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Research Article

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A polynomial bound for the number of maximal systems of imprimitivity of a finite transitive permutation group

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Abstract: We show that there exists a constant a such that, for every subgroup H of a finite group G, the number of maximal subgroups of G containing H is bounded above by $a|G : H|^{3/2}$. In particular, a transitive permutation group of degree n has at most $an^{3/2}$ maximal systems of imprimitivity. When G is soluble, generalizing a classic result of Tim Wall, we prove a much stronger bound, that is, the number of maximal subgroups of G containing H is at most $|G : H| - 1$.

Keywords: Wall conjecture, maximal subgroups, permutation groups, systems of imprimitivity

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1 Introduction

In 1961, Tim Wall [12] has conjectured that the number of maximal subgroups of a finite group G is less than the group order $|G|$. Wall himself proved the conjecture under the additional hypothesis that G is soluble. The first remarkable progress towards a good understanding of Wall's conjecture is due to Liebeck, Pyber and Shalev [9]; they proved that all, but (possibly) finitely many, simple groups satisfy Wall's conjecture. Actually, Liebeck, Pyber and Shalev prove [9, Theorem 1.3] a polynomial version of Wall's conjecture: there exists an absolute constant c such that every finite group G has at most $c|G|^{3/2}$ maximal subgroups. Based on the conjecture of Guralnick on the dimension of certain first cohomology groups [6] and on some computer computations of Frank Lübeck, Wall's conjecture was disproved in 2012 by the participants of an AIM workshop; see [7].

The question of Wall can be generalized in the context of finite permutation groups and this was done by Peter Cameron; see [3] (also for the motivation of this question).

Question 1.1 (Cameron [3]). Is the number of maximal blocks of imprimitivity through a point for a transitive group G of degree n bounded above by a polynomial of degree n? Find the best bound!

To see that this question extends naturally the question of Wall, we fix some notation. Given a finite group G and a subgroup H of G, we denote by

$$\max(H, G) := \big|\{M \mid M \text{ maximal subgroup of } G \text{ with } H \leq M\}\big|$$

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the number of maximal subgroups of G containing H. Now, if Ω is the domain of a transitive permutation group G and $\omega \in \Omega$, then there exists a one-to-one correspondence between the maximal systems of imprimitivity of G and the maximal subgroups of G containing the point stabilizer G_ω, and hence Question 1.1 asks for a polynomial upper bound for $\max(G_\omega, G)$ as a function of $n = |G : G_\omega|$. When $n = |G|$, that is, G acts regularly on itself, the question of Cameron reduces to the question of Wall and [9, Theorem 1.3] yields a positive solution in this case, with exponent $\frac{3}{2}$.

The main result of this paper is a positive solution to Question 1.1.

Theorem 1.2. *There exists a constant a such that, for every finite group G and for every subgroup H of G, we have* $\max(H, G) \leq a|G : H|^{3/2}$. *In particular, a transitive permutation group of degree n has at most* $an^{3/2}$ *maximal systems of imprimitivity.*

In the case of soluble groups, we actually obtain a much tighter bound, which extends the result of Wall [12, (8.6), p. 58] for soluble groups on his own conjecture.

Theorem 1.3. *If G is a finite soluble group and H is a proper subgroup of G, then* $\max(H, G) \leq |G : H| - 1$. *In particular, a soluble transitive permutation group of degree* $n \geq 2$ *has at most* $n - 1$ *maximal systems of imprimitivity.*

2 Preliminaries

We start by reviewing some basic results on G-groups, on monolithic primitive groups and on crowns tailored to our proof of Theorem 1.2. For the first part we follow [5], for the second part we follow [8] and for the third part we follow [1, Chapter 1] and [5]. This section will also help for setting some notation. All groups in this paper are finite.

2.1 Monolithic primitive groups and crown-based power

Recall that an *abstract* group L is said to be *primitive* if it has a maximal subgroup with trivial core. Incidentally, given a group G and a subgroup M we denote by

$$\mathrm{core}_G(M) := \bigcap_{g \in G} M^g$$

the core of M in G. The *socle* $\mathrm{soc}(L)$ of a primitive group L is either a minimal normal subgroup, or the direct product of two non-abelian minimal normal subgroups. A primitive group L is said to be *monolithic* if the first case occurs, that is, $\mathrm{soc}(L)$ is a minimal normal subgroup of L and hence (necessarily) L has a unique minimal normal subgroup.

Let L be a monolithic primitive group and let $A := \mathrm{soc}(L)$. For each positive integer k, let L^k be the k-fold direct product of L. The *crown-based power* of L of size k is the subgroup L_k of L^k defined by

$$L_k := \{(l_1, \dots, l_k) \in L^k \mid l_1 \equiv \cdots \equiv l_k \pmod{A}\}.$$

Equivalently, if we denote by $\mathrm{diag}(L^k)$ the diagonal subgroup of L^k, then $L_k = A^k \mathrm{diag}(L^k)$.

For the proof of the next lemma we need some basic terminology, which we borrow from [11, Sections 4.3 and 4.4]. Let κ be a positive integer and let A be a direct product $S_1 \times \cdots \times S_\kappa$, where the S_i are pairwise isomorphic non-abelian simple groups. We denote by $\pi_i : A \to S_i$ the natural projection onto S_i. A subgroup X of A is said to be a *strip* if $X \neq 1$ and, for each $i \in \{1, \dots, \kappa\}$, either $X \cap \mathrm{Ker}(\pi_i) = 1$ or $\pi_i(X) = 1$. The support of the strip X is the set $\{i \in \{1, \dots, \kappa\} \mid \pi_i(X) \neq 1\}$. The strip X is said to be *full* if $\pi_i(X) = S_i$ for all i in the support of X. Two strips X and Y are *disjoint* if their supports are disjoint. A subgroup X of A is said to be a *subdirect* subgroup if $\pi_i(X) = S_i$ for each $i \in \{1, \dots, \kappa\}$.

Scott's lemma (see for instance [11, Theorem 4.16]) shows (among other things) that if X is a subdirect subgroup of A, then X is a direct product of pairwise disjoint full strips of A.

Lemma 2.1. *Let $L_{k'}$ be a crown-based power of L of size k' having non-abelian socle $N^{k'}$ and let H' be a core-free subgroup of $L_{k'}$ contained in $N^{k'}$. Then $|N^{k'} : H'| \geq 5^{k'}$.*

Proof. We argue by induction on k'. If $k' = 1$, then the result is clear because $N^{k'} = N$ has no proper subgroups having index less then 5. Suppose that $k' \geq 2$ and write $N := N_1 \times \cdots \times N_{k'}$, where $N_1, \dots, N_{k'}$ are the minimal normal subgroups of $L_{k'}$ contained in $N^{k'}$. For each $i \in \{1, \dots, k'\}$, we denote by $\pi_i : N^{k'} \to N_i$ the natural projection onto N_i.

Suppose that there exists $i \in \{1, \dots, k'\}$ with $\pi_i(H') < N_i$. Then $N_i H'/N_i$ is a core-free subgroup of $L_{k'}/N_i \cong L_{k'-1}$ and is contained in $N^{k'}/N_i$. Therefore, by induction, $|N^{k'} : H' N_i| = |N^{k'}/N_i : H' N_i/N_i| \geq 5^{k'-1}$. Furthermore, $|H' N_i : H'| = |N_i : H' \cap N_i| \geq 5$ because N_i has no proper subgroups having index less then 5. Therefore, $|N^{k'} : H'| \geq 5^{k'}$.

Suppose that $\pi_i(H') = N_i$ for every $i \in \{1, \dots, k'\}$. Since N is non-abelian, we may write $N_i = S_{i,1} \times \cdots \times S_{i,\ell}$ for some pairwise isomorphic non-abelian simple groups $S_{i,j}$ of cardinality s. For each $i \in \{1, \dots, k'\}$ and $j \in \{1, \dots, \ell\}$, we denote by $\pi_{i,j} : N^{k'} \to S_{i,j}$ the natural projection onto $S_{i,j}$. Since $\pi_i(H') = N_i$, we deduce $\pi_{i,j}(H') = S_{i,j}$ for every $i \in \{1, \dots, k'\}$ and $j \in \{1, \dots, \ell\}$. In particular, H' is a subdirect subgroup of $S_{1,1} \times \cdots \times S_{k',\ell}$, and hence (by Scott's lemma) H' is a direct product of pairwise disjoint full strips. Since no N_i is contained in H', there exist two distinct indices $i_1, i_2 \in \{1, \dots, k'\}$ and $j_1, j_2 \in \{1, \dots, \ell\}$ such that (i_1, j_1) and (i_2, j_2) are involved in the same full strip of H'. If we now consider the projection $\pi_{i_1,i_2} : N^{k'} \to N_{i_1} \times N_{i_2}$, we obtain $|N_{i_1} \times N_{i_2} : \pi_{i_1,i_2}(H')| \geq s \geq 60 \geq 5^2$. The inductive hypothesis applied to $\mathrm{Ker}(\pi_{i_1,i_2}) \cap H'$ yields $|\mathrm{Ker}(\pi_{i_1,i_2}) : \mathrm{Ker}(\pi_{i_1,i_2}) \cap H'| \geq 5^{k'-2}$, and hence $|N^{k'} : H'| \geq 5^{k'}$. □

In the proofs of Theorem 1.2 and 1.3, we use without mention the following basic fact.

Lemma 2.2. *Let M be a normal subgroup of a crown-based power L_k with socle N^k. Then either $M \leq N^k$ or $N^k \leq M$.*

Proof. For each $i \in \{1, \dots, k\}$, we write $N_i := \{(n_1, \dots, n_k) \in N^k \mid n_j = 1 \text{ for all } j \in \{1, \dots, k\} \setminus \{i\}\}$. In particular, $N = N_1 \times \cdots \times N_k$.

Let M be a normal subgroup of the crown-based power L_k with socle N^k and with $M \nleq N^k$. Let $m \in M \setminus N^k$. For each $i \in \{1, \dots, k\}$, since M does not centralize N_i, we deduce $1 \neq [M, N_i] \leq M \cap N_i$. As N_i is one of the minimal normal subgroups of L_k, we must have $N_i \leq M$. Therefore, $N^k = N_1 \times \cdots \times N_k \leq M$. □

2.2 Basic facts on *G*-groups

Given a group G, a *G-group* A is a group A together with a group homomorphism $\theta : G \to \mathrm{Aut}(A)$ (for simplicity, we write a^g for the image of $a \in A$ under the automorphism $\theta(g)$). Given a G-group A, we have the corresponding *semi-direct product* $A \rtimes_\theta G$ (or simply $A \rtimes G$ when θ is clear from the context), where the multiplication is given by

$$g_1 a_1 \cdot g_2 a_2 = g_1 g_2 a_1^{g_2} a_2$$

for every $a_1, a_2 \in A$ and for every $g_1, g_2 \in G$. A G-group A is said to be *irreducible* if G leaves no non-identity proper normal subgroup of A invariant.

Two G-groups A and B are said to be *G-isomorphic* (and we write $A \cong_G B$) if there exists an isomorphism $\varphi : A \to B$ such that

$$(a^g)^\varphi = (a^\varphi)^g$$

for every $a \in A$ and for every $g \in G$. Similarly, we say that A and B are *G-equivalent* (and we write $A \sim_G B$) if there exist two isomorphisms $\varphi : A \to B$ and $\Phi : A \rtimes G \to B \rtimes G$ such that the following diagram commutes:

$$\begin{tikzcd}
1 \arrow[r, hook] & A \arrow[r, hook] \arrow[d, "\varphi"] & A \rtimes G \arrow[r, two heads] \arrow[d, "\Phi"] & G \arrow[r, two heads] \arrow[d, equal] & 1 \\
1 \arrow[r, hook] & B \arrow[r, hook] & B \rtimes G \arrow[r, two heads] & G \arrow[r, two heads] & 1.
\end{tikzcd}$$

Being "G-equivalent" is an equivalence relation among G-groups coarser than the "G-isomorphic" equivalence relation, that is, two G-isomorphic G-groups are necessarily G-equivalent. The converse is not necessarily true: for instance, if A and B are two isomorphic non-abelian simple groups and $G := A \times B$ acts on A and on B by conjugation, then $A \not\cong_G B$ and $A \sim_G B$. However, when A and B are abelian, the converse is true, that is, if A and B are abelian, then $A \sim_G B$ if and only if $A \cong_G B$; see [8, page 178].

Let G be a group and let $A := X/Y$ be a chief factor of G, where X and Y are normal subgroups of G. Clearly, the action by conjugation of G endows A with the structure of a G-group and, in fact, A is an irreducible G-group. On the set of chief factors, the G-equivalence relation is easily described. Indeed, it is proved in [8, Proposition 1.4] that two chief factors A and B of G are G-equivalent if and only if either

- A and B are G-isomorphic, or
- there exists a maximal subgroup M of G such that $G/\mathrm{core}_G(M)$ has two minimal normal subgroups N_1 and N_2 that are G-isomorphic to A and B, respectively.

(The example in the previous paragraph witnesses that the second possibility does arise.) From this, it follows that, for every monolithic primitive group L and for every $k \in \mathbb{N}$, the minimal normal subgroups of the crown-based power L_k are all L_k-equivalent.

2.3 Crowns of a finite group

Let X and Y be normal subgroups of G with $A = X/Y$ being a chief factor of G. A *complement* U to A in G is a subgroup U of G such that

$$G = UX \quad \text{and} \quad Y = U \cap X.$$

We say that $A = X/Y$ is a *Frattini chief factor* if X/Y is contained in the Frattini subgroup of G/Y; this is equivalent to saying that A is abelian and there is no complement to A in G. The number $\delta_G(A)$ of non-Frattini chief factors G-equivalent to A in any chief series of G does not depend on the series, and hence $\delta_G(A)$ is a well-defined integer depending only on the chief factor A.

We denote by L_A the *monolithic primitive group associated* to A, that is,

$$L_A := \begin{cases} A \rtimes (G/C_G(A)) & \text{if } A \text{ is abelian,} \\ G/C_G(A) & \text{otherwise.} \end{cases}$$

If A is a non-Frattini chief factor of G, then L_A is a homomorphic image of G. More precisely, there exists a normal subgroup N of G such that

$$G/N \cong L_A \quad \text{and} \quad \mathrm{soc}(G/N) \sim_G A.$$

Consider now the collection $\mathcal{N}_A$ of all normal subgroups N of G with $G/N \cong L_A$ and $\mathrm{soc}(G/N) \sim_G A$: the intersection

$$R_G(A) := \bigcap_{N \in \mathcal{N}_A} N$$

has the property that $G/R_G(A)$ is isomorphic to the crown-based power $(L_A)_{\delta_G(A)}$, that is, $G/R_G(A) \cong (L_A)_{\delta_G(A)}$.

The socle $I_G(A)/R_G(A)$ of $G/R_G(A)$ is called the *A-crown* of G and it is a direct product of $\delta_G(A)$ minimal normal subgroups all G-equivalent to A.

We conclude this preliminary section with two technical lemmas and one of the main results from [9].

Lemma 2.3 ([1, Lemma 1.3.6]). *Let G be a finite group with trivial Frattini subgroup. There exists a chief factor A of G and a non-identity normal subgroup D of G with $I_G(A) = R_G(A) \times D$.*

Lemma 2.4 ([5, Proposition 11]). *Let G be a finite group with trivial Frattini subgroup, let $I_G(A)$, $R_G(A)$ and D be as in the statement of Lemma 2.3 and let K be a subgroup of G. If $G = KD = KR_G(A)$, then $G = K$.*

Theorem 2.5 ([9, Theorem 1.4]). *There exists a constant c such that every finite group has at most $cn^{3/2}$ core-free maximal subgroups of index n.*

Theorem 2.5 is an improvement of [10, Corollary 2]. We warn the reader that the statement of Theorem 2.5 is slightly different from that of [9, Theorem 1.4]: to get Theorem 2.5 one should take into account [9, Theorem 1.4] and the remark following its statement.

3 Proofs of Theorems 1.2 and 1.3

In this section, we prove Theorems 1.2 and 1.3. Our proofs are inspired from some ideas developed in [4]. Moreover, our proofs have some similarities, and hence we start by deducing some general facts holding for both.

We start by defining the universal constant a. Observe that the series $\sum_{u=1}^{\infty} u^{-3/2}$ converges. We write

$$a' := \sum_{u=1}^{\infty} \frac{1}{u^{3/2}}.$$

Let c be the universal constant arising from Theorem 2.5. We define

$$a := \frac{11ca'}{1 - 1/2^{3/2}}.$$

Recall that $\max(H, G)$ is the number of maximal subgroups of G containing H. For the proofs of Theorems 1.2 and 1.3 we argue by induction on $|G : H| + |G|$. The case $|G : H| = 1$ for the proof of Theorem 1.2 is clear because $\max(H, G) = 0$. Similarly, the case that H is maximal in G for the proof of Theorem 1.3 is clear because $\max(H, G) = 1$. In particular, for the proof of Theorem 1.2, we suppose $|G : H| > 1$ and, for the proof of Theorem 1.3, we suppose that H is not maximal in G.

Consider

$$\tilde{H} := \bigcap_{\substack{H \leq M < G \\ M \text{ max. in } G}} M.$$

Observe that $\max(H, G) = \max(\tilde{H}, G)$. In particular, when $H < \tilde{H}$, we have $|G : \tilde{H}| < |G : H|$ and hence, by induction, we have $\max(H, G) = \max(\tilde{H}, G) \leq a|G : \tilde{H}|^{3/2} < a|G : H|^{3/2}$. Moreover, when G is soluble, we have $\max(H, G) = \max(\tilde{H}, G) \leq |G : \tilde{H}| - 1 < |G : H| - 1$. Therefore, we may suppose $H = \tilde{H}$, that is,

$$H \text{ is an intersection of maximal subgroups of } G. \tag{3.1}$$

Suppose that H contains a non-identity normal subgroup N of G. Since $\max(H, G) = \max(H/N, G/N)$ and $|G/N| < |G|$, by induction, we have $\max(H, G) = \max(H/N, G/N) \leq a|G/N : H/N|^{3/2} = a|G : H|^{3/2}$. Moreover, when G is soluble, we have $\max(H, G) = \max(H/N, G/N) \leq |G/N : H/N| - 1 = |G : H| - 1$. Therefore, we may suppose

$$\mathrm{core}_G(H) = 1. \tag{3.2}$$

Let F be the Frattini subgroup of G. From (3.1) we have $F \leq H$ and hence, from (3.2), $F = 1$. In particular, we may now apply Lemma 2.3 to the group G.

Choose I, R and D as in Lemma 2.3. By (3.1), we may write

$$H = X_1 \cap \cdots \cap X_\rho \cap Y_1 \cap \cdots \cap Y_\sigma,$$

where $X_1, \dots, X_\rho$ are the maximal subgroups of G not containing D and $Y_1, \dots, Y_\sigma$ are the maximal subgroups of G containing D. We define

$$X := X_1 \cap \cdots \cap X_\rho \quad \text{and} \quad Y := Y_1 \cap \cdots \cap Y_\sigma.$$

Thus $H = X \cap Y$.

For every $i \in \{1, \dots, \rho\}$, since $D \nleq X_i$, we have $G = DX_i$, and hence Lemma 2.4 (applied with $K := X_i$) yields $R \leq X_i$. In particular,

$$R \leq X. \tag{3.3}$$

Since $R = R_G(A)$ for some chief factor A of G, Section 2.3 yields

$$G/R \cong L_k$$

for some monolithic primitive group L and for some positive integer k. We let N denote the minimal normal subgroup (a.k.a. the socle) of L. From the definition of I and R we have $I/R = \mathrm{soc}(G/R) \cong \mathrm{soc}(L_k) = N^k$. Finally, let $T := X \cap I$. In particular,

$$\frac{T}{R} = \frac{X}{R} \cap \frac{I}{R}.$$

We have

$$H \cap D = (X \cap Y) \cap D = X \cap (Y \cap D) = X \cap D = X \cap (I \cap D) = (X \cap I) \cap D = T \cap D.$$

It follows

$$|G : HD| = \frac{|G : H|}{|HD : H|} = \frac{|G : H|}{|D : H \cap D|} = \frac{|G : H|}{|D : T \cap D|}.$$

If $D \leq T$, then $D \leq X$, and hence $D \leq X \cap Y = H$ because $D \leq Y$. However, this is a contradiction because $D \neq 1$ and hence, from (3.2), $D \nleq H$. Therefore, $D \nleq T$ and $|D : T \cap D| > 1$.

Applying our inductive hypothesis, we obtain

$$\sigma = \max(HD/D, G/D) \leq a|G/D : HD/D|^{3/2} = a|G : HD|^{3/2} = a\Big(\frac{|G : H|}{|D : D \cap T|}\Big)^{3/2} \leq \frac{a}{2^{3/2}}|G : H|^{3/2}. \tag{3.4}$$

Moreover, when G is soluble and HD is a proper subgroup of G, we obtain

$$\sigma = \max(HD/D, G/D) \leq |G/D : HD/D| - 1 = |G : HD| - 1 = \frac{|G : H|}{|D : D \cap T|} - 1 \leq \frac{|G : H|}{2} - 1. \tag{3.5}$$

(Observe that, when G is soluble and $G = HD$, we have $\sigma = 0$, and hence the inequality $\sigma \leq |G : H|/2 - 1$ is valid also in this degenerate case.)

From (3.3) we deduce $\rho \leq \max(HR, G)$. If $R \nleq H$, then $|G : HR| < |G : H|$ and hence, applying our inductive hypothesis, we obtain

$$\rho \leq \max(HR, G) \leq a|G : HR|^{3/2} = a\Big(\frac{|G : H|}{|HR : H|}\Big)^{3/2} \leq \frac{a}{2^{3/2}}|G : H|^{3/2}. \tag{3.6}$$

Moreover, when G is soluble and HR is a proper subgroup of G, we obtain

$$\rho \leq \max(HR, G) \leq |G : HR| - 1 = \frac{|G : H|}{|HR : H|} - 1 \leq \frac{|G : H|}{2} - 1. \tag{3.7}$$

(As above, when G is soluble and $G = HR$, we have $\rho = 0$, and hence the inequality $\rho \leq |G : H|/2 - 1$ is valid also in this degenerate case.)

Now, from (3.4) and (3.6), we have

$$\max(H, G) = \sigma + \rho \leq \frac{2a}{2^{3/2}} \cdot |G : H|^{3/2} < a|G : H|^{3/2}.$$

Similarly, when G is soluble, from (3.5) and (3.7) we have

$$\max(H, G) = \sigma + \rho \leq \frac{|G : H|}{2} - 1 + \frac{|G : H|}{2} - 1 < |G : H| - 1.$$

In particular, for the rest of the proof, we may assume that $R \leq H$. Now, (3.2) yields $R = 1$, and hence $G \cong L_k$ and $D = I$. Therefore, we may identify G with L_k and D with N^k.

Set

$$\mathcal{C} := \{\mathrm{core}_G(X_i) \mid i \in \{1, \dots, \rho\}\}$$

and, for every $C \in \mathcal{C}$, set

$$\mathcal{M}_C := \{X_i \mid i \in \{1, \dots, \rho\},\ C = \mathrm{core}_G(X_i)\}.$$

For the rest of our argument for proving Theorems 1.2 and 1.3, we prefer to keep the proofs separate.

Proof of Theorem 1.2. In this proof, we distinguish two cases.

Case 1: Suppose that N is non-abelian. Since N is non-abelian, the group $G = L_k$ has exactly k minimal normal subgroups. We denote by $N_1, \dots, N_k$ the minimal normal subgroups of G. In particular, $I = N^k = N_1 \times N_2 \times \cdots \times N_k$.

First, we claim that, for every $i \in \{1, \dots, \rho\}$, there exist $x, y \in \{1, \dots, k\}$ such that $N_\ell \leq X_i$ for every $\ell \in \{1, \dots, k\} \setminus \{x, y\}$, that is, X_i contains all but possibly at most two minimal normal subgroups of G.

We argue by induction on k. The statement is clearly true when $k \leq 2$. Suppose then $k \geq 3$ and let $C := \mathrm{core}_G(X_i)$. If $C = 1$, then X_i is a maximal core-free subgroup of G, and hence the action of G on the right cosets of X_i gives rise to a faithful primitive permutation representation. Since a primitive permutation group has at most two minimal normal subgroups [2, Theorem 4.4] and since G has exactly k minimal normal subgroups, we deduce that $k \leq 2$, which is a contradiction. Therefore, $C \neq 1$.

Since $N_1, \ldots, N_k$ are the minimal normal subgroups of L_k, we deduce that there exists $\ell \in \{1, \ldots, k\}$ with $N_\ell \leq C$. Now, the proof of the claim follows applying the inductive hypothesis to $G/N_\ell \cong L_{k-1}$ and to its maximal subgroup X_i/N_ℓ.

The previous claim shows that, for every $C \in \mathcal{C}$, C contains all but possibly at most two minimal normal subgroups of $N^k = I$. Therefore,

$$|\mathcal{C}| \leq k^2.$$

Let $C \in \mathcal{C}$ and let $M \in \mathcal{M}_C$. The reader might find it useful to see Figure 1, where we have drawn a fragment of the subgroup lattice of G relevant to our argument.

Let k' be the number of minimal normal subgroups of G contained in M. In particular, $I \cap M \cong N^{k'}$. Observe that $I \cap H$ is contained in $I \cap M$ and is core-free in G. Applying Lemma 2.1 (with H' replaced by $I \cap H$ in a crowned-based group isomorphic to $L_{k'}$), we get $|I \cap M : I \cap H| \geq 5^{k'}$. As $k' \geq k-2$, we deduce $t \geq 5^{k-2}$.

Now, M/C is a core-free maximal subgroup of G/C. From Theorem 2.5, when $C = \mathrm{core}_G(M)$ and $z = |G : C|$ are fixed, we have at most $cz^{3/2}$ choices for M. As $t \geq 5^{k-2}$, we have $z \leq |G : H|/5^{k-2}$. Thus

$$\rho = \sum_{C\in\mathcal{C}} |\mathcal{M}_C| \leq \sum_{C\in\mathcal{C}} \sum_{\substack{z\,||G:H| \\ z\leq |G:H|/5^{k-2}}} cz^{3/2} \leq ck^2 \sum_{\substack{z\,||G:H| \\ z\leq |G:H|/5^{k-2}}} z^{3/2} = ck^2\Big(\frac{|G:H|}{5^{k-2}}\Big)^{3/2} \sum_{\substack{z\,||G:H| \\ z\leq |G:H|/5^{k-2}}} \Big(\frac{5^{k-2}z}{|G:H|}\Big)^{3/2}.$$

Therefore,

$$\sum_{\substack{z\,||G:H| \\ z\leq |G:H|/5^{k-2}}} \Big(\frac{5^{k-2}z}{|G:H|}\Big)^{3/2} \leq \sum_{u=1}^{\infty} \frac{1}{u^{3/2}} = a'.$$

Finally, it is easy to verify that $k^2/5^{3(k-2)/2} \leq 11$ for every k. Summing up,

$$\rho \leq 11ca'|G : H|^{3/2}. \tag{3.8}$$

From (3.4), (3.8) and from the definition of a, we have

$$\max(H, G) = \sigma + \rho \leq \frac{a}{2^{3/2}}|G : H|^{3/2} + 11ca'|G : H|^{3/2} = a|G : H|^{3/2}.$$

Case 2: Suppose that N is abelian. As N is abelian, the action of L by conjugation on N endows N with the structure of an L-module. Since L is primitive, N is irreducible. Set $q := |\mathrm{End}_L(N)|$. Now, N is a vector space over the finite field $\mathbb{F}_q$ with q elements, and hence $|N| = q^{k'}$ for some positive integer k'.

Let $C \in \mathcal{C}$ and let $M \in \mathcal{M}_C$. By Lemma 2.2, $C \leq I$. Now, the action of G/C on the right cosets of M/C is a primitive permutation group with point stabilizer M/C. Observe that in this primitive action, I/C is the socle of G/C. In particular, G/C acts irreducibly as a linear group on I/C, and hence C is a maximal L-submodule of I. Since I is the direct sum of k pairwise isomorphic irreducible L-modules, we deduce that we have at most $(q^k - 1)/(q - 1)$ choices for C. Moreover, $|G : M| = |G/C : M/C| = |N| = q^{k'}$. From Theorem 2.5, when C is fixed, we have at most $c|G : M|^{3/2} = c(q^{k'})^{3/2}$ choices for $M \in \mathcal{M}_C$. This yields

$$\rho \leq |\mathcal{C}| \cdot \max_{C\in\mathcal{C}} |\mathcal{M}_C| \leq \frac{q^k - 1}{q - 1} \cdot cq^{3k'/2} < cq^{k+3k'/2}. \tag{3.9}$$

As we have observed above, $M \cap I = C$ is an L-submodule of G. Since an intersection of L-submodules is an L-submodule, we deduce that

$$H \cap I = (X_1 \cap \cdots \cap X_\rho) \cap I$$

is an L-submodule of G and hence $H \cap I \trianglelefteq G$. Since H is core-free in G, we deduce $H \cap I = 1$, and hence $|I| = |N|^k = q^{kk'}$ divides $|G : H|$. In particular, $|G : H| \geq q^{kk'}$. Therefore, from (3.9) we obtain

$$\rho \leq c|G : H|^{\frac{k+3k'/2}{kk'}}.$$

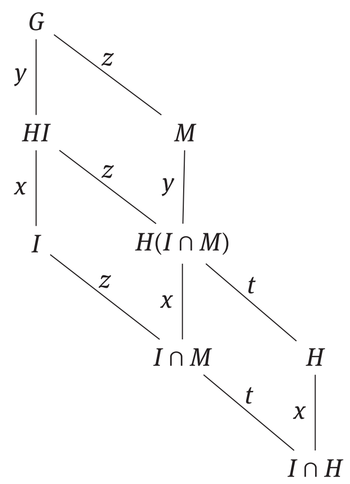

Figure 1: Subgroup lattice for G.

When $k \neq 1$ or when $(k, k') \neq (2, 1)$, we have $\frac{k+3k'/2}{kk'} \leq \frac{3}{2}$. When $k = 1$, by refining (3.9), we obtain the sharper bound $\rho \leq cq^{3k'/2} \leq c|G : H|^{3/2}$. When $(k, k') = (2, 1)$, we may again refine (3.9):

$$\rho \leq c(q+1)q^{3/2} \leq c \cdot 2q \cdot q^{3/2} = 2cq^{5/2} \leq 2c|G : H|^{5/4} \leq 2c|G : H|^{3/2}.$$

Summing up, in all cases we have

$$\rho \leq 2c|G : H|^{3/2}. \tag{3.10}$$

From (3.4) and (3.10) we have

$$\max(H, G) = \sigma + \rho \leq \frac{a}{2^{3/2}}|G : H|^{3/2} + 2c|G : H|^{3/2} < a|G : H|^{3/2},$$

as desired. □

The rest of the proof of Theorem 1.3 follows the same idea as in Case 2 above, but taking in account that the whole group G is soluble.

Proof of Theorem 1.3. Since $G = L_k$ and $I = N^k$, we may write $G = I \rtimes K$, where K is a complement of N in L. As in the proof of Theorem 1.2 for the case that N is abelian, we have that the action of L by conjugation on N endows N with the structure of an L-module. Since L is primitive, N is irreducible. Set $q := |\mathrm{End}_L(N)|$. Now, N is a vector space over the finite field $\mathbb{F}_q$ with q elements, and hence $|N| = q^{k'}$ for some positive integer k'.

Let $C \in \mathcal{C}$ and let $M \in \mathcal{M}_C$. As we have observed above (for the proof of Case 2), $M \cap I = C$ is a maximal L-submodule of G, $H \cap I = 1$ and $|I| = |N|^k = q^{kk'}$ divides $|G : H|$. In particular, $|G : H| = \ell q^{kk'}$ for some positive integer ℓ.

Since G is soluble and since M is a maximal subgroup of G supplementing I, we have $M = C \rtimes K^x$ for some maximal L-submodule C of I and some $x \in I$. Arguing as in the proof of Theorem 1.2 for the case that N is abelian, we deduce that we have at most $(q^k - 1)/(q - 1)$ choices for C. Moreover, we have at most $|I/C| = |G : M| = |N| = q^{k'}$ choices for x. This yields

$$\rho \leq \frac{q^k - 1}{q - 1} q^{k'}.$$

Now, (3.5) gives $\sigma \leq |G : H|/|D : D \cap T| - 1$: recall that $D = I = N^k$ and $D \cap T = D \cap H = I \cap H = 1$. Thus $\sigma \leq |G : H|/|D| - 1 = |G : H|/q^{kk'} - 1 = \ell - 1$. Therefore,

$$\max(H, G) = \sigma + \rho \leq \ell - 1 + \frac{q^k - 1}{q - 1} q^{k'}. \tag{3.11}$$

When $\ell \geq 2$, a computation shows that the right-hand side of (3.11) is less than or equal to

$$\ell q^{kk'} - 1 = |G : H| - 1.$$

In particular, we may suppose that $\ell = 1$. In this case, $|G : H| = q^{kk'} = |I|$ and hence $G = IH = I \rtimes H$. Moreover, $\sigma = 0$. Since H is not a maximal subgroup of G (recall the base case for our inductive argument), $k \geq 2$.

Assume also $k' = 1$. Since $|\mathrm{End}_L(N)| = q = |N|$, we deduce that L/N is isomorphic to a subgroup of the multiplicative group of the field $\mathbb{F}_q$ and hence $|L : N|$ is relatively prime to q. Therefore, $|G : I|$ is relatively prime to q and hence so is $|H|$. Therefore, replacing H by a suitable G-conjugate, we may suppose that $K = H$. Using this information, we may now refine our earlier argument bounding ρ. Let $C \in \mathcal{C}$ and let $M \in \mathcal{M}_C$. Since $G = I \rtimes H$ is soluble, M is a maximal subgroup of G supplementing I and $H \leq M$, we have $M = C \rtimes H$ for some maximal L-submodule C of I. We deduce that we have at most $(q^k - 1)/(q - 1)$ choices for C and hence we have at most $(q^k - 1)/(q - 1)$ choices for M. This yields

$$\max(H, G) = \sigma + \rho = \rho \leq \frac{q^k - 1}{q - 1} \leq q^k - 1 = |G : H| - 1,$$

and the result is proved in this case.

Assume $k' \geq 2$. A computation (using $\ell = 1$ and $k, k' \geq 2$) shows that the right-hand side of (3.11) is less than or equal to $q^{kk'} - 1 = |G : H| - 1$. □

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