ORIENTED PRO-ℓ GROUPS WITH THE BOGOMOLOV-POSITSELSKI PROPERTY

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Abstract. For a prime number ℓ we say that an oriented pro-ℓ group (G,θ) has the Bogomolov-Positselski property if the kernel of the canonical projection on its maximal θ-abelian quotient $\pi^{\mathrm{ab}}_{G,\theta}\colon G \to G(\theta)$ is a free pro-ℓ group contained in the Frattini subgroup of G. We show that oriented pro-ℓ groups of elementary type have the Bogomolov-Positselski property (cf. Theorem 1.2). This shows that Efrat's Elementary Type Conjecture implies a positive answer to Positselski's version of Bogomolov's Conjecture on maximal pro-ℓ Galois groups of a field $\mathbb{K}$ in case that $\mathbb{K}^\times/(\mathbb{K}^\times)^\ell$ is finite. Secondly, it is shown that for an $H^\bullet$-quadratic oriented pro-ℓ group (G,θ) the Bogomolov-Positselski property can be expressed by the injectivity of the transgression map $d_2^{2,1}$ in the Hochschild-Serre spectral sequence (cf. Theorem 1.4).

1. Introduction

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By an *ℓ-oriented profinite group* for a prime number ℓ we understand a profinite group G together with a continuous homomorphism of profinite groups $\theta\colon G \to \mathbb{Z}_\ell^\times$, where $\mathbb{Z}_\ell^\times$ denotes the group of units of the ring of ℓ-adic integers $\mathbb{Z}_\ell$. An ℓ-oriented pro-ℓ group (G,θ) will be simply called an *oriented pro-ℓ group*. For a field $\mathbb{K}$, we denote by $G_{\mathbb{K}} = \mathrm{Gal}(\bar{\mathbb{K}}^{\mathrm{sep}}/\mathbb{K})$ its absolute Galois group, where $\bar{\mathbb{K}}^{\mathrm{sep}}$ denotes a separable closure of $\mathbb{K}$. For any prime number ℓ, $G_{\mathbb{K}}$ carries naturally the cyclotomic ℓ-orientation $\tilde{\theta}_{\mathbb{K},\ell}\colon G_{\mathbb{K}} \to \mathbb{Z}_\ell^\times$ (cf. Example 2.1 and [28, (1.3)]). The following conjecture formulated by L. Positselski in [21, Conjecture 2] was motivated by an earlier conjecture of F. Bogomolov (cf. [2] and [21, Conjecture 1], see also Remark 3.3 below).

Conjecture 1.1. *Let $\mathbb{K}$ be a field containing a primitive ℓ^{th}-root of unity, and also $\sqrt{-1}$ if $\ell = 2$, and set*
$$\sqrt[\ell^\infty]{\mathbb{K}} = \mathbb{K}\left(\sqrt[\ell^n]{a}, a \in \mathbb{K}, n \geq 1\right).$$
Then the maximal pro-ℓ Galois group of $\sqrt[\ell^\infty]{\mathbb{K}}$ is a free pro-ℓ group.

A profinite group G admits a maximal pro-ℓ quotient $G(\ell) = G/O^\ell(G)$, where $O^\ell(G)$ is the closed normal subgroup of G being generated by all pro-q Sylow subgroups for all

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prime numbers $q \neq \ell$. Apart from $\ker(\theta)$, an oriented pro-ℓ group (G,θ) contains the distinguished closed subgroups

$$K_\theta(G) = \mathrm{cl}\left(\left\langle\, h^{-\theta(g)}ghg^{-1} \;\mid\; g\in G, h\in\ker(\theta)\,\right\rangle\right) \tag{1.1}$$

and

$$I_\theta(G) = \mathrm{cl}\left(\left\langle\, h\in\ker(\theta) \;\mid\; \exists k\in\mathbb{N}_0 : h^{\ell^k}\in K_\theta(G)\,\right\rangle\right) \tag{1.2}$$

— the former introduced in [11] — i.e., $I_\theta(G)$ is the closure of the isolator (cf. [13, §66]) of $K_\theta(G)$ in $\ker(\theta)$. An oriented pro-ℓ group (G,θ) is said to be *θ-abelian*, if the subgroup $K_\theta(G)$ is trivial and if $\ker(\theta)$ is a free abelian pro-ℓ group (in this case G is a free abelian-by-cyclic pro-ℓ group for $\ell\neq 2$, cf. Remark 2.2).By definition $K_\theta(G)$ is a closed normal subgroup of G contained in the Frattini subgroup $\Phi(G) = \mathrm{cl}(G^\ell\cdot[G,G])$ of G. Note that

$$[\ker(\theta),\ker(\theta)] \subseteq K_\theta(G) \subseteq \ker(\theta), \tag{1.3}$$

so that the quotient $\ker(\theta)/K_\theta(G)$ is an abelian pro-ℓ group, and $I_\theta(G)/K_\theta(G)$ is its torsion subgroup. In particular, if $\theta\colon G\to\mathbb{Z}_\ell^\times$ is trivial (i.e., θ is identically equal to 1), then $K_\theta(G)$ coincides with the closure of the commutator subgroup of G.

Every oriented pro-ℓ group (G,θ) admits a maximal θ-abelian quotient $(G(\theta),\bar\theta)$, where $G(\theta) = G/I_\theta(G)$ and $\bar\theta\colon G(\theta)\to\mathbb{Z}_\ell^\times$ is the homomorphism induced by θ. Namely, $(G(\theta),\bar\theta)$ is $\bar\theta$-abelian and one has a canonical surjective homomorphism

$$\pi^{\mathrm{ab}}_{G,\theta}\colon (G,\theta)\longrightarrow (G(\theta),\bar\theta)$$

of oriented pro-ℓ groups satisfying the following: for every homomorphism $\psi\colon (G,\theta)\to (A,\theta^\circ)$ of oriented pro-ℓ groups onto a θ°-abelian pro-ℓ group (A,θ°) there exists a unique homomorphism of oriented pro-ℓ groups $\psi^{\mathrm{ab}}_\theta\colon (G(\theta),\bar\theta)\to(A,\theta^\circ)$ such that $\psi = \psi^{\mathrm{ab}}_\theta\circ\pi^{\mathrm{ab}}_{G,\theta}$ (cf. Proposition 2.3).

The hypothesis of Conjecture 1.1 on the primitive ℓ^{th}-roots lying in $\mathbb{K}$ implies that the maximal pro-ℓ quotient $G_{\mathbb{K}}(\ell)$ of the absolute Galois group $G_{\mathbb{K}}$ carries naturally an ℓ-orientation

$$\tilde\theta_{\mathbb{K},\ell}\colon G_{\mathbb{K}}(\ell)\longrightarrow\mathbb{Z}_\ell^\times. \tag{1.4}$$

So, Conjecture 1.1 predicts that $I_{\tilde\theta_{\mathbb{K},\ell}}(G_{\mathbb{K}}(\ell))$ is a free pro-ℓ group contained in the Frattini subgroup $\Phi(G_{\mathbb{K}}(\ell))$ of $G_{\mathbb{K}}(\ell)$ (cf. Proposition 2.6 and § 3.1). At this point it should be mentioned that in fact one has to deal with two properties of oriented pro-ℓ groups. The oriented pro-ℓ group (G,θ) is said to be *Kummerian*, if $I_\theta(G) = \ker(\pi^{\mathrm{ab}}_{G,\theta})$ is contained in the Frattini subgroup $\Phi(G)$ of G. This property can be reformulated in several different ways (cf. Proposition 2.6). Bearing this fact in mind we say that the Kummerian (cf. §2.3) oriented pro-ℓ group (G,θ) has the *Bogomolov-Positselski property*, if $I_\theta(G) = \ker(\pi^{\mathrm{ab}}_{G,\theta})$ is a free pro-ℓ group. E.g., the oriented pro-ℓ group $(G,\mathbf{1})$, where $\mathbf{1}$ is the trivial ℓ-orientation on G, is Kummerian if, and only if, the maximal abelian pro-ℓ quotient $G^{\mathrm{ab}} = G/G'$ is a free abelian pro-ℓ group, and has the Bogomolov-Positselski property if, and only if, it is Kummerian and the closure of the commutator subgroup of G is a free pro-ℓ group.

The class of oriented pro-ℓ groups $\mathbf{ET}_\ell$ of *elementary type* is the smallest class of oriented pro-ℓ groups containing $\mathbb{Z}_\ell$ with all its ℓ-orientations, all Demushkin pro-ℓ

groups with their natural ℓ-orientation (cf. [28, Proposition 5.2]) and which is closed with respect to free products in the category of oriented pro-ℓ groups and fibre products (cf. § 5.3). The Elementary Type Conjecture formulated by Ido Efrat in [8] predicts that for every field $\mathbb{K}$ containing an ℓ^{th}-root of unity (and also $\sqrt{-1}$ if $\ell = 2$) satisfying $|\mathbb{K}^\times/(\mathbb{K}^\times)^\ell| < \infty$ the oriented pro-ℓ group $(G_{\mathbb{K}}(\ell), \tilde{\theta}_{\mathbb{K},\ell})$ must be of elementary type. The first main purpose of this paper is to establish the following theorem relating the Elementary Type Conjecture with Conjecture 1.1.

Theorem 1.2. *Every oriented pro-ℓ group of elementary type has the Bogomolov-Positselski property.*

From Theorem 1.2 one concludes the following (cf. Proposition 5.13):

Corollary 1.3. *Let $\mathbb{K}$ be a field containing a primitive ℓ^{th}-root of 1 (and also $\sqrt{-1}$ if $\ell = 2$), such that the quotient $\mathbb{K}^\times/(\mathbb{K}^\times)^\ell$ is finite. Then Conjecture 1.1 holds true in the following cases:*

(a) $\mathbb{K}$ *is finite;*
(b) $\mathbb{K}$ *is a pseudo algebraically closed (PAC) field, or an extension of relative trascendence degree 1 of a PAC field;*
(c) $\mathbb{K}$ *is an extension of trascendence degree 1 of a local field;*
(d) $\mathbb{K}$ *is ℓ-rigid (for the definition of ℓ-rigid field see* [36, p. 722]*);*
(e) $\mathbb{K}$ *is an algebraic extension of a global field of characteristic not ℓ.*

By the Norm Residue Theorem (cf. [12,35,37,38]), the mod ℓ-Milnor K-ring $K^M_\bullet(\mathbb{K})/\ell$ of a field $\mathbb{K}$ is isomorphic to the cohomology algebra $H^\bullet(G_{\mathbb{K}}(\ell), \mathbb{F}_\ell)$ provided $\ell \neq \text{char}(\mathbb{K})$ and $\mathbb{K}$ contains a primitive ℓ^{th}-root of unity. Moreover, L. Positselski showed in [21, Theorem 1.4] that Conjecture 1.1 is a consequence of a strong Koszulity property of the cohomology algebra $H^\bullet(G_{\mathbb{K}}(\ell), \mathbb{F}_\ell)$.

Our second objective is to establish the following criterion ensuring the Bogomolov-Positselski property of an abstract oriented pro-ℓ group (G, θ). Surprisingly, it only depends on low-dimensional group cohomology, but in a sophisticated way (cf. Theorem 4.5).

Theorem 1.4. *Let (G, θ) be a Kummerian oriented pro-ℓ group with a quadratic $\mathbb{F}_\ell$-cohomology algebra $H^\bullet(G, \mathbb{F}_\ell)$, and let*

$$\mathbf{s}\colon \quad \{1\} \longrightarrow I_\theta(G) \longrightarrow G \longrightarrow G(\theta) \longrightarrow \{1\} \tag{1.5}$$

be the canonical extension of pro-ℓ groups. Then G has the Bogomolov-Positselski property if, and only if, the transgression map

$$d_2^{2,1}\colon H^2(G(\theta), H^1(I_\theta(G), \mathbb{F}_\ell)) \longrightarrow H^4(G(\theta), \mathbb{F}_\ell) \tag{1.6}$$

is injective.

Remark 1.5. As $\mathbf{s}$ is a *Frattini pro-ℓ cover* (i.e., $I_\theta(G)$ is contained in the Frattini subgroup of G, cf. § 3.2), inflation yields an isomorphism $j^1\colon H^1(G(\theta), \mathbb{F}_\ell) \to H^1(G, \mathbb{F}_\ell)$. Since $H^\bullet(G, \mathbb{F}_\ell)$ is quadratic, inflation may also be considered as a surjective homomorphism of graded $\mathbb{F}_\ell$-algebras

$$j^\bullet\colon H^\bullet(G(\theta), \mathbb{F}_\ell) \longrightarrow H^\bullet(G, \mathbb{F}_\ell), \tag{1.7}$$

where the left-side term of (1.7) is the exterior algebra generated by $H^1(G(\theta),\mathbb{F}_\ell)$ (cf. § 4.1). By [21, Theorem 1.4], (G,θ) has the Bogomolov-Positselski property provided $H^\bullet(G,\mathbb{F}_\ell)$ is a Koszul $\mathbb{F}_\ell$-algebra and $\ker(j^\bullet)$ is a Koszul $H^\bullet(G,\mathbb{F}_\ell)$-module (cf. [21, §3.3]). Hence the natural question arising in this context is, whether one can express $\ker(d_2^{2,1})$ in terms of $\mathrm{Ext}^{s,t}_{H^\bullet(G,\mathbb{F}_\ell)}(\mathbb{F}_\ell,\ker(j^\bullet))$, $s\neq t$.

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2. Oriented pro-ℓ groups

For a pro-ℓ group G and a positive integer n, G^n will denote the closed subgroup of G generated by the n-th powers of all elements of G. Moreover, for two elements $g,h\in G$, we set

$$ {}^g h = ghg^{-1}, \qquad \text{and} \qquad [g,h] = {}^g h\cdot h^{-1}, $$

and for two subgroups H_1,H_2 of G, $[H_1,H_2]$ will denote the closed subgroup of G generated by all commutators $[g,h]$ with $g\in H_1$ and $h\in H_2$. In particular, G' will denote the closure of the commutator subgroup of G.

2.1. **ℓ-Orientations of profinite groups.** Let $\mathbb{Z}_\ell$ denote the ring of ℓ-adic integers, and let $\mathbb{Z}_\ell^\times$ denote its group of units. Note that $\mathbb{Z}_\ell^\times$ is a virtual pro-ℓ group, in more detail:

(a) if $\ell\neq 2$ then the Sylow pro-ℓ subgroup of $\mathbb{Z}_\ell^\times$ is $1+\ell\mathbb{Z}_\ell=\{1+\ell\lambda \mid \lambda\in\mathbb{Z}_\ell\}$, which is free pro-$\ell$ cyclic;
(b) if $\ell=2$ then $\mathbb{Z}_2^\times = 1+2\mathbb{Z}_2\simeq \mathbb{Z}/2\times(1+4\mathbb{Z}_2)$, and the factor $1+4\mathbb{Z}_2$ is isomorphic to $\mathbb{Z}_2$.

An oriented pro-ℓ group (G,θ) is a pro-ℓ group G together with a continuous group homomorphism $\theta\colon G\to\mathbb{Z}_\ell^\times$. Moreover, (G,θ) is said to be *torsion-free* if $\ell\neq 2$, or if $\ell=2$ and $\mathrm{im}(\theta)\subseteq 1+4\mathbb{Z}_2$ — observe that in a torsion-free oriented pro-ℓ group (G,θ), G need not be a torsion free pro-ℓ group, e.g., $(\mathbb{Z}/\ell,\mathbf{1})$ is a torsion-free oriented pro-ℓ as $\mathrm{im}(\mathbf{1})=\{1\}$.

Oriented pro-ℓ groups where introduced by I. Efrat in [8] under the name "cyclotomic pro-ℓ pairs". For an oriented pro-ℓ group (G,θ), $\mathbb{Z}_\ell(1)$ will denote the continuous left $\mathbb{Z}_\ell[\![G]\!]$-module which is isomorphic to $\mathbb{Z}_\ell$ as an abelian pro-ℓ group, such that $g\cdot v=\theta(g)\cdot v$ for every $g\in G$ and $v\in\mathbb{Z}_\ell(1)$ (cf. [28, § 1]). Conversely, if a pro-ℓ group G comes endowed with a continuous left $\mathbb{Z}_\ell[\![G]\!]$-module M which is isomorphic to $\mathbb{Z}_\ell$ as an abelian pro-ℓ group, then M induces an orientation $\theta\colon G\to\mathbb{Z}_\ell^\times$ by $\theta(g)\cdot v=g\cdot v$ for every $g\in G$ and $v\in M$, such that $M\simeq\mathbb{Z}_\ell(1)$.

The fundamental examples of oriented pro-ℓ groups arise in Galois theory (cf. [11, § 4]).

Example 2.1. For a field $\mathbb{K}$, let $\bar{\mathbb{K}}^{\mathrm{sep}}$ denote a separable closure of $\mathbb{K}$, and let μ_{ℓ^∞} denote the group of roots of 1 of ℓ-power order lying in $\bar{\mathbb{K}}^{\mathrm{sep}}$. If $\mathbb{K}$ contains a primitive ℓ^{th}-root of unity, then μ_{ℓ^∞} is contained in the maximal pro-ℓ extension $\mathbb{K}(\ell)$ of $\mathbb{K}$.As $\mu_{\ell^\infty}\simeq\mathbb{Z}[\frac{1}{\ell}]/\mathbb{Z}$ and $\mathrm{Aut}(\mathbb{Z}[\frac{1}{\ell}]/\mathbb{Z})$ is isomorphic to $\mathbb{Z}_\ell^\times$, the action of the *maximal pro-ℓ*

Galois group $G_{\mathbb{K}}(\ell) = \mathrm{Gal}(\mathbb{K}(\ell)/\mathbb{K})$ of $\mathbb{K}$ on μ_{ℓ^∞} fixes the primitive ℓ^{th}-roots of unity, and induces the *ℓ-cyclotomic character*

$$\tilde{\theta}_{\mathbb{K},\ell} \colon G_{\mathbb{K}}(\ell) \longrightarrow \mathbb{Z}_\ell^\times.$$

In particular,

$$\sigma(\zeta) = \zeta^{\tilde{\theta}_{\mathbb{K},\ell}(\sigma)} \qquad \text{for all } \sigma \in G_{\mathbb{K}}(\ell), \zeta \in \mu_{\ell^\infty}.$$

Furthermore, one has $\mathrm{im}(\tilde{\theta}_{\mathbb{K},\ell}) = 1 + \ell^f \mathbb{Z}_\ell$ — where f is the positive integer satisfying $|\mu_{\ell^\infty} \cap \mathbb{K}^\times| = \ell^f$ — in case $\mu_{\ell^\infty} \cap \mathbb{K}^\times$ is non-empty and finite, and $\mathrm{im}(\tilde{\theta}_{\mathbb{K},\ell}) = \{1\}$ if $\mu_{\ell^\infty} \subseteq \mathbb{K}^\times$. The continuous $G_{\mathbb{K}}(\ell)$-module $\mathbb{Z}_\ell(1)$ induced by the cyclotomic character is called the 1st Tate twist of $\mathbb{Z}_\ell$ (cf. [20, Def. 7.3.6]), and for every $n \geq 1$, $\mathbb{Z}_\ell(1)/\ell^n$ is isomorphic to the $G_{\mathbb{K}}(\ell)$-module of the ℓ^n-th roots of 1.

Note that oriented pro-ℓ groups form a category $\mathbf{Or}_\ell$, i.e., for $(G,\theta), (H,\theta') \in \mathrm{ob}(\mathbf{Or}_\ell)$ a morphism of oriented pro-ℓ groups $\phi\colon (G,\theta) \to (H,\theta')$ is a continuous group homomorphism $\phi\colon G \to H$ of pro-ℓ groups satisfying $\theta' \circ \phi = \theta$.

For an oriented pro-ℓ group (G,θ) one has the following constructions.

(a) Let N be a normal subgroup of G such that $N \subseteq \ker(\theta)$. Then one has an oriented pro-ℓ group

$$(G,\theta)/N := (G/N, \bar{\theta}),$$

where $\bar{\theta}\colon G/N \to \mathbb{Z}_\ell^\times$ is the orientation induced by θ.

(b) Let A be an abelian pro-ℓ group. Then one has an oriented pro-ℓ group

$$A \rtimes (G,\theta) := (A \rtimes G, \tilde{\theta}),$$

where $gag^{-1} = a^{\theta(g)}$ for all $g \in G$ and $a \in A$, and $\tilde{\theta} = \theta \circ \pi$, where $\pi\colon A \rtimes G \to G$ is the canonical projection.

2.2. **The maximal θ-abelian quotient of an oriented pro-ℓ group.** Let (G,θ) be a torsion-free oriented pro-ℓ group. Then $G/\ker(\theta) \simeq \mathrm{im}(\theta)$ is torsion-free, and thus either trivial or isomorphic to $\mathbb{Z}_\ell$. Therefore, the epimorphism $G \twoheadrightarrow G/\ker(\theta)$ splits, and since $ghg^{-1} \equiv h^{\theta(g)} \bmod K_\theta(G)$ for every $g \in G$ and $h \in \ker(\theta)$, one concludes that

$$(G,\theta)/K_\theta(G) \simeq \frac{\ker(\theta)}{K_\theta(G)} \rtimes (\mathrm{im}(\theta), \mathrm{Id}_{\mathrm{im}(\theta)}). \tag{2.1}$$

Remark 2.2. By (2.1), if (G,θ) is a torsion-free θ-abelian oriented pro-ℓ group, then it is isomorphic to the oriented pro-ℓ group $\ker(\theta) \rtimes (\mathrm{im}(\theta), \mathrm{Id}_{\mathrm{im}(\theta)})$. Conversely, if A is a free abelian pro-ℓ group, and $(\bar{G},\theta)$ is an oriented pro-ℓ group satisfying $\ker(\theta) = \{1\}$, then the oriented pro-ℓ group $(G,\tilde{\theta}) = A \rtimes (\bar{G},\theta)$ is $\tilde{\theta}$-abelian, since $\ker(\tilde{\theta}) = A$ is a free abelian pro-ℓ group, and as $ghg^{-1} = h^{\tilde{\theta}(g)}$ for every $g \in \bar{G}$ and $h \in A$ and thus $K_{\tilde{\theta}}(G) = \{1\}$.

Let (G,θ) be an oriented pro-ℓ group. Put $\bar{G} = G/I_\theta(G)$ and let $\bar{\theta}\colon \bar{G} \to \mathbb{Z}_\ell^\times$ denote the induced orientation. Since the quotient $\ker(\theta)/I_\theta(G)$ is torsion-free (cf. § 1), the oriented pro-ℓ group $(G(\theta), \bar{\theta})$ is $\bar{\theta}$-abelian. This group together with the canonical projection

$$\pi^{\mathrm{ab}}_{G,\theta} \colon G \longrightarrow G(\theta) \tag{2.2}$$

has the following universal property.

Proposition 2.3. *Let (G,θ) be an oriented pro-ℓ group, let (A,θ°) be an oriented θ°-abelian pro-ℓ group, and let $\psi\colon (G,\theta)\to(A,\theta^\circ)$ be a continuous homomorphism of oriented pro-ℓ groups. Then ψ factors through $\pi^{ab}_{G,\theta}$, i.e., there exists a (unique) continuous group homomorphism*

$$\psi^{\mathrm{ab}}_{G,\theta}\colon (G(\theta),\bar{\theta}) \longrightarrow (A,\theta^\circ)$$

satisfying $\psi=\psi^{\mathrm{ab}}_{G,\theta}\circ\pi^{\mathrm{ab}}_{G,\theta}$.

Proof. As ψ is a homomorphism of oriented pro-ℓ groups, and as (A,θ°) is θ°-abelian, one has

$$\psi(\ker(\theta))\subseteq\ker(\theta^\circ)\qquad\text{and}\qquad \psi(K_\theta(G))\subseteq K_{\theta^\circ}(A)=\{1\}. \tag{2.3}$$

As $\ker(\theta^\circ)$ is torsion-free, this implies that $\psi(I_\theta(G))=\{1\}$. Hence the induced homomorphism $\psi^{\mathrm{ab}}_{G,\theta}\colon G(\theta)\to A$ of oriented pro-ℓ groups has the desired properties. □

Remark 2.4. Let $(G,\theta)\simeq A\rtimes((G,\theta)/\ker(\theta))$ be a torsion-free θ-abelian oriented pro-ℓ group. Then for every subgroup H of G one has

$$H\simeq(H\cap A)\rtimes(H/\ker(\theta|_H)),$$

and thus the oriented pro-ℓ group $(H,\theta|_H)$ is split $\theta|_H$-abelian (cf. [28, Remark 3.12]).

2.3. **Kummerian oriented pro-ℓ groups.** Let (G,θ) be an oriented torsion-free pro-ℓ group. Since $\mathrm{im}(\theta)\subseteq 1+\ell\mathbb{Z}_\ell$, the action of G on the quotient $\mathbb{Z}_\ell(1)/\ell$ of the continuous G-module $\mathbb{Z}_\ell(1)$ is trivial, i.e., $\mathbb{Z}_\ell(1)/\ell\simeq\mathbb{F}_\ell$ as a trivial left $\mathbb{Z}_\ell[\![G]\!]$-module. In the proof of the subsequent proposition we will make use of the following

Fact 2.5. *Let A be an abelian pro-ℓ group, and let B be a closed subgroup of A which is a direct summand of A satisfying $B\subseteq A^\ell$. Then $B=\{0\}$.*

Proof. Let $A=B\oplus C$. Then $A^\ell=B^\ell\oplus C^\ell$, and as $B^\ell\subseteq B$, and $B\cap C=\{0\}$ one concludes that $B\subseteq B^\ell$, i.e., $B=B^\ell=\Phi(B)$. Hence $B=\{0\}$. □

A torsion-free oriented pro-ℓ group (G,θ) is said to be *Kummerian* if the following equivalent properties are satisfied.

Proposition 2.6. *Let (G,θ) be a torsion-free oriented pro-ℓ group. Then the following are equivalent:*

(i) *the map $H^1(G,\mathbb{Z}_\ell(1)/\ell^n)\longrightarrow H^1(G,\mathbb{F}_\ell)$ induced by the epimorphism of discrete left G-modules $\mathbb{Z}_\ell(1)/\ell^n\twoheadrightarrow\mathbb{Z}_\ell(1)/\ell\simeq\mathbb{F}_\ell$, is surjective for every $n\geq 1$ (cf. [11]).*
(ii) *The quotient $\ker(\theta)/K_\theta(G)$ is a free abelian pro-ℓ group.*
(iii) *The oriented pro-ℓ group $(G,\theta)/K_\theta(G)=(G/K_\theta(G),\bar{\theta})$ is $\bar{\theta}$-abelian.*
(iv) *$K_\theta(G)$ is isolated in $\ker(\theta)$, i.e., $I_\theta(G)=K_\theta(G)$.*
(v) *The group $H^2_{\mathrm{cts}}(G,\mathbb{Z}_\ell(1))$ is a torsion-free $\mathbb{Z}_\ell$-module.*
(vi) *$I_\theta(G)\subseteq\Phi(G)$.*

(Here H^*_{cts} denotes continuous cochain cohomology as defined by J. Tate in [34]).

Proof. For G finitely generated the equivalences between (i) and (ii) was shown in [11, Thm. 5.6], and the equivalence between (ii) and (iii) follows from Remark 2.2. For general G the equivalences were shown in [26, Thm. 1.2]. The equivalence between (i) and (v) is shown in [28, Prop. 2.1], and (iii) $\Leftrightarrow$ (iv) is a direct consequence of (2.1) and Remark 2.2. Hence (i)–(v) are equivalent. As $K_\theta(G) \subseteq \Phi(G)$ one has (iv) $\Rightarrow$ (vi). Thus it remains to show that (vi) $\Rightarrow$ (iv). Let $\pi\colon G \to G/\Phi(G)$ denote the canonical projection, and let

$$\pi_*\colon \ker(\theta)/K_\theta(G)\cdot\ker(\theta)^\ell \longrightarrow G/\Phi(G) \tag{2.4}$$

denote the induced map — note that $K_\theta(G)\ker(\theta)^\ell = \ker(\theta)^\ell[G,\ker(\theta)]$, by (1.1). As $\mathrm{im}(\theta)$ — which is isomorphic to either $\mathbb{Z}_\ell$ or $\{1\}$ — is projective, the 5-term exact sequence associated to the Hochschild-Serre spectral sequence yields an exact sequence

$$H^1(G,\mathbb{F}_\ell) \xrightarrow{\pi_*^\vee} H^1(\ker(\theta),\mathbb{F}_\ell)^G \longrightarrow \{0\} \tag{2.5}$$

Thus, by Pontrjagin duality, π_* is injective. Note that

$$\mathrm{tor}(\ker(\theta)/K_\theta(G)) = I_\theta(G)/K_\theta(G) \tag{2.6}$$

is a direct summand of the abelian pro-ℓ group $\ker(\theta)/K_\theta(G)$ (cf. § 1). Since $\pi(I_\theta(G)) = \{1\}$ by (vi), and since π_* is injective, one concludes that $I_\theta(G) \subseteq K_\theta(G)\cdot\ker(\theta)^\ell$. Hence $I_\theta(G)/K_\theta(G) = \{1\}$ by Fact 2.5. □

Example 2.7. (a) If (G,θ) is a torsion-free θ-abelian pro-ℓ group, then, by Proposition 2.6–(ii), (G,θ) is Kummerian, as $K_\theta(G) = \{1\}$ and $\ker(\theta)$ is free abelian by definition.

(b) If G is a free pro-ℓ group, then by Proposition 2.6–(v) the oriented pro-ℓ group (G,θ) is Kummerian for any orientation $\theta\colon G \to \mathbb{Z}_\ell^\times$, as $\mathrm{cd}(G) = 1$ (cf. [20, Prop. 3.5.17]).

(c) If (G,θ) is an oriented pro-ℓ group with trivial orientation $\theta \equiv \mathbf{1}$, then (G,θ) is Kummerian if, and only if, the abelianization G^{ab} is a free abelian pro-ℓ group (cf. [11, Example 3.5–(1)]).

The following result is a consequence of Kummer theory (cf. [11, Thm. 4.2]).

Theorem 2.8. *Let $\mathbb{K}$ be a field containing a primitive ℓ^{th}-root of 1 (and also $\sqrt{-1}$ if $\ell = 2$). Then $(G_\mathbb{K}(\ell),\theta_{\mathbb{K},\ell})$ is a torsion-free Kummerian oriented pro-ℓ group.*

From Proposition 2.3 and Proposition 2.6–(iv), one concludes the following fact.

Corollary 2.9. *Let (G,θ) be a Kummerian torsion-free oriented pro-ℓ group. Then $(G/K_\theta(G),\bar\theta)$ is the maximal θ-abelian quotient of G.*

3. The Bogomolov-Positselski property

3.1. **Bogomolov's conjecture.** Let $\mathbb{K}$ be a field containing a primitive ℓ^{th}-root of 1 (and also $\sqrt{-1}$ if $\ell = 2$), and let $\mathbb{L} = \sqrt[\ell^\infty]{\mathbb{K}}$ denote the compositum of all radical extensions $\mathbb{K}(\sqrt[\ell^n]{a})$, with $a \in \mathbb{K}^\times$ and $n \geq 1$, i.e.,

$$\mathbb{L} = \sqrt[\ell^\infty]{\mathbb{K}} = \mathbb{K}\left(\sqrt[\ell^n]{a} \mid a \in \mathbb{K}^\times, n \geq 1\right). \tag{3.1}$$

The maximal pro-ℓ Galois group $G_\mathbb{L}(\ell)$ of the field $\mathbb{L}$ is equal to the pro-ℓ group $K_{\tilde\theta_{\mathbb{K},\ell}}(G_\mathbb{K}(\ell))$ associated to the oriented pro-ℓ group $(G_\mathbb{K}(\ell),\tilde\theta_{\mathbb{K},\ell})$ (cf. [11, Thm. 4.2]). Observe that the ℓ-cyclotomic character associated to the maximal pro-ℓ Galois group of $\mathbb{L}$ is the trivial ℓ-orientation $\mathbf{1}\colon K_{\tilde\theta_{\mathbb{K},\ell}}(G_\mathbb{K}(\ell)) \to \{1\} \subseteq \mathbb{Z}_\ell^\times$.

Motivated by a conjecture formulated by F. Bogomolov in [2] — see Remark 3.3 below —, L. Positselski stated the following conjecture on the pro-ℓ group $G_{\mathbb{L}}(\ell) = K_{\tilde{\theta}_{\mathbb{K},\ell}}(G_{\mathbb{K}}(\ell))$ (cf. [21, Conjecture 1.2]).

Conjecture 3.1. *Let $\mathbb{K}$ be a field containing a primitive ℓ^{th}-root of 1, and also $\sqrt{-1}$ if $\ell = 2$. Then the maximal pro-ℓ Galois group $G_{\mathbb{L}}(\ell)$ of $\mathbb{L} = \sqrt[\ell^\infty]{\mathbb{K}}$ is a free pro-ℓ group.*

Conjecture 3.1 is the motivation for the following definition.

Definition 3.2. A Kummerian oriented pro-ℓ group (G, θ) is said to have the *Bogomolov-Positselski property* if the subgroup $K_\theta(G)$ is a free pro-ℓ group.

Hence, Conjecture 3.1 may be restated as follows: if $\mathbb{K}$ is a field containing a primitive ℓ^{th}-root of 1 (and also $\sqrt{-1}$ if $\ell = 2$), then the oriented pro-ℓ group $(G_{\mathbb{K}}(\ell), \tilde{\theta}_{\mathbb{K},\ell})$ has the Bogomolov-Positselski property.

Remark 3.3. The original formulation of Bogomolov's conjecture states that if $\mathbb{K}$ is a field containing an algebraically closed field then the (closure of the) commutator subgroup of the Sylow pro-ℓ subgroup of the absolute Galois group $G_{\mathbb{K}}$ of $\mathbb{K}$ is a free pro-ℓ group. Furthermore, the (closure of the) commutator subgroup of the maximal pro-ℓ Galois group $G_{\mathbb{K}}(\ell)$ should be a free pro-ℓ group as well (see also [3, Conjecture 6.2] and [19, § 3.1.2], where the conjecture is stated for function fields).
In [21], Positselski observed that the only essential part of the condition about the algebraically closed subfield of $\mathbb{K}$ is that $\mathbb{K}$ should contain all the roots of 1 of ℓ-power order. Consequently, he formulated the following conjecture (cf. [21, Conjecture 1.1]): the pro-ℓ Sylow subgroup of the absolute Galois group $G_{\mathbb{L}}$, with $\mathbb{L} = \sqrt[\ell^\infty]{\mathbb{K}}$ and $\mathbb{K}$ an arbitrary field, is a free pro-ℓ group, i.e., $\mathrm{cd}_\ell(G_{\mathbb{L}}) \leq 1$ (or, equivalently, $G_{\mathbb{L}}$ is ℓ-projective, cf. [32, §I.3.4, Proposition 16]). Note that this conjecture is stronger than Conjecture 3.1, and likely hard to approach, while — as stated by Positselski himself, cf. [21, § 1.3] — the latter is closer to Bogomolov's original conjecture.

Example 3.4. (a) Let (G, θ) be a torsion-free θ-abelian oriented pro-ℓ group. Then (G, θ) is Kummerian (cf. Example 2.7–(a)), and by Proposition 2.6–(iv) one has

$$I_\theta(G) = K_\theta(G) = \{1\}.$$

So, (G, θ) has the Bogomolov-Positselski property.

(b) Let (G, θ) be an oriented pro-ℓ group with G being a free pro-ℓ group. Then (G, θ) is Kummerian by Example 2.7–(b), and it has the Bogomolov-Positselski property as every closed subgroup of G is again a free pro-ℓ group.

(c) Let

$$\begin{aligned} G &= \langle\, x, y, z \mid [x, y] = z, [x, z] = [y, z] = 1 \,\rangle \\ &= \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{Z}_\ell \right\} \end{aligned}$$

be the *Heisenberg group over $\mathbb{Z}_\ell$*, and set $(G, \mathbf{1})$, where $\mathbf{1}\colon G \to \mathbb{Z}_\ell^\times$ is the trivial orientation. Then $K_\theta(G) = G' \simeq \mathbb{Z}_\ell$ is the cyclic pro-ℓ subgroup generated by z, and $G^{\text{ab}} \simeq \mathbb{Z}_\ell^2$. Hence (G, θ) is Kummerian by Example 2.7–(c), and it has the Bogomolov-Positselski property. Nevertheless, G does not occur as the maximal pro-ℓ Galois group of any field containing μ_{ℓ^∞} (cf. [25, Ex. 5.4]).

3.2. **Self-isolated pro-ℓ groups and Frattini pro-ℓ covers.** Let G be a pro-ℓ group, and let $H \subseteq G$ be a subgroup. The *isolator of* H is the subgroup

$$\mathrm{Iso}(H) = \mathrm{cl}\left(\langle\, g \in G \;\mid\; g^n \in H \text{ for some } n \geq 1 \,\rangle\right)$$

(cf. [13, § 66]). We say that H is *self-isolated* if $\mathrm{Iso}(H) = H$. In particular, if N is a normal subgroup of G, then G is self-isolated if, and only if, the quotient G/N is a torsion-free pro-ℓ group. The following fact is almost straightforward.

Fact 3.5. *Let (G, θ) be a torsion-free θ-abelian oriented pro-ℓ group. Let N be a normal subgroup of G contained in both $\ker(\theta)$ and $\Phi(G)$. If N is self-isolated, then $N = \{1\}$.*

Proof. By Remark 2.2, $\Phi(G) \cap \ker(\theta) = \ker(\theta)^{\ell}$. As $N \subseteq \ker(\theta)$ is an isolated subgroup, it is a direct summand of $\ker(\theta)$. Thus by Fact 2.5, N is trivial. □

Fact 3.5 has the following consequence.

Proposition 3.6. *Let (G, θ) be a torsion-free Kummerian oriented pro-ℓ group. Let $N \trianglelefteq G$ be a closed normal, self-isolated, subgroup of G contained in $\ker(\theta)$ satisfying*

$$K_\theta(G) \subseteq N \subseteq \Phi(G).$$

Then $N = K_\theta(G)$.

A *Frattini pro-ℓ cover* of pro-ℓ groups is a short exact sequence of pro-ℓ groups

$$\{1\} \longrightarrow N \longrightarrow G \xrightarrow{\;\tau\;} \bar{G} \longrightarrow \{1\} \tag{3.2}$$

satisfying $N \subseteq \Phi(G)$. One also says that $\tau\colon G \to \bar{G}$ is a Frattini pro-ℓ cover of $\bar{G}$. One may characterize those pro-ℓ groups which may be completed into Kummerian oriented pro-ℓ groups with the Bogomolov-Positselski property as follows.

Theorem 3.7. *A pro-ℓ group G may be completed into a Kummerian oriented pro-ℓ group (G, θ) with the Bogomolov-Positselski property if, and only if, G is a Frattini pro-ℓ cover (3.2) of $\bar{G}$, where $(\bar{G}, \bar{\theta})$ is a $\bar{\theta}$-abelian oriented pro-ℓ group and N is a free pro-ℓ group.*

Proof. If (G, θ) is Kummerian with the Bogomolov-Positselski property, then, by Proposition 2.6, $(G/K_\theta(G), \bar{\theta}) = (G, \theta)/K_\theta(G)$ is $\bar{\theta}$-abelian and $N = K_\theta(G)$ is a free pro-ℓ group by Definition 3.2. This shows one implication.

Conversely, if $(\bar{G}, \bar{\theta})$ is $\bar{\theta}$-abelian, then the epimorphism of oriented pro-ℓ groups $(G, \theta) \to (\bar{G}, \bar{\theta})$ factors through $(G, \theta)/I_\theta(G)$ by Proposition 2.3. Hence $I_\theta(G) \subseteq N$, while $N \subseteq \Phi(G)$ by hypothesis, thus (G, θ) is Kummerian by Proposition 2.6:(vi). Thus, $I_\theta(G) = K_\theta(G)$ by Proposition 2.6:(iv), and since

$$K_\theta(G) = I_\theta(G) \subseteq N \subseteq \Phi(G),$$

Proposition 3.6 yields $N = K_\theta(G)$, i.e., (G, θ) has the Bogomolov-Positselski property. □

4. The Bogomolov-Positselski property and cohomology

4.1. **Quadratic cohomology and the Norm Residue Theorem.** Let G be a pro-ℓ group. The cohomology groups $H^n(G,\mathbb{F}_\ell)$, $n \geq 1$, where $\mathbb{F}_\ell$ is the trivial G-module isomorphic — as abelian group — to $\mathbb{F}_\ell = \mathbb{Z}/\ell\mathbb{Z}$, come endowed with the bilinear *cup-product*

$$H^s(G,\mathbb{F}_\ell) \times H^t(G,\mathbb{F}_\ell) \xrightarrow{\cup} H^{s+t}(G,\mathbb{F}_\ell), \qquad s,t \geq 0,$$

which is associative and graded-commutative, i.e., $\beta\cup\alpha = (-1)^{st}\alpha\cup\beta$ for $\alpha \in H^s(G,\mathbb{F}_\ell)$ and $\beta \in H^t(G,\mathbb{F}_\ell)$ (cf. [20, Ch. I, § 4]). Thus,

$$H^\bullet(G,\mathbb{F}_\ell) = \coprod_{n\geq 0} H^n(G,\mathbb{F}_\ell)$$

is a connected $\mathbb{N}_0$-graded, graded-commutative, associative $\mathbb{F}_\ell$-algebra.

For an $\mathbb{F}_\ell$-vector space V, let $\mathbf{T}^\bullet V$ denote the $\mathbb{F}_\ell$-tensor algebra, i.e.,

$$\mathbf{T}^\bullet V = \coprod_{n\in\mathbb{N}_0} \mathbf{T}^n V \qquad \text{where} \qquad \mathbf{T}^n V = V^{\otimes^n}. \tag{4.1}$$

The $\mathbb{N}_0$-graded associative $\mathbb{F}_\ell$-algebra $\mathbf{A}_\bullet$ is said to be generated in degree 1, if the canonical homomorphism $\phi_\bullet\colon \mathbf{T}^\bullet\mathbf{A}_1 \to \mathbf{A}_\bullet$ of $\mathbb{N}_0$-graded associative $\mathbb{F}_\ell$-algebras is surjective. Moreover, $\mathbf{A}_\bullet$ is said to be quadratic, if it is 1-generated and $\ker(\phi_\bullet) = \langle\ker(\phi_2)\rangle$, i.e., the ideal $\ker(\phi_\bullet)$ is generated in degree 2.

Definition 4.1. A pro-ℓ group G is said to be *$H^\bullet$-quadratic* if $H^\bullet(G,\mathbb{F}_\ell)$ is a quadratic algebra.

For an $\mathbb{F}_\ell$-vector space V, let $\mathbf{\Lambda}^\bullet V = \mathbf{T}^\bullet V/\langle\, v\otimes v \mid v \in V\,\rangle$ denote the *exterior $\mathbb{F}_\ell$-algebra* spanned by V, and $\mathbf{S}^\bullet V = \mathbf{T}^\bullet V/\langle\, v\otimes w - w\otimes v \mid v,w \in V\,\rangle$ denote the *symmetric $\mathbb{F}_\ell$-algebra* spanned by V. Then G is $H^\bullet$-quadratic if the cup-product induces an isomorphism of graded $\mathbb{F}_\ell$-algebras

$$\mathbf{\Xi}^\bullet H^1(G,\mathbb{F}_\ell)/\langle\, W\,\rangle \xrightarrow{\sim} H^\bullet(G,\mathbb{F}_\ell), \tag{4.2}$$

where $\mathbf{\Xi}^\bullet = \mathbf{\Lambda}^\bullet$ if ℓ is odd, and $\mathbf{\Xi}^\bullet = \mathbf{S}^\bullet$ if $\ell = 2$. Moreover,

$$W = \ker\left(\Xi^2(H^1(G,\mathbb{F}_\ell)) \xrightarrow{\cup} H^2(G,\mathbb{F}_\ell)\right). \tag{4.3}$$

By the Norm Residue Theorem, if the field $\mathbb{K}$ contains a primitive ℓ^{th}-root of unity, then the maximal pro-ℓ Galois group $G_{\mathbb{K}}(\ell)$ is $H^\bullet$-quadratic (cf. [23] or [28]).

Remark 4.2. Let $\ell = 2$ and let G be a pro-2 group. Then one has $\alpha\cup\alpha = 0$ for every $\alpha \in H^1(G,\mathbb{F}_2)$ if, and only if, the map

$$H^1(G,\mathbb{Z}/4) \longrightarrow H^1(G,\mathbb{F}_2),$$

induced by the epimorphism of trivial G-modules $\mathbb{Z}/4 \to \mathbb{F}_2$, is surjective (cf. [28, Fact 7.1]). In particular, if (G,θ) is a torsion-free Kummerian oriented pro-2 group, one concludes that $\alpha\cup\alpha = 0$ for all $\alpha \in H^1(G,\mathbb{F}_2)$. This is the case for $(G_{\mathbb{K}}(2),\tilde{\theta}_{\mathbb{K},2})$, with $\mathbb{K}$ a field containing $\sqrt{-1}$, i.e., $H^\bullet(G_{\mathbb{K}}(2),\mathbb{F}_2)$ is quadratic and also a quotient of the exterior algebra $\mathbf{\Lambda}^\bullet H^1(G_{\mathbb{K}}(2),\mathbb{F}_2)$.

Example 4.3. Let (G,θ) be torsion-free θ-abelian oriented pro-ℓ group. Then G is a torsion free powerful pro-ℓ group (cf. [5, Ch. 4, § 1]), and

$$G \simeq \varprojlim_{i\in I} A_i \rtimes \mathrm{im}(\theta)$$

for finitely generated free abelian pro-ℓ groups A_i. Thus by M. Lazard's theorem (cf. [15]) one has $\mathbf{\Lambda}_\bullet H^1(G,\mathbb{F}_\ell)$ (see, e.g., [28, Thm. 3.13]), and hence G is $H^\bullet$-quadratic.

4.2. **Quadratic cohomology and the Bogomolov-Positselski property.** Let (G,θ) be a torsion-free Kummerian oriented pro-ℓ group. The short exact sequence of pro-ℓ groups

$$\{1\} \longrightarrow I_\theta(G) \longrightarrow G \longrightarrow G(\theta) \longrightarrow \{1\} \tag{4.4}$$

induces the 5-terms exact sequence in cohomology

$$0 \longrightarrow H^1(G(\theta),\mathbb{F}_\ell) \xrightarrow{\mathrm{inf}^1_{G(\theta),G}} H^1(G,\mathbb{F}_\ell) \xrightarrow{\mathrm{res}^1_{G,I_\theta(G)}} H^1(I_\theta(G),\mathbb{F}_\ell)^{G(\theta)} \xrightarrow{d_2^{0,1}} H^2(G(\theta),\mathbb{F}_\ell) \xrightarrow{\mathrm{inf}^2_{G(\theta),G}} H^2(G,\mathbb{F}_\ell) \tag{4.5}$$

(cf. [20, Prop. 1.6.7]). As (G,θ) is Kummerian, one has $I_\theta(G){=}\ K_\theta(G) \subseteq \Phi(G)$ (cf. Proposition 2.6(iv)). Hence $\mathrm{inf}^1_{G(\theta),G}$ is an isomorphism and $\mathrm{res}^1_{G,I_\theta(G)}$ is the 0-map. As $(G(\theta),\bar\theta) = (G,\theta)/I_\theta(G)$ is $\bar\theta$-abelian, one has

$$H^\bullet(G(\theta),\mathbb{F}_\ell) \simeq \mathbf{\Lambda}^\bullet H^1(G,\mathbb{F}_\ell) \tag{4.6}$$

(cf. Example 4.3). If in addition G is $H^\bullet$-quadratic, then $H^\bullet(G,\mathbb{F}_\ell)$ is a quotient of $\mathbf{\Lambda}^\bullet H^1(G,\mathbb{F}_\ell)$ (cf. Remark 4.2). In particular, the inflation map $\psi^\bullet = \mathrm{inf}^\bullet_{G(\theta),G}$ induces a surjective homomorphism of $\mathbb{N}_0$-graded $\mathbb{F}_\ell$-algebras

$$H^\bullet(G(\theta),\mathbb{F}_\ell) \simeq \Lambda^\bullet H^1(G,\mathbb{F}_\ell) \xrightarrow{\psi_\bullet} H^\bullet(G,\mathbb{F}_\ell) \tag{4.7}$$

satisfying

$$\ker(\psi_n) \simeq \ker(\psi_2) \wedge \left(\Lambda^{n-2} H^1(G,\mathbb{F}_\ell)\right) \quad \text{for all } n \geq 2. \tag{4.8}$$

Since $\mathrm{res}^1_{G,K_\theta(G)}$ is trivial, one concludes from (4.5) that $d_2^{0,1}$ is injective, $\mathrm{im}(d_2^{0,1}) = \ker(\psi_2)$, and $H^2(G,\mathbb{F}_\ell) \simeq H^2(G(\theta),\mathbb{F}_\ell)/\mathrm{im}(d_2^{0,1})$. Thus, as $H^\bullet(G,\mathbb{F}_\ell)$ is quadratic, one has

$$H^\bullet(G,\mathbb{F}_\ell) \simeq H^\bullet(G(\theta),\mathbb{F}_\ell)/\langle\, \mathrm{im}(d_2^{0,1})\,\rangle. \tag{4.9}$$

4.3. **A cohomological criterion.** Let (G,θ) be a Kummerian torsion-free oriented pro-ℓ group which is $H^\bullet$-quadratic. Let $(E_r^{s,t}, d_r^{s,t})$ denote the Hochschild-Serre spectral sequence with coefficients in $\mathbb{F}_\ell$ associated to the short exact sequence (4.4), i.e.,

$$E_2^{s,t} = H^s(G(\theta), H^t(I_\theta(G),\mathbb{F}_\ell)) \Longrightarrow E_\infty^{s,t}, \qquad s,t \geq 0, \tag{4.10}$$

with differentials $d_r^{s,t}\colon E_r^{s,t} \to E_r^{s+r,t-r+1}$ satisfying $d_r \circ d_r = 0$ (cf. [20, Ch. II, § 4]). In particular, by (4.7) one has $E_2^{\bullet,0} \simeq \Lambda^\bullet H^1(G,\mathbb{F}_\ell)$. For $\alpha \in E_2^{s,0} = H^s(G(\theta),\mathbb{F}_\ell)$, $s \geq 0$, and $\beta \in E_2^{0,1} = H^1(I_\theta(G),\mathbb{F}_\ell)^{G(\theta)}$, one has

$$\begin{aligned} &\alpha \cup \beta \in H^s(G(\theta), \mathbb{F}_\ell \otimes H^1(I_\theta(G),\mathbb{F}_\ell)) = E_2^{s,1}, \\ &d_2^{s,1}(\alpha\cup\beta) = (-1)^{s+1}\alpha \cup d_2^{0,1}(\beta) \in E_2^{s+2,0} \end{aligned} \tag{4.11}$$

(cf. [20, Ch. II, Ex. 4.5]).

Proposition 4.4. *Let (G,θ) be a torsion-free Kummerian oriented pro-ℓ group with G being $H^\bullet$-quadratic. Then*

(i) *$E_\infty^{s,t}$ is concentrated on the 0th line, i.e., $E_\infty^{s,t} = 0$ for every $s \geq 0$ and $t \geq 1$;*
(ii) *$E_3^{s,0} \simeq E_\infty^{s,0} \simeq H^s(G,\mathbb{F}_\ell)$ for every $s \geq 0$.*

Proof. Since (G,θ) is Kummerian, by (4.6) one has $E_2^{\bullet,0} \simeq \Lambda^\bullet H^1(G,\mathbb{F}_\ell)$. For every $t \geq 0$ there exists a descending separating filtration $(F^kH^t(G,\mathbb{F}_\ell))_{0\leq k\leq t}$ satisfying $F^0H^t(G,\mathbb{F}_\ell) = H^t(G,\mathbb{F}_\ell)$ and

$$F^sH^{s+t}(G,\mathbb{F}_\ell)/F^{s+1}H^{s+t}(G,\mathbb{F}_\ell) \simeq E_\infty^{s,t} \tag{4.12}$$

where $F^{s+t+1}H^{s+t}(G,\mathbb{F}_\ell) = \{0\}$ (cf. [1, p. 99]). By [20, Ch. II, § 4, Ex. 1], the composite of the maps

$$E_2^{s,0} = H^s(G(\theta),\mathbb{F}_p) \twoheadrightarrow E_3^{s,0} \twoheadrightarrow \cdots \twoheadrightarrow E_\infty^{s,0} \rightarrowtail H^s(G,\mathbb{F}_p)$$

is the *s-th left edge morphism* (cf. [20, p. 99]) and hence coincides with the inflation map $\inf^s_{G(\theta),G}$, which is surjective by (4.7). Thus $F^0H^t(G,\mathbb{F}_\ell) = H^t(G,\mathbb{F}_\ell)$ for all $t \geq 0$, i.e., $E_\infty^{\bullet,0} \simeq H^\bullet(G,\mathbb{F}_p)$, and consequently $E_\infty^{k,t} = 0$ for every $1 \leq k \leq t$. This shows (i).

By (4.11), one has canonical homomorphisms of $\mathbb{N}_0$-graded $\mathbb{F}_\ell$-algebras

$$\begin{aligned} \sigma^\bullet &\colon H^\bullet(G(\theta),\mathbb{F}_\ell)/\langle\, \mathrm{im}(d_2^{0,1})\,\rangle \longrightarrow E_3^{\bullet,0}, \\ \tau^\bullet &\colon E_3^{\bullet,0} \longrightarrow H^\bullet(G,\mathbb{F}_\ell). \end{aligned} \tag{4.13}$$

Moreover, $\sigma^\bullet$ and $\tau^\bullet$ are surjective, σ^k and τ^k are isomorphisms for $k \in \{0,1,2\}$, and their composition is an isomorphism of quadratic $\mathbb{F}_\ell$-algebras by (4.9). Thus $\sigma^\bullet$ and $\tau^\bullet$ are isomorphisms which shows (ii). □

Let (G,θ) be a Kummerian torsion-free oriented pro-ℓ group, and put $K_\theta(G)^{\mathrm{ab}} = K_\theta(G)/K_\theta(G)'$. Recall that if $\mathbb{K}$ is a field containing a primitive ℓ^{th}-root of 1 and $(G,\theta) = (G_{\mathbb{K}}(\ell),\tilde{\theta}_{\mathbb{K},\ell})$, then $K_\theta(G)^{\mathrm{ab}}$ is a free abelian pro-ℓ group, as the oriented pro-ℓ group $(K_\theta(G),\theta|_{K_\theta(G)})$ is again Kummerian, and since $\theta|_{K_\theta(G)}$ is trivial. The short exact sequence of pro-ℓ groups

$$\{1\} \longrightarrow K_\theta(G)^{\mathrm{ab}} \xrightarrow{\ \iota\ } G/K_\theta(G)' \xrightarrow{\ \pi\ } G(\theta) \longrightarrow \{1\}\ , \tag{4.14}$$

where $G(\theta) = G/K_\theta(G)$, defines a cohomology class $u \in H^2_{\mathrm{cts}}(G(\theta),K_\theta(G)^{\mathrm{ab}})$ (cf. [20, p. 143]), where $K_\theta(G)^{\mathrm{ab}}$ is considered as a topological left $\mathbb{Z}_\ell[\![G(\theta)]\!]$-module and H^*_{cts} denotes continuous cochain cohomology (cf. [20, Ch. II, § 7]). Since $[G,K_\theta(G)] \subseteq \Phi(G)$, one has

$$\mathrm{Hom}(K_\theta(G),\mathbb{F}_\ell) = \mathrm{Hom}(K_\theta(G),\mathbb{F}_\ell)^{G(\theta)} = E_2^{0,1}.$$

Thus, the pairing

$$K_\theta(G)^{\mathrm{ab}} \times E_2^{0,1} \longrightarrow \mathbb{F}_\ell,$$
$$(hK_\theta(G)', \beta) \longmapsto \beta(h), \qquad \text{for } h \in K_\theta(G),$$

induces a map

$$\phi_u \colon E_2^{2,1} = H^2(G(\theta), \mathrm{Hom}(K_\theta(G), \mathbb{F}_\ell)) \longrightarrow E_2^{4,0} = H^4(G(\theta), \mathbb{F}_\ell) \tag{4.15}$$

given by $\phi_u(\alpha) = u \cup \alpha$ (cf. [20, p. 114]).

Theorem 4.5. *Let (G, θ) be a Kummerian torsion-free oriented pro-ℓ group with G an $H^\bullet$-quadratic pro-ℓ group. Then the following are equivalent.*

(i) *(G, θ) has the Bogomolov-Positselski property;*
(ii) *the differential map $d_2^{2,1} \colon E_2^{2,1} \to E_2^{4,0} \simeq \Lambda^4 H^1(G(\theta), \mathbb{F}_\ell)$ is injective;*
(iii) *the map ϕ_u is injective, i.e., $u \cup \alpha \neq 0$ for every non-trivial $\alpha \in E_2^{2,1}$.*

If these conditions hold, then the spectral sequence $E_2^{s,t} \Rightarrow E_\infty^{s,t}$ collapses at the E_3-page, i.e., $E_3 = E_\infty$.

Proof. By Proposition 4.4(ii), for every $s \geq 0$ one has $E_3^{s,0} \simeq E_4^{s,0} \simeq \ldots \simeq E_\infty^{s,0}$. Since, by definition, $E_4^{s,0} = E_3^{s,0}/\mathrm{im}(d_3^{s-3,2})$, one concludes that the maps

$$d_3^{s-3,2} \colon E_3^{s-3,2} \longrightarrow E_3^{s,0} \simeq H^s(G, \mathbb{F}_\ell)$$

must be the 0-maps for every $s \geq 3$. In particular, $E_4^{0,2} = \ker(d_3^{0,2})$ is equal to $E_3^{0,2}$, which is $\ker(d_2^{0,2})$ by definition. As $E_r^{s,t}$ is a first-quadrant spectral sequence, one has $E_{r+1}^{0,2} = \ker(d_r^{0,2})$ and the map $d_r^{0,2} \colon E_r^{0,2} \to E_r^{r,3-r} = 0$ is the 0-map for every $r \geq 4$. This implies that $E_3^{0,2} = E_4^{0,2} = \ldots = E_\infty^{0,2}$. Thus, applying Proposition 4.4(i), yields

$$0 = E_\infty^{0,2} = E_3^{0,2} = \ker(d_2^{0,2}), \tag{4.16}$$

i.e., $d_2^{0,2} \colon E_2^{0,2} \to E_2^{2,1}$ is injective.

Moreover, one has $E_3^{2,1} = E_4^{2,1} = E_\infty^{2,1}$, as $E_{r+1}^{2,1} = \ker(d_r^{2,1})/\mathrm{im}(d_r^{2-r,r})$ and both maps

$$d_r^{2,1} \colon E_r^{2,1} \longrightarrow E_r^{2+r,2-r} = 0 \qquad \text{and} \qquad d_r^{2-r,r} \colon E_r^{2-r,r} = 0 \longrightarrow E_r^{2,1}$$

are the 0-maps for every $r \geq 3$. Applying Proposition 4.4(i) again yields

$$0 = E_\infty^{2,1} = E_3^{2,1} = \ker(d_2^{2,1})/\mathrm{im}(d_2^{0,2}), \tag{4.17}$$

i.e., $\ker(d_2^{2,1}) = \mathrm{im}(d_2^{0,2})$.

Thus, if (G, θ) has the Bogomolov-Positselski property, then $I_\theta(G) = K_\theta(G)$ is a free pro-ℓ group. Then $H^t(K_\theta(G), \mathbb{F}_\ell) = 0$ for every $t \geq 2$ (cf. [20, Prop. 3.5.17]), and thus $E_r^{0,t} = 0$ for all $r \geq 2$ and $t \geq 2$. In particular, the map $d_2^{0,2} \colon H^2(I_\theta(G), \mathbb{F}_\ell)^{G(\theta)} \to E_2^{2,1}$ is trivial, and hence by (4.17), one has $\ker(d_2^{2,1}) = 0$. This proves the implication (i)⇒(ii).

Conversely, if $d_2^{2,1}$ is injective, then, by (4.17), one has $\mathrm{im}(d_2^{0,2}) = \ker(d_2^{2,1}) = 0$. Since $d_2^{0,2}$ is injective by (4.16), this implies that $E_2^{0,2} = H^2(I_\theta(G), \mathbb{F}_\ell)^{G(\theta)} = 0$. Since G is a pro-ℓ group, the equality $H^2(I_\theta(G), \mathbb{F}_\ell)^{G(\theta)} = 0$ implies that $H^2(I_\theta(G), \mathbb{F}_\ell) = 0$, and thus $I_\theta(G)$ is free by [20, Prop. 3.5.17]. This proves the implication (ii)⇒(i). The equivalence between (ii) and (iii) follows from [20, Thm. 2.4.4].

Finally, if $I_\theta(G)$ is a free pro-ℓ group, one has $E_r^{s,t} = 0$ for all $s \geq 0$, $t \geq 2$, and $r \geq 2$. Hence, all maps $d_3^{s,t}$ are trivial, for all $s, t \geq 0$, so that $E_3^{s,t} = E_\infty^{s,t}$. □

Question 4.6. Let (G,θ) be a Kummerian torsion-free pro-ℓ group with G being an $H^\bullet$-quadratic pro-ℓ group, and let $(E_r^{s,t}, d_r^{s,t})$ be the Hochschild-Serre spectral sequence associated to (4.4). By Proposition 4.4, for every $s \geq 0$ one has $E_3^{s,0} \simeq E_\infty^{s,0}$, and $E_\infty^{s,t} = 0$ for $s \geq 0$ and $t \geq 1$. Moreover, by Theorem 4.5, if (G,θ) has the Bogomolov-Positselski property, then

$$E_3^{s,t} \simeq E_\infty^{s,t} \qquad \text{for every } s,t \geq 0, \tag{4.18}$$

i.e., $E_r^{s,t}$ collapses at the E_3-page. It would be interesting to understand whether (4.18) implies the Bogomolov-Positselski property for (G,θ). We suspect that the answer should be affirmative. However, we could not find any evidence for this speculation.

Remark 4.7. Let $\mathbb{K}$ be a field containing a primitive ℓ^{th}-root of unity (and also $\sqrt{-1}$ if $\ell = 2$), put $\mathbb{L} = \sqrt[\ell^\infty]{\mathbb{K}}$ and consider the torsion-free Kummerian oriented pro-ℓ group $(G_{\mathbb{K}}(\ell), \tilde{\theta}_{\mathbb{K},\ell})$. The oriented pro-$\ell$ group $(I_{\tilde{\theta}_{\mathbb{K},\ell}}(G_{\mathbb{K}}(\ell)), \mathbf{1})$ is again Kummerian and torsion free, and thus one has

$$I_{\tilde{\theta}_{\mathbb{K},\ell}}(G_{\mathbb{K}}(\ell))' = K_{\mathbf{1}}(G_{\mathbb{L}}(\ell)) = G_{\sqrt[\ell^\infty]{\mathbb{L}}}(\ell) \tag{4.19}$$

$$K_{\tilde{\theta}_{\mathbb{K},\ell}}(G_{\mathbb{K}})^{\text{ab}} = G_{\mathbb{L}}(\ell)^{\text{ab}} = \text{Gal}(\sqrt[\ell^\infty]{\mathbb{L}}/\mathbb{L}), \tag{4.20}$$

where the latter is a free abelian pro-ℓ group (cf. Example 2.7–(c)). Hence, the short exact sequence (4.14) translates into

$$\{1\} \longrightarrow \text{Gal}(\sqrt[\ell^\infty]{\mathbb{L}}/\mathbb{L}) \xrightarrow{\ \iota\ } \text{Gal}(\sqrt[\ell^\infty]{\mathbb{L}}/\mathbb{K}) \xrightarrow{\ \pi\ } \text{Gal}(\mathbb{L}/\mathbb{K}) \longrightarrow \{1\}\ . \tag{4.21}$$

Recall that by Kummer theory one has an isomorphism of (discrete) ℓ-elementary abelian groups $H^1(\text{Gal}(\sqrt[\ell^\infty]{\mathbb{L}}/\mathbb{L}), \mathbb{F}_\ell) \simeq \mathbb{L}^\times/(\mathbb{L}^\times)^\ell$, where $\mathbb{L}^\times = \mathbb{L} \smallsetminus \{0\}$ denotes the multiplicative group of the field $\mathbb{L}$. Then by Theorem 4.5 the cohomology element $u \in H^2_{\text{cts}}(\text{Gal}(\mathbb{L}/\mathbb{K}), \text{Gal}(\sqrt[p^\infty]{\mathbb{L}}/\mathbb{L}))$ associated to the extension of pro-ℓ groups (4.21) induces a homomorphism

$$\phi_{u,\mathbb{L}} \colon H^2\left(\text{Gal}(\mathbb{L}/\mathbb{K}), \mathbb{L}^\times/(\mathbb{L}^\times)^\ell\right) \longrightarrow H^4(\text{Gal}(\mathbb{L}/\mathbb{K}), \mathbb{F}_\ell)$$

which is injective if, and only if, $\mathbb{L}$ satisfies Conjecture 3.1. In view of Theorem 4.5, the knowledge of the structure of $\mathbb{L}^\times/(\mathbb{L}^\times)^\ell$ as continuous $\text{Gal}(\mathbb{L}/\mathbb{K})$-module, or an arithmetic interpretation of the map $\phi_{u,\mathbb{L}}$, may contribute to the solution of Conjecture 3.1.

5. Oriented pro-ℓ groups of elementary type

5.1. **Demushkin groups and one-relator pro-ℓ groups.** A *Demushkin group* is a Poincaré duality pro-ℓ group of dimension 2, namely, a pro-ℓ group G whose $\mathbb{F}_\ell$-cohomology satisfies the following conditions:

(i) $\dim(H^1(G,\mathbb{F}_\ell)) < \infty$;
(ii) $H^2(G,\mathbb{F}_\ell) \simeq \mathbb{F}_\ell$;
(iii) cup-product induces a perfect pairing $H^1(G,\mathbb{F}_\ell) \times H^1(G,\mathbb{F}_\ell) \to H^2(G,\mathbb{F}_\ell)$

(cf. [20, Def. 3.9.9]). Note that by condition (ii) such a pro-ℓ group G has a single defining relation, namely, G may be defined as the quotient F/N of a free pro-ℓ group F over a normal subgroup $N \subseteq F$ generated as a normal subgroup of F by a single element contained in $\Phi(F)$ (cf., e.g., [20, p. 231–232]).

A Demushkin group comes equipped with a distinguished orientation $\eth_G \colon G \to \mathbb{Z}_\ell^\times$, induced by the action of G on its dualizing module, described in [14, Thm. 4]. The orientation $\eth_G\colon G \to \mathbb{Z}_\ell^\times$ is the only orientation which completes G into a Kummerian oriented pro-ℓ group $(G, \eth_G)$ (cf. [28, Prop. 5.2]). The oriented pro-ℓ group $(G, \eth_G)$ enjoys also the Bogomolov-Positselski property.

Theorem 5.1. *Let G be a Demushkin group, endowed with the canonical orientation $\eth_G\colon G \to \mathbb{Z}_\ell^\times$, and suppose that $\mathrm{im}(\eth_G) \subseteq 1 + 4\mathbb{Z}_2$ if $\ell = 2$. Then the oriented pro-ℓ group $(G, \eth_G)$ has the Bogomolov-Positselski property.*

Proof. Since $(G, \eth_G)$ is Kummerian, by [28, Prop. 5.2], Proposition 2.6(iii) and Remark 2.2, one has $G/I_{\eth_G}(G) \simeq \mathbb{Z}_\ell^{d-1} \rtimes \mathbb{Z}_\ell$, with $d = \dim(H^1(G, \mathbb{F}_\ell))$. Therefore, $I_{\eth_G}(G) = K_{\eth_G}(G)$ is a subgroup of G of infinite index, and thus it is a free pro-ℓ group by [32, § I.4.5, Exercise 5(b)]. □

As mentioned above, Demushkin groups have a single defining relation. One may prove the Bogomolov-Positselski property also for 1-relator pro-ℓ groups G with quadratic $\mathbb{F}_\ell$-cohomology which can be completed into a Kummerian oriented pro-ℓ group $(G, \mathbf{1})$ with a trivial orientation.

Proposition 5.2. *Let G be a finitely generated pro-ℓ group with a single defining relation such that*

(i) *$H^\bullet(G, \mathbb{F}_\ell)$ is a quadratic algebra;*
(ii) *$(G, \mathbf{1})$ is Kummerian.*

Then $(G, \mathbf{1})$ has the Bogomolov-Positselski property.

Proof. Since $(G, \mathbf{1})$ is Kummerian, the quotient G^{ab} is a free abelian pro-ℓ group (cf. Example 2.7(c)). We need to show that $G' = K_{\mathbf{1}}(G) = I_{\mathbf{1}}(G)$ is a free pro-ℓ group.

Since G has a single defining relation, $H^2(G, \mathbb{F}_\ell) \simeq \mathbb{F}_\ell$ (cf. [20, Cor. 3.9.5]). Moreover, since $H^\bullet(G, \mathbb{F}_\ell)$ is quadratic, $H^2(G, \mathbb{F}_\ell)$ is generated by cup products $\chi \cup \psi$ with $\chi, \psi \in H^1(G, \mathbb{F}_\ell)$, so that the cup product from $H^1(G, \mathbb{F}_\ell)$ to $H^2(G, \mathbb{F}_\ell)$ is not trivial (see also [24, Prop. 4.2]). Consequently, [39, Cor. 2] yields a short exact sequence of pro-ℓ groups

$$\{1\} \longrightarrow N \longrightarrow G \longrightarrow \bar{G} \longrightarrow \{1\}$$

which satisfies the following three properties: N is a free pro-ℓ group; $\bar{G}$ is a Demushkin group; and for every subgroup S of G containing N, the inflation map

$$\mathrm{inf}^2_{S,N}\colon H^2(S/N, \mathbb{F}_\ell) \longrightarrow H^2(S, \mathbb{F}_\ell) \tag{5.1}$$

is an isomorphism (this last property is shown to hold in the proof of [39, Cor. 2]).

Since G is finitely generated, also $\bar{G}$ is finitely generated. Moreover, by (5.1) the inflation map $H^2(\bar{G}, \mathbb{F}_\ell) \to H^2(G, \mathbb{F}_\ell)$ is an isomorphism, and thus by the five-terms exact sequence (cf. [20, Prop. 1.6.7]) the restriction map

$$\mathrm{res}^1_{G,N}\colon H^1(G, \mathbb{F}_\ell) \longrightarrow H^1(N, \mathbb{F}_\ell)^G$$

is surjective. Since $(G, \mathbf{1})$ is Kummerian, and since $\mathrm{res}^1_{G,N}$ is surjective, [26, Thm. 1.2] implies that also the oriented pro-ℓ group $(\bar{G}, \mathbf{1}) = (G, \mathbf{1})/N$ is Kummerian. Hence, the canonical orientation $\eth_{\bar{G}}\colon \bar{G} \to \mathbb{Z}_\ell^\times$ must coincide with the trivial orientation $\mathbf{1}$ (cf. [28, Proposition 5.2]). By Theorem 5.1, the oriented pro-ℓ group $(\bar{G}, \mathbf{1})$ has the Bogomolov-Positselski property, and thus $K_{\mathbf{1}}(\bar{G})$ — which coincides with $\bar{G}'$ — is a free pro-ℓ group.

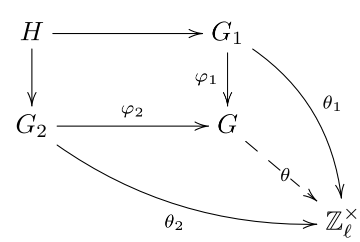
Let S be the normal subgroup of G containing N such that $S/N \simeq \bar{G}'$. Thus, $G/S \simeq \bar{G}/\bar{G}'$ is abelian, and therefore $S \supseteq G'$. By (5.1), one has $H^2(S/N, \mathbb{F}_\ell) \simeq H^2(S, \mathbb{F}_\ell)$, and the term on the left-hand side is trivial as S/N is a free pro-ℓ group. Hence, also $H^2(S, \mathbb{F}_\ell) = 0$, and S is a free pro-ℓ group (cf. [20, Prop. 3.5.17]). Since $G' \subseteq S$, and $\mathrm{cd}_\ell(G') \leq \mathrm{cd}_\ell(S) = 1$, G' must be free (cf. [32, § 3.3, Proposition 14]). □

Remark 5.3. Let F be a finitely generated free pro-ℓ group, let r be an element of $\Phi(F)$ and let R denote the normal subgroup of F generated by r. Suppose that $\ell \neq 2$. By [24, Prop. 4.2] and Example 2.7(c), the pro-ℓ group $G = F/R$ satisfies the conditions (i)–(ii) in Proposition 5.2 if, and only if, $r \in F'$ and $r \notin F^p \cdot [F', F]$.

5.2. **Free constructions.** By [8, § 3], the *free product* of two oriented pro-ℓ groups (G_1, θ_1) and (G_2, θ_2) is the oriented pro-ℓ group (G, θ) where G is the free pro-ℓ product of G_1, G_2, and $\theta \colon G \to \mathbb{Z}_\ell^\times$ is the orientation induced by θ_1, θ_2 via the universal property of G (see also [28, § 3.4]).

One may extend the above definition to *free amalgamated pro-ℓ products* of oriented pro-ℓ groups (we refer to [31, § 9.2] for the definition of free amalgamated pro-ℓ products).

Definition 5.4. Let (G_1, θ_1) and (G_2, θ_2) be two oriented pro-ℓ groups such that G_1 and G_2 have a common subgroup $H \subseteq G_1, G_2$ satisfying $\theta_1|_H = \theta_2|_H$. The *amalgamated pro-$\ell$ product of oriented pro-ℓ groups* of (G_1, θ_1) and (G_2, θ) with amalgamation in H is the oriented pro-ℓ group $(G, \theta) = (G_1, \theta_1) \amalg_H^{\hat{\ell}} (G_2, \theta_2)$, where $G = G_1 \amalg_H^{\hat{\ell}} G_2$ is the free amalgamated pro-ℓ product of G_1 and G_2 over H, and $\theta \colon G \to \mathbb{Z}_\ell^\times$ is the orientation which makes the diagram



commute.

Note that the morphisms φ_1 and φ_2 may not be injective (cf. [31, p. 369]). If they are, the free amalgamated pro-ℓ product is said to be *proper*.

If $H = \{1\}$, then $(G_1, \theta_1) \amalg_H^{\hat{\ell}} (G_2, \theta_2)$ coincides with the free product of oriented pro-ℓ groups. In this case we simply write $(G_1, \theta_1) \amalg^{\hat{\ell}} (G_2, \theta_2)$, instead of $(G_1, \theta_1) \amalg_{\{1\}}^{\hat{\ell}} (G_2, \theta_2)$. Free products of oriented pro-ℓ groups preserve Kummerianity (cf. [11, Prop. 7.5]).

Proposition 5.5. *Let (G_1, θ_1) and (G_2, θ_2) be two Kummerian oriented pro-ℓ groups. Then the free product $(G_1, \theta_1) \amalg^{\hat{\ell}} (G_2, \theta_2)$ is again Kummerian.*

We prove that — under certain conditions — if the free amalgamated pro-ℓ product of two Kummerian oriented pro-ℓ groups with the Bogomolov-Positselski property is again Kummerian, then it has also the Bogomolov-Positselski property.

Theorem 5.6. *Let (G_1, θ_1) and (G_2, θ_2) be torsion free Kummerian oriented pro-ℓ groups with the Bogomolov-Positselski property, with common finitely generated subgroup*

$U = G_1 \cap G_2$ such that $\theta_1|_U = \theta_2|_U$ and that (U, θ_U) is θ_U-abelian, where $\theta_U = \theta_i|_U$ for $i = 1, 2$. Suppose that

(i) *the amalgamated pro-ℓ product $(G, \theta) = (G_1, \theta_1) \amalg_U^{\hat{\ell}} (G_2, \theta_2)$ is Kummerian;*

(ii) *the restriction maps*

$$\mathrm{res}^1_{G,G_i} \colon H^1(G, \mathbb{F}_\ell) \to H^1(G_i, \mathbb{F}_\ell) \qquad \text{and} \qquad \mathrm{res}^1_{G_i,U} \colon H^1(G_i, \mathbb{F}_\ell) \to H^1(U, \mathbb{F}_\ell)$$

are surjective for both $i = 1, 2$.

Then (G, θ) has the Bogomolov-Positselski property.

Remark 5.7. (a) If U in the statement of Theorem 5.6 is the trivial group, then (G, θ) is the usual free product of oriented pro-ℓ groups, and the two conditions are satisfied by (G, θ). For condition (i), see Proposition 5.5, and condition (ii) is trivially satisfied. Hence, the Bogomolov-Positselski property is preserved by free products of oriented pro-ℓ groups.

(b) By duality, for $i \in \{1, 2\}$ the map res^1_{G,G_i}, respectively the map $\mathrm{res}^1_{G_i,U}$, is surjective if, and only if, the map $\bar{\iota}_i \colon G_i/\Phi(G_i) \to G/\Phi(G)$ induced by the inclusion $\iota_i \colon U \hookrightarrow G_i$, respectively the map $\bar{\iota}_{U,i} \colon U/\Phi(U) \to G_i/\Phi(G_i)$ induced by the inclusion $\iota_{U,i} \colon U \hookrightarrow G_i$, is injective.

Proof. By [23, Thm A], U is a *uniformly powerful pro-ℓ group*, and therefore [27, Prop. 5.22] implies that $G = G_1 \amalg_U^{\hat{\ell}} G_2$ is a proper amalgam. Moreover, by hypothesis one has the monomorphisms of ℓ-elementary abelian groups $\bar{\iota}_i$ and $\bar{\iota}_{U,i}$, with $i = 1, 2$ (cf. Remark 5.7(b)). Hence, also $\bar{\iota}_U = \bar{\iota}_i \circ \bar{\iota}_{U,i} \colon U/\Phi(U) \to G/\Phi(G)$ is injective for both $i = 1, 2$.

Let $\iota_U \colon U \hookrightarrow G$ be the inclusion of U in G, and for $i = 1, 2$, set

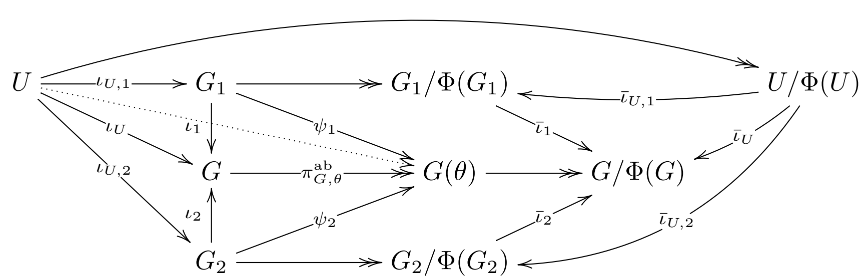
$$\psi_U = \pi^{\mathrm{ab}}_{G,\theta} \circ \iota_U \colon U \longrightarrow G(\theta) = G/K_\theta(G),$$
$$\psi_i = \pi^{\mathrm{ab}}_{G,\theta} \circ \iota_i \colon G_i \longrightarrow G(\theta) = G/K_\theta(G).$$

Then

$$\ker(\psi_U) = U \cap K_\theta(G) \qquad \text{and} \qquad \ker(\psi_i) = G_i \cap K_\theta(G). \tag{5.2}$$

Now consider the commutative diagram

(5.3)



where the dotted arrow from U to $G(\theta)$ is ψ_U. By Remark 2.4, the oriented pro-ℓ groups $(\mathrm{im}(\psi_U), \theta|_{\mathrm{im}(\psi_U)})$ and $(\mathrm{im}(\psi_i), \theta|_{\mathrm{im}(\psi_i)})$ are $\theta|_{\mathrm{im}(\psi_U)}$- and $\theta|_{\mathrm{im}(\psi_i)}$-abelian, respectively. In particular,

$$\ker(\psi_i) \supseteq I_{\theta_i}(G_i) = K_{\theta_i}(G_i), \tag{5.4}$$

where the left-hand side inclusion follows by Proposition 2.3, and the right-side equality follows by Proposition 2.6(iv), as (G_i, θ_i) is Kummerian for $i \in \{1, 2\}$ by hypothesis. Consequently, the pro-ℓ groups $\mathrm{im}(\psi_U)$ and $\mathrm{im}(\psi_i)$ are torsion-free, so that $\ker(\psi_U)$ and $\ker(\psi_i)$ are self-isolated subgroups of U and G_i respectively. On the other hand, by duality one has $\ker(\psi_U) \subseteq \Phi(U)$ and $\ker(\psi_i) \subseteq \Phi(G_i)$, as the maps $\bar{\iota}_U$ and $\bar{\iota}_i$ are injective. Altogether, by (5.2) and (5.4) one has

$$K_{\theta|_U}(U) = \{1\} \subseteq U \cap K_\theta(G) \subseteq \Phi(U) \qquad \text{and} \qquad K_{\theta_i}(G_i) \subseteq G_i \cap K_\theta(G) \subseteq \Phi(G_i),$$

and thus $\{1\} = U \cap K_\theta(G)$ and $K_{\theta_i}(G_i) = G_i \cap K_\theta(G)$ by Proposition 3.6.

Now, let $\mathcal{T} = (\mathscr{V}(\mathcal{T}), \mathscr{E}(\mathcal{T}))$ be the pro-ℓ tree whose vertices and edges are given by

$$\mathscr{V}(\mathcal{T}) = \{\, gG_1, gG_2 \mid g \in G\} \quad \text{and} \quad \mathscr{E}(\mathcal{T}) = \{\, gU, \overline{gU} \mid g \in G\,\},$$

respectively. In particular, every edge $gU \in \mathscr{E}(\mathcal{T})$ defines an origin, the G_1-coset gG_1 and a terminus, the G_2-coset gG_2. For $\overline{gU} \in \mathscr{E}(\mathcal{T})$ the roles of the terminus and origin are interchanged. Then $\mathcal{T}$ is a second countable pro-ℓ tree, with a natural G-action (cf. [30, Example 6.2.3]). For $v = gG_i \in \mathscr{V}(\mathcal{T})$ and $\mathbf{e} = hU \in \mathscr{E}(\mathcal{T})$, with $g, h \in G$ and $i \in \{1, 2\}$, let K_v and $K_{\mathbf{e}}$ denote the stabilizers of v and $\mathbf{e}$ in $K_\theta(G)$, respectively. Hence

$$K_v = \{\, x \in K_\theta(G) \mid xg \in gG_i \,\} = K_\theta(G) \cap gG_ig^{-1},$$
$$K_{\mathbf{e}} = \{\, x \in K_\theta(G) \mid xh \in hU \,\} = K_\theta(G) \cap hUh^{-1}.$$

Since $K_\theta(G)$ is a normal subgroup of G, for every $v = gG_i \in \mathscr{V}(\mathcal{T})$ the subgroup K_v is isomorphic to $K_\theta(G) \cap G_i = K_{\theta_i}(G_i)$, which is free by hypothesis; while for every $\mathbf{e} = hU \in \mathscr{E}(\mathcal{T})$ the subgroup $K_{\mathbf{e}}$ is equal to $\{1\}$, and hence no non-trivial element of $K_\theta(G)$ stabilizes an edge. Therefore, by [17, Thm. 5.6], $K_\theta(G)$ has the following decomposition as free pro-ℓ product:

$$K_\theta(G) = \left(\coprod_{v \in \mathscr{V}'} K_v \right) \amalg F, \tag{5.5}$$

for some subset $\mathscr{V}'$ of $\mathscr{V}(\mathcal{T})$, where F is a free pro-ℓ group. Hence $K_\theta(G)$ is the free pro-ℓ product of free pro-ℓ groups, and thus it is a free pro-ℓ group as well. □

Example 5.8. Let (G_1, θ_1) and (G_2, θ_2) be the oriented pro-ℓ groups with

$$G_1 = \left\langle\, x, y_1, y_3 \mid [y_1, y_3] = 1,\ {}^x y_j = y_j^{1+\ell},\ \forall\, j \in \{1, 3\} \,\right\rangle \simeq \mathbb{Z}_\ell^2 \rtimes \mathbb{Z}_\ell,$$
$$G_2 = \left\langle\, x, y_2, y_3 \mid [y_2, y_3] = 1,\ {}^x y_j = y_j^{1+\ell},\ \forall\, j \in \{2, 3\} \,\right\rangle \simeq \mathbb{Z}_\ell^2 \rtimes \mathbb{Z}_\ell,$$

and such that $\theta_i(x) = 1 + \ell$ and $\theta_i(y_i) = \theta_i(y_3) = 1$ for both $i = 1, 2$. By Remark 2.2, these two oriented pro-ℓ groups are respectively θ_1- and θ_2-abelian. Set $U = G_1 \cap G_2$ — i.e. U is the subgroup generated by x, y_3. Clearly, $\theta_1|_U = \theta_2|_U$, and

$$(U, \theta_i|_U) = \langle\, y_3 \,\rangle \rtimes (\langle\, x \,\rangle, \theta_i|_{\langle\, x \,\rangle}) \qquad \text{for both } i = 1, 2,$$

which is $\theta_i|_U$-abelian by Remark 2.4. Moreover, it is straightforward to see that the maps $\bar{\iota}_{U,i} \colon U/\Phi(U) \to G_i/\Phi(G_i)$ are injective for both $i = 1, 2$. Now let (G, θ) be the oriented pro-ℓ group $(G_1, \theta_1) \amalg_U^{\hat{\ell}} (G_2, \theta_2)$. Then

$$G = \left\langle\, x, y_1, y_2, y_3 \mid [y_1, y_3] = [y_2, y_3] = 1,\ {}^x y_i = y_i^{1+\ell},\ \forall\, i \in \{1, 2, 3\} \,\right\rangle$$

and $\theta(x) = 1+\ell$, $\theta(y_j) = 1$ for $j = 1, 2, 3$. Moreover, one has an epimorphism of oriented pro-ℓ groups $\tau\colon (G, \theta) \to (\bar{G}, \bar{\theta})$, where

$$\bar{G} = \left\langle\ \bar{x}, \bar{y}_1, \bar{y}_2, \bar{y}_3 \ \mid\ [\bar{y}_j, \bar{y}_{j'}] = 1,\ {}^{\bar{x}}\bar{y}_j = \bar{y}_j^{1+\ell},\ \forall\, j, j' \in \{1, 2, 3\}\ \right\rangle \simeq \mathbb{Z}_\ell^3 \rtimes \mathbb{Z}_\ell,$$

and $\bar{x} = \tau(x)$, $\bar{y}_j = \tau(y_j)$ for $j = 1, 2, 3$. By Remark 2.2, $(\bar{G}, \bar{\theta})$ is $\bar{\theta}$-abelian, and thus $\ker(\tau) \supseteq I_\theta(G)$ by Proposition 2.3. On the other hand, it is straightforward to see that $\Phi(G) \supseteq \ker(\tau)$, and hence (G, θ) is Kummerian by Proposition 2.6–(vi). Since (G_1, θ_1) and (G_2, θ_2) have the Bogomolov-Positselski property by Example 3.4–(a), Theorem 5.6 implis that also (G, θ) has the Bogomolov-Positselski property. Observe that G is $H^\bullet$-quadratic (cf. [27, Rem. 5.25–(c)]).

5.3. **Pro-ℓ groups of elementary type.** Let (G, θ) be an oriented pro-ℓ group, and let A be a free abelian pro-ℓ group. Recall that the *semidirect product* $A \rtimes (G, \theta) = (A \rtimes G, \theta \circ \pi)$ is the oriented pro-ℓ group where $gag^{-1} = a^{\theta(g)}$ for all $a \in A$ and $g \in G$, and $\pi\colon A \rtimes G \to G$ is the canonical projection (cf. [8, § 3]).

The following is straightforward (cf., e.g., [11, Prop. 3.6]).

Proposition 5.9. *Given an oriented pro-ℓ group (G, θ) and a free abelian pro-ℓ group A, one has $K_{\theta\circ\pi}(A \rtimes G) = K_\theta(G)$. In particular, $A \rtimes (G, \theta)$ is Kummerian if, and only if, (G, θ) is Kummerian; and $A \rtimes (G, \theta)$ has the Bogomolov-Positselski property if, and only if, (G, θ) has the Bogomolov-Positselski property.*

The family $\mathbf{ET}_\ell$ of *oriented pro-ℓ groups of elementary type* is the smallest class of finitely generated oriented pro-ℓ groups satisfying (cf. [8, § 3])

(a) the oriented pro-ℓ group $(G, \eth_G)$, with G a Demushkin group, is of elementary type;
(b) the oriented pro-ℓ group $(\mathbb{Z}_\ell, \theta)$, with $\theta\colon \mathbb{Z}_\ell \to \mathbb{Z}_\ell^\times$ arbitrary, is of elementary type;
(c) if the oriented pro-ℓ group (G, θ) is of elementary type and A is a finitely generated free abelian pro-ℓ group, then also the semidirect product $A \rtimes (G, \theta)$ is of elementary type;
(d) if (G_1, θ_1) and (G_2, θ_2) are oriented pro-ℓ groups of elementary type then also the free pro-ℓ product $(G_1, \theta_1) \amalg^{\hat{\ell}} (G_2, \theta_2)$ is of elementary type.

Remark 5.10. (a) In the original definition of oriented pro-2 groups of elementary type one has that also the cyclic group C_2 of order 2, endowed with the non-trivial orientation $\theta_{C_2}\colon C_2 \twoheadrightarrow \{\pm 1\} \subset \mathbb{Z}_2^\times$, is a pro-2 group of elementary type (cf. [8, p. 242]). Since our results always assume oriented pro-ℓ groups to be torsion-free, we may safely exclude (C_2, θ_{C_2}) from the above definition of oriented pro-ℓ groups of elementary type.
(b) From the results in [28, § 3.3–3.4], one may deduce that a finitley generated subgroup H of an oriented pro-ℓ groups of elementary type (G, θ) gives rise to a pro-ℓ groups of elementary type $(H, \theta|_H)$.
(c) If (F, θ) is a torsion-free oriented pro-ℓ group with F a finitely generated free pro-ℓ group and $\theta\colon F \to \mathbb{Z}_\ell^\times$ any orientation, then (F, θ) is of elementary type. Indeed, if $\theta = \mathbf{1}$, then (F, θ) is isomorphic to the free pro-ℓ product of d copies of the oriented pro-ℓ group $(\mathbb{Z}_\ell, \mathbf{1})$, where d is the minimal number of generators

of F. Otherwise, $\mathrm{im}(\theta)\simeq\mathbb{Z}_\ell$, and the short exact sequence of pro-ℓ groups

$$\{1\}\longrightarrow\ker(\theta)\longrightarrow F\longrightarrow\mathrm{im}(\theta)\longrightarrow\{1\}$$

splits. In this case, let $\{x_1,\ldots,x_d\}$ be a minimal generating set where $\theta(x_1)\neq 1$ and $\theta(x_i)=1$ for $i\geq 2$, and let H be the subgroup of F generated by $\{x_2,\ldots,x_d\}$, which is free. Then, $(F,\theta)\simeq(H,\mathbf{1})\amalg^{\hat{\ell}}(\mathrm{im}(\theta),\mathrm{id}_{\mathrm{im}(\theta)})$, where both factors are oriented pro-ℓ groups of elementary type.

From Example 2.7–(b), § 5.1, and Propositions 5.5 and 5.9, one concludes that oriented pro-ℓ groups of elementary type are Kummerian. I. Efrat's *Elementary Type Conjecture* states that if $\mathbb{K}$ is a field containing a primitive ℓ^{th}-root of 1 (and also $\sqrt{-1}$ if $\ell=2$) and if the maximal pro-ℓ Galois group $G_{\mathbb{K}}(\ell)$ is finitely generated, then $(G_{\mathbb{K}}(\ell),\tilde{\theta}_{\mathbb{K},\ell})$ is of elementary type (cf. [6,7], see also [16, § 10] and [28, § 7.5]).

Example 5.11. The oriented pro-ℓ group (G,θ) as in Example 5.8 is not of elementary type. Indeed, the subgroup of G generated by $\{x,y_1,y_2\}$ contains a finitely generated subgroup which does not complete into a Kummerian oriented pro-ℓ group (cf. [26, Ex. 5.3]) — in particular, G does not occur as the maximal pro-ℓ Galois group of a field containing a primitive ℓ^{th}-root of unity (and also $\sqrt{-1}$ if $\ell=2$). Therefore, (G,θ) is not of elementary type by Remark 5.10–(b).

Theorem 5.12. *Let (G,θ) be an oriented pro-ℓ group of elementary type. Then (G,θ) has the Bogomolov-Positselski property.*

Proof. If G is a free pro-ℓ group, then (G,θ) has the Bogomolov-Positselski property by Example 3.4–(a). If G is a Demushkin group and $\theta=\eth_G$, then $(G,\eth_G)$ has the Bogomolov-Positselski property by Theorem 5.1.

By Proposition 5.9, if $(G,\theta)=A\rtimes(G_0,\theta|_{G_0})$ where A is a free abelian pro-ℓ group and the right side factor is an oriented pro-ℓ group of elementary type, then (G,θ) has the Bogomolov-Positselski property — provided that $(G_0,\theta|_{G_0})$ has the Bogomolov-Positselski property.

Finally, by Theorem 5.6, if $(G,\theta)=(G_1,\theta_1)\amalg^{\hat{\ell}}(G_2,\theta_2)$ and both (G_1,θ_1) and (G_2,θ_2) have the Bogomolov-Positselski property, then also (G,θ) has the Bogomolov-Positselski property. □

Let $\mathbb{K}$ be a field containing a primitive ℓ^{th}-root of unity, and set $\mathbb{K}^\times=\mathbb{K}\smallsetminus\{0\}$. Since Kummer theory yields an isomorphism of (discrete) ℓ-elementary abelian pro-ℓ groups $H^1(G_{\mathbb{K}}(\ell),\mathbb{F}_\ell)^\vee\simeq\mathbb{K}^\times/(\mathbb{K}^\times)^\ell$, the pro-$\ell$ group $G_{\mathbb{K}}(\ell)$ is finitely generated if, and only if, the quotient $\mathbb{K}^\times/(\mathbb{K}^\times)^\ell$ is finite. One has the following (see [18, Thm. D], and [9] for item (f)).

Proposition 5.13. *Let $\mathbb{K}$ be a field containing a primitive ℓ^{th}-root of 1 (and also $\sqrt{-1}$ if $\ell=2$), such that the quotient $\mathbb{K}^\times/(\mathbb{K}^\times)^\ell$ is finite. Then the oriented pro-ℓ group $(G_{\mathbb{K}}(\ell),\theta_{\mathbb{K},\ell})$ is of elementary type in the following cases:*

(a) $\mathbb{K}$ *is finite;*
(b) $\mathbb{K}$ *is a pseudo algebraically closed (PAC) field, or an extension of relative trascendence degree 1 of a PAC field;*
(c) $\mathbb{K}$ *is an extension of trascendence degree 1 of a local field;*

(d) $\mathbb{K}$ *is ℓ-rigid* (*cf.* [36, p. 722], *see also* [4, § 3]);
(e) $\mathbb{K}$ *is algebraic extension of a global field of characteristic not ℓ;*
(f) $\mathbb{K} = \Bbbk(\!(T)\!)$, *where* $(G_{\Bbbk}(\ell), \theta_{\Bbbk,\ell})$ *is of elementary type.*

Corollary 1.3 follows from Theorem 5.12 and Proposition 5.13.

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