11) holds since the problem is autonomous. Clearly (3.9) and (3.10) imply

$$\lambda_i(s) = \lambda_i(0) + 2ky_i(s), \quad \lambda_{n+i}(s) = \lambda_{n+i}(0) - 2kx_i(s), \quad i = 1, \dots, n \quad (3.12)$$

and hence

$$a_i(s) = \lambda_i(0) + 4ky_i(s), \quad a_{n+i}(s) = \lambda_{n+i}(0) - 4kx_i(s), \quad i = 1, \dots, n. \quad (3.13)$$

In particular we obtain

$$a_i(0) = \lambda_i(0), \quad a_{n+i}(0) = \lambda_{n+i}(0), \quad i = 1, \dots, n. \quad (3.14)$$

We would like to emphasize the following:

Remark 3.15. For every optimal control v , the multiplier (λ_0^v, λ^v) associated to v , in general, is not unique. By construction, the function a depends on the choice of the multiplier (λ_0^v, λ^v) . Such function a is absolutely continuous.

Let us start our investigation. First we study the case of a general norm and later on we add further assumptions of strict convexity and smoothness.

For our optimal control we have the following normality property:

Proposition 3.16. *Assume that $N : \mathbb{R}^{2n} \rightarrow \mathbb{R}_{\geq 0}$ is a norm. Let v be an optimal control for the problem (3.2). Then v is a normal control, i.e. $\lambda_0^v = \lambda_0 = 1$.*

The statement follows by Goh's condition (see chapter 20 in [2]). We give a simple direct proof for the convenience of the reader.

Proof: Let us assume by contradiction that $\lambda_0 = 0$. Since v is optimal, the Maximum Principle (3.7) guarantees that the min exists, for every $s \in [0, T]$.

On the other hand, the function $u \mapsto a(s) \cdot u$ is affine. This implies that the above min exists only if $a = 0$ in $[0, T]$. But then (3.13) implies that $v = 0$ in $[0, T]$. This gives $x_i(0) = x_i(s)$ and $y_i(0) = y_i(s)$ for every $s \in [0, T]$. Clearly this implies that $T = 0$ as well. This gives a contradiction to $\gamma(0) \neq \gamma(T)$. Hence, we conclude that $\lambda_0 = 1$. \square

The following result follows essentially from Proposition 2.4 and will prove to be useful in the sequel.

Proposition 3.17. *Assume that $N : \mathbb{R}^{2n} \rightarrow \mathbb{R}_{\geq 0}$ is a norm. Let v be an optimal control for the problem (3.2). Then*

a. *the Maximum Principle (3.7) is equivalent to*

$$a(s) \in \partial F_N(v(s)) = N(v(s))\partial N(v(s)), \quad \forall s \in [0, T]; \quad (3.18)$$

which is equivalent to

$$v(s) \in \partial(F_N)^*(a(s)) = N_*(a(s))\partial N_*(a(s)), \quad \forall s \in [0, T]. \quad (3.19)$$

b. *there exists a unique constant $R(v) = R > 0$ such that*

$$N(v(s)) = N_*(a(s)) = R, \quad \forall s \in [0, T]. \quad (3.20)$$

Proof: Since v is optimal, we have $\lambda_0 = 1$ and the Maximum Principle (3.7) guarantees that the min exists, for every s . Using the definition of subdifferential, it is easy to see that a point $v(s)$ realizes the min in (3.7) if and only if (3.18) holds. The equality in (3.18) follows from (2.7).

On the other hand, by (2.2), (3.18) is equivalent to

$$v(s) \in \partial F_N^*(a(s)), \quad \forall s \in [0, T].$$

and again by (2.7) we obtain (3.19). This proves a.

The convexity of F_N implies, by (2.2), that (3.18) is equivalent to

$$F_N(v(s)) + F_N^*(a(s)) = v(s) \cdot a(s), \quad \forall s \in [0, T]. \quad (3.21)$$

Note that Proposition 2.4 gives $F_N^*(a(s)) = \frac{1}{2}(N_*(a(s)))^2$. Now (3.11) gives that the function $s \mapsto F_N^*(a(s))$ is in fact constant, i.e.

$$N_*(a(s)) = R, \quad \forall s \in [0, T], \quad (3.22)$$

for some fixed $R \geq 0$. If $R = 0$, relation (3.19) implies $v(s) = 0$ in $[0, T]$ contradicting the fact that $\gamma(T) = g \neq e$. Thus we have that $R > 0$.

Now (3.19) becomes $v(s) \in R\partial N_*(a(s))$, for all $s \in [0, T]$. This implies $\frac{v(s)}{R} \in \partial N_*(a(s))$ and taking into account (2.6) we have $\frac{v(s)}{R} \cdot a(s) = N_*(a(s))$, i.e.

$$v(s) \cdot a(s) = R^2. \quad (3.23)$$

Equation (3.21) becomes

$$\frac{1}{2}(N(v(s)))^2 + \frac{1}{2}R^2 = R^2, \quad \forall s.$$

Clearly R depends only on v . This proves b. \square

It is clear that (3.20) is equivalent to

$$v(s) \in \partial B_N(0, R), \quad a(s) \in \partial B_{N_*}(0, R), \quad \forall s \in [0, T].$$

In what follows we would like to clarify the relation between the solutions of (3.2) and the concept of geodesics as defined in the introduction of this paper. In our problem (3.2) we fix $T > 0$, the initial point e and a final point $g \in \mathbb{H}^n$. Let us consider an optimal control v and the associated trajectory $\gamma : [0, T] \rightarrow \mathbb{H}^n$ such that $v = -\dot{\gamma}_I$, $\gamma(0) = e$ and $\gamma(T) = g$. The Pontryagin Maximum Principle gives us the multiplier $(1, \lambda^v)$ and hence a function a . The tern (v, λ^v, a) is in general not unique. It satisfies relations (3.7)–(3.11) and the previous Proposition 3.17, for the same $R > 0$. This implies that

$$d_{SF}(e, g) = \int_0^T N(v(s)) ds = TR,$$

where d_{SF} is, as in (1.6), the sub-Finsler distance from the points e and g .

Now we can distinguish two cases:

- (1) if $R = 1$, then γ is a geodesic according to (1.3).
- (2) if $R \neq 1$, then γ is not a geodesic according to (1.3). In this case we have to change the parametrization of γ to obtain a geodesic. In order to do that we define $\tilde{\gamma} : [0, TR] \rightarrow \mathbb{H}^n$ by

$$\tilde{\gamma}(s) = \gamma(s/R). \quad (3.24)$$

Clearly $\text{Im}(\tilde{\gamma}) = \text{Im}(\gamma)$, $\tilde{\gamma}(TR) = \gamma(T) = g$ and $\tilde{v}(s) = v(s/R)/R$. It is easy to see that such \tilde{v} is an optimal control for the problem in (3.2), with the same final point g and final time TR . Since N is homogeneous we obtain, for every $s \in [0, TR]$

$$R = N(v(s)) = N(R\tilde{v}(sR)) = RN(\tilde{v}(sR)).$$

Hence the “ R ” associated to \tilde{v} is 1. Now we obtain

$$d_{SF}(0, g) = \int_0^{TR} N(\tilde{v}(s)) ds = TR.$$

The new curve $\tilde{\gamma}$ is really the geodesic in the sense of (1.3).

Conversely, if we have a geodesic $\gamma : [0, T] \rightarrow \mathbb{H}^n$ from 0 to g , then $v = -\dot{\gamma}_I$ is an optimal control for (3.2) and the uniqueness of R in Proposition 3.17 implies easily that $R = 1$.

We would like to formulate the above observation in the following:

Remark 3.25. A horizontal curve $\gamma : [0, T] \rightarrow \mathbb{H}^n$, with initial point the origin, is a geodesic if and only if the associated $v = -\dot{\gamma}_I$ is a optimal control for (1.7) with $\gamma(T) = g \neq e$ and $R = 1$.

Moreover, if for a horizontal curve $\gamma : [0, T] \rightarrow \mathbb{H}^n$, with initial point the origin, the associated $v = -\dot{\gamma}_I$ is an optimal control for (3.2) with $\gamma(T) = g \neq e$ and $R \neq 1$, after a reparametrization of γ (as in (3.24)) we obtain a geodesic.

In the sequel, when we deal with a finite geodesic γ , then $N(-\dot{\gamma}_I(s)) = R = 1$. Furthermore, in the case when $\gamma : [0, \infty) \rightarrow \mathbb{H}^n$ is an infinite geodesic, then for every fixed $T > 0$ the curve $\gamma|_{[0, T]}$ is a finite geodesic, which implies that we have that $N(-\dot{\gamma}_I(s)) = R = 1$ for all $s \in [0, \infty)$.

In the following we shall add the assumption of strict convexity of the norm N .

3.3. N is a strictly convex norm. Let recall that the strict convexity of N implies that N_* is smooth (see [21, Proposition 5.4.7]).

Proposition 3.26. *Assume that $N : \mathbb{R}^{2n} \rightarrow \mathbb{R}_{\geq 0}$ is a strictly convex norm. Let v be an optimal control for the problem (3.2). Then*

- c. *for every $a(s)$ there exists a unique $v(s)$ such that inclusion (3.19) holds in the form of an equality; more precisely we obtain*

$$v(s) = R \nabla N_*(a(s)), \quad (3.27)$$

where $R > 0$ is the constant from (3.20).

- d. *If $k = 0$, then the unique solutions $\gamma = (x_1, \dots, x_n, y_1, \dots, y_n, t)$ of (3.7)–(3.11) are horizontal segments.*

Proof: The strict convexity of N as a norm implies the strict convexity of the function F_N (see Proposition 2.11): this gives that for every $a(s) \in \mathbb{R}^{2n}$ the value of

$$\min_{u \in \mathbb{R}^{2n}} (F_N(u) - a(s) \cdot u)$$

is achieved at a unique point $v(s) \in \mathbb{R}^{2n}$. This proves c.

Let $k = 0$. It is easy to see, by (3.13), that we obtain $a_i(s) = \lambda_i(0)$ for $s \in [0, T]$, $i = 1, \dots, 2n$. In the Maximum Principle (3.7) the min exists, is unique and since $a(s)$ does not depend on s , we obtain for such min that $v = v(s)$ does not depend on s . Hence we have that γ is a horizontal segment. \square

This argument does not work without the strict convexity assumption on N : more precisely if $k = 0$ and if N and thus F_N is only a convex function, it is not always possible to guarantee that for every fixed $a(s)$, the function

$$u \mapsto F_N(u) - a(s) \cdot u$$

has a unique minimum, for every fixed s .

Let us recall that in general the subdifferential as a set-valued mapping $a \mapsto \partial N_*(a)$ is upper continuous. In our case the subdifferential is single valued $\partial N_*(\cdot) = \{\nabla N_*(\cdot)\}$ and consequently we obtain that $a \rightarrow \nabla N_*(a)$ is continuous. Furthermore, using the fact that the function $s \rightarrow a(s)$ is absolutely continuous, we obtain by (3.27), the following:

Remark 3.28. If $N : \mathbb{R}^{2n} \rightarrow \mathbb{R}_{\geq 0}$ is a strictly convex norm and v is an optimal control for the problem (3.2), then v is continuous.

4. PROOF OF THE MAIN THEOREM

In this section we give the proof of Theorem 1.2. For the sake of a better understanding we have decided to give the proof in two stages. In the first stage we consider the special case when N is a smooth norm. In this case the proof is easier as we can show uniqueness of the multiplier coming from Pontryagin Maximum Principle.

The general case, when N is strictly convex but possibly non-smooth, some technical complications arise as in general the multiplier is not unique. In the second step we show how to overcome this difficulty and prove Theorem 1.2 also in this case.

4.1. N is a strictly convex and smooth norm.

Proposition 4.1. *Assume that $N : \mathbb{R}^{2n} \rightarrow \mathbb{R}_{\geq 0}$ is a strictly convex and smooth norm. Let $v : [0, T] \rightarrow \mathbb{R}^{2n}$ be an optimal control for the problem (3.2). Then we have that*

- e. *for every $v(s)$ there exists a unique $a(s)$ such that inclusion (3.18) holds;*
- f. *the associated multiplier λ^v (and hence the function a in (3.13)) is unique.*

Proof: Since N is differentiable except at the origin, we know that $\partial N(v(s)) = \{\nabla N(v(s))\}$ for every $s \in [0, T]$. More precisely (3.18) becomes the equality

$$a(s) = R\nabla N(v(s)), \quad \forall s \in [0, T].$$

Hence e. is proved.

Let $v : [0, T] \rightarrow \mathbb{R}^{2n}$ be the optimal control of the problem (3.2) and let us assume that there exist two multipliers $\lambda^v, \tilde{\lambda}^v : [0, T] \rightarrow \mathbb{R}^{2n+1}$ associated to v . Let a and \tilde{a} defined by (3.13) by λ^v and $\tilde{\lambda}^v$ respectively. Now e. implies

$$a(s) = \tilde{a}(s), \quad s \in [0, T] \tag{4.2}$$

and hence, a.e. $\dot{a}(s) = \dot{\tilde{a}}(s)$. By (3.13) this implies that $4\lambda_{2n+1}\dot{y}_i = 4\tilde{\lambda}_{2n+1}\dot{y}_i$ and $4\lambda_{2n+1}\dot{x}_i = 4\tilde{\lambda}_{2n+1}\dot{x}_i$. Since the associated trajectory is not a constant curve, there exists a value $s \in [0, T]$ and an index i with the property that $\dot{x}_i(s)$ or $\dot{y}_i(s)$ does not vanish. This implies $k = \lambda_{2n+1} = \tilde{\lambda}_{2n+1}$. In (4.2) we obtain

$$\lambda_i(0) + 4ky_i(s) = \tilde{\lambda}_i(0) + 4ky_i(s), \quad \lambda_{n+i}(0) - 4kx_i(s) = \tilde{\lambda}_{n+i}(0) - 4kx_i(s), \quad i = 1, \dots, n.$$

Relation (3.12) gives f. □

As we will show, if N is only a strictly convex norm, then Proposition 4.1 fails (see Example 5.1).

In what follows we shall consider infinite geodesics $\gamma : \mathbb{R} \rightarrow \mathbb{H}^n$. We restrict γ to the half-line $[0, \infty)$ and we call this restriction also an infinite geodesic. All our considerations can be repeated to the restriction of γ to negative parameter values and so it is enough to consider only the restriction $\gamma : [0, \infty) \rightarrow \mathbb{H}^n$.

As a next step we shall restrict γ to the finite interval $[0, T]$ for some $T > 0$ we can conclude our curve is a solution to the optimal control problem (3.2). The main technical difficulty in our analysis is the possible dependence on the value of T of the multiplier λ^v from Pontryagin's theorem. Indeed, Pontryagin's theorem does not guarantee the uniqueness of the multiplier λ^v for a given optimal control v and it can happen that for two different values of say $T' > T$ we obtain two different multipliers $\lambda^{v,T}$ and $\lambda^{v,T'}$ that do not necessarily coincide on the the interval $[0, T]$. The following proposition shows how to deal with this problem if the norm N is smooth.

Let $\gamma : [0, \infty) \rightarrow \mathbb{H}^n$ be a infinite geodesic. We say that λ is an *infinite multiplier* if for every fixed $T > 0$ the function $\lambda|_{[0, T]}$ is a multiplier associated to the optimal control $-\dot{\gamma}|_{[0, T]}$ for the problem (3.2) with $\gamma(T) = g$.

Proposition 4.3. *Let $N : \mathbb{R}^{2n} \rightarrow \mathbb{R}_{\geq 0}$ be a strictly convex and smooth norm. Let $\gamma : [0, \infty) \rightarrow \mathbb{H}^n$ be a infinite geodesic with respect to the sub-Finsler metric associated to N . Then there exists a unique infinite multiplier $\lambda : [0, \infty) \rightarrow \mathbb{R}^{2n+1}$, and hence a unique function $a : [0, \infty) \rightarrow \mathbb{R}^{2n}$ via (3.13) such that $N_*(a(s)) = 1$.*

Proof: Let $\gamma : [0, \infty) \rightarrow \mathbb{H}^n$ be an infinite sub-Finsler geodesic associated to N by (1.3) and (1.6).

For some positive $T > 0$, the function $\gamma|_{[0, T]}$ is a geodesic from $\gamma(0) = e$ to $\gamma(T)$, with horizontal velocity $v|_{[0, T]}$. Since N is smooth (see Proposition 4.1), there exists a unique pair (λ^T, a^T) associated to such horizontal velocity such that

$$a^T(s) = \nabla N(v(s)), \quad s \in [0, T].$$

Notice that if we take $T' > T$ then the above relation will also be satisfied by $a^{T'}$ on $[0, T]$ i.e.

$$a^{T'}(s) = \nabla N(v(s)), \quad s \in [0, T].$$

From the above relation we conclude that $a^T = a^{T'}$ on the common interval $[0, T]$ showing that in fact a^T does not really depend on T and we are in the position to define the a in $[0, \infty)$ such that

$$a(s) = \nabla N(v(s)), \quad s \in [0, \infty).$$

Uniqueness of a can be used to conclude the uniqueness of the multiplier as follows. Notice first that relation (3.13) gives

$$a_i(s) = \lambda_i^T(0) + 4\lambda_{2n+1}^T y_i(s), \quad a_{n+i}(s) = \lambda_{n+i}^T(0) - 4\lambda_{2n+1}^T x_i(s), \quad i = 1, \dots, n, \quad s \in [0, T].$$

Taking derivatives w.r.t. s we obtain

$$\dot{a}(s) = 4\lambda_{2n+1}^T J_n v(s), \quad \text{a.e. } s \in [0, T],$$

where J_n is the standard symplectic matrix in \mathbb{R}^{2n} . Since there exists a value $s_0 \in (0, \infty)$ such that $v(s_0) \neq 0$ we conclude that λ_{2n+1}^T does not depend on T for $T > s_0$. Setting $\lambda_{2n+1}^T = k$ in the previous relation, it is easy to see the λ^T does not depend on T for $T > s_0$ proving the claim. \square

The following statement is a special case of Theorem 1.2 under the additional assumption that N is smooth.

Proposition 4.4. *Let $N : \mathbb{R}^{2n} \rightarrow \mathbb{R}_{\geq 0}$ be a strictly convex and smooth norm. Let*

$$\gamma = (x_1, \dots, x_n, y_1, \dots, y_n, t) : \mathbb{R} \rightarrow \mathbb{H}^n$$

be an infinite geodesic with respect to the associated sub-Finsler metric. Then γ is a horizontal line.

Proof: We consider again the restriction $\gamma : [0, \infty) \rightarrow \mathbb{H}^n$. Proposition 4.3 implies that we can associate a unique function a satisfying (3.13) with $N_*(a(s)) = 1$ for $s \geq 0$.

From Proposition 4.3 we can associate a values $\lambda_i(0)$ for $i = 1, \dots, 2n$ and $\lambda_{2n+1} = k$ satisfying

$$a_i(s) = \lambda_i(0) + 4ky_i(s), \quad a_{n+i}(s) = \lambda_{n+i}(0) - 4kx_i(s), \quad i = 1, \dots, n, \quad s \in [0, \infty). \quad (4.5)$$

There are two cases to consider: $k = 0$ and $k \neq 0$.

In the first case we obtain that a is constant vector. On the other hand we have the equation $a = a(s) = \nabla N(v(s))$, $s \in [0, \infty)$. Using this relation we obtain that v must also be a constant since N is a strictly convex norm. This implies that $\{\gamma(s) : s \in [0, \infty)\}$ is a horizontal half-line. The same argument can be done for negative values of s as well and we can conclude that γ is a horizontal line.

In what follows we shall prove that the other possibility: $k \neq 0$ will lead to a contradiction. To see this we use first that $N_*(a(s)) = 1$ and therefore a takes its values in a compact set. Then, the relation (4.5) can be used to express γ in terms of a . More precisely we obtain:

$$\frac{a_i(s) - \lambda_i(0)}{4k} = y_i(s), \quad \frac{a_{n+i}(s) - \lambda_{n+i}(0)}{-4k} = x_i(s), \quad i = 1, \dots, n, \quad s \in [0, \infty). \quad (4.6)$$

This implies that there exists a constant $C > 0$ such that

$$\|(x_1(s), \dots, x_n(s), y_1(s), \dots, y_n(s))\| = \|(\gamma_1(s), \dots, \gamma_{2n}(s))\| \leq C, \quad \forall s \geq 0. \quad (4.7)$$

This means that the restriction of our geodesic γ to $[0, \infty)$ lies in an infinite Euclidean cylinder centered around the vertical axis of a fixed radius $C > 0$. Let us define $\{\gamma_k\}_{k \geq 1}$, $\gamma_k : [0, \infty) \rightarrow \mathbb{H}^n$ by

$$\gamma_k(s) = \delta_{\frac{1}{k}}(\gamma(sk)), \quad s \in [0, \infty).$$

In what follows we shall check that $\gamma_k : [0, \infty) \rightarrow \mathbb{H}^n$ is a infinite geodesic for every k .

In order to do that observe first that since $\gamma : [0, \infty) \rightarrow \mathbb{H}^n$ is a geodesic with respect to d_{SF} we have

$$d_{SF}(\gamma(s), \gamma(s')) = |s - s'|, \quad \text{for all } s, s' \in [0, \infty).$$

Using the homogeneity of the metric d_{SF} we can infer that

$$\begin{aligned} d_{SF}(\gamma_k(s), \gamma_k(s')) &= d_{SF}(\delta_{\frac{1}{k}}\gamma(ks), \delta_{\frac{1}{k}}\gamma(ks')) = \frac{1}{k}d_{SF}(\gamma(ks), \gamma(ks')) = \\ &= \frac{1}{k}|ks - ks'| = |s - s'| \quad \text{for all } s, s' \in [0, \infty). \end{aligned}$$

This relation shows that $\gamma_k : [0, \infty) \rightarrow \mathbb{H}^n$ is an infinite geodesic for every k as claimed. At the same time this also shows that the family of functions $\{\gamma_k\}_{k \geq 1}$, $\gamma_k : [0, 1] \rightarrow \mathbb{H}^n$ is uniformly bounded and equicontinuous. Uniform boundedness of the family follows from the fact that $\gamma_k(0) = 0$ for all k . Applying the theorem of Arzelá-Ascoli we obtain a subsequence $\gamma_{n_k} : [0, 1] \rightarrow \mathbb{H}^n$ converging uniformly to a function $\hat{\gamma} : [0, 1] \rightarrow \mathbb{H}^n$. Using the fact that

$$d_{SF}(\gamma_{n_k}(s), \gamma_{n_k}(s')) = |s - s'| \quad \text{for all } s, s' \in [0, 1],$$

and taking the limit in this equation as $k \rightarrow \infty$ we obtain

$$d_{SF}(\hat{\gamma}(s), \hat{\gamma}(s')) = |s - s'| \quad \text{for all } s, s' \in [0, 1].$$

This shows that $\hat{\gamma} : [0, 1] \rightarrow \mathbb{H}^n$ is a geodesic.

On the other hand, relation (4.7) implies that $\hat{\gamma} = (\hat{\gamma}_1, \dots, \hat{\gamma}_{2n+1})$ will satisfy $\hat{\gamma}_i = 0$ for $i = 1 \dots 2n$. This means that the curve $\hat{\gamma}$ is contained in the vertical axis. It is clear also

that the the image $\hat{\gamma}$ is not reduced to a single point and so it must contain a non-trivial interval of the vertical axis. However, it is well known that the Hausdorff dimension with respect to d_{SF} of any non-trivial interval of the vertical axis is equal to 2. (For this statement and more general related results we refer to [9]). This clearly contradicts the fact that $\hat{\gamma}$ is a geodesic. \square

4.2. N is a strictly convex and non-smooth norm. In this subsection we show the modifications necessary to prove Theorem 1.2 in the general case when the norm N is strictly convex and not necessarily smooth.

In fact the proof follows along the same lines as the one of Proposition 4.4. The only missing piece from the puzzle is the following boundedness property of infinite geodesics.

Proposition 4.8. *Let $N : \mathbb{R}^{2n} \rightarrow \mathbb{R}_{\geq 0}$ be a strictly convex and possibly non-smooth norm. Let*

$$\gamma = (x_1, \dots, x_n, y_1, \dots, y_n, t) : \mathbb{R} \rightarrow \mathbb{H}^n$$

be an infinite geodesic with respect to the sub-Finsler metric associated to N . Assume in addition that γ is not a horizontal line. Then there exists a constant $C > 0$ such that

$$\|(x_1(s), \dots, x_n(s), y_1(s), \dots, y_n(s))\| = \|(\gamma_1(s), \dots, \gamma_{2n}(s))\| \leq C, \quad \forall s \in \mathbb{R}. \quad (4.9)$$

Proof: We shall prove the estimate of (4.9) only for values $s \geq 0$ For negative values of s the argument is similar. Let us restrict γ to the finite interval $[0, T]$ for some $T > 0$.

Let us recall that relation (3.13) gives

$$a_i^T(s) = \lambda_i^T(0) + 4\lambda_{2n+1}^T y_i(s), \quad a_{n+i}^T(s) = \lambda_{n+i}^T(0) - 4\lambda_{2n+1}^T x_i(s), \quad i = 1, \dots, n, \quad s \in [0, T];$$

note that the choice of the functions a^T and λ^T is not unique. Setting $s = 0$ in the above relations and using that $\gamma(0) = e$ we conclude that $a_i^T(0) = \lambda_i^T(0)$, for every i , and so we can write these relations in shorthand as

$$a^T(s) = a^T(0) + 4k^T J_n \gamma_I(s), \quad s \in [0, T], \quad (4.10)$$

where $k^T = \lambda_{2n+1}^T$ and J_n is the standard symplectic matrix in \mathbb{R}^{2n} . Let us choose a parameter value s_0 for which $v(s_0) \neq v(0)$ and $\gamma_I(s_0) \neq 0$. Since γ is not a line, such value s_0 can be found for a sufficiently large T . Let us write the above relation (4.10) at s_0 as

$$a^T(s_0) - a^T(0) = 4k^T J_n \gamma_I(s_0), \quad s \in [0, T]. \quad (4.11)$$

Notice that $a^T(s_0) \in \partial F_N(v(s_0))$, $a^T(0) \in \partial F_N(v(0))$. Furthermore, observe that $\partial F_N(v(s_0))$ and $\partial F_N(v(0))$ are two compact and convex sets. By the strict convexity of N the intersection of $\partial F_N(v(s_0))$ and $\partial F_N(v(0))$ is empty according to Proposition 2.11. This implies that

$$c := \min\{\|u - u'\| : u \in \partial F_N(v(s_0)), u' \in \partial F_N(v(0))\} > 0, \quad (4.12)$$

which yields the estimate

$$0 < c \leq \|a^T(s_0) - a^T(0)\| \quad (4.13)$$

for all values of $T > s_0$. Taking the norm in (4.11) we obtain

$$c \leq 4|k^T| \|\gamma_I(s_0)\|,$$

which implies that

$$|k^T| \geq \frac{c}{4\|\gamma_I(s_0)\|}, \quad T > s_0. \quad (4.14)$$

Now let us turn back to the relation (4.10): using that $N(v(s)) = 1$, $a^T(s) \in \partial F_N(v(s))$ and the fact that the operator $v \rightarrow \partial F_N(v)$ is upper semicontinuous (see [12, Proposition 4.2.3] and our section 2) we obtain that $a^T(s)$ takes values in a fixed compact set that is independent on T and s . We can now write (4.10) in the form

$$\frac{a^T(s) - a^T(0)}{4k^T} = J_n \gamma_I(s), \quad s \in [0, T].$$

Taking into consideration the above discussion and the estimate (4.14) we can conclude that the quantity on the left hand side of the above relation takes values in a fixed compact set independently on s and T proving our claim. \square

5. EXAMPLES AND FINAL REMARKS

Let $p \in [1, \infty]$, and let $\|\cdot\|_p$ be the p -norm on \mathbb{R}^{2n} and $a \in (0, \infty)$. Then one can check (see [10]) that the function

$$\hat{N} = N_{p,a} : \mathbb{H}^n \rightarrow \mathbb{R}, (z, t) \mapsto \max \left\{ \|z\|_p, a\sqrt{|t|} \right\},$$

defines a left-invariant norm on \mathbb{H}^n , if either $1 \leq p \leq 2$ and $0 < a \leq 1$ or if $2 < p \leq \infty$ and $0 < a \leq n^{1/p-1/2}$.

In [10] it was proved that $\hat{N} = N_{p,a}$ has the *GLP* if and only if $p \in (1, +\infty)$. This can be obtained also as an immediate corollary of Theorem 1.2 observing that the norm N associated to $\hat{N} = N_{p,a}$ is the usual p -norm: $N(z) = \|z\|_p$. Since $\|\cdot\|_p$ is strictly convex if and only if $p \in (1, +\infty)$ we obtain this result as an immediate corollary of Theorem 1.2. Notice, that for $p = \infty$ or $p = 1$, $\hat{N} = N_{p,a}$ admits infinite geodesics which are not lines, and in fact there exist in this setting isometric embeddings which are non-linear.

The above example is a particular case of a more general phenomena. Observe first that non-strict convexity of a given norm N in \mathbb{R}^{2n} is equivalent with the existence of non-linear infinite geodesics in the normed space (\mathbb{R}^{2n}, d_N) . Next, we notice that horizontal lifts of infinite geodesics in (\mathbb{R}^{2n}, d_N) will be infinite geodesics in associated sub-Finsler space (\mathbb{H}^n, d_{SF}) . This shows that there exists an abundance of non-linear infinite geodesics that arise naturally in (\mathbb{H}^n, d_{SF}) whenever the original norm N is not strictly convex. For more details about the above argument we refer to the proof of Proposition 3.14 in [10].

In the first Heisenberg group \mathbb{H}^1 , the classification of the geodesics with respect to a sub-Finsler distance associated to a norm N is related to the following *isoperimetric problem* on the Minkowski plane (\mathbb{R}^2, N) : given a number A find a closed path through 0 of minimal N -length which encloses (Euclidean) area A . To describe the solution, we recall the following notation for the closed unit ball and dual ball in (\mathbb{R}^2, N) ,

$$B := \{z \in \mathbb{R}^2 : N(z) \leq 1\} \quad \text{and} \quad B^\circ := \{w : w \cdot z \leq 1, z \in B\}.$$

The *isoperimetrix* I is the boundary of B° rotated by $\pi/2$, and it can be parameterized as a closed curve. Buseman [13] proved that the solution to the above stated isoperimetric problem is given by (appropriate dilation and translation) of the isoperimetrix. Note that if N is strictly convex, then I is of class \mathcal{C}^1 . Considering the associated sub-Finsler space (\mathbb{H}^1, d_{SF}) we can conclude that horizontal lifts of the isoperimetrix will be in fact geodesics in (\mathbb{H}^1, d_{SF}) . Let us note that this classical result of Busemann follows rather

easily from the control-theoretical approach, in particular Proposition 3.17. In fact we can observe that according to Proposition 3.17 the geodesics in (\mathbb{H}^n, d_{SF}) are also curves on the boundary of the dual ball, B° where

$$B := \{z \in \mathbb{R}^{2n} : N(z) \leq 1\} \quad \text{and} \quad B^\circ := \{w : w \cdot z \leq 1, z \in B\}.$$

To be clear, in the case $n = 1$, if we have a geodesic then in Proposition 3.17 we obtain $N_*(a(s)) = N(v(s)) = R = 1$; this implies, using (3.13),

$$(\lambda_1(0) + 4ky(s), \lambda_2(0) - 4kx(s)) \in \partial B_{N_*}, \quad \forall s.$$

If $k \neq 0$, this is equivalent to

$$(y(s), -x(s)) \in \frac{1}{4k} \left(\partial B_{N_*} - (\lambda_1(0), \lambda_2(0)) \right), \quad \forall s.$$

And this is equivalent to the fact that $(x(s), y(s))$ lies in the set described by the Busseman result, i.e. considering the set ∂B_{N_*} , a translation, a dilation and a rotation of $\pi/2$. For $n = 1$ the whole boundary is a curve and so the geodesics in (\mathbb{H}^1, d_{SF}) can be explicitly characterised.

In higher dimensions such explicit characterisation cannot be expected for a general norm N . In certain particular cases this is still possible. An example of this explicit characterisation is the well-known case of the sub-Riemannian geodesics when N is the usual Euclidean norm $N(z) = \|z\|_2$ given by Monti [22] who also used a control theoretical approach (note that a different proof of the expressions of geodesics in \mathbb{H}^n is given in [18]). In this case the optimal control problem (3.2) can be explicitly solved and the expressions of the sub-Riemannian geodesics are precisely computed [22].

As we have showed, the case of N a strictly convex and non-smooth norm may lead to the non-uniqueness and ambiguity of the multiplier coming from Pontryagin's theorem. In the next example we show that this situation actually happens. We construct a solution of the optimal control problem (3.2) such that there exist multiple choices for $a(s)$ such that inclusion (3.18) holds, since the multiplier λ^v is not unique. From this we can see that without the assumption on the smoothness of the norm, Proposition 4.1 does not hold.

Example 5.1 (strictly convex and non-smooth norm). *Let $N : \mathbb{R}^2 \rightarrow \mathbb{R}_{\geq 0}$ defined by ,*

$$N(z) = |x| + \sqrt{2x^2 + y^2},$$

for $z = (x, y) \in \mathbb{R}^2$.

It is an exercise to show that the shape of ∂B_N is a sort of an American football ball, non smooth in the points $(0, \pm 1)$: more precisely we have

$$\partial B_N = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 + 2|x| - 1 = 0\}.$$

We can calculate the dual norm using its definition via some rather tedious calculation to obtain

$$N_*(x, y) = \begin{cases} -|x| + \sqrt{2}\sqrt{x^2 + y^2} & \text{if } |x| \geq |y| \\ |y| & \text{if } |x| < |y| \end{cases} \quad (5.1)$$

with

$$\partial B_{N_*} = \left\{ (x, y) \in \mathbb{R}^2 : |x| \leq 1, |y| = 1 \right\} \cup \left\{ (x, y) \in \mathbb{R}^2 : |x| \geq 1, x^2 + 2y^2 - 2|x| = 1 \right\}$$

and

$$\nabla N_*(x, y) = \begin{cases} \left(-\operatorname{sgn}(x) + \frac{\sqrt{2}x}{\sqrt{x^2 + y^2}}, \frac{\sqrt{2}y}{\sqrt{x^2 + y^2}} \right) & \text{if } |x| \geq |y|, \\ (0, \operatorname{sgn}(y)) & \text{if } |x| < |y|. \end{cases} \quad (5.2)$$

We claim that the curve $\gamma : [0, \tau] \rightarrow \mathbb{H}^1$ defined by

$$\gamma(s) = \begin{cases} (0, -s, 0) & \text{if } s \in [0, 1] \\ \left(-1 + \sqrt{(1 + 2\theta(s) - \theta^2(s))/2}, -\theta(s), \gamma_3(s) \right) & \text{if } s \in (1, \tau] \end{cases} \quad (5.3)$$

is a geodesic. The function γ_3 is defined via the horizontality condition on γ , while the point τ and the function θ are defined as follows. Consider the Cauchy problem:

$$\begin{cases} \dot{\theta} = \frac{\sqrt{2 + 4\theta - 2\theta^2}}{1 + \theta} \\ \theta(1) = 1. \end{cases} \quad (5.4)$$

The solution is given by the relation

$$s - \sqrt{2} \arcsin\left(\frac{\theta(s) - 1}{\sqrt{2}}\right) + \frac{1}{2} \sqrt{2 + 4\theta(s) - 2\theta(s)^2} = 2. \quad (5.5)$$

This relation defines a differentiable, increasing function $\theta : [1, \tau] \rightarrow \mathbb{R}$, with $\tau = 2 + \pi/\sqrt{2}$, such that $\theta(\tau) = 1 + \sqrt{2}$.

The first step in order to show that γ is a geodesic is to prove that it is a solution of the Pontryagin system (3.7)–(3.11). Equivalently γ is a curve such that the triple (v, λ, a) satisfies (3.7)–(3.11) for some a and λ .

Clearly, by (5.3), we have

$$-\dot{\gamma}_I(s) = v(s) = \begin{cases} (0, 1) & \text{if } s \in [0, 1] \\ \left(\frac{\dot{\theta}(\theta - 1)}{\sqrt{2 + 4\theta - 2\theta^2}}, \dot{\theta}(s) \right) & \text{if } s \in (1, \tau]. \end{cases} \quad (5.6)$$

Let us consider the function $\lambda : [0, \tau] \rightarrow \mathbb{R}^3$ defined as in (3.12) choosing $\lambda_1(0) = 0$, $\lambda_2(0) = 1$ and $\lambda_3 = k = -1/4$, i.e.

$$\lambda(s) = \left(-\frac{1}{2}y(s), 1 + \frac{1}{2}x(s), -\frac{1}{4} \right),$$

where $\gamma_I(s) = (x(s), y(s))$. It is immediate to see that for such λ conditions (3.8)–(3.10) are satisfied.

Now, we define the function $a : [0, \tau] \rightarrow \mathbb{R}^2$ via (3.13):

$$a(s) = \begin{cases} (s, 1) & \text{if } s \in [0, 1] \\ \left(\theta(s), \sqrt{(1 + 2\theta(s) - \theta^2(s))/2} \right) & \text{if } s \in (1, \tau]. \end{cases} \quad (5.7)$$

We have to prove that relations (3.7) and (3.11) hold. First, it is easy to see, using the first line in (5.1) and $\theta(s) > 0$, that

$$N_*(a(s)) = 1, \quad \forall s \in [0, \tau]. \quad (5.8)$$

Now the Maximum Principle (3.7) is equivalent to (3.19), i.e. we have to prove that

$$v(s) \in N_*(a(s))\partial N_*(a(s)) = \{\nabla N_*(a(s))\}, \quad s \in [0, \tau],$$

taking into account (5.6), (5.7) and (5.2). The previous equality for $s \in [0, 1]$ is obvious; for $s \in [1, \tau]$ we have to obtain

$$\left(\frac{\dot{\theta}(\theta - 1)}{\sqrt{2 + 4\theta - 2\theta^2}}, \dot{\theta}(s) \right) = \left(-1 + \frac{2\theta}{1 + \theta}, \frac{\sqrt{2 + 4\theta - 2\theta^2}}{1 + \theta} \right).$$

The two components of this equation are equivalent; together with the condition $\theta(1) = 1$, we obtain exactly the Cauchy problem (5.4).

Hence (v, λ, a) solves the Pontryagin system (3.7)–(3.11) and γ is candidate to be a geodesic. Now, from the mentioned Buseman's theorem [13], γ is indeed a finite geodesic.

In the spirit of Proposition 4.8, let us consider $T = 1$ and the restriction of γ to $[0, T]$: we have the geodesic $\gamma|_{[0,1]} = \tilde{\gamma}$, i.e. the horizontal segment from the origin to the point $(0, -1, 0)$. Hence $\tilde{v} = -\tilde{\gamma}_I = (0, 1)$ is a optimal solution for (3.2) with $g = (0, -1, 0)$. Let us prove that for every (ℓ, k) such that

$$\max(|\ell|, |\ell - 4k|) \leq 1 \tag{5.9}$$

the function $\lambda^{\ell,k} : [0, 1] \rightarrow \mathbb{R}^3$ defined by

$$\lambda^{\ell,k}(s) = (\ell - 2ks, 1, k)$$

is a multiplier associated to the optimal control \tilde{v} .

First, for every (ℓ, k) , via (3.13), we have the function $a^{\ell,k} : [0, 1] \rightarrow \mathbb{R}^2$ defined as

$$a^{\ell,k}(s) = (\ell - 4ks, 1).$$

Conditions (3.8)–(3.10) are easily satisfied. Condition (5.9) gives that $N_*(a^{\ell,k}(s)) = 1$. Hence (3.19) becomes

$$\tilde{v}(s) = (0, 1) = \nabla N_*(a^{\ell,k}(s)), \quad \forall s \in [0, 1].$$

Using (5.2), condition (5.9) guarantees the previous equality. Hence \tilde{v} admits infinitely many multipliers.

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