The limits of decidability in fuzzy description logics with general concept inclusions

Stefan Borgwardt [a,*], Felix Distel [a,*], Rafael Peñaloza [a,b,*]

[a] *Institute for Theoretical Computer Science, Technische Universität Dresden, 01062 Dresden, Germany*
[b] *Center for Advancing Electronics Dresden, Germany*

A B S T R A C T

Fuzzy description logics (DLs) can be used to represent and reason with vague knowledge. This family of logical formalisms is very diverse, each member being characterized by a specific choice of constructors, axioms, and triangular norms, which are used to specify the semantics. Unfortunately, it has recently been shown that the consistency problem in many fuzzy DLs with general concept inclusion axioms is undecidable. In this paper, we present a proof framework that allows us to extend these results to cover large classes of fuzzy DLs. On the other hand, we also provide matching decidability results for most of the remaining logics. As a result, we obtain a near-universal classification of fuzzy DLs according to the decidability of their consistency problem.

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1. Introduction

Description logics (DLs) [1] are a family of knowledge representation formalisms, designed to represent the terminological knowledge of a domain in a formally well-understood way. They form the base language for many large-scale knowledge bases, like SNOMED CT[1] and the Gene Ontology,[2] but arguably their largest success to date is the recommendation by the W3C of the DL-based language OWL as the standard ontology language for the Semantic Web.[3] DLs essentially allow to state relations between *concepts*, which represent subsets of a specific domain containing exactly those domain elements that share certain properties. *Roles* correspond to binary relations that allow to state connections between concepts. For example, the concept of a human father can be expressed as

$$\mathsf{Human} \sqcap \mathsf{Male} \sqcap \exists \mathsf{hasChild}.\top,$$

which describes the set of all humans that are male and have a child. Here, Human and Male are atomic concept names, whereas hasChild is a role name. Domain-specific relations between concepts can be expressed in *axioms* such as

$$\mathsf{bob} : \mathsf{Male}, \qquad \mathsf{Human} \sqsubseteq \forall \mathsf{hasChild}.\mathsf{Human},$$

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* Corresponding authors. Tel.: +49 351 463 38231; fax: +49 351 463 37959.
  E-mail addresses: stefborg@tcs.inf.tu-dresden.de (S. Borgwardt), felix@tcs.inf.tu-dresden.de (F. Distel), penaloza@tcs.inf.tu-dresden.de (R. Peñaloza).
  [1] http://www.ihtsdo.org/snomed-ct/.
  [2] http://www.geneontology.org/.
  [3] http://www.w3.org/TR/owl2-overview/.

saying that bob is a male individual, and that every human can only have human children, respectively. The former axiom is called an *assertion*, the latter a *general concept inclusion (GCI)*. In DLs, various reasoning problems over a set of such axioms, called an *ontology* or *knowledge base*, are studied. The most fundamental one is to decide whether an ontology is *consistent*; that is, if the restrictions expressed by its axioms can actually be realized in a model. Different sets of constructors for expressing concepts, such as conjunctions ($\sqcap$) or value restrictions ($\forall$), lead to logics of varying expressivity, resulting in differences between the computational complexity of their consistency problems. For example, in the inexpressive DL $\mathcal{EL}$, consistency is trivial, whereas other reasoning problems such as subsumption have only polynomial complexity. In the more expressive $\mathcal{ALC}$, consistency without GCIs is PSpace-complete, and is ExpTime-complete in the presence of GCIs. The very expressive $\mathcal{SROIQ}$, the formalism underlying the *OWL 2 Direct Semantics*, has a 2-NExpTime-complete consistency problem.

In their classical form, however, DLs are not well-suited for representing and reasoning with the vagueness and imprecision that are endemic to many knowledge domains, e.g. in the bio-medical fields. For example, one of the most common symptoms of diseases is the presence of *fever*, which is characterized by a *high* body temperature. Clearly, it is not possible to precisely distinguish high body temperatures from non-high body temperatures. In order to appropriately represent this knowledge, it is necessary to use a formalism capable of handling imprecision. Fuzzy variants of DLs have been introduced as a means of handling imprecise terminological knowledge. This is achieved by interpreting concepts as *fuzzy sets*. In a nutshell, a fuzzy set associates with every element of the universe a value from the interval $[0, 1]$, which expresses its *degree of membership* to the set. This makes it possible to express, e.g. that 38 °C is a high body temperature to degree 0.7, while 39 °C belongs to the same concept with degree 1.

Compared to classical DLs, fuzzy DLs have an additional degree of freedom for choosing how to interpret the logical constructors. A standard approach, inherited from mathematical fuzzy logic [2,3], is to use a continuous triangular norm (t-norm) [4] to interpret conjunction. The three most commonly used t-norms, called *Gödel*, *Łukasiewicz*, and *product*, have the interesting property that all other continuous t-norms can be represented by composing copies of them in a certain way. From the chosen t-norm $\otimes$, the semantics of all other logical constructors is determined, generalizing the properties of the classical operators. Ontologies of fuzzy DLs generalize classical ontologies by annotating each axiom with a fuzzy value that specifies the degree to which the axiom holds. For example, a fuzzy assertion like $\langle \mathsf{bob} : \exists \mathsf{hasFever}.\mathsf{High} \geq 0.6 \rangle$ can specify that an individual (in this case bob) belongs to a fuzzy concept ($\exists\mathsf{hasFever}.\mathsf{High}$) at least to a certain degree (e.g. 0.6).

For the last two decades, research on fuzzy DLs has covered many different logics, from the inexpressive $\mathcal{EL}$ [5] to the expressive $\mathcal{SROIQ}(D)$ [6], from simple fuzzy semantics [7] to ones covering all continuous t-norms [8], from acyclic terminologies [9] to GCIs [10]. Fuzzy reasoning algorithms were implemented [11,12] and the use of fuzziness in practical applications was studied [13,14]. Recently, the focus in the area changed when some tableau-based algorithms for DLs allowing general concept inclusions were shown to be incorrect [15,16]. This raised doubts about the decidability of the consistency problem in these logics, and eventually led to a plethora of undecidability results for fuzzy DLs [16–19]. In particular, one does not need to go beyond the expressivity of $\otimes$-$\mathcal{ALC}$ to get undecidability [18,19].

The main goal of this paper is to characterize the limits of decidability in fuzzy DLs; in other words, we want to partition the family of fuzzy DLs according to the decidability of consistency in them. For the cases where the problem is decidable, we are also interested in finding precise complexity bounds. Given the sheer number of fuzzy DLs available, identified by the set of constructors, types of axioms, and t-norm that they use, it is infeasible to study each of them independently. Instead, we develop general methods for proving (un)decidability of these logics.

Most of the known undecidability results [16,17,19] focus on one specific fuzzy DL; that is, undecidability is proven for a specific set of constructors, axioms, and chosen semantics. The papers [16,17] show undecidability of (extensions of) $\otimes$-$\mathcal{ALC}_{f,\geq}$, where $\otimes$ is the product t-norm, while [19] shows the same for the Łukasiewicz t-norm. The only exception is [18], where undecidability is shown for $\otimes$-$\mathcal{IAL}_{f,=}$ for all t-norms $\otimes$ "starting" with the product t-norm. Abstracting from the details of each specific logic, all these proofs of undecidability follow the same basic pattern. In essence, it is shown that the logic satisfies a series of properties that allows it to encode the Post Correspondence Problem [20].

In the first part of this paper, we generalize these ideas and describe a set of properties that together imply undecidability of a fuzzy DL. We use this general framework to strengthen all previously known undecidability results to cover all continuous t-norms except the Gödel t-norm, for which the problem is decidable [21]. Additionally, we present some variants on the same ideas that allow us to prove undecidability of fuzzy DLs that do not fit precisely into the main framework. For instance, we show that the fairly inexpressive fuzzy DL $\otimes$-$\mathcal{IEL}_{=}$ is undecidable for any continuous t-norm $\otimes$ except the Gödel t-norm. This can be strengthened to the even less expressive $\otimes$-$\mathfrak{N}\mathcal{EL}$ if $\otimes$ starts with the Łukasiewicz t-norm. These logics are of interest since they correspond to fuzzy variants of the prototypical classical DL $\mathcal{ALC}$. Indeed, they have the same expressivity as $\mathcal{ALC}$ when their semantics is restricted to the two classical truth values.

In the second part of the paper, we complement these results by considering fuzzy DLs based on t-norms that do not start with the Łukasiewicz t-norm, which in particular includes the product and Gödel t-norms. Under this assumption, we show that consistency is decidable even for the very expressive logic $\otimes$-$\mathcal{SROIQ}_{f,\geq}$ if axioms are not allowed to express upper bounds. We show an even stronger result: under these conditions, an ontology is consistent w.r.t. fuzzy semantics iff it is consistent w.r.t. *crisp* semantics, i.e. using only the classical truth values 0 and 1. Thus, ontology consistency in $\otimes$-$\mathcal{SHOI}$ is ExpTime-complete, and in $\otimes$-$\mathcal{SROIQ}$ it is 2-NExpTime-complete. If these restrictions are not met, then the problem is undecidable, as shown in the first part of the paper.

Table 1
Gödel, product, and Łukasiewicz t-norms.

Name	$x \otimes y$	$x \Rightarrow y$	$\ominus x$	$x \oplus y$
Gödel (G)	$\min\{x, y\}$	$\begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$	$\begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$	$\max\{x, y\}$
Product (Π)	$x \cdot y$	$\begin{cases} 1 & \text{if } x \leq y \\ y/x & \text{otherwise} \end{cases}$	$\begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$	$x + y - x \cdot y$
Łukasiewicz (Ł)	$\max\{x + y - 1, 0\}$	$\min\{1 - x + y, 1\}$	$1 - x$	$\min\{x + y, 1\}$

Some of the results in this paper have appeared in a preliminary form in conference papers [22,23]. Here, we not only combine those previous publications, but include more detailed proofs, add new undecidability results (see Section 3.5), and discuss (un-)decidability results for fuzzy DLs under general model semantics (see Section 5.1). In particular, we

- use the framework for showing undecidability from [22] to prove these results here in more detail;
- add some explanatory material (examples, figures) to aid understanding of these proofs;
- describe additions to the framework that allow us to show more undecidability results for fuzzy DLs of the form $\otimes$-$\mathcal{IEL}$ (Section 3.5);
- extend the proof from [23] that shows decidability for many of the remaining fuzzy DLs to deal with $\otimes$-$\mathcal{SROIQ}_{\mathsf{f},\geq}$ instead of only $\otimes$-$\mathcal{SHOI}_{\mathsf{f},\geq}$; and
- discuss related semantics and reasoning problems and present related work in more detail (Section 5).

2. Preliminaries

We start with a brief introduction to t-norms and mathematical fuzzy logic, which will be useful for defining fuzzy extensions of description logics.

2.1. Triangular norms and mathematical fuzzy logic

Mathematical fuzzy logic can be used to express imprecise or vague information [2]. It extends classical logic by interpreting predicates as fuzzy sets over an interpretation domain. Given a non-empty domain $\mathcal{D}$, a *fuzzy set* is a function $F : \mathcal{D} \rightarrow [0, 1]$ from $\mathcal{D}$ into the real unit interval $[0, 1]$, with the intuition that an element $x \in \mathcal{D}$ belongs to F with *degree* $F(x)$. The interpretation of the logical constructors is based on appropriate truth functions that generalize the properties of the connectives of classical logic to the interval $[0, 1]$. The most prominent truth functions used in the fuzzy logic literature are based on triangular norms (or t-norms) [4].

A *t-norm* is a binary operator $\otimes : [0, 1] \times [0, 1] \rightarrow [0, 1]$ that is associative, commutative, and monotone, and has 1 as its unit element. The t-norm is used to generalize classical conjunction. We will only consider *continuous* t-norms in this paper, which means that they are continuous as a function, i.e. we have for all convergent sequences $(x_n)_{n \geq 0}, (y_n)_{n \geq 0}$ that

$$\left(\lim_{n \to \infty} x_n \right) \otimes \left(\lim_{n \to \infty} y_n \right) = \lim_{n \to \infty} (x_n \otimes y_n).$$

The *residuum* of a t-norm $\otimes$ is a binary operator $\Rightarrow : [0, 1] \times [0, 1] \rightarrow [0, 1]$ that satisfies $z \leq x \Rightarrow y$ iff $x \otimes z \leq y$ for all $x, y, z \in [0, 1]$. If $\otimes$ is continuous, then this equivalence determines the unique residuum

$$x \Rightarrow y := \sup\{z \in [0, 1] \mid x \otimes z \leq y\}.$$

The residuum behaves like classical implication on the truth values 0 and 1 and is used to generalize the implication to fuzzy logics. Given a residuum $\Rightarrow$, the *residual negation* is the unary operator $\ominus : [0, 1] \rightarrow [0, 1]$ defined by $\ominus x := x \Rightarrow 0$. As implied by its name, this operator generalizes classical negation. Finally, the disjunction can be fuzzified using the *t-conorm* of a t-norm $\otimes$, which is a binary operator $\oplus : [0, 1] \times [0, 1] \rightarrow [0, 1]$ defined as $x \oplus y := 1 - ((1 - x) \otimes (1 - y))$. Like $\otimes$, it is associative, commutative, and monotone in both arguments, but its unit element is 0.

Whenever we have a continuous t-norm $\otimes$, then we denote by $\Rightarrow$, $\ominus$, and $\oplus$ the corresponding residuum, residual negation, and t-conorm, respectively. Three important continuous t-norms are the Gödel (G), product (Π), and Łukasiewicz (Ł) t-norms. They are listed in Table 1 together with their induced operators. Fuzzy logics are sometimes extended with the involutive negation operator, defined as $\sim x := 1 - x$ [24,25]. Observe that for $\otimes = $ Ł the involutive negation and the residual negation coincide; that is, the equality $\sim x = x \Rightarrow 0$ holds. However, for any other continuous t-norm $\otimes$, the involutive negation is not expressible in terms of $\otimes$ and its residuum $\Rightarrow$.

The following are simple consequences of the above definitions [4].

Proposition 1. *For every continuous t-norm $\otimes$ and $x, y \in [0, 1]$,*

- $x \Rightarrow y = 1$ *iff* $x \leq y$,

- $1 \Rightarrow y = y$, *and*
- $x \oplus y = 0$ *iff* $x = 0$ *and* $y = 0$.

From the three fundamental t-norms listed in Table 1, all continuous t-norms can be constructed as described next. For any $a, b \in [0, 1]$ with $a < b$, we define the *scaling function* $\sigma_{a,b} : [0, 1] \to [a, b]$ by $\sigma_{a,b}(x) := a + (b - a)x$ for all $x \in [0, 1]$. This linear function is bijective with the inverse given by $\sigma_{a,b}^{-1}(x) := \frac{x-a}{b-a}$. Let now $((a_i, b_i))_{i \in I}$ be a (possibly infinite) family of non-empty, mutually disjoint open subintervals of $[0, 1]$ and $(\otimes_i)_{i \in I}$ be a family of continuous t-norms over the same index set I. The *ordinal sum* of $(((a_i, b_i), \otimes_i))_{i \in I}$ is the t-norm $\otimes$, defined for every $x, y \in [0, 1]$ by

$$x \otimes y := \begin{cases} \sigma_{a_i, b_i}(\sigma_{a_i, b_i}^{-1}(x) \otimes_i \sigma_{a_i, b_i}^{-1}(y)) & \text{if } x, y \in [a_i, b_i] \text{ for some } i \in I; \\ \min\{x, y\} & \text{otherwise.} \end{cases}$$

This construction always yields a continuous t-norm, whose residuum is given by

$$x \Rightarrow y := \begin{cases} 1 & \text{if } x \leq y; \\ \sigma_{a_i, b_i}(\sigma_{a_i, b_i}^{-1}(x) \Rightarrow_i \sigma_{a_i, b_i}^{-1}(y)) & \text{if } a_i \leq y < x \leq b_i; \\ y & \text{otherwise,} \end{cases}$$

where $\Rightarrow_i$ denotes the residuum of $\otimes_i$, for each $i \in I$. Intuitively, this means that the t-norm $\otimes$ and its residuum "behave like" $\otimes_i$ and its residuum in each of the intervals $[a_i, b_i]$, and like the Gödel t-norm and residuum everywhere else.

Two t-norms $\otimes$, $\otimes'$ are *isomorphic* if there exists a strictly increasing mapping $\iota : [0, 1] \to [0, 1]$ such that $\iota(x \otimes y) = \iota(x) \otimes' \iota(y)$. It has been shown that, up to isomorphism, every continuous t-norm can be represented as the ordinal sum of copies of the Łukasiewicz and product t-norms.

Theorem 2. *(See [26].) Every continuous t-norm is an ordinal sum of t-norms isomorphic to the Łukasiewicz t-norm or the product t-norm.*

In the following, let $\otimes$ be a continuous t-norm and $(((a_i, b_i), \otimes_i))_{i \in I}$ be its (unique) representation as ordinal sum given by Theorem 2. For ease of presentation, we assume without loss of generality that the isomorphisms occurring in this theorem are the identity mapping, which means that each $\otimes_i$ is either the Łukasiewicz or product t-norm. We call the tuples $(((a_i, b_i), \otimes_i))_{i \in I}$ the *components* of $\otimes$. We say that $\otimes$ (a, b)-*contains* Łukasiewicz or product if it has a component of the form $((a, b), Ł)$ or $((a, b), \Pi)$, respectively. Similarly, $\otimes$ *starts with Łukasiewicz* if it has a component $((0, b), Ł)$. Whenever the exact location of the interval (a, b) is irrelevant, we will omit it.

An element $x \in [0, 1]$ is called *idempotent* (w.r.t. $\otimes$) if $x \otimes x = x$. Note that the idempotent elements are exactly those that are not in (a_i, b_i) for any $i \in I$. In particular, 0 and 1 are always idempotent, as are a_i and b_i for any $i \in I$. It is easy to see that a continuous t-norm has infinitely many non-idempotent elements if and only if it is not the Gödel t-norm.

An element $x \in (0, 1]$ is called a *zero divisor* (of $\otimes$) if there exists a $y \in (0, 1]$ such that $x \otimes y = 0$. Of the three fundamental continuous t-norms from Table 1, only the Łukasiewicz t-norm has zero divisors: every element $x \in (0, 1)$ is a zero divisor for this t-norm since $1 - x > 0$ and $x \otimes (1 - x) = 0$. In fact, a continuous t-norm has zero divisors if and only if it starts with Łukasiewicz.

Lemma 3. *(See [4].) A continuous t-norm has zero divisors iff it starts with the Łukasiewicz t-norm.*

All continuous t-norms that do not start with Łukasiewicz define the same residual negation, known as the *Gödel negation* (cf. Table 1).

Lemma 4. *(See [4].) For any t-norm $\otimes$ without zero divisors and every $x \in [0, 1]$,*

(a) $x \Rightarrow y = 0$ *iff* $x > 0$ *and* $y = 0$*; and*
(b) $\ominus x = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{otherwise.} \end{cases}$

Example 5. The continuous t-norm $\otimes$ defined by

$$x \otimes y := \begin{cases} 2xy & \text{if } x, y \in [0, 0.5], \\ \max\{x + y - 1, 0.5\} & \text{if } x, y \in [0.5, 1], \\ \min\{x, y\} & \text{otherwise,} \end{cases}$$

is the ordinal sum of the two components $((0, 0.5), \Pi)$ and $((0.5, 1), Ł)$. In particular, it has no zero divisors, and therefore its residual negation is the Gödel negation. Its only idempotent elements are 0, 0.5, and 1.

Triangular norms are the basis for defining the semantics of fuzzy description logics, which are introduced in the following section.

Table 2
Some relevant DLs and their expressivity.

Name	$\top$	$\sqcap$	$\exists$	$\forall$	$\bot$	$\rightarrow$	$\boxminus$	$\neg$
$\mathcal{EL}$	✓	✓	✓					
$\mathfrak{I}\mathcal{EL}$	✓	✓	✓		✓	✓	(✓)	
$\mathfrak{N}\mathcal{EL}$	✓	✓	✓		(✓)		✓	
$\mathcal{ELC}$	✓	✓	✓		(✓)			✓
$\mathcal{AL}$	✓	✓	✓	✓				
$\mathfrak{I}\mathcal{AL}$	✓	✓	✓	✓	✓	✓	(✓)	
$\mathfrak{N}\mathcal{AL}$	✓	✓	✓	✓	(✓)		✓	
$\mathcal{ALC}$	✓	✓	✓	✓	(✓)			✓

2.2. Fuzzy description logics

The syntax and semantics of a fuzzy description logic $\otimes\text{-}\mathcal{L}$ are determined by two components: the language $\mathcal{L}$ and the t-norm $\otimes$. We first introduce the syntactical part $\mathcal{L}$, which is determined by a choice of logical constructors and axioms, and usually extends the syntax of an underlying classical description logic. As there exists a large variety of constructors available in description logics, we start defining a generic syntax of $\mathcal{L}$ and later describe instantiations to more specific logics.

The central notion of DLs is that of concepts, which are built from atomic concepts (so-called *concept names*) using different constructors, like conjunction, implication, or existential restrictions.

Definition 6 *(Concepts).* Let N_C, N_R, and N_I be mutually disjoint sets of *concept names*, *role names*, and *individual names*, respectively. The set of *(complex) concepts* is defined inductively as follows:

- every concept name $A \in N_C$ is a concept; and
- if C, D are concepts and r is a role name, then $\top$ (*top concept*), $\bot$ (*bottom concept*), $C \sqcap D$ (*conjunction*), $C \rightarrow D$ (*implication*), $\neg C$ (*strong negation*), $\boxminus C$ (*residual negation*), $\exists r.C$ (*existential restriction*), and $\forall r.C$ (*value restriction*) are also concepts.

For $n \in \mathbb{N}$, we define C^n as the n-ary conjunction of a concept C with itself. More formally, if C is a concept, then we set

- $C^0 := \top$, and
- $C^{n+1} := C \sqcap C^n$ for all $n \in \mathbb{N}$.

As mentioned before, different description logics $\mathcal{L}$ are determined by the constructors they allow. In the DL $\mathcal{EL}$, concepts are built using only the constructors $\top, \sqcap$, and $\exists$. Extending $\mathcal{EL}$ with value restrictions yields the DL $\mathcal{AL}$. Following the notation from [27], the letters $\mathcal{C}$ and $\mathfrak{N}$ denote the presence of the strong negation ($\neg$) and residual negation ($\boxminus$), respectively. The prefix $\mathfrak{I}$ expresses that the implication ($\rightarrow$) and bottom ($\bot$) constructors are allowed. Table 2 summarizes this nomenclature for the logics that we will investigate in this paper. Constructors that can be simulated by others in the same logic are indicated in parentheses. In Section 4 we further extend the set of constructors to prove decidability of more expressive fuzzy DLs.

The second component defining the expressivity of a fuzzy DL $\otimes\text{-}\mathcal{L}$ are its axioms. Axioms are the means to represent domain knowledge, by describing relations between individuals, roles, and concepts. In contrast to classical DLs, in fuzzy DLs axioms often include a lower bound for the degree to which the axiom should hold. This lower bound provides a larger flexibility for the interpretations that satisfy the axiom.

Definition 7 *(Axioms).* An *axiom* is either a general concept inclusion (GCI) or an assertion, where

- a *GCI* is an expression of the form $\langle C \sqsubseteq D \geq p \rangle$, where C, D are concepts and $p \in [0, 1]$; and
- an *assertion* is of the form $\langle e : C \rhd p \rangle$ or $\langle (d, e) : r \rhd p \rangle$, where C is a concept, r is a role name, d, e are individual names, $p \in [0, 1]$, and $\rhd \in \{\geq, =\}$. It is an *inequality assertion* if $\rhd$ is $\geq$ and an *equality assertion* if $\rhd$ is $=$.

An axiom is called *crisp* if $p = 1$. An *ontology* is a finite set of axioms. It is called a *classical ontology* if it contains only crisp axioms.

For crisp axioms, we will usually remove the part "$\rhd 1$", and simply write, e.g. $\langle C \sqsubseteq D \rangle$. As with the choice of the constructors, the axioms influence the expressivity of the logic. We always assume that our logics allow at least classical ontologies. Given a DL $\mathcal{L}$, we will use the subscripts f, $\geq$, and $=$ to denote that arbitrary GCIs, inequality assertions, and equality assertions are allowed, respectively. For instance, $\mathcal{EL}_{\mathsf{f},\geq}$ denotes the logic $\mathcal{EL}$ where ontologies may contain arbitrary GCIs and inequality assertions, but no equality assertions. Table 3 summarizes the expressivity of these subscripts.

Table 3
The possible subscripts of a fuzzy DL.

Subscript	Crisp GCIs	Fuzzy GCIs	Crisp assertions	$\geq$-assertions	$=$-assertions
None	✓		✓		
f	✓	✓	✓		
$\geq$	✓		✓	✓	
$=$	✓		✓	(✓)	✓

The semantics of a fuzzy DL $\otimes$-$\mathcal{L}$ is defined by interpreting concepts as fuzzy sets, and roles as fuzzy binary relations. Compared to classical DLs, fuzzy DLs have an additional degree of freedom in the selection of their semantics since the interpretation of the constructors depends on the continuous t-norm $\otimes$ that was chosen. The semantics of fuzzy DLs is usually obtained by viewing the DL part as a fragment of first-order logic [1] and lifting the first-order expression to the fuzzy semantics used in fuzzy predicate logics [2]. In particular, this means that existential and value restrictions are interpreted by suprema and infima, respectively, over the whole interpretation domain.

Definition 8 (*Interpretations*). An *interpretation* $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ consists of a non-empty *domain* $\Delta^{\mathcal{I}}$ and an *interpretation function* $\cdot^{\mathcal{I}}$ that assigns to every $A \in \mathsf{N_C}$ a fuzzy set $A^{\mathcal{I}} : \Delta^{\mathcal{I}} \to [0,1]$, to every $r \in \mathsf{N_R}$ a fuzzy binary relation $r^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \to [0,1]$, and to every $d \in \mathsf{N_I}$ an element $d^{\mathcal{I}} \in \Delta^{\mathcal{I}}$ of the domain. The interpretation function is extended to complex concepts as follows for all $x \in \Delta^{\mathcal{I}}$:

- $\top^{\mathcal{I}}(x) = 1$,
- $(C \sqcap D)^{\mathcal{I}}(x) = C^{\mathcal{I}}(x) \otimes D^{\mathcal{I}}(x)$,
- $(C \to D)^{\mathcal{I}}(x) = C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x)$,
- $(\neg C)^{\mathcal{I}}(x) = {\sim} C^{\mathcal{I}}(x)$,
- $(\boxminus C)^{\mathcal{I}}(x) = \ominus C^{\mathcal{I}}(x)$,
- $(\exists r.C)^{\mathcal{I}}(x) = \sup_{y \in \Delta^{\mathcal{I}}} (r^{\mathcal{I}}(x,y) \otimes C^{\mathcal{I}}(y))$,
- $(\forall r.C)^{\mathcal{I}}(x) = \inf_{y \in \Delta^{\mathcal{I}}} (r^{\mathcal{I}}(x,y) \Rightarrow C^{\mathcal{I}}(y))$.

The interpretation $\mathcal{I}$ is *finite* if its domain is finite, and *crisp* if $A^{\mathcal{I}}(x) \in \{0,1\}$ and $r^{\mathcal{I}}(x,y) \in \{0,1\}$ for all concept names A, role names r, and domain elements x, y. We say that an interpretation $\mathcal{I}'$ is an *extension* of $\mathcal{I}$ if it has the same domain as $\mathcal{I}$, agrees with $\mathcal{I}$ on the interpretation of $\mathsf{N_C}$, $\mathsf{N_R}$, and $\mathsf{N_I}$ and additionally defines values for some new concept names not appearing in $\mathsf{N_C}$.

Notice that the semantics of existential and value restrictions require the computation of a supremum or infimum of the membership degrees of a possibly infinite set of elements of the interpretation domain. As is customary for fuzzy DLs, we therefore restrict reasoning to a special kind of models, called witnessed models [8,28]. An interpretation $\mathcal{I}$ is called *witnessed* if for every concept C, role name r, and $x \in \Delta^{\mathcal{I}}$ there exist $y, y' \in \Delta^{\mathcal{I}}$ such that

- $(\exists r.C)^{\mathcal{I}}(x) = r^{\mathcal{I}}(x,y) \otimes C^{\mathcal{I}}(y)$, and
- $(\forall r.C)^{\mathcal{I}}(x) = r^{\mathcal{I}}(x,y') \Rightarrow C^{\mathcal{I}}(y')$.

This means that the suprema and infima in the semantics of existential and value restrictions are actually maxima and minima, respectively. Without this restriction, the value of $(\exists r.C)^{\mathcal{I}}(x)$ might, e.g. be 1 without x actually having a single r-successor with degree 1 that belongs to C with degree 1. Such a behavior is usually unwanted in description logics, where an existential restriction is intended to express the existence of an adequate successor.

The main reasoning problem that we consider in this paper is (witnessed) ontology consistency; that is, deciding whether one can find a witnessed interpretation satisfying all the axioms of an ontology.

Definition 9 (*Consistency*). A witnessed interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ *satisfies* the GCI $\langle C \sqsubseteq D \geq p \rangle$ if for all $x \in \Delta^{\mathcal{I}}$, we have $C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x) \geq p$. It *satisfies* the assertion $\langle e : C \rhd p \rangle$ (resp., $\langle (d, e) : r \rhd p \rangle$) if $C^{\mathcal{I}}(e^{\mathcal{I}}) \rhd p$ (resp., $r^{\mathcal{I}}(d^{\mathcal{I}}, e^{\mathcal{I}}) \rhd p$). It is a *model* of an ontology $\mathcal{O}$ if it satisfies all the axioms in $\mathcal{O}$.

An ontology is *consistent* if it has a model.

According to this semantics, the crisp GCIs $\langle C \sqsubseteq D \rangle$ and $\langle D \sqsubseteq C \rangle$ are satisfied iff $C^{\mathcal{I}}(x) = D^{\mathcal{I}}(x)$ for every $x \in \Delta^{\mathcal{I}}$. It thus makes sense to abbreviate them by the expression $\langle C \equiv D \rangle$, as we will do for the rest of this paper. Note that the restriction to witnessed interpretations is not without loss of generality since there exist ontologies that have general models, but no witnessed models [28]. In Section 5.1, we comment on the importance of this restriction and the consequences of dropping it.

We now relate some of the introduced fuzzy DLs according to their expressive power. For every choice of constructors $\mathcal{L}$ and t-norm $\otimes$, the inequality concept assertion $\langle e : C \geq q \rangle$ can be expressed in $\otimes$-$\mathcal{L}_=$ using the two axioms $\langle e : A = q \rangle$ and
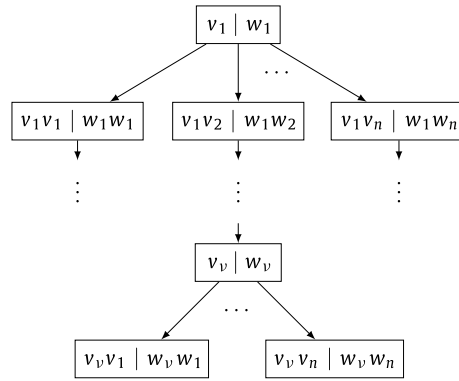
Fig. 1. The search tree for an instance $\mathcal{P}$ of the PCP.

$\langle A \sqsubseteq C \rangle$, where A is a new concept name, and thus $\otimes\text{-}\mathcal{L}_=$ is at least as expressive as $\otimes\text{-}\mathcal{L}_\geq$. Furthermore, since the residual negation can be expressed using the implication and bottom constructors, we know that $\otimes\text{-}\mathfrak{I}\mathcal{AL}$ is as least as expressive as $\otimes\text{-}\mathfrak{N}\mathcal{AL}$ and the same holds for $\otimes\text{-}\mathfrak{I}\mathcal{EL}$ and $\otimes\text{-}\mathfrak{N}\mathcal{EL}$.

If we restrict the semantics to the Łukasiewicz t-norm, for which involutive and residual negation coincide, we obtain that Ł-$\mathcal{ELC}$, Ł-$\mathfrak{N}\mathcal{EL}$, Ł-$\mathfrak{I}\mathcal{EL}$, Ł-$\mathcal{ALC}$, Ł-$\mathfrak{N}\mathcal{AL}$, and Ł-$\mathfrak{I}\mathcal{AL}$ are all equivalent [2]. Indeed, under this semantics value and existential restrictions are dual to each other $((\forall r.C)^{\mathcal{I}} = (\neg\exists r.\neg C)^{\mathcal{I}})$ and the implication can be expressed by negation and conjunction $((C \rightarrow D)^{\mathcal{I}} = (\neg(C \sqcap \neg D))^{\mathcal{I}})$. However, for arbitrary t-norms these equalities need not hold. For instance, if any t-norm different from Łukasiewicz is used, then $(\neg\exists r.\neg C)^{\mathcal{I}} \neq (\forall r.C)^{\mathcal{I}}$.

In the next section, we describe a general framework to show undecidability of consistency in fuzzy description logics. Subsequently, we show that consistency in many of the logics for which we cannot show undecidability is equivalent to consistency in the underlying crisp description logics, and thus decidable. Intuitively, a fuzzy DL is undecidable whenever it can express upper bounds for the membership degrees of concepts, e.g. through the involutive negation or the implication constructor. On the other hand, our decidability results exploit the fact that some fuzzy DLs cannot express such upper bounds except for 0. At the end of this paper, we comment on reasoning w.r.t. general (non-witnessed) models and on the decidability of reasoning problems other than consistency.

3. Undecidable fuzzy DLs

We now describe a general approach for proving that the consistency problem for a fuzzy DL $\otimes\text{-}\mathcal{L}$ is undecidable. It is based on a reduction from a variant of the Post correspondence problem (PCP) which is known to be undecidable [20].

Definition 10 *(PCP).* Let $\mathcal{P} = \{(v_1, w_1), \ldots, (v_n, w_n)\}$ be a finite set of pairs of words over the alphabet $\Sigma = \{1, \ldots, s\}$ with $s > 1$. The *Post correspondence problem* asks whether there is a finite sequence $i_1 \ldots i_k \in \{1, \ldots, n\}^*$ such that $v_1 v_{i_1} \ldots v_{i_k} = w_1 w_{i_1} \ldots w_{i_k}$. If this sequence exists, it is called a *solution* for $\mathcal{P}$.

Notice that in this variant of the PCP, a solution always starts with the first pair of words (v_1, w_1). We will abbreviate $\{1, \ldots, n\}$ by $\mathcal{N}$. For $\nu = i_1 \ldots i_k \in \mathcal{N}^*$, we use the notation $v_\nu := v_1 v_{i_1} \ldots v_{i_k}$ and $w_\nu := w_1 w_{i_1} \ldots w_{i_k}$. In order to solve an instance $\mathcal{P} = \{(v_1, w_1), \ldots, (v_n, w_n)\}$ of the PCP, we consider its *search tree*, which has one node for every $\nu \in \mathcal{N}^*$, where ε is the root, and νi is the i-th successor of ν for each $i \in \mathcal{N}$. Every node ν in this tree is labeled with the words $v_\nu, w_\nu \in \Sigma^*$, as shown in Fig. 1. Obviously, the instance $\mathcal{P}$ has a solution iff its search tree contains a node labeled by two equal words.

Correspondingly, our reduction of the PCP to the consistency problem of a fuzzy DL consists of two parts. Given an instance $\mathcal{P}$ of the PCP, we first construct an ontology $\mathcal{O}_\mathcal{P}$ that describes the search tree of $\mathcal{P}$, and then check whether this tree contains a solution for $\mathcal{P}$. More precisely, we enforce that for every model $\mathcal{I}$ of $\mathcal{O}_\mathcal{P}$ and every $\nu \in \mathcal{N}^*$, there is an $x_\nu \in \Delta^{\mathcal{I}}$ such that $V^{\mathcal{I}}(x_\nu) = \mathrm{enc}(v_\nu)$ and $W^{\mathcal{I}}(x_\nu) = \mathrm{enc}(w_\nu)$, where $\mathrm{enc}\colon \Sigma^* \rightarrow [0, 1]$ is an injective function that encodes words over Σ into the interval $[0, 1]$ (see Theorem 12). Once we have encoded the words v_ν and w_ν using V and W, we add axioms that restrict the models to those that satisfy $V^{\mathcal{I}}(x_\nu) \neq W^{\mathcal{I}}(x_\nu)$ for all $\nu \in \mathcal{N}^*$. This ensures that $\mathcal{P}$ has a solution if and only if the ontology is inconsistent (see Theorem 13).

3.1. A special case

We first describe the construction on the relatively easy example of the fuzzy DL $\Pi\text{-}\mathfrak{I}\mathcal{AL}_=$. This is essentially the proof from [18], divided into several small steps. Later, we present a general framework that allows us to prove undecidability of many fuzzy DLs at the same time. This framework consists of several properties that a fuzzy DL can have, which together lead to undecidability. We label each part of the following construction by the name of the property of the general framework it corresponds to (see Section 3.2).

Let in the following $\mathcal{P} = \{(v_1, w_1), \ldots, (v_n, w_n)\}$ be an instance of the PCP over the alphabet Σ. Recall that Σ consists of the first s positive integers. We can thus view every word in Σ^* as a natural number represented in base $s + 1$. On the other hand, every natural number n has a unique representation in base $s + 1$, which can be seen as a word over the alphabet $\Sigma_0 := \Sigma \cup \{0\} = \{0, \ldots, s\}$. This is not a bijection since, e.g. the words 001202 and 1202 represent the same number. However, it is a bijection between the set $\Sigma \Sigma_0^*$ and the positive natural numbers. In the following, we interpret the empty word ε as 0, thereby extending this bijection to $\{\varepsilon\} \cup \Sigma \Sigma_0^*$ and all non-negative integers.

In the following constructions and proofs, we view elements of Σ_0^* both as words and as natural numbers in base $s+1$. It is usually clear from the context which interpretation is used. However, to avoid confusion, we sometimes use the notation $\underline{u}$ to express that u is seen as a word. Thus, for instance, if $s = 3$, then $3 \cdot 2^2 = 30$ (in base 4), but $\underline{3} \cdot \underline{2}^2 = \underline{322}$. Furthermore, $\underline{000}$ is a word of length 3, whereas 000 is simply the number 0. We extend this notation to rational numbers, and may use, e.g. the expression $0.\underline{0}^3 \cdot \underline{1}$ to denote the number 0.0001 (again, in base 4). For a word $u = \alpha_1 \cdots \alpha_m$ with $\alpha_i \in \Sigma_0$, $1 \le i \le m$, we denote by $\overleftarrow{u}$ the word $\alpha_m \cdots \alpha_1 \in \Sigma_0^*$.

For the case of $\Pi\text{-}\mathfrak{I}\mathcal{AL}_=$, we use the encoding function enc: $\Sigma^* \to [0, 1]$ given by $\mathrm{enc}(u) := 2^{-u}$ to encode words as values from the interval $[0, 1]$, and thus we have, e.g. $\mathrm{enc}(\varepsilon) = 2^{-0} = 1$ and $\mathrm{enc}(2) = 2^{-2} = 1/4$.

The initialization property   The first step in constructing the ontology $\mathcal{O}_\mathcal{P}$ that describes the search tree of $\mathcal{P}$ is to initialize the root of this search tree. The root is represented by the individual name e_0, for which we have to initialize the values for V and W, as well as several other auxiliary concept names. Due to the presence of equality assertions, this step is particularly easy in $\Pi\text{-}\mathfrak{I}\mathcal{AL}_=$:

$$\langle e_0 : V = \mathrm{enc}(v_1) \rangle, \ \langle e_0 : W = \mathrm{enc}(w_1) \rangle, \ \langle e_0 : M = 1/2 \rangle,$$
$$\langle e_0 : V_1 = \mathrm{enc}(v_1) \rangle, \ \ldots, \ \langle e_0 : V_n = \mathrm{enc}(v_n) \rangle,$$
$$\langle e_0 : W_1 = \mathrm{enc}(w_1) \rangle, \ \ldots, \ \langle e_0 : W_n = \mathrm{enc}(w_n) \rangle. \tag{1}$$

The concept names $V_1, \ldots, V_n, W_1, \ldots, W_n$ are intended to be constants that hold the above values at every node of the search tree, and are used in each step to concatenate the words $v_1, \ldots, v_n, w_1, \ldots, w_n$ to the words currently encoded by V and W. Similarly, the value of M is always $1/2$ throughout the search tree, and is used to compare the values of V and W at each node.

The concatenation property   The next step is to compute the values $\mathrm{enc}(v_1 v_i)$ and $\mathrm{enc}(w_1 w_i)$ for the successors $i \in \mathcal{N}$ of the root node. We introduce additional auxiliary concept names $D_{V \circ v_i}$ and $D_{W \circ w_i}$ to hold these values. We can achieve the correct concatenation using the equivalence

$$\langle D_{V \circ v_i} \equiv V^{(s+1)^{|v_i|}} \sqcap V_i \rangle \tag{2}$$

for every $i \in \mathcal{N}$, and similarly for $D_{W \circ w_i}$. Indeed, since V has the value $\mathrm{enc}(v_1) = 2^{-v_1}$ and V_i has the value $\mathrm{enc}(v_i) = 2^{-v_i}$ at e_0, $D_{V \circ v_i}$ is evaluated to $2^{-(v_1(s+1)^{|v_i|} + v_i)} = 2^{-\underline{v_1 v_i}} = \mathrm{enc}(v_1 v_i)$. In general, whenever V has the value $\mathrm{enc}(v_\nu)$ for some $\nu \in \mathcal{N}^*$, then $D_{V \circ v_i}$ has the value $\mathrm{enc}(v_{\nu i})$.

The successor property   We now construct the successors of the root node, which are labeled by the role names $r_1, \ldots, r_n$, using the axioms

$$\langle \top \sqsubseteq \exists r_1. \top \rangle, \ \ldots, \ \langle \top \sqsubseteq \exists r_n. \top \rangle. \tag{3}$$

Every (witnessed) model of these axioms has an r_i-successor for every domain element and every $i \in \mathcal{N}$.

The transfer property   To finish the construction of the search tree of $\mathcal{P}$, it remains to transfer the values of $D_{V \circ v_i}$ to the value of V at the r_i-successors. We also have to transfer the values of $D_{W \circ w_i}$ and the auxiliary constants $M, V_1, \ldots, V_n, W_1, \ldots, W_n$. This is accomplished using the axioms

$$\langle \exists r_i. V \sqsubseteq D_{V \circ v_i} \rangle, \ \langle D_{V \circ v_i} \sqsubseteq \forall r_i. V \rangle$$
$$\langle \exists r_i. W \sqsubseteq D_{W \circ w_i} \rangle, \ \langle D_{W \circ w_i} \sqsubseteq \forall r_i. W \rangle$$
$$\langle \exists r_i. M \sqsubseteq M \rangle, \ \langle M \sqsubseteq \forall r_i. M \rangle$$
$$\cdots \tag{4}$$

for each $i \in \mathcal{N}$. It can be shown that the axioms in (1)–(4) restrict all their models to "embed" an encoding of the search tree of $\mathcal{P}$. This is summarized in the *canonical model property* in the next section (for details, see the proof of Theorem 12).

The solution property Finally, to ensure that V and W always encode different words, we employ the axiom

$$\langle \top \sqsubseteq \big((V \to W) \sqcap (W \to V)\big) \to M \rangle. \tag{5}$$

This ensures that at each node $v \in \mathcal{N}^*$ of the search tree one of the concepts $V \to W$ or $W \to V$ has a value smaller than or equal to that of M, i.e. 1/2. This means that $\mathrm{enc}(v_v)$ and $\mathrm{enc}(w_v)$ differ by at least a factor of 2, which is equivalent to the fact that $v_v \neq w_v$ (for details, see Lemmata 14 and 19). Axiom (5) is of a simpler form than the ones used in previous undecidability proofs [18,19] since we consider here the variant of the PCP where all solutions must start with the first pair of words (v_1, w_1), and thus we do not need to exclude the root node ε from consideration.

If we collect all the axioms in (1)–(5), the resulting ontology is consistent iff $\mathcal{P}$ has no solution. Therefore, the consistency problem in Π-$\mathfrak{ZAL}_=$ is undecidable. For different fuzzy DLs, different steps of this construction are more or less difficult, depending on the t-norm and the allowed constructors. In the next section, we present a generalized description of how to show undecidability by a reduction of the PCP, which we then instantiate to yield undecidability results for a variety of fuzzy description logics.

3.2. The framework

In the following, let $\mathcal{P}$ be an instance of the PCP and $\otimes$-$\mathcal{L}$ be any fuzzy DL as introduced in Section 2. We first formalize the requirements for the encoding function enc. Recall from the previous section that we have to be able to concatenate constant words (i.e. v_i) to already computed encodings of words (i.e. v_v). Furthermore, we need to be able to test equality of words by comparing the residua of their encodings. When enc satisfies the latter property, we call it a *valid encoding function*. The former requirement is formalized later in the concatenation property.

Recall that for every $p, q \in [0, 1]$, we have $p = q$ iff $p \Rightarrow q = q \Rightarrow p = 1$ (see Lemma 1). Thus, to decide whether $\mathcal{P}$ has a solution, we have to check whether $\mathrm{enc}(v_v) \Rightarrow \mathrm{enc}(w_v) < 1$ or $\mathrm{enc}(w_v) \Rightarrow \mathrm{enc}(v_v) < 1$ holds for every $v \in \mathcal{N}^*$. In the special case in Section 3.1, it is clear that these residua are either 1 or smaller or equal to 1/2. Thus, the test simplifies to checking whether $\mathrm{enc}(v_v) \Rightarrow \mathrm{enc}(w_v) \leq 1/2$ or $\mathrm{enc}(w_v) \Rightarrow \mathrm{enc}(v_v) \leq 1/2$ holds. However, in general it is not possible to put a constant bound on these residua in case they are smaller than 1. Instead, we can often construct a word whose encoding bounds these residua. Clearly, the precise word and encoding must depend on the t-norm used. Another difference to the special case of Section 3.1 is that we allow a word u to be encoded by a *set* of values $\mathrm{Enc}(u) \subseteq [0, 1]$. This simplifies some of the proofs, but requires us to ensure that these encodings remain unique, i.e. that no two words can be encoded by the same value.

Definition 11 *(Valid encoding function).* A function $\mathrm{Enc} \colon \Sigma_0^* \to 2^{[0,1]}$ is called a *valid encoding function* for $\otimes$ if

(a) for every $u \in \{\varepsilon\} \cup \Sigma \Sigma_0^*$ and every $v \in \{0\}^*$, we have $\mathrm{Enc}(vu) = \mathrm{Enc}(u)$,
(b) the sets $\mathrm{Enc}(u)$ and $\mathrm{Enc}(u')$ are nonempty and disjoint for any two different words $u, u' \in \{\varepsilon\} \cup \Sigma \Sigma_0^*$, and
(c) there exist two words $u_\varepsilon, u_+ \in \Sigma_0^*$ such that for every $v \in \mathcal{N}^*$, $p \in \mathrm{Enc}(v_v)$, $q \in \mathrm{Enc}(w_v)$, and $m \in \mathrm{Enc}(\underline{u_\varepsilon} \cdot \underline{u_+}^{|v|})$ it holds that $\underline{u_\varepsilon} \cdot \underline{u_+}^{|v|} \in \{\varepsilon\} \cup \Sigma \Sigma_0^*$ and

$$v_v \neq w_v \quad \text{iff} \quad \min\{p \Rightarrow q, q \Rightarrow p\} \leq m.$$

Condition (a) is due to the fact that we often view the words of Σ_0^* as natural numbers in base $s + 1$ (cf. Section 3.1), and thus words that differ only in the number of leading zeros should have the same encoding. Condition (b) ensures that one can uniquely identify a word from its encoding, modulo leading zeros. Finally, Condition (c) requires that every value in $\mathrm{Enc}(\underline{u_\varepsilon} \cdot \underline{u_+}^{|v|})$ can be used to check whether encodings of v_v and w_v are equal by comparing the above residua to this value.

In the following, Enc represents a valid encoding function for $\otimes$, and u_ε, u_+ are the words required by Condition (c). We additionally assume that we have a function $\mathrm{enc} \colon \Sigma_0^* \to [0, 1]$ that chooses a representative $\mathrm{enc}(u) \in \mathrm{Enc}(u)$ for each $u \in \Sigma_0^*$. Such a function must always exist due to the Conditions (a) and (b) of Definition 11.

As in the previous section, we use the concept names V, W to represent the values of the words v_v and w_v at the nodes of the search tree for $\mathcal{P}$. We designate the concept name M to represent the bounding word $\underline{u_\varepsilon} \cdot \underline{u_+}^{|v|}$ from Definition 11, and M_+ to represent u_+. We also use the concept names V_i, W_i to encode the words v_i, w_i from $\overline{\mathcal{P}}$, and the role names r_i to distinguish the different successors in the search tree, for each $i \in \mathcal{N}$. The individual name e_0 is used to distinguish the root node. Formally, the search tree for $\mathcal{P}$ is represented by the canonical model $\mathcal{I}_\mathcal{P} = (\mathcal{N}^*, \cdot^{\mathcal{I}_\mathcal{P}})$ of the ontology $\mathcal{O}_\mathcal{P}$ we will construct. It is defined as follows for every $v \in \mathcal{N}^*$ and $i \in \mathcal{N}$:

- $e_0^{\mathcal{I}_\mathcal{P}} := \varepsilon$,
- $V^{\mathcal{I}_\mathcal{P}}(v) := \mathrm{enc}(v_v)$, $W^{\mathcal{I}_\mathcal{P}}(v) := \mathrm{enc}(w_v)$,
- $V_i^{\mathcal{I}_\mathcal{P}}(v) := \mathrm{enc}(v_i)$, $W_i^{\mathcal{I}_\mathcal{P}}(v) := \mathrm{enc}(w_i)$,
- $M^{\mathcal{I}_\mathcal{P}}(v) := \mathrm{enc}(\underline{u_\varepsilon} \cdot \underline{u_+}^{|v|})$, $M_+^{\mathcal{I}_\mathcal{P}}(v) := \mathrm{enc}(u_+)$,
- $r_i^{\mathcal{I}_\mathcal{P}}(v, vi) := 1$ and $r_i^{\mathcal{I}_\mathcal{P}}(v, v') := 0$ if $v' \neq vi$.
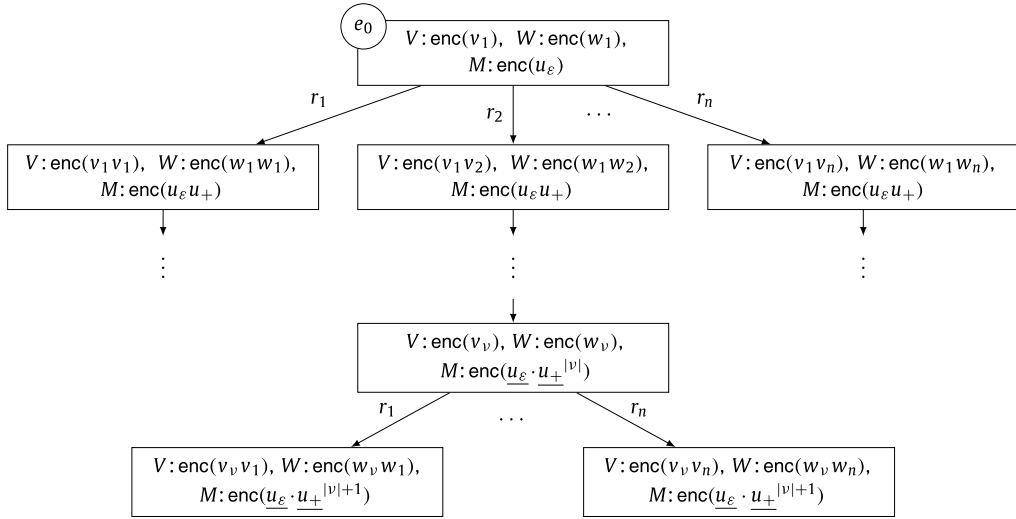
Fig. 2. The canonical model $\mathcal{I}_\mathcal{P}$ for an instance $\mathcal{P}$ of the PCP.

Since every element of $\mathcal{N}^*$ has exactly one r_i-successor with degree greater than 0, $\mathcal{I}_\mathcal{P}$ is a witnessed interpretation. This model is depicted in Fig. 2 and clearly represents the search tree for $\mathcal{P}$ (cf. Fig. 1).

The goal is to construct an ontology $\mathcal{O}_\mathcal{P}$ that can only be satisfied by interpretations that "include" the search tree of $\mathcal{P}$. Given that the interpretation $\mathcal{I}_\mathcal{P}$ represents this tree, we want the logic to satisfy the following property. Here, we use the expression $p \sim q$ for $p, q \in [0, 1]$ to denote the fact that $p, q \in \mathsf{Enc}(u)$ for some word $u \in \Sigma_0^*$. By Conditions (a) and (b) of Definition 11, this word is unique except for the number of leading zeros. But Condition (a) ensures that leading zeros are irrelevant for the encoding, and thus from $p \sim q$ and $p \in \mathsf{Enc}(u)$ for some $u \in \Sigma_0^*$, we can always infer that $q \in \mathsf{Enc}(u)$.

The canonical model property ($\mathrm{P}_\triangle$)

> The logic $\otimes\text{-}\mathcal{L}$ has the *canonical model property* if there is an ontology $\mathcal{O}_\mathcal{P}$ such that for every model $\mathcal{I}$ of $\mathcal{O}_\mathcal{P}$ there is a mapping $g: \Delta^{\mathcal{I}_\mathcal{P}} \to \Delta^\mathcal{I}$ with
>
> $$A^{\mathcal{I}_\mathcal{P}}(v) \sim A^\mathcal{I}\big(g(v)\big)$$
>
> for every $A \in \{V, W, M, M_+\} \cup \bigcup_{i=1}^n \{V_i, W_i\}$, and $v \in \mathcal{N}^*$.

As in the previous section, rather than trying to prove this property directly for some fuzzy DL, we provide several simpler properties that together imply the canonical model property. We often motivate the following constructions using only the concept V and the words v_v; however, all arguments apply analogously to W, w_v and M, $\underline{u_\varepsilon} \cdot u_+^{|v|}$.

As illustrated in Section 3.1, we construct the search tree in an inductive way. First, we restrict every interpretation $\mathcal{I}$ to satisfy that $A^{\mathcal{I}_\mathcal{P}}(\varepsilon) \sim A^\mathcal{I}(e_0^\mathcal{I})$ for every relevant concept name. This makes sure that the root ε of the search tree is properly represented at the individual $g(\varepsilon) := e_0^\mathcal{I}$. Let now $g(v)$ be a node satisfying this property, and $i \in \mathcal{N}$. We ensure that there is a node $g(vi)$ that also satisfies the property in three steps: first, we force the existence of an individual y with $r_i^\mathcal{I}(g(v), y) = 1$ and set $g(vi) := y$. Then, we compute a value in $\mathsf{Enc}(v_v v_i)$ from $V^\mathcal{I}(g(v)) \in \mathsf{Enc}(v_v)$ and $V_i^\mathcal{I}(g(v)) \in \mathsf{Enc}(v_i)$. Finally, we transfer this value to the previously created successor to ensure that $V^\mathcal{I}(g(vi)) \sim \mathsf{enc}(v_v v_i)$. The value of $V_j^\mathcal{I}(g(v))$ for every $j \in \mathcal{N}$ is similarly transferred to $V_j^\mathcal{I}(g(vi))$.

Each step of the previous construction is guaranteed by a property of the logic $\otimes\text{-}\mathcal{L}$. These properties, which are ultimately used to produce the ontology $\mathcal{O}_\mathcal{P}$, are described next.

The initialization property ($\mathrm{P}_{\mathsf{ini}}$)

> The logic $\otimes\text{-}\mathcal{L}$ has the *initialization property* if for every concept C, individual name e, and $u \in \Sigma_0^*$ there is an ontology $\mathcal{O}_{C(e)=u}$ such that for every model $\mathcal{I}$ of $\mathcal{O}_{C(e)=u}$ it holds that $C^\mathcal{I}(e^\mathcal{I}) \in \mathsf{Enc}(u)$.

Assume now that $\otimes$-$\mathcal{L}$ satisfies $\mathsf{P_{ini}}$. Then, to initialize the search tree, we can set the values of V and W at e_0 to valid encodings of v_1 and w_1, respectively, and the value of M to an encoding of u_ε. Moreover, we need that M_+ encodes u_+ and every V_i and W_i encodes the word v_i and w_i, respectively, for every $i \in \mathcal{N}$. We thus define the ontology

$$\mathcal{O}_{\mathcal{P},\mathsf{ini}} := \mathcal{O}_{M(e_0)=u_\varepsilon} \cup \mathcal{O}_{M_+(e_0)=u_+} \cup \mathcal{O}_{V(e_0)=v_1} \cup \mathcal{O}_{W(e_0)=w_1} \cup \bigcup_{i=1}^{n}(\mathcal{O}_{V_i(e_0)=v_i} \cup \mathcal{O}_{W_i(e_0)=w_i}).$$

This is an abstract version of the axioms (1) presented in Section 3.1 for Π-$\mathfrak{I}\mathcal{AL}_=$. Note that there we had $u_+ = \varepsilon$, and thus the concept name M_+ was not needed.

The successor property ($\mathsf{P}_\rightarrow$)

> The logic $\otimes$-$\mathcal{L}$ has the *successor property* if for all role names r there is an ontology $\mathcal{O}_{\exists r}$ such that for every model $\mathcal{I}$ of $\mathcal{O}_{\exists r}$ and every $x \in \Delta^{\mathcal{I}}$ there is a $y \in \Delta^{\mathcal{I}}$ with $r^{\mathcal{I}}(x, y) = 1$.

If a logic satisfies this property, then the ontology

$$\mathcal{O}_{\mathcal{P},\rightarrow} := \bigcup_{i \in \mathcal{N}} \mathcal{O}_{\exists r_i}$$

ensures the existence of an r_i-successor with value 1 for every node of the search tree and every $i \in \mathcal{N}$, corresponding to the r_i-connections in the canonical model. For our initial example of Π-$\mathfrak{I}\mathcal{AL}_=$, this task was achieved by the axioms in (3).

The concatenation property ($\mathsf{P}_\circ$)

> The logic $\otimes$-$\mathcal{L}$ has the *concatenation property* if for all words $u \in \Sigma_0^*$, and concepts C and C_u, there is an ontology $\mathcal{O}_{C\circ u}$ and a concept name $D_{C\circ u}$ such that for every model $\mathcal{I}$ of $\mathcal{O}_{C\circ u}$ and every $x \in \Delta^{\mathcal{I}}$, if $C_u^{\mathcal{I}}(x) \in \mathsf{Enc}(u)$ and $C^{\mathcal{I}}(x) \in \mathsf{Enc}(u')$ for some $u' \in \{\varepsilon\} \cup \Sigma\Sigma_0^*$, then $D_{C\circ u}^{\mathcal{I}}(x) \in \mathsf{Enc}(\underline{u'u})$.

The goal of this property is to ensure that at every node where $V^{\mathcal{I}}(x) \in \mathsf{Enc}(u)$ for some $u \in \{\varepsilon\} \cup \Sigma\Sigma_0^*$, and $C_{v_i}^{\mathcal{I}}(x) \in \mathsf{Enc}(v_i)$, then $D_{V\circ v_i}^{\mathcal{I}}(x) \in \mathsf{Enc}(\underline{u v_i})$, and similarly for W, w_i and M, u_+. Thus, we define the ontology

$$\mathcal{O}_{\mathcal{P},\circ} := \bigcup_{i=1}^{n}(\mathcal{O}_{V\circ v_i} \cup \mathcal{O}_{W\circ w_i} \cup \mathcal{O}_{M\circ u_+}).$$

To simplify the notation, we use the concept names V_i, W_i, M_+ instead of $C_{v_i}, C_{w_i}, C_{u_+}$ in this ontology. This corresponds to the axioms given for Π-$\mathfrak{I}\mathcal{AL}_=$ in (2). Note that by construction, the values of $V^{\mathcal{I}}(x)$, $W^{\mathcal{I}}(x)$, and $M^{\mathcal{I}}(x)$ should always be encodings of words from $\{\varepsilon\} \cup \Sigma\Sigma_0^*$.

3.2.1. The transfer property ($\mathsf{P}_\rightsquigarrow$)

> The logic $\otimes$-$\mathcal{L}$ has the *transfer property* if for all concepts C, D and role names r there is an ontology $\mathcal{O}_{C\overset{r}{\rightsquigarrow}D}$ such that for every model $\mathcal{I}$ of $\mathcal{O}_{C\overset{r}{\rightsquigarrow}D}$ and every $x, y \in \Delta^{\mathcal{I}}$, if $r^{\mathcal{I}}(x, y) = 1$ and $C^{\mathcal{I}}(x) \in \mathsf{Enc}(u)$ for some $u \in \Sigma_0^*$, then $D^{\mathcal{I}}(y) \in \mathsf{Enc}(u)$.

To ensure that the values of $\mathsf{enc}(\underline{u_\varepsilon \cdot u_+}^{|v|})$, $\mathsf{enc}(u_+)$, $\mathsf{enc}(v_{vi})$, and $\mathsf{enc}(v_j)$ for every $j \in \mathcal{N}$ are transferred from x to the r_i-successor y_i for every $i \in \mathcal{N}$, we use the ontology

$$\mathcal{O}_{\mathcal{P},\rightsquigarrow} := \bigcup_{i \in \mathcal{N}} \mathcal{O}_{D_{M\circ u_+}\overset{r_i}{\rightsquigarrow}M} \cup \mathcal{O}_{M_+\overset{r_i}{\rightsquigarrow}M_+} \cup \mathcal{O}_{D_{V\circ v_i}\overset{r_i}{\rightsquigarrow}V} \cup \mathcal{O}_{D_{W\circ w_i}\overset{r_i}{\rightsquigarrow}W} \cup \bigcup_{i,j \in \mathcal{N}} \mathcal{O}_{V_j\overset{r_i}{\rightsquigarrow}V_j} \cup \mathcal{O}_{W_j\overset{r_i}{\rightsquigarrow}W_j}.$$

This was accomplished by the Π-$\mathfrak{I}\mathcal{AL}_=$-axioms in (4).

As argued before, if we combine these four properties, then we obtain the canonical model property.

Theorem 12. *Let* Enc *be a valid encoding function for* $\otimes$. *If the logic* $\otimes$-$\mathcal{L}$ *satisfies* P_{ini}, $P_{\rightarrow}$, P_{o}, *and* $P_{\rightsquigarrow}$, *then it also satisfies* $P_{\triangle}$.

Proof. We show that the ontology $\mathcal{O}_{\mathcal{P}} := \mathcal{O}_{\mathcal{P},\mathrm{ini}} \cup \mathcal{O}_{\mathcal{P},\mathrm{o}} \cup \mathcal{O}_{\mathcal{P},\rightarrow} \cup \mathcal{O}_{\mathcal{P},\rightsquigarrow}$ satisfies the conditions of $P_{\triangle}$. For a model $\mathcal{I}$ of $\mathcal{O}_{\mathcal{P}}$, we construct the function $g \colon \mathcal{N}^* \to \Delta^{\mathcal{I}}$ inductively as follows.

We first set $g(\varepsilon) := e_0^{\mathcal{I}}$. The fact that $\mathcal{I}$ is a model of $\mathcal{O}_{\mathcal{P},\mathrm{ini}}$ implies that $V^{\mathcal{I}}(g(\varepsilon)) = V^{\mathcal{I}}(e_0^{\mathcal{I}}) \in \mathsf{Enc}(v_1)$, and thus $V^{\mathcal{I}}(g(\varepsilon)) \sim \mathsf{enc}(v_1) = V^{\mathcal{I}_{\mathcal{P}}}(\varepsilon)$, and likewise for W, M, M_+, V_i, and W_i for all $i \in \mathcal{N}$.

Let now v be such that $g(v)$ has already been defined, $V^{\mathcal{I}}(g(v)) \sim \mathsf{enc}(v_v)$, and $V_i^{\mathcal{I}}(g(v)) \sim \mathsf{enc}(v_i)$. Since Enc is a valid encoding function and by the definition of $\sim$, we know that $V^{\mathcal{I}}(g(v)) \in \mathsf{Enc}(v_v)$ and $V_i^{\mathcal{I}}(g(v)) \in \mathsf{Enc}(v_i)$ hold. Thus, from the fact that $\mathcal{I}$ is a model of $\mathcal{O}_{\mathcal{P},\mathrm{o}}$ we infer that $D_{V \circ v_i}^{\mathcal{I}}(g(v)) \in \mathsf{Enc}(v_{vi})$. Since $\mathcal{I}$ satisfies $\mathcal{O}_{\mathcal{P},\rightarrow}$, for each $i \in \{1, \ldots, n\}$ there must be an element $y_i \in \Delta^{\mathcal{I}}$ with $r_i^{\mathcal{I}}(g(v), y_i) = 1$. Define now $g(vi) := y_i$. The restrictions of $\mathcal{O}_{\mathcal{P},\rightsquigarrow}$ ensure that $V^{\mathcal{I}}(g(vi)) \sim D_{V \circ v_i}^{\mathcal{I}}(g(v)) \sim V^{\mathcal{I}_{\mathcal{P}}}(vi)$ and $V_i^{\mathcal{I}}(g(vi)) \sim V_i^{\mathcal{I}_{\mathcal{P}}}(vi)$ for all $i \in \mathcal{N}$, and analogously for W, W_i and M, M_+. □

We now describe how the property $P_{\triangle}$ can be used to prove undecidability of $\otimes$-$\mathcal{L}$. Recall that the idea is to add a set $\mathcal{O}_{V \neq W}$ of axioms (as in (5)) to $\mathcal{O}_{\mathcal{P}}$ so that every model $\mathcal{I}$ is restricted to satisfy $V^{\mathcal{I}}(g(v)) \napprox W^{\mathcal{I}}(g(v))$ for every $v \in \mathcal{N}^*$, thus obtaining an ontology that is consistent if and only if $\mathcal{P}$ has no solution. More formally, we have to show that (i) every model of $\mathcal{O}_{\mathcal{P}} \cup \mathcal{O}_{V \neq W}$ witnesses the non-existence of a solution for $\mathcal{P}$, and (ii) if $\mathcal{P}$ has no solution, then we can find a model of $\mathcal{O}_{\mathcal{P}} \cup \mathcal{O}_{V \neq W}$. Part (i) uses the fact that every model of $\mathcal{O}_{\mathcal{P}}$ encodes the canonical model by $P_{\triangle}$. For part (ii), the idea is to show that $\mathcal{I}_{\mathcal{P}}$ can be extended to a model of $\mathcal{O}_{\mathcal{P}} \cup \mathcal{O}_{V \neq W}$. However, for this to work, $\mathcal{I}_{\mathcal{P}}$ has to be a model of $\mathcal{O}_{\mathcal{P}}$ in the first place.

For the rest of this section, we thus assume that $\mathcal{I}_{\mathcal{P}}$ can actually be extended to a model of $\mathcal{O}_{\mathcal{P}}$; while $\mathcal{O}_{\mathcal{P}}$ might define additional concept names, it should not contradict the information about $V, W, M, \ldots$ represented by $\mathcal{I}_{\mathcal{P}}$. It is important to keep in mind for the subsequent sections that this constitutes an additional condition that has to be verified before we can show undecidability of a given fuzzy DL $\otimes$-$\mathcal{L}$. We also assume that $\otimes$-$\mathcal{L}$ satisfies $P_{\triangle}$, and for a given model $\mathcal{I}$ of $\mathcal{O}_{\mathcal{P}}$, g denotes the function mapping the nodes of $\mathcal{I}_{\mathcal{P}}$ to elements of $\Delta^{\mathcal{I}}$ given by the property. In Section 3.3, we show that these assumptions actually hold for a variety of fuzzy description logics.

Recall that the key to showing undecidability of $\otimes$-$\mathcal{L}$ is to be able to express the restriction that V and W encode different words at every node $v \in \mathcal{N}^*$ of the search tree. Since Enc is a valid encoding function and the concept name M encodes the word $u_\varepsilon \cdot u_+^{|v|}$ at every $v \in \mathcal{N}^*$, it suffices to check whether, for all $v \in \mathcal{N}^*$, either $(V \to W)^{\mathcal{I}_{\mathcal{P}}}(v) \leq M^{\mathcal{I}_{\mathcal{P}}}(v)$ or $(W \to V)^{\mathcal{I}_{\mathcal{P}}}(v) \leq M^{\mathcal{I}_{\mathcal{P}}}(v)$ holds (see Condition (c) of Definition 11).

The solution property $(P_{\neq})$

If the logic $\otimes$-$\mathcal{L}$ satisfies $P_{\triangle}$ with $\mathcal{O}_{\mathcal{P}}$, and $\mathcal{I}_{\mathcal{P}}$ can be extended to a model of $\mathcal{O}_{\mathcal{P}}$, then $\otimes$-$\mathcal{L}$ has the *solution property* if there is an ontology $\mathcal{O}_{V \neq W}$ such that the following conditions are satisfied:

1. For every model $\mathcal{I}$ of $\mathcal{O}_{\mathcal{P}} \cup \mathcal{O}_{V \neq W}$ and every $v \in \mathcal{N}^*$,

$$\min\{V^{\mathcal{I}}(g(v)) \Rightarrow W^{\mathcal{I}}(g(v)), \; W^{\mathcal{I}}(g(v)) \Rightarrow V^{\mathcal{I}}(g(v))\} \leq M^{\mathcal{I}}(g(v)).$$

2. If for every $v \in \mathcal{N}^*$ we have

$$\min\{V^{\mathcal{I}_{\mathcal{P}}}(v) \Rightarrow W^{\mathcal{I}_{\mathcal{P}}}(v), \; W^{\mathcal{I}_{\mathcal{P}}}(v) \Rightarrow V^{\mathcal{I}_{\mathcal{P}}}(v)\} \leq M^{\mathcal{I}_{\mathcal{P}}}(v),$$

then $\mathcal{I}_{\mathcal{P}}$ can be extended to a model of $\mathcal{O}_{\mathcal{P}} \cup \mathcal{O}_{V \neq W}$.

Notice that for any instance $\mathcal{P}$ of the PCP, the ontologies $\mathcal{O}_{\mathcal{P}}$ and $\mathcal{O}_{V \neq W}$ are both finite. We now show that if a fuzzy DL satisfies this property, then consistency of ontologies is undecidable.

Theorem 13. *If* $\otimes$-$\mathcal{L}$ *satisfies* $P_{\neq}$, *then* $\mathcal{P}$ *has a solution iff* $\mathcal{O}_{\mathcal{P}} \cup \mathcal{O}_{V \neq W}$ *is inconsistent.*

Proof. If $\mathcal{O}_{\mathcal{P}} \cup \mathcal{O}_{V \neq W}$ is inconsistent, then in particular no extension of $\mathcal{I}_{\mathcal{P}}$ can satisfy this ontology. By $P_{\neq}$, there is a $v \in \mathcal{N}^*$ such that

$$V^{\mathcal{I}_{\mathcal{P}}}(v) \Rightarrow W^{\mathcal{I}_{\mathcal{P}}}(v) > M^{\mathcal{I}_{\mathcal{P}}}(v) \quad \text{and} \quad W^{\mathcal{I}_{\mathcal{P}}}(v) \Rightarrow V^{\mathcal{I}_{\mathcal{P}}}(v) > M^{\mathcal{I}_{\mathcal{P}}}(v).$$

By the definition of $\mathcal{I}_{\mathcal{P}}$ and Condition (c) of Definition 11, we have $v_v = w_v$, and thus $\mathcal{P}$ has a solution.

For the converse direction, assume that $\mathcal{O}_{\mathcal{P}} \cup \mathcal{O}_{V \neq W}$ has a model $\mathcal{I}$. By $P_{\neq}$, for every $v \in \mathcal{N}^*$ we have

$$V^{\mathcal{I}}(g(v)) \Rightarrow W^{\mathcal{I}}(g(v)) \leq M^{\mathcal{I}}(g(v)) \quad \text{or} \quad W^{\mathcal{I}}(g(v)) \Rightarrow V^{\mathcal{I}}(g(v)) \leq M^{\mathcal{I}}(g(v)).$$
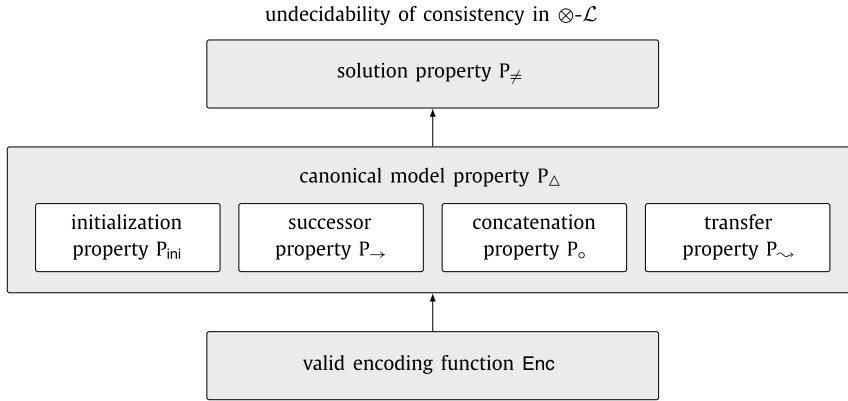
undecidability of consistency in $\otimes$-$\mathcal{L}$

Fig. 3. The framework for showing undecidability of consistency in a fuzzy DL $\otimes$-$\mathcal{L}$.

By $P_\triangle$, the definition of $\mathcal{I}_\mathcal{P}$, and Condition (c) of Definition 11, it follows that $v_\nu \neq w_\nu$. Since this holds for all $\nu \in \mathcal{N}^*$, we know that $\mathcal{P}$ has no solution. $\quad\square$

Fig. 3 informally depicts the relationships between all notions introduced in this section. The existence of a valid encoding function is the basic condition for all our properties. The canonical model property is implied by the conjunction of the smaller properties. Finally, the solution property depends on the canonical model property and guarantees undecidability of consistency in the given logic $\otimes$-$\mathcal{L}$.

3.3. First results

We use the properties of the previous section to show undecidability results for consistency in numerous fuzzy DLs. In Sections 3.4 and 3.5, we develop extensions of the framework to prove undecidability of this problem in a wider class of logics. The technical proofs of the following lemmata can be found in Appendix A.

The first step is to find a valid encoding function for our continuous t-norm $\otimes$. We assume in the following that $\otimes$ is not the Gödel t-norm. The reason for this is that our encoding function and the subsequent constructions depend on the choice of one component $((a, b), \otimes')$ of $\otimes$ where $\otimes'$ is either Ł or Π. If $\otimes$ is different from the Gödel t-norm, such a component must exist by Theorem 2. It is important that the component that we choose remains fixed throughout the whole construction. In the case that $\otimes' = $ Ł, we denote our choice by $Ł^{(a,b)}$, and similarly for $\otimes' = \Pi$. Correspondingly, we denote the fuzzy description logic by $Ł^{(a,b)}$-$\mathcal{L}$ or $\Pi^{(a,b)}$-$\mathcal{L}$.

We now use the chosen component to encode the words from Σ_0^*. For $u \in \{0\}^*$ (in particular for $u = \varepsilon$) we always use the encoding $\mathsf{Enc}(u) := [b, 1]$, i.e. all values from the upper bound of our component to 1 are valid encodings for ε. For these words, we define $\mathsf{enc}(u) := b$. For the remaining words $u \in \Sigma_0^* \Sigma \Sigma_0^*$, we use only a singleton set $\mathsf{Enc}(u) := \{\mathsf{enc}(u)\}$, where $\mathsf{enc}(u)$ depends on the chosen component. For the case of $\Pi^{(a,b)}$, we define

$$\mathsf{enc}(u) := \sigma_{a,b}\big(2^{-u}\big) \in (a, b),$$

and for $Ł^{(a,b)}$ we use

$$\mathsf{enc}(u) := \sigma_{a,b}(1 - 0.\widetilde{u}) \in (a, b).$$

Recall that we defined $\sigma_{a,b}(x) := a + (b - a)x$ for all $x \in [0, 1]$ (see Section 2).

Lemma 14. *The functions* Enc *described above are valid encoding functions for t-norms of the form* $\Pi^{(a,b)}$ *or* $Ł^{(a,b)}$.

Variants of these encoding functions and words u_ε, u_+ have been used before to show undecidability of fuzzy description logics based on the product [18] and Łukasiewicz [19] t-norms.

We now present several results about instances of $\otimes$-$\mathcal{L}$ that satisfy the properties introduced in the previous section. Recall that one precondition for the property $P_{\neq}$ is that $\mathcal{I}_\mathcal{P}$ can be extended to a model of $\mathcal{O}_\mathcal{P}$. Thus, in the following constructions of $\mathcal{O}_{C(e)=u}$, $\mathcal{O}_{\exists r}$, $\mathcal{O}_{C \circ u}$, and $\mathcal{O}_{C \overset{r}{\rightsquigarrow} D}$, it is important keep in mind that the resulting ontology $\mathcal{O}_\mathcal{P}$ (as defined in the previous section) should not contradict information in $\mathcal{I}_\mathcal{P}$. However, we are allowed to define values for auxiliary concept names like $D_{V \circ v_i}$.

First, we present several cases for $\otimes$-$\mathcal{L}$ in which the initialization property holds. For the case of the logic $Ł^{(0,b)}$-$\mathfrak{NEL}$, note first that for every $x \in (0, b]$ we have that $x \Rightarrow 0 = b - x$; that is, the residual negation yields a "local involutive negation" over the interval $[0, b]$. Thus, the concept $\boxminus C$ is interpreted as the local involutive negation of the interpretation

of C, whenever the latter is in this interval. In this logic, we use the short-hand $C \rightharpoonup D$ for $\boxminus(C \sqcap \boxminus D)$ to express a function similar to the residuum. In fact, for all $x, y \in [0, 1]$, we have

$$\big(x \otimes (y \Rightarrow 0)\big) \Rightarrow 0 = \begin{cases} y & \text{if } y < b \leq x \\ b - x + y & \text{if } y < x < b \\ 1 & \text{otherwise.} \end{cases}$$

In particular, $(C \rightharpoonup D)^{\mathcal{I}}(x) = (C \rightarrow D)^{\mathcal{I}}(x)$ holds whenever $D^{\mathcal{I}}(x) < b$ for an interpretation $\mathcal{I}$ and $x \in \Delta^{\mathcal{I}}$.

Lemma 15. *For every continuous t-norm $\otimes$, the logics $\otimes\text{-}\mathcal{EL}_{=}$, $\otimes\text{-}\mathcal{ELC}_{\geq}$, and $\text{Ł}^{(0,b)}\text{-}\mathfrak{NEL}$ satisfy* $\mathrm{P_{ini}}$.

It turns out that the successor and concatenation properties hold for all logics $\otimes\text{-}\mathcal{L}$ that we consider. In particular, the successor property only needs the constructors $\top$ and $\exists$ and the restriction to witnessed models, whereas the concatenation property only requires the constructors $\top$ and $\sqcap$.

Lemma 16. *For every continuous t-norm $\otimes$, the logic $\otimes\text{-}\mathcal{EL}$ satisfies* $\mathrm{P_{\rightarrow}}$.

For the concatenation property, it is necessary to have non-idempotent elements. Since we have assumed at the beginning of this section that $\otimes$ is not the Gödel t-norm, this restriction is always satisfied.

Lemma 17. *For every continuous t-norm $\otimes$ except the Gödel t-norm, the logic $\otimes\text{-}\mathcal{EL}$ satisfies* $\mathrm{P_{\circ}}$.

This leaves only one property required for the canonical model property, namely the transfer property. We prove in Appendix A that this can be satisfied using existential restrictions in combination with either value restrictions, involutive negation, or residual negation under Łukasiewicz semantics.

Lemma 18. *For every continuous t-norm $\otimes$ except the Gödel t-norm, the logics $\otimes\text{-}\mathcal{AL}$, $\otimes\text{-}\mathcal{ELC}$, and $\text{Ł}^{(0,b)}\text{-}\mathfrak{NEL}$ satisfy* $\mathrm{P_{\rightsquigarrow}}$.

Together with Theorem 12, the previous lemmata show that the logics $\otimes\text{-}\mathcal{AL}_{=}$, $\otimes\text{-}\mathcal{ELC}_{\geq}$, and $\text{Ł}^{(0,b)}\text{-}\mathfrak{NEL}$ have the canonical model property. We will see that the last two logics also satisfy the solution property, while for $\otimes\text{-}\mathcal{AL}_{=}$ we additionally need the implication constructor.

Recall that a necessary condition for the solution property is that the canonical model $\mathcal{I}_{\mathcal{P}}$ can be extended to a model of the ontology $\mathcal{O}_{\mathcal{P}}$ constructed from the individual parts in Lemmata 15 to 18. It is a simple task to verify that this holds in all the cases described above. We only need to assume that a unique new concept name is used for every auxiliary concept name appearing in the different ontologies, such as $D_{V \circ v_i}$. In fact, the values of these auxiliary concept names at each node v are uniquely determined by the values of the concept names V, W, V_i, W_i, M, M_+ at v. Moreover, since every v has exactly one r_i-successor with degree greater than 0 for every $i \in \mathcal{N}$, it follows that $\mathcal{I}_{\mathcal{P}}$ can be extended to a *witnessed* model of $\mathcal{O}_{\mathcal{P}}$.

Lemma 19. *Let $\otimes$ be any continuous t-norm except the Gödel t-norm. If one of the logics $\otimes\text{-}\mathfrak{IEL}_{=}$, $\otimes\text{-}\mathcal{ELC}_{\geq}$, or $\text{Ł}^{(0,b)}\text{-}\mathfrak{NEL}$ satisfies $\mathrm{P_{\triangle}}$ with $\mathcal{O}_{\mathcal{P}}$ and $\mathcal{I}_{\mathcal{P}}$ can be extended to a model of $\mathcal{O}_{\mathcal{P}}$, then this logic also satisfies* $\mathrm{P_{\neq}}$.

This concludes the first round of undecidability proofs using the framework presented in Section 3.2. Using Theorem 13, we get the following results.

Corollary 20. *For every continuous t-norm $\otimes$ except the Gödel t-norm, ontology consistency is undecidable in the logics $\otimes\text{-}\mathfrak{IAL}_{=}$, $\otimes\text{-}\mathcal{ELC}_{\geq}$, and furthermore it is undecidable for $\text{Ł}^{(0,b)}\text{-}\mathfrak{NEL}$.*

Table 4 summarizes the results and distinguishes between classical ontologies, inequality assertions, and equality assertions on the vertical axis, and different combinations of constructors on the horizontal axis. An entry "$\otimes$" stands for every continuous t-norm except the Gödel t-norm. Note that $\text{Ł-}\mathcal{ELC}$ is as expressive as $\text{Ł-}\mathfrak{NEL} = \text{Ł}^{(0,1)}\text{-}\mathfrak{NEL}$, and thus consistency $\text{Ł-}\mathcal{ELC}$ is also undecidable. This already subsumes the previously known undecidability results for consistency in

- $\Pi\text{-}\mathcal{ALC}_{f,\geq}$ with strict GCIs [16],
- $\Pi^{(0,b)}\text{-}\mathfrak{IAL}_{f,=}$ [18], and
- $\text{Ł-}\mathcal{ELC}_{f,\geq}$ [19].

We have strengthened the first and the last result to cover all fuzzy DLs $\otimes\text{-}\mathcal{ELC}_{\geq}$ with any continuous t-norm except the Gödel t-norm. Moreover, consistency in $\text{Ł-}\mathcal{ELC}$ is undecidable even for classical ontologies. The second result was similarly extended to cover (almost) all continuous t-norms. Interestingly, all logics considered so far are fuzzy extensions of classical $\mathcal{ALC}$, and indeed equivalent to $\mathcal{ALC}$ when restricted to two truth values.

Table 4
The undecidability results of Corollary 20.

	$\mathfrak{N}\mathcal{EL}$	$\mathfrak{I}\mathcal{AL}$	$\mathcal{ELC}$
Classical	Ł$^{(0,b)}$	Ł$^{(0,b)}$	Ł
$\geq$	Ł$^{(0,b)}$	Ł$^{(0,b)}$	$\otimes$
$=$	Ł$^{(0,b)}$	$\otimes$	$\otimes$

We will show in Section 4 that ontology consistency is *decidable* in $\otimes$-$\mathfrak{I}\mathcal{AL}_{\mathsf{f},\geq}$ if $\otimes$ does not start with Łukasiewicz. Furthermore, for the Gödel t-norm, consistency even in G-$\mathfrak{I}\mathcal{ALC}_{\mathsf{f},=}$ is decidable [21]. Together with Corollary 20, this already covers many fuzzy description logics. Only two gaps remain for which the decidability status of consistency is still open.

The first gap concerns the fuzzy DLs above $\otimes$-$\mathfrak{N}\mathcal{EL}_=$, where $\otimes$ does not start with Łukasiewicz. For such t-norms, we show in Section 3.5 that consistency is undecidable for $\otimes$-$\mathfrak{I}\mathcal{EL}_=$. Unfortunately, we must leave open the decidability status of consistency in $\otimes$-$\mathfrak{N}\mathcal{EL}_=$ and $\otimes$-$\mathfrak{N}\mathcal{AL}_=$ if $\otimes$ does not start with Łukasiewicz.

The second gap is about fuzzy DLs $\otimes$-$\mathcal{ELC}$ with involutive negation over classical ontologies. In addition to the Łukasiewicz t-norm, in Section 3.4 we show that consistency is also undecidable for the product t-norm. However, apart from the fundamental t-norms, not much is known about the decidability of consistency in $\otimes$-$\mathcal{ELC}$.

3.4. The case of Π-$\mathcal{ELC}$

To prove that consistency in Π-$\mathcal{ELC}$ is also undecidable, we extend the framework of Section 3.2 by allowing a different version of the PCP to be reduced. In this section, the compared words do not start with v_1/w_1, but with the empty word. More formally, we consider a solution to an instance $\mathcal{P} = \{(v_1, w_1), \ldots, (v_n, w_n)\}$ of the PCP to be a non-empty sequence $\nu = i_1 \ldots i_k \in \{1, \ldots, n\}^+$ for which $v_{i_1} \ldots v_{i_k} = w_{i_1} \ldots w_{i_k}$ holds. Correspondingly, we redefine here the abbreviations $v_\nu := v_{i_1} \ldots v_{i_k}$ and $w_\nu := w_{i_1} \ldots w_{i_k}$. We call the canonical model resulting from these modified definitions $\mathcal{I}_{\mathcal{P}'}$. It can be defined just as in Section 3.2, but the values it holds are now different. This also leads to a modified canonical model property $\mathrm{P}'_\triangle$, which is defined exactly as before, except that $\mathcal{I}_\mathcal{P}$ is replaced by $\mathcal{I}_{\mathcal{P}'}$. Observe that Enc, as defined in Section 3.3 for $\Pi^{(a,b)}$, remains a valid encoding function, and we can use $u_\varepsilon = 1$ and $u_+ = \varepsilon$ as before.

Unfortunately, we cannot show the initialization property, and instead directly construct an ontology for the canonical model property. The full construction is presented in Appendix A (p. 49).

Lemma 21. *The logic* Π-$\mathcal{ELC}$ *satisfies* $\mathrm{P}'_\triangle$.

As before, it is easily verified that $\mathcal{I}_{\mathcal{P}'}$ can be extended to a witnessed model ontology constructed in this proof. In the light of the different version of the PCP we consider here, it is clear that we also need a different solution property. It has to be ensured that V and W encode different words at every node of the search tree *except the root node*, where they both encode ε. We denote by $\mathrm{P}'_{\neq}$ the solution property in which $\mathcal{N}^*$ has been replaced by $\mathcal{N}^+$ to reflect this change. It is easy to see that Theorem 13 also holds under these changes.

Lemma 22. *The logic* Π-$\mathcal{ELC}$ *satisfies* $\mathrm{P}'_{\neq}$.

We thus obtain the following result.

Corollary 23. *Ontology consistency in* Π-$\mathcal{ELC}$ *is undecidable.*

The proofs of undecidability for both Ł-$\mathcal{ELC}$ and Π-$\mathcal{ELC}$ use the fact that one can construct the constant $1/2$ using the axiom $\langle H \equiv \neg H \rangle$. We conjecture that these proofs can be lifted to $\otimes$-$\mathcal{ELC}$, where $\otimes$ is any continuous t-norm for which $1/2$ is not an idempotent element. This condition ensures that $1/2$ lies in a component of norm that uses either the Łukasiewicz or the product t-norm. Starting from this value, one can construct encodings of the words v_i and w_i. However, the encoding has to be adapted since $1/2$ need not lie in the exact center of the component interval.

3.5. The case of $\otimes$-$\mathfrak{I}\mathcal{EL}_=$

We have shown so far that for every continuous t-norm $\otimes$, except the Gödel t-norm, $\otimes$-$\mathfrak{I}\mathcal{EL}_=$ satisfies the properties $\mathrm{P}_{\mathsf{ini}}$, P_o, and $\mathrm{P}_\rightarrow$. By Theorem 12, we need only to show the transfer property to know that these logics satisfy $\mathrm{P}_\triangle$ and, due to Lemma 19, that consistency is undecidable. Rather than showing that $\otimes$-$\mathfrak{I}\mathcal{EL}_=$ satisfies $\mathrm{P}_\rightsquigarrow$, in this section we strengthen Theorem 12 by showing that a weaker property, which we call the *simultaneous transfer property*, together with the other properties, implies the canonical model property. This extends our framework by another method to verify $\mathrm{P}_\triangle$. We then show that for every continuous t-norm except the Gödel t-norm $\otimes$-$\mathfrak{I}\mathcal{EL}$ satisfies the simultaneous transfer property.
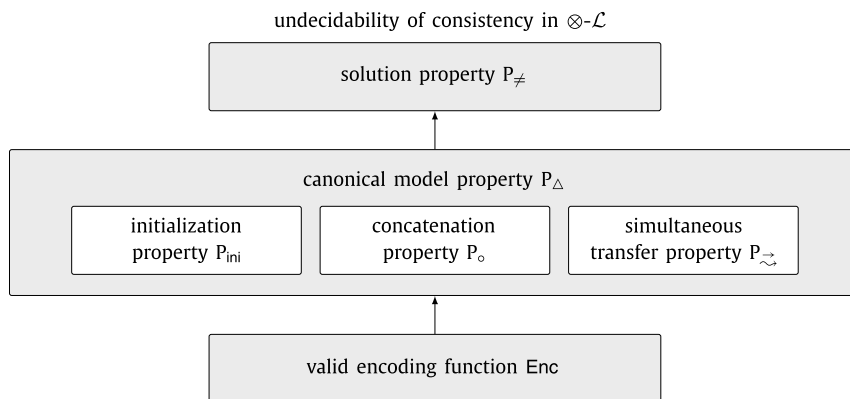
Fig. 4. A new way to prove the canonical model property.

Recall that the transfer property ensures that it is possible to transfer a membership degree from any domain element to all its r-successors. In the reduction from the PCP, this property is used to copy several degrees. It thus makes sense to allow for all these degrees to be transferred simultaneously, as described in the following property.

Simultaneous transfer property ($\mathrm{P}_{\overset{\to}{\leadsto}}$)

> The logic $\otimes$-$\mathcal{L}$ has the *simultaneous transfer property* if for every finite set $\{(C_1, D_1), \ldots, (C_k, D_k)\}$ of pairs of concept names there is an ontology $\mathcal{O}_{(C_j)\leadsto(D_j)}$ such that for every model $\mathcal{I}$ of $\mathcal{O}_{(C_j)\leadsto(D_j)}$ and every $x \in \Delta^{\mathcal{I}}$, if for every $j, 1 \leq j \leq m$, there is a $u_j \in \Sigma_0^*$ such that $C_j^{\mathcal{I}}(x) \in \mathsf{Enc}(u_j)$ and $u_1 \notin \{0\}^*$, then there exists a $y \in \Delta^{\mathcal{I}}$ such that for all $j, 1 \leq j \leq m$, it holds that $D_j^{\mathcal{I}}(y) \in \mathsf{Enc}(u_j)$.

Given an instance $\mathcal{P}$ of the PCP with words $(v_1, w_1), \ldots, (v_n, w_n)$, we can assume w.l.o.g. that $v_1 \neq \varepsilon$, and thus $v_1 \notin \{0\}^*$. Then, we can choose for every $i, 1 \leq i \leq n$, the set

$$\big\{(V_1, V_1), \ldots, (V_n, V_n), (W_1, W_1), \ldots, (W_n, W_n), (M_+, M_+), (D_{M \circ u_+}, M), (D_{V \circ v_i}, V), (D_{W \circ w_i}, W)\big\},$$

which ensures the existence of the i-th successor of every node; i.e., that the concatenation with the pair (v_i, w_i) is considered. The last three pairs are used to transfer the computed concatenations to the i-th successors, while the remaining pairs ensure that all constants are available for the next round of concatenations. We then define the ontology $\mathcal{O}_{\mathcal{P}, \overset{\to}{\leadsto}}$ as the union of the resulting ontologies $\mathcal{O}_{(C_j^{(i)})\leadsto(D_j^{(i)})}$ to transfer all the needed values to the correct successors.

It is easy to see that any logic that satisfies $\mathrm{P}_\to$ and $\mathrm{P}_\leadsto$ must also satisfy $\mathrm{P}_{\overset{\to}{\leadsto}}$. Indeed, $\mathrm{P}_\to$ ensures that there is an r-successor with degree 1, and $\mathrm{P}_\leadsto$ states that each $C_j^{\mathcal{I}}(x)$ can be copied to $D_j^{\mathcal{I}}(y)$ if $r^{\mathcal{I}}(x, y) = 1$. Moreover, the ontology $\mathcal{O}_{\mathcal{P}} := \mathcal{O}_{\mathcal{P}, \overset{\to}{\leadsto}} \cup \mathcal{O}_{\mathcal{P}, \mathsf{ini}} \cup \mathcal{O}_{\mathcal{P}, \circ}$ satisfies the conditions in the definition of $\mathrm{P}_\triangle$.

Theorem 24. *If a logic $\otimes$-$\mathcal{L}$ satisfies the properties $\mathrm{P}_{\mathsf{ini}}$, $\mathrm{P}_\circ$, and $\mathrm{P}_{\overset{\to}{\leadsto}}$, then it also satisfies $\mathrm{P}_\triangle$.*

Proof. The function g for a model $\mathcal{I}$ of $\mathcal{O}_{\mathcal{P}}$ can be constructed as in the proof of Theorem 12, with the exception that we define as $g(vi)$ that element $y \in \Delta^{\mathcal{I}}$ whose existence is guaranteed by $\mathcal{O}_{\mathcal{P}, \overset{\to}{\leadsto}}$ when we consider $x = g(v)$. □

Fig. 4 depicts the alternative way of showing undecidability using the simultaneous transfer property instead of the successor and transfer properties (cf. Fig. 3).

Lemma 25. *For every continuous t-norm except the Gödel t-norm, the logic $\otimes$-$\mathfrak{IEL}$ satisfies $\mathrm{P}_{\overset{\to}{\leadsto}}$.*

Together with Theorem 24, this implies that $\otimes$-$\mathfrak{IEL}_=$ satisfies the canonical model property whenever $\otimes$ is not the Gödel t-norm.

It is also easy to see that $\mathcal{I}_{\mathcal{P}}$ can be extended to a model of the ontology $\mathcal{O}_{\mathcal{P}}$ constructed from the ontologies provided by the initialization, concatenation, and simultaneous transfer properties: as before, the values of the auxiliary variables are uniquely determined by the values of the concept names V, W, V_i, W_i at each node v. By Lemma 19, we know that $\otimes$-$\mathfrak{IEL}_=$ satisfies the solution property, which yields the final undecidability result of this paper.

Corollary 26. *For any continuous t-norm $\otimes$ except the Gödel t-norm, ontology consistency in $\otimes$-$\mathcal{IEL}_=$ is undecidable.*

4. Decidable fuzzy DLs

It remains to show the decidability results claimed in the previous section. We will prove that ontology consistency is decidable in $\otimes$-$\mathcal{IAL}_{f,\geq}$ if $\otimes$ has no zero divisors, i.e. it does not start with Łukasiewicz. We show this by means of a straightforward reduction of fuzzy to classical ontologies. We present this reduction for a much more expressive description logic, namely $\otimes$-$\mathcal{SROIQ}_{f,\geq}^{+}$, which extends $\otimes$-$\mathcal{IAL}_{f,\geq}$ by several concept and role constructors and complex role inclusion axioms.

Note that even in the classical case the syntax of $\mathcal{SROIQ}^{+}$ we introduce below leads to undecidability of consistency. Decidability is regained for $\mathcal{SROIQ}$, which imposes several restrictions on the form of number restrictions and complex role inclusions [29,30]. However, our reduction also works for the more expressive logic and is easier to present without the restrictions of $\mathcal{SROIQ}$.

4.1. Fuzzy $\mathcal{SROIQ}^{+}$

Formally, in addition to the syntax of $\otimes$-$\mathcal{IAL}_{f,\geq}$, the fuzzy DL $\otimes$-$\mathcal{SROIQ}_{f,\geq}^{+}$ allows the role constructors u (*universal role*), s^{-} (*inverse*), and $\boxminus s$ (*residual negation*) to build *complex roles s* from role names. An interpretation $\mathcal{I}$ is extended to complex roles as follows for all $x, y \in \Delta^{\mathcal{I}}$:

- $u^{\mathcal{I}}(x, y) = 1$,
- $(s^{-})^{\mathcal{I}}(x, y) = s^{\mathcal{I}}(y, x)$,
- $(\boxminus s)^{\mathcal{I}}(x, y) = \ominus s^{\mathcal{I}}(x, y)$.

We also introduce the new concept constructors $\exists s.\mathsf{Self} f$, $\{d\}$ (*nominal*), $C \sqcup D$ (*disjunction*), $\geq n\, s.C$ (*at-least restriction*), and $\leq n\, s.C$ (*at-most restriction*) for $d \in \mathsf{N_I}$, a complex role s, and $n \in \mathbb{N}$. Moreover, complex roles are allowed in existential and value restrictions and role assertions. The new constructors are interpreted by an interpretation $\mathcal{I}$ as follows for all $x \in \Delta^{\mathcal{I}}$:

- $(\exists s.\mathsf{Self})^{\mathcal{I}}(x) = s^{\mathcal{I}}(x, x)$,
- $\{d\}^{\mathcal{I}}(x) = 1$ if $d^{\mathcal{I}} = x$ and 0 otherwise,
- $(C \sqcup D)^{\mathcal{I}}(x) = C^{\mathcal{I}}(x) \oplus D^{\mathcal{I}}(x)$,
- $(\geq n\, s.C)^{\mathcal{I}}(x) = \sup\{p \in [0, 1] \mid |\{y \in \Delta^{\mathcal{I}} \mid s^{\mathcal{I}}(x, y) \otimes C^{\mathcal{I}}(y) \geq p\}| \geq n\}$,
- $(\leq n\, s.C)^{\mathcal{I}}(x) = \sup\{p \in [0, 1] \mid |\{y \in \Delta^{\mathcal{I}} \mid s^{\mathcal{I}}(x, y) \Rightarrow \ominus C^{\mathcal{I}}(y) < p\}| \leq n\}$.

The last two expressions are equivalent to the semantics of number restrictions used in [31]. Notice that whenever $\otimes$ has no zero divisors, the residual negation $\ominus$ is the Gödel negation (see Lemma 4) and for every interpretation $\mathcal{I}$ and $x \in \Delta^{\mathcal{I}}$, it holds that

$$(\leq n\, s.C)^{\mathcal{I}}(x) = \big(\boxminus\big(\geq (n+1)\, s.C\big)\big)^{\mathcal{I}}(x) = \begin{cases} 1 & \text{if } |\{y \in \Delta^{\mathcal{I}} \mid s^{\mathcal{I}}(x, y) \otimes C^{\mathcal{I}}(y) > 0\}| \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

This means that under such a t-norm *at-most* restrictions are always crisp.

We additionally allow *complex role inclusions* to occur in ontologies, which are axioms of the form $\langle s_1 \circ \cdots \circ s_n \sqsubseteq t \geq p \rangle$, where $n \in \mathbb{N}$, $s_1, \ldots, s_n, t$ are complex roles, and $p \in [0, 1]$. An interpretation $\mathcal{I}$ *satisfies* this role inclusion if

$$\big(s_1^{\mathcal{I}}(x_0, x_1) \otimes \cdots \otimes s_n^{\mathcal{I}}(x_{n-1}, x_n)\big) \Rightarrow t^{\mathcal{I}}(x_0, x_n) \geq p$$

for all $x_0, \ldots, x_n \in \Delta^{\mathcal{I}}$. If $n = 0$, then we write $\langle \mathsf{id} \sqsubseteq t \geq p \rangle$ and the semantics simplifies to $t^{\mathcal{I}}(x_0, x_0) \geq p$ for all $x_0 \in \Delta^{\mathcal{I}}$.

Those axioms of classical $\mathcal{SROIQ}$ that are not included in $\mathcal{SROIQ}^{+}$ can be simulated as follows:

- negated role assertions: $\langle (d, e) : \boxminus s \rangle$;
- inequality assertions between individual names: $\langle \{d\} \sqsubseteq \boxminus\{e\} \rangle$;
- transitivity: $\langle s \circ s \sqsubseteq s \rangle$;
- symmetry: $\langle s \sqsubseteq s^{-} \rangle$;
- asymmetry: $\langle s \sqsubseteq \boxminus s^{-} \rangle$;
- reflexivity: $\langle \mathsf{id} \sqsubseteq s \rangle$;
- irreflexivity: $\langle \mathsf{id} \sqsubseteq \boxminus s \rangle$;
- role disjointness: $\langle s \sqsubseteq \boxminus t \rangle$.

4.2. The crisp model property

The undecidability results of Section 3 all rely heavily on the fact that one can design ontologies that allow only models with infinitely many truth values. We shall see that one cannot construct such an ontology in $\otimes\text{-}\mathcal{SROIQ}^+_{f,\geq}$ if $\otimes$ has no zero divisors. It is even true that all consistent $\otimes\text{-}\mathcal{SROIQ}^+_{f,\geq}$-ontologies have a crisp model; that is, a model using only the values 0 and 1.

Definition 27. A fuzzy DL $\otimes\text{-}\mathcal{L}$ has the *crisp model property* if every ontology that is consistent in $\otimes\text{-}\mathcal{L}$ has a crisp model.

As mentioned in the beginning of this section, we now consider only continuous t-norms without zero divisors. Our main result is based on the function $\mathbb{1}$ that maps fuzzy truth values to crisp truth values by defining, for all $x \in [0,1]$,

$$\mathbb{1}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Since $\otimes$ is a continuous t-norm without zero divisors, it follows from Lemma 4 that $\mathbb{1}(x) = \ominus \ominus x$ for all $x \in [0,1]$. This function is compatible with the residual negation, the t-norm, the corresponding t-conorm, implication, and suprema. It is also compatible with minima, provided that they exist. The proofs of the following lemmata can be found in Appendix B.

Lemma 28. *Let $\otimes$ be a continuous t-norm without zero divisors. For all $x, y \in [0,1]$ and all non-empty sets $X \subseteq [0,1]$ it holds that*

1. $\mathbb{1}(\ominus x) = \ominus\mathbb{1}(x)$,
2. $\mathbb{1}(x \otimes y) = \mathbb{1}(x) \otimes \mathbb{1}(y)$,
3. $\mathbb{1}(x \oplus y) = \mathbb{1}(x) \oplus \mathbb{1}(y)$,
4. $\mathbb{1}(x \Rightarrow y) = \mathbb{1}(x) \Rightarrow \mathbb{1}(y)$,
5. $\mathbb{1}(\sup\{x \mid x \in X\}) = \sup\{\mathbb{1}(x) \mid x \in X\}$, *and*
6. *if $\min\{x \mid x \in X\}$ exists, then $\mathbb{1}(\min\{x \mid x \in X\}) = \min\{\mathbb{1}(x) \mid x \in X\}$.*

Notice that in general the function $\mathbb{1}$ is not compatible with the infimum. Consider for example the set $X = \{\frac{1}{n} \mid n \in \mathbb{N}\}$. Then $\inf X = 0$ and hence $\mathbb{1}(\inf X) = 0$, but $\inf\{\mathbb{1}(\frac{1}{n}) \mid n \in \mathbb{N}\} = \inf\{1\} = 1$. However, under witnessed model semantics all infima needed to interpret universal restrictions are minima, which eliminates this problem.

We use Lemma 28 to construct a crisp interpretation from a fuzzy interpretation by simply applying the function $\mathbb{1}$. Given a (witnessed) interpretation $\mathcal{I}$, we construct the interpretation $\mathcal{J}$ over the domain $\Delta^{\mathcal{J}} := \Delta^{\mathcal{I}}$ by defining, for all concept names $A \in \mathsf{N_C}$, role names $r \in \mathsf{N_R}$, individual names $d \in \mathsf{N_I}$, and $x, y \in \Delta^{\mathcal{I}}$,

$$A^{\mathcal{J}}(x) := \mathbb{1}\big(A^{\mathcal{I}}(x)\big), \quad r^{\mathcal{J}}(x, y) := \mathbb{1}\big(r^{\mathcal{I}}(x, y)\big), \quad \text{and} \quad d^{\mathcal{J}} := d^{\mathcal{I}}.$$

We now show that $\mathcal{J}$ preserves the compatibility shown in Lemma 28 w.r.t. complex roles and concepts.

Lemma 29. *For all complex concepts C, complex roles s, and $x, y \in \Delta^{\mathcal{I}}$, it holds that $C^{\mathcal{J}}(x) = \mathbb{1}(C^{\mathcal{I}}(x))$ and $s^{\mathcal{J}}(x, y) = \mathbb{1}(s^{\mathcal{I}}(x, y))$.*

With the help of this lemma, we can show that the crisp interpretation $\mathcal{J}$ satisfies all the axioms that are satisfied by $\mathcal{I}$.

Lemma 30. *If $\mathcal{I}$ is a witnessed model of an ontology $\mathcal{O}$ in $\otimes\text{-}\mathcal{SROIQ}^+_{f,\geq}$, then $\mathcal{J}$ is also a witnessed model of $\mathcal{O}$.*

Proof. Observe first that axioms with value $p = 0$ are trivially satisfied by $\mathcal{J}$. Let now $\langle e{:}C \geq p \rangle$ be a concept assertion in $\mathcal{O}$ with $p \in (0,1]$. Since it is satisfied by $\mathcal{I}$, we have $C^{\mathcal{I}}(e^{\mathcal{I}}) \geq p > 0$. Lemma 29 yields $C^{\mathcal{J}}(e^{\mathcal{J}}) = 1 \geq p$. The same argument can be used for role assertions.

Let now $\langle C \sqsubseteq D \geq p \rangle$ be a GCI in $\mathcal{O}$ with $p \in (0,1]$ and consider any $x \in \Delta^{\mathcal{I}}$. As the GCI is satisfied by $\mathcal{I}$, we have $C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x) \geq p > 0$. By Lemmata 28 and 29, we obtain

$$C^{\mathcal{J}}(x) \Rightarrow D^{\mathcal{J}}(x) = \mathbb{1}\big(C^{\mathcal{I}}(x)\big) \Rightarrow \mathbb{1}\big(D^{\mathcal{I}}(x)\big) = \mathbb{1}\big(C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x)\big) = 1 \geq p,$$

and thus $\mathcal{J}$ satisfies the GCI.

A similar argument shows that $\mathcal{J}$ satisfies all complex role inclusions in $\mathcal{O}$. $\quad\square$

Thus, by applying $\mathbb{1}$ to the truth degrees we obtain a crisp model $\mathcal{J}$ from any fuzzy model $\mathcal{I}$ of a $\otimes\text{-}\mathcal{SROIQ}^+_{f,\geq}$-ontology $\mathcal{O}$.

Theorem 31. *If $\otimes$ is a continuous t-norm without zero divisors, then the logic $\otimes\text{-}\mathcal{SROIQ}^+_{f,\geq}$ has the crisp model property.*

In the next section, we use this result to show that ontology consistency in sublogics of $\otimes\text{-}\mathcal{SROIQ}^+_{f,\geq}$ can be decided using classical reasoning algorithms.

4.3. Consistency

For a given ontology $\mathcal{O}$ in $\otimes\text{-}\mathcal{SROIQ}^+_{f,\geq}$, we define crisp($\mathcal{O}$) to be the classical $\mathcal{SROIQ}^+$-ontology that is obtained from $\mathcal{O}$ by replacing all the non-zero truth values appearing in the axioms by 1. Axioms with value 0 can be removed without affecting the semantics of $\mathcal{O}$. For example, for the ontology

$$\mathcal{O} = \big\{ \langle a : C \geq 0.2 \rangle, \langle (a, b) : \boxminus r \geq 0.8 \rangle, \langle C \sqsubseteq D \geq 0.5 \rangle, \langle r \circ s \sqsubseteq s \geq 0.1 \rangle \big\}$$

we obtain

$$\mathsf{crisp}(\mathcal{O}) = \big\{ \langle a : C \rangle, \langle (a, b) : \boxminus r \rangle, \langle C \sqsubseteq D \rangle, \langle r \circ s \sqsubseteq s \rangle \big\}.$$

Lemma 32. *Let $\otimes$ be a continuous t-norm without zero divisors, $\mathcal{O}$ be a $\otimes\text{-}\mathcal{SROIQ}^+_{f,\geq}$-ontology and $\mathcal{I}$ be a crisp interpretation. Then $\mathcal{I}$ is a model of $\mathcal{O}$ iff it is a model of* crisp($\mathcal{O}$).

Proof. In the proof of both directions we can ignore axioms with truth value 0. Assume that crisp($\mathcal{O}$) has a model $\mathcal{I}$ and let $\langle C \sqsubseteq D \geq p \rangle$ be an axiom from $\mathcal{O}$ with $p \in (0, 1]$. Since $\mathcal{I}$ is a model of crisp($\mathcal{O}$), it must satisfy $\langle C \sqsubseteq D \rangle$; that is, $C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x) \geq 1 \geq p$ holds for all $x \in \Delta^{\mathcal{I}}$. Thus, $\mathcal{I}$ satisfies $\langle C \sqsubseteq D \geq p \rangle$. The proof that $\mathcal{I}$ satisfies assertions and complex role inclusions is analogous. Hence $\mathcal{I}$ is also a model of $\mathcal{O}$.

For the other direction, assume that $\mathcal{I}$ satisfies a GCI $\langle C \sqsubseteq D \geq p \rangle$ from $\mathcal{O}$ with $p \in (0, 1]$. As $\mathcal{I}$ is a crisp interpretation, we have $C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x) \in \{0, 1\}$ for all $x \in \Delta^{\mathcal{I}}$. Together with $C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x) \geq p > 0$, this implies that $C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x) = 1$, and thus $\mathcal{I}$ satisfies the crisp GCI $\langle C \sqsubseteq D \rangle$. The same argument can be used for complex role inclusions and assertions. Thus $\mathcal{I}$ is also a model of crisp($\mathcal{O}$). □

In particular, a $\otimes\text{-}\mathcal{SROIQ}^+_{f,\geq}$-ontology $\mathcal{O}$ has a crisp model iff crisp($\mathcal{O}$) has a crisp model. Together with Theorem 31, this shows that a $\otimes\text{-}\mathcal{SROIQ}^+_{f,\geq}$-ontology $\mathcal{O}$ is consistent iff crisp($\mathcal{O}$) has a crisp model. The latter is a classical reasoning problem.

Consistency in classical $\mathcal{SROIQ}^+$ is undecidable in general, as the number restrictions and role axioms we introduced are too powerful. However, one can use reasoning algorithms for any sublogic of $\mathcal{SROIQ}^+$, for example $\mathcal{SROIQ}$ or $\mathcal{SHOI}$, to decide consistency of ontologies in the corresponding fuzzy DL over a t-norm without zero divisors. For example, reasoning with GCIs in crisp $\mathcal{SHOI}$ is known to be ExpTime-complete [32], while in $\mathcal{SROIQ}$ it is 2-NExpTime-complete [30].

Corollary 33. *If $\otimes$ is a continuous t-norm without zero divisors, then the complexity of deciding consistency in any sublogic of $\otimes\text{-}\mathcal{SROIQ}^+_{f,\geq}$ is the same as in the underlying classical description logic.*

This result is different from previous work on reducing reasoning in finite-valued fuzzy DLs to classical reasoning [6, 31,33,34]. There, the authors simulate fuzzy concepts by using linearly many *cut-concepts and -roles* of the form A_p and r_p for $A \in \mathsf{N_C}$, $r \in \mathsf{N_R}$, and $p \in [0, 1]$. They then recursively translate the fuzzy ontology into a classical one, which may be exponentially larger. In contrast, our reduction shows that infinite-valued fuzzy DLs of the form $\otimes\text{-}\mathcal{SROIQ}^+_{f,\geq}$ for a t-norm $\otimes$ without zero divisors are too weak to support actual fuzzy consistency reasoning—one can simply remove all fuzzy values from the input ontology. However, this is not true for other reasoning problems supported by the algorithms in [6,31,33,34] (see Section 5.3).

Theorem 31 and Lemma 32 still hold when we restrict the semantics to the less expressive logics $\otimes\text{-}\mathcal{SHO}_{f,\geq}$, or $\otimes\text{-}\mathcal{SI}_{f,\geq}$. The crisp DLs $\mathcal{SHO}$ and $\mathcal{SI}$ are known to have the finite model property [35,36], and $\otimes\text{-}\mathcal{SHO}_{f,\geq}$ and $\otimes\text{-}\mathcal{SI}_{f,\geq}$ inherit the finite model property from their crisp ancestors.

Theorem 34. *If $\otimes$ is a continuous t-norm without zero divisors, then the logics $\otimes\text{-}\mathcal{SHO}_{f,\geq}$ and $\otimes\text{-}\mathcal{SI}_{f,\geq}$ and their sublogics have the finite model property.*

At this point, we want to correct a typing error in [15, Theorem 3.8]. There, it appears to be shown that $\Pi\text{-}\mathcal{AL}_{\geq}$ does *not* have the finite model property, contradicting the above result. However, the assertion $\langle a : A \geq 0.5 \rangle$ used in the proof must in fact be $\langle a : A = 0.5 \rangle$ in order for the arguments to work.

5. Discussion and related work

Table 5 summarizes the results obtained in Sections 3 and 4. As in Table 4, columns describe the class of logical constructors allowed in the logic, while the rows denote the types of assertions allowed: classical, inequality assertions, or equality assertions. The content of a cell then shows the class of continuous t-norms for which consistency has been shown to be undecidable, where $\otimes$ stands for any non-idempotent t-norm; that is, any t-norm except the Gödel t-norm. Thus,

Table 5
Undecidability of consistency in fuzzy description logics.

	$\mathfrak{NEL}$	$\mathfrak{NAL}$	$\mathfrak{IEL}$	$\mathcal{SROIQ}$	$\mathcal{ELC}$	$\mathfrak{IALC}$
Classical	Ł$^{(0,b)}$	Ł$^{(0,b)}$	Ł$^{(0,b)}$	Ł$^{(0,b)}$	Π, Ł	Π, Ł$^{(0,b)}$
$\geq$	Ł$^{(0,b)}$	Ł$^{(0,b)}$	Ł$^{(0,b)}$	Ł$^{(0,b)}$	$\otimes$	$\otimes$
$=$	Ł$^{(0,b)}$	Ł$^{(0,b)}$	$\otimes$	$\otimes$	$\otimes$	$\otimes$

for instance, the upper–left cell states that Ł$^{(0,b)}$-$\mathfrak{NEL}$ is undecidable. Cells with gray background mark logics for which decidability of consistency has been fully characterized, either between t-norms with or without zero divisors, or between the Gödel t-norm and all other t-norms. For the other logics, only the stated undecidability results are known. Regarding the Gödel t-norm, one could think that papers like [6,31] show decidability even for the very expressive fuzzy DL G-$\mathcal{SROIQ}_{f,=}$. However, these papers explicitly restrict reasoning to a finite set of truth values. Thus, decidability of consistency for the Gödel t-norm is only known up to G-$\mathfrak{IALC}_{f,=}$ [21]. However, we strongly believe also G-$\mathcal{SROIQ}_{f,=}$ to have a decidable consistency problem. Notice that we have shown undecidability using only classical ontologies, while the decidability results hold also in the presence of fuzzy GCIs ($_f$). Thus, the results depicted in Table 5 are independent of whether we use crisp or fuzzy GCIs.

In the rest of this section, we present a short survey on other kinds of reasoning problems in fuzzy DLs. We start by describing the known results regarding reasoning w.r.t. general models, i.e., removing the restriction to witnessed models. Afterwards, we briefly discuss the other standard reasoning tasks in fuzzy DLs, namely satisfiability, subsumption, and instance checking.

5.1. General models

The general framework for proving undecidability presented in Section 3 is independent of the class of models used for reasoning, and hence applies also for reasoning w.r.t. general models. However, when instantiating the general framework to specific fuzzy DLs, we have used the properties of witnessed models to prove the successor property. Indeed, in the proof of Lemma 16, we use that if $(\exists r.\top)^{\mathcal{I}}(x) = 1$ for some model $\mathcal{I}$ and $x \in \Delta^{\mathcal{I}}$, then there must exist a $y \in \Delta^{\mathcal{I}}$ such that $r^{\mathcal{I}}(x, y) = 1$, which cannot be guaranteed for general models. If this assumption is dropped, it is possible to modify the fuzzy DL in question to show (a variant of) the successor property.

First, one can see from the proof of Lemma 16 that it is only necessary to witness existential restrictions of the form $\exists r.\top$. In [22], interpretations that fulfill the witnessing condition only for this kind of existential restrictions are called $\top$-*witnessed*. Restricting the semantics to $\top$-witnessed interpretations thus also leads to undecidability in the logics of Corollaries 20 and 23. A second possibility is to allow axioms of the form crisp(r), asserting that the role name r can take only the values 0 or 1. Indeed, in such a logic one can use the two axioms crisp(r) and $\exists r.\top$ to show the successor property. The corresponding results have also been shown in [22].

Consider now a continuous t-norm $\otimes$ that has a component $((a, b), Ł)$ or $((a, b), Π)$ with $b < 1$. Using this component for the encoding function, it is possible to relax the successor property to only create an r-successor with value $\geq b$ rather than $= 1$, similar to the construction used in Section 3.5. Indeed, if we modify the successor and the transfer properties to say $r^{\mathcal{I}}(x, y) \geq b$ instead of $r^{\mathcal{I}}(x, y) = 1$, we can show a result analogous to Theorem 12 to obtain the canonical model property. For the successor property, we can still use the axiom $\exists r.\top$ since for every model $\mathcal{I}$ of this axiom and every $x \in \Delta^{\mathcal{I}}$ there must be at least one $y \in \Delta^{\mathcal{I}}$ such that $r^{\mathcal{I}}(x, y) \geq b$, as the supremum of all these values is 1 (recall the semantics of $\exists$ from Definition 8). For the transfer property, it is often irrelevant whether $r^{\mathcal{I}}(x, y)$ has the value 1 or any value $\geq b$ since both behave in the same way when combined with encodings in the interval $[a, b]$ using $\otimes$ and $\Rightarrow$. Unfortunately, under the mentioned modification we cannot show the transfer property for $\otimes$-$\mathcal{ELC}$ anymore. The reason is that we cannot guarantee that the involutive negation remains in the interval $[a, b]$, and hence $r^{\mathcal{I}}(x, y)$ need not behave as a neutral element when computing its t-norm with $(\neg D)^{\mathcal{I}}(y)$ (see the proof of Lemma 18).

For the special case of Ł-$\mathfrak{NEL}$, it is also possible to show undecidability of consistency w.r.t. general models [37], but this requires greater modifications. The main idea is not to encode the words $u \in \Sigma^+$ by single values, but to allow a certain error bound in the encoding. Thus, the encoding Enc(u) of each $u \in \Sigma^+$ is a subinterval of $[0, 1]$. To obtain a valid encoding function, one has to ensure that intervals encoding different words do not overlap. Since the axiom $\exists r.\top$ can only ensure, for each $p < 1$, the existence of an r-successor with value greater than or equal to p, the transfer of values using the axioms from the proof of Lemma 18 might incur some additional error. By always choosing p large enough, one can ensure that the resulting value stays inside the prescribed error bounds. A final difference is that for the solution property, instead of a single value M one has to use *two* values that keep track of the length of the words v_ν and w_ν encoded in V and W, respectively. For further details, see [37].

For the special case of $\otimes$-$\mathfrak{IEL}_=$ treated in Section 3.5, the restriction to witnessed models is fundamental in the proofs of the simultaneous transfer property (Lemmata 36 and 37). In this case, undecidability can still be shown if roles can be restricted to be crisp; unfortunately, the same is not true if we restrict only to $\top$-witnessed models, or to t-norms containing a component $((a, b), Ł)$ or $((a, b), Π)$ with $b < 1$.

For the decidability results from Section 4, the restriction to witnessed models was only used in the proof of Lemma 29 to show that the semantics of value restrictions $\forall s.C$ is compatible with the function $\mathbb{1}$. From this it is easy to see that,

under general model semantics, the result of Corollary 33 still holds for all sublogics of $\otimes\text{-}\mathcal{SROIQ}^+_{f,\geq}$ that do not have the $\forall$ constructor [38].

Furthermore, the decidability result for G-$\mathfrak{I}\mathcal{ALC}_{f,=}$ also holds under general model semantics [39].

5.2. Satisfiability and local consistency

Given an ontology $\mathcal{O}$ without assertions, a concept C, and a degree $p > 0$, we say that C is *p-satisfiable* w.r.t. $\mathcal{O}$ if there is a model $\mathcal{I}$ of $\mathcal{O}$ where $C^{\mathcal{I}}(x) \geq p$ holds for some $x \in \Delta^{\mathcal{I}}$. The undecidability results for consistency of *classical* ontologies (the first row of Table 5) immediately carry over to this problem since the reduction uses only one individual name and consistency of a finite set $\mathcal{O}$ of GCIs together with the crisp assertions $\langle e_0 : C_1 \rangle, \ldots, \langle e_0 : C_n \rangle$ is equivalent to the 1-satisfiability of the concept $C_1 \sqcap \cdots \sqcap C_n$ w.r.t. $\mathcal{O}$.

For the other undecidability results of Section 3, such an adaptation is not so straightforward. The reason is that several concept names need to be initialized to different values. However, this shows undecidability of *local consistency* [40,41], which is a decision problem between concept satisfiability and ontology consistency that asks for a model with an individual that has different degrees for several concepts at the same time. In the presence of fuzzy GCIs, at least the consistency of *inequality* assertions $\langle e_0 : C_1 \geq p_1 \rangle, \ldots, \langle e_0 : C_n \geq p_n \rangle$ can be reduced to the 1-satisfiability of a new concept name A w.r.t. the original GCIs and the axioms $\langle A \sqsubseteq C_1 \geq p_1 \rangle, \ldots, \langle A \sqsubseteq C_n \geq p_n \rangle$. This shows that all undecidability results in the first row of Table 5 also apply to concept satisfiability w.r.t. classical ontologies. Similarly, the second row can be used to determine the decidability of concept satisfiability in the presence of fuzzy GCIs.

On the other hand, the decidability results of Section 4 also hold for concept satisfiability since C is *p*-satisfiable w.r.t. $\mathcal{O}$ iff $\mathcal{O} \cup \{\langle a : C \geq p \rangle\}$ is consistent, where a is a fresh individual name. Furthermore, it follows from the construction of Section 4 that the *best satisfiability degree* of a concept C, i.e. the supremum over all p for which C is *p*-satisfiable, is always either 0 or 1 (for details, see [23]).

Also note that in the smaller logics $\otimes\text{-}\mathcal{EL}$ and $\otimes\text{-}\mathcal{AL}$, both consistency and satisfiability are trivial problems since all ontologies written in these logics are consistent.

5.3. Subsumption and instance checking

Little is known about subsumption, another fundamental reasoning problem for fuzzy DLs, in the presence of GCIs. Formally, for $p \in [0, 1]$, a concept C is *p-subsumed* by a concept D w.r.t. an ontology $\mathcal{O}$ if the fuzzy GCI $\langle C \sqsubseteq D \geq p \rangle$ is satisfied in every model of $\mathcal{O}$. A related problem is to find the *best subsumption degree* of C and D w.r.t. $\mathcal{O}$, i.e. the supremum over all p for which C is *p*-subsumed by D w.r.t. $\mathcal{O}$.

Even though $\otimes\text{-}\mathcal{SROIQ}^+_{f,\geq}$ has the crisp model property, *p*-subsumption in this logic cannot be decided using only crisp models. In fact, the GCI $\langle \top \sqsubseteq A \geq p \rangle$ for $p \in (0, 1)$ forces the best subsumption degree of $\top$ and A to be p, whereas $\top$ is even 1-subsumed by A when only crisp models are considered. Thus, in every fuzzy DL of the form $\otimes\text{-}\mathcal{L}_f$, *p*-subsumption cannot be decided using only crisp models. A similar example is used in [23] to show the same for all fuzzy DLs $\otimes\text{-}\mathcal{L}$ where $\otimes$ has no zero divisors and $\mathcal{L}$ contains the residual negation. If $\otimes$ is the product t-norm, then *p*-subsumption cannot even be decided over the class of all finite models [23].

The listed negative results also hold for the related problem of deciding *p-instances*, i.e. whether an assertion $\langle a : C \geq p \rangle$ holds in every model of a given ontology [23].

On the positive side, *p*-subsumption in G-$\mathcal{EL}$ can be decided in polynomial time in the size of the input ontology [5]. However, *p*-subsumption is co-NP-hard in $\otimes\text{-}\mathcal{EL}$ whenever $\otimes$ contains the Łukasiewicz t-norm [42].

5.4. Related work

Fuzzy description logics were first considered in [43], where a sublanguage of $\mathcal{ALC}$ was fuzzified using the so-called *Zadeh* semantics. This approach has its origin in fuzzy set theory [24] and uses the Gödel t-norm and t-conorm, but the *S-implication* $\sim x \oplus y$ rather than the residuum $x \Rightarrow y$. In fact, the first algorithms for deciding consistency and entailment in fuzzy variants of $\mathcal{ALC}$ were based on the Zadeh semantics [9,44,45]. Later, it was discovered that reasoning in this logic can be restricted to the finitely many values occurring in the input ontology (and their negations). Based on this idea, a reduction to reasoning in classical DLs was presented [7].

As later noticed by Hájek [28], all the previously developed tableau algorithms implicitly restricted reasoning to witnessed models, without making this assumption explicit. In that paper, Hájek also introduced general t-norm based fuzzy semantics for fuzzy $\mathcal{ALC}$ and proved that 1-satisfiability and 1-subsumption under these semantics are decidable if the background ontology is empty. For the Łukasiewicz t-norm, this is true even without the restriction to witnessed models since in this case the two semantics coincide. In [46], it is proved that 1-subsumption is also decidable in $\Pi\text{-}\mathcal{ALC}$ without a background ontology. If one restricts reasoning to so-called *quasi-witnessed* models, satisfiability is also decidable in this setting. In [47], the framework from [28] is extended to include ontologies, and axiomatizations of t-norm based fuzzy DLs are investigated.

After the introduction of t-norm based semantics for fuzzy DLs, several tableau algorithms were developed to decide consistency and subsumption in these new logics. The main idea is that the tableau rules generate a system of constraints

that has to be solved at the end. The variables in these constraints are either binary variables or range over $[0, 1]$. The constraints themselves are either linear or quadratic, depending on the t-norm used. Such tableau algorithms are presented in [48] for an extension of Π-$\mathcal{ALC}_{f,=}$, and in [49] for an extension of Ł-$\mathcal{ALC}_{f,=}$. In [50], the latter algorithm is extended to deal with qualified cardinality restrictions (denoted by $\mathcal{Q}$). In [51], a tableau algorithm for $\otimes$-$\mathcal{SI}_=$ is developed for arbitrary continuous t-norms $\otimes$, as long as the fuzzy operators can be expressed by finite systems of quadratic equations. Finally, in [52] the authors propose a tableau algorithm for $\otimes$-$\mathcal{ALC}_{f,=}$, but using S-implications instead of residua, and even allow for truth values from an arbitrary complete lattice instead of $[0, 1]$.

However, all of the above tableau algorithms were shown to be incorrect in the presence of (crisp) GCIs [15,16]. The reason is that the blocking conditions employed by these algorithms are too greedy, and might lead to a satisfiable set of constraints, even though the given ontology has no model. A sound, complete and terminating tableau-based algorithm for consistency of $\otimes$-$\mathcal{ALC}_{f,\geq}$ was proposed in [53,54]; however, it requires the solution of a finitely-represented, but infinite, system of (linear or quadratic) inequalities. Following these revelations, many t-norm based fuzzy description logics were shown to have an undecidable consistency problem, e.g. Π-$\mathcal{ALC}_{f,\geq}$ with strict GCIs [16], $\Pi^{(0,b)}$-$\mathcal{JAL}_{f,=}$ [18], and Ł-$\mathcal{ELC}_{f,\geq}$ [19]. In [22], a first version of the framework presented in Section 3 was described, which subsumed all previously known undecidability results.

Restricting to Zadeh semantics, decidability of consistency has been shown even for very expressive description logics like $\mathcal{SHOIN}$ [7,55]. The main reason for these results is that reasoning can be limited to the values occurring in the input ontology (and their negations). Under Gödel semantics, consistency is ExpTime-complete in G-$\mathcal{JALC}_{f,=}$ w.r.t. both witnessed [21] and non-witnessed semantics [39]. Here, one additionally needs to keep track of the *order* between the values of concepts, but not the values themselves. Moreover, we first proved in [23] that in $\otimes$-$\mathcal{SHOI}_{f,\geq}$ for any t-norm $\otimes$ without zero divisors, the values in the input ontology do not have any effect in the consistency of the ontology, and can simply be removed (see Section 4). The inexpressive DL $\mathcal{EL}$ also keeps its polynomial complexity for subsumption under Gödel semantics [5].

Some work has also considered fuzzy DLs that are restricted *a priori* to finitely many degrees of truth. The idea to reduce consistency to consistency in classical DLs has been used for arbitrary finite chains of truth values with combination functions similar to ordinal sums of the Gödel and Łukasiewicz t-norms, and even for very expressive DLs such as $\mathcal{SROIQ}$ [33,34]; however, the reduction often increases the size of the input ontology by an exponential factor. In contrast, tight complexity bounds were shown for consistency and subsumption in fuzzy DLs below $\mathcal{SHOI}$ over an *arbitrary* finite lattice using a combination of automata-based constructions [40,56] and tableaux rules [41].

6. Conclusions

We have studied the limits of decidability of ontology consistency in fuzzy DLs. On one hand, we have presented several undecidability results that strengthen all previously-known cases of fuzzy DLs with an undecidable consistency problem. To do this, we have developed a general framework for proving undecidability, which is based on a set of relatively simple properties. Using this framework, we were able to show, for instance, that consistency is undecidable in the very simple DL $\mathfrak{NEL}$ if the semantics are based on a t-norm with zero divisors. Extensions of this framework with different ways to prove the canonical model property also allowed us to prove that the problem is undecidable in Π-$\mathcal{ELC}$ and $\otimes$-$\mathcal{JEL}_=$ for any t-norm different from the Gödel t-norm. All of these logics are equivalent to classical $\mathcal{ALC}$ when their semantics is restricted to two truth values.

An analysis of these results suggests that the culprit for undecidability of a fuzzy DL is the capacity of expressing specific upper bounds within a non-idempotent component of the t-norm. Indeed, fuzzy GCIs usually provide a lower bound for the interpretation of a concept. If the involutive negation is allowed, then a lower bound for the concept $\neg C$ corresponds to an upper bound for C. Similarly, the implication constructor can be used to propagate upper bounds through concepts, and the residual negation defines a "local" involutive negation in every t-norm that contains zero divisors. Conversely, our proofs of decidability exploit the fact that for any continuous t-norm $\otimes$ without zero divisors, upper bounds different from 0 cannot be expressed in $\otimes$-$\mathcal{SROIQ}_{f,\geq}$. If $\mathcal{I}$ is a witnessed model of an ontology $\mathcal{O}$, then mapping all the positive truth degrees given by $\mathcal{I}$ to 1 yields a crisp model of $\mathcal{O}$ (see Lemma 30). If this intuition is correct, then it suggests that for any t-norm without zero divisors, consistency in $\otimes$-$\mathfrak{NEL}_=$ and $\otimes$-$\mathfrak{NAL}_=$ is decidable.

Our analysis of the limits of decidability for fuzzy DLs is almost complete. As can be seen from Table 5, there are only a few remaining gaps, which we plan to cover in future work. In this work, we consider mainly standard constructors studied for classical DLs. Other fuzzy constructors like *hedges* [44,57–59], or *aggregation operators* [60,61] may require a different analysis. Note also that we have considered here only the ontology consistency problem. The decidability and complexity of other standard reasoning tasks, such as subsumption or instance checking, are other topics for future research. We also intend to find the precise complexity, and optimal algorithms, for reasoning in light-weight fuzzy DLs, such as $\otimes$-$\mathcal{EL}$ and $\otimes$-*DL-Lite*, over arbitrary continuous t-norms $\otimes$.

As has been noted by several authors [13,14], the ability to manage vague and imprecise knowledge is a desired feature of intelligent systems to be used in the biological and medical domains, among many others. Studying the complexity of reasoning with different fuzzy DLs allows us to discern which of these may be suitable formalisms for implementing a fuzzy knowledge representation and reasoning system. It is clearly desirable to stay in the decidable part of Table 5. However, the decidability results of Section 4 are also not helpful since they show that consistency can be decided using classical

reasoners without any modification of the input ontology. This leaves only the Gödel t-norm as a promising candidate for an implementation. As an alternative, one could use many-valued DLs that support only a *finite* set of truth values, arranged in a residuated lattice or a total order [31,33,34,40,41]. The complexity of reasoning in such logics is often the same as for the underlying classical DLs. Sometimes, highly-optimized reasoners [62,63] for classical reasoning problems can be reused after a suitable reduction.

Acknowledgements

Appendix A. Proofs for Section 3

Lemma 14. *The functions* Enc *described above are valid encoding functions for t-norms of the form* $\Pi^{(a,b)}$ *or* $Ł^{(a,b)}$.

Proof. In both cases described in Section 3.3, the encodings of different words $u, u' \in \Sigma \Sigma_0^*$ are different, and in particular smaller than b, and thus are not included in Enc(ε). Furthermore, the encodings do not depend on the number of leading zeros. Thus, the first two conditions of Definition 11 are satisfied. For Condition (c), we analyze the two cases of $\Pi^{(a,b)}$ and $Ł^{(a,b)}$ separately.

For $\Pi^{(a,b)}$, consider two different words $v, w \in \Sigma^*$ and assume w.l.o.g. that $v < w$. Then $v + 1 \leq w$ and hence $2^{-w} \leq 2^{-(v+1)} = 2^{-v}/2$. If $v \neq \varepsilon$, this implies that

$$\text{enc}(v) \Rightarrow \text{enc}(w) = \sigma_{a,b}(2^{-w}/2^{-v}) \leq \sigma_{a,b}(1/2) = \text{enc}(1) < 1.$$

For $v = \varepsilon$, we similarly have $p \Rightarrow \text{enc}(w) = \text{enc}(w) \leq \text{enc}(1) < 1$ for any $p \in \text{Enc}(\varepsilon) = [b, 1]$. Conversely, if $v = w$, then $\text{enc}(v) \Rightarrow \text{enc}(w) = 1 = \text{enc}(w) \Rightarrow \text{enc}(v)$. Thus, the words $u_\varepsilon := 1$ and $u_+ := \varepsilon$ satisfy Condition (c) of Definition 11.

For the case of $Ł^{(a,b)}$, let $k = \max\{|v_i|, |w_i| \mid i \in \mathcal{N}\}$ be the maximal length of a word occurring in $\mathcal{P}$. Then, for every $v \in \mathcal{N}^*$, we have $|v_\nu| \leq (|v|+1)k$ and $|w_\nu| \leq (|v|+1)k$. If $v_\nu \neq w_\nu$, these words must differ in one of the first $\ell := (|v|+1)k$ letters. Thus, if $v_\nu \neq \varepsilon$ and $w_\nu \neq \varepsilon$, then either $\text{enc}(v_\nu) > \text{enc}(w_\nu)$, and thus

$$\begin{aligned}
\text{enc}(v_\nu) \Rightarrow \text{enc}(w_\nu) &= \sigma_{a,b}\big(\min\{1, 1 + 0.\overleftarrow{v_\nu} - 0.\overleftarrow{w_\nu}\}\big) \\
&= \min\big\{b, \sigma_{a,b}(1 + 0.\overleftarrow{v_\nu} - 0.\overleftarrow{w_\nu})\big\} \\
&\leq \sigma_{a,b}\big(1 - (s+1)^{-(\ell+1)}\big) \\
&= \sigma_{a,b}\big(1 - 0.\underleftarrow{1 \cdot 0^\ell}\big) \\
&= \text{enc}(\underline{1} \cdot \underline{0}^\ell) < 1,
\end{aligned}$$

or, similarly, $\text{enc}(v_\nu) < \text{enc}(w_\nu)$ and $\text{enc}(w_\nu) \Rightarrow \text{enc}(v_\nu) \leq \text{enc}(\underline{1} \cdot \underline{0}^\ell) < 1$. Note that again this also holds if $v_\nu = \varepsilon$, since w_ν also differs from $\underline{0}^\ell$ in one of the first ℓ letters, and similarly if $w_\nu = \varepsilon$. Conversely, if $v_\nu = w_\nu$, then both residua yield 1 as result, which is greater than $\text{enc}(\underline{1} \cdot \underline{0}^\ell)$. Thus, setting $u_\varepsilon := \underline{1} \cdot \underline{0}^k$ and $u_+ := \underline{0}^k$ satisfies Condition (c) of Definition 11. $\square$

Lemma 15. *For every continuous t-norm $\otimes$, the logics $\otimes$-$\mathcal{EL}_=$, $\otimes$-$\mathcal{ELC}_\geq$, and $Ł^{(0,b)}$-$\mathfrak{NEL}$ satisfy* P_{ini}.

Proof. In the case of $\otimes$-$\mathcal{EL}_=$, we can use the simple ontology $\mathcal{O}_{C(e)=u} := \{\langle e : C = \text{enc}(u)\rangle\}$ to enforce that $C^{\mathcal{I}}(e^{\mathcal{I}}) = \text{enc}(u) \in \text{Enc}(u)$ is satisfied by every model $\mathcal{I}$.

In $\otimes$-$\mathcal{ELC}_\geq$, the two axioms $\langle e : C \geq \text{enc}(u)\rangle$ and $\langle e : \neg C \geq 1 - \text{enc}(u)\rangle$ express the same restriction. The first axiom ensures that $C^{\mathcal{I}}(e^{\mathcal{I}}) \geq \text{enc}(u)$, while the second requires that $1 - C^{\mathcal{I}}(e^{\mathcal{I}}) \geq 1 - \text{enc}(u)$, i.e. $C^{\mathcal{I}}(e^{\mathcal{I}}) \leq \text{enc}(u)$, holds.

For the logic $Ł^{(0,b)}$-$\mathfrak{NEL}$, a more involved construction is necessary. We first ensure that a fresh auxiliary concept name A has a value from Enc(u) at all domain elements, and then require that C and A have the same value at e. For the first part, we use the two axioms

$$\big\langle H^{(s+1)^{|u|}} \equiv \boxminus H^{(s+1)^{|u|}}\big\rangle, \ \big\langle A \equiv H^{2\overleftarrow{u}}\big\rangle.$$

Observe that, whenever $H^{\mathcal{I}}(x) \in [0, b]$ for some interpretation $\mathcal{I}$ and $x \in \Delta^{\mathcal{I}}$, then for every $m \in \mathbb{N}$ we have by linearity of $\sigma_{0,b}$ that

$$\big(H^m\big)^{\mathcal{I}}(x) = \sigma_{0,b}\big(\max\{0, m(\sigma_{0,b}^{-1}(H^{\mathcal{I}}(x)) - 1) + 1\}\big) = \max\{0, m(H^{\mathcal{I}}(x) - b) + b\}.$$

Let now $\mathcal{I}$ be an interpretation that satisfies these axioms and $x \in \Delta^{\mathcal{I}}$. If $u \in \{0\}^*$, then the second axiom enforces that $A^{\mathcal{I}}(x) = \top^{\mathcal{I}}(x) = 1 \in \mathsf{Enc}(u)$ holds. If $u \notin \{0\}^*$, then by the first axiom we have

$$\max\big\{0, (s+1)^{|u|}\big(H^{\mathcal{I}}(x) - b\big) + b\big\} = b - \max\big\{0, (s+1)^{|u|}\big(H^{\mathcal{I}}(x) - b\big) + b\big\}.$$

This shows that $-b = 2(s+1)^{|u|}(H^{\mathcal{I}}(x) - b)$, and thus $H^{\mathcal{I}}(x) = b - \frac{b}{2(s+1)^{|u|}}$. From the second axiom it follows that

$$A^{\mathcal{I}}(x) = \max\left\{0, 2\widetilde{u}\left(-\frac{b}{2(s+1)^{|u|}}\right) + b\right\}.$$

Since $\frac{\widetilde{u}}{(s+1)^{|u|}} = 0.\widetilde{u} < 1$, we obtain $A^{\mathcal{I}}(x) = b - b(0.\widetilde{u}) = \sigma_{0,b}(1 - 0.\widetilde{u}) = \mathsf{enc}(u)$.

For the second part, we use the axiom

$$\big\langle e : (C \rightharpoonup A) \sqcap (A \rightharpoonup C)\big\rangle.$$

If $u \in \{0\}^*$, the semantics of $\rightharpoonup$ and the fact that $A^{\mathcal{I}}(e^{\mathcal{I}}) \in \mathsf{Enc}(u) = \mathsf{Enc}(\varepsilon) = [b, 1]$ imply that $C^{\mathcal{I}}(e^{\mathcal{I}}) \in [b, 1] = \mathsf{Enc}(u)$. If $u \notin \{0\}^*$, then $A^{\mathcal{I}}(e^{\mathcal{I}}) = \mathsf{enc}(u) < b$, which implies that $C^{\mathcal{I}}(e^{\mathcal{I}}) < b$, and thus $C^{\mathcal{I}}(e^{\mathcal{I}}) = A^{\mathcal{I}}(e^{\mathcal{I}}) = \mathsf{enc}(u)$. $\square$

Lemma 16. *For every continuous t-norm $\otimes$, the logic $\otimes$-$\mathcal{EL}$ satisfies* $\mathrm{P}_{\rightarrow}$.

Proof. Consider the ontology $\mathcal{O}_{\exists r} := \{\langle \top \sqsubseteq \exists r.\top \rangle\}$. Any model $\mathcal{I}$ of this axiom satisfies $(\exists r.\top)^{\mathcal{I}}(x) = 1$ for every $x \in \Delta^{\mathcal{I}}$. Since reasoning is restricted to witnessed models, there must exist a $y \in \Delta^{\mathcal{I}}$ with $r^{\mathcal{I}}(x, y) = 1$, as required for the successor property. $\square$

Lemma 17. *For every continuous t-norm $\otimes$ except the Gödel t-norm, the logic $\otimes$-$\mathcal{EL}$ satisfies* $\mathrm{P}_{\circ}$.

Proof. By assumption, $\otimes$ must contain either the product or the Łukasiewicz t-norm in some interval. We divide the proof depending on the representative chosen for the encoding function.

For the case of $\Pi^{(a,b)}$-$\mathcal{EL}$, observe that for every $u \in \Sigma_0^*$ and $u' \in \Sigma \Sigma_0^*$, we have $u'(s+1)^{|u|} + u = \underline{u'u}$. Given $u \in \Sigma_0^*$, we define the ontology

$$\mathcal{O}_{C \circ u} := \big\{\big\langle D_{C \circ u} \equiv C^{(s+1)^{|u|}} \sqcap C_u\big\rangle\big\}.$$

Observe that for every interpretation $\mathcal{I}$ and $x \in \Delta^{\mathcal{I}}$, if $C^{\mathcal{I}}(x) = \sigma_{a,b}(p)$ and $m \in \mathbb{N}$, then

$$\big(C^m\big)^{\mathcal{I}}(x) = \sigma_{a,b}\big(p^m\big).$$

Let now $\mathcal{I}$ be a model of $\mathcal{O}_{C \circ u}$, $x \in \Delta^{\mathcal{I}}$, and $u' \in \{\varepsilon\} \cup \Sigma \Sigma_0^*$ such that $C_u^{\mathcal{I}}(x) \in \mathsf{Enc}(u)$ and $C^{\mathcal{I}}(x) \in \mathsf{Enc}(u')$. If $u \notin \{0\}^*$ and $u' \neq \varepsilon$, then we have

$$D_{C \circ u}^{\mathcal{I}}(x) = \sigma_{a,b}\big(2^{-(u'(s+1)^{|u|}+u)}\big) = \mathsf{enc}\big(\underline{u'u}\big).$$

If $u \in \{0\}^*$ and $u' \neq \varepsilon$, we have $C_u^{\mathcal{I}}(x) \in [b, 1]$, and thus

$$D_{C \circ u}^{\mathcal{I}}(x) = \big(C^{(s+1)^{|u|}}\big)^{\mathcal{I}}(x) = \sigma_{a,b}\big(2^{-(u'(s+1)^{|u|}+0)}\big) = \mathsf{enc}\big(\underline{u'u}\big).$$

Similarly, for $u \notin \{0\}^*$ and $u' = \varepsilon$ we get $(C^{(s+1)^{|u|}})^{\mathcal{I}}(x) \in [b, 1]$, which implies that

$$D_{C \circ u}^{\mathcal{I}}(x) = C_u^{\mathcal{I}}(x) = \mathsf{enc}(\varepsilon \underline{u}).$$

Finally, if $u \in \{0\}^*$ and $u' = \varepsilon$, then $D^{\mathcal{I}}(x) = (C^{(s+1)^{|u|}} \sqcap C_u)^{\mathcal{I}}(x) \in [b, 1] = \mathsf{Enc}(\varepsilon \underline{u})$.

For the case of $Ł^{(a,b)}$-$\mathcal{EL}$, we define the ontology

$$\mathcal{O}_{C \circ u} := \big\{C'^{(s+1)^{|u|}} \equiv C,\ D_{C \circ u} \equiv C' \sqcap C_u\big\}.$$

Let $\mathcal{I}$ be a model of $\mathcal{O}_{C \circ u}$, $x \in \Delta^{\mathcal{I}}$, and assume that $C_u^{\mathcal{I}}(x) \in \mathsf{Enc}(u)$ and $C^{\mathcal{I}}(x) \in \mathsf{Enc}(u')$ for some $u' \in \{\varepsilon\} \cup \Sigma \Sigma_0^*$. If $u' \neq \varepsilon$, then from the first axiom it follows that

$$\big(C'^{(s+1)^{|u|}}\big)^{\mathcal{I}}(x) = C^{\mathcal{I}}(x) = \sigma_{a,b}\big(1 - 0.\overleftarrow{u'}\big) \in (a, b).$$

Since $\otimes(a, b)$-contains Łukasiewicz, this implies that $C'^{\mathcal{I}}(x) \in (a, b)$. Thus,

$$\sigma_{a,b}\big(\max\{0, (s+1)^{|u|}\big(\sigma_{a,b}^{-1}\big(C'^{\mathcal{I}(x)}\big) - 1\big) + 1\}\big) = C^{\mathcal{I}}(x) = \sigma_{a,b}\big(1 - 0.\overleftarrow{u'}\big),$$

which shows that

$$C'^{\mathcal{I}}(x) = \sigma_{a,b}\big(1 - (s+1)^{-|u|}0.\overleftarrow{u'}\big).$$

If $u \notin \{0\}^*$, then it follows that

$$\begin{aligned}D_{C \circ u}^{\mathcal{I}}(x) &= \sigma_{a,b}\big(\max\{0, (1 - 0.\widetilde{u}) + \big(1 - (s+1)^{-|u|}0.\overleftarrow{u'}\big) - 1\}\big)\\ &= \sigma_{a,b}\big(1 - 0.\widetilde{u} - (s+1)^{-|u|}0.\overleftarrow{u'}\big) = \mathsf{enc}(\underline{u'u}).\end{aligned}$$

If $u \in \{0\}^*$, then $C_u^{\mathcal{I}}(x) \in [b, 1]$, and thus

$$D_{C \circ u}^{\mathcal{I}}(x) = C'^{\mathcal{I}(x)} = \sigma_{a,b}\big(1 - (s+1)^{-|u|}0.\overleftarrow{u'}\big) = \mathsf{enc}(\underline{u'u}).$$

It remains to consider the case that u' is the empty word, and thus $C^{\mathcal{I}}(x) \in [b, 1]$. By the first axiom, we also have $C'^{\mathcal{I}}(x) \in [b, 1]$. If $u \notin \{0\}^*$, then

$$D_{C \circ u}^{\mathcal{I}}(x) = C_u^{\mathcal{I}}(x) = \mathsf{enc}(u) = \mathsf{enc}(\varepsilon\underline{u}).$$

On the other hand, if $u \in \{0\}^*$, then we have $D_{C \circ u}^{\mathcal{I}}(x) \in [b, 1] = \mathsf{Enc}(\varepsilon\underline{u})$. $\quad\square$

Lemma 18. *For every continuous t-norm $\otimes$ except the Gödel t-norm, the logics $\otimes$-$\mathcal{AL}$, $\otimes$-$\mathcal{ELC}$, and Ł$^{(0,b)}$-$\mathfrak{NEL}$ satisfy* $\mathrm{P}_{\rightsquigarrow}$.

Proof. Let $\mathcal{I}$ be an interpretation and $x, y \in \Delta^{\mathcal{I}}$ such that $C^{\mathcal{I}}(x) \in \mathsf{Enc}(u)$ for some $u \in \Sigma_0^*$ and $r^{\mathcal{I}}(x, y) = 1$. Regardless of whether we have chosen $\Pi^{(a,b)}$ or Ł$^{(a,b)}$, if $u \notin \{0\}^*$, then the goal is to have $D^{\mathcal{I}}(y) = C^{\mathcal{I}}(x)$. On the other hand, if $u \in \{0\}^*$, then $C^{\mathcal{I}}(x) \geq b$, and we only need to ensure that $D^{\mathcal{I}}(x) \geq b$.

In all fuzzy DLs based on $\mathcal{EL}$, we can formulate the axiom $\langle \exists r.D \sqsubseteq C \rangle$. If $\mathcal{I}$ satisfies this axiom, then

$$D^{\mathcal{I}}(y) = r^{\mathcal{I}}(x, y) \otimes D^{\mathcal{I}}(y) \leq (\exists r.D)^{\mathcal{I}}(x) \leq C^{\mathcal{I}}(x).$$

We now add an axiom ensuring that also $D^{\mathcal{I}}(y) \geq C^{\mathcal{I}}(x)$ holds if $u \notin \{0\}^*$, and $D^{\mathcal{I}}(y) \geq b$ holds if $u \in \{0\}^*$. The precise form of this axiom depends on the expressivity of the logic used.

In $\otimes$-$\mathcal{AL}$, we can use the axiom $\langle C \sqsubseteq \forall r.D \rangle$ to restrict $\mathcal{I}$ to satisfy

$$C^{\mathcal{I}}(x) \leq (\forall r.D)^{\mathcal{I}}(x) \leq r^{\mathcal{I}}(x, y) \Rightarrow D^{\mathcal{I}}(y) = D^{\mathcal{I}}(y),$$

and thus also $D^{\mathcal{I}}(y) \geq C^{\mathcal{I}}(x) \geq b$ if $u \in \{0\}^*$.

In the case of $\otimes$-$\mathcal{ELC}$, if $\mathcal{I}$ is a model of $\langle \exists r.\neg D \sqsubseteq \neg C \rangle$, then

$$1 - D^{\mathcal{I}}(y) = r^{\mathcal{I}}(x, y) \otimes \big(1 - D^{\mathcal{I}}(y)\big) \leq (\exists r.\neg D)^{\mathcal{I}}(x) \leq 1 - C^{\mathcal{I}}(x),$$

and thus $C^{\mathcal{I}}(x) \leq D^{\mathcal{I}}(y)$ as in the previous case.

Finally, for Ł$^{(0,b)}$-$\mathfrak{NEL}$, we use the axiom $\langle \exists r.\boxminus D \sqsubseteq \boxminus C \rangle$, similar to the one for $\otimes$-$\mathcal{ELC}$. If $\mathcal{I}$ satisfies this axiom, then

$$\ominus D^{\mathcal{I}}(y) = r^{\mathcal{I}}(x, y) \otimes \big(\ominus D^{\mathcal{I}}(y)\big) \leq (\exists r.\boxminus D)^{\mathcal{I}}(x) \leq \ominus C^{\mathcal{I}}(x).$$

If $u \notin \{0\}^*$, then $D^{\mathcal{I}}(y) \leq C^{\mathcal{I}}(x) < b$, which shows that $b - D^{\mathcal{I}}(y) \leq b - C^{\mathcal{I}}(x)$, and thus $C^{\mathcal{I}}(x) \leq D^{\mathcal{I}}(y)$. If $u \in \{0\}^*$, then $\ominus D^{\mathcal{I}}(y) \leq \ominus C^{\mathcal{I}}(x) = 0$, and thus $D^{\mathcal{I}}(y) \geq b$ as required. $\quad\square$

Lemma 19. *Let $\otimes$ be any continuous t-norm except the Gödel t-norm. If one of the logics $\otimes$-$\mathfrak{IEL}_=$, $\otimes$-$\mathcal{ELC}_{\geq}$, or Ł$^{(0,b)}$-$\mathfrak{NEL}$ satisfies* $\mathrm{P}_{\triangle}$ *with $\mathcal{O}_{\mathcal{P}}$ and $\mathcal{I}_{\mathcal{P}}$ can be extended to a model of $\mathcal{O}_{\mathcal{P}}$, then this logic also satisfies* $\mathrm{P}_{\neq}$.

Proof. For $\otimes$-$\mathfrak{IEL}_=$, we define the ontology

$$\mathcal{O}_{V \neq W} := \big\{\big\langle \top \sqsubseteq \big((V \rightarrow W) \sqcap (W \rightarrow V)\big) \rightarrow M \big\rangle\big\}.$$

For every model $\mathcal{I}$ of $\mathcal{O}_{\mathcal{P}} \cup \mathcal{O}_{V \neq W}$ and every $v \in \mathcal{N}^*$, we have

$$\big(V^{\mathcal{I}}\big(g(v)\big) \Rightarrow W^{\mathcal{I}}\big(g(v)\big)\big) \otimes \big(W^{\mathcal{I}}\big(g(v)\big) \Rightarrow V^{\mathcal{I}}\big(g(v)\big)\big) \leq M^{\mathcal{I}}\big(g(v)\big).$$

Since at least one of the two residua must be 1, this implies

$$\min\big\{V^{\mathcal{I}}\big(g(v)\big) \Rightarrow W^{\mathcal{I}}\big(g(v)\big), W^{\mathcal{I}}\big(g(v)\big) \Rightarrow V^{\mathcal{I}}\big(g(v)\big)\big\} \leq M^{\mathcal{I}}\big(g(v)\big)$$

as required.

For the second condition, assume that $\mathcal{I}_\mathcal{P}$ cannot be extended to a model of $\mathcal{O}_\mathcal{P} \cup \mathcal{O}_{V \neq W}$. Since there is an extension $\mathcal{I}$ of $\mathcal{I}_\mathcal{P}$ that satisfies $\mathcal{O}_\mathcal{P}$, we know that $\mathcal{I}$ must violate $\mathcal{O}_{V \neq W}$. This means that there is a $\nu \in \mathcal{N}^*$ such that

$$1 = \top^{\mathcal{I}_\mathcal{P}}(\nu) > \left(\left((V \to W) \sqcap (W \to V)\right) \to M\right)^{\mathcal{I}_\mathcal{P}}(\nu).$$

This implies that

$$M^{\mathcal{I}_\mathcal{P}}(\nu) < \left(V^{\mathcal{I}_\mathcal{P}}(\nu) \Rightarrow W^{\mathcal{I}_\mathcal{P}}(\nu)\right) \otimes \left(W^{\mathcal{I}_\mathcal{P}}(\nu) \Rightarrow V^{\mathcal{I}_\mathcal{P}}(\nu)\right)$$
$$= \min\left\{V^{\mathcal{I}_\mathcal{P}}(\nu) \Rightarrow W^{\mathcal{I}_\mathcal{P}}(\nu), W^{\mathcal{I}_\mathcal{P}}(\nu) \Rightarrow V^{\mathcal{I}_\mathcal{P}}(\nu)\right\}.$$

For $\otimes$-$\mathcal{ELC}_\geq$, consider the ontology

$$\mathcal{O}_{V \neq W} := \left\{\langle X \sqsubseteq X \sqcap X\rangle, \langle \top \sqsubseteq \neg(X \sqcap \neg X)\rangle,\right. \tag{A.1}$$

$$\langle X \sqcap V \sqsubseteq X \sqcap W \sqcap M\rangle, \tag{A.2}$$

$$\left.\langle \neg X \sqcap W \sqsubseteq \neg X \sqcap V \sqcap M\rangle\right\}. \tag{A.3}$$

For every model of the axioms in (A.1) and every $x \in \Delta^\mathcal{I}$, we know that $X^\mathcal{I}(x) \leq X^\mathcal{I}(x) \otimes X^\mathcal{I}(x)$ and hence, $X^\mathcal{I}(x)$ must be an idempotent element w.r.t. $\otimes$. Recall that $X^\mathcal{I}(x)$ can thus not lie in any component of $\otimes$, which implies that $\otimes$ behaves like the Gödel t-norm on $X^\mathcal{I}(x)$. In particular, we get $0 \geq (X \sqcap \neg X)^\mathcal{I}(x) = \min\{X^\mathcal{I}(x), 1 - X^\mathcal{I}(x)\}$, and thus $X^\mathcal{I}(x) \in \{0, 1\}$.

Let now $\mathcal{I}$ be a model of $\mathcal{O}_\mathcal{P} \cup \mathcal{O}_{V \neq W}$ and $\nu \in \mathcal{N}^*$. If $X^\mathcal{I}(g(\nu)) = 1$, then axiom (A.2) states that $V^\mathcal{I}(g(\nu)) \leq W^\mathcal{I}(g(\nu)) \otimes M^\mathcal{I}(g(\nu))$. We consider which representative was chosen for the encoding function:

$\Pi^{(a,b)}$: Since $W^\mathcal{I}(g(\nu)) \in \mathsf{Enc}(w_\nu)$, we know in particular that $W^\mathcal{I}(g(\nu)) > a$. Furthermore, since $M^\mathcal{I}(g(\nu)) = \mathsf{enc}(1) < b$ and product is a strict t-norm,[4] for every $z > M^\mathcal{I}(g(\nu))$, we have that $W^\mathcal{I}(g(\nu)) \otimes z > W^\mathcal{I}(g(\nu)) \otimes M^\mathcal{I}(g(\nu)) \geq V^\mathcal{I}(g(\nu))$.

$Ł^{(a,b)}$: If $w_\nu \neq \varepsilon$, then since the length of w_ν is bounded by $\ell := (|\nu| + 1)k$ and

$$W^\mathcal{I}\big(g(\nu)\big) \otimes M^\mathcal{I}\big(g(\nu)\big) = \sigma_{a,b}\big(\max\{0, 1 - 0.\overleftarrow{w_\nu} - \big(0.\underline{0}^\ell \cdot \underline{1}\big)\}\big),$$

we have

$$W^\mathcal{I}\big(g(\nu)\big) \otimes M^\mathcal{I}\big(g(\nu)\big) = \sigma_{a,b}\big(1 - 0.\overleftarrow{w_\nu} - \big(0.\underline{0}^\ell \cdot \underline{1}\big)\big) \in (a, b).$$

For $w_\nu = \varepsilon$, it also follows that

$$W^\mathcal{I}\big(g(\nu)\big) \otimes M^\mathcal{I}\big(g(\nu)\big) = M^\mathcal{I}\big(g(\nu)\big) = \sigma_{a,b}\big(1 - \big(0.\underline{0}^\ell \cdot \underline{1}\big)\big) \in (a, b).$$

Thus, by the properties of the Łukasiewicz t-norm we again have that for any $z > M^\mathcal{I}(g(\nu))$, $W^\mathcal{I}(g(\nu)) \otimes z > W^\mathcal{I}(g(\nu)) \otimes M^\mathcal{I}(g(\nu)) \geq V^\mathcal{I}(g(\nu))$ holds.

In both cases, we get

$$W^\mathcal{I}\big(g(\nu)\big) \Rightarrow V^\mathcal{I}\big(g(\nu)\big) = \sup\{z \in [0, 1] \mid W^\mathcal{I}\big(g(\nu)\big) \otimes z \leq V^\mathcal{I}\big(g(\nu)\big)\}$$
$$= \inf\{z \in [0, 1] \mid W^\mathcal{I}\big(g(\nu)\big) \otimes z > V^\mathcal{I}\big(g(\nu)\big)\}$$
$$\leq \inf\{z \in [0, 1] \mid z > M^\mathcal{I}\big(g(\nu)\big)\}$$
$$= M^\mathcal{I}\big(g(\nu)\big).$$

On the other hand, if $X^\mathcal{I}(g(\nu)) = 0$, then we know that $V^\mathcal{I}(g(\nu)) \Rightarrow W^\mathcal{I}(g(\nu)) \leq M^\mathcal{I}(g(\nu))$ by similar arguments, using axiom (A.3) instead of (A.2). Thus, we always have

$$\min\{V^\mathcal{I}\big(g(\nu)\big) \Rightarrow W^\mathcal{I}\big(g(\nu)\big), W^\mathcal{I}\big(g(\nu)\big) \Rightarrow V^\mathcal{I}\big(g(\nu)\big)\} \leq M^\mathcal{I}\big(g(\nu)\big).$$

To show the second point of $P_\neq$, assume that

$$\min\{V^{\mathcal{I}_\mathcal{P}}(\nu) \Rightarrow W^{\mathcal{I}_\mathcal{P}}(\nu), W^{\mathcal{I}_\mathcal{P}}(\nu) \Rightarrow V^{\mathcal{I}_\mathcal{P}}(\nu)\} \leq M^{\mathcal{I}_\mathcal{P}}(\nu) < 1$$

and consider an extension $\mathcal{I}$ of $\mathcal{I}_\mathcal{P}$ that satisfies $\mathcal{O}_\mathcal{P}$, which exists by assumption. We show that $\mathcal{I}$ can be further extended to a model of $\mathcal{O}_{V \neq W}$.

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[4] A continuous t-norm is *strict* if it is strictly monotone [4].

To find the values for X, consider any element $\nu \in \mathcal{N}^*$. By assumption, exactly one of the residua $V^{\mathcal{I}_{\mathcal{P}}}(\nu) \Rightarrow W^{\mathcal{I}_{\mathcal{P}}}(\nu)$ and $W^{\mathcal{I}_{\mathcal{P}}}(\nu) \Rightarrow V^{\mathcal{I}_{\mathcal{P}}}(\nu)$ is equal to 1. If $V^{\mathcal{I}_{\mathcal{P}}}(\nu) \Rightarrow W^{\mathcal{I}_{\mathcal{P}}}(\nu) = 1$, we set $X^{\mathcal{I}}(\nu) := 1$, which trivially satisfies axiom (A.3) at ν. By assumption, we must then have $W^{\mathcal{I}_{\mathcal{P}}}(\nu) \Rightarrow V^{\mathcal{I}_{\mathcal{P}}}(\nu) \leq M^{\mathcal{I}_{\mathcal{P}}}(\nu)$. By the definition of the residuum, we know that $W^{\mathcal{I}_{\mathcal{P}}}(\nu) \otimes m' > V^{\mathcal{I}_{\mathcal{P}}}(\nu)$ for all $m' > M^{\mathcal{I}_{\mathcal{P}}}(\nu)$. Since $\otimes$ is continuous and monotone, this means that $V^{\mathcal{I}_{\mathcal{P}}}(\nu) \leq W^{\mathcal{I}_{\mathcal{P}}}(\nu) \otimes M^{\mathcal{I}_{\mathcal{P}}}(\nu)$, i.e. axiom (A.2) is also satisfied at ν.

If the other residuum is equal to 1, we set $X^{\mathcal{I}}(\nu) := 0$ and can use dual arguments to show that axioms (A.2) and (A.3) are satisfied at ν. We have thus constructed an extension of $\mathcal{I}_{\mathcal{P}}$ that satisfies both $\mathcal{O}_{\mathcal{P}}$ and $\mathcal{O}_{V \neq W}$.

The last case is that of $\textup{Ł}^{(0,b)}$-$\mathcal{NEL}$, for which we can use the ontology

$$\mathcal{O}_{V \neq W} := \left\{ \left\langle \top \sqsubseteq \left((V \rightarrowtail W) \sqcap (W \rightarrowtail V) \right) \rightarrowtail M \right\rangle \right\},$$

which is similar to the one for $\otimes$-$\mathcal{IEL}_=$. In any model $\mathcal{I}$ of $\mathcal{O}_{\mathcal{P}} \cup \mathcal{O}_{V \neq W}$ it holds that for every $\nu \in \mathcal{N}^*$, $\left(\left((V \rightarrowtail W) \sqcap (W \rightarrowtail V) \right) \rightarrowtail M \right)^{\mathcal{I}}(g(\nu)) \geq 1$.

If $V^{\mathcal{I}}(g(\nu)) \leq W^{\mathcal{I}}(g(\nu))$, then $((W \rightarrowtail V) \rightarrowtail M)^{\mathcal{I}}(g(\nu)) \geq 1$. This can only be the case if $M^{\mathcal{I}}(g(\nu)) \geq b$ or $M^{\mathcal{I}}(g(\nu)) \geq (W \rightarrowtail V)^{\mathcal{I}}(g(\nu))$. The former is impossible since $M^{\mathcal{I}}(g(\nu)) = \text{enc}(1 \cdot \underline{0}^{(|\nu|+1)k}) < b$ by construction of $\mathcal{O}_{\mathcal{P}}$. By the definition of $\rightarrowtail$, the latter implies that $V^{\mathcal{I}}(g(\nu)) < b$, and thus

$$W^{\mathcal{I}}(g(\nu)) \Rightarrow V^{\mathcal{I}}(g(\nu)) = (W \rightarrowtail V)^{\mathcal{I}}(g(\nu)) \leq M^{\mathcal{I}}(g(\nu)).$$

Similarly, if $W^{\mathcal{I}}(g(\nu)) \leq V^{\mathcal{I}}(g(\nu))$, then $V^{\mathcal{I}}(g(\nu)) \Rightarrow W^{\mathcal{I}}(g(\nu)) \leq M^{\mathcal{I}}(g(\nu))$. In both cases, we have

$$\min\left\{ V^{\mathcal{I}}(g(\nu)) \Rightarrow W^{\mathcal{I}}(g(\nu)), W^{\mathcal{I}}(g(\nu)) \Rightarrow V^{\mathcal{I}}(g(\nu)) \right\} \leq M^{\mathcal{I}}(g(\nu)).$$

To show the second condition of P$_{\neq}$, assume that $\mathcal{I}_{\mathcal{P}}$ cannot be extended to a model of $\mathcal{O}_{\mathcal{P}} \cup \mathcal{O}_{V \neq W}$. Since there is an extension $\mathcal{I}$ of $\mathcal{I}_{\mathcal{P}}$ that satisfies $\mathcal{O}_{\mathcal{P}}$, we know that $\mathcal{I}$ violates $\mathcal{O}_{V \neq W}$. This means that there is a $\nu \in \mathcal{N}^*$ such that

$$\left(\left((V \rightarrowtail W) \sqcap (W \rightarrowtail V) \right) \rightarrowtail M \right)^{\mathcal{I}_{\mathcal{P}}}(\nu) < 1,$$

and thus

$$(V \rightarrowtail W)^{\mathcal{I}_{\mathcal{P}}}(\nu) \otimes (W \rightarrowtail V)^{\mathcal{I}_{\mathcal{P}}}(\nu) > M^{\mathcal{I}_{\mathcal{P}}}(\nu).$$

As above, the value $(V \rightarrowtail W)^{\mathcal{I}_{\mathcal{P}}}(\nu) \otimes (W \rightarrowtail V)^{\mathcal{I}_{\mathcal{P}}}(\nu)$ is either $V^{\mathcal{I}_{\mathcal{P}}}(\nu) \Rightarrow W^{\mathcal{I}_{\mathcal{P}}}(\nu)$ or $W^{\mathcal{I}_{\mathcal{P}}}(\nu) \Rightarrow V^{\mathcal{I}_{\mathcal{P}}}(\nu)$, depending on which of the values $V^{\mathcal{I}_{\mathcal{P}}}(\nu)$ and $W^{\mathcal{I}_{\mathcal{P}}}(\nu)$ is greater. Thus, both $V^{\mathcal{I}_{\mathcal{P}}}(\nu) \Rightarrow W^{\mathcal{I}_{\mathcal{P}}}(\nu)$ and $W^{\mathcal{I}_{\mathcal{P}}}(\nu) \Rightarrow V^{\mathcal{I}_{\mathcal{P}}}(\nu)$ must be greater than $M^{\mathcal{I}_{\mathcal{P}}}(\nu)$, showing that

$$\min\left\{ V^{\mathcal{I}_{\mathcal{P}}}(\nu) \Rightarrow W^{\mathcal{I}_{\mathcal{P}}}(\nu), W^{\mathcal{I}_{\mathcal{P}}}(\nu) \Rightarrow V^{\mathcal{I}_{\mathcal{P}}}(\nu) \right\} > M^{\mathcal{I}_{\mathcal{P}}}(\nu). \qquad \square$$

Lemma 21. *The logic* Π-$\mathcal{ELC}$ *satisfies* $P'_{\triangle}$.

Proof. We use the following modified ontology instead of $\mathcal{O}_{\mathcal{P}}$ from Section 3.3:

$$\mathcal{O}_{\mathcal{P}'} := \mathcal{O}_{\mathcal{P}, \circ} \cup \mathcal{O}_{\mathcal{P}, \rightarrow} \cup \left\{ \langle e_0 : V \rangle, \langle e_0 : W \rangle \right\} \cup$$

$$\bigcup_{i=1}^{n} \mathcal{O}_{D_{V \circ v_i} \overset{r_i}{\leadsto} V} \cup \mathcal{O}_{D_{W \circ w_i} \overset{r_i}{\leadsto} W} \cup$$

$$\left\{ \langle M \equiv \neg M \rangle, \langle M_+ \equiv \top \rangle \right\} \cup \bigcup_{i=1}^{n} \left\{ \langle V_i \equiv M^{v_i} \rangle, \langle W_i \equiv M^{w_i} \rangle \right\}.$$

Here, $\mathcal{O}_{\mathcal{P}, \circ}$, $\mathcal{O}_{\mathcal{P}, \rightarrow}$, and $\mathcal{O}_{C \overset{r}{\leadsto} D}$ are as defined in Section 3.2 and in the proofs of the corresponding lemmata in Section 3.3. As before, the values of the concept names V and W are initialized at e_0 to an encoding of $v_\varepsilon = w_\varepsilon = \varepsilon$, namely 1. But instead of initializing all the constants M, M_+, V_i, W_i at e_0 and then transferring their values to all successors, we define the values to be constant at all domain elements and need to transfer only the new values of V and W. In particular, M always has the value $\text{enc}(1) = 1/2$, while M_+ is always $\text{enc}(\varepsilon) = 1$. The axioms for V_i and W_i ensure that they get the values $(1/2)^{v_i} = 2^{-v_i} = \text{enc}(v_i)$ and $\text{enc}(w_i)$, respectively, at all domain elements.

It can now be shown similarly to Theorem 12 that this ontology satisfies the conditions of the canonical model property. $\square$

Lemma 22. *The logic* Π-$\mathcal{ELC}$ *satisfies* $P'_{\neq}$.

Proof. The proof is essentially the same as that of Lemma 19; we only describe the differences here.

The ontology $\mathcal{O}_{V \neq W}$ is similar to the one used for $\otimes\text{-}\mathcal{ELC}_{\geq}$, with the addition of a flag Y to distinguish the root node ε of $\mathcal{I}_{\mathcal{P}'}$. We define

$$\mathcal{O}_{V \neq W} := \big\{ \langle \exists r_i. \neg Y \sqsubseteq \neg \top \rangle \mid 1 \leq i \leq n \big\} \cup$$
$$\big\{ \langle X \sqsubseteq X \sqcap X \rangle, \langle \top \sqsubseteq \neg(X \sqcap \neg X) \rangle, \langle e_0 : \neg Y \rangle,$$
$$\langle Y \sqcap X \sqcap V \sqsubseteq Y \sqcap X \sqcap W \sqcap M \rangle,$$
$$\langle Y \sqcap \neg X \sqcap W \sqsubseteq Y \sqcap \neg X \sqcap V \sqcap M \rangle \big\}. \tag{A.4}$$

Every model of the axioms in (A.4) has to satisfy that every r_i-successor with degree 1 must belong to Y with degree 1, for every $i \in \mathcal{N}$. In particular, because of the construction of $\mathcal{O}_{\mathcal{P}, \rightarrow}$ (see the proof of Lemma 16), this means that for every model $\mathcal{I}$ of $\mathcal{O}_{\mathcal{P}'} \cup \mathcal{O}_{V \neq W}$ and every $v \in \mathcal{N}^+$, we have $Y^{\mathcal{I}}(g(v)) = 1$. On the other hand, $Y^{\mathcal{I}}(g(\varepsilon))$ must be 0. The role of X is the same as before. The remainder of the first condition of $\mathrm{P}'_{\neq}$ can thus be shown as in the proof of Lemma 19, but using $\mathcal{N}^+$ instead of $\mathcal{N}^*$.

For the second condition of $\mathrm{P}'_{\neq}$, consider an extension $\mathcal{I}$ of $\mathcal{I}_{\mathcal{P}'}$ that satisfies $\mathcal{O}_{\mathcal{P}'}$. To extend $\mathcal{I}$ to a model of $\mathcal{O}_{V \neq W}$, we first set $Y^{\mathcal{I}}(v) := 1$ for every $v \in \mathcal{N}^+$ and $X^{\mathcal{I}}(\varepsilon) := Y^{\mathcal{I}}(\varepsilon) := 0$. The remaining values $X^{\mathcal{I}}(v)$ for $v \in \mathcal{N}^+$ can be chosen exactly as in the proof of Lemma 19. Again, the proof is the same as before, with $\mathcal{N}^+$ instead of $\mathcal{N}^*$. □

Lemma 25. *For every continuous t-norm except the Gödel t-norm, the logic $\otimes\text{-}\mathfrak{I}\mathcal{EL}$ satisfies* $\mathrm{P}_{\overset{r}{\leadsto}}$.

The proof is divided into the following three lemmata. The first one provides an auxiliary result that is similar to the successor property.

Lemma 35. *Let $\otimes$ be a continuous t-norm of the form $\Pi^{(a,b)}$ or $\text{Ł}^{(a,b)}$. In $\otimes\text{-}\mathcal{EL}$, for every role name r and all concept names C, D, there is a classical ontology $\mathcal{O}_{C \overset{r}{\rightarrow} D}$ such that for every model $\mathcal{I}$ of this ontology and every $x \in \Delta^{\mathcal{I}}$ with $C^{\mathcal{I}}(x) \otimes C^{\mathcal{I}}(x) \in (a,b)$ there is a $y \in \Delta^{\mathcal{I}}$ such that $r^{\mathcal{I}}(x, y) \geq b$ and $C^{\mathcal{I}}(x) = D^{\mathcal{I}}(y)$.*

Proof. We can use the ontology $\mathcal{O}_{C \overset{r}{\rightarrow} D} := \{ \langle C \sqsubseteq \exists r.D \rangle, \langle \exists r.(D \sqcap D) \sqsubseteq C \sqcap C \rangle \}$ to achieve this behavior. To see this, consider a model $\mathcal{I}$ of this ontology and some $x \in \Delta^{\mathcal{I}}$ with $C^{\mathcal{I}}(x) \otimes C^{\mathcal{I}}(x) \in (a,b)$. Since $\mathcal{I}$ is witnessed, the first axiom ensures that there is an element $y \in \Delta^{\mathcal{I}}$ such that

$$C^{\mathcal{I}}(x) \leq \sup_{z \in \Delta^{\mathcal{I}}} r^{\mathcal{I}}(x, z) \otimes D^{\mathcal{I}}(z) = r^{\mathcal{I}}(x, y) \otimes D^{\mathcal{I}}(y),$$

while the second axiom implies that

$$r^{\mathcal{I}}(x, y) \otimes D^{\mathcal{I}}(y) \otimes D^{\mathcal{I}}(y) \leq \sup_{z \in \Delta^{\mathcal{I}}} r^{\mathcal{I}}(x, z) \otimes D^{\mathcal{I}}(z) \otimes D^{\mathcal{I}}(z) \leq C^{\mathcal{I}}(x) \otimes C^{\mathcal{I}}(x).$$

From these two inequalities and the monotonicity of $\otimes$, we then have

$$r^{\mathcal{I}}(x, y) \otimes D^{\mathcal{I}}(y) \otimes D^{\mathcal{I}}(y) \leq C^{\mathcal{I}}(x) \otimes C^{\mathcal{I}}(x) \leq r^{\mathcal{I}}(x, y) \otimes r^{\mathcal{I}}(x, y) \otimes D^{\mathcal{I}}(y) \otimes D^{\mathcal{I}}(y). \tag{A.5}$$

Since $C^{\mathcal{I}}(x) \otimes C^{\mathcal{I}}(x) \in (a,b)$, from this it follows that $r^{\mathcal{I}}(x, y) \otimes D^{\mathcal{I}}(y) \otimes D^{\mathcal{I}}(y)$ is also in (a,b). This means that $r^{\mathcal{I}}(x, y)$ must be greater than or equal to b since otherwise we would have

$$r^{\mathcal{I}}(x, y) \otimes \big(r^{\mathcal{I}}(x, y) \otimes D^{\mathcal{I}}(y) \otimes D^{\mathcal{I}}(y) \big) < r^{\mathcal{I}}(x, y) \otimes D^{\mathcal{I}}(y) \otimes D^{\mathcal{I}}(y),$$

by the definitions of ordinal sums and the product and the Łukasiewicz t-norms, in contradiction to (A.5). This implies that $D^{\mathcal{I}}(y) \otimes D^{\mathcal{I}}(y) \in (a,b)$, and thus (A.5) can be simplified to $D^{\mathcal{I}}(y) \otimes D^{\mathcal{I}}(y) = C^{\mathcal{I}}(x) \otimes C^{\mathcal{I}}(x)$.

If $\otimes$ contains the product t-norm in (a,b), then we obtain $(D^{\mathcal{I}}(y))^2 = (C^{\mathcal{I}}(x))^2$, i.e. $D^{\mathcal{I}}(y) = C^{\mathcal{I}}(x)$. On the other hand, if $\otimes$ contains the Łukasiewicz t-norm in (a,b), then the fact that $C^{\mathcal{I}}(x) \otimes C^{\mathcal{I}}(x) > a$ implies that $C^{\mathcal{I}}(x)$ must be strictly greater than $\frac{a+b}{2}$, and similarly for $D^{\mathcal{I}}(y)$. We obtain $2 \cdot D^{\mathcal{I}}(y) - b = 2 \cdot C^{\mathcal{I}}(x) - b$, which again shows that $D^{\mathcal{I}}(y) = C^{\mathcal{I}}(x)$. □

We divide the main proof of Lemma 25 in two cases, depending on whether $\otimes$ contains the product or the Łukasiewicz t-norm.

Lemma 36. $\Pi^{(a,b)}\text{-}\mathfrak{I}\mathcal{EL}$ *satisfies* $\mathrm{P}_{\overset{r}{\leadsto}}$.

Proof. We know that for every word $u \in \Sigma_0^*$, $\mathrm{enc}(u) = \sigma_{a,b}(2^{-u}) > a$. In particular, for every interpretation $\mathcal{I}$, $x \in \Delta^{\mathcal{I}}$, and j, $1 \le j \le k$, we have $C_j^{\mathcal{I}}(x) > a$. We define the ontology $\mathcal{O}_{(C_j \rightsquigarrow (D_j)}$ as follows:

$$\mathcal{O}_{(C_j \rightsquigarrow (D_j)} := \mathcal{O}_{H \xrightarrow{r} H'} \cup$$
$$\{\langle H \equiv C_1^2 \sqcap \cdots \sqcap C_k^2 \rangle\} \cup \tag{A.6}$$
$$\{\langle \exists r.D_j \sqsubseteq C_j \rangle, \langle \exists r.(D_j \to H') \sqsubseteq C_j \to H \rangle \mid 1 \le j \le k\}, \tag{A.7}$$

where r is a fresh role name, H and H' are fresh concept names, and $\mathcal{O}_{H \xrightarrow{r} H'}$ is the ontology given by Lemma 35.

We show that this ontology satisfies the conditions for the simultaneous transfer property. Let $\mathcal{I}$ be a model of this ontology and $x \in \Delta^{\mathcal{I}}$. By assumption, we know that there exists a word $u \in \Sigma_0^* \setminus \{0\}^*$ such that $C_1^{\mathcal{I}}(x) = \mathrm{enc}(u) \in (a, b)$, and furthermore, $C_j^{\mathcal{I}}(x) \in (a, 1]$ for all j, $2 \le j \le k$. Using the axiom from (A.6), we get $H^{\mathcal{I}}(x) \in (a, b)$. Since $\otimes$ behaves as the product t-norm in (a, b), this implies $H^{\mathcal{I}}(x) \otimes H^{\mathcal{I}}(x) \in (a, b)$, and thus by Lemma 35 there exists an element $y \in \Delta^{\mathcal{I}}$ with $r^{\mathcal{I}}(x, y) \ge b$ and $H'^{\mathcal{I}}(y) = H^{\mathcal{I}}(x)$.

We need only show that, for every j, $1 \le j \le k$, $D_j^{\mathcal{I}}(y) = C_j^{\mathcal{I}}(x) = \mathrm{enc}(u_j)$ if $u_j \notin \{0\}^*$ and $D_j^{\mathcal{I}}(y) \ge b$ if $C_j^{\mathcal{I}}(x) \ge b$. Let j be an arbitrary index $1 \le j \le k$ and suppose first that $C_j^{\mathcal{I}}(x) \ge b$ holds. Since $H^{\mathcal{I}}(x) < b$, then it also follows that $C_j^{\mathcal{I}}(x) \Rightarrow H^{\mathcal{I}}(x) = H^{\mathcal{I}}(x) < b$. The second axiom from (A.7) ensures that

$$r^{\mathcal{I}}(x, y) \otimes \left(D_j^{\mathcal{I}}(y) \Rightarrow H'^{\mathcal{I}}(y)\right) \le \left(\exists r.(D_j \to H')\right)^{\mathcal{I}}(x) \le C_j^{\mathcal{I}}(x) \Rightarrow H^{\mathcal{I}}(x) = H^{\mathcal{I}}(x) < b.$$

Since $r^{\mathcal{I}}(x, y) \ge b$ and $H'^{\mathcal{I}}(y) = H^{\mathcal{I}}(x)$, this implies $a < D_j^{\mathcal{I}}(y) \Rightarrow H^{\mathcal{I}}(x) \le H^{\mathcal{I}}(x) < b$, and thus by the definition of the residuum $\Rightarrow$ of an ordinal sum, it must be the case that $D_j^{\mathcal{I}}(y) \ge b$.

For the other case, suppose now that $C_j^{\mathcal{I}}(x) = \mathrm{enc}(u_j) < b$ for some $u_j \in \Sigma_0^* \setminus \{0\}^*$. We show that the two axioms from (A.7) ensure that $D_j^{\mathcal{I}}(y) = C_j^{\mathcal{I}}(x)$. The first axiom restricts $\mathcal{I}$ to satisfy

$$r^{\mathcal{I}}(x, y) \otimes D_j^{\mathcal{I}}(y) \le C_j^{\mathcal{I}}(x) < b,$$

and since $r^{\mathcal{I}}(x, y) \ge b$, it follows that $D_j^{\mathcal{I}}(y) \le C_j^{\mathcal{I}}(x)$. Analogously, from the second axiom, we derive that $D_j^{\mathcal{I}}(y) \Rightarrow H'^{\mathcal{I}}(y) \le C_j^{\mathcal{I}}(x) \Rightarrow H^{\mathcal{I}}(x)$. Recall that $a < H'^{\mathcal{I}}(y) = H^{\mathcal{I}}(x) < b$, and thus by the axiom in (A.6) we have $C_j^{\mathcal{I}}(x) > H^{\mathcal{I}}(x)$. We can infer that $D_j^{\mathcal{I}}(y) \Rightarrow H^{\mathcal{I}}(x) \le C_j^{\mathcal{I}}(x) \Rightarrow H^{\mathcal{I}}(x) < b$, and thus we also have $D_j^{\mathcal{I}}(x) > H^{\mathcal{I}}(x)$. From the definition of the residuum of an ordinal sum, we obtain

$$\frac{\sigma_{a,b}^{-1}(H^{\mathcal{I}}(x))}{\sigma_{a,b}^{-1}(D_j^{\mathcal{I}}(y))} \le \frac{\sigma_{a,b}^{-1}(H^{\mathcal{I}}(x))}{\sigma_{a,b}^{-1}(C_j^{\mathcal{I}}(x))},$$

and since $H^{\mathcal{I}}(x) > a$ and $\sigma_{a,b}$ is a strictly monotone bijection between $[0, 1]$ and $[a, b]$, we get $\sigma_{a,b}^{-1}(H^{\mathcal{I}}(x)) > 0$ and $D_j^{\mathcal{I}}(y) \ge C_j^{\mathcal{I}}(x)$. As this holds for every j, it is possible to transfer all the values simultaneously. □

The novel idea in this construction is to exploit the fact that the residuum is antitone in its first argument to provide a lower bound for $D_j^{\mathcal{I}}(y)$. For this construction to work, it is necessary that $a < H^{\mathcal{I}}(x) < C_j^{\mathcal{I}}(y)$ since otherwise the implication $C_j^{\mathcal{I}}(x) \Rightarrow H^{\mathcal{I}}(x)$ will simply be a or 1. This restriction is ensured by the axiom in (A.6).

For the case in which $\otimes$ contains the Łukasiewicz t-norm in the interval (a, b), we use the same idea for showing that the simultaneous transfer property holds. However, in this case we cannot ensure that H, which is interpreted as the conjunction of all the concepts C_j^2, has a degree strictly greater than a. Thus, we need to add some additional restrictions to handle the case where $H^{\mathcal{I}}(x) = a$.

Lemma 37. $\text{Ł}^{(a,b)}\text{-}\exists\mathcal{EL}$ *satisfies* $\text{P}_{\rightsquigarrow}$.

Proof. Define the ontology $\mathcal{O}_{(C_j \rightsquigarrow (D_j)}$ as follows:

$$\mathcal{O}_{(C_j \rightsquigarrow (D_j)} := \mathcal{O}_{G \xrightarrow{r} G'} \cup \mathcal{O}_{E \xrightarrow{r} E'} \cup$$
$$\{\langle H \equiv C_1^2 \sqcap \cdots \sqcap C_k^2 \rangle, \langle H \equiv G \sqcap G \rangle, \langle H' \equiv G' \sqcap G' \rangle \tag{A.8}$$
$$\langle C_1 \equiv E \sqcap E \rangle, \langle \exists r.H' \sqsubseteq H \rangle, \langle \exists r.((E' \to H') \to H') \sqsubseteq E \rangle\} \cup \tag{A.9}$$
$$\{\langle \exists r.D_j \sqsubseteq C_j \rangle, \langle \exists r.(D_j \to H') \sqsubseteq C_j \to H \rangle \mid 1 \le j \le k\}, \tag{A.10}$$

where r is a fresh role name, H, H', G, G', E, and E' are fresh concept names, and $\mathcal{O}_{G \xrightarrow{r} G'}$, $\mathcal{O}_{E \xrightarrow{r} E'}$ are the ontologies given by Lemma 35.

Let $\mathcal{I}$ be a model of this ontology and $x \in \Delta^{\mathcal{I}}$. It is easy to see that $H^{\mathcal{I}}(x) \leq C_j^{\mathcal{I}}(x)$ holds for all j, $1 \leq j \leq k$. Additionally, we know that $H^{\mathcal{I}}(x) \in [a, b]$. Using Lemma 35, we first show that there exists a $y \in \Delta^{\mathcal{I}}$ such that $r^{\mathcal{I}}(x, y) \geq b$ and $H^{\mathcal{I}}(x) = H'^{\mathcal{I}}(y)$. The proof is divided in two cases: (1) if $H^{\mathcal{I}}(x) > a$ and (2) if $H^{\mathcal{I}}(x) = a$.

Case (1) If $H^{\mathcal{I}}(x) > a$, then from the second axiom in (A.8) we get $G^{\mathcal{I}}(x) \otimes G^{\mathcal{I}}(x) = H^{\mathcal{I}}(x) \in (a, b)$. Thus, Lemma 35 yields the existence of an element $y \in \Delta^{\mathcal{I}}$ with $r^{\mathcal{I}}(x, y) \geq b$ and $G'^{\mathcal{I}}(y) = G^{\mathcal{I}}(x)$. The third axiom in (A.8) then implies that $H'^{\mathcal{I}}(y) = G'^{\mathcal{I}}(y) \otimes G'^{\mathcal{I}}(y) = G^{\mathcal{I}}(x) \otimes G^{\mathcal{I}}(x) = H^{\mathcal{I}}(x)$.

Case (2) If $H^{\mathcal{I}}(x) = a$, then we use the axioms from (A.9). By assumption, $E^{\mathcal{I}}(x) \otimes E^{\mathcal{I}}(x) = C_1^{\mathcal{I}}(x) \in (a, b)$, and hence as before Lemma 35 shows the existence of an element $y \in \Delta^{\mathcal{I}}$ such that $r^{\mathcal{I}}(x, y) \geq b$ and $E^{\mathcal{I}}(x) = E'^{\mathcal{I}}(y)$. The second axiom in (A.9) states that $r^{\mathcal{I}}(x, y) \otimes H'^{\mathcal{I}}(y) \leq H^{\mathcal{I}}(x) = a$. Since $r^{\mathcal{I}}(x, y) \geq b$, it follows that $H'^{\mathcal{I}}(y) \leq a$. From the third axiom we then have that

$$\left(E'^{\mathcal{I}}(y) \Rightarrow H'^{\mathcal{I}}(y)\right) \Rightarrow H'^{\mathcal{I}}(y) \leq E^{\mathcal{I}}(x) < b.$$

In particular, this means that $E'^{\mathcal{I}}(y) \Rightarrow H'^{\mathcal{I}}(y) > H'^{\mathcal{I}}(y)$ since otherwise the residuum would be $1 \geq b$. But since $E'^{\mathcal{I}}(y) > a$ and by the definition of the residuum of an ordinal sum, this can only be the case if $H'^{\mathcal{I}}(y) = a = H^{\mathcal{I}}(x)$.

In both cases, we have shown the existence of a $y \in \Delta^{\mathcal{I}}$ with $r^{\mathcal{I}}(x, y) \geq b$ and $H^{\mathcal{I}}(x) = H'^{\mathcal{I}}(y) \in [a, b)$. As in Lemma 36, we need to show that, whenever $C_j^{\mathcal{I}}(x) \geq b$, then also $D_j^{\mathcal{I}}(y) \geq b$, and if $u_j \notin \{0\}^*$, then $D_j^{\mathcal{I}}(y) = C_j^{\mathcal{I}}(x) = \text{enc}(u_j)$. The former case can be shown as in the proof of Lemma 36. In the latter case, the first axiom from (A.10) again ensures that $D_j^{\mathcal{I}}(y) \leq C_j^{\mathcal{I}}(x)$ since $C_j^{\mathcal{I}}(x) < b$ and $r^{\mathcal{I}}(x, y) \geq b$. From the second axiom and the fact that $H'^{\mathcal{I}}(y) = H^{\mathcal{I}}(x)$ it similarly follows that $D_j^{\mathcal{I}}(y) \Rightarrow H^{\mathcal{I}}(x) \leq C_j^{\mathcal{I}}(x) \Rightarrow H^{\mathcal{I}}(x) < b$. We now know that $H^{\mathcal{I}}(x) < C_j^{\mathcal{I}}(x) < b$ and $H^{\mathcal{I}}(x) < D_j^{\mathcal{I}}(y) < b$, and therefore

$$1 - \sigma_{a,b}^{-1}\left(D_j^{\mathcal{I}}(y)\right) + \sigma_{a,b}^{-1}\left(H^{\mathcal{I}}(x)\right) \leq 1 - \sigma_{a,b}^{-1}\left(C_j^{\mathcal{I}}(x)\right) + \sigma_{a,b}^{-1}\left(H^{\mathcal{I}}(x)\right).$$

Thus, we have $D_j^{\mathcal{I}}(y) \geq C_j^{\mathcal{I}}(x)$, which finishes the proof.

This concludes the proof of Lemma 25.

Appendix B. Proofs for Section 4

Lemma 28. *Let $\otimes$ be a continuous t-norm without zero divisors. For all $x, y \in [0, 1]$ and all non-empty sets $X \subseteq [0, 1]$ it holds that*

1. $\mathbb{1}(\ominus x) = \ominus \mathbb{1}(x)$,
2. $\mathbb{1}(x \otimes y) = \mathbb{1}(x) \otimes \mathbb{1}(y)$,
3. $\mathbb{1}(x \oplus y) = \mathbb{1}(x) \oplus \mathbb{1}(y)$,
4. $\mathbb{1}(x \Rightarrow y) = \mathbb{1}(x) \Rightarrow \mathbb{1}(y)$,
5. $\mathbb{1}(\sup\{x \mid x \in X\}) = \sup\{\mathbb{1}(x) \mid x \in X\}$, *and*
6. *if* $\min\{x \mid x \in X\}$ *exists, then* $\mathbb{1}(\min\{x \mid x \in X\}) = \min\{\mathbb{1}(x) \mid x \in X\}$.

Proof. It holds that $\mathbb{1}(\ominus x) = \ominus\ominus\ominus x = \ominus \mathbb{1}(x)$, which proves 1. Since $\otimes$ does not have zero divisors, it holds that $x \otimes y = 0$ iff $x = 0$ or $y = 0$. This yields $\mathbb{1}(x \otimes y) = 0$ iff $\mathbb{1}(x) = 0$ or $\mathbb{1}(y) = 0$. Because there are no zero divisors, this shows that

$$\mathbb{1}(x \otimes y) = 0 \quad \text{iff} \quad \mathbb{1}(x) \otimes \mathbb{1}(y) = 0.$$

Since both $\mathbb{1}(x \otimes y)$ and $\mathbb{1}(x) \otimes \mathbb{1}(y)$ can only have the values 0 or 1, this is sufficient to prove the second statement. Since 0 is a unit for $\oplus$, we have $x \oplus y = 0$ iff $x = y = 0$, and thus $\mathbb{1}(x \oplus y) = 0$ holds iff $\mathbb{1}(x) \oplus \mathbb{1}(y) = 0$. This suffices to prove 3. We use Proposition 4 to prove 4:

$$\mathbb{1}(x \Rightarrow y) = \begin{cases} 1 & \text{if } x = 0 \text{ or } y > 0 \\ 0 & \text{if } x > 0 \text{ and } y = 0 \end{cases}$$
$$= \begin{cases} 1 & \text{if } \mathbb{1}(x) = 0 \text{ or } \mathbb{1}(y) = 1 \\ 0 & \text{if } \mathbb{1}(x) = 1 \text{ and } \mathbb{1}(y) = 0 \end{cases}$$
$$= \mathbb{1}(x) \Rightarrow \mathbb{1}(y).$$

To prove 5, observe that $\sup X = 0$ iff $X = \{0\}$, which yields

$$\mathbb{1}(\sup X) = 0 \Leftrightarrow \sup X = 0 \Leftrightarrow X = \{0\} \Leftrightarrow \{\mathbb{1}(x) \mid x \in X\} = \{0\} \Leftrightarrow \sup\{\mathbb{1}(x) \mid x \in X\} = 0.$$

Assume now that $\min X = x_{\min}$ exists. Then we have

$$\mathbb{1}(\min X) = 0 \Leftrightarrow x_{\min} = 0 \Leftrightarrow 0 \in \big\{\mathbb{1}(x) \,\big|\, x \in X\big\} \Leftrightarrow \min\big\{\mathbb{1}(x) \,\big|\, x \in X\big\} = 0,$$

which proves 6. $\quad\square$

Lemma 29. *For all complex concepts C, complex roles s, and $x, y \in \Delta^{\mathcal{I}}$, it holds that $C^{\mathcal{J}}(x) = \mathbb{1}(C^{\mathcal{I}}(x))$ and $s^{\mathcal{J}}(x, y) = \mathbb{1}(s^{\mathcal{I}}(x, y))$.*

Proof. We first prove the claim for complex roles by induction over the structure of s. For role names, this follows directly from the definition of $\mathcal{J}$. If s is the universal role u, then $u^{\mathcal{J}}(x, y) = 1 = \mathbb{1}(1) = \mathbb{1}(u^{\mathcal{I}}(x, y))$ holds for all $x, y \in \Delta^{\mathcal{I}}$. If $s = t^{-}$, then we have $s^{\mathcal{J}}(x, y) = t^{\mathcal{J}}(y, x) = \mathbb{1}(t^{\mathcal{I}}(y, x)) = \mathbb{1}(s^{\mathcal{I}}(x, y))$ by induction. Finally, if $s = \boxminus t$, then $(\boxminus t)^{\mathcal{J}}(x, y) = \ominus t^{\mathcal{J}}(x, y) = \ominus\mathbb{1}(t^{\mathcal{I}}(x, y)) = \mathbb{1}((\boxminus t)^{\mathcal{I}}(x, y))$ by Lemma 28.

For the complex concepts, we also use induction over the structure of C. The claim obviously holds for $C = \bot$ and $C = \top$. For $C = A \in \mathsf{N_C}$ it follows immediately from the definition of $\mathcal{J}$. It also holds for $C = \{a\}$ with $a \in \mathsf{N_I}$, because $\{a\}^{\mathcal{I}}(x)$ can only take the values 0 or 1 for all $x \in \Delta^{\mathcal{I}}$. Furthermore, we have $(\exists s.\mathsf{Self})^{\mathcal{J}}(x) = s^{\mathcal{J}}(x, x) = \mathbb{1}(s^{\mathcal{I}}(x, x)) = \mathbb{1}((\exists s.\mathsf{Self})^{\mathcal{I}}(x))$ by the claim for complex roles.

Assume now that the concepts D and E satisfy $D^{\mathcal{J}}(x) = \mathbb{1}(D^{\mathcal{I}}(x))$ and $E^{\mathcal{J}}(x) = \mathbb{1}(E^{\mathcal{I}}(x))$ for all $x \in \Delta^{\mathcal{I}}$. For the case of $C = D \sqcap E$, Lemma 28 yields that for all $x \in \Delta^{\mathcal{I}}$

$$\begin{aligned} C^{\mathcal{J}}(x) = D^{\mathcal{J}}(x) \otimes E^{\mathcal{J}}(x) &= \mathbb{1}\big(D^{\mathcal{I}}(x)\big) \otimes \mathbb{1}\big(E^{\mathcal{I}}(x)\big) \\ &= \mathbb{1}\big(D^{\mathcal{I}}(x) \otimes E^{\mathcal{I}}(x)\big) = \mathbb{1}\big(C^{\mathcal{I}}(x)\big). \end{aligned}$$

Likewise, the compatibility of $\mathbb{1}$ with the t-conorm, residuum, and residual negation entails the result for the cases $C = D \sqcup E$, $C = D \to E$, and $C = \boxminus D$.

For $C = {\geq}n\,s.D$, where s is a complex role and $n \in \mathbb{N}$, we obtain

$$\begin{aligned} \mathbb{1}\big(C^{\mathcal{I}}(x)\big) &= \mathbb{1}\big(\sup\big\{p \in [0, 1] \,\big|\, \big|\{y \in \Delta^{\mathcal{I}} \mid s^{\mathcal{I}}(x, y) \otimes D^{\mathcal{I}}(y) \geq p\}\big| \geq n\big\}\big) \\ &= \sup\big\{\mathbb{1}(p) \,\big|\, p \in [0, 1], \big|\{y \in \Delta^{\mathcal{I}} \mid s^{\mathcal{I}}(x, y) \otimes D^{\mathcal{I}}(y) \geq p\}\big| \geq n\big\} \\ &= \begin{cases} 1 & \text{if } |\{y \in \Delta^{\mathcal{I}} \mid s^{\mathcal{I}}(x, y) \otimes D^{\mathcal{I}}(y) > 0\}| \geq n \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 1 & \text{if } |\{y \in \Delta^{\mathcal{I}} \mid s^{\mathcal{J}}(x, y) \otimes D^{\mathcal{J}}(y) = 1\}| \geq n \\ 0 & \text{otherwise} \end{cases} \\ &= \sup\big\{p \in [0, 1] \,\big|\, \big|\{y \in \Delta^{\mathcal{I}} \mid s^{\mathcal{J}}(x, y) \otimes D^{\mathcal{J}}(y) \geq p\}\big| \geq n\big\} \\ &= C^{\mathcal{J}}(x) \end{aligned}$$

from Lemma 28 and the claim for complex roles. The claim for existential restrictions follows from rewriting $\exists r.C$ as ${\geq}1\,r.C$ and observing that the restriction to witnessed models is irrelevant in this situation.

If $C = {\leq}n\,s.D$, we have

$$\begin{aligned} \mathbb{1}\big(C^{\mathcal{I}}(x)\big) &= \mathbb{1}\big(\sup\big\{p \in [0, 1] \,\big|\, \big|\{y \in \Delta^{\mathcal{I}} \mid s^{\mathcal{I}}(x, y) \Rightarrow \ominus D^{\mathcal{I}}(y) < p\}\big| \leq n\big\}\big) \\ &= \mathbb{1}\big(\sup\big\{p \in [0, 1] \,\big|\, \big|\{y \in \Delta^{\mathcal{I}} \mid s^{\mathcal{I}}(x, y) \otimes D^{\mathcal{I}}(y) > \ominus p\}\big| \leq n\big\}\big) \\ &= \begin{cases} 1 & \text{if } |\{y \in \Delta^{\mathcal{I}} \mid s^{\mathcal{I}}(x, y) \otimes D^{\mathcal{I}}(y) > 0\}| \leq n \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 1 & \text{if } |\{y \in \Delta^{\mathcal{I}} \mid s^{\mathcal{J}}(x, y) \otimes D^{\mathcal{J}}(y) > 0\}| \leq n \\ 0 & \text{otherwise} \end{cases} \\ &= \sup\big\{p \in [0, 1] \,\big|\, \big|\{y \in \Delta^{\mathcal{I}} \mid s^{\mathcal{J}}(x, y) \otimes D^{\mathcal{J}}(y) > \ominus p\}\big| \leq n\big\} \\ &= \sup\big\{p \in [0, 1] \,\big|\, \big|\{y \in \Delta^{\mathcal{I}} \mid s^{\mathcal{J}}(x, y) \Rightarrow \ominus D^{\mathcal{J}}(y) < p\}\big| \leq n\big\} \\ &= C^{\mathcal{J}}(x) \end{aligned}$$

by the same arguments.

Finally, if $C = \forall s.D$, we have

$$\mathbb{1}\big(C^{\mathcal{I}}(x)\big) = \mathbb{1}\Big(\inf_{y \in \Delta^{\mathcal{I}}} s^{\mathcal{I}}(x, y) \Rightarrow D^{\mathcal{I}}(y)\Big).$$

Since $\mathcal{I}$ is witnessed, there must be some $y_0 \in \Delta^{\mathcal{I}}$ such that

$$s^{\mathcal{I}}(x, y_0) \Rightarrow D^{\mathcal{I}}(y_0) = \inf_{y \in \Delta^{\mathcal{I}}} s^{\mathcal{I}}(x, y) \Rightarrow D^{\mathcal{I}}(y);$$

that is, $\min_{y \in \Delta^{\mathcal{I}}} s^{\mathcal{I}}(x, y) \Rightarrow D^{\mathcal{I}}(y)$ exists. Thus, as in the above cases, we can apply Lemma 28 and the claim for complex roles to derive that $\mathbb{1}(C^{\mathcal{I}}(x)) = C^{\mathcal{J}}(x)$. □

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