# Linear perturbations of metrics with holonomy Spin(7)

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**Abstract**

We apply the method of linear perturbations to the case of Spin(7)-structures, showing that the only nontrivial perturbations are those determined by a rank one nilpotent matrix.

We consider linear perturbations of the Bryant-Salamon metric on the spin bundle over $S^4$ that retain invariance under the action of Sp(2), showing that the metrics obtained in this way are isometric.

Riemannian metrics with holonomy Spin(7) have been studied in differential geometry since the celebrated theorem of Berger [3], listing the possible holonomy groups of an irreducible, nonsymmetric simply connected Riemannian manifold. Metrics with holonomy contained in Spin(7) are known to be Ricci-flat [4], and they imply the presence of a parallel spinor [24]. They are also relevant for string theory (see [14]).

The first local examples of metrics with holonomy Spin(7) were constructed in [5], and the first complete metric was obtained in [6]; the latter takes the form of an explicit Sp(2)-invariant metric on the spinor bundle over $S^4$. It was later shown in [10] that this metric belongs to a one-parameter family of invariant metrics.

We note that the metrics of [6] are of cohomogeneity one; other cohomogeneity one metrics with holonomy contained in Spin(7) have been constructed in [11, 18, 14, 23, 9, 7, 1, 2]. Outside of the cohomogeneity one setting other constructions exist, but the metrics they determine are not explicit (see [16, 17, 13, 20]).

As observed in [5], a metric with holonomy contained in Spin(7) is defined by a closed form $\Omega$ which is pointwise linearly equivalent to a reference 4-form on $\mathbb{R}^8$ with stabilizer Spin(7). It is then possible to define perturbations of a Spin(7)-metric by replacing $\Omega$ with a perturbed form $\Omega + \delta$ which remains pointwise linearly equivalent to $\Omega$. Notice that for the parallel 3-forms $\varphi$ arising in the context of holonomy $G_2$ the form $\varphi + \delta$ is always linearly equivalent to $\varphi$ for $\delta$ sufficiently small; in other terms, $\varphi$ is stable in the sense of [15]. The Spin(7) form $\Omega$ is not stable, however, so more work is needed in order to obtain a perturbation.

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One possible approach was considered in [19, Section 5.2] by taking

$$\delta = v^\flat \wedge (w \lrcorner \Omega) - w^\flat \wedge (v \lrcorner \Omega), \tag{1}$$

for $v, w$ vector fields on $M$. In terms of the infinitesimal action $\rho$ of $\mathfrak{gl}(T_xM)$ on $\Lambda^4 T_x^* M$, this amounts to setting $\delta = \rho(A)\Omega$, where $A$ is the skew-symmetric endomorphism $A = v^\flat \otimes w - w^\flat \otimes v$. We recall that under Spin(7) the bundle of four-forms splits as

$$\Lambda^4_1 \oplus \Lambda^4_7 \oplus \Lambda^4_{27} \oplus \Lambda^4_{35}; \tag{2}$$

the skew-symmetric $A$ determines a perturbation term $\delta$ in $\Lambda^4_7$. Whilst this construction gives nontrivial perturbations of the original metric in the case of $G_2$ (mutatis mutandis: the relevant decomposition is $\Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27}$ and the perturbation $\delta$ an element of $\Lambda^3_7$), it turns out that in the Spin(7) case the perturbed form never defines a Spin(7)-structure ([19]).

A different ansatz was considered in [8] in the context of Sp(2)Sp(1)-structures, which amounts to imposing that $A$ be nilpotent, rather than skew-symmetric. The key observation, working at a point, is that when

$$\rho(A)(\rho(A)\Omega) = 0, \tag{3}$$

the form

$$\Omega + t\delta, \quad \delta = \rho(A)\Omega$$

is always in the same $GL(8, \mathbb{R})$-orbit as $\Omega$ for any $t$; one then says that $\delta$ is a linear perturbation of $\Omega$. It turns out (see [8]) that one can assume $A$ to be nilpotent without loss of generality.

In this paper we study nilpotent perturbations of the Spin(7)-form $\Omega$. By a case-by-case analysis of the possible Jordan forms of a nilpotent matrix in $\mathfrak{gl}(8, \mathbb{R})$, and making use of Spin(7)-invariance of (3), we prove that any linear perturbation of the Spin(7) form $\Omega$ is defined by a rank one nilpotent matrix, i.e. it has the form

$$\delta = v^\flat \wedge (w \lrcorner \Omega),$$

with $v, w$ orthogonal vector fields. In terms of (2), the resulting perturbations of the Spin(7) form turn out to be elements of $\Lambda^4_7 \oplus \Lambda^4_{35}$.

We apply the method of linear perturbations to the Bryant-Salamon metric; we construct a family of linear perturbations parameterized by three functions of one variable. However, it turns out that the resulting metrics are isometric; due to the fact that nilpotent perturbations preserve volumes, we do not recover the squashed deformations of [10].

Our result complements the result of [21], stating that the Bryant-Salamon is rigid in the class of asymptotically conical Spin(7) metrics.

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# 1 Linear perturbations

In this section we classify linear perturbation at a point of 4-forms defining a Spin(7)-structure, proving that they are in one-to-one correspondence with nilpotent matrices of rank one in $\mathfrak{gl}(8, \mathbb{R})$.

We first recall some results from [8]. For a lighter notation, we shall write $\mathbb{R}^{n*}$ instead of $(\mathbb{R}^n)^*$. Denote by

$$\mathfrak{gl}(\mathbb{R}^{n*}) \times \Lambda^k\mathbb{R}^{n*} \to \Lambda^k\mathbb{R}^{n*}, \quad (A,\omega) \mapsto \rho(A)\omega$$

the natural action of $\mathfrak{gl}(\mathbb{R}^{n*})$ on $\Lambda^k\mathbb{R}^{n*}$. We shall write $\rho(A)^2\omega$ for $\rho(A)(\rho(A)\omega)$.

**Proposition 1.1** ([8])**.** *Fix $\omega \in \Lambda^k\mathbb{R}^{n*}$ and a solution $A \in \mathfrak{gl}(\mathbb{R}^{n*})$ of*

$$\rho(A)^2\omega = 0. \tag{4}$$

*Then*

$$\beta_t = \omega + t\rho(A)\omega$$

*lies in the same* $\mathrm{GL}(n,\mathbb{R})$*-orbit as $\omega$ for all $t \in \mathbb{R}$.*

It turns out that there is no loss of generality in assuming that $A$ is nilpotent. Indeed, we can apply the Jordan decomposition and write $A = S + N$, where $S$ is semisimple and $N$ is nilpotent. We have the following:

**Proposition 1.2** ([8])**.** *Let $\omega \in \Lambda^k\mathbb{R}^{n*}$ and $A \in \mathfrak{gl}(\mathbb{R}^{n*})$ a solution of* (4) *with Jordan decomposition $A = S + N$. Then*

$$\rho(N)\omega = \rho(A)\omega,\ \rho(N)^2\omega = 0.$$

*Remark* 1.3. Let $v \in \mathbb{R}^n$, $\alpha \in V^*$ and $\omega \in \Lambda^p\mathbb{R}^{n*}$. Then $\rho(v\otimes\alpha)\omega = \alpha \wedge (v \lrcorner\, \omega)$.

Indeed it suffices to prove the claim for $p = 2$: let $\varepsilon_1, \varepsilon_2 \in \mathbb{R}^{n*}$, then

$$\begin{aligned}\rho(v\otimes\alpha)\varepsilon_1\wedge\varepsilon_2 &= (v\otimes\alpha)(\varepsilon_1)\wedge\varepsilon_2 + \varepsilon_1\wedge(v\otimes\alpha)(\varepsilon_2)\\ &= \varepsilon_1(v)\alpha\wedge\varepsilon_2 + \varepsilon_1\wedge\varepsilon_2(v)\alpha = \alpha\wedge(v\lrcorner\,\varepsilon_1)\wedge\varepsilon_2 - \alpha\wedge\varepsilon_1\wedge(v\lrcorner\,\varepsilon_2)\\ &= \alpha\wedge\big(v\lrcorner\,(\varepsilon_1\wedge\varepsilon_2)\big)\end{aligned}$$

where last equality follows from Leibnitz's rule.

*Remark* 1.4. Let $A$ be a nilpotent, rank-one endomorphism of $\mathbb{R}^{n*}$, then

$$\rho(A)^2 = 0.$$

In particular $A$ is a solution of (4) for all $\omega$.

Indeed if $A$ has rank 1 there exists a basis $v^1, \dots, v^n$ of $\mathbb{R}^{n*}$ such that

$$Av^1 = v^2, \quad Av^2 = \cdots = Av^n = 0.$$

We can write $A$ in tensorial form as $A = v_1 \otimes v^2$, where $v_1, \dots, v_n$ is the corresponding dual basis in $\mathbb{R}^n$. Let $\omega \in \Lambda^p\mathbb{R}^{n*}$; we have

$$\begin{aligned}\rho(v_1\otimes v^2)^2\omega = v^2\wedge\Big(v_1\lrcorner\,\big(v^2\wedge(v_1\lrcorner\,\omega)\big)\Big)&\\ = v^2\wedge\big((v_1\lrcorner\, v^2)\wedge(v_1\lrcorner\,\omega) - v^2\wedge(v_1\lrcorner\, v_1\lrcorner\,\omega)\big) &= 0\end{aligned}$$

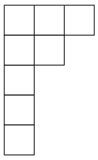
where the first identity follows from Remark 1.3 and the second one holds by the Leibnitz rule for $\lrcorner$.

Recall that if $e_1, \dots, e_8$ is the standard basis of $\mathbb{R}^8$ and $\alpha, \beta, \Omega$ are the linear forms defined by

$$\alpha = e^{12} + e^{34} + e^{56} + e^{78},$$
$$\beta = (e^1 + ie^2) \wedge (e^3 + ie^4) \wedge (e^5 + ie^6) \wedge (e^7 + ie^8),$$
$$\Omega = \frac{\alpha^2}{2} + Re(\beta),$$

then the stabilizer in $\mathrm{GL}(8, \mathbb{R})$ of the 4-form $\Omega$ is a subgroup of $\mathrm{SO}(8)$ isomorphic to $\mathrm{Spin}(7)$ (see [4, 5]). Moreover, $\mathrm{Spin}(7)$ acts transitively on the sphere $S^7 \subset \mathbb{R}^8$, and the stabilizer of $e_8$ is isomorphic to $G_2$, which acts transitively on the sphere $S^6 \subset \mathbb{R}^7 \cong \mathbb{R}^7 \times \{0\}$. From now on we shall make the identifications $\mathrm{Spin}(7) = \mathrm{Stab}(\Omega)$, $G_2 = \mathrm{Stab}(\Omega) \cap \mathrm{Stab}(e_8)$. Giving a $\mathrm{Spin}(7)$-structure on a 8-manifold amounts to giving a 4-form linearly equivalent to $\Omega$ at each point.

Thus, we are interested in linear perturbations of $\Omega$; in particular, we set $n = 8$ and $k = 4$. Up to change of basis, nilpotent matrices are classified over the reals by partitions with weight 8, giving 22 possibilities that can be encoded in terms of Young diagrams. For example, the diagram

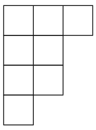
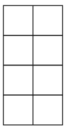
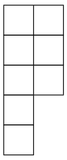
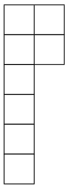
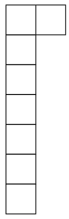
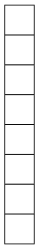
$\Gamma =$  .

describes an endomorphism of $\mathbb{R}^{8^*}$ with Jordan blocks of size $(3, 2, 1, 1, 1)$, which, with respect to some basis $\{w^1, v^2, v^3, w^4, v^5, v^6, v^7, v^8\}$, satisfies

$$w^1 \mapsto v^2 \mapsto v^3 \mapsto 0$$
$$w^4 \mapsto v^5 \mapsto 0$$
$$v^6, v^7, v^8 \mapsto 0.$$

In the rest of this paper we will use the notation illustrated in the last example: for each Jordan block $J_i$ of dimension $r \geq 2$ we fix an element $w^i$ such that $w^i, Aw^i, \dots, A^{r-1}w^i$ are linearly indipendent, and denote the other basis elements by $v^j$. The dual basis of $\mathbb{R}^n$ will be denoted by $\{w_i, v_j\}$.

In the following, we will need to consider the Young diagrams

$\Gamma_1 =$  , $\Gamma_2 =$  , $\Gamma_3 =$  , $\Gamma_4 =$  , $\Gamma_5 =$  , $\Gamma_6 =$  .

describing six particular configurations of Jordan blocks. Notice that $\Gamma_5$ corresponds to rank-one nilpotent endomorphisms and $\Gamma_6$ to zero.

Given a four-form $\omega$ on $\mathbb{R}^8$ and two vectors $u, v \in \mathbb{R}^8$, we will say that the contraction $u \lrcorner v \lrcorner \omega$ is degenerate if so is the bilinear form induced on the quotient $\mathbb{R}^8 / \operatorname{Span}\{u, v\}$, i.e.

$$(u \lrcorner v \lrcorner \omega)^3 \neq 0.$$

**Lemma 1.5.** *Fix $\omega \in \Lambda^4\mathbb{R}^{8^*}$ and let $A \in \mathfrak{gl}(\mathbb{R}^{n*})$ be a nilpotent solution of* (4)*. If $A$ has diagram $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ and $\{w^i, v^k\}$ is a Jordan basis of $A$, then*

$$w_i \lrcorner\, w_j \lrcorner\, \omega \text{ is degenerate for all } i, j. \tag{5}$$

*Proof. Case* $\Gamma_2$: writing $A$ in tensorial form we have

$$A = \sum_{i=1}^{4} w_i \otimes v^i.$$

The following hold:

$$\begin{aligned}\rho(A)^2\omega = \rho\Big(\sum_{i=1}^{4} w_i \otimes v^i\Big)^2 \omega &= \sum_{1\le i,j\le 4} \rho(w_i \otimes v^i)\rho(w_j \otimes v^j)\omega \\ &= 2\sum_{1\le i<j\le 4} v^i \wedge v^j \wedge (w_i \lrcorner\, w_j \lrcorner\, \omega).\end{aligned}$$

The second equality follows from the identities

$$\begin{aligned}\rho(w_i \otimes v^i)\rho(w_j \otimes v^j) &= \rho(w_j \otimes v^j)\rho(w_i \otimes v^i), \\ \rho(w_i \otimes v^i)^2 &= 0,\end{aligned}$$

(easy consequences of Remark 1.3 and Remark 1.4), and the last equality holds because of Remark 1.3. Thus, we can write (4) in the form

$$\sum_{1\le i<j\le 4} v^{ij} \wedge (w_i \lrcorner\, w_j \lrcorner\, \omega) = 0. \tag{6}$$

Contracting by $w_k$, multiplying with $v^l$ and using Remark 1.3 and Remark 1.4 we obtain the following identities:

$$v^{lij} \wedge (w_k \lrcorner\, w_i \lrcorner\, w_j \lrcorner\, \omega) = 0 \qquad \forall i,j,k,l \quad : \quad \{i,j,l,k\} = \{1,2,3,4\}. \tag{7}$$

We can decompose $\omega$ as

$$\omega = \sum_{i=1}^{4} w^i \wedge \alpha_i + \sum_{1\le i<j\le 4} w^{ij} \wedge \beta_{ij} + \sum_{1\le i<j<k\le 4} w^{ijk} \wedge \gamma_{ijk} + \delta w^{1234} + \varepsilon \tag{8}$$

$$\alpha_i, \beta_{ij}, \gamma_{ijk}, \varepsilon \in \Lambda \ \operatorname{Span}\{v^1, \dots, v^4\} \quad , \quad \delta \in \mathbb{R}.$$

We have that (7) implies

$$\delta = 0, \quad \gamma_{ijk} = c^l v^l \qquad \forall\, \{i,j,l,k\} = \{1,2,3,4\}, \quad c^l \in \mathbb{R}. \tag{9}$$

Notice that in order to prove the degeneracy of $w_i \lrcorner\, w_j \lrcorner\, \omega$ it is sufficient to prove $c^l = 0$ for $l = 1,2,3,4$. Substituting (8) and (9) in (6) and writing (6) in the form $w^1 \wedge I_1 + w^2 \wedge I_2 + w^3 \wedge I_3 + w^4 \wedge I_4 = 0$ it turns out that $I_1 = 0 = I_2 = I_3 = I_4$; this implies the linear system

$$\begin{pmatrix} 1 & -1 & 0 & -1 \\ 0 & -1 & 1 & -1 \\ 1 & 0 & -1 & 1 \\ -1 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} c^1 \\ c^2 \\ c^3 \\ c^4 \end{pmatrix} = 0;$$

by nonsingularity of the matrix, we have $c^l = 0$, $l = 1, \dots, 4$.

*Case* $\Gamma_1$: this time we have

$$A = v_4 \otimes v^1 + \sum_{i=1}^{3} w_i \otimes v^i.$$

Arguing as in case $\Gamma_2$, Equation (4) can be written as

$$2 \sum_{1 \le i < j \le 3} v^{ij} \wedge (w_i \lrcorner w_j \lrcorner \omega) + v^4 \wedge \left( w_1 \lrcorner \omega + 2 \sum_{i=1}^{3} v^i \wedge (w_i \lrcorner v_1 \lrcorner \omega) \right) = 0. \quad (10)$$

Multiplying by $v^4$ and contracting by $w_k$ with $k = 1, 2, 3$ gives

$$v^{ij4} \wedge (w_3 \lrcorner w_2 \lrcorner w_1 \lrcorner \omega) = 0 \quad \forall\, 1 \le i < j \le 3. \quad (11)$$

Similarly as in the case of $\Gamma_2$, we write

$$\omega = \sum_{i=1}^{3} w^i \wedge \alpha_i + \sum_{1 \le i < j \le 3} w^{ij} \wedge \beta_{ij} + w^{123} \wedge \gamma_{123} + \delta w^{123} + \varepsilon, \quad (12)$$

and (11) gives

$$\gamma_{123} = \lambda v^4 \quad , \quad \lambda \in \mathbb{R}. \quad (13)$$

It is sufficient to prove $\lambda = 0$: substituting (12), (13) in (11) and writing

$$\beta_{12} = \sum_{i<j} y_{ij} v^{ij}; \quad \beta_{13} = \sum_{i<j} x_{ij} v^{ij}; \quad \beta_{23} = \sum_{i<j} z_{ij} v^{ij},$$

we obtain an equation of the form

$$I_1 v^{124} \wedge w^3 + I_2 v^{134} \wedge w^2 + I_3 v^{234} \wedge w^1 + \cdots = 0 \qquad I_1, I_2, I_3 \in \mathbb{R},$$

resulting in $I_1 = 0 = I_2 = I_3$; explicitly, we have the linear system

$$\begin{pmatrix} 1 & 3 & 0 \\ -1 & 0 & 3 \\ 1 & 2 & -2 \end{pmatrix} \begin{pmatrix} \lambda \\ x_{12} \\ y_{13} \end{pmatrix} = 0,$$

with nonsingular matrix, so $\lambda = x_{12} = y_{13} = 0$.

Cases $\Gamma_3, \Gamma_4$ are similar (and easier). □

We will need the following:

**Proposition 1.6.** *Let $u, v \in \mathbb{R}^8$ be linearly indipendent. Then $u \lrcorner v \lrcorner \Omega$ is non-degenerate.*

*Proof.* It is sufficient to prove the thesis with $u, v$ orthogonal and normalized, because the following hold:

$$(u \lrcorner v \lrcorner \Omega)^3 = \|u\|^3 \|v\|^3 \left( \frac{u}{\|u\|} \lrcorner \frac{v}{\|v\|} \lrcorner \Omega \right)^3,$$
$$u \lrcorner v \lrcorner \Omega = u \lrcorner \big(v - P_u v\big) \lrcorner \Omega,$$

where $P_u$ is the orthogonal projection onto the subspace generated by $u$. So let $u, v$ be orthogonal vectors in $S^7$; since Spin(7) acts transitively on $S^7$, there exists $R_1 \in \mathrm{Spin}(7)$ such that $R_1 v = e_8$; in particular $R_1$ is an isometry, so $R_1 u \perp R_1 v = e_8$ and $R_1 u \in \mathbb{R}^7$. It follows that $R_1 u \in S^6$, but $G_2$ is transitive on $S^6$ so there exists $R_2 \in G_2$ such that $R_2 R_1 u = e_7$. Setting $R = R_1^{-1} R_2^{-1}$ we have $u = Re_7$ and $v = Re_8$. For all $x, y \in \mathbb{R}^8$ we have

$$\begin{aligned}(u \lrcorner v \lrcorner \Omega)(x, y) &= \Omega(Re_7, Re_8, x, y) = \Omega(e_7, e_8, R^{-1}x, R^{-1}y) \\ &= (e_7 \lrcorner e_8 \lrcorner \Omega)(R^{-1}x, R^{-1}y) = (R^{-1})^*(e_7 \lrcorner e_8 \lrcorner \Omega)(x, y); \quad (14)\end{aligned}$$

the second equality holds by the Spin(7)-invariance of $\Omega$. So from (14) we have

$$(u \lrcorner v \lrcorner \Omega)^3 = (R^{-1})^*(e_7 \lrcorner e_8 \lrcorner \Omega)^3,$$

but

$$(e_7 \lrcorner e_8 \lrcorner \Omega)^3 = (e^{35} + e^{48} + e^{67})^3 = 6e^{354867} \neq 0. \qquad \square$$

We can finally prove:

**Theorem 1.7.** *If $\rho(A)\Omega$ is a linear perturbation of $\Omega$, i.e. $\rho(A)^2\Omega = 0$, then the nilpotent part of $A$ has rank at most one.*

*Proof.* For each diagram $\Gamma$, we can fix a representative endomorphism $A_\Gamma$ and compute the space

$$K_\Gamma = \left\{ \omega \in \Lambda^4 \mathbb{R}^{8*} \;\middle|\; \rho(A_\Gamma)^2 \omega = 0 \right\}.$$

The equation $\rho(A)^2\Omega = 0$ has a solution with diagram $\Gamma$ if $\rho(A_\Gamma)^2\omega = 0$ for some $\omega$ in the same $\mathrm{GL}(8, \mathbb{R})$-orbit as $\Omega$; by Proposition 1.6, this implies that for any linearly indipendent vectors $u, v \in \mathbb{R}^8$ the map

$$\begin{aligned} K_\Gamma &\to \Lambda^4 \mathbb{R}^{8*} \\ \omega &\mapsto (u \lrcorner v \lrcorner \omega)^3 \end{aligned}$$

is not identically zero. As observed in [8], this rules out all cases except $\Gamma_1, \dots, \Gamma_6$. Let $A$ be a solution with $\Gamma$ one of the remaining diagrams. Using again Proposition 1.6, we have that $u \lrcorner v \lrcorner \Omega$ is nondegenerate for any choice of linearly independent vectors $u, v \in \mathbb{R}^8$; it follows from Lemma 1.5 that all nilpotent solutions of $\rho(A)^2\Omega = 0$ are either zero or rank-one nilpotent endomorphisms. $\square$

*Remark* 1.8. Linear perturbations of a Spin(7) form lie in the module $\Lambda^4_7 \oplus \Lambda^4_{35}$. Indeed, the map

$$\mathfrak{sl}(8, \mathbb{R}) \to \Lambda^4 \mathbb{R}^8, \quad A \mapsto \rho(A)\Omega,$$

is Spin(7)-equivariant and its kernel $\mathfrak{spin}(7)$ has dimension 21; the image is therefore the only Spin(7)-module of dimension 42 inside $\Lambda^4 \mathbb{R}^8$.

Notice that we cosider $\mathfrak{sl}(8, \mathbb{R})$ instead of $\mathfrak{gl}(8, \mathbb{R})$ because we assume $A$ to be nilpotent.

## 2 A cohomogeneity one description of the Bryant-Salamon metric

Recall from [6] that the spinor bundle $S$ over $S^4$ carries a cohomogeneity one metric with holonomy Spin(7); this metric has cohomogeneity one under the action of Sp(2). In this section we give a description of these metrics in terms of cohomogeneity one actions which will be needed in order to study the linear perturbations.

Explicitly, the Lie group $\mathrm{Sp}(2) = \{g \in \mathrm{GL}(2,\mathbb{H}) \mid gg^* = I\}$ contains two copies of Sp(1), i.e.

$$\mathrm{Sp}(1)_+ = \left\{ \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} |p \in \mathrm{Sp}(1) \right\}, \quad \mathrm{Sp}(1)_- = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix} |q \in \mathrm{Sp}(1) \right\}.$$

At the Lie algebra level,

$$\mathfrak{sp}(2) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & c \end{pmatrix} \right\}, \mathfrak{sp}_+ = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \right\}, \mathfrak{sp}_- = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix} \right\},$$

with $a, c \in \operatorname{Im}\mathbb{H}$, $b \in \mathbb{H}$.

The spinor bundle $S$ has the form

$$S = (\mathrm{Sp}(2) \times \mathbb{H})/(\mathrm{Sp}(1)_+ \times \mathrm{Sp}(1)_-),$$

where $(p, q) \in \mathrm{Sp}(1)_+ \times \mathrm{Sp}(1)_-$ acts on the right by

$$(g, v)(p, q) = (g(p, q), p^{-1}vq).$$

$S$ is of cohomogeneity one under the action of Sp(2); there is one singular orbit, namely $\mathrm{Sp}(2)/\mathrm{Sp}(1)_+ \times \mathrm{Sp}(1)_- = S^4$, and the complement of the singular orbit has the form

$$S \setminus S^4 = \mathrm{Sp}(2)/\mathrm{Sp}(1)_+ \times \mathbb{R}_+.$$

Notice that the following

$$\begin{aligned} A_1 &= \frac{1}{\sqrt{12}} \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}, A_2 = \frac{1}{\sqrt{12}} \begin{pmatrix} j & 0 \\ 0 & 0 \end{pmatrix}, A_3 = \frac{1}{\sqrt{12}} \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix}, \\ A_4 &= \frac{1}{\sqrt{12}} \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix}, A_5 = \frac{1}{\sqrt{12}} \begin{pmatrix} 0 & 0 \\ 0 & j \end{pmatrix}, A_6 = \frac{1}{\sqrt{12}} \begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix}, \\ X_1 = \frac{1}{\sqrt{24}} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}&, X_2 = \frac{1}{\sqrt{24}} \begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix}, X_3 = \frac{1}{\sqrt{24}} \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}, X_4 = \frac{1}{\sqrt{24}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \end{aligned} \tag{15}$$

is an orthonormal basis of $\mathfrak{sp}(2)$ with respect to the Killing metric. Let $a = a_0 + ia_1 + ja_2 + ka_3$ be the standard real coordinates in $\mathbb{H}$; following [6], we define $\mathbb{H}$-valued one-forms on $\mathrm{Sp}(2) \times \mathbb{H}$

$$\phi = iA^4 + jA^5 + kA^6, \quad \omega = X^4 + iX^1 + jX^2 + kX^3, \quad \alpha = da - a\phi;$$

we then define $\operatorname{Im}\mathbb{H}$-valued two-forms

$$B = \frac{1}{2}(\bar{\alpha} \wedge \alpha), \quad \Omega = \frac{1}{2}(\bar{\omega} \wedge \omega).$$

When needed, we will use indices to indicate components in $\mathbb{H}$, i.e.

$iB_1 + jB_2 + kB_3 = i(\alpha_0 \wedge \alpha_1 - \alpha_2 \wedge \alpha_3) + j(\alpha_0 \wedge \alpha_2 - \alpha_3 \wedge \alpha_1) + k(\alpha_0 \wedge \alpha_3 - \alpha_1 \wedge \alpha_2).$

The Bryant-Salamon 4-form is a linear combination of the forms

$$\psi_1 = \alpha_0 \wedge \alpha_1 \wedge \alpha_2 \wedge \alpha_3, \quad \psi_= B_1 \wedge \Omega_1 + B_2 \wedge \Omega_2 + B_3 \wedge \Omega_3,$$
$$\psi_3 = \omega_0 \wedge \omega_1 \wedge \omega_2 \wedge \omega_3,$$

with coefficients determined by the smooth functions on $\mathbb{H}$

$$f(r) = 4\bigl(1 + r\bigr)^{-2/5}, \quad g(r) = 5k\bigl(1 + r\bigr)^{3/5},$$

where we have set $r = a\bar{a} = \|a\|^2$. More precisely, the Bryant-Salamon 4-form $\Phi \in \Omega^4\bigl(\mathrm{Sp}(2) \times \mathbb{H}\bigr)$ is defined as

$$\Phi = f^2\psi_1 + fg\psi_2 + g^2\psi_3. \tag{16}$$

Since $\Phi$ is basic relative to the action of $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$, it induces a form on the quotient $S = \mathrm{Sp}(2) \times \mathbb{H}/\mathrm{Sp}(1) \times \mathrm{Sp}(1)$, also to be denoted by $\Phi$.

**Proposition 2.1.** *Under the inclusion*

$$\tilde{\chi} \colon \mathrm{Sp}(2) \times \mathbb{R}_+ \to \mathrm{Sp}(2) \times \mathbb{H}, \quad (g, t) \mapsto (g, \sqrt{t}),$$

*the Bryant-Salamon 4-form pulls back to*

$$\tilde{\chi}^*\Phi = -\frac{dt}{2} \wedge \Bigl(tf(t)^2 A^{456} + f(t)g(t)\bigl(A^4 \wedge (-X^{14} - X^{23}) + A^5 \wedge (-X^{24} + X^{13}) + A^6 \wedge (-X^{34} - X^{12})\bigr)\Bigr) - tf(t)g(t)\bigl(A^{56} \wedge (-X^{14} - X^{23}) + A^{64} \wedge (-X^{24} + X^{13}) + A^{45} \wedge (-X^{34} - X^{12})\bigr) - g(t)^2 X^{1234}.$$

*Proof.* By definition we have $\tilde{\chi}^* a = \sqrt{t}$, $\tilde{\chi}^* r = t$, so

$$\tilde{\chi}^*\alpha = \frac{dt}{2\sqrt{t}} - i\sqrt{t}\phi_1 - j\sqrt{t}\phi_2 - k\sqrt{t}\phi_3.$$

We obtain

$$\begin{aligned}
\tilde{\chi}^*\psi_1 &= -t\frac{dt}{2} \wedge \phi_1 \wedge \phi_2 \wedge \phi_3 = -t\frac{dt}{2} \wedge A^{456};\\
\tilde{\chi}^*\psi_2 &= -\frac{dt}{2} \wedge (\phi_1 \wedge \Omega_1 + \phi_2 \wedge \Omega_2 + \phi_3 \wedge \Omega_3) - t(\phi_2 \wedge \phi_3 \wedge \Omega_1 + \phi_3 \wedge \phi_1 \wedge \Omega_2 + \phi_1 \wedge \phi_2 \wedge \Omega_3)\\
&= -\frac{dt}{2} \wedge \bigl(A^4 \wedge (-X^{14} - X^{23}) + A^5 \wedge (-X^{24} + X^{13}) + A^6 \wedge (-X^{34} - X^{12})\bigr)\\
&\quad - t\,\bigl(A^{56} \wedge (-X^{14} - X^{23}) + A^{64} \wedge (-X^{24} + X^{13}) + A^{45} \wedge (-X^{34} - X^{12})\bigr);\\
\tilde{\chi}^*\psi_3 &= \omega_0 \wedge \omega_1 \wedge \omega_2 \wedge \omega_3 = -X^{1234}.
\end{aligned}$$

The statement follows immediately. □

# 3 Linear perturbations of the Bryant-Salamon metric

In this section we study Sp(2)-invariant linear perturbations of the Bryant-Salamon metric.

By Theorem 1.7, a linear perturbation of a Spin(7)-structure is obtained by the choice of a rank one nilpotent endomorphism of the tangent bundle at each point. Thus, the global data for a linear perturbation is given by the choice of a vector field $X$ and a one form $\alpha$ with $\alpha(X) = 0$. Since we work in the Sp(2)-invariant setting, we will require both vector field and form to be invariant.

Thus, the first step is to construct an Sp(2)-invariant vector field on the cohomogeneity one manifold $\mathrm{Sp}(2)/\mathrm{Sp}(1)_+ \times \mathbb{R}_+$. We will need the following observation:

**Lemma 3.1.** *Let a Lie group $G$ act transitively on $M$, and let $H$ be the stabilizer at a point $m$. Then $X \in \mathfrak{g}$ defines a $G$-invariant vector field on $M$ of the form*

$$X^+_{gm} = g^* \frac{d}{dt}|_{t=0}\ \exp(tX)m$$

*if and only if $X$ belongs to*

$$\mathfrak{n}(H) = \{\ X \in \mathfrak{g} \mid \mathrm{Ad}_h X - X \in \mathfrak{h} \quad \forall h \in H\ \} \tag{17}$$

*All $G$-invariant vector fields on $M$ are of this form.*

*Proof.* The vector field $X^+$ is well defined and invariant if and only if $X^+_{gm} = X^+_{g'm}$ whenever $gm = g'm$; in other words, we need $X^+_{gm} = X^+_{ghm}$ for all $g \in G, h \in H$. Since

$$\begin{aligned} X^+_{ghm} &= g^* h^* \frac{d}{dt}|_{t=0}\ \exp(tX)m = g^* \frac{d}{dt}|_{t=0}\ h\, exp(tX)h^{-1}m \\ &= g^* \frac{d}{dt}|_{t=0}\ \exp(t\,\mathrm{Ad}_h X)m, \end{aligned}$$

we obtain that $X^+$ is well defined when

$$\frac{d}{dt}|_{t=0}\ exp(tX)m = \frac{d}{dt}|_{t=0}\ \exp(t\,\mathrm{Ad}_h X)m \quad \forall h \in H,$$

which is equivalent to $X$ lying in $\mathfrak{n}(H)$.

Conversely, given an invariant vector field $Y$ on $M$, we have $Y_{gm} = g^* Y_m$, where

$$Y_m = \frac{d}{dt}|_{t=0} \exp(tX)m, \quad X \in \mathfrak{g}.$$

By invariance, $Y = X^+$, and $X$ lies in $\mathfrak{n}(H)$ by the first part. □

**Proposition 3.2.** *Relative to the action of $G = Sp(2)$ on $Sp(2)/\mathrm{Sp}(1)_+$ we have*

$$\mathfrak{n}(\mathrm{Sp}(1)_+) = \mathfrak{sp}(1)_+ \times \mathfrak{sp}(1)_-. \tag{18}$$

*Proof.* An element

$$\begin{pmatrix} x & y \\ -\bar{y} & w \end{pmatrix} \in \mathfrak{sp}(2)$$

lies in $\mathfrak{n}(\mathrm{Sp}(1)_+)$ if and only if for all $p$ in $Sp(1)$ we have

$$\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & y \\ -\bar{y} & w \end{pmatrix} \begin{pmatrix} \bar{p} & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} x & y \\ -\bar{y} & w \end{pmatrix} \in \mathfrak{sp}(1)_+.$$

This is equivalent to $py = y$ for all $p$, i.e. $y = 0$, so the statement is proved. □

Summing up, we have a linear map

$$\mathfrak{sp}(1)_+ \times \mathfrak{sp}(1)_- \to \mathfrak{X}_{Sp(2)}(Sp(2)/Sp(1)_+), \quad X \mapsto X^+;$$

its kernel is $\mathfrak{sp}(1)_+$, showing that invariant vector fields on $\mathrm{Sp}(2)/\mathrm{Sp}(1)_+$ take the form

$$X = x_4 A_4 + x_5 A_5 + x_6 A_6 \in \mathfrak{sp}(1)_-.$$

**Lemma 3.3.** *Every* $\mathrm{Sp}(2)$*-invariant vector field* $Y$ *on* $\mathrm{Sp}(2)/\mathrm{Sp}(1)_+$ *satisfies*

$$\mathcal{L}_Y \left(A^{56} \wedge (-X^{14} - X^{23}) + A^{64} \wedge (-X^{24} + X^{13}) + A^{45} \wedge (-X^{34} - X^{12})\right) = 0$$

$$\mathcal{L}_Y X^{1234} = 0$$

*Proof.* As an $\mathrm{Sp}(1)_-$-module, $\mathfrak{sp}(2)$ decomposes as

$$3\mathbb{R} + \mathbb{H} + \mathfrak{sp}(1)_- = \mathrm{Span}\{A_1, A_2, A_3\} + \mathrm{Span}\{X_1, X_2, X_3, X_4\} + \mathrm{Span}\{A_4, A_5, A_6\},$$

with $\Lambda^2\mathbb{H}$ splitting as $3\mathbb{R} + \mathfrak{sp}(1)_-$. The inclusion of $\mathfrak{sp}(1)_-$ in $\Lambda^2\mathbb{H}$ is realized by the $\mathrm{Sp}(1)_-$-equivariant map

$$A_4 \mapsto \Omega_1, A_5 \mapsto \Omega_2, A_6 \mapsto \Omega_3.$$

It follows that $A^{56} \wedge \Omega_1 + A^{64} \wedge \Omega_2 + A^{45} \wedge \Omega_3$ is $\mathrm{Sp}(1)_-$-invariant. As an element of $\Lambda^4\mathbb{H}$, $X^{1234}$ is also $\mathrm{Sp}(1)_-$-invariant.

Writing $Y = aA_4^+ + bA_5^+ + cA_6^+$, the statement follows. □

**Theorem 3.4.** *Given smooth even functions* $a, b, c \colon \mathbb{R} \to \mathbb{R}$*, the* 4*-form*

$$\Phi + dt \wedge (a(t)A_4^+ + b(t)A_5^+ + c(t)A_6^+) \lrcorner \Phi \tag{19}$$

*is closed and defines an* $\mathrm{Sp}(2)$*-invariant metric with holonomy contained in* $\mathrm{Spin}(7)$.

*Proof.* Observe first that the vector field $a(t)A_4^+ + b(t)A_5^+ + c(t)A_6^+$ is globally defined and vanishes on the special orbit, i.e. the zero section of the spinor bundle.

The 4-form (19) is a linear perturbation of the Bryant-Salamon form by a nilpotent endomorphism of rank one (see Remark 1.4), so it defines again a Spin(7)-structure. In order to check that it is closed, write $Y = a(t)A_4^+ + b(t)A_5^+ + c(t)A_6^+$; we have

$$d(dt \wedge Y \lrcorner \Phi) = -dt \wedge d(Y \lrcorner \Phi) = -dt \wedge \mathcal{L}_Y \Phi.$$

Since $Y$ is $\mathrm{Sp}(2)$-invariant, by Lemma 3.3 we have that $\mathcal{L}_Y$ annihilates the restriction of $\Phi$ to each principal orbit $\{t = t_0\}$, and therefore $dt \wedge \mathcal{L}_Y \Phi = 0$. By [12], the metric defined by the perturbed form has holonomy contained in Spin(7). □

It is now natural to ask whether the perturbed metrics are isometric to the Bryant-Salamon metric. It turns out that they are isometric under an Sp(2)-equivariant diffeomorphism, due to the following:

**Lemma 3.5.** *Any* Sp(2)*-invariant vector field on* $S$ *is a Killing field for the Bryant-Salamon metric.*

*Proof.* The Bryant-Salamon metric takes the form

$$\begin{aligned}&f(\alpha_0^2+\dots+\alpha_3^2)+g(\omega_0^2+\dots+\omega_3^2)\\&=f(\frac{1}{4t}dt^2+t((A^4)^2+(A^5)^2+(A^6)^2))+g((X^1)^2+(X^2)^2+(X^3)^2+(X^4)^2).\end{aligned}$$

Since $(A^4)^2+(A^5)^2+(A^6)^2$ and $(X^1)^2+(X^2)^2+(X^3)^2+(X^4)^2$ are $\mathrm{Sp}(1)_-$-invariant, the claim follows. □

Arguing as in [8, Proposition 5.2], we obtain:

**Proposition 3.6.** *The* Sp(2)*-invariant linear perturbations of the Bryant-Salamon metric are obtained from the Bryant-Salamon metric via an* Sp(2)*-equivariant diffeomorphism.*

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