SAMPLING IN SPACES OF ENTIRE FUNCTIONS OF EXPONENTIAL TYPE IN $\mathbb{C}^{n+1}$

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ABSTRACT. In this paper we consider the question of sampling for spaces of entire functions of exponential type in several variables. The novelty resides in the growth condition we impose, that is, that their restriction to a hypersurface is square integrable with respect to a natural measure. The hypersurface we consider is the boundary $b\mathcal{U}$ of the Siegel upper half-space $\mathcal{U}$ and it is fundamental that $b\mathcal{U}$ can be identified with the Heisenberg group $\mathbb{H}_n$. We consider entire functions in $\mathbb{C}^{n+1}$ of exponential type with respect to the hypersurface $b\mathcal{U}$ whose restriction to $b\mathcal{U}$ are square integrable with respect to the Haar measure on $\mathbb{H}_n$. For these functions we prove a version of the Whittaker–Kotelnikov–Shannon Theorem. Instrumental in our work are spaces of entire functions in $\mathbb{C}^{n+1}$ of exponential type with respect to the hypersurface $b\mathcal{U}$ whose restrictions to $b\mathcal{U}$ belong to some homogeneous Sobolev space on $\mathbb{H}_n$. For these spaces, using the group Fourier transform on $\mathbb{H}_n$, we prove a Paley–Wiener type theorem and a Plancherel–Pólya type inequality.

1. Introduction and statement of the main results

The classical Paley–Wiener theorem characterizes the entire functions of exponential type a in the complex plane whose restriction to the real line is square integrable as the space of L^2-functions whose Fourier transform is supported in the interval $[-a, a]$. For such functions perhaps the most far reaching result is the Whittaker–Kotelnikov–Shannon sampling theorem. These results have been extended to several variables for functions in $\mathbb{C}^n$ whose restrictions to the surface $\{\operatorname{Im} z = 0\} = \mathbb{R}^n$ have Fourier transform supported in a compact set Ω, see e.g. [SW71] and [Lan67].

In this paper we take a different approach. We consider a hypersurface M that separates the whole space $\mathbb{C}^{n+1}$ into two unbounded connected components and entire functions in $\mathbb{C}^{n+1}$ that satisfy some exponential growth condition adapted to M – namely, of quadratic order in the complex tangential directions to M and of linear growth in the transversal direction. We also require that these functions have restriction to M which is square integrable, or more generally is in some homogeneous Sobolev space. For such functions we prove a Paley–Wiener type theorem and a Whittaker–Kotelnikov–Shannon sampling theorem.

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Consider the complex spaces $\mathbb{C}^{n+1}$, with $n \geqslant 1$, and the strongly pseudoconvex hypersurface

$$b\mathcal{U} = \left\{\zeta = (\zeta', \zeta_{n+1}) \in \mathbb{C}^n \times \mathbb{C} : \operatorname{Im}\zeta_{n+1} = \tfrac{1}{4}|\zeta'|^2\right\},$$

which is the topological boundary of the *Siegel upper-half space*

$$\mathcal{U} = \Big\{\zeta = (\zeta', \zeta_{n+1}) \in \mathbb{C}^n \times \mathbb{C} : \varrho(\zeta) := \operatorname{Im}\zeta_{n+1} - \tfrac{1}{4}|\zeta'|^2 > 0\Big\}. \tag{1}$$

It is well known that $\mathcal{U}$ is biholomorphic to the unit ball B in $\mathbb{C}^{n+1}$. The $(1,1)$-form $\theta = \frac{i}{2}(\overline{\partial} - \partial)\rho$ is a pseudo-hermitian structure on $M = b\mathcal{U}$, and it is non-degenerate. Then, $\theta \wedge (d\theta)^n$ is a volume form on $b\mathcal{U}$, i.e. θ is a contact form. Then, there exists a natural Riemannian metric g_θ on $b\mathcal{U}$. The boundary $b\mathcal{U}$ of $\mathcal{U}$ can be endowed with the structure of a nilpotent Lie group, namely the Heisenberg group $\mathbb{H}_n$. It turns out that the volume form $\theta \wedge (d\theta)^n$ coincides with the Haar measure on $\mathbb{H}_n$. The Haar measure on $\mathbb{H}_n$ coincides with two canonical measures defined on the strongly pseudoconvex manifold M, namely the Webster metric [DT06] and the Fefferman metric [Fef79, p. 259], resp. The volume forms constructed starting from such metrics coincide with the Haar measure on $\mathbb{H}_n$. We also remark that the induced metric from $\mathbb{C}^{n+1}$ differs from the metric $g_{\lambda\theta}$, for every smooth $\lambda : b\mathcal{U} \to (0, +\infty)$.

In $\mathbb{C}^{n+1}$ we introduce coordinates by means of a foliation of copies of $b\mathcal{U}$. Given $\zeta = (\zeta', \zeta_{n+1}) \in \mathbb{C}^{n+1}$, we define $\Psi : \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^n \times \mathbb{R} \times \mathbb{R}$ by

$$\Psi(\zeta', \zeta_{n+1}) = \big(\zeta', \operatorname{Re}\zeta_{n+1}, \operatorname{Im}\zeta_{n+1} - \tfrac{1}{4}|\zeta'|^2\big) =: (z, t, h).$$

Then, Ψ is a C^∞-diffeomorphism, and

$$\Psi^{-1}(z, t, h) = \big(z, t + i\tfrac{1}{4}|z|^2 + ih\big) =: (\zeta', \zeta_{n+1}).$$

Notice that $h = \varrho(\zeta', \zeta_{n+1})$, where ϱ is as in (1). Clearly, the boundary $b\mathcal{U}$ is characterized by the points of $\mathbb{C}^{n+1}$ such that $\Psi(\zeta) = (z, t, 0)$, that is, $h = \varrho(\zeta) = 0$, and

$$\mathcal{U} = \big\{\zeta \in \mathbb{C}^{n+1} : \Psi(\zeta) = (z, t, h) \text{ is such that } h > 0\big\}.$$

When $h = 0$, we write $[z, t]$ in place of $(z, t, 0)$. Then, the boundary $b\mathcal{U}$ can be identified with the Heisenberg group $\mathbb{H}_n$, which is the set $\mathbb{C}^n \times \mathbb{R}$ endowed with product

$$[w, s][z, t] = \big[w + z, s + t - \tfrac{1}{2}\operatorname{Im}(w \cdot \bar{z})\big].$$

Recall that $\mathbb{H}_n$ is a nilpotent Lie group and the Lebesgue measure on $\mathbb{C}^n \times \mathbb{R}$ coincides with both the right and left Haar measure on $\mathbb{H}_n$.

On $\mathbb{H}_n$ we consider the standard (positive) sub-Laplacian Δ and its fractional powers $\Delta^{s/2}$ (see Section 2 for details). Let $\mathcal{S} = \mathcal{S}(\mathbb{H}_n)$ denote the space of Schwartz functions on $\mathbb{H}_n$. Then, we have the following definition.

Definition 1.1. For $1 < p < \infty$ and $s > 0$ we define the *homogeneous Sobolev space* $\dot{W}^{s,p} = \dot{W}^{s,p}(\mathbb{H}_n)$ as the completion of $\mathcal{S}$ with respect to the norm $\|\Delta^{s/2}\varphi\|_{L^p}$, $\varphi \in \mathcal{S}$. More precisely, given the equivalence relation on the space of L^p-Cauchy sequences of Schwartz functions, $\{\varphi_k\} \sim \{\psi_k\}$ if $\Delta^{s/2}(\varphi_k - \psi_k) \to 0$, as $k \to +\infty$, and denoting by $[\{\varphi_k\}]$ the equivalence classes, then

$$\dot{W}^{s,p} = \Big\{\big[\{\varphi_k\}\big] : \{\varphi_k\} \subseteq \mathcal{S}, \{\Delta^{s/2}\varphi_k\} \text{ is a Cauchy sequence in } L^p, \\ \text{with } \big\|\big[\{\varphi_k\}\big]\big\|_{\dot{W}^{s,p}} = \lim_{k\to+\infty}\|\Delta^{s/2}\varphi_k\|_{L^p}\Big\}.$$

It is easy to characterize the homogeneous spaces $\dot{W}^{s,p}(\mathbb{H}_n)$ when $1 < p < \infty$ and $0 < s < (2n+2)/p$, see Section 2.[1]

On $\mathbb{H}_n$ we define a *homogeneous* norm (with respect to the natural anisotropic dilations) by setting

$$|[z,t]| := \left(\tfrac{1}{16}|z|^4 + t^2\right)^{1/4}.$$

Then, we introduce a "$\mathcal{U}$-adapted norm" in $\mathbb{C}^{n+1}$,

$$\|\zeta\|_{\mathcal{U}} = |[z,t]|^2 + |h|, \qquad \text{where} \quad \Psi(\zeta) = (z,t,h).$$

Notice that $\|\zeta\|_{\mathcal{U}}$ grows like $|[z,t]|^2$ in the complex tangential directions of $\mathbb{H}_n$ and like $|h|$ in the transversal directions.

We now introduce the spaces of entire functions we deal with. For a function F defined on $\mathbb{C}^{n+1}$, we set $\widetilde{F} = F \circ \Psi^{-1}$ and $\widetilde{F}_h[z,t] := \widetilde{F}(z,t,h)$. Then, in particular, $\widetilde{F}_0 = F_{|b\mathcal{U}}$.

Definition 1.2. Let $a > 0$ be given. We define the space of entire functions of exponential type *a with respect to the hypersurface $b\mathcal{U}$* as

$$\mathcal{E}_a = \Big\{ F \in \mathrm{Hol}(\mathbb{C}^{n+1}) : \text{ for every } \varepsilon > 0 \text{ there exists } C_\varepsilon > 0 \text{ such that } \ |F(\zeta)| \leqslant C_\varepsilon e^{(a+\varepsilon)\|\zeta\|_{\mathcal{U}}} \Big\}.$$

We define the corresponding Paley–Wiener spaces as

$$\mathcal{PW}_a = \big\{ F \in \mathcal{E}_a : \ \widetilde{F}_0 \in L^2(\mathbb{H}_n) \text{ with norm } \|F\|_{\mathcal{PW}_a} = \|\widetilde{F}_0\|_{L^2(\mathbb{H}_n)} \big\}.$$

For $0 < s < n+1$ we define the *fractional* Paley–Wiener spaces $\mathcal{PW}_a^s$ as

$$\mathcal{PW}_a^s = \big\{ F \in \mathcal{E}_a : \ \widetilde{F}_0 \in \dot{W}^{s,2} \text{ with norm } \|F\|_{\mathcal{PW}_a^s} = \|\widetilde{F}_0\|_{\dot{W}^{s,2}} \big\}.$$

We point out that the fractional Paley–Wiener spaces $\mathcal{PW}_a^s$ arise naturally in our setting, since they constitute a tool for proving our main results. In the 1-dimensional setting these spaces were introduced in [MPS20b], see also [MPS20a].

In order to state our main results we recall the basic facts about the Fourier transform on the Heisenberg group (see also [AMPS19, Fol89, Ric92]). For $\lambda \in \mathbb{R}^*$ the Fock space is defined as

$$\mathcal{F}^\lambda = \left\{ F \in \mathrm{Hol}(\mathbb{C}^n) : \ \left(\frac{|\lambda|}{2\pi}\right)^n \int_{\mathbb{C}^n} |F(z)|^2 \, e^{-\frac{|\lambda|}{2}|z|^2} dz < +\infty \right\}$$

so that $\mathcal{F}^\lambda = \mathcal{F}^{|\lambda|}$. For $\lambda \in \mathbb{R}^*$ and $[z,t] \in \mathbb{H}_n$, the Bargmann representation $\beta_\lambda[z,t]$ is the operator on $\mathcal{F}^\lambda$

$$\beta_\lambda[z,t]F(w) = \begin{cases} e^{i\lambda t - \frac{\lambda}{2} w\cdot\overline{z} - \frac{\lambda}{4}|z|^2} F(w+z) & \text{if } \lambda > 0, \\ e^{i\lambda t + \frac{\lambda}{2} w\cdot z + \frac{\lambda}{4}|z|^2} F(w+\bar{z}) & \text{if } \lambda < 0. \end{cases} \tag{2}$$

Notice that $\beta_\lambda[z,t] = \beta_{-\lambda}[\overline{z},-t]$, when $\lambda < 0$. Given $f \in L^1(\mathbb{H}_n)$ its Fourier transform consists of a family of operators $\{\beta_\lambda(f)\}_{\lambda\in\mathbb{R}^*}$ where $\beta_\lambda(f)$ is a Hilbert–Schmidt operator on $\mathcal{F}^\lambda$ given by

$$\beta_\lambda(f)F(w) = \int_{\mathbb{H}_n} f[z,t]\beta_\lambda[z,t]F(w)\,dzdt.$$

If $f \in L^2(\mathbb{H}_n)$ and $\|\cdot\|_{\mathrm{HS}}$ denotes the Hilbert–Schmidt norm on $\mathcal{F}^\lambda$, we have Plancherel's formula

$$\|f\|^2_{L^2(\mathbb{H}_n)} = \frac{1}{(2\pi)^{n+1}} \int_{\mathbb{R}} \|\beta_\lambda(f)\|^2_{\mathrm{HS}} |\lambda|^n \, d\lambda, \tag{3}$$

[1] We point out that $2n+2$ is the *homogeneous* dimension of $\mathbb{H}_n$.

and, if $f \in L^1 \cap L^2(\mathbb{H}_n)$, the inversion formula

$$f[z,t] = \frac{1}{(2\pi)^{n+1}} \int_{\mathbb{R}} \operatorname{tr}\big(\beta_\lambda(f)\beta_\lambda[z,t]^*\big)|\lambda|^n \, d\lambda \,. \tag{4}$$

Definition 1.3. For $s \in \mathbb{R}$ we define the space $\mathcal{L}_s^2$ as the space of measurable fields of operators

$$\tau : \mathbb{R}^* \to \prod_{\lambda \in \mathbb{R}^*} \mathscr{L}(\mathcal{F}^\lambda)$$

where $\mathscr{L}(\mathcal{F}^\lambda)$ denotes the bounded operators on $\mathcal{F}^\lambda$, such that

$$\|\tau\|^2_{\mathcal{L}_s^2} := \frac{1}{(2\pi)^{n+1}} \int_{\mathbb{R}^*} \|\tau(\lambda)\|^2_{\mathrm{HS}} \, |\lambda|^{n+s} d\lambda < +\infty,$$

where $\|\cdot\|_{\mathrm{HS}} := \|\cdot\|_{\mathrm{HS}(\mathcal{F}^\lambda)}$. We also define $\mathcal{H}_s^2$ as the subspace of $\tau \in \mathcal{L}_s^2$ such that

(i) $\tau(\lambda) = 0$ for $\lambda > 0$;

(ii) $\operatorname{ran}(\tau(\lambda)) \subseteq \operatorname{span}\{1\}$, when $\lambda < 0$.

If $E \subseteq (-\infty, 0)$, then $\mathcal{H}_s^2(E) = \big\{\tau \in \mathcal{H}_s^2 : \operatorname{supp}\tau \subseteq E\big\}$. Finally, when $s = 0$ we simply write $\mathcal{L}^2$ and $\mathcal{H}^2$ in place of $\mathcal{L}_0^2$ and $\mathcal{H}_0^2$, resp.

Our first result provide the expected characterization of the spaces $\mathcal{PW}_a$ and $\mathcal{PW}_a^s$, $0 \leqslant s < n+1$.

Theorem 1.4. *Let $0 \leqslant s < n+1$. If $F \in \mathcal{PW}_a^s$, then $\beta_\lambda(\widetilde{F}_0) \in \mathcal{H}_s^2\big([-a,0)\big)$,*

$$F(\zeta) = \widetilde{F}_h[z,t] = \frac{1}{(2\pi)^{n+1}} \int_{-a}^{0} e^{\lambda h} \operatorname{tr}\big(\beta_\lambda(\widetilde{F}_0)\beta_\lambda[z,t]^*\big)\,|\lambda|^n d\lambda\,,$$

and $\|F\|_{\mathcal{PW}_a^s} = \|\beta_\lambda(\widetilde{F}_0)\|_{\mathcal{L}_s^2}$. Conversely, let $\tau \in \mathcal{H}_s^2\big([-a,0)\big)$, and define

$$F(\zeta) = \widetilde{F}_h[z,t] = \frac{1}{(2\pi)^{n+1}} \int_{-a}^{0} e^{\lambda h} \operatorname{tr}\big(\tau(\lambda)\beta_\lambda[z,t]^*\big)\,|\lambda|^n d\lambda\,. \tag{5}$$

Then $F \in \mathcal{PW}_a^s$, $\beta_\lambda(\widetilde{F}_0) = \tau(\lambda)$ and $\|F\|_{\mathcal{PW}_a^s} = \|\tau\|_{\mathcal{L}_s^2}$.

We show that for functions in $\mathcal{PW}_a$ a sampling result holds true, extending the classical Whittaker–Kotelnikov–Shannon Theorem in dimension 1. We recall that given a reproducing kernel Hilbert space $\mathcal{K}$ of holomorphic functions on a domain Ω, a sequence $\Gamma = \{\gamma\} \subseteq \Omega$ is a sampling sequence for $\mathcal{K}$ if there exist constants $C, C' > 0$ such that for all $F \in \mathcal{K}$

$$C\|F\|^2_{\mathcal{K}} \leqslant \sum_{\gamma\in\Gamma} |F(\gamma)|^2 \|K_\gamma\|^{-1}_{\mathcal{K}} \leqslant C'\|F\|^2_{\mathcal{K}}\,,$$

where K_γ denotes the reproducing kernel at $\gamma \in \Omega$.

We now define the sequences for which we establish our sampling theorem. For $b > 0$ let $L_b \subseteq \mathbb{C}$ be the square lattice

$$L_b = \Big\{\gamma_{\ell,m} \in \mathbb{C} : \gamma_{\ell,m} = \sqrt{\tfrac{2\pi}{b}}(\ell + im), (\ell,m) \in \mathbb{Z}^2\Big\}. \tag{6}$$

Definition 1.5. For $(b_1, \dots, b_n) \in \mathbb{R}^n_+$ and $a \in \mathbb{R}_+$, consider the sequence of points $\Gamma \subseteq b\mathcal{U}$

$$\Gamma = \Big\{\big(\gamma, \tfrac{\pi}{a}k + \tfrac{i}{4}|\gamma|^2\big) \in \mathbb{C}^n \times \mathbb{C} : \ \gamma \in L_{b_1} \times \cdots \times L_{b_n},\, k \in \mathbb{Z}\Big\}.$$

Theorem 1.6. *Let $\Gamma \subseteq b\mathcal{U}$ be as in Definition 1.5 and suppose $b_j > a$, $j = 1, \ldots, n$. Then, there exist constants $C_\Gamma, C'_\Gamma > 0$ such that for all $F \in \mathcal{PW}^n_a$ we have*

$$\|F\|^2_{\mathcal{PW}^n_a} \leqslant C_\Gamma \sum_{\gamma \in \Gamma} |\partial_t^{n/2} F(\gamma)|^2 \leqslant C'_\Gamma \|F\|^2_{\mathcal{PW}^n_a},$$

where $t = \operatorname{Re} \zeta_{n+1}$. As a consequence, for every $G \in \mathcal{PW}_a$ we have the sampling

$$\|G\|^2_{\mathcal{PW}_a} \leqslant C_\Gamma \sum_{\gamma \in \Gamma} |G(\gamma)|^2 \leqslant C'_\Gamma \|G\|^2_{\mathcal{PW}_a}.$$

We point out that Theorem 1.6 most likely could be generalized to more general sequences of points, see the comments in the last Section 7.

We emphasize here one peculiar difference between the 1-dimensional and the several variables settings: in the 1-dimensional setting the Fourier transform of a function in the fractional Paley–Wiener space PW^s_a is supported on a symmetric interval $[-a, a]$ ([MPS20b, Theorem 1]), whereas in several variables the non-commutative Fourier transform of a function in $\mathcal{PW}^s_a$ is supported on an interval of the form $[-a, 0]$. In fact, the upper half-plane $U = \{\zeta = x + iy \in \mathbb{C} : y > 0\}$ and its complement $U^c = \{\zeta = x + iy \in \mathbb{C} : y \leqslant 0\}$ have the same geometry. However, our result is modeled on the Siegel upper-half space $\mathcal{U}$ and the geometries of $\mathcal{U}$ and $\mathcal{U}^c$ are clearly different. We will see (Lemma 3.2) that a function $F \in \mathcal{PW}^s_a$ is bounded in $\mathcal{U}$, whereas grows exponentially in $\mathcal{U}^c$.

The classical Paley–Wiener and Whittaker–Kotelnikov–Shannon Theorems have a natural and straightforward extension in several variables at least when we consider entire functions in $\mathbb{C}^d$ of exponential type with respect to the cube $[-a, a]^d$ and the sampling on the lattice $\frac{\pi}{a}\mathbb{Z}^d$. More generally, it is possible to consider entire functions of exponential type with respect to a symmetric body K. A symmetric body is a convex, compact and symmetric subset of $\mathbb{R}^d$ with non-empty interior. Then, if $f \in \operatorname{Hol}(\mathbb{C}^d)$ and $f|_{\mathbb{R}^d} \in L^2$, f is of exponential type with respect to K if and only if the Fourier transform of $f|_{\mathbb{R}^d}$ is supported on K, see [SW71, Chapter III]. Notice that in this case $\mathbb{R}^d$ is a totally real submanifold of real co-dimension d. Sampling theorems for entire functions in several variables in more general sets than a dilation of the lattice $\mathbb{Z}^d$ have drawn considerable interest in recent times. In [OU12] the authors discuss the sampling for the Paley–Wiener space of entire functions in several variables with convex spectrum. In [GL20] the authors obtain very interesting results concerning some necessary and some sufficient condition for sampling in Fock spaces in $\mathbb{C}^2$ in connection with existence of Gabor frames in $\mathbb{R}^2$. In [GHOCR19] the authors prove strict density inequalities for sampling and interpolation in Fock spaces in $\mathbb{C}^d$ defined by a plurisubharmonic weight. See also [GJM20] for related results, and the references in the cited papers.

The paper is organized as follows. In Section 2 we recall some facts about about representation theory and the fractional laplacian on the Heisenberg group. In Section 3 we prove a Plancherel–Pólya type inequality and the Paley–Wiener type result, Theorem 1.4. We also compute the reproducing kernels for the spaces $\mathcal{PW}^s_a$, $0 \leqslant s < n+1$ and show in Section 3 a resemble between the classical Paley–Wiener space $PW_a(\mathbb{C})$ in 1 variable and the fractional space $\mathcal{PW}^n_a$. In Section 4 we prove a representation theorem for functions in $\mathcal{PW}^s_a$ that we shall use in the proof of our sampling theorem, but that we believe is of interest in its own. Section 5 is devoted to a careful estimate of the sampling constants for the Fock space $\mathcal{F}^\lambda$ when λ varies in the bounded interval $(0, a]$. In Section 6 we prove our main result and we conclude with some final remarks and open questions in Section 7.

2. Preliminaries and basic facts

2.1. **The Sobolev space $\dot{W}^{s,p}$ and the Fourier transform.** We consider the fractional operator $\Delta^{s/2}$ defined following [Kom66, Fol75] as

$$\Delta^{s/2}\varphi = \lim_{\varepsilon\to 0}\frac{1}{\Gamma(k-\frac{s}{2})}\int_{\varepsilon}^{\infty} r^{k-\frac{s}{2}-1}e^{-r\Delta}\Delta^k\varphi\,dr\,, \tag{7}$$

where $k > s/2$ is an integer, whose domain is the set of $\varphi \in L^p(\mathbb{H}_n)$ for which the limit exists in L^p. Then $\Delta^{s/2}$ is a closed operator on L^p, $1 < p < \infty$ and its domain contains the Schwartz space $\mathcal{S}$, see [Fol75, Thm. (3.15)]. When $0 < s < 2n+2$, the operator $\Delta^{s/2}$ has an inverse given by convolution with a locally integrable homogeneous function. We denote such convolution operator by $\mathcal{I}_s$. The following result in contained in [Fol75, (1.11), (3.18)]. Recall that the homogeneous Sobolev spaces $\dot{W}^{s,p}$ is the completion of $\mathcal{S}$ with respect to the norm $\|\Delta^{s/2}\varphi\|_{L^p}$.

Proposition 2.1. *Let $1 < p < \infty$, $0 < s < (2n+2)/p$ and p^* given by $\frac{1}{p^*} = \frac{1}{p} - \frac{s}{2n+2}$. Then, $\mathcal{I}_s : L^p \to L^{p^*}$ is bounded. Therefore, if $1 < p < \infty$, $0 < s < (2n+2)/p$,*

$$\dot{W}^{s,p} = \big\{ f \in L^{p^*} : \Delta^{s/2} f \in L^p,\ \|f\|_{\dot{W}^{s,p}} := \|\Delta^{s/2} f\|_{L^p} \big\}\,.$$

In particular, we emphasize that if $f \in \dot{W}^{s,p}$, then there exists a sequence $\{\varphi_k\}_k \subseteq \mathcal{S}$ such that $\varphi_k \to f$ in L^{p^*} and $\{\Delta^{s/2}\varphi_k\}_k$ admits a limit in L^p. The fractional laplacian $\Delta^{s/2} f$ of f is set by definition to be such a limit.

Our goal now is to extend the definition of the Fourier transform to $\dot{W}^{s,2}$ when $0 < s < n+1$. The differentials of the Bargmann representations, that are defined in (2) , can be computed to give:

(i) for all $\lambda \neq 0$, $d\beta_\lambda(T) = i\lambda$;
(ii) for $\lambda > 0$, $d\beta_\lambda(Z_j) = \partial_{w_j}$, $d\beta_\lambda(\overline{Z}_j) = -\frac{\lambda}{2}w_j$;
(iii) for $\lambda < 0$, $d\beta_\lambda(Z_j) = \frac{\lambda}{2}w_j$, and $d\beta_\lambda(\overline{Z}_j) = \partial_{w_j}$;

see [Fol89] or [Tha93]. It is important to recall that, with our choice of normalization of the Fourier transform, if $f, g \in L^1(\mathbb{H}_n)$, $\beta_\lambda(f * g) = \beta_\lambda(f)\beta_\lambda(g)$, so that for any right-invariant vector field D

$$\beta_\lambda(Df) = -d\beta_\lambda(D)\beta_\lambda(f)\,. \tag{8}$$

Since $\Delta = -\frac{2}{n}\sum_{j=1}^n (Z_j\overline{Z}_j + \overline{Z}_j Z_j)$, we obtain that $d\beta_\lambda(\Delta)$ is the operator on $\mathcal{F}^\lambda$ such that $d\beta_\lambda(\Delta)e_\alpha = |\lambda|(1+|\alpha|/n)e_\alpha$. In particular, $d\beta_\lambda(\Delta)$ is a diagonal operator on $\mathcal{F}^\lambda$ with respect to the standard basis $\{e_\alpha\}$. Therefore, from (7) it follows that

$$\begin{aligned} d\beta_\lambda(\Delta^{s/2})e_\alpha &= \frac{1}{\Gamma(k-\frac{s}{2})}\int_0^\infty r^{k-\frac{s}{2}-1} d\beta_\lambda\big(e^{-r\Delta}\Delta^k\big)e_\alpha\,dr \\ &= \frac{1}{\Gamma(k-\frac{s}{2})}\int_0^\infty r^{k-\frac{s}{2}-1} e^{-r d\beta_\lambda(\Delta)} d\beta_\lambda(\Delta^k)e_\alpha\,dr \\ &= \frac{1}{\Gamma(k-\frac{s}{2})}\int_0^\infty r^{k-\frac{s}{2}-1} e^{-r|\lambda|(1+|\alpha|/n))}|\lambda|^k(1+|\alpha|/n)^k e_\alpha\,dr \\ &= \big[|\lambda|(1+|\alpha|/n)\big]^{s/2}e_\alpha\,. \end{aligned} \tag{9}$$

Lemma 2.2. *Let $0 \leqslant s < n+1$. Then, the Fourier transform on $\mathbb{H}_n$ defines a bounded operator $\beta : \dot{W}^{s,2} \to \mathcal{L}^2_s$.*

Proof. If $\varphi \in \mathcal{S}$, then $\varphi, \Delta^{s/2}\varphi \in L^2$, so that both $\beta_\lambda(\varphi), \beta_\lambda(\Delta^{s/2}\varphi) \in \mathrm{HS}(\mathcal{F}^\lambda)$. The identities (8) and (9) now give that

$$\beta_\lambda(\varphi) = |\lambda|^{-s/2} M_\lambda \beta_\lambda(\Delta^{s/2}\varphi), \tag{10}$$

where M_λ is the bounded operator on $\mathcal{F}^\lambda$ such that $M_\lambda(e_\alpha) = (1+|\alpha|/n)^{-s/2} e_\alpha$. Then,

$$\begin{aligned} \|\beta(\varphi)\|^2_{\mathcal{L}^2_s} &= \int_{\mathbb{R}^*} \|\beta_\lambda(\varphi)\|^2_{\mathrm{HS}} \, |\lambda|^{n+s} d\lambda \\ &= \int_{\mathbb{R}^*} \|M_\lambda \beta_\lambda(\Delta^{s/2}\varphi)\|^2_{\mathrm{HS}} \, |\lambda|^n d\lambda \\ &\leqslant \int_{\mathbb{R}^*} \|\beta_\lambda(\Delta^{s/2}\varphi)\|^2_{\mathrm{HS}} \, |\lambda|^n d\lambda \\ &= \|\varphi\|^2_{\dot{W}^{s,2}}. \quad \square \end{aligned}$$

Following [Ric92] and using the above differentials (i-iii) it is possible to see how the holomorphicity forces some constraints on the support of the Fourier transform. In particular the following lemma holds, see also [AMPS19, Section 2.3] and [CPar].

Lemma 2.3. *Let $c \in \mathbb{R}$ and set*

$$\mathcal{U}_c = \Big\{ \zeta = (\zeta', \zeta_{n+1}) : \operatorname{Im} \zeta_{n+1} > \tfrac{1}{4}|\zeta'|^2 + c \Big\}.$$

Let $F \in \mathrm{Hol}(\mathcal{U}_c)$, $\widetilde{F} = F \circ \Psi^{-1}$ and set $\widetilde{F}_h[z,t] = \widetilde{F}(z,t,h)$. If $h > c$ and $\widetilde{F}_h \in L^2(\mathbb{H}_n)$, then $\beta_\lambda(\widetilde{F}_h) = 0$ for $\lambda > 0$ and $\operatorname{ran}\big(\beta_\lambda(\widetilde{F}_h)\big) \subseteq \operatorname{span}\{e_0\}$.

Remark 2.4. As a consequence of the lemma, if

$$\mathcal{V} = \Big\{ \zeta \in \mathbb{C}^{n+1} : \tfrac{|\zeta'|^2}{4} + c - \delta < \operatorname{Im} \zeta_{n+1} < \tfrac{|\zeta'|^2}{4} + c + \delta \Big\}$$

is a neighborhood of $b\mathcal{U}_c$, $F \in \mathrm{Hol}(\mathcal{V})$ and $\widetilde{F}_t \in L^2(\mathbb{H}_n)$ for $t \in (c-\delta, c+\delta)$, then

$$\sigma_\lambda(\widetilde{F}_{c+h}) = e^{\lambda h} \sigma_\lambda(\widetilde{F}_c)$$

for any $h \in (-\delta, \delta)$, see e.g. [AMPS19, PS16, OV79].

We now recall that the operators $i^{-1}T$ and Δ admit commuting self-adjoint extensions on $L^2(\mathbb{H}_n)$, see [Str91] or [Tha98, Ch. 2]. If $F \in \mathrm{Hol}(\mathcal{V})$, where $\mathcal{V}$ is a tubular neighborhood of $b\mathcal{U}$, and $\widetilde{F}_0 \in L^2(\mathbb{H}_n)$, then $(\Delta + iT)\widetilde{F}_0 = 0$, that is, $\Delta \widetilde{F}_0 = -iT\widetilde{F}_0$. Therefore,

$$\Delta^{s/2} \widetilde{F}_0 = (-iT)^{s/2} \widetilde{F}_0 \tag{11}$$

for all such F's and $s > 0$. Hence,

$$\beta_\lambda(\Delta^{s/2}\widetilde{F}_0) = \beta_\lambda((-iT)^{s/2}\widetilde{F}_0) = |\lambda|^{s/2} \beta_\lambda(\widetilde{F}_0). \tag{12}$$

2.2. **Fock spaces, lattices and Weierstrass σ-functions.** We now recall some facts on lattices and the associated Weierstrass σ-functions.

For $b > 0$, we let L_b to be the square lattice

$$L_b = \big\{ \gamma_{\ell m} \in \mathbb{C} : \gamma_{\ell m} = \sqrt{\tfrac{2\pi}{b}}(\ell + im), \, (\ell, m) \in \mathbb{Z}^2 \big\}, \tag{13}$$

For such a lattice L_b we consider the Weierstrass σ-function associated to L_b,

$$\sigma_{L_b}(z) = z \prod_{(\ell,m)\in\mathbb{Z}^2\setminus(0,0)} \left(1-\frac{z}{\gamma_{\ell m}}\right)\exp\left\{\frac{z}{\gamma_{\ell m}}+\frac{z^2}{2\gamma_{\ell m}^2}\right\}.$$

We recall a few well-known properties of the the Weierstrass σ-function σ_{L_b} for any $b>0$, see e.g. [Zhu12, Ch.1]:

(i) σ_{L_b} is an entire function of order 2 and type $\frac{b}{4}$ that vanishes exactly at the points of L_b;

(ii) for all $z\in\mathbb{C}$, $|\sigma_{L_b}(z)|e^{-\frac{b}{4}|z|^2}$ is double periodic with periods $\sqrt{2\pi/b}$ and $i\sqrt{2\pi/b}$ and it is bounded above and below by constants C_b, c_b resp., depending only on b times $d(z,L_b)$, the euclidean distance of z from the lattice L_b, for all $z\in\mathbb{C}$;

(iii) there exists a constant $c_b'>0$ depending only on b, such that for all $\gamma_{\ell m}\in L_b$,

$$|\sigma_{L_b}'(\gamma_{\ell m})|e^{-\frac{b}{4}|\gamma_{\ell m}|^2}\geqslant c_b'. \tag{14}$$

We also recall that given the lattice L_b, then for any any $f\in\mathcal{F}^{b'}$ with $b'<b$ we have the decomposition

$$f(z)=\sum_{\gamma_{\ell m}\in L_b}\frac{f(\gamma_{\ell m})}{\sigma_{L_b}'(\gamma_{\ell m})}\frac{\sigma_{L_b}(z)}{z-\gamma_{\ell m}} \tag{15}$$

where the series converges in $\mathrm{Hol}(\mathbb{C})$, see e.g. [Zhu12, Prop. 4.24].

3. The Plancherel–Pólya inequality

In this section we prove our first results. We begin with a Plancherel–Pólya type inequality adapted to the Siegel half-space. This result implies in particular that the spaces $\mathcal{PW}_a^s$ are complete, for $0\leqslant s<n+1$.

3.1. **The Plancherel–Pólya inequality.** We now prove a Phrangém–Lindelöf type result for the Siegel half-space. We first need the following modified version of the classical result in the complex plane.

Lemma 3.1. *Let $g\in\mathrm{Hol}(\mathbb{C})$ and suppose that there exist constants $c,a,M>0$ such that:*

(i) $|g(t)|\leqslant M$,

(ii) *for every $\varepsilon>0$ there exists $C_\varepsilon>0$ such that*

$$|g(w)|\leqslant C_\varepsilon e^{(a+\varepsilon)(c|t|+|u|)},$$

where $w=t+iu$.

Then,

$$|g(w)|\leqslant Me^{a|u|}.$$

The classical proof applies also here and we skip the details; see, for instance, [You01]. In the Siegel half-space we have the following variation.

Lemma 3.2. *Let $F\in\mathrm{Hol}(\mathbb{C}^{n+1})$ and suppose that there exists constant $c,a,M>0$ such that:*

(i) $|F_{|b\mathcal{U}}(\zeta)|=|\widetilde{F}_0[z,t]|\leqslant M$,

(ii) *for every $\varepsilon>0$ there exists $C_\varepsilon>0$ such that*

$$|F(\zeta)|=|\widetilde{F}(z,t,h)|\leqslant C_\varepsilon e^{(a+\varepsilon)(c|[z,t]|^2+|h|)}.$$

Then, setting $h_- = -\min(0,h)$, we have

$$|F(\zeta)| = |\widetilde{F}(z,t,h)| \leqslant M e^{ah_-}\,.$$

Proof. For $w = t + iu \in \mathbb{C}$ and for every fixed $\zeta' \in \mathbb{C}^n$ we define

$$g_{\zeta'}(w) = F(\zeta', w + \tfrac{i}{4}|\zeta'|^2) = \widetilde{F}(z,t,u).$$

Then, $g_{\zeta'}$ is entire and from (ii) above we get

$$\begin{aligned}|g_{\zeta'}(w)| &\leqslant C_\varepsilon e^{(a+\varepsilon)(c|[z,t]|^2+|u|)} \leqslant C_\varepsilon e^{(a+\varepsilon)(c\frac{|z|^2}{4}+c|t|+|u|)}\\ &\leqslant C'_\varepsilon(\zeta') e^{(a+\varepsilon)(c|t|+|u|)}\end{aligned}$$

where the constant $C'_\varepsilon(\zeta')$ depends on the fixed $\zeta' \in \mathbb{C}^n$. Moreover,

$$|g_{\zeta'}(t)| = |F(\zeta', t + \tfrac{i}{4}|\zeta'|^2)| \leqslant M,$$

where M is an absolute constant not depending on ζ'. Lemma 3.1 implies

$$|g_{\zeta'}(w)| \leqslant M e^{a|u|}\,.$$

Thus, setting $w = \zeta_{n+1} - \frac{i}{4}|\zeta'|^2$ we have $h = \rho(\zeta) = u$ and

$$|F(\zeta', \zeta_{n+1})| = |F(\zeta', w + \tfrac{i}{4}|\zeta'|^2)| = |g_{\zeta'}(w)| \leqslant M e^{a|h|}\,.$$

In order to complete the proof, we need to show that we can improve the above inequality when $h > 0$, by showing that in fact $|F(\zeta)| \leqslant M$ when $\zeta \in \overline{\mathcal{U}}$. For each ζ_{n+1} fixed we have

$$\sup_{\zeta' \colon \frac{|\zeta'|^2}{4} \leqslant \operatorname{Im}\zeta_{n+1}} |F(\zeta', \zeta_{n+1})| = \sup_{\zeta' \colon \frac{|\zeta'|^2}{4} = \operatorname{Im}\zeta_{n+1}} |F(\zeta', \zeta_{n+1})| \leqslant \sup_{\zeta \in b\mathcal{U}} |F(\zeta', \zeta_{n+1})|\,.$$

Therefore,

$$\sup_{\zeta \in \overline{\mathcal{U}}} |F(\zeta)| = \sup_{(\zeta', \zeta_{n+1}) \colon \frac{|\zeta'|^2}{4} \leqslant \operatorname{Im}\zeta_{n+1}} |F(\zeta)| \leqslant \sup_{\zeta \in b\mathcal{U}} |F(\zeta', \zeta_{n+1})| \leqslant M\,,$$

as we wished to show. □

From this Phragmén–Lindelöf principle we deduce a version of the Plancherel–Pólya inequality in this setting. For $c \in \mathbb{R}$ we set $\mathcal{U}_c = \big\{\zeta = (\zeta', \zeta_{n+1}) : \operatorname{Im}\zeta_{n+1} > \frac{|\zeta'|^2}{4} + c\big\}$.

Proposition 3.3. (Plancherel–Pólya Inequality) *Let $F \in \mathcal{E}_a$ be such that $\widetilde{F}_0 \in L^p(\mathbb{H}_n)$, $1 < p < \infty$. Then, for all $h \in \mathbb{R}$,*

$$\int_{\mathbb{H}_n} |\widetilde{F}_h[z,t]|^p\, dzdt \leqslant e^{aph_-} \|\widetilde{F}_0\|^p_{L^p(\mathbb{H}_n)}\,,$$

where $(z,t,h) = \Psi(\zeta)$ and $h_- = -\min(0,h)$. In particular, $F \in H^p(\mathcal{U}_c)$, the Hardy space on $\mathcal{U}_c$, for all $c \in \mathbb{R}$.

Proof. Let $\varphi \in C_c^\infty(\mathbb{H}_n)$, $0 \leqslant \varphi \leqslant 1$, $\|\varphi\|_{L^{p'}(\mathbb{H}_n)} \leqslant 1$, where $\frac{1}{p} + \frac{1}{p'} = 1$, and define

$$G(\zeta) = \int_{\mathbb{H}_n} \widetilde{F}_h\big([z,t][w,s]\big)\varphi[w,s]\, dwds\,.$$

Then, if $\zeta \in b\mathcal{U}$, i.e. $\Psi(\zeta) = (z,t,0) \equiv [z,t] \in \mathbb{H}_n$,

$$|G(\zeta)| \leqslant \|\widetilde{F}_0\|_{L^p(\mathbb{H}_n)} \|\varphi\|_{L^{p'}(\mathbb{H}_n)} \leqslant \|\widetilde{F}_0\|_{L^p(\mathbb{H}_n)}\,.$$

Moreover, for $\zeta \in \mathbb{C}^{n+1}$, using [Cyg81] (with a slight abuse of notation) we have

$$\begin{aligned}\|\big([z,t][w,s],h\big)\|_{\mathcal{U}} &= \big|[z,t][w,s]\big|^2 + |h| \leqslant \big(|[z,t]| + |[w,s]|\big)^2 + |h| \\ &\leqslant 2\big([z,t]|^2 + |[w,s]|^2\big) + |h|\,.\end{aligned}$$

Therefore,

$$\begin{aligned}|G(\zeta)| &\leqslant C_\varepsilon \int_{\mathbb{H}_n} e^{(a+\varepsilon)(2[z,t]|^2+|h|+2|[w,s]|^2)} |\varphi[w,s]|\, dwds \\ &\leqslant C'_\varepsilon e^{(a+\varepsilon)(2|[z,t]|^2+|h|)}\,, \end{aligned} \tag{16}$$

since φ has compact support. By Lemma 3.2 we obtain

$$|G(\zeta)| \leqslant e^{ah_-} \|\widetilde{F}_0\|_{L^p(\mathbb{H}_n)}\,,$$

that is,

$$\Big|\int_{\mathbb{H}_n} \widetilde{F}_h\big([z,t][w,s]\big)\varphi[w,s]\, dwds\Big| \leqslant e^{ah_-} \|\widetilde{F}_0\|_{L^p(\mathbb{H}_n)}\,,$$

for every $\varphi \in C_c^\infty(\mathbb{H}_n)$, $\|\varphi\|_{L^{p'}(\mathbb{H}_n)} \leqslant 1$. Therefore,

$$\|\widetilde{F}_h\|_{L^p(\mathbb{H}_n)} = \|\widetilde{F}_h([\cdot,\cdot][w,s])\|_{L^p(\mathbb{H}_n)} \leqslant e^{ah_-} \|\widetilde{F}_0\|_{L^p(\mathbb{H}_n)}\,.$$

The conclusions follow. □

Proof of Theorem 1.4. We begin with the case $s = 0$. Let $F \in \mathcal{PW}_a$. By Prop. 3.3 it follows that $\|\widetilde{F}_h\|_{L^2(\mathbb{H}_n)} \leqslant e^{ah_-}\|\widetilde{F}_0\|_{L^2(\mathbb{H}_n)}$ and $F \in H^2(\mathcal{U}_c)$ for all $c \in \mathbb{R}$. In particular, arguing as in Remark 2.4 we obtain that

$$\beta_\lambda(\widetilde{F}_h) = e^{\lambda h}\beta_\lambda(\widetilde{F}_0)$$

for all $h \in \mathbb{R}$. Thus, thanks to the Paley–Wiener characterization of H^2, we have $\sigma_\lambda(\widetilde{F}_0) = 0$ for $\lambda > 0$. By Plancherel's formula it then follows that

$$\|\widetilde{F}_h\|^2_{L^2(\mathbb{H}_n)} = \int_{-\infty}^0 \|\beta_\lambda(\widetilde{F}_h)\|^2_{\mathrm{HS}} |\lambda|^n\, d\lambda = \int_{-\infty}^0 e^{2\lambda h} \|\beta_\lambda(\widetilde{F}_0)\|^2_{\mathrm{HS}} |\lambda|^n\, d\lambda\,,$$

whereas, by Prop. 3.3

$$\|\widetilde{F}_h\|^2_{L^2(\mathbb{H}_n)} \leqslant e^{2ah_-} \int_{-\infty}^0 \|\beta_\lambda(\widetilde{F}_0)\|^2_{\mathrm{HS}} |\lambda|^n\, d\lambda\,.$$

Therefore, for all $h \in \mathbb{R}$,

$$\int_{-\infty}^0 e^{2\lambda h} \|\beta_\lambda(\widetilde{F}_0)\|^2_{\mathrm{HS}} |\lambda|^n\, d\lambda \leqslant e^{2ah_-} \int_{-\infty}^0 \|\beta_\lambda(\widetilde{F}_0)\|^2_{\mathrm{HS}} |\lambda|^n\, d\lambda\,,$$

and, by letting $h \to -\infty$, this easily implies that $\operatorname{supp} \beta_\lambda(\widetilde{F}_0) \subseteq [-a, 0)$.

To prove the converse direction, given $\tau \in \mathcal{H}^2\big([-a,0)\big)$, arguing as [AMPS19, Lemma 3.1], we see that for every $\lambda < 0$, $\big|\operatorname{tr}\big(\tau(\lambda)\beta_\lambda[z,t]^*\big)\big| \leqslant \|\tau(\lambda)\|_{\mathrm{HS}}$. Therefore,

$$\begin{aligned}\int_{-a}^0 e^{\lambda h}\big|\operatorname{tr}\big(\tau(\lambda)\beta_\lambda[z,t]^*\big)\big||\lambda|^n\,d\lambda &\leqslant \int_{-a}^0 e^{\lambda h}\|\tau(\lambda)\|_{\mathrm{HS}}|\lambda|^n\,d\lambda\\ &\leqslant \|\tau\|_{\mathcal{L}^2}\bigg(\int_{-a}^0 e^{2\lambda h}|\lambda|^n\,d\lambda\bigg)^{1/2}\\ &\leqslant C\|\tau\|_{\mathcal{L}^2}e^{ah_-}\,,\end{aligned}$$

This shows that the integral in (5) converges absolutely. Let F be given by (5). The same argument as in [AMPS19, Lemma 3.1] now shows that F is entire, hence in $\mathcal{E}_a$ by the previous estimate. Moreover, F is such that $\beta_\lambda(\widetilde{F}_0) = \tau(\lambda) \in \mathcal{L}^2$, so that by Plancherel's formula $\widetilde{F}_0 \in L^2(\mathbb{H}_n)$, that is, $F \in \mathcal{PW}_a$.

We now consider the case $s > 0$. If $F \in \mathcal{PW}_a^s$, then by Lemma 2.2 $\beta_\lambda(\widetilde{F_0})$ is well defined and $\beta(\widetilde{F_0}) \in \mathcal{L}^2_s$. Let φ be a Schwartz function on $\mathbb{H}_n$ such that $\beta_\lambda(\varphi) = \varphi_0(\lambda)\langle\cdot, e_0\rangle e_0$ for $\varphi_0 \in C^\infty$ having support in $[-N,-1/N]$, for some $N > 0$. We define $\widetilde{G}_h[z,t] = (\widetilde{F}_h * \varphi)[z,t]$, so that

$$G(\zeta) = \int_{\mathbb{H}_n}\widetilde{F}_h\big([z,t][w,s]^{-1}\big)\varphi[w,s]\,dwds\,.$$

We claim that $G \in \mathcal{E}_a$. Indeed, observe that $\beta_\lambda(\widetilde{G}_0) = \beta_\lambda(\widetilde{F}_0)\beta_\lambda(\varphi) = \varphi_0(\lambda)\beta_\lambda(\widetilde{F}_0)$, has compact support contained in $[-N,-1/N]$. Since $\beta(\widetilde{F}_0) \in \mathcal{L}^2_s$, it follows that $\beta(\widetilde{G}_0) \in \mathcal{L}^2$, which in turns gives $\widetilde{G}_0 \in L^2(\mathbb{H}_n)$. The first part of the theorem now shows that G is entire function of exponential type at most N. However, since $F \in \mathcal{E}_a$, arguing as in (16) we also have

$$\begin{aligned}|G(\zeta)| &\leqslant \int_{\mathbb{H}_n}|\widetilde{F}_h\big([z,t][w,s]^{-1}\big)|\,|\varphi[w,s]|\,dwds\\ &\leqslant Ce^{(a+\varepsilon)(2|[z,t]|^2+|h|)}\,.\end{aligned}$$

Since $\widetilde{G}_0 \in L^2(\mathbb{H}_n)$, by the previous case $s = 0$, $G \in \mathcal{PW}_a$ and $\operatorname{supp}(\beta_\lambda(\widetilde{G}_0)) \subseteq [-a,0)$. By the choice of φ, this easily implies that also $\operatorname{supp}(\beta_\lambda(\widetilde{F}_0)) \subseteq [-a,0)$, and that $\operatorname{ran}\beta_\lambda(\widetilde{F}_0) \subseteq \operatorname{span}\{e_0\}$.

By Lemma 2.2, in particular by (10), we have

$$\beta_\lambda\big(\Delta^{s/2}\widetilde{F}_0\big)e_\alpha = |\lambda|^{s/2}[1+|\alpha|/n]^{s/2}\beta_\lambda(\widetilde{F}_0)e_\alpha\,.$$

Then,

$$\|\beta_\lambda\big(\Delta^{s/2}\widetilde{F}_0\big)\|^2_{\mathrm{HS}} = \sum_\alpha\big|\langle\beta_\lambda\big(\Delta^{s/2}\widetilde{F}_0\big)e_\alpha, e_\alpha\rangle_{\mathcal{F}^\lambda}\big|^2 = |\lambda|^s\|\beta_\lambda(\widetilde{F}_0)\|^2_{\mathrm{HS}}\,.$$

Hence,

$$\begin{aligned}\|F\|^2_{\mathcal{PW}_a^s} &= \int_{\mathbb{H}_n}|\Delta^{s/2}\widetilde{F}_0[z,t]|^2\,dzdt\\ &= \int_{-\infty}^0\|\beta_\lambda(\Delta^{s/2}\widetilde{F}_0)\|^2_{\mathrm{HS}}|\lambda|^n\,d\lambda\\ &= \int_{-\infty}^0\|\beta_\lambda(\widetilde{F}_0)\|^2_{\mathrm{HS}}|\lambda|^{n+s}\,d\lambda\,.\end{aligned}$$

In particular, $\beta_\lambda(\widetilde{F}_0) \in \mathcal{H}^2_s$, with equality of norms.

Conversely, let $\tau \in \mathcal{H}^2_s\big([-a,0)\big)$ and F be given by (5). We have that

$$\begin{aligned}\int_{-a}^0 e^{\lambda h}\big|\operatorname{tr}\big(\tau(\lambda)\beta_\lambda[z,t]^*\big)\big||\lambda|^n\,d\lambda &\leqslant \int_{-a}^0 e^{\lambda h}\|\tau(\lambda)\|_{\mathrm{HS}}|\lambda|^n\,d\lambda\\ &\leqslant \|\tau\|_{\mathcal{L}^2_s}\bigg(\int_{-a}^0 e^{2\lambda h}|\lambda|^{n-s}\,d\lambda\bigg)^{1/2}\\ &\leqslant C\|\tau\|_{\mathcal{L}^2_s}e^{ah_-}\,, \end{aligned} \tag{17}$$

where $C < +\infty$ if and only if $s < n+1$. In this case we can conclude that $F \in \mathcal{E}_a$. Now, we claim that $\beta_\lambda(\Delta^{s/2}\widetilde{F}_0) = |\lambda|^{s/2}\tau(\lambda)$, so that

$$\begin{aligned}\|\Delta^{s/2}\widetilde{F}_0\|^2_{L^2(\mathbb{H}_n)} &= \int_{-a}^0 \|\beta_\lambda(\Delta^{s/2}\widetilde{F}_0)\|^2_{\mathrm{HS}}|\lambda|^n\,d\lambda\\ &= \int_{-a}^0 \|\tau(\lambda)\|^2_{\mathrm{HS}}|\lambda|^{n+s}\,d\lambda\\ &= \|\tau\|^2_{\mathcal{L}^2_s}\,,\end{aligned}$$

as we wished to show. It remains to prove the claim. It is easy to construct fields of operators η_ε such that $\eta_\varepsilon \in \mathcal{L}^2(-a+\varepsilon,-\varepsilon)$ be smooth in λ and $\eta_\varepsilon \to \tau$ in $\mathcal{H}^2_s(-a,0)$ as $\varepsilon \to 0$. Then, the function

$$G_\varepsilon(\zeta) = \widetilde{G}_{\varepsilon,h}[z,t] := \frac{1}{(2\pi)^{n+1}}\int_{-a}^{-\varepsilon} e^{\lambda h}\operatorname{tr}\big(\eta_\varepsilon(\lambda)\beta_\lambda[z,t]^*\big)\,|\lambda|^n d\lambda$$

is in $\mathcal{S}(\mathbb{H}_n)$. Hence, using (12) and (4), we have that

$$\beta_\lambda(\Delta^{s/2}(\widetilde{G}_{\varepsilon,0})) = |\lambda|^{s/2}\beta_\lambda(\widetilde{G}_{\varepsilon,0}) = |\lambda|^{s/2}\eta_\varepsilon(\lambda).$$

Since $\eta_\varepsilon \to \tau$ in $\mathcal{H}^2_s(-a,0)$, $|\lambda|^{s/2}\eta_\varepsilon \to |\lambda|^{s/2}\tau$ in $\mathcal{H}^2(-a,0)$, so that $\Delta^{s/2}(\widetilde{G}_{\varepsilon,0})$ converges in $L^2(\mathbb{H}_n)$ to a function G such that $\beta_\lambda(G) = |\lambda|^{s/2}\tau(\lambda)$. Moreover, since $\dot{W}^{2,s}$ embeds continously in L^{2^*}, we also have that $\widetilde{G}_{\varepsilon,0}$ is a Cauchy in L^{2^*} and its limit is $\widetilde{F}_0$. Then, by definition, $\Delta^{s/2}\widetilde{F}_0 = G$ and the claim follows.

□

As a consequence of the Paley–Wiener theorems we obtain that the space $\mathcal{PW}^s_a, 0 \leqslant s < n+1$, is a reproducing kernel Hilbert space and we explicitly compute its kernel. We set

$$Q(\omega,\zeta) = \tfrac{1}{2i}(\omega_{n+1} - \overline{\zeta}_{n+1}) - \tfrac14\omega'\cdot\overline{\zeta}'\,,$$

so that, by writing $\zeta = \big(z, t+i(h+\frac{|z|^2}{4})\big)$, $\omega = \big(w, u+i(k+\frac{|w|^2}{4})\big)$,

$$\widetilde{Q}(z,t,h;w,u,k) = \tfrac{1}{2i}\Big(u - t + \tfrac12\operatorname{Im}(w\cdot\bar z) + i\big(h+k+\tfrac14|w-z|^2\big)\Big)\,.$$

Corollary 3.4. *Let $s \in [0,n+1)$. Then, the space $\mathcal{PW}^s_a$ is a reproducing kernel Hilbert space with reproducing kernel*

$$K(\omega,\zeta) = K_\zeta(\omega) = \frac{1}{(2\pi)^{n+1}}\int_{-a}^0 e^{2i\lambda Q(\omega,\zeta)}|\lambda|^{n-s}\,d\lambda\,, \tag{18}$$

and $\beta_\lambda\big(\widetilde{K_{\zeta,0}}\big) = \chi_{[-a,0)}(\lambda)e^{\lambda h}|\lambda|^{-s}P_0\beta_\lambda[z,t]$, where P_0 denotes the orthogonal projection onto the subspace generated by e_0.

Proof. The Plancherel–Pólya Inequality, Proposition 3.3, implies that $\mathcal{PW}_a^s$ continuously embeds into $\mathrm{Hol}(\mathbb{C}^{n+1})$. Hence, the completeness of $\mathcal{PW}_a^s$ and the boundedness of the point-evaluation functionals follow.

The explicit computation of the kernel follows from a standard argument. Let τ denote the element of $\mathcal{H}_s^2([-a,0))$ and define

$$
\begin{aligned}
F(\zeta) = \widetilde{F}_h[z,t] &= \frac{1}{(2\pi)^{n+1}} \int_{-a}^{0} e^{\lambda h} \operatorname{tr}\left(\tau(\lambda)\beta_\lambda[z,t]^*\right) |\lambda|^n d\lambda \\
&= \frac{1}{(2\pi)^{n+1}} \int_{-a}^{0} e^{\lambda h} \operatorname{tr}\left(\tau(\lambda)P_0\beta_\lambda[z,t]^*\right) |\lambda|^n d\lambda.
\end{aligned}
$$

The last identity holds since $\operatorname{ran}\tau_F(\lambda) \subseteq \operatorname{span}\{e_0\}$. We also have

$$
F(\zeta) = \widetilde{F}_h[z,t] = \langle F, K_\zeta\rangle_{\mathcal{PW}_a^s} = \frac{1}{(2\pi)^{n+1}} \int_{-a}^{0} \operatorname{tr}\left(\tau(\lambda)\beta_\lambda((\widetilde{K}_\zeta)_0)^*\right) |\lambda|^{n+s}\, d\lambda.
$$

Since the above identities hold for all $\tau \in \mathcal{H}_s^2([-a,0))$ it follows that

$$
\beta_\lambda((\widetilde{K}_\zeta)_0) = e^{\lambda h}\chi_{[-a,0)}(\lambda)|\lambda|^{-s}P_0\beta_\lambda[z,t].
$$

From the inversion formula (5), and arguing as in the proof of [AMPS19, Corollary 4.3] to compute $\operatorname{tr}\left(P_0\beta_\lambda[z,t]\beta_\lambda[w,s]^*\right)$, we obtain that

$$
\begin{aligned}
K_\zeta(\omega) &= \frac{1}{(2\pi)^{n+1}} \int_{-a}^{0} e^{\lambda(h+k)} \operatorname{tr}\left(P_0\beta_\lambda[z,t]\beta_\lambda[w,s]^*\right) |\lambda|^{n-s} d\lambda \\
&= \frac{1}{(2\pi)^{n+1}} \int_{-a}^{0} e^{\lambda(h+k+\frac14|w-z|^2+i(t-s-\frac12 \operatorname{Im} w\cdot\bar z)} |\lambda|^{n-s} d\lambda
\end{aligned}
$$

and the conclusion follows. □

Remark 3.5. In particular, in the case $s=n$ the reproducing kernel $K(\omega,\zeta)$ of $\mathcal{PW}_a^n$ takes a more familiar expression, that involves the sinc function, $\operatorname{sinc} z = \frac{\sin z}{z}$. Namely,

$$
K(\omega,\zeta) = (2\pi)^{-n-1} a e^{-iaQ(\omega,\zeta)} \operatorname{sinc}\left(aQ(\omega,\zeta)\right).
$$

We also observe that

$$
\begin{aligned}
\|K_\zeta\|^2_{\mathcal{PW}_a^s} &= \frac{1}{(2\pi)^{n+1}} \int_{-a}^{0} e^{2h\lambda} \| \,|\lambda|^{-s} P_0\beta_\lambda[z,t]\|^2_{\mathrm{HS}} \,|\lambda|^{n+s} d\lambda = \frac{1}{(2\pi)^{n+1}} \int_{-a}^{0} e^{2h\lambda} \,|\lambda|^{n-s} d\lambda \\
&\asymp_{s,a} e^{2ah_-}\,.
\end{aligned}
$$

In particular, $\|K_\zeta\|_{\mathcal{PW}_a^s} \asymp_{s,a} 1$ for $\zeta \in b\mathcal{U}$.

4. A representation theorem for $\mathcal{PW}_a^s$

In this section we prove a representation theorem for functions in $\mathcal{PW}_a^s$, for $s\in[0,n+1)$. We denote by $\mathscr{F}$ the 1-dimensional Euclidean Fourier transform, that is, for $f\in L^1(\mathbb{R})$,

$$
\mathscr{F} f(\xi) = \int_{\mathbb{R}} f(x) e^{-ix\xi}\, dx.
$$

Recall that $\mathscr{F}$ extends to a surjective isomorphism $\mathscr{F}: L^2(\mathbb{R}) \to L^2(\mathbb{R})$ where

$$\|f\|_{L^2(\mathbb{R})} = \frac{1}{2\pi}\|\mathscr{F}f\|_{L^2(\mathbb{R})}$$

and the inverse $\mathscr{F}^{-1}$ is defined as

$$\mathscr{F}^{-1}f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} f(\xi)e^{i\xi x}\,d\xi.$$

Theorem 4.1. *Let $F \in \mathcal{PW}_a^s$, $0 \leqslant s < n+1$. For $\zeta' \in \mathbb{C}^n$ fixed, define $f_{\zeta'}(\kappa) = F(\zeta',\kappa)$, where $\kappa = x+iy \in \mathbb{C}$. Then there exists $\phi : \mathbb{C}^n \times \mathbb{R} \to \mathbb{C}$ such that the function $\mathscr{F}^{-1}\phi(\zeta',\cdot)(x)$ extends to an entire function in the variable κ and it holds that*

$$F(\zeta',\zeta_{n+1}) = \mathscr{F}^{-1}\phi(\zeta',\cdot)(-\zeta_{n+1})\,, \tag{19}$$

Moreover, the function ϕ satisfies the following:

(i) *$\phi(\cdot,\lambda) \in \mathcal{F}^\lambda$ for a.e. $\lambda \in [-a,0)$;*

(ii) *$\|\phi(\cdot,\lambda)\|_{\mathcal{F}^\lambda} \in L^2\big([-a,0),|\lambda|^{s-n}d\lambda\big)$;*

(iii) *$\|F\|^2_{\mathcal{PW}_a^s} = \dfrac{1}{(2\pi)^{n+1}}\displaystyle\int_{-a}^0 \|\phi(\cdot,\lambda)\|^2_{\mathcal{F}^\lambda}\,|\lambda|^{s-n}\,d\lambda.$*

If $s = n$ we also have

(iv) *$\phi(\zeta',\cdot) \in L^2([-a,0))$ for all $\zeta' \in \mathbb{C}^n$; in particular $F(\zeta',\cdot)$ belongs to the one-dimensional Paley–Wiener space $PW_a(\mathbb{C})$ for all $\zeta' \in \mathbb{C}^n$.*

Proof. Observe that from Theorem 1.4 it follows that $\mathcal{PW}_a \cap \mathcal{PW}_a^s$ is dense in both spaces. Then, if $F \in \mathcal{PW}_a \cap \mathcal{PW}_a^s$, the computations that follow are all justified. By Theorem 1.4 we have

$$\begin{aligned}
F(\zeta',\zeta_{n+1}) = \widetilde{F}_h[z,t] &= \frac{1}{(2\pi)^{n+1}}\int_{-a}^0 e^{\lambda h}\,\mathrm{tr}\,\big(\beta_\lambda(\widetilde{F}_0)\beta_\lambda[z,t]^*\big)|\lambda|^n\,d\lambda\\
&= \frac{1}{2\pi}\int_{-a}^0 \Big(\frac{|\lambda|}{2\pi}\Big)^n \mathrm{tr}\,\big(\beta_\lambda(\widetilde{F}_0)\beta_\lambda[z,0]^*\big)e^{-i\lambda(t+ih)}\,d\lambda\,,\\
&= \frac{1}{2\pi}\int_{-a}^0 \Big(\frac{|\lambda|}{2\pi}\Big)^n \mathrm{tr}\,\big(\beta_\lambda(\widetilde{F}_0)\beta_\lambda[\zeta',0]^*\big)e^{-\frac{\lambda}{4}|\zeta'|^2}e^{-i\lambda\zeta_{n+1}}\,d\lambda\,.
\end{aligned}$$

Therefore, setting

$$\phi(\zeta',\lambda) = \big(|\cdot|/(2\pi)\big)^n\,\mathrm{tr}\,\big(\beta_\lambda(\widetilde{F}_0)\beta_\lambda[\zeta',0]^*\big)e^{-\frac{\lambda}{4}|\zeta'|^2}\chi_{[-a,0)}(\lambda)\,,$$

it follows that $F(\zeta', \zeta_{n+1}) = \mathscr{F}^{-1}\phi(\zeta', \cdot)(-\zeta_{n+1})$, that is, (19) holds. From (2) we deduce that $\phi(\cdot, \lambda)$ is entire, and using (19) and (11) it follows that

$$\begin{aligned}
\|F\|^2_{\mathcal{PW}^s_a} &= \int_{\mathbb{H}_n} |\Delta^{s/2}\widetilde{F}_0[z,t]|^2 \, dzdt \\
&= \int_{\mathbb{H}_n} | \, |T|^{s/2}\widetilde{F}_0[z,t]|^2 \, dzdt \\
&= \int_{\mathbb{C}^n}\int_{\mathbb{R}} | \, |T|^{s/2}F(z, t+\tfrac{i}{4}|z|^2)|^2 \, dt \, dz \\
&= \int_{\mathbb{C}^n}\int_{\mathbb{R}} |\mathscr{F}\big(|T|^{s/2}F(z, \cdot + \tfrac{i}{4}|z|^2)\big)(\lambda)|^2 \, d\lambda \, dz \\
&= \frac{1}{2\pi}\int_{-a}^{0} |\lambda|^s \int_{\mathbb{C}^n} \big|\phi(\zeta',\lambda)\big|^2 e^{-\frac{|\lambda|}{2}|z|^2} \, dz \, d\lambda \\
&= \frac{1}{(2\pi)^{n+1}}\int_{-a}^{0} \|\phi(\cdot,\lambda)\|^2_{\mathcal{F}^\lambda} \, |\lambda|^{s-n} \, d\lambda.
\end{aligned}$$

The conclusions (i-iii) now follow. About (iv), if P_0 denotes the orthogonal projection onto the subspace generated by e_0, we have

$$\begin{aligned}
|\big(\phi(\zeta',\cdot)|^2 &= |\big(|\cdot|/(2\pi)\big)^n \operatorname{tr}\big(\beta_\lambda(\widetilde{F}_0)\beta_\lambda[\zeta',0]^*\big)e^{-\frac{\lambda}{4}|\zeta'|^2}\chi_{[-a,0)}(\lambda)|^2 \\
&= |\big(|\cdot|/(2\pi)\big)^n \operatorname{tr}\big(\beta_\lambda(\widetilde{F}_0)P_0\beta_\lambda[\zeta',0]^*\big)e^{-\frac{\lambda}{4}|\zeta'|^2}\chi_{[-a,0)}(\lambda)|^2 \\
&\leqslant \|\beta_\lambda(\widetilde{F}_0)\|^2_{\mathrm{HS}}\|P_0\beta_\lambda[\zeta',0]\|^2_{\mathrm{HS}} e^{-\frac{\lambda}{2}|\zeta'|^2}|\big(|\cdot|/(2\pi)\big)^n\chi_{[-a,0)}(\lambda)|^2 \\
&\leqslant e^{\frac{a}{2}|\zeta'|^2}\|\beta_\lambda(\widetilde{F}_0)\|^2_{\mathrm{HS}}|\big(|\cdot|/(2\pi)\big)^n\chi_{[-a,0)}(\lambda)|^2
\end{aligned}$$

where we used $\lambda < 0$ and the identity $\|P_0\beta_\lambda[\zeta',0]\|^2_{\mathrm{HS}} = 1$. Since $F \in \mathcal{PW}^n_a$

$$\int_{-a}^{0} \|\beta_\lambda(\widetilde{F}_0)\|^2_{\mathrm{HS}}\big(|\lambda|/(2\pi)\big)^{2n} < \infty,$$

and this completes the proof. □

As a consequence, we have the following. For $0 \leqslant s < n+1$ we set

$$\Upsilon_s = \Big\{\phi : \mathbb{C}^n \times [-a,0) \to \mathbb{C} : \ (i)\, \phi(\cdot,\lambda) \in \mathcal{F}^\lambda \text{ for all } \lambda \in [-a,0), \\ (ii)\, \|\phi\|^2_\Upsilon := \frac{1}{(2\pi)^{n+1}}\int_{-a}^{0} \|\phi(\cdot,\lambda)\|^2_{\mathcal{F}^\lambda}|\lambda|^{s-n} \, d\lambda < \infty\Big\}.$$

Corollary 4.2. *For $0 \leqslant s < n+1$, denoting by t the real part of ζ_{n+1}, then the mapping*

$$\mathscr{U} : \mathcal{PW}^s_a \ni F \mapsto \mathscr{F}_{n+1}\big(F_{|_{\mathrm{Im}\,\zeta_{n+1}=0}}\big) \in \Upsilon_s$$

is a unitary map, where $\mathscr{F}_{n+1}$ denotes the Euclidean Fourier transform in the variable t; in particular

$$\|F\|^2_{\mathcal{PW}^s_a} = \frac{1}{(2\pi)^{n+1}}\int_{-a}^{0} \|\phi(\cdot,\lambda)\|^2_{\mathcal{F}^\lambda} \, |\lambda|^{s-n} \, d\lambda.$$

Proof. We only need to prove that the mapping is onto. Given $\phi \in \Upsilon_s$, setting $F(\zeta', \zeta_{n+1}) = \mathscr{F}^{-1}\phi(\zeta', \cdot)(-\zeta_{n+1})$ it is easy to see that $F \in \mathcal{PW}_a^s$ and the conclusion follows. □

5. Sampling in the Fock space

In this section we prove a result that it may be considered as *folklore*. We consider the 1-dimensional case and make explicit the dependence on λ of the sampling constant in the case of square lattices for the Fock space $\mathcal{F}^\lambda(\mathbb{C})$. However, we believe that the result is not completely obvious and it is key for our Theorem 1.6.

Lemma 5.1. *Let $a > 0$ be given, let $b > a$ and let L_b be the square lattice (6). Let $f \in \mathcal{F}^\lambda$ with $0 < \lambda \leqslant a$. Then, for any $\eta \in L_b$, the function*

$$F_\eta^\lambda(z) := e^{\frac{\lambda}{2} z\overline{\eta}} f(z - \eta)$$

belongs to the Fock space $\mathcal{F}^{b'}$ with $a < b' < b$.

Proof. We have

$$\begin{aligned}\int_{\mathbb{C}} |F_\eta^\lambda(z)|^2 e^{-\frac{b'}{2}|z|^2}\, dz &= \int_{\mathbb{C}} |e^{\frac{\lambda}{2}(z+\eta)\overline{\eta}} f(z)|^2 e^{-\frac{b'}{2}|z+\eta|^2}\, dz \\ &= e^{\frac{\lambda}{2}|\eta|^2} \int_{\mathbb{C}} |f(z)|^2 e^{-\frac{b'-\lambda}{2}|z+\eta|^2} e^{-\frac{\lambda}{2}|z|^2}\, dz \\ &\leqslant e^{\frac{\lambda}{2}|\eta|^2} \int_{\mathbb{C}} |f(z)|^2 e^{-\frac{\lambda}{2}|z|^2}\, dz\end{aligned}$$

and the conclusion follows. □

Lemma 5.2. *Let $a > 0$ be given, let $b > a$ and let L_b be the square lattice (6). For $0 < \lambda \leqslant a$ define the positive measure*

$$\mu_{L_b}^\lambda := \sum_{\gamma \in L_b} e^{-\frac{\lambda}{2}|\gamma|^2} \delta_\gamma$$

where δ_γ is the unit point measure at γ and consider the integral operator

$$f \mapsto Tf = \int_{\mathbb{C}} K_t(\cdot, \eta) f(\eta)\, d\mu_{L_b}^\lambda(\eta)$$

with positive kernel

$$K_t(\gamma, \eta) = e^{\frac{\lambda}{4}(|\gamma|^2 + |\eta|^2)} e^{-\frac{t-\lambda}{4}|\gamma+\eta|^2}, \qquad a < t < b.$$

Then, the operator T extends to a bounded operator $T : L^2(L_b, \mu_{L_b}^\lambda) \to L^2(L_b, \mu_{L_b}^\lambda)$ with operator norm uniformly bounded for $0 < \lambda \leqslant a$.

Proof. For $\gamma \in L_b$ we have

$$Tf(\gamma) = \sum_{\eta \in L_b} K_t(\gamma, \eta) f(\eta) e^{-\frac{\lambda}{2}|\eta|^2}.$$

By Schur's test [Gra14, Appendix A.2] it is enough to find $\varphi > 0$ and $C > 0$ such that

$$\sum_{\eta \in L_b} K_t(\gamma, \eta) \varphi(\eta) e^{-\frac{\lambda}{2}|\eta|^2} \leqslant C \varphi(\gamma).$$

This would also guarantee that the operator norm of T is bounded by C. Choosing $\varphi(\gamma) = e^{\frac{\lambda}{4}|\gamma|^2}$ the conclusion follows with a constant C independent of λ as we wished to show. ☐

Theorem 5.3. *Let $a > 0$ be given, let $b > a$ and let L_b be the square lattice (6). Then there exist constants $C_b, C_b' > 0$ such that for all $0 < \lambda \leqslant a$ and all $f \in \mathcal{F}^\lambda(\mathbb{C})$ we have*

$$\|f\|^2_{\mathcal{F}^\lambda} \leqslant C_b \lambda \sum_{\gamma\in L_b} |f(\gamma)|^2 e^{-\frac{\lambda}{2}|\gamma|^2} \leqslant C_b' \|f\|^2_{\mathcal{F}^\lambda}.$$

Proof. Let $0 < \lambda \leqslant a$ and let $f \in \mathcal{F}^\lambda$ be given. Let R_{L_b} be the fundamental region of the square lattice L_b and let $R_{L_b,\eta}$ be the translated region $R_{L_b} + \eta$ where $\eta \in L_b$. Then, $\mathbb{C} = \bigcup_{\eta\in L_b} R_{L_\lambda,\eta}$ and the intersections of the $R_{L_\lambda,\eta}$'s have Lebesgue measure zero. Therefore,

$$\begin{aligned} \|f\|^2_{\mathcal{F}^\lambda} &= \frac{\lambda}{2\pi} \sum_{\eta\in L_b} \int_{R_{L_b,\eta}} |f(z)|^2 e^{-\frac{\lambda}{2}|z|^2}\, dz \\ &= \frac{\lambda}{2\pi} \sum_{\eta\in L_b} \int_{R_{L_b}} |f(z-\eta)|^2 e^{-\frac{\lambda}{2}|z-\eta|^2}\, dz \\ &= \frac{\lambda}{2\pi} \sum_{\eta\in L_b} e^{-\frac{\lambda}{2}|\eta|^2} \int_{R_{L_b}} |e^{\frac{\lambda}{2}z\overline{\eta}} f(z-\eta)|^2 e^{-\frac{\lambda}{2}|z|^2}\, dz. \end{aligned} \tag{20}$$

In particular the factor $e^{-\frac{\lambda}{2}|z|^2}$ is bounded above and below on the region R_{L_b} with positive constants uniformly on z and λ. Hence, we conclude that

$$\|f\|^2_{\mathcal{F}^\lambda} \approx \lambda \sum_{\eta\in L_b} e^{-\frac{\lambda}{2}|\eta|^2} \int_{R_{L_b}} |e^{\frac{\lambda}{2}z\overline{\eta}} f(z-\eta)|^2\, dz, \tag{21}$$

that is, the two quantities are comparable up to some positive constants which do not depend on λ.

Now, setting $F^\lambda_\eta(z) = e^{\frac{\lambda}{2}z\overline{\eta}} f(z-\eta)$, Lemma 5.1 guarantees the decomposition

$$F^\lambda_\eta(z) = \sum_{\gamma\in L_b} \frac{F^\lambda_\eta(\gamma)}{\sigma'_{L_b}(\gamma)} \frac{\sigma_{L_b}(z)}{z-\gamma}.$$

Since $\left|\frac{\sigma_{L_b}(z)}{z-\gamma}\right| \leqslant C$ for a constant C which does not depend on $z \in R_{L_b}$ and $\gamma \in L_b$, we have

$$\begin{aligned} \|f\|^2_{\mathcal{F}^\lambda} &\leqslant C\lambda \sum_{\eta\in L_b} e^{-\frac{\lambda}{2}|\eta|^2} \int_{R_{L_b}} |e^{\frac{\lambda}{2}z\overline{\eta}} f(z-\eta)|^2\, dz \\ &\leqslant C\lambda \sum_{\eta\in L_b} e^{-\frac{\lambda}{2}|\eta|^2} \int_{R_{L_b}} \left(\sum_{\gamma\in L_b} \left| \frac{F^\lambda_\eta(\gamma)}{\sigma'_{L_b}(\gamma)} \right| \right)^2 dz \\ &\leqslant C \frac{2\pi\lambda}{b} \sum_{\eta\in L_b} e^{-\frac{\lambda}{2}|\eta|^2} \left(\sum_{\gamma\in L_b} \left| F^\lambda_\eta(\gamma) e^{-\frac{t}{4}|\gamma|^2} \right| \right)^2 \end{aligned}$$

where $a < t < b$ and we used the estimate (14). Now,

$$\frac{2\pi\lambda}{b}\sum_{\eta\in L_b} e^{-\frac{\lambda}{2}|\eta|^2}\Big(\sum_{\gamma\in L_b}\big|F^\lambda_\eta(\gamma)e^{-\frac{t}{4}|\gamma|^2}\big|\Big)^2 = \sum_{\eta\in L_b} e^{-\frac{\lambda}{2}|\eta|^2}\Big(\sum_{\gamma\in L_b}\big|e^{\frac{\lambda}{2}|\eta|^2+\frac{\lambda}{2}\gamma\overline{\eta}-\frac{t}{4}|\gamma+\eta|^2}f(\gamma)\big|\Big)^2$$
$$= \frac{2\pi\lambda}{b}\sum_{\eta\in L_b} e^{-\frac{\lambda}{2}|\eta|^2}\Big(\sum_{\gamma\in L_b} K_t(\gamma,\eta)|f(\gamma)|e^{-\frac{\lambda}{2}|\gamma|^2}\Big)^2,$$

where we have set

$$K_t(\gamma,\eta) = e^{\frac{\lambda}{4}(|\gamma|^2+|\eta|^2))}e^{-\frac{t-\lambda}{4}|\gamma+\eta|^2}.$$

Thus, from Lemma 5.2 we get

$$\sum_{\eta\in L_b} e^{-\frac{\lambda}{2}|\eta|^2}\Big(\sum_{\gamma\in L_b} K_t(\gamma,\eta)|f(\gamma)|e^{-\frac{\lambda}{2}|\gamma|^2}\Big)^2 \leqslant C\sum_{\eta\in L_b}|f(\eta)|^2 e^{-\frac{\lambda}{2}|\eta|^2}$$

where C does not depend on λ. In conclusion, we have

$$\|f\|^2_{\mathcal{F}^\lambda} \leqslant C_b\lambda\sum_{\eta\in L_b} e^{-\frac{\lambda}{2}|\eta|^2}\Big(\sum_{\gamma\in L_b}\big|F^\lambda_\eta(\gamma)e^{-\frac{t}{4}|\gamma|^2}\big|\Big)^2 \leqslant C_b\lambda\sum_{\eta\in L_b}|f(\eta)|^2 e^{-\frac{\lambda}{2}|\eta|^2}$$

with C_b independent of λ as we wished to show.

Next, denoting by $D(\gamma,r)$ the disk centered at $\gamma\in\mathbb{C}$ with radius $r>0$, we show that for all $f\in\mathrm{Hol}(\mathbb{C})$ and $d>0$ we have

$$|f(\gamma)|^2 e^{-d|\gamma|^2} \leqslant \frac{d}{\pi\big(1-e^{-dr^2}\big)}\int_{D(\gamma,r)}|f(w)|^2 e^{-d|w|^2}\,dw\,. \tag{22}$$

For, by the mean value formula we have that

$$|f(\gamma)|^2 \leqslant \frac{1}{2\pi}\int_0^{2\pi}|f(\gamma+re^{i\theta})|^2\,d\theta\,,$$

so that,

$$\begin{aligned}\int_{D(\gamma,r)}|f(w)|^2 e^{-d|w|^2}\,dw &= \int_{D(0,r)}|f(w+\gamma)|^2 e^{-d|w+\gamma|^2}\,dw\\ &= \int_{D(0,r)}\big|f(w+\gamma)e^{-dw\overline{\gamma}}\big|^2 e^{-d(|w|^2+|\gamma|^2)}\,dw\\ &= \int_0^r e^{-d(s^2+|\gamma|^2)}\int_0^{2\pi}\big|f(se^{i\theta}+\gamma)e^{-dse^{i\theta}\overline{\gamma}}\big|^2\,d\theta s ds\\ &\geqslant 2\pi|f(\gamma)|^2 e^{-d|\gamma|^2}\int_0^r e^{-ds^2}\,sds\\ &= \frac{\pi}{d}\big(1-e^{-dr^2}\big)|f(\gamma)|^2 e^{-d|\gamma|^2},\end{aligned}$$

and (22) follows. Finally, given the lattice L_b, we let $0<r<\inf\{|\gamma_1-\gamma_2| : \gamma_1,\gamma_2\in L_b\}$. Then, the disks $\{D(\gamma,r) : \gamma\in L_b\}$ are disjoint so that, by (22) we have

$$\begin{aligned}\lambda\sum_{\gamma\in L_b}|f(\gamma)|^2 e^{-\frac{\lambda}{2}|\gamma|^2} &\leqslant \frac{\lambda}{\big(1-e^{-\frac{\lambda}{2}r^2}\big)}\sum_{\gamma\in L_b}\frac{\lambda}{2\pi}\int_{D(\gamma,r)}|f(w)|^2 e^{-\frac{\lambda}{2}|w|^2}\,dw\\ &\leqslant C_b'\|f\|^2_{\mathcal{F}^\lambda},\end{aligned}$$

where we have set

$$C_b' = \sup_{\lambda\in(0,a]} \frac{\lambda}{\big(1-e^{-\frac{\lambda}{2}r^2}\big)}. \quad \square$$

The following result now follows easily.

Corollary 5.4. *For $j=1,\dots,n$ let $a>0$ be given, let $b_j>a$ and set $\widetilde{L}=L_{b_1}\times\cdots\times L_{b_n}$. Then, there exist constants $C_b, C_b'>0$ such that for all $\lambda\in(0,a]$ and $f\in\mathcal{F}^\lambda(\mathbb{C}^n)$ we have*

$$\|f\|^2_{\mathcal{F}^\lambda} \leqslant C_b\lambda^n \sum_{\gamma\in\widetilde{L}} |f(\gamma)|^2 e^{-\frac{|\lambda|}{2}|\gamma|^2} \leqslant C_b'\|f\|^2_{\mathcal{F}^\lambda}.$$

6. Sampling in $\mathcal{PW}_a$

Before proving Theorem 1.6, we study a few properties of $\mathcal{PW}_a^n$. In particular we present some elements and produce an explicit orthonormal basis of such space. We also remark that because of the Fourier transform characterization of $\mathcal{PW}_a^n$ the Fock spaces $\mathcal{F}^\lambda$ that will appear in this section are defined for negative λ in $[-a,0)$ and that, by definition, $\mathcal{F}^\lambda=\mathcal{F}^{|\lambda|}$.

We use both the notation $\mathscr{F}g$ and $\widehat{g}$ to denote the 1-dimensional Euclidean Fourier transform of $g\in L^2(\mathbb{R})$. Let $g\in L^2(\mathbb{R})$ such that $\operatorname{supp}\widehat{g}\subseteq[-a,0]$, we set $G(\zeta',\zeta_{n+1})=g(\zeta_{n+1})$, where we denote by g its entire extension to $\mathbb{C}$ (notice that G is indipendent of $\zeta'\in\mathbb{C}^n$). Then we compute

$$\begin{aligned}
\|G\|^2_{\mathcal{PW}_a^n} &= \int_{\mathbb{H}_n} \big|\Delta^{n/2}\widetilde{G}_0[z,t]\big|^2\,dzdt = \int_{\mathbb{C}^n}\int_{\mathbb{R}} \big|\partial_t^{n/2}G\big(z,t+\tfrac{i}{4}|\zeta'|^2\big)\big|^2\,dzdt\\
&= \int_{\mathbb{C}^n}\int_{-a}^0 |\lambda|^n\big|(\mathscr{F}G)(z,\lambda)\big|^2 e^{\frac{\lambda}{2}|z|^2}\,d\lambda dz = \int_{\mathbb{C}^n}\int_{-a}^0 |\lambda|^n|\widehat{g}(\lambda)|^2 e^{\frac{\lambda}{2}|z|^2}\,d\lambda dz\\
&= (2\pi)^n\int_{-a}^0 |\widehat{g}(\lambda)|^2\,d\lambda.
\end{aligned}$$

Hence, $G\in\mathcal{PW}_a^n$ and $\|G\|^2_{\mathcal{PW}_a^n}=(2\pi)^{n+1}\|g\|^2_{L^2(\mathbb{R})}$. More generally, given a multiindex α, we set

$$G_\alpha(\zeta',\zeta_{n+1}) = \frac{1}{\sqrt{2^{|\alpha|}\alpha!}}(\zeta')^\alpha\partial_t^{|\alpha|/2}g(\zeta_{n+1}).$$

Lemma 6.1. *The following properties hold.*

(i) *For every α we have*

$$\|G_\alpha\|^2_{\mathcal{PW}_a^n} = (2\pi)^{n+1}\|g\|^2_{L^2(\mathbb{R})} = C_n\sum_{k\in\mathbb{Z}}\big|g\big(\tfrac{\pi}{a}k\big)\big|^2.$$

(ii) *Let $g_\ell\in L^2(\mathbb{R})$ be such that $\operatorname{supp}(\widehat{g}_\ell)\subseteq[-a,0]$, $\{\widehat{g}_\ell:\ell\in\mathbb{Z}\}$ is an orthonormal basis of $L^2(-a,0)$ and set*

$$G_{\alpha,\ell}(\zeta',\zeta_{n+1}) = \frac{1}{\sqrt{2^{|\alpha|}\alpha!}}(\zeta')^\alpha\partial_t^{|\alpha|/2}g_\ell(\zeta_{n+1}).$$

Then $\big\{G_{\alpha,\ell}:\alpha\in\mathbb{N}^n,\ \ell\in\mathbb{Z}\big\}$ is an orthonormal basis of $\mathcal{PW}_a^n$.

Proof. We observe that for any $r > 0$, $(\partial_t)^r g = \mathscr{F}^{-1}\big(|\lambda|^r \widehat{g}\big) \in PW_a$, see [MPS20b]. Then,

$$\begin{aligned}
\|G_\alpha\|^2_{\mathcal{PW}^n_a} &= \int_{\mathbb{C}^n}\int_{\mathbb{R}} \big|\partial_t^{(n+|\alpha|)/2} G_\alpha\big(z, t+\tfrac{i}{4}|\zeta'|^2\big)\big|^2\, dzdt \\
&= \frac{1}{2^{|\alpha|}\alpha!}\int_{\mathbb{C}^n} |z^\alpha|^2 \int_{-a}^0 |\lambda|^{n+|\alpha|}|\widehat{g}(\lambda)|^2 e^{\frac{\lambda}{2}|z|^2}\, d\lambda dz \\
&= (2\pi)^n \int_{-a}^0 |\widehat{g}(\lambda)|^2\, d\lambda = (2\pi)^{n+1}\|g\|^2_{L^2(\mathbb{R})}\,.
\end{aligned}$$

Conclusion (i) now follows from the classical Whittaker–Kotelnikov–Shannon theorem. In order to prove (ii) we argue in a similar fashion:

$$\begin{aligned}
\langle G_\alpha,\, G_\beta\rangle &= \int_{\mathbb{H}_n} \Delta^{n/2}\widetilde{(G_\alpha)_0}[z,t]\overline{\Delta^{n/2}\widetilde{(G_\beta)_0}[z,t]}\, dzdt \\
&= \int_{\mathbb{C}^n}\int_{\mathbb{R}} \big|\partial_t^{(n+|\alpha|)/2} G_\alpha\big(z, t+\tfrac{i}{4}|z|^2\big)\overline{\partial_t^{(n+|\beta|)/2} G_\beta\big(z, t+\tfrac{i}{4}|z|^2\big)}\, dzdt \\
&= \frac{1}{\sqrt{2^{|\alpha|+|\beta|}\alpha!\beta!}}\int_{\mathbb{C}^n} z^\alpha \overline{z^\beta}\int_{\mathbb{R}} \partial_t^{(n+|\alpha|)/2} g_\ell\big(t+\tfrac{i}{4}|z|^2\big)\overline{\partial_t^{(n+|\beta|)/2} g_m\big(t+\tfrac{i}{4}|z|^2\big)}\, dtdz \\
&= \frac{1}{\sqrt{2^{|\alpha|+|\beta|}\alpha!\beta!}}\int_{\mathbb{C}^n} z^\alpha \overline{z^\beta}\int_{\mathbb{R}} \partial_t^{(n+|\alpha|)/2} g_\ell\big(t+\tfrac{i}{4}|z|^2\big)\overline{\partial_t^{(n+|\beta|)/2} g_m\big(t+\tfrac{i}{4}|z|^2\big)}\, dtdz \\
&= c_n\frac{1}{\sqrt{2^{|\alpha|+|\beta|}\alpha!\beta!}}\int_{\mathbb{C}^n} z^\alpha \overline{z^\beta}\int_{-a}^0 |\lambda|^{n+(|\alpha|+|\beta|)/2}\widehat{g}_\ell(\lambda)\overline{\widehat{g}_m(\lambda)} e^{\frac{\lambda}{2}|z|^2}\, d\lambda dz \\
&= \delta_{\alpha,\beta}\langle g_\ell,\, g_m\rangle_{L^2(\mathbb{R})}\,.
\end{aligned}$$

Thus, $\{G_{\alpha,\ell}\}$ is an orthonormal system. We show that it also complete. Let $F \in \mathcal{PW}^n_a$ be orthogonal to $\{G_{\alpha,\ell} :\, \alpha \in \mathbb{N}^n, k \in \mathbb{Z}\}$. Using the same computation as above we see that

$$\begin{aligned}
\langle F,\, G_{\alpha,\ell}\rangle &= c_n\frac{1}{\sqrt{2^{|\alpha|}\alpha!}}\int_{\mathbb{C}^n}\overline{z^\alpha}\int_{-a}^0 |\lambda|^{n+|\alpha|/2}(\mathscr{F}F)(z,\lambda)\overline{\widehat{g}_\ell(\lambda)} e^{\frac{\lambda}{2}|z|^2}\, d\lambda dz \\
&= c_n\frac{1}{\sqrt{2^{|\alpha|}\alpha!}}\int_{-a}^0 |\lambda|^{n+|\alpha|/2}\int_{\mathbb{C}^n}(\mathscr{F}F)(z,\lambda)\overline{z^\alpha}e^{\frac{\lambda}{2}|z|^2}\, dz\,\overline{\widehat{g}_\ell(\lambda)}\, d\lambda \\
&= C_n\int_{-a}^0 (\mathscr{F}F)_{\alpha,\lambda}(\lambda)\overline{\widehat{g}_\ell(\lambda)}\, d\lambda,
\end{aligned}$$

where we denote by $(\mathscr{F}F)_{\alpha,\lambda}(\lambda)$ the Fourier coefficient of $\mathscr{F}F(\cdot,\lambda)$ in $\mathcal{F}^\lambda$ w.r.t. the basis $\{e_{\alpha,\lambda}\}$, that is, $e_{\alpha,\lambda} = z^\alpha/\|z^\alpha\|_{\mathcal{F}^\lambda}$. Since F is orthogonal to $\{G_{\alpha,\ell}\}$ for all $\ell \in \mathbb{Z}$, it follows that $(\mathscr{F}F)_{\alpha,\lambda}(\lambda) = 0$ λ-a.e., and then by Proposition 4.2 that $F = 0$. □

Proof of Theorem 1.6. For $F \in \mathcal{PW}^n_a$, let $\phi(\zeta',\lambda) = \mathscr{F}_{n+1}F(\zeta',\lambda)$. By Lemma 4.1

$$\|F\|^2_{\mathcal{PW}^n_a} = (2\pi)^{-n-1}\int_{-a}^0 \|\phi(\cdot,\lambda)\|^2_{\mathcal{F}^\lambda}\, d\lambda\,.$$

Given the sequence of points Γ as in the statement, by Corollary 5.4 we have

$$\begin{aligned}\|F\|^2_{\mathcal{PW}^n_a} &\leqslant \frac{C_\Gamma}{(2\pi)^{n+1}}\int_{-a}^0 |\lambda|^n \sum_{\gamma\in\Gamma} |\phi(\gamma,\lambda)|^2 e^{-\frac{|\lambda|}{2}|\gamma|^2}\, d\lambda \\ &= \frac{C_\Gamma}{(2\pi)^{n+1}}\sum_{\gamma\in\Gamma}\int_{-a}^0 |\lambda|^n |\phi(\gamma,\lambda)|^2 e^{-\frac{|\lambda|}{2}|\gamma|^2}\, d\lambda \\ &= \frac{C_\Gamma}{(2\pi)^{n}}\sum_{\gamma\in\Gamma}\int_{\mathbb{R}} |\partial_t^{n/2} F(\gamma, t+\tfrac{i}{4}|\gamma|^2)|^2\, dt \\ &= \frac{C_\Gamma}{(2\pi)^{n}}\sum_{\gamma\in\Gamma}\sum_{\ell\in\mathbb{Z}} |\partial_t^{n/2} F(\gamma, \tfrac{\pi}{a}\ell+\tfrac{i}{4}|\gamma|^2)|^2 ,\end{aligned}$$

where the last identity follows from (iv) in Lemma 4.1, the fact that $PW(\mathbb{C})$ is closed under (fractional) differentiation and the classical Whittaker–Kotelnikov–Shannon Theorem. Conversely, by the same sequences of equalities,

$$\begin{aligned}\sum_{\gamma\in\Gamma}\sum_{\ell\in\mathbb{Z}} |\partial_t^{n/2} F(\gamma, \tfrac{\pi}{a}\ell+\tfrac{i}{4}|\gamma|^2)|^2 &= (2\pi)^{-2n-1}\int_{-a}^0 \sum_{\gamma\in\Gamma} |\lambda|^n |\phi(\gamma,\lambda)|^2 e^{-\frac{|\lambda|}{2}|\gamma|^2}\, d\lambda \\ &\leqslant C_2'(2\pi)^{-n}\int_{-a}^0 \|\phi(\cdot,\lambda)\|^2_{\mathcal{F}^\lambda}\, d\lambda \\ &= C_2'(2\pi)\|F\|^2_{\mathcal{PW}^n_a} .\end{aligned}$$

This proves the theorem, with $C_1 = C_1'/(2\pi)^n$ and $C_2 = C_2'(2\pi)$.

Finally, let $G \in \mathcal{PW}_a$ be given. Consider $\widetilde{G}_0$ and for $\varepsilon > 0$ define

$$\Psi_\varepsilon[z,t] = \frac{1}{(2\pi)^{n+1}}\int_{-a}^{-\varepsilon} \operatorname{tr}\big(\beta_\lambda(\widetilde{G}_0)\beta_\lambda[z,t]^*\big)|\lambda|^{n/2}\, d\lambda .$$

By (4) it follows that $\Psi_\varepsilon \in L^2(\mathbb{H}_n)$ and that

$$\beta_\lambda(\Psi_\varepsilon) = \chi_{[-a,-\varepsilon]}(\lambda)|\lambda|^{-n/2}\beta_\lambda(\widetilde{G}_0) \quad \text{and} \quad \beta_\lambda(\Delta^{n/2}\Psi_\varepsilon) = \alpha\chi_{[-a,-\varepsilon]}(\lambda)\beta_\lambda(\widetilde{G}_0)$$

for some constant α, $|\alpha| = 1$. Since $\mathscr{F}_{n+1}\widetilde{G}_0 \in \mathcal{H}^2\big([-a,-\varepsilon)\big)$, Theorem 1.4 implies that Ψ_ε extends to a function $F_\varepsilon \in \mathcal{PW}_a \cap \mathcal{PW}^n_a$. Moreover, the sequence $\{\Delta^{n/2}\Psi_\varepsilon\}$ is a Cauchy sequence in $L^2(\mathbb{H}_n)$, that is, $\{F_\varepsilon\}$ is a Cauchy sequence in $\mathcal{PW}^n_a$. Let F be its limit. It is clear that $\widetilde{F}_0 = \mathcal{I}_n\widetilde{G}_0$, where $\mathcal{I}_n$ is the inverse of $\Delta^{n/2}$ (see Proposition 2.1), that is, $\Delta^{n/2}\widetilde{F}_0 = \widetilde{G}_0$. Therefore, by the first part of the theorem,

$$\begin{aligned}\|G\|^2_{\mathcal{PW}_a} = \|F\|^2_{\mathcal{PW}^n_a} &\leqslant C_\Gamma\sum_{\gamma\in\Gamma}|\partial_t^{n/2}F(\gamma)|^2 = C_\Gamma\sum_{\gamma\in\Gamma}|\Delta_t^{n/2}F(\gamma)|^2 \\ &= C_\Gamma\sum_{\gamma\in\Gamma}|G(\gamma)|^2 = C_\Gamma\sum_{\gamma\in\Gamma}|\Delta_t^{n/2}F(\gamma)|^2 \\ &\leqslant C_\Gamma'\|F\|^2_{\mathcal{PW}^n_a} = C_\Gamma'\|G\|^2_{\mathcal{PW}_a}.\end{aligned}$$

This proves the theorem. □

As a consequence we have

Corollary 6.2. *The space $\mathcal{PW}_a^n$ admits a frame of reproducing kernels, namely $\{K_\gamma : \gamma \in \Gamma\}$, where Γ is a lattice as in Theorem 1.6.*

7. Final remarks and open questions

We believe that the spaces we introduced are worth investigating and, for instance in light of formula (18), arise quite naturally in our multi-dimensional setting. The present work leaves some open questions. First of all, it should be proved a more general version of Theorem 1.6 by combining the characterization of sampling sequences for the 1-dimensional Paley–Wiener space PW_a and some sufficient conditions for sampling sequences for the Fock space $\mathcal{F}(\mathbb{C}^n)$ as in [GL20].

Moreover, in this paper we essentially dealt with the Hilbert case and we left the case $p \neq 2$ for future studies.

Finally, our formulas and results suggest that the space $\mathcal{PW}_a^n$ might have a privileged role, as for the case of the Drury–Arveson space, see [ACM+21] for a study of such space on the Siegel domain $\mathcal{U}$.

References

[ACM+21] N. Arcozzi, N. Chalmoukis, A. Monguzzi, M. M. Peloso, and M. Salvatori, *The Drury–Arveson space on the Siegel upper half-space and a von Neumann type inequality*, 2021.

[AMPS19] N. Arcozzi, A. Monguzzi, M. M. Peloso, and M. Salvatori, *Paley-Wiener theorems on the Siegel upper half-space*, J. Fourier Anal. Appl. **25** (2019), no. 4, 1958–1986.

[CPar] M. Calzi and M. M. Peloso, *Holomorphic function spaces on Homogeneous Siegel Domains*, Diss. Math. (to appear), pp. 173.

[Cyg81] J. Cygan, *Subadditivity of homogeneous norms on certain nilpotent Lie groups*, Proc. Amer. Math. Soc. **83** (1981), no. 1, 69–70.

[DT06] S. Dragomir and G. Tomassini, *Differential geometry and analysis on CR manifolds*, Progress in Mathematics, vol. 246, Birkhäuser Boston, Inc., Boston, MA, 2006.

[Fef79] C. Fefferman, *Parabolic invariant theory in complex analysis*, Adv. in Math. **31** (1979), no. 2, 131–262.

[Fol75] G. B. Folland, *Subelliptic estimates and function spaces on nilpotent Lie groups*, Ark. Mat. **13** (1975), no. 2, 161–207.

[Fol89] ———, *Harmonic analysis in phase space*, Annals of Mathematics Studies, vol. 122, Princeton University Press, Princeton, NJ, 1989.

[GHOCR19] K. Gröchenig, A. Haimi, J. Ortega-Cerdà, and J. L. Romero, *Strict density inequalities for sampling and interpolation in weighted spaces of holomorphic functions*, J. Funct. Anal. **277** (2019), no. 12, 108282, 34.

[GJM20] K. Gröchenig, P. Jaming, and E. Malinnikova, *Zeros of the Wigner distribution and the short-time Fourier transform*, Rev. Mat. Complut. **33** (2020), no. 3, 723–744.

[GL20] K. Gröchenig and Y. Lyubarskii, *Sampling of entire functions of several complex variables on a lattice and multivariate Gabor frames*, Complex Var. Elliptic Equ. **65** (2020), no. 10, 1717–1735.

[Gra14] L. Grafakos, *Modern Fourier analysis*, third ed., Graduate Texts in Mathematics, vol. 250, Springer, New York, 2014.

[Kom66] H. Komatsu, *Fractional powers of operators*, Pacific J. Math. **19** (1966), 285–346.

[Lan67] H. J. Landau, *Necessary density conditions for sampling and interpolation of certain entire functions*, Acta Math. **117** (1967), 37–52. MR 222554

[MPS20a] A. Monguzzi, M. M. Peloso, and M. Salvatori, *Fractional Laplacian, homogeneous Sobolev spaces and their realizations*, Ann. Mat. Pura Appl. (4) **199** (2020), no. 6, 2243–2261. MR 4165680

[MPS20b] ———, *Fractional Paley-Wiener and Bernstein spaces*, Coll. Math. (2020), 1–29.

[OU12] A. Olevskii and A. Ulanovskii, *On multi-dimensional sampling and interpolation*, Analysis and Mathematical Physics **2** (2012), no. 2, 149–170.

[OV79] R. D. Ogden and S. Vági, *Harmonic analysis of a nilpotent group and function theory of Siegel domains of type* II, Adv. in Math. **33** (1979), no. 1, 31–92. MR 540636

[PS16] Marco M. Peloso and Maura Salvatori, *On some spaces of holomorphic functions of exponential growth on a half-plane*, Concr. Oper. **3** (2016), no. 1, 52–67. MR 3491883

[Ric92] F. Ricci, *Harmonic Analysis on the Heisenberg Group*, Unpublished notes, Cortona (1992), ii+67.

[Str91] Robert S. Strichartz, *L^p harmonic analysis and Radon transforms on the Heisenberg group*, J. Funct. Anal. **96** (1991), no. 2, 350–406. MR 1101262

[SW71] E. M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton University Press, Princeton, N.J., 1971, Princeton Mathematical Series, No. 32.

[Tha93] S. Thangavelu, *On Paley-Wiener theorems for the Heisenberg group*, J. Funct. Anal. **115** (1993), no. 1, 24–44.

[Tha98] ———, *Harmonic analysis on the Heisenberg group*, Progress in Mathematics, vol. 159, Birkhäuser Boston, Inc., Boston, MA, 1998.

[You01] R. M. Young, *An introduction to nonharmonic Fourier series*, first ed., Academic Press, Inc., San Diego, CA, 2001.

[Zhu12] K. Zhu, *Analysis on Fock spaces*, Graduate Texts in Mathematics, vol. 263, Springer, New York, 2012.

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