FREE GROUP REPRESENTATIONS FROM VECTOR-VALUED MULTIPLICATIVE FUNCTIONS, III

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Abstract. Let π be an irreducible unitary representation of a finitely generated nonabelian free group Γ; suppose π is weakly contained in the regular representation. In 2001 the first and third authors conjectured that such a representation must be either *odd* or *monotonous* or *duplicitous*. In 2004 they introduced the class of *multiplicative representations*: this is a large class of representations obtained by looking at the action of Γ on its Cayley graph. In the second paper of this series we showed that some of the multiplicative representations were monotonous. Here we show that all the other multiplicative representations are either odd or duplicitous. The conjecture is therefore established for multiplicative representations.

1. Introduction

Let Γ be a free group on a finite set of generators, Ω its Gromov boundary and $C(\Omega)$ the C^*-algebra of complex continuous functions on Ω. Given a unitary representation $(\pi, \mathcal{H})$ of Γ, we say that ι is a boundary realization of π if ι is an isometric Γ-inclusion of $\mathcal{H}$ into $\mathcal{H}'$ where

- $\mathcal{H}'$ is the representation space of a $\Gamma \ltimes C(\Omega)$-representation π',
- $\iota(\mathcal{H})$ is cyclic for the action of $C(\Omega)$ on $\mathcal{H}'$.

Any representation space of $\Gamma \ltimes C(\Omega)$ can be seen as an L^2 space on Ω where $C(\omega)$ is acting by pointwise multiplication. Indeed, any representation of $C(\Omega)$ can be so seen. This is one version of the spectral theorem (see [?]). So the space could be $L^2(\Omega, d\nu)$ for any Borel measure ν. But the L^2-space is not necessarily scalar valued. It could also be $L^2(\Omega, d\nu, \mathcal{H}_0)$ for any Hilbert space $\mathcal{H}_0$.

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There is the further complication that the space might be the direct sum of such L^2-spaces for disjoint measures $d\nu$ and spaces $\mathcal{H}_0$ of different dimensions. This does not happen in any of our cases because we work with irreducible representations of $\Gamma \ltimes C(\Omega)$. we refer to [21, Chapter X, Theorem 3.8] for a complete description of representations of $\Gamma \ltimes C(\Omega)$.

In the examples dealt with in this paper the dimension of $\mathcal{H}_0$ is always finite, but can be arbitrarily large. The authors know of a method to determine $\dim(\mathcal{H}_0)$ starting with the multiplicative system which is used to define π'. The answer, not presented here, is not too difficult, but neither is it entirely obvious. In particular, even when the matrices of the multiplicative system are of high dimension, the dimension of $\mathcal{H}_0$ can be 1.

The measure ν is necessarily Γ-quasiinvariant and is determined only up to class. The class varies. For instance, for the representations from [22] it is the class of a counting measure; for the representations from [7] it is the class of the "obvious" measure on the boundary of the tree. The authors do not know how the rather wide collection of measure classes which can occur compares to other such collections. For instance, are all these measures comparable to hitting probabilities of random walks on Γ? As of now, the authors have no plans to study these questions, so they are open to all.

For brevity we shall say that a representation is *tempered* if it is weakly contained in the regular representation. Note that π has a boundary realization if and only if it is tempered (see [10]). Observe that ι is an isometry, but it needn't be surjective! Call a boundary realization $\iota : \mathcal{H} \to \mathcal{H}'$ *perfect* if ι is a unitary equivalence, i.e. a bijection and not just an injection.

Fix a tempered irreducible unitary representation π of Γ. We say that π satisfies *monotony* if

- Up to equivalence π has a single boundary realization.
- That realization is perfect.

π satisfies *duplicity* if

- Up to equivalence π has exactly two perfect boundary realizations, (ι_1, π'_1) and (ι_2, π'_2) where π'_1 and π'_2 are not equivalent as $\Gamma \ltimes C(\Omega)$ representations.
- Let π' acting on $\mathcal{H}'_1 \oplus \mathcal{H}'_2$ be the direct sum of π'_1 and π'_2. Up to equivalence all the imperfect boundary realizations of π are given by the maps $v \mapsto (t_1\iota_1(v), t_2\iota_2(v))$ where $t_1, t_2 > 0$ and $t_1^2 + t_2^2 = 1$.

π satisfies *oddity* if

- Up to equivalence π has exactly one boundary realization, (ι, π') where π' acts on $\mathcal{H}'$.
- That realization is not perfect.

The principal series representations considered by Figà-Talamanca and Picardello [7] and Figà-Talamanca and Steger [8] satisfy duplicity, except for the two at the endpoints, which satisfy monotony. These questions about duplicity are answered in [19]. That paper also uses the Duplicity and Oddity theorems from [10]. It states them without proof and then applies them. A very general proof that endpoints representations satisfy monotony can be found in [1].

Here are a few known representations satisfying oddity: the non-endpoint representations from [17] and [18] as described in Example 6.5 of [16]; the midpoint principal unitary representation from [7], *upon restriction* to the subgroup of even elements of the free group.

In [16] the first and third authors constructed a class of *multiplicative representations*. This class is large enough to contain more or less all previously constructed tempered irreducible representations whose construction uses the action of Γ on its Cayley graph: in Section 6 of the same paper it is shown how multiplicative representations can cover most of the above mentioned representations. In the same paper it was proved that the representations constructed from our multiplicative functions are irreducible *as representations of* $\Gamma \ltimes C(\Omega)$. The second paper in this series [13] and the present third paper are devoted to the study of irreducibility and inequivalence of multiplicative representations as *representations of* Γ.

It turns out that the growth of matrix coefficients plays a central role both for the classification above conjectured and for the proof of irreducibility.

In 1979 Haagerup [9] showed that tempered representations can be characterized by the growth of their matrix coefficients, namely he proved that for a representation π having a cyclic vector v the following conditions are equivalent:

- π is tempered;
- The map $\phi^v_\varepsilon(x) = \langle v, \pi(x)v\rangle e^{-\varepsilon|x|}$ is square integrable for every positive ε;
-

$$\sum_{|x|=n} |\langle v, \pi(x)v\rangle|^2 \le (n+1)^2\|v\|^4. \tag{1}$$

Haagerup's inequality implies

$$\|\phi_\varepsilon^v\|_2^2 = \sum_{x\in\Gamma} |\langle v, \pi(x)v\rangle|^2 e^{-2\varepsilon|x|} \le C\|v\|^4 \left(\frac{1}{\varepsilon}\right)^3 . \tag{2}$$

The problem of finding the correct asymptotic for $\sum_{|x|=n} |\langle v, \pi(x)v\rangle|^2$ is nontrivial and has been treated by many authors, not only for a free group: see for example [1], [2], [3], [7], [8], [13].

In [13] it is shown if π is a multiplicative representation then $\|\phi_\varepsilon^v\|_2^2$ can be explicitly calculated for v in a dense set of *smooth vectors*. It turns out to be asymptotically proportional to either $1/\varepsilon$ or $1/\varepsilon^2$ or $1/\varepsilon^3$ as $\varepsilon \to 0$.

The exponent 3 for $1/\varepsilon$ in Haagerup's inequality (2) is an upper bound for the growth of the ℓ^2 norm of ϕ_ε^v which is attained only in rather special cases: endpoint representations of the isotropic/anisotropic principal series of Figà-Talamanca and Picardello/Figà-Talamanca and Steger [7], [8] have these maximal asymptotics; likewise for other multiplicative representations constructed from very special matrix systems (see [13]). Recently Boyer and Garncarek [1] constructed a huge family of irreducible representations with maximal asymptotics.

Every multiplicative representation provides a perfect boundary realization of itself (see Proposition 2.6), but what happens when we restrict this representation to Γ? Is this representation still irreducible? Are there other boundary realizations of this Γ-representation?

In [15] we conjectured that any irreducible unitary representation of Γ weakly contained in the regular representation is monotonous, or duplicitous, or odd.

In [13] we characterized, within the class of multiplicative representations, those which satisfy monotony: they are exactly those for which either $\|\phi_\varepsilon^v\|_2^2 \simeq 1/\varepsilon^2$ or $\|\phi_\varepsilon^v\|_2^2 \simeq 1/\varepsilon^3$, as $\varepsilon \to 0$. This paper is devoted to the study of the case

$$\|\phi_\varepsilon^v\|_2^2 \simeq 1/\varepsilon\,.$$

We shall prove that in this case there are only two possibilities:

- Either the multiplicative Γ-representation is irreducible and satisfies duplicity (Theorem 5.4);
- or the multiplicative Γ-representation decomposes into two inequivalent irreducible Γ-representations and each of them satisfies oddity (Theorem 5.5).

We may conclude that our conjecture is true for the irreducible components of multiplicative Γ-representations.

The techniques used here are completely different from those of [13]: we shall use the Duplicity and Oddity Theorems of [10] and new investigations into the eigenspace of 1 of the transition matrix used to compute $\|\phi_\varepsilon^v\|_2^2$.

2. Preliminary

Γ will stand for a non-abelian free group on a finite set A^+ of generators. We also let $A = A^+ \cup A^-$ for the set of generators and their inverses. Recall that the Cayley graph of Γ with respect to A is a tree of degree $q+1 = |A|$, and that this tree has a standard compactification which is obtained by adjoining a boundary, which we denote Ω. This boundary can be described as the space of ends of the tree; it also coincides with the boundary of Γ considered as a Gromov hyperbolic group.

Concretely, if we identify Γ with the set of finite reduced words

$$\{a_1 a_2 \dots a_n\,;\ a_j \in A, a_j a_{j+1} \neq 1\},$$

then we can identify Ω with the set of infinite reduced words

$$\{a_1 a_2 a_3 \dots\,;\ a_j \in A, a_j a_{j+1} \neq 1\}.$$

For $x \in \Gamma$ let $\Gamma(x)$ be the set of finite reduced words which start with the reduced word for x; let $\Omega(x)$ be the set of infinite reduced words which start with the reduced word for x. A basis for the topology on the compactification $\Gamma \sqcup \Omega$ is given by the singletons $\{x\}$ and the sets $\Gamma(x) \sqcup \Omega(x)$, as x varies through Γ. The left-action of Γ on Γ extends to a continuous action on the compactification.

Let $a \in A$. For any directed edge (x, xa) of the tree define

$$\Gamma(x, xa) = \{y \in \Gamma\,;\ d(y, xa) < d(y, x)\},$$

where d counts the number of the edges joining two vertices. Upon removing that edge, the tree decomposes as $\Gamma = \Gamma(x, xa) \amalg \Gamma(xa, x)$.

Suppose that the length $|xa| = |x| + 1$, that is, suppose that a is the last letter in the reduced word for xa. Then $\Gamma(x, xa) = \Gamma(xa)$ (while if $|xa| = |x| - 1$, then $\Gamma(x, xa) = \Gamma \sim \Gamma(x)$). If $|y| < |xa| = |x| + 1$, then also $|yxa| = |yx| + 1$, and so

$$\begin{aligned} y\Gamma(xa) &= y\Gamma(x, xa) = \Gamma(yx, yxa) = \Gamma(yxa) \\ y\Omega(xa) &= y(\overline{\Gamma(xa)} \cap \Omega) = \overline{y\Gamma(xa)} \cap \Omega = \overline{\Gamma(yxa)} \cap \Omega = \Omega(yxa)\ . \end{aligned}$$

Considering, by contrast, the case $y = (xa)^{-1}$, one finds

$$(xa)^{-1}\Gamma(xa) = (xa)^{-1}\Gamma(x, xa) = \Gamma(a^{-1}, e) = \Gamma \sim \Gamma(a^{-1}).$$

In order to construct a multiplicative representation as described in [16] one needs:

- A matrix system (V_a, H_{ba}): it consists of finite dimensional complex vector spaces V_a, for each $a \in A$, and linear maps $H_{ba} : V_a \to V_b$ for each pair $a, b \in A$, where $H_{ba} = 0$ whenever $ba = e$.
- A collection of positive definite sesquilinear forms (B_a) on each V_a satisfying, for each $a \in A$ and $v_a \in V_a$ the following *compatibility condition:*

$$B_a(v_a, v_a) = \sum_{b \in A} B_b(H_{ba}v_a, H_{ba}v_a). \tag{3}$$

With these ingredients we are going to construct the (Hilbert) space of *multiplicative functions* on which the multiplicative representation will act by translation.

Definition 2.1. A multiplicative function is a function $f : \Gamma \to \sqcup_{a \in A} V_a$ satisfying the following condition: there exists $N = N(f)$ such that for every $x \in \Gamma$ with $|x| \geq N$

$$\begin{array}{ll} f(xa) \in V_a & \text{if } |xa| = |x| + 1 \\ f(xab) = H_{ba} f(xa) & \text{if } |xab| = |x| + 2 \end{array} \tag{4}$$

We declare that two multiplicative functions *are equivalent* if they differ only on a finite set. We shall denote by $\mathcal{H}^\infty$ the space of equivalence classes of multiplicative functions.

Definition 2.2. Let $x \in \Gamma$ and $a \in A$. An *elementary multiplicative function* is defined as follows:

$$\begin{cases} \mu[x, xa, v_a](y) = 0, \quad \text{for } y \notin \Gamma(x, xa), \\ \mu[x, xa, v_a](xa) = v_a, \\ \mu[x, xa, v_a](ybc) = H_{cb}\, \mu[x, xa, v_a](yb), \\ \text{if } yb, ybc \in \Gamma(x, xa), \text{ and } d(ybc, x) = d(y, x) + 2. \end{cases} \tag{5}$$

It is clear that every multiplicative function f is equivalent to a finite linear combination of functions $\mu[y, yb, v_b]$ for $|y| \geq N(f)$ and $|yb| = |y| + 1$.

Remark 2.3. Observe that in the above definition (5) we do not require that $|xa| = |x| + 1$. For example, if $x = a_1 \dots a_n$ is the reduced word for x and $a = {a_n}^{-1}$, then $e \in \Gamma(x, xa)$ and

$$\mu[x, x{a_n}^{-1}, v_{a_n^{-1}}](e) = H_{a_1^{-1}a_2^{-1}} \dots H_{a_{n-1}^{-1}a_n^{-1}} v_{a_n^{-1}} .$$

Note that $y\Gamma(x, xa) = \Gamma(yx, yxa)$ and that

$$\mu[x, xa, v_a](y^{-1} \cdot) = \mu[yx, yxa, v_a](\cdot) .$$

In the past literature the following notion of multiplicative function has been considered by many Authors: say that $h : \Gamma \to C$ is multiplicative if $h(xy) = h(x)h(y)$ whenever $|xy| = |x| + |y|$. It was first proved by Haagerup [9] that the function $q^{-\lambda|x|}$ is positive definite for all positive λ. De Michele–Figá-Talamanca [6] generalized Haagerup's result to a wider class of multiplicative functions, while Pytlik and Szwarc [20] gave, for all positive λ, an explicit description of a representation π_λ having the function $q^{-\lambda|x|}$ as matrix coefficient. Since the free group doesn't admit any irreducible square integrable representation (see Cecchini–Figá-Talamanca [4]), every multiplicative function in $\ell^2(\Gamma)$ corresponds to a reducible representation. Even if you consider multiplicative functions in $\ell^{2+\epsilon}(\Gamma)$, (that is weakly associated with the regular representation), you end up with a *reducible* Γ-representation, unless the number of generators is infinite (see [20] and [12]).

Our multiplicative functions are different from those considered in the above mentioned papers, even in the scalar case. First of all we allow a multiplicative function to be zero on a full set $\Gamma(x)$, second we identify functions that differ on a finite set. One of our tools is to define a *norm* for multiplicative functions consistent with the equivalence relation. This norm preserves the unitarity of the Γ-action and allows the possibility of obtaining an irreducible representation of Γ.

Definition 2.4. Let $\mu[x, xa, v_a]$ and $\mu[y, yb, v_b]$ be two elementary multiplicative functios. Define

$$\begin{aligned} &\langle \mu[x, xa, v_a], \mu[y, yb, v_b] \rangle = 0, \quad \text{if } \Gamma(x, xa) \cap \Gamma(y, yb) = \emptyset, \\ &\langle \mu[x, xa, v_a], \mu[x, xa, v'_a] \rangle = B_a(v_a, v'_a). \end{aligned} \tag{6}$$

The compatibility condition (3) ensures that $\langle \cdot, \cdot \rangle$ is well defined and can be extended by linearity to a scalar product in $\mathcal{H}^\infty$ (see [16] for details).

Given all these ingredients one sets, for every $y \in \Gamma$ and $f \in \mathcal{H}^\infty$,

$$(\pi(y)f)(x) = f(y^{-1}x).$$

It can be shown that π is unitary with respect to $\langle \cdot, \cdot \rangle$ and hence extends to $\mathcal{H}$, the completion of $\mathcal{H}^\infty$, to a unitary representation that we shall call *multiplicative*. We aware the reader that the multiplicative representation hitherto constructed need not be *irreducible*, as it is pointed out in [11]. In order to hope for an irreducible representation of Γ we need to impose the following irreducibility condition on the matrix system:

Definition 2.5. An invariant subsystem of (V_a, H_{ba}) is a collection of subspaces $W_a \subseteq V_a$ such that $H_{ba}(W_a) \subseteq W_b$, for all $a, b \in A$. The system (V_a, H_{ba}) is called *irreducible* if it is non-zero and there are no invariant subsystems except for itself and the zero subsystem.

In [16] it is proved that any irreducible matrix system admits, up to a normalization, a unique (up to scalar multiples) tuple of strictly positive definite forms (B_a) satisfying (3).

From this point on we shall assume that all systems are irreducible and normalized so that (3) holds for a given tuple (B_a) of positive definite forms.

For brevity we shall call such systems (V_a, H_{ba}, B_a) *matrix systems with inner product*.

Now we need to specify in which sense a multiplicative representation gives raise to a $\Gamma \ltimes C(\Omega)$ representation.

Proposition 2.6. *Let* $(\pi, \mathcal{H})$ *be a multiplicative representation as described above. Then* $(\mathrm{Id}, \mathcal{H})$ *is a perfect boundary realization of* π.

Proof. Let $f \in \mathcal{H}$, $x \in \Gamma$ and let $\mathbf{1}_x$ be the characteristic function of the set $\Omega(x)$. Set

$$(\pi(\mathbf{1}_x)f)(y) = \begin{cases} f(y), & \text{if } y \in \Gamma(x), \\ 0, & \text{otherwise,} \end{cases}$$

and extend this action by linearity and continuity to all of $C(\Omega)$.

Observe that one has

$$\pi(x)\pi(G)\pi(x)^{-1} = \pi(\lambda(x)G), \qquad \text{for } x \in \Gamma \text{ and } G \in C(\Omega). \tag{7}$$

where $\lambda : \Gamma \to \mathrm{Aut}(C(\Omega))$ is given by $(\lambda(x)G)(\omega) = G(x^{-1}\omega)$. In fact, a pair of actions which satisfy (7) fit together to give a representation of the *crossed-product* C^**-algebra*, denoted $\Gamma \ltimes C(\Omega)$.

Vice versa, any $\Gamma \ltimes C(\Omega)$-representation comes from a pair of actions which fit together as per (7). One can consult [5] for the definition of the crossed-product. Hence $(\mathrm{Id}, \mathcal{H})$ is a boundary realization of π which is obviously surjective. ☐

Hence any multiplicative representation provides a perfect boundary realization of itself. Suppose π is *irreducible* and weakly contained in the regular representation. How many different boundary realizations does it have? To make this question precise, we need

Definition 2.7. Let $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ be a unitary representation of Γ. Two boundary realizations $\iota_j : \mathcal{H} \to \mathcal{H}'_j$ are *equivalent* if there is a unitary equivalence of $\Gamma \ltimes C(\Omega)$-representations $J : \mathcal{H}'_1 \to \mathcal{H}'_2$, such that $J\iota_1 = \iota_2$.

The above definition concerns equivalence of $\Gamma \ltimes C(\Omega)$ representations, while the next definition concerns the *equivalence of matrix systems:*

Definition 2.8. A *map* from the system (V_a, H_{ba}) to $(V_a^\sharp, H_{ba}^\sharp)$, is a tuple (J_a), where $J_a : V_a \to V_a^\sharp$, is a linear map and

$$H_{ba}^\sharp J_a = J_b H_{ba} \quad \text{for all } a, b \in A\,.$$

The tuple is called an *equivalence* if each J_a is a bijection. Two systems are called *equivalent* if there is an equivalence between them.

In [16] it is proved that two multiplicative representations arising from irreducible matrix systems are equivalent as $\Gamma \ltimes C(\Omega)$ representations if and only if the two systems are equivalent; hence for irreducible $\Gamma \ltimes C(\Omega)$ representations the two definitions 2.7 and 2.8 are equivalent. Obviously equivalent systems give raise to equivalent Γ-representations, but the converse is no longer true as we shall see in Section 4.

3. Good Vectors and Boundary Realizations.

We begin with the following important

Definition 3.1. Given a representation $(\pi, \mathcal{H})$, we say that a non-zero vector $w \in \mathcal{H}$ is a *good vector* if it satisfies the *Good Vector Bound*, namely if there exists a constant C, depending only on w, such that

$$\text{(GVB)} \qquad \sum_{|x|=n} |\langle v, \pi(x)w\rangle|^2 \le C\|v\|^2, \quad \text{for all } \ n \in \mathbb{N}, v \in \mathcal{H}.$$

We observe that, if w is a good vector, then, for every v, as $\varepsilon \to 0$,

$$\sum_{x\in\Gamma} |\langle v, \pi(x)w\rangle|^2 e^{-\varepsilon|x|} = \sum_{n=0}^{\infty} \sum_{|x|=n} |\langle v, \pi(x)w\rangle|^2 e^{-\varepsilon n} \le C\frac{\|v\|^2}{1-e^{-\varepsilon}} \simeq \frac{1}{\varepsilon}.$$

The first key observation is that the existence of "good vectors" is deeply related to the existence of imperfect boundary realizations, as the following proposition shows:

Proposition 3.2. [15] *If a representation* $(\pi, \mathcal{H})$ *of* Γ *admits an* imperfect *boundary realization, then some non-zero vector* $w \in \mathcal{H}$ *satisfies* (GVB).

For an arbitrary representation it is really very hard to understand whether there exists or not a good vector! See for example the paper of Boyer and Garncareck [1] where the existence of good vectors is ruled out!

For a representation constructed from an irreducible matrix system there is the possibility to calculate

$$\sum_{x\in\Gamma} |\langle v, \pi(x)w\rangle|^2 e^{-\varepsilon|x|} \tag{8}$$

explicitly at least for a dense set of vectors v and w. The calculations in the following subsection are taken from [13].

3.1. **The Twin of a System.** Given a finite dimensional vector space V, $\overline{V}$ will stand for its complex conjugate, V' for its dual, the space of linear functionals, while $V^* = \overline{V}'$ will stand for the space of antilinear functionals. We recall some identifications that will be used in this paper. The space $V_1 \otimes V_2$ will be identified with the space of linear maps $v_1 \otimes v_2 : V_2' \to V_1$ given by $(v_1 \otimes v_2)(f) = f(v_2)\, v_1 = \langle v_2, f\rangle\, v_1$.

It follows that, given $T_1 \in \mathscr{L}(V_1, V_3)$ and $T_2 \in \mathscr{L}(V_2, V_4)$, the map

$$T_1 \otimes T_2 : V_1 \otimes V_2 \to V_3 \otimes V_4$$

corresponds to the operator

$$\mathscr{L}(V_2', V_1) \to \mathscr{L}(V_4', V_3), \quad S \mapsto T_1\, S\, T_2'.$$

So we shall write $(T_1 \otimes T_2)S = T_1 S T_2'$. The duality isomorphism

$$\mathcal{L} : \mathscr{L}(V, V^*) \to \mathscr{L}(V^*, V)'$$

defines a bilinear form which can be written explicitly by means of the trace function

$$\mathcal{B} : \mathscr{L}(V, V^*) \times \mathscr{L}(V^*, V) \to \mathbb{C},$$

$$\mathcal{B}(T, S) := (\mathcal{L}(T))(S) = \operatorname{tr}(TS) = \operatorname{tr}(ST). \tag{9}$$

Positive definite sesquilinear forms B_a on V_a are identified with maps $B_a \in \mathscr{L}(V_a, V_a^*)$; under this identification one also has $B_a^* = B_a$.

The compatibility condition (3) may be rewritten as:

$$(TB)_a = \sum_{b\in A} H_{ba}^* B_b H_{ba} = \sum_{b\in A} H_{ba}^* \otimes H_{ba}' B_b = B_a. \tag{10}$$

The above equation says that the tuple (B_a) is a right eigenvector for the matrix $T = (H_{ba}^* \otimes H_{ba}')_{a,b}$ corresponding to eigenvalue 1.

For every $a \in A$ set $\widehat{V}_a := V_{a^{-1}}^* = \overline{V}'_{a^{-1}}$.

A system of linear maps $H_{ba} : V_a \to V_b$ induces an obvious system of maps $H_{ba}^* : V_b^* \to V_a^*$, by the rule $H_{ba}^*(f) = f \circ H_{ba}$, and also maps

$$\widehat{H}_{ba} := H_{a^{-1}b^{-1}}^* : \widehat{V}_a \to \widehat{V}_b.$$

Hence the matrix system (V_a, H_{ba}) induces another matrix system $(\widehat{V}_a, \widehat{H}_{ba})$, which is irreducible if so is (V_a, H_{ba}).

Proposition 3.3. [13] *Assume that* (V_a, H_{ba}, B_a) *is a matrix system with inner product. Then there exists a unique (up to multiple scalars) positive definite tuple* $(\widehat{B}_a)$, $\widehat{B}_a : \widehat{V}_a \to \widehat{V}_a^*$ *on* $\widehat{V}_a$ *such that the matrix system* $(\widehat{V}_a, \widehat{H}_{ba}, \widehat{B}_a)$ *is a system with inner products.*

Definition 3.4. The system $(\widehat{V}_a, \widehat{H}_{ba}, \widehat{B}_a)$ above constructed is called the *twin of the system* (V_a, H_{ba}, B_a). Since the twin of the twin of a system is the system itself, we shall briefly say that the two systems $(\widehat{V}_a, \widehat{H}_{ba}, \widehat{B}_a)$ and (V_a, H_{ba}, B_a) are *twin*.

We shall see later that twin systems *may not be equivalent* as matrix systems. Nontheless th twin of the system will play a central role in the computation of (8). Let $\mu[e, a, v_a]$ and $\mu[e, b, v_b]$ be elementary multiplicative functions constructed from the system (V_a, H_{ba}, B_a) as per (5) and let $(\widehat{V}_a, \widehat{H}_{ba}, \widehat{B}_a)$ be the twin system. For any $a, b \in A$, we define maps $E_{ab} : V_b \to \widehat{V}_a$ by

$$E_{ab} = \sum_{\substack{c\in A \\ c\neq a,b^{-1}}} H^*_{ca^{-1}} B_c H_{cb} = \sum_{\substack{c\in A \\ c\neq a,b^{-1}}} \widehat{H}_{ac^{-1}} B_c H_{cb},$$

where $E_{ab} = 0$ whenever $ab = e$.

The quantity

$$\|\phi_\varepsilon^{v_a,v_b}\|_2^2 = \sum_{x\in\Gamma} |\langle \mu[e, a, v_a], \pi(x)\mu[e, b, v_b]\rangle|^2 e^{-\varepsilon|x|}$$

can be calculated using the following block matrix: $\mathcal{D} = (D_{i,j})_{i,j=1,\dots,4}$

(11)

$$\mathcal{D} = \begin{pmatrix} \left(\widehat{H}_{ab} \otimes \overline{\widehat{H}}_{ab}\right)_{a,b} & \left(E_{ab} \otimes \overline{\widehat{H}}_{ab}\right)_{a,b} & \left(\widehat{H}_{ab} \otimes \overline{E}_{ab}\right)_{a,b} & \left(E_{ab} \otimes \overline{E}_{ab}\right)_{a,b} \\ 0 & \left(H_{ab} \otimes \overline{\widehat{H}}_{ab}\right)_{a,b} & 0 & \left(H_{ab} \otimes \overline{E}_{ab}\right)_{a,b} \\ 0 & 0 & \left(\widehat{H}_{ab} \otimes \overline{H}_{ab}\right)_{a,b} & \left(E_{ab} \otimes \overline{H}_{ab}\right)_{a,b} \\ 0 & 0 & 0 & \left(H_{ab} \otimes \overline{H}_{ab}\right)_{a,b} \end{pmatrix}$$

Indeed $\mathcal{D}$ is a transition matrix which allows to pass from

$$\sum_{|x|=n} |\langle \mu[e, a, v_a], \pi(x)\mu[e, b, v_b]\rangle|^2$$

to the same quantity summed on all $|x| = n + 1$.

It can be shown that, under our assumptions, the spectral radius of $\mathcal{D}$ is always one, [14]. Moreover

A) If the two systems (H_{ba}, V_a, B_a) and $(\widehat{V}_a, \widehat{H}_{ba}, \widehat{B}_a)$ are not equivalent, 1 is an eigenvalue of multiplicity two;
B) If the two systems (H_{ba}, V_a, B_a) and $(\widehat{V}_a, \widehat{H}_{ba}, \widehat{B}_a)$ are equivalent, 1 is an eigenvalue of multiplicity four.

Hence the growth of the quantity $\|\phi_\varepsilon^{v_a,v_b}\|_2^2$ depends on the eigenspace of 1 of $\mathcal{D}$. This space can have, in general, dimension 1, 2, 3 or 4, depending on many facts.

The study for inequivalent systems has been done in [13] and it is summarized in the following Theorem:

Theorem 3.5. *If the two systems* (V_a, H_{ba}, B_a) *and* $(\widehat{V}_a, \widehat{H}_{ba}, \widehat{B}_a)$ *are not equivalent, then*

AI *The dimension of the eigenspace of* 1 *is* 2 *if and only if, for every* $a, b, \in A$ *and* $v_a, v_b \in V_a, V_b$*, one has*

$$\|\phi_\varepsilon^{v_a,v_b}\|_2^2 \simeq \frac{1}{\varepsilon}, \quad as \quad \varepsilon \to 0.$$

In this case there exists a unique tuple of linear maps $Q_a : V_a \to \widehat{V}_a$ *satisfying*

$$\widehat{H}_{ab} Q_b + E_{ab} = Q_a H_{ab}, \quad a, b \in A. \tag{12}$$

AII *The dimension of the eigenspace of* 1 *is* 1 *if and only if for every* $a, b, \in A$ *and* $v_a, v_b \in V_a, V_b$*, one has*

$$\|\phi_\varepsilon^{v_a,v_b}\|_2^2 \simeq \frac{1}{\varepsilon^2}, \quad as \quad \varepsilon \to 0.$$

In this case no vector in $\mathcal{H}$ *satisfies the good vector bound* (GVB) *and no system of* $Q_a : V_a \to \widehat{V}_a$ *can satisfy* (12).

3.2. **Equivalent Systems.** Now we pass to the study of equivalent systems, therefore we shall assume, till the end of this section, that the two systems (V_a, H_{ba}, B_a) and $(\widehat{V}_a, \widehat{H}_{ba}, \widehat{B}_a)$ are irreducible, normalized, and equivalent.

We are interested in the case where the dimension of the eigenspace of 1 is 4, since this is linked to the growth $1/\varepsilon$, as shown in the following

Theorem 3.6. [13, Theorem 1]*Let* $\mathcal{D}$ *be the matrix constructed by equivalent systems as in* (11) *and let* d *be the dimension of the eigenspace of* 1 *. Then* $d = 4$ *if and only if*

$$\|\phi_\varepsilon^{v_a,v_b}\|_2^2 \simeq \frac{1}{\varepsilon}, \quad as \quad \varepsilon \to 0.$$

and the result follows.

Remark 3.7. Let $(K_a)_a$, be an equivalence between the two systems

$$K_a : V_a \to \widehat{V}_a, \quad \widehat{H}_{ab}K_b = K_a H_{ab}, \quad \text{for all}\, a, b \in A. \tag{13}$$

One can always assume $K_a^* = K_{a^{-1}}$. Indeed, passing to adjoint in (13), we get

$$H_{ab}^* K_a^* = K_b^* \widehat{H}_{ab}^*, \quad \text{that is} \quad \widehat{H}_{b^{-1}a^{-1}} K_a^* = K_b^* H_{b^{-1}a^{-1}}\,.$$

Write a and b for a^{-1} and b^{-1}, to see that the tuple $K_{a^{-1}}^*$ gives another valid system. Since

$$K_a = \frac{(K_a + K_{a^{-1}}^*)}{2} + i\frac{(K_a - K_{a^{-1}}^*)}{2i},$$

anyone of the two addends will satisfy the required condition.

Lemma 3.8. *The following tuples are, up to constants, the only right eigenvectors corresponding to the eigenvalue* 1 *for the indicated submatrices of the main matrix* (11)

(i) $U_1 = (B_{a^{-1}})_a$ *for* D_{11};
(ii) $U_2 = (K_a^{-1} B_{a^{-1}})_a$ *for* D_{22};
(iii) $U_3 = (B_{a^{-1}} K_{a^{-1}}^{-1})_a$ *for* D_{33};
(iv) $U_4 = (\widehat{B}_{a^{-1}})_a = (K_a^{-1} B_{a^{-1}} K_{a^{-1}}^{-1})_a$ *for* D_{44}.

Proof. The result follows by the compatibility condition (3), the property (13) of the equivalence (K_a), and the identity $K_a^* = K_{a^{-1}}$. □

Now, (10) shows that $V = (B_a)_a$ is an eigenvector corresponding to 1 for the dual of D_{44}

$$D'_{44} = \big((H_{ab} \otimes \overline{H_{ab}})_{a,b}\big)' = (H_{ba}^* \otimes H'_{ba})_{a,b}.$$

Moreover, the space $\mathrm{Ker}((D_{44} - I)')$ is one-dimensional and generated by V. The equality $\mathrm{Im}(D_{44} - I) = \mathrm{Ker}((D_{44} - I)')^{\perp}$ therefore yields the following property involving the trace (9), for any arbitrary X:

$$X \in \mathrm{Im}(D_{44} - I) \Leftrightarrow \mathrm{tr}(XV) = \mathrm{tr}(VX) = 0. \tag{14}$$

Lemma 3.9. *Under the hypotheses hitherto assumed on the matrix systems, the following are equivalent:*

a) *The dimension of the eigenspace corresponding to the eigenvalue* 1 *of* $\mathcal{D}$ *is* $d \geq 3$;
b) $\displaystyle\sum_{a,b \in A} \mathrm{tr}(K_a^{-1} E_{ab}\, \widehat{B}_{b^{-1}} H_{ab}^*\, B_a) = 0.$

Proof. In case of equivalence of matrix systems, the equality (13) yields a similarity between all elements in the main diagonal in $\mathcal{D}$, indeed

$$D_{ii} = S_i\, D_{44}\, S_i^{-1} \quad i = 1, 2, 3,$$

where $S_1 = \operatorname{diag}(K_a \otimes \overline{K_a},\ a \in A)$, $\quad S_2 = \operatorname{diag}(\mathrm{Id} \otimes \overline{K_a},\ a \in A)$, and $S_3 = \operatorname{diag}(K_a \otimes \mathrm{Id},\ a \in A)$.

Hence the full matrix will be similar to a matrix having all the diagonal elements equal to $D = D_{4,4}$.

To analyze the dimension of the eigenspace corresponding to the eigenvalue 1 of $\mathcal{D}$ means to find all solutions of $\mathcal{D}P = P$, therefore we can assume that $\mathcal{D} - I$ is equal to

$$\begin{pmatrix} D-I & A & B & C \\ 0 & D-I & 0 & B \\ 0 & 0 & D-I & A \\ 0 & 0 & 0 & D-I \end{pmatrix},$$

where $A = S_3^{-1} D_{34}$, and $B = S_2^{-1} D_{24}$.

Let $Z = (Z_a)_a$ denote a (normalized) eigenvector corresponding to the eigenvalue 1 of D, (note that $Z_a^* = Z_a$); it verifies the equality

$$(A\,Z)_a^* = \sum_b (K_a^{-1} E_{ab} Z_b H_{ab}^*)^* = \sum_b H_{ab} Z_b (K_a^{-1} E_{ab})^* = (B\,Z)_a.$$

The eigenspace corresponding to the eigenvalue 1 of $\mathcal{D}$ has dimension greater or equal to 3 if and only if there exist 2 linear independent eigenvectors $0 \neq P = (P_j)_{j=1,\dots,4}$, not proportional to $u = (\delta_{j1} Z)_{j=1,\dots,4}$, such that $\mathcal{D}P = P$.

If $P_2 = P_3 = P_4 = 0$, and $\mathcal{D}P = P$ then P_1 is either zero or proportional to Z. Hence P not proportional to u means $(P_2, P_3, P_4) \neq (0,0,0)$.

Observe that, for the discussion before (14), b) means $\operatorname{tr}(AZV) = 0$ or, equivalently, $AZ \in \operatorname{Im}(D - I)$, where $Z = (\widehat{B}_{a^{-1}})_a$ and $V = (B_a)_a$.

We now prove a) implies b). Assume $d \geq 3$.

Equality $\mathcal{D}P = P$ is verified by at least 2 linear independent vectors not proportional to u. They verify

$$\begin{cases} (D-I)P_1 + AP_2 + BP_3 + CP_4 = 0 \\ (D-I)P_2 + BP_4 = 0 \\ (D-I)P_3 + AP_4 = 0 \\ (D-I)P_4 = 0 \end{cases} \tag{15}$$

The last equation in (15) implies P_4 is either zero or proportional to Z.

If $P_4 = cZ$, without loss of generality we can assume $c = 1$. From the third equation in (15) we get $AZ \in \operatorname{Im}(D - I)$ which is equivalent, by (14), to b).

If $P_4 = 0$, the second and third equation in (15) imply both P_2 and P_3 are either zero or proportional to Z, but in any case $(P_2, P_3) \neq (0,0)$.

If $(P_2, P_3) = (0, Z), (Z, 0)$, from the first equation in (15), we get, passing to adjoint if needed, that $AZ \in \mathrm{Im}(D-I)$, which is equivalent, by (14), to b).

Finally, assume each solution of (15) verifies $(P_2, P_3, P_4) = (cZ, Z, 0)$ for some $0 \neq c \in \mathbb{C}$, (we can always assume one of the constants is equal to 1).

Since $d \geq 3$, there are at least two such solutions, say P and P'. With obvious meaning of symbols, by the first equation in (15)

$$(D-I)(P_1' - P_1) + (c' - c)AZ = 0.$$

If $c' \neq c$, the latter implies b).

On the other hand $c' = c$ is impossible. On the contrary, if $c' = c$ we get $P_1' = P_1 + \alpha Z$, $\alpha \in \mathbb{C}$. In other words, up to constant, there is only one other vector not proportional to u in a basis of $\mathrm{Ker}(\mathcal{D} - I)$, in contradiction with $d \geq 3$.

We now show that b) implies a).

As already noted, b) means that $AZ \in \mathrm{Im}(D-I)$, and passing to adjoint, $BZ \in \mathrm{Im}(D-I)$, too. Therefore there are vectors $W_2 = (W_{2,a})_a$, and $W_3 = (W_{3,a})_a$, such that $W_{2,a} = W_{3,a}^*$, and

$$(D-I)W_2 + BZ = 0, \quad (D-I)W_3 + AZ = 0. \tag{16}$$

It follows that $(W_3, Z, 0, 0)^\top$ and $(W_2, 0, Z, 0)^\top$ are in $\mathrm{Ker}(\mathcal{D} - I)$, and together with u they form a set of three linear independent vectors. Hence $\dim \mathrm{Ker}(\mathcal{D} - I) \geq 3$. □

Lemma 3.10. *Assume that the dimension of the eigenspace corresponding to the eigenvalue* 1 *of* $\mathcal{D}$ *is* $d \geq 3$.

Then there exist linear maps $Q_b : V_b \to \widehat{V}_b$, $b \in A$, *such that the vector*

$$\begin{pmatrix} (\widehat{B}_{b^{-1}}\, Q_b^*)_b \\ (Q_b\, \widehat{B}_{b^{-1}})_b \\ (\widehat{B}_{b^{-1}})_b \end{pmatrix}$$

is a (right) eigenvector corresponding to the eigenvalue 1 *of the the principal submatrix* $\mathcal{D}_1$ *of* $\mathcal{D}$, *obtained by deleting the rows and columns of* $D_{1,1}$.

Proof. Following the notation of Lemma 3.9, set $S_2 = \mathrm{diag}(\mathrm{Id} \otimes \overline{K_a},\ a \in A)$, and $S_2 = \mathrm{diag}(K_a \otimes \mathrm{Id},\ a \in A)$.

Since $d \geq 3$, from the proof of Lemma 3.9, we get that there are vectors $W_2 = (W_{2,a})_a$, and $W_3 = (W_{3,a})_a$, such that $W_{2,a} = W_{3,a}^*$ and

$$\begin{cases} (D_{22} - I)S_2 W_2 + D_{24} Z = 0 \\ (D_{33} - I)S_3 W_3 + D_{34} Z = 0 \\ (D_{44} - I)Z = 0, \end{cases}$$

where Z is an eigenvector corresponding to the eigenvalue 1 for D_{44}, and, without loss of generality we can assume $Z = U_4 = (\widehat{B}_{a^{-1}})_a$, see Lemma 3.8. Also note that

$$(S_3 W_3)_b = K_b W_{3,b} : \widehat{V}_{b^{-1}} = V_b^* \to \widehat{V}_b.$$

Since $\widehat{B}_{b^{-1}}$ is strictly positive definite, it is invertible as a linear map $\widehat{B}_{b^{-1}} : V_b^* \to V_b$. Hence the map

$$Q_b = K_b \, W_{3,b} \, \widehat{B}_{b^{-1}}^{-1} : V_b \to \widehat{V}_b,$$

is well defined and linear. It follows $K_b \, W_{3,b} = Q_b \, \widehat{B}_{b^{-1}}$ and

$$(S_2 W_2)_b = W_{2,b} K_b^* = (K_b \, W_{3,b})^* = \widehat{B}_{b^{-1}} \, Q_b^*.$$

□

Theorem 3.11. *Under the hypotheses hitherto assumed on the matrix systems, the dimension of the eigenspace corresponding to the eigenvalue* 1 *of* $\mathcal{D}$ *is* $d = 4$ *if and only if the following conditions hold*

1) $\displaystyle\sum_{a,b \in A} \operatorname{tr}(\widehat{H}_{ab} \, B_{b^{-1}} \, K_{b^{-1}}^{-1} \, E_{ab}^* \, \widehat{B}_a) = 0;$
2) *Given the linear maps* $Q_b : V_b \to \widehat{V}_b$*, provided by Lemma 3.10, the following identity holds:*

$$\widehat{H}_{ab} Q_b + E_{ab} = Q_a H_{ab}, \quad \textit{for all } a, b \in A.$$

Proof. Condition 1) is equivalent to $d \geq 3$ by Lemma 3.9. Condition 2) is equivalent, by the same argument used in [13, Theorem 5.13] to

$$0 = \sum_{a,b \in A} \operatorname{tr}(\widehat{B}_a \, E_{ab} \, \widehat{B}_{b^{-1}} \, Q_b^* \, \widehat{H}_{ab}^* + \widehat{B}_a \, \widehat{H}_{ab} \, Q_b \, \widehat{B}_{b^{-1}} \, E_{ab}^* + \widehat{B}_a \, E_{ab} \, \widehat{B}_{b^{-1}} \, E_{ab}^*)$$

$$= \operatorname{tr}\left(\sum_{a \in A} \widehat{B}_a \sum_{b \in A} \left[E_{ab} \widehat{B}_{b^{-1}} Q_b^* \widehat{H}_{ab}^* + \widehat{H}_{ab} Q_b \widehat{B}_{b^{-1}} E_{ab}^* + E_{ab} \widehat{B}_{b^{-1}} E_{ab}^* \right] \right),$$

meaning that the vector

$$W = (W_j)_{j=2,3,4} = \begin{pmatrix} (\widehat{B}_{b^{-1}} \, Q_b^*)_b \\ (Q_b \, \widehat{B}_{b^{-1}})_b \\ (\widehat{B}_{b^{-1}})_b \end{pmatrix}$$

is such that $D_{1,2}W_2 + D_{1,3}W_3 + D_{1,4}W_4 \in \mathrm{Im}(D_{11} - I)$.

Hence conditions 1) and 2) are equivalent to the existence of a fourth linear independent vector $W = (W_j)_{j=1,\dots,4}$ in (the basis of) $\mathrm{Ker}(\mathcal{D} - I)$ besides those provided in the proof of Lemma 3.9. □

We summarize the results for equivalent systems in the following Theorem:

Theorem 3.12. *If the two systems* (V_a, H_{ba}, B_a) *and* $(\widehat{V}_a, \widehat{H}_{ba}, \widehat{B}_a)$ *are equivalent, then* 1 *is an eigenvalue of multiplicity four for* $\mathcal{D}$ *and the dimension of the eigenspace of* 1 *is two, three or four. In particular*

BI *The dimension of the eigenspace of* 1 *is* 4 *if and only if, for every* $a, b, \in A$ *and* $v_a, v_b \in V_a, V_b$*, one has*

$$\|\phi_\varepsilon^{v_a,v_b}\|_2^2 \simeq \frac{1}{\varepsilon}, \quad \text{as} \quad \varepsilon \to 0. \tag{17}$$

As well as in the case ***AI*** *there exists a tuple of linear maps* $Q_a : V_a \to \widehat{V}_a$ *satisfying* (12)

$$\widehat{H}_{ab} Q_b + E_{ab} = Q_a H_{ab},\ a, b \in A,$$

but in this case the tuple is not unique.

BII *The dimension of the eigenspace of* 1 *is* 2 *(respectively* 3*) if and only if*

$$\|\phi_\varepsilon^{v_a,v_b}\|_2^2 \simeq \frac{1}{\varepsilon^3} \quad (\textit{respectively } \frac{1}{\varepsilon^2}), \quad \text{as} \quad \varepsilon \to 0.$$

In both cases no vector in $\mathcal{H}$ *satisfies the good vector bound* (GVB) *and no system of* $Q_a : V_a \to \widehat{V}_a$ *can satisfy* (12).

Proof. The only thing to prove is that the dimension of the eigenspace of 1 cannot be 1. If this were the case one should have a cycle of length four in the generalized eigenspace of 1. This implies that the growth of the quantity is

$$\sum_{|x|=n} |\langle \mu[e, a, v_a], \pi(x)\mu[e, b, v_b]\rangle|^2 e^{-\varepsilon|x|} \simeq n^3 e^{-\varepsilon n}$$

which contradicts Haagerup's inequality (1). □

In [13] we proved that all the representations arising from **AII** satisfy monotony. The calculation and the techniques presented in [13] can be carried out virtually unchanged to prove that also the representations arising from **BII** satisfy monotony.

This paper is devoted to the study of cases **AI, BI**.

4. Equivalent multiplicative Γ-representations

In this section the reigning hypotheses are as follows:

• π and $\widehat{\pi}$ are multiplicative representations built up from irreducible twin systems (V_a, H_{ba}, B_a) and $(\widehat{V}_a, \widehat{H}_{ba}, \widehat{B}_a)$: this implies that π and $\widehat{\pi}$ are irreducible as representations of $\Gamma \ltimes C(\Omega)$ [16, Theorem 5.3];

• For every $a, b, \in A$ and $v_a, v_b \in V_a, V_b$, one has (17) and hence there exists a tuple of linear maps $Q_a : V_a \to \widehat{V}_a$ satisfying:

$$\widehat{H}_{ab} Q_b + E_{ab} = Q_a H_{ab},\ a, b \in A. \tag{18}$$

We begin with the following

Lemma 4.1. *The maps $Q_a : V_a \to \widehat{V}_a$ appearing in* (18) *also satisfy*

$$Q_a^* + Q_{a^{-1}} = 0, \quad a \in A$$

Proof. Take the adjoint in $\widehat{H}_{ab} Q_b + E_{ab} = Q_a H_{ab}$, and write b^{-1} for a and a^{-1} for b:

$$Q^*_{a^{-1}} \widehat{H}^*_{b^{-1}a^{-1}} + E^*_{b^{-1}a^{-1}} = H^*_{b^{-1}a^{-1}} Q^*_{b^{-1}}.$$

Remember that $E^*_{b^{-1}a^{-1}} = E_{ab}$ and write

$$\widehat{H}_{ab} Q^*_{b^{-1}} = Q^*_{a^{-1}} H_{ab} + E_{ab}$$
$$\widehat{H}_{ab} Q_b + E_{ab} = Q_a H_{ab}$$

Adding up to the two equations gives

$$\widehat{H}_{ab}(Q_b + Q^*_{b^{-1}}) = (Q_a + Q^*_{a^{-1}}) H_{ab},\ a, b \in A.$$

If the two systems are inequivalent $(Q_a + Q^*_{a^{-1}})$ must be zero by Remark 3.4 in [16]. If the two systems are equivalent one can replace Q_a by $\tilde{Q}_a = \frac{1}{2}(Q_a - Q^*_{a^{-1}})$. □

Theorem 4.2. *Assume the existence of a tuple of linear maps Q_a : $V_a \to \widehat{V}_a$ satisfying* (18).

Then there exists a linear bijection $J : \mathcal{H}^\infty \to \widehat{\mathcal{H}}^\infty$ that intertwines π to $\widehat{\pi}$.

Proof. We shall first define J for functions $\mu[x, xa, v_a]$ with $|xa| = |x| + 1$.

Recall that the tree decomposes as the disjoint union of the sets $\Gamma(x, xa)$ and $\Gamma(xa, x)$. Write $\mu[xa, x, v_{a^{-1}}]$ for $\mu[xa, xaa^{-1}, v_{a^{-1}}]$ and observe that the maps $\mu[x, xa, v_a]$ and $\mu[xa, x, v_{a^{-1}}]$ are orthogonal.

For all $a \in A$ and $v_a \in V_a$ define

$$J(\mu[x, xa, v_a]) = \widehat{\mu}[x, xa, -Q_a v_a] + \widehat{\mu}[xa, x, B_a v_a], \tag{19}$$

where $\widehat{\mu}$ is the multiplicative function constructed from the system $(\widehat{V}_a, \widehat{H}_{ab}, \widehat{B}_a)$ according to (5). Note that $Q_a v_a \in \widehat{V}_a$, $B_a v_a \in V_a^* = \widehat{V}_{a^{-1}}$.

Since $\pi(y)\mu[x, xa, v_a] = \mu[yx, yxa, v_a]$ regardless of whether yx or xa are reduced or not, it is clear that J will intertwine π to $\widehat{\pi}$.

To see that J is well defined write

$$\mu[e, a, v_a] = \sum_{b\neq a^{-1}} \mu[a, ab, H_{ba} v_a]$$

and compute

$$J(\sum_{b\neq a^{-1}} \mu[a, ab, H_{ba} v_a]) = \sum_{b\neq a^{-1}} (\widehat{\mu}[a, ab, -Q_b H_{ba} v_a] + \widehat{\mu}[ab, a, B_b H_{ba} v_a]).$$

We need

$$J(\mu[e, a, v_a]) = \widehat{\mu}[e, a, -Q_a v_a] + \widehat{\mu}[a, e, B_a v_a] \tag{20}$$

$$= \sum_{b\neq a^{-1}} (\widehat{\mu}[a, ab, -Q_b H_{ba} v_a] + \widehat{\mu}[ab, a, B_b H_{ba} v_a])\,. \tag{21}$$

Compute (20) at $x = e$ to get $B_a v_a$, while (21) evaluated at e gives

$$\sum_{b\neq a^{-1}} \widehat{H}_{a^{-1}b^{-1}} B_b H_{ba} v_a = \sum_{b\neq a^{-1}} H^*_{ba} B_b H_{ba} v_a,$$

they are equal since (B_a) is a right eigenvector of the matrix $(H^*_{ba} \otimes H'_{ba})_{a,b}$. Proceed now to compute (20) at $x = ac$: we get $-\widehat{H}_{ca} Q_a v_a$ while (21) gives

$$-Q_c H_{ca} v_a + \sum_{b\neq c, a^{-1}} \widehat{H}_{cb^{-1}} B_b H_{ba} v_a\,.$$

Equality holds if and only if

$$Q_c H_{ca} = \widehat{H}_{ca} Q_a \ + \sum_{b\neq c, a^{-1}} \widehat{H}_{cb^{-1}} B_b H_{ba},$$

which is (18) where we wrote a for c, b for a and c for b.

We shall now provide the expression for $J(\mu[a, e, v_{a^{-1}}])$. Observe that

$$\begin{aligned} J(\pi(a^{-1})\mu[e, a, v_a]) &= \widehat{\pi}(a^{-1}) J\mu[e, a, v_a] \\ &= \widehat{\mu}[a^{-1}, e, -Q_a v_a] + \widehat{\mu}[e, a^{-1}, B_a v_a], \end{aligned}$$

write now a for a^{-1}: we get

$$J\mu[a, e, v_{a^{-1}}] = \widehat{\mu}[e, a, B_{a^{-1}} v_{a^{-1}}] + \widehat{\mu}[a, e, -Q_{a^{-1}} v_{a^{-1}}].$$

Identify $V_a \oplus V_{a^{-1}}$ with the space of multiplicative functions of the form $\mu[e, a, v_a] + \mu[a^{-1}, e, v_{a^{-1}}]$.

With this identification we may think of J as acting on each $V_a \oplus V_{a^{-1}}$ via the matrix

$$\begin{pmatrix} -Q_a & B_{a^{-1}} \\ B_a & -Q_{a^{-1}} \end{pmatrix} : V_a \oplus V_{a^{-1}} \to \widehat{V}_a \oplus \widehat{V}_{a^{-1}} , \tag{22}$$

where $\widehat{V}_a \oplus \widehat{V}_{a^{-1}}$ is identified with the space of functions of the form $\widehat{\mu}[e, a, v_a] + \widehat{\mu}[a^{-1}, e, v_{a^{-1}}]$. Moreover, the matrix (22) has as right inverse

(23)

$$\begin{pmatrix} -B_a^{-1}Q_a^*(B_{a^{-1}} + Q_a B_a^{-1} Q_a^*)^{-1} & (B_a + Q_{a^{-1}} B_{a^{-1}}^{-1} Q_{a^{-1}}^*)^{-1} \\ (B_{a^{-1}} + Q_a B_a^{-1} Q_a^*)^{-1} & -B_{a^{-1}}^{-1} Q_{a^{-1}}^* (B_a + Q_{a^{-1}} B_{a^{-1}}^{-1} Q_{a^{-1}}^*)^{-1} \end{pmatrix}.$$

Indeed, a simple multiplication and Lemma 4.1 show that it suffices to prove that the matrix (23) is well defined, i.e. $B_a + Q_{a^{-1}} B_{a^{-1}}^{-1} Q_{a^{-1}}^*$ is invertible. But the latter is positive definite, hence invertible.

Extend now J by linearity to $\mathcal{H}^\infty$ by (**??**).

We shall now prove that J is invertible from $\mathcal{H}^\infty \to \widehat{\mathcal{H}}^\infty$.

To prove surjectivity is sufficient, by linearity and the intertwining property, to check that all the functions of the type $\widehat{\mu}[e, a, \hat{v}_a] + \widehat{\mu}[a, e, \hat{v}_{a^{-1}}]$ are in the image of J.

By (23) we know that, given $\hat{v}_a \in \widehat{V}_a$ and $\hat{v}_{a^{-1}} \in \widehat{V}_{a^{-1}}$, there exist $w_a \in V_a$, $w_{a^{-1}} \in V_{a^{-1}}$ such that

$$\begin{pmatrix} -Q_a & B_{a^{-1}} \\ B_a & -Q_{a^{-1}} \end{pmatrix} \begin{pmatrix} w_a \\ w_{a^{-1}} \end{pmatrix} = \begin{pmatrix} \hat{v}_a \\ \hat{v}_{a^{-1}} \end{pmatrix},$$

and so

$$J(\mu[e, a, w_a] + \mu[a, e, w_{a^{-1}}]) = \widehat{\mu}[e, a, \hat{v}_a] + \widehat{\mu}[a, e, \hat{v}_{a^{-1}}] .$$

We proof that J is injective as follows. For any $n \in \mathbb{N}$ let W_n be the finite dimensional subspace of $\mathcal{H}^\infty$ consisting of multiplicative functions of the form

$$f = \sum_{|x|=n} \sum_{\substack{a\in A \\ |xa|=|x|+1}} \mu[x, xa, f(xa)].$$

Let J_n be the restriction of J on this subspace. It is easy to see that $J_n : W_n \to \widehat{W_n}$ is onto, therefore J_n is also one-to-one. Now, let $Jf = 0$ for some multiplicative function f. By (**??**) there exists n such that $f \in W_n$ and $J_n f = 0$, therefore $f = 0$ and the theorem is proved. □

Incidentally we have proved the following

Proposition 4.3. *Any intertwiner T between π and $\widehat{\pi}$ which sends functions of the form $\mu[e, a, v_a] + \mu[a^{-1}, e, v_{a^{-1}}]$ into functions of the same type lying in $\widehat{\mathcal{H}}^\infty$, must be given by a matrix*

$$\begin{pmatrix} X_a & Y_{a^{-1}} \\ Y_a & X_{a^{-1}} \end{pmatrix}$$

where $X_a : V_a \to \widehat{V}_a$, $Y_a : V_a \to \widehat{V}_{a^{-1}}$, must be chosen so that

$$Y_a = \lambda B_a \quad \textit{for some } \lambda \in \mathbb{C},$$

$$X_a H_{ab} + \lambda E_{ab} = \widehat{H}_{ab} X_b\,.$$

In particular the maps $-X_a$ must satisfy (18) *with respect to the choice of λB_a .*

Proof. Assume that $J(\mu[e, a, v_a]) = \widehat{\mu}[e, a, X_a v_a] + \widehat{\mu}[a, e, Y_a v_a]$ and proceed as in the proof of Theorem 4.2 to see that

$$J(\mu[e, a, v_a]) = \widehat{\mu}[e, a, X_a v_a] + \widehat{\mu}[a, e, Y_a v_a]$$

must be equal to

$$\sum_{b \neq a^{-1}} (\widehat{\mu}[a, ab, X_b H_{ba} v_a] + \widehat{\mu}[ab, a, Y_b H_{ba} v_a])\;.$$

The two expressions will be equal if and only if Y_a is a multiple of B_a, say λB_a, and the maps $-X_a$ satisfy (18) with respect to the choice of λB_a.

If the two systems (V_a, H_{ba}, B_a) and $(\widehat{V}_a, \widehat{H}_{ba}, \widehat{B}_a)$ are not equivalent then there is only one possibility for the maps Q_a and hence the matrix for T will be

$$\begin{pmatrix} -\lambda Q_a & \lambda B_{a^{-1}} \\ \lambda B_a & -\lambda Q_a^* \end{pmatrix}.$$

If the two systems (V_a, H_{ba}, B_a) and $(\widehat{V}_a, \widehat{H}_{ba}, \widehat{B}_a)$ **are** equivalent then there exists a tuple of linear maps $K_a : V_a \to \widehat{V}_a$ so that

$$K_a H_{ab} = \widehat{H}_{ab} K_b$$

and this will increase the possibilities for the Q_a. Having in mind that we want to keep the condition $Q_a^* + Q_{a^{-1}} = 0$, observe that we may always assume that $K_a^* = K_{a^{-1}}$ (see Remark 3.7).

In this case, for a fixed given choice of λB_a all the possible choices for X_a satisfying $X_a^* + X_{a^{-1}} = 0$ are of the form

$$X_a = -\lambda Q_a + icK_a \quad \text{for real } c,$$

and the matrix for T will be

$$\begin{pmatrix} -\lambda Q_a + icK_a & \lambda B_{a^{-1}} \\ \lambda B_a & -\lambda Q_a^* + icK_a^* \end{pmatrix}.$$

□

We are now ready to prove the following

Theorem 4.4. *Assume the existence of a tuple of linear maps* $Q_a : V_a \to \widehat{V}_a$ *satisfying* (18).

Let J *be as in* (19)*. Then* $J : \mathcal{H}^\infty \to \widehat{\mathcal{H}}^\infty$ *preserves the inner product and hence it extends to a unitary equivalence between* π *and* $\widehat{\pi}$*.*

Proof. Let Q_a be a tuple of maps satisfying (18). Construct J as in Theorem 4.2 and let

$$\begin{pmatrix} -Q_a & B_{a^{-1}} \\ B_a & -Q_{a^{-1}} \end{pmatrix}$$

be the matrix representing it.

At the algebraic level we know that J is invertible and intertwines π to $\widehat{\pi}$, from $\mathcal{H}^\infty$ to $\widehat{\mathcal{H}}^\infty$. Since the twin of the twin system is the original system and $J^{-1} : \widehat{\mathcal{H}}^\infty \to \mathcal{H}^\infty$ intertwines the representations as well, Proposition 4.3 applied to the twin system, says that the matrix (23) for J^{-1} must be of the form

$$\begin{pmatrix} -\widehat{Q}_a & \widehat{B}_{a^{-1}} \\ \widehat{B}_a & -\widehat{Q}_{a^{-1}} \end{pmatrix} \tag{24}$$

where $\widehat{B}_a$ is a right eigenvector of the matrix $(\widehat{H}^*_{ba} \otimes \widehat{H}'_{ba})_{a,b}$ and the maps $\widehat{Q}_a : \widehat{V}_a \to V_a$ satisfy

$$H_{ab}\widehat{Q}_b + \widehat{E}_{ab} = \widehat{Q}_a\widehat{H}_{ab}, \quad a, b \in A, \tag{25}$$

with $\widehat{E}_{ab}$ the analogue of E_{ab} with respect to the twin system.

Hence

$$J^{-1}(\widehat{\mu}[e, a, \hat{v}_a]) = \mu[e, a, -\widehat{Q}_a\hat{v}_a] + \mu[a, e, \widehat{B}_a\hat{v}_a].$$

Remember that (24) is also a left inverse for the matrix (22) and use Lemma 4.1 to get

$$\widehat{Q}_aQ_a + \widehat{B}_{a^{-1}}B_a = \mathrm{Id} \tag{26}$$

$$\widehat{B}_aQ_a = -\widehat{Q}_{a^{-1}}B_a \tag{27}$$

$$Q_{a^{-1}}\widehat{B}_a = -B_a\widehat{Q}_a \tag{28}$$

Now we prove that J preserves the inner product in $\mathcal{H}^\infty$ defined in (6).

Due to the structure of multiplicative functions, by linearity and sesquilinearity, since π and $\widehat{\pi}$ are unitary representations intertwined by J and J^{-1}, it is sufficient to show that

$$\langle Jf, Jg\rangle = \langle f, g\rangle$$

when either $f = \mu[x, xa, v_a]$ and $g = \mu[x, xa, w_a]$ or $f = \mu[a_1, e, w_{a_1^{-1}}]$ and $g = \mu[x, xa, v_a]$, where a_1 is the first letter of x, and xa is reduced. In the first case one has

$$\langle \mu[x, xa, v_a], \mu[x, xa, w_a]\rangle = B_a(v_a, w_a) \tag{29}$$

while

$$\begin{aligned} &\langle J\mu[x, xa, v_a], J\mu[x, xa, w_a]\rangle \\ = \;& \langle \widehat{\mu}[e, a, -Q_a v_a] + \widehat{\mu}[a, e, B_a v_a], \widehat{\mu}[e, a, -Q_a w_a] + \widehat{\mu}[a, e, B_a w_a]\rangle \\ = \;& \widehat{B}_{a^{-1}}(B_a v_a, B_a w_a) + \widehat{B}_a(Q_a v_a, Q_a w_a) \;. \end{aligned} \tag{30}$$

Hence (29) will be equal to (30) if and only if

$$B_a = B_a^* \widehat{B}_{a^{-1}} B_a + Q_a^* \widehat{B}_a Q_a = B_a \widehat{B}_{a^{-1}} B_a + B_a \widehat{Q}_a Q_a,$$

(after an application of (27) and Lemma 4.1), which is true by (26).

In the second case, since $\mu[a_1, e, w_{a_1^{-1}}]$ and $\mu[x, xa, v_a]$ are orthogonal, we need to prove $\langle Jf, Jg\rangle = 0$. Let us suppose that $xa = a_1 \dots a_n a_{n+1}$ is reduced. We have, again by orthogonality and since $a_1^{-1} xa = a_2 \dots a_{n+1}$,

$$\begin{aligned} \langle Jf, Jg\rangle &= \langle J\mu[a_1, e, w_{a_1^{-1}}], J\mu[x, xa, v_a]\rangle \\ &= \langle \widehat{\mu}[e, a_1^{-1}, -Q_{a_1^{-1}} w_{a_1^{-1}}], \widehat{\pi}(a_2 \dots a_n a_{n+1})\, \widehat{\mu}[e, a_{n+1}^{-1}, B_{a_{n+1}} v_{a_{n+1}}]\rangle \\ &\quad + \langle \widehat{\mu}[e, a_1, B_{a_1^{-1}} w_{a_1^{-1}}], \widehat{\pi}(a_1 \dots a_n)\, \widehat{\mu}[e, a_{n+1}, -Q_{a_{n+1}} v_{a_{n+1}}]\rangle \\ &\quad + \langle \widehat{\mu}[e, a_1, B_{a_1^{-1}} w_{a_1^{-1}}], \widehat{\pi}(a_1 \dots a_n a_{n+1})\, \widehat{\mu}[e, a_{n+1}^{-1}, B_{a_{n+1}} v_{a_{n+1}}]\rangle. \end{aligned}$$

Each of the above quantities have been already calculated in the proof of [13, Lemma 5.5] and are respectively equal to

$$\begin{aligned} &-\widehat{B}_{a_1^{-1}}(Q_{a_1^{-1}} w_{a_1^{-1}}\, ,\, \widehat{H}_{a_1^{-1} a_2^{-1}} \dots \widehat{H}_{a_n^{-1} a_{n+1}^{-1}} B_{a_{n+1}} v_{a_{n+1}}) \\ &-\widehat{B}_{a_{n+1}}(\widehat{H}_{a_{n+1} a_n} \dots \widehat{H}_{a_2 a_1} B_{a_1^{-1}} w_{a_1^{-1}}\, ,\, Q_{a_{n+1}} v_{a_{n+1}}) \\ &+\sum_{j=0}^{n-1} \widehat{E}_{a_{j+2} a_{j+1}}(\widehat{H}_{a_{j+1} a_j} \dots \widehat{H}_{a_2 a_1} B_{a_1^{-1}} w_{a_1^{-1}}, \widehat{H}_{a_{j+2}^{-1} a_{j+3}^{-1}} \dots \widehat{H}_{a_n^{-1} a_{n+1}^{-1}} B_{a_{n+1}} v_{a_{n+1}}). \end{aligned}$$

Therefore, after an application of (27), (28), and Lemma 4.1, everything is proved if we show that, for every $n \geq 1$,

$$\begin{aligned} & H_{a_{n+1} a_n} \dots H_{a_2 a_1} \widehat{Q}_{a_1} - \widehat{Q}_{a_{n+1}} \widehat{H}_{a_{n+1} an} \dots \widehat{H}_{a_2 a_1} \\ & + \sum_{j=0}^{n-1} H_{a_{n+1} a_n} \dots H_{a_{j+3} a_{j+2}} \widehat{E}_{a_{j+2} a_{j+1}} \widehat{H}_{a_{j+1} a_j} \dots \widehat{H}_{a_2 a_1} = 0. \end{aligned} \tag{31}$$

The latter can be easily proved by induction on $n \geq 1$, by means of (25). We omit the details, just note that if $n = 1$, (31) is indeed

$$H_{a_2 a_1} \widehat{Q}_{a_1} - \widehat{Q}_{a_2} \widehat{H}_{a_2 a_1} + \widehat{E}_{a_2 a_1} = 0.$$

□

Theorem 4.5. *Assume the reigning hypotheses of this Section. If the two systems* (V_a, H_{ba}, B_a) *and* $(\widehat{V}_a, \widehat{H}_{ba}, \widehat{B}_a)$ *are equivalent then* π *splits into the sum of two* Γ*-representations.*

Remark 4.6. It will be proved in the next Section that these two Γ-representations are indeed irreducible and inequivalent.

Proof. Let (K_a) be an equivalence between the two systems. By Lemma 5.2 of [16] we may assume that $K_a : V_a \to \widehat{V}_a$ is a A-tuple of unitary operators satisfying, by Remark 3.7, $K_{a^{-1}} = K_a^*$. Fix a system of Q_a satisfying (18). Normalize the $\widehat{B}_a$ so that

$$J = \begin{pmatrix} -Q_a & B_{a^{-1}} \\ B_a & -Q_a^* \end{pmatrix} \quad J^{-1} = \begin{pmatrix} -\widehat{Q}_a & \widehat{B}_{a^{-1}} \\ \widehat{B}_a & -\widehat{Q}_a^* \end{pmatrix}$$

and set

$$\mathcal{K} = \begin{pmatrix} K_a & 0 \\ 0 & K_{a^{-1}} \end{pmatrix}.$$

Consider the operator $\mathcal{K}J^{-1}\mathcal{K} : \mathcal{H}^\infty \to \widehat{\mathcal{H}}^\infty$. We may assume that J is unitary as well, so that $\mathcal{K}J^{-1}\mathcal{K}$ is also unitary and, by Theorem 4.2, intertwines the two Γ-representations π and $\widehat{\pi}$. By Proposition 4.3 it must be of the form

$$\begin{pmatrix} -\lambda Q_a + icK_a & \lambda B_{a^{-1}} \\ \lambda B_a & -\lambda Q_a^* + icK_a^* \end{pmatrix}.$$

We pass to calculate the product of the three matrices. Since the term $K_{a^{-1}}\widehat{B}_a K_a = K_a^* \widehat{B}_a K_a$ is positive, the same must be true for λB_a, so that λ must be positive. Hence $\mathcal{K}J^{-1}\mathcal{K}$ must satisfy the following equation:

$$\mathcal{K}J^{-1}\mathcal{K} = \lambda J + ic\mathcal{K}$$

for some positive λ and real c. Multiply both sides by $\mathcal{K}^{-1}J\mathcal{K}^{-1}$ to get the equation:

$$\mathrm{Id} = \lambda(\mathcal{K}^{-1}J)^2 + ic(\mathcal{K}^{-1}J).$$

Hence the unitary operator $(\mathcal{K}^{-1}J)$ has two complex eigenvalues, say $\lambda_\pm$, which are distinct since $(\mathcal{K}^{-1}J)$ is not a scalar. Some elementary algebra shows that λ must be 1 and $\lambda_\pm = \frac{-ic\pm\sqrt{4-c^2}}{2}$. Hence the Γ-representation splits into the direct sum of two representations each corresponding to an eigenspace of $(\mathcal{K}^{-1}J)$. □

Remark 4.7. We remark that if we replace J by

$$\tilde{J} = \frac{2}{\sqrt{4-c^2}}(J - \frac{ic}{2}\mathcal{K})$$

we still obtain a valid non trivial intertwiner for π and $\widehat{\pi}$, but this choice of $\tilde{J}$ will lead to the unitary operator

$$\mathcal{J} = (\mathcal{K}^{-1})\tilde{J}$$

having the simpler eigenvalues $+1$ and -1.

5. Γ-irreducibility of multiplicative representations

We begin this section by recalling the Duplicity and the Oddity Theorems from [10]. Denote by $\|\cdot\|_{HS}$ the Hilbert-Schmidt norm of an operator.

Duplicity Theorem 5.1. Let $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ be a unitary representation of Γ. Suppose

- $(\pi'_\pm, \mathcal{H}'_\pm)$ are two irreducible $\Gamma \ltimes C(\Omega)$-representations, inequivalent as $\Gamma \ltimes C(\Omega)$-representations.
- $\iota_\pm : \mathcal{H} \to \mathcal{H}'_\pm$ are two perfect boundary realizations of π.
- The following Finite Trace Condition holds

$$\text{(FTC)} \qquad \|(\iota_-^* \pi'_-(\mathbf{1}_a)\iota_-)(\iota_+^* \pi'_+(\mathbf{1}_b)\iota_+)\|_{HS} < \infty, \quad a, b \in A, a \neq b.$$

Then

- π is irreducible as a Γ-representation.
- Up to equivalence, ι_+ and ι_- are the only perfect boundary realizations of π.
- Any imperfect boundary realization of π is equivalent to $\sqrt{t_+}\,\iota_+ \oplus \sqrt{t_-}\,\iota_- : \mathcal{H} \to \mathcal{H}'_+ \oplus \mathcal{H}'_-$ for constants $t_+, t_- > 0$ with $t_+ + t_- = 1$.

Oddity Theorem 5.2. Let $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ be a unitary representation of Γ. Suppose

- $(\pi', \mathcal{H}')$ is an irreducible $\Gamma \ltimes C(\Omega)$-representation.
- $\iota : \mathcal{H} \to \mathcal{H}'$ is an imperfect realization of π.
- The following Finite Trace Condition holds

$$\text{(FTC)} \qquad \|(\iota^* \pi'(\mathbf{1}_a)\iota)(\iota^* \pi'(\mathbf{1}_b)\iota)\|_{HS} < \infty, \quad a, b \in A, a \neq b.$$

Then

- π is irreducible as a Γ-representation.
- Up to equivalence, ι is the only boundary realization of π.

Observe that the unitary Γ-action which π' gives on $\mathcal{H}'$ stabilizes $\mathcal{H}_1 = \iota(\mathcal{H})$, so it also stabilizes the orthogonal complement $\mathcal{H}_2 = \mathcal{H} \ominus \mathcal{H}_1$. Let $\pi_2 : \Gamma \to \mathcal{U}(\mathcal{H}_2)$ denote the Γ-action on $\mathcal{H}_2$. One can also conclude:

- π_2 is irreducible.
- π_2 is inequivalent to π.

In order to use the above cited Theorems we need to investigate the Finite Trace Condition (FTC) for multiplicative representations.

We have the fundamental

Lemma 5.3. *Let (V_a, H_{ba}, B_a) be a normalized irreducible system and consider its twin $(\widehat{V}_a, \widehat{H}_{ba}, \widehat{B}_a)$. Assume the existence of a tuple of linear maps $Q_a : V_a \to \widehat{V}_a$ satisfying* (18).

Let J be the intertwining operator defined in (19)*. Then, for every $a \neq b \in A$, the operator $\widehat{\pi}(\mathbf{1}_b) J \pi(\mathbf{1}_a)$ has finite rank.*

Proof. We claim that, for fixed $a \neq b$, $\widehat{\pi}(\mathbf{1}_b)\, J\, \pi(\mathbf{1}_a)$ maps $\mathcal{H}^\infty$ in a subspace of $\widehat{\mathcal{H}}^\infty$ of finite dimension. Since finite dimensional subspaces are closed, this implies that $\widehat{\pi}(\mathbf{1}_b)\, J\, \pi(\mathbf{1}_a)$ has finite rank. More precisely, let $\widehat{\mathcal{E}}_b$ be the subspace of $\widehat{\mathcal{H}}^\infty$ generated by the set of (equivalence class of) multiplicative functions $\{\widehat{\mu}[e, b, \widehat{v}_b],\ v_b \in \widehat{V}_b\}$.

It is clear that $\widehat{\mathcal{E}}_b$ is itself a finite dimensional space. We are going to show that

$$\widehat{\pi}(\mathbf{1}_b)\, J\, \pi(\mathbf{1}_a)(\mathcal{H}^\infty) \subset \widehat{\mathcal{E}}_b. \tag{32}$$

By linearity and modulo the equivalence relation, it is sufficient to prove (32) for functions like $\mu[x, xc, v_c]$ with $|xc| = |x| + 1$.

If $x \notin \Gamma(a)$, we have $\mathbf{1}_{\Gamma(a)}\mu[x, xc, v_c] = 0$, therefore

$$\widehat{\pi}(\mathbf{1}_b)\, J\, \pi(\mathbf{1}_a)(\mu[x, xc, v_c]) = \widehat{\pi}(\mathbf{1}_b)\, J\, (\mathbf{1}_{\Gamma(a)}\mu[x, xc, v_c]) = 0.$$

If $x \in \Gamma(a)$, we have $\mathbf{1}_{\Gamma(a)}\mu[x, xc, v_c] = \mu[x, xc, v_c]$, and, since $a \neq b$,

$$\begin{aligned} & \widehat{\pi}(\mathbf{1}_b)\, J\, \pi(\mathbf{1}_a)(\mu[x, xc, v_c]) = \widehat{\pi}(\mathbf{1}_b)\, J\, (\mu[x, xc, v_c]) \\ = \; & \mathbf{1}_{\Gamma(b)}(\widehat{\mu}[x, xc, -Q_c v_c] + \widehat{\mu}[xc, x, B_c v_c]) = \mathbf{1}_{\Gamma(b)}(\widehat{\mu}[xc, x, B_c v_c]). \end{aligned}$$

It is easy to show by induction on $|x|$ that $x \in \Gamma(a)$ implies, for all $w \in \widehat{V}_{c^{-1}}$, $\mathbf{1}_{\Gamma(b)}(\widehat{\mu}[xc, x, w]) \in \widehat{\mathcal{E}}_b$.

Indeed, if $|x| = 1$, then $x = a$, and since ac is reduced

$$\begin{aligned} & \mathbf{1}_{\Gamma(b)}(\widehat{\mu}[ac, a, w]) = \mathbf{1}_{\Gamma(b)}(\sum_{a' \neq c} \widehat{\mu}[a, aa', \widehat{H}_{a'c^{-1}} w]) \\ = \; & \mathbf{1}_{\Gamma(b)}(\widehat{\mu}[a, e, \widehat{H}_{a^{-1}c^{-1}} w]) = \mathbf{1}_{\Gamma(b)}(\sum_{a' \neq a} \widehat{\mu}[e, a', \widehat{H}_{a'a^{-1}} \widehat{H}_{a^{-1}c^{-1}} w]) \\ = \; & \mathbf{1}_{\Gamma(b)}(\widehat{\mu}[e, b, \widehat{H}_{ba^{-1}} \widehat{H}_{a^{-1}c^{-1}} w]) = \widehat{\mu}[e, b, \widehat{H}_{ba^{-1}} \widehat{H}_{a^{-1}c^{-1}} w] \in \widehat{\mathcal{E}}_b. \end{aligned}$$

Next we suppose the statement is true for $|x| = N-1$ and we consider $|x| = N$, $x = x_1 \ldots x_N$. Repeating twice the previous argument, we have if $|xc| = |x| + 1$, by the induction hypothesis

$$\begin{aligned} & \mathbf{1}_{\Gamma(b)}(\widehat{\mu}[xc, x, w]) = \mathbf{1}_{\Gamma(b)}(\widehat{\mu}[x, xx_N^{-1}, \widehat{H}_{x_N^{-1}c^{-1}}w]) \\ = \; & \mathbf{1}_{\Gamma(b)}(\widehat{\mu}[x_1 \dots x_{N-1}, x_1 \dots x_{N-2}, \widehat{H}_{x_{N-1}^{-1}x_N^{-1}}\widehat{H}_{x_N^{-1}c^{-1}}w]) \in \widehat{\mathcal{E}}_b. \end{aligned}$$

□

We shall now consider the case of inequivalent twin systems:

Theorem 5.4. *Let (V_a, H_{ba}, B_a) be a normalized irreducible system and let $(\widehat{V}_a, \widehat{H}_{ba}, \widehat{B}_a)$ be its twin. Assume that the two systems are not equivalent. Assume moreover that the Γ-representations π and $\widehat{\pi}$ arising from (V_a, H_{ba}, B_a) and $(\widehat{V}_a, \widehat{H}_{ba}, \widehat{B}_a)$ are equivalent. Then π satisfies duplicity. In particular π is irreducible as Γ-representation.*

Proof. Let $\iota_+ = \mathrm{Id} : \mathcal{H} \to \mathcal{H}$, and $\iota_- = \mathrm{Id} \circ J : \mathcal{H} \to \widehat{\mathcal{H}}$ where $J : \mathcal{H} \to \widehat{\mathcal{H}}$ is as in (19). It is clear that ι_+ is perfect and, since J is a bijection, the same is true for ι_- (see Proposition 2.6). Since both ι_- and ι_+ are unitary operators the Finite Trace Condition becomes simply:

$$\|\widehat{\pi}(\mathbf{1}_b)\iota_-\iota_+^*\pi(\mathbf{1}_a)\|_{HS} < +\infty\,,$$

for $a, b \in A$, $a \neq b$. Since $\iota_-\iota_+^*$ is nothing but J the result follows from Lemma 5.3 and the Duplicity Theorem 5.1. □

Let us turn now to equivalent twin systems:

Theorem 5.5. *Let (V_a, H_{ba}, B_a) be a normalized irreducible system and let $(\widehat{V}_a, \widehat{H}_{ba}, \widehat{B}_a)$ be its twin. Assume that the two systems are equivalent. Then π splits into the direct sum of two irreducible Γ-representations both satisfying oddity.*

Proof. Let, as in Remark 4.7,

$$\tilde{J} = \frac{2}{\sqrt{4-c^2}}(J - \frac{ic}{2}\mathcal{K})$$

where J is the intertwining operator defined in 19. It is more convenient to work with

$$\mathcal{J} = \mathcal{K}^{-1}\tilde{J}$$

since its eigenvalues are ± 1. Let $\mathcal{H}_1, \mathcal{H}_2$, be the eigenspaces corresponding to 1 and -1. One has $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. Let π_1 be the restriction of π to $\mathcal{H}_1$. It is obvious that the inclusion map $\iota : \mathcal{H}_1 \hookrightarrow \mathcal{H}$ gives an imperfect realization for π_1. We shall prove that π_1 satisfies the hypothesis of the Oddity Theorem 5.2. For every $a \neq b$ we need to compute

$$\|(\iota^*\pi(\mathbf{1}_a)\iota\iota^*\pi(\mathbf{1}_b)\iota)\|_{HS}\ .$$

Observe that $\iota\iota^*$ is the projection onto $\mathcal{H}_1$, which is equal to $(\mathrm{Id}+\mathcal{J})/2$. Since $\pi(\mathbf{1}_a)\pi(\mathbf{1}_b)=0$ we have

$$\begin{aligned} \pi(\mathbf{1}_a)\iota\iota^*\pi(\mathbf{1}_b)\iota) &= \pi(\mathbf{1}_a)\left(\frac{\mathrm{Id}+\mathcal{J}}{2}\right)\pi(\mathbf{1}_b) = \pi(\mathbf{1}_a)\left(\frac{\mathcal{J}}{2}\right)\pi(\mathbf{1}_b) \\ &= \frac{1}{2}\pi(\mathbf{1}_a)\mathcal{K}^{-1}\tilde{J}\pi(\mathbf{1}_b)\ . \qquad (33) \end{aligned}$$

The operator in the last line (33) is a scalar multiple of

$$\pi(\mathbf{1}_a)\left(\mathcal{K}^{-1}J - \frac{ic}{2}\mathrm{Id}\right)\pi(\mathbf{1}_b) = \pi(\mathbf{1}_a)\left(\mathcal{K}^{-1}J\right)\pi(\mathbf{1}_b).$$

Since $\mathcal{K}$ intertwines π to $\widehat{\pi}$ we have

$$\pi(\mathbf{1}_a)\left(\mathcal{K}^{-1}J\right)\pi(\mathbf{1}_b) = \mathcal{K}^{-1}\widehat{\pi}(\mathbf{1}_a)J\pi(\mathbf{1}_b)\ .$$

By Lemma 5.3, $\widehat{\pi}'(\mathbf{1}_a)J\pi'(\mathbf{1}_b)$ is a finite rank operator, and the same is true for each of the operators appearing at every step, up to the operator $\iota^*\pi'(\mathbf{1}_a)\iota\iota^*\pi'(\mathbf{1}_b)\iota$, so that its Hilbert–Schmidt norm is finite. Apply now Theorem 5.2. □

Remark 5.6. The other representation arising from the eigenspace of -1 of $\mathcal{J}$ also satisfies the hypothesis of the Oddity Theorem: one can consult [10] or can calculate directly the FTC condition since the projection onto the other eigenspace is $(\mathrm{Id}-\mathcal{J})/2$.

Theorem 5.7. *Assume that π_1 and π_2 are multiplicative representations built up from two irreducible inequivalent systems (V_a^1, H_{ba}^1, B_a^1) and (V_a^2, H_{ba}^2, B_a^2). Assume that π_1 and π_2 are equivalent as Γ-representations. Then the two systems are twin.*

Proof. Assume first that at least one of the two systems, let us say (V_a^1, H_{ba}^1, B_a^1), is not equivalent to its twin.

As per Proposition 2.6 each π_i $(i=1,2)$ provides a perfect realization of itself. Assume that $T:\mathcal{H}_1\to\mathcal{H}_2$ intertwines π_1 to π_2.

Define $\iota:\mathcal{H}_1\to\mathcal{H}_1\oplus\mathcal{H}_2$ by $\iota=\frac{1}{\sqrt{2}}(\mathrm{Id}\oplus\mathrm{Id}\,T)$ and $\pi'=\pi_1\oplus\pi_2$.

Let $\mathcal{H}'$ be the closure in $\mathcal{H}_1\oplus\mathcal{H}_2$ of the image of $\pi'(C(\Omega))[\pi'(\mathcal{H}_1)]$.

It follows that ι is a boundary realization for π_1. Moreover $(\pi',\mathcal{H}')$ will be perfect if and only if the operator T, which intertwines π_1 to π_2 intertwines also the two actions of $C(\Omega)$ on $\mathcal{H}_1$ and $\mathcal{H}_2$. Since this is impossible because π_1 and π_2 are inequivalent as $\Gamma\ltimes C(\Omega)$ representations, Proposition 3.2 applied to π_1 ensures that $\mathcal{H}_1$ contains some nonzero vector satisfying (GVB).

On the other hand we know from Theorem 3.5 that in this case there exists a tuple of linear maps $Q_a: V_a^1\to\widehat{V^1}_a$ satisfying (18).

By Theorem 4.2 π_1 and $\widehat{\pi_1}$ are equivalent, by Lemma 5.3 they also satisfy (FTC).

By Theorem 5.1 $(\mathrm{Id}, \mathcal{H}_1)$ and $(\mathrm{Id}\, J, \widehat{\mathcal{H}}_1)$ are the only perfect boundary realizations, hence the latter must be equivalent to $(\mathrm{Id}\, T, \mathcal{H}_2)$.

Assume now that both systems (V_a^1, H_{ba}^1, B_a^1) and (V_a^2, H_{ba}^2, B_a^2) are equivalent to their twin: we shall see that this is impossible.

If (V_a^1, H_{ba}^1, B_a^1) is equivalent to its twin by Theorem 5.5, π_1 splits into the direct sum of two irreducible inequivalent Γ-representations $\pi_1^{\pm}$ and the same is true for π_2. Since π_1 and π_2 are equivalent, one of the addends of π_1, say π_1^+ is equivalent to one of the addends of π_2, say π_2^+. By the Oddity Theorem 5.2 π_1^+ and π_2^+ admit **exactly one** boundary realization, and this implies that (V_a^1, H_{ba}^1, B_a^1) and (V_a^2, H_{ba}^2, B_a^2) are equivalent: a contradiction. □

References

[1] A. Boyer, Ł. Garncarek, Asymptotic Schur orthogonality in hyperbolic groups with application to monotony, Trans. Amer. Math. Soc. **371** 6815–6841 (2019)

[2] A. Boyer, G. Link, Ch. Pittet, Ergodic boundary representations, Ergodic Theory Dynam. Systems **39** no. 8, 2017–2047 (2019)

[3] A. Boyer, L.A. Pinochet, An ergodic theorem for the quasi-regular representation of the free group, Bull. Belg. Math. Soc. Simon Stevin **24** no. 2, 243–255 (2017)

[4] C. Cecchini, A. Figá-Talamanca, Projections of uniqueness for Lp(G). Pacific J. Math. **51**, 3747 (1974)

[5] K. R. Davidson, C^*-algebras by example. Fields Institute Monographs, **6**. American Mathematical Society, Providence, RI, (1996)

[6] L. De Michele, A. Figá-Talamanca, Positive definite functions on free groups. Amer. J. Math. **102** no. 3, 503509 (1980),

[7] A. Figà-Talamanca, A. M. Picardello, *Spherical functions and harmonic analysis on free groups* J. Funct. Anal. **47** 281–304 (1982)

[8] A. Figà-Talamanca, T. Steger, *Harmonic analysis for anisotropic random walks on homogeneous trees*, Mem. Amer. Math. Soc., **531** 1–68 (1994)

[9] U. Haagerup, *An Example of a nonnuclear $C^\star$-algebra which has the metric approximation property*, Invent. Math., **50** 279–293 (1979)

[10] W. Hebish, M. G. Kuhn, and T. Steger, *Free group representations: duplicity on the boundary* Preprint:arXiv:1905.03011.

[11] A. Iozzi, M. G. Kuhn, and T. Steger, *Stability properties of multiplicative representations of the free group* Trans. Amer. Math. Soc. **371** no. 12, 86998731 (2019)

[12] M.G. Kuhn, T. Steger, *Multiplicative functions on free groups and irreducible representations* Pacific J. Math. **169**, no. 2, 311334 (1995).

[13] M. G. Kuhn, S. Saliani, and T. Steger, *Free group representations from vector-valued multiplicative functions, II*, Math. Z. **284** 1137–1162 (2016)

[14] M. G. Kuhn, T. Steger, *More irreducible boundary representations of free groups*, Duke Math. J. **82** 381–436 (1996)

[15] M. G. Kuhn, T. Steger, *Monotony of certain free group representations*, J. Funct. Anal. **179** 1–17 (2001)
[16] M. G. Kuhn, T. Steger, *Free group representations from vector-valued multiplicative functions, I*, Israel J. Math. **144** 317–341 (2004)
[17] W.L. Paschke, *Pure eigenstates for the sum of generators of the free group*, Pacific J. Math. **197** 151–171 (2001)
[18] W.L. Paschke, *Some irreducible free group representations in which a linear combination of the generators has an eigenvalue*, J. Australian Math. Soc. **72** 257–286 (2002)
[19] C. Pensavalle, T. Steger *Restriction problem for anisotropic principal series of free groups* J. Funct. Anal. **140**, no. 1, 122 (1996).
[20] T. Pytlik, R. Szwarc *An analytic family of uniformly bounded representations of free groups* Acta Math. **157**, no. 3-4, 287309 (1986).
[21] M. Takesaki *Theory of operator algebras. II.* Encyclopaedia of Mathematical Sciences, 125. Operator Algebras and Non-commutative Geometry, 6. Springer-Verlag, Berlin, (2003).
[22] H. Yoshizawa, *Some remarks on unitary representations of the free group* Osaka Math. J. **3**, 5563, (1951).

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